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Stochastic functional differential equations on manifolds

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Abstract. In this paper, we study stochastic functional differential equations (sfde's) whose solutions are constrained to live on a smooth compact Riemannian manifold. We prove the existence and uniqueness of solutions to such sfde's. We consider examples of geometrical sfde's and establish the smooth dependence of the solution on finite-dimensional parameters.

1. Introduction

The theory of stochastic functional differential equations (sfde's) in Euclidean space was developed by Itô and Nisio ([I.N]), Kushner ([Ku]), Mizel and Trutzer ([M.T]), Mohammed ([Mo_2], [Mo_3]) and Mohammed and Scheutzow ([Mo.S_1], [Mo.S_2]). The purpose of this work is to constrain solutions of such sfde's to stay on a smooth compact submanifold of Euclidean space, or more generally, to construct solutions of sfde's which live on any smooth compact Riemannian manifold M. Indeed, we wish to define and study sfde's on M of the form

$$dx_t = F(t, x) \circ dw_t, \quad t > 0,$$

and driven by Brownian motion $w_t \in \mathbf{R}^k$, on a probability space $(\Omega, \mathscr{F}, P)$.

The main difficulty in this study is that the tangent space along a solution path is random, unlike in the flat case. To elaborate on this question, we shall designate entities pertaining to the "curved" manifold M by the subscript c and the corresponding ones in "flat" space by the subscript f. We shall use this notation throughout the

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article. Let $C([-\delta, 0], M)$ be the space of all continuous paths $\gamma_{c,\cdot} : [-\delta, 0] \to M$. Denote by $e_s : C([-\delta, 0], M) \to M$, $s \in [-\delta, 0]$, the family of evaluation maps

$$e_s(\gamma_{c,\cdot}) := \gamma_{c,s}, \quad \gamma_{c,\cdot} \in C([-\delta, 0], M).$$

Let $T(M)$ be the tangent bundle of M and denote by $e_{-\delta}^* T(M)$ and $e_0^* T(M)$ the pullback vector bundles of $T(M)$ over $C([-\delta, 0], M)$ by the evaluation maps $e_{-\delta}$ and e_0, respectively. A deterministic functional differential equation (fde) is an (everywhere defined) section of the bundle $e_0^* T(M) \to C([-\delta, 0], M)$. Given the Riemannian structure on M, deterministic parallel transport is well defined everywhere on the space of differentiable paths on M. Therefore, if the evaluations e_s are restricted to differentiable paths on M, then we can identify the pull-backs $e_{-\delta}^* T(M)$ and $e_0^* T(M)$ by using deterministic parallel transport $\tau_{0,-\delta}(\gamma_{c,\cdot})$ from $\gamma_{c,-\delta}$ to $\gamma_{c,0}$ along each differentiable path $\gamma_{c,\cdot} : [-\delta, 0] \to M$.

However, the above setting is inadequate in the stochastic case. In this case, one may wish to "randomize" the path $\gamma_{c,\cdot}$ by giving $C([-\delta, 0], M)$ a semimartingale measure. Under such a measure the set of differentiable paths is negligible. If the noise w is one-dimensional, one may define a stochastic functional differential equation (sfde) as an almost everywhere defined section of the pull-back bundle $e_0^* T(M)$ over $C([-\delta, 0], M)$. An identification of the bundles $e_{-\delta}^* T(M)$ and $e_0^* T(M)$ is effected by *stochastic parallel transport* along semimartingale paths, which is almost surely defined with respect to the underlying semimartingale measure. These considerations show that it is necessary to change the function space of initial paths in order to study sfde's on manifolds. We will therefore work in a space of semimartingales from $[-\delta, 0]$ into M, with a convenient topology and with a filtration depending on time.

Deterministic functional differential equations on Hilbert manifolds and the existence of their semiflows were studied by Mohammed in [Mo_1]. The present work is motivated in part by a conjecture in [Mo_1] (Chapter 5, p. 143).

Now let us recall some aspects of the theory of sfde's on flat space. The state space is the set of continuous paths $C([-\delta, 0], R^d)$ or some other Banach space of paths on $\mathbf{R}^d$, and the trajectory of the sfde constitutes an infinite dimensional Feller process on the state space. The problem of existence of a stochastic semiflow was studied by Mohammed [Mo_3], and Mohammed and Scheutzow ([Mo.S_1], [Mo.S_2]). See [Mo_3] and the references therein. In this paper, we will not address this issue for sfde's on manifolds.

A theory of differential equations in a space of semimartingales on a manifold was developed by B. Driver ([Dr], [Cr], [E.S], [Hs], [No], [Le_1], [Ci.Cr], [Li]). It is useful to compare our theory with that of Driver:

- Driver's theory yields a deterministic flow on the space of semimartingales on the manifold. Some of the techniques which we use in this paper are similar to those used in the study of Driver's flow. For instance, we use stochastic parallel transport to "pull back" the calculus on the manifold onto the tangent space at the starting point of the initial semimartingale. This gives a sfde in a linear space of semimartingales with values in the tangent space $T_x(M)$ at a given fixed point $x \in M$. In the *delay* case when the coeffcient of the equation does not depend on the present state of the solution, the structure of our equation is simpler in some sense than Driver's. In this case, our formulas are less involved than their counterparts in Driver's theory, because it is not necessary to differentiate the stochastic parallel transport with respect to the semimartingale path.

- Throughout its evolution, Driver's flow maintains the same filtration as that of the initial semimartingale process. In our sfde, the state of the trajectory at any time is adapted to a different filtration than that of the initial process.
- In Driver's theory, there is only one source of randomness, which arises from stochastic parallel transport along Brownian paths. Our theory involves two sources of randomness: One which arises from the initial semimartingale (via stochastic parallel transport), and the other from the driving Brownian motion.
- Wiener measure on the manifold is quasi-invariant under Driver's flow; that is, the law of the solution of Driver's ode at any subsequent time is absolutely continuous with respect to that of the initial Brownian motion on the manifold ([Dr]). This is not the case in our context. For a sfde on a manifold, one does not expect the law of the solution at any given time to be absolutely continuous with respect to the law of the initial semimartingale.

The present article falls into two parts.

In the first part, we define a large class of sfde's on the manifold. Using parallel transport, we "pull back" the sfde onto the tangent space at the starting point of the initial semimartingale. This procedure yields a non geometric sfde defined on flat path space, which can then be solved via Picard's iteration method. In this part, we study a geometrical example of a stochastic delay equation on the manifold, and show that it possesses a Markov property in a suitably defined space of semimartingales.

In the second part, we examine the regularity in the initial semimartingale of the solution of the geometric stochastic delay equation introduced in the first part. The analysis uses the stochastic Chen-Souriau calculus developed by Léandre in [Le_2] and [Le_3]. It turns out that the function space of semimartingales used in the first part does not appear to give smoothness of the solution of the geometric stochastic delay equation in the initial semimartingale. We therefore use a Fréchet space of semimartingales generated by a countable family of semimartingale norms rather than a single norm. The techniques used in this part are similar to those of Léandre [Le_1].

II. A general existence theorem

In this section, we shall define a large class of sfde's on a compact Riemannian manifold. We then state and prove an existence theorem for this class of sfde's.

We begin by fixing notation. Let M be a smooth compact d-dimensional Riemannian manifold, $\delta > 0$ and $T > 0$. Suppose $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq -\delta}, P)$ is a complete filtered probability space satisfying the usual conditions.

Let $w_t,\ t \geq -\delta$, be a k-dimensional Brownian motion on $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq -\delta}, P)$ adapted to the filtration $(\mathscr{F}_t)_{t\geq -\delta}$. Suppose that $w_{-\delta} = 0$.

For any (finite-dimensional) manifold N, we will denote by $L^0(\Omega, N)$ the space of all N-valued ($\mathscr{F}$-measurable) random variables $\Omega \to N$, given the topology of convergence in probability.

If N is any smooth finite-dimensional Riemannian manifold and $x \in N$, denote by $\mathscr{S}([-\delta, T], N; -\delta, x)$ the space of all N-valued $(\mathscr{F}_t)_{t \geq -\delta}$-adapted continuous semimartingales $\gamma : [-\delta, T] \times \Omega \to N$ with $\gamma_{-\delta} = x$.

Fix $x \in M$. Define the *Itô map* by the association

$$\mathscr{S}([-\delta, T], M; -\delta, x) \ni \gamma_{c,\cdot} \mapsto \gamma_{f,\cdot} \in \mathscr{S}([-\delta, T], T_x(M); -\delta, 0)$$

where

$$\begin{cases} d\gamma_{f,t} = \tau_{t,-\delta}^{-1}(\gamma_{c,\cdot}) \circ d\gamma_{c,t}, & -\delta < t < T, \\ \gamma_{f,-\delta} = 0. \end{cases} \tag{2.1}$$

The differential in the above equation is in the Stratonovich sense, and $\tau_{t,-\delta}(\gamma_{c,\cdot})$ denotes stochastic parallel transport from $x = \gamma_{c,-\delta}$ to $\gamma_{c,t}$ along the semimartingale $\gamma_{c,\cdot}$ ([E.E], [Em]). Observe that the Itô map is a bijection.

Denote by $\mathscr{S}_{2,f}^T$ the Hilbert space of all semimartingales $\gamma_{f,\cdot} \in \mathscr{S}([-\delta, T], T_x(M); -\delta, 0)$ such that

$$\gamma_{f,t} = \int_{-\delta}^{t} A_s\, dw_s + \int_{-\delta}^{t} B_s\, ds, \quad -\delta \leq t \leq T, \tag{2.2}$$

and

$$\|\gamma_{f,\cdot}\|_2^2 := E[\int_{-\delta}^{T} \|A_s\|^2\, ds] + E[\int_{-\delta}^{T} |B_s|^2\, ds] < \infty, \tag{2.3}$$

where $A : [-\delta, T] \times \Omega \to L(\mathbf{R}^k, T_x(M))$ and $B : [-\delta, T] \times \Omega \to T_x(M)$ are adapted, previsible processes. In the sequel, we shall refer to the pair (A, B) as the *characteristics* of $\gamma_{f,\cdot}$ (or $\gamma_{c,\cdot}$). Note that the Hilbert norm $\|\cdot\|_2$ induces a topology on $\mathscr{S}_{2,f}^T$ slightly different from the traditional semimartingale topologies that are often used in stochastic analysis (cf. [D.M]).

Denote by $\mathscr{S}_{2,c}^T$ the image of $\mathscr{S}_{2,f}^T$ under the Itô map with the induced topology.

Let $\gamma_{c,\cdot} \in \mathscr{S}_{2,c}^T$ and fix any $t \in [-\delta, T]$. Set $\gamma_{c,s}^t := \gamma_{c,s\wedge t}$, $s \in [-\delta, T]$. Then $\gamma_{c,\cdot}^t \in \mathscr{S}_{2,c}^T$ and $(\gamma_{c,\cdot}^t)_f = (\gamma_{f,\cdot})^t$.

Consider the evaluation map $e : [0, T] \times \mathscr{S}_{2,c}^T \to L^0(\Omega, M)$ defined by

$$e(t, \gamma_{c,\cdot}) := \gamma_{c,t}, \quad (t, \gamma_{c,\cdot}) \in [0, T] \times \mathscr{S}_{2,c}^T.$$

The tangent bundle $T(M) \to M$ induces the k-frame vector bundle $L(\underline{\mathbf{R}^k}, T(M)) \to M$ whose fiber at each $z \in M$ is given by $L(\underline{\mathbf{R}^k}, T(M))_z := L(\mathbf{R}^k, T_z(M))$. Furthermore, the frame bundle $L(\underline{\mathbf{R}^k}, T(M)) \to M$ induces a vector bundle $L^0(\Omega, L(\underline{\mathbf{R}^k}, T(M))) \to L^0(\Omega, M)$ whose fiber $L^0(\Omega, L(\underline{\mathbf{R}^k}, T(M)))_Z$ over each $Z \in L^0(\Omega, M)$ is given by

$$L^0(\Omega, L(\underline{\mathbf{R}^k}, T(M)))_Z := \{Y : Y(\omega) \in L(\mathbf{R}^k, T_{Z(\omega)}(M)) \text{ a.a. } \omega \in \Omega\}.$$

Denote by $e^*L^0(\Omega, L(\underline{\mathbf{R}^k}, T(M)))$ the pull-back bundle of $L^0(\Omega, L(\underline{\mathbf{R}^k}, T(M))) \to L^0(\Omega, M)$ by e over $[0, T] \times \mathscr{S}_{2,c}^T$. A section of the bundle $e^*L^0(\Omega, L(\underline{\mathbf{R}^k}, T(M))) \to [0, T] \times \mathscr{S}_{2,c}^T$ is a map $F_c : [0, T] \times \mathscr{S}_{2,c}^T \to L^0(\Omega, L(\underline{\mathbf{R}^k}, T(M)))$ such that $F_c(t, \gamma_{c,\cdot}^t) \in L(\mathbf{R}^k, T_{\gamma_{c,t}}(M))$ for each $(t, \gamma_{c,\cdot}) \in [0, T] \times \mathscr{S}_{2,c}^T$ a.s.. Each such section has a *flat version* $F_f : [0,T] \times \mathscr{S}^T_{2,f} \to L^0(\Omega, L(\mathbf{R}^k, T_x(M)))$ given by

$$F_f(t, \gamma_{f,.}) := \tau^{-1}_{t,-\delta}(\gamma_{c,.}) F_c(t, \gamma_{c,.})$$

for all $(t, \gamma_{f,.}) \in [0,T] \times \mathscr{S}^T_{2,f}$. In the above relation, $\tau^{-1}_{t,-\delta}(\gamma_{c,.})$ denotes stochastic parallel transport of k-linear frames over $T_{\gamma_{c,t}}(M)$ to k-linear frames over $T_{\gamma_{c,-\delta}}(M)$.

A *stochastic functional differential equation (sfde)* on M is a section $F_c : [0,T] \times \mathscr{S}^T_{2,c} \to L^0(\Omega, L(\underline{\mathbf{R}}^k, T(M)))$ of $e^* L^0(\Omega, L(\underline{\mathbf{R}}^k, T(M))) \to [0,T] \times \mathscr{S}^T_{2,c}$ satisfying the following properties:

(i) F_c is "non-anticipating": $F_c(t, \gamma_{c,.}) = F_c(t, \gamma^t_{c,.})$ for all $(t, \gamma_{c,.}) \in [0,T] \times \mathscr{S}^T_{2,c}$, a.s..
(ii) For each $\gamma_{f,.} \in \mathscr{S}^T_{2,f}$, the process $[0,T] \ni t \mapsto F_f(t, \gamma^t_{f,.}) \in L(\mathbf{R}^k, T_x(M))$ is an $(\mathscr{F}_t)_{0 \le t \le T}$-semimartingale.

Consider the Stratonovich sfde

$$\begin{cases} dx_{c,t} = F_c(t, x^t_{c,.}) \circ dw_t, \quad 0 < t < T, \\ x^0_{c,.} = \gamma^0_{c,.} \end{cases} \tag{I.c}$$

In general, the above sfde does not have a solution. In order to establish the existence of a unique solution, we will impose a Lipschitz-type condition on F_c. For this purpose, we will use the Itô map to pullback the sfde (I.c) to an sfde on the flat space $T_x(M)$. This induces the following Stratonovich sfde on $T_x(M)$:

$$\begin{cases} dx_{f,t} = F_f(t, x^t_{f,.}) \circ dw_t, \quad 0 < t < T, \\ x^0_{f,.} = \gamma^0_{f,.} \end{cases} \tag{I.f}$$

where $F_f : [0,T] \times \mathscr{S}^T_{2,f} \to L^0(\Omega, L(\mathbf{R}^k, T_x(M)))$ is the flat version of F_c. In order to establish existence and uniqueness of the solution to (I.c), we will impose "boundedness" and "Lipschitz conditions" on F_c that will be expressed in terms of its flat version F_f. First, we convert (I.f) into the equivalent Itô form

$$\begin{cases} dx_{f,t} = F_f(t, x^t_{f,.}) dw_t + \Delta F_f(t, x^t_{f,.})\, dt, \quad 0 < t < T, \\ x^0_{f,.} = \gamma^0_{f,.} \end{cases} \tag{I.if}$$

In the above sfde, $\Delta F_f(., x^t_{f,.}) : [0,T] \times \mathscr{S}^T_{2,f} \to L^0(\Omega, L(\mathbf{R}^k, T_x(M)))$ is the Stratonovich correction term defined below.

In order to compute the Stratonovich correction terms for our examples, we will use the following notation. For any $\gamma_{f,.} \in \mathscr{S}^T_{2,f}$, define the joint quadratic variation $\langle F_f(., \gamma^{\cdot}_{f,.}), w \rangle$ of the semimartingale $[0,T] \ni t \mapsto F_f(t, \gamma^t_{f,.}) \in L(\mathbf{R}^k, T_x(M))$ and Brownian motion w by setting

$$\langle F_f(., \gamma^{\cdot}_{f,.}), w \rangle_t := \sum_{i=1}^k \langle F_f(., \gamma^{\cdot}_{f,.})(e_i), w^i \rangle_t \in T_x(M), \quad 0 \le t \le T,$$

where $\{e_i\}_{i=1}^k$ is the canonical orthonormal basis for $\mathbf{R}^k$, $w_t = \sum_{i=1}^k w_t^i e_i$, $t \geq 0$, and w^i, $1 \leq i \leq k$, are k independent one-dimensional standard Brownian motions. We now set

$$\Delta F_f(t, x_{f,.}^t) := \frac{1}{2}\langle F_f(., x_{f,.}^{\cdot}), w\rangle_t', \quad t > 0$$

where x_f is the solution of (I.f).

Hypotheses (H).

(i) *Boundedeness*. There exists a deterministic constant C_1 such that

$$|F_f(t, \gamma_{f,.}^t)| + |\Delta F_f(t, \gamma_{f,.}^t)| < C_1 < \infty, \quad \text{a.s.} \tag{2.4}$$

for all $(t, \gamma_{f,.}) \in [0, T] \times \mathscr{S}_{2,f}^T$.

(ii) *Lipschitz condition*. Assume that for each positive real number R there is a positive deterministic constant $L := L(R)$ such that

$$\begin{aligned} E[|F_f(t, \gamma_{f,.}^t) - F_f(t, (\gamma')_{f,.}^t)|^2 + |\Delta F_f(t, \gamma_{f,.}^t) - \Delta F_f(t, (\gamma')_{f,.}^t)|^2] \\ \leq L\|\gamma_{f,.}^t - (\gamma')_{f,.}^t\|_2^2 \end{aligned} \tag{2.5}$$

for all $t \in [0, T]$, and whenever $\gamma_{f,.}, \gamma_{f,.}' \in \mathscr{S}_{2,f}^T$ have characteristics (A, B) and (A', B') (resp.) a.s. bounded by R.

Remark. Assume that the sfde F_c satisfies the *delay condition*

$$F_f(t, \gamma_{f,.}^t) = F_f(t, \gamma_{f,.}^{t-\delta}) \tag{2.6}$$

for all $(t, \gamma_{f,.}) \in [0, T] \times \mathscr{S}_{2,f}^T$. Note that (2.6) is equivalent to

$$F_c(t, \gamma_{c,.}^t) = \tau_{t,t-\delta}(\gamma_{c,.}^t) F_c(t, \gamma_{c,.}^{t-\delta}) \tag{2.6$'$}$$

for all $(t, \gamma_{c,.}) \in [0, T] \times \mathscr{S}_{2,c}^T$. It is easy to see that (2.6) implies that $\langle F_f(., \gamma_{f,.}^{\cdot}), w\rangle(t) = 0$ for all $t \in [0, T]$. Therefore, under the delay condition (2.6), the *Stratonovich* equation $(I.f)$ now coincides with the *Itô* equation:

$$\begin{cases} dx_{f,t} = F_f(t, x_f^{(t-\delta)})\, dw_t, & 0 < t < T, \\ x_f^0 = \gamma_f^0, \end{cases} \tag{2.7}$$

with no correction term! (cf. [Mo3], p. 5). Thus for equation (2.7) one may drop the Stratonovich correction term in (2.4) and (2.5) of Hypotheses (H).

We now give some geometrical examples of sfde's that satisfy Hypotheses (H) above.

Examples. Let X_1, X_2 be smooth sections of the k-frame bundle $L(\underline{\mathbf{R}}^k, T(M)) \to M$. Consider the geometrical sfde's

$$dx_{c,t} = \left\{\int_{t-\delta}^t \tau_{t,s}(x_{c,.}) X_1(x_{c,s}) ds + X_2(x_{c,t})\right\} \circ dw_t, \quad 0 < t < T, \tag{$I.g_1$}$$

$$dx_{c,t} = \tau_{t,t-\delta}(x_{c,.})X_1(x_{c,t-\delta}) \circ dw_t, \quad 0 < t < T, \qquad (I.g_2)$$

with corresponding functionals

$$\begin{cases} F_c^1(t, \gamma_{c,.}) := \int_{t-\delta}^t \tau_{t,s}(\gamma_{c,.})X_1(\gamma_{c,s})\, ds + X_2(\gamma_{c,t}), \\ F_c^2(t, \gamma_{c,.}) := \tau_{t,t-\delta}(\gamma_{c,.})X_1(\gamma_{c,t-\delta}), \\ F_f^1(t, \gamma_{f,.}) = \int_{t-\delta}^t \tau_{-\delta,s}(\gamma_{c,.})X_1(\gamma_{c,s})\, ds + \tau_{-\delta,t}(\gamma_{c,.})X_2(\gamma_{c,t}), \\ F_f^2(t, \gamma_{f,.}) = \tau_{-\delta,t-\delta}(\gamma_{c,.})X_1(\gamma_{c,t-\delta}), \end{cases} \tag{2.8}$$

for $t \in [0, T]$, $\gamma_{c,.} \in \mathscr{S}_{2,c}^T$, $\gamma_{f,.} \in \mathscr{S}_{2,f}^T$. In the above relations, $\tau_{t,s}(x_{c,.})$ denotes stochastic parallel transport along $x_{c,.}$ of k-linear frames over $T_{x_{c,s}}(M)$ to k-linear frames over $T_{x_{c,t}}(M)$.

We will verify that the functionals $F_c^i, i = 1, 2$, are sfde's satisfying hypotheses (H). Since these hypotheses are intrinsic, we may embed M (isometrically) in $R^{d'}$ (where $d' > d$) and extend the Riemannian structure to the whole of $\mathbf{R}^{d'}$ in such a way that the extended Riemannian metric has bounded derivatives of all orders and is uniformly non-degenerate. Extend the Levi-Civita connection on M to a connection on $\mathbf{R}^{d'}$ which preserves the metric on $\mathbf{R}^{d'}$, and with Christoffel symbols having bounded derivatives of all orders. The pair $(\gamma_{c,t}, \tau_{t,-\delta}(\gamma_{c,.}))$ then corresponds to a pathwise continuous process $\hat{x}_t \in \mathbf{R}^{d'} \times \mathbf{R}^{d' \times d'}$ which solves the following Stratonovitch sde:

$$\begin{cases} d\hat{x}_t = \hat{Z}(\hat{x}_t) \circ A_t\, dw_t + \hat{Z}(\hat{x}_t)B_t\, dt, \quad -\delta < t < T, \\ \hat{x}_{-\delta} = (x, Id_{\mathbf{R}^d})(\equiv (x, Id_{T_x(M)})) \end{cases} \tag{2.9}$$

on $\mathbf{R}^{d'} \times \mathbf{R}^{d' \times d'}$, where (A, B) are the characteristics of $\gamma_{c,.}$, with $A_t \in L(\mathbf{R}^k, \mathbf{R}^d)$, $B_t \in \mathbf{R}^d$. The coefficient $\hat{Z} : \mathbf{R}^{d'} \times R^{d' \times d'} \to L(\mathbf{R}^d, \mathbf{R}^{d'} \times \mathbf{R}^{d' \times d'})$ is C^∞ (and hence locally Lipschitz with derivatives of all orders bounded on bounded sets, uniformly in the characteristics (A, B) of $\gamma_{c,.}$).

We next convert (2.9) into Itô form. To do this, let $\{e_i\}_{i=1}^d$ be the standard basis for $\mathbf{R}^d$. Define

$$\hat{Y}^{i,j}(\cdot) := \frac{1}{2}[D\hat{Z}(\cdot) \circ (\hat{Z}(\cdot))](e_i, e_j), \ i, j = 1, \cdots, d.$$

Then, for each $t \in [-\delta, T]$, $(\hat{Y}^{i,j}(\hat{x}_t))_{i,j=1}^d$ may be viewed as a $(d \times d)$-matrix with entries in $\mathbf{R}^{d'} \times \mathbf{R}^{d' \times d'}$. This matrix will also be denoted by $\hat{Y}(\hat{x}_t)$. With this notation, (2.9) takes the *Itô* form

$$\begin{cases} d\hat{x}_t = \hat{Z}(\hat{x}_t)A_t\, dw_t + \mathrm{trace}(\hat{Y}(\hat{x}_t)A_t A_t^*)\, dt + \hat{Z}(\hat{x}_t)B_t\, dt, \quad -\delta < t < T, \\ \hat{x}_{-\delta} = (x, Id_{\mathbf{R}^d}). \end{cases} \tag{2.10}$$

Observe that, by its definition, $\hat{Y}$ is C^∞. The vector fields X_i, $i = 1, 2$, may be extended to smooth vector fields on $\mathbf{R}^{d'}$ with all derivatives globally bounded. These extensions will be denoted by the same symbols.

Note first that

$$\langle F_f^1(., \gamma_{f,.}^{\cdot}), w\rangle_t = \langle \tau_{-\delta,.}(\gamma_{c,.})X_2(\gamma_{c,.}), w\rangle_t, \quad \langle F_f^2(., \gamma_{f,.}^{\cdot}), w\rangle_t = 0,$$

for all $t \in [0, T]$. Denote by $p_2 : \mathbf{R}^{d'} \times \mathbf{R}^{d'\times d'} \to \mathbf{R}^{d'\times d'}$ the projection of $\mathbf{R}^{d'} \times \mathbf{R}^{d'\times d'}$ onto the second factor. If x_f is the solution of the sfde

$$\begin{cases} dx_{f,t} = F_f^1(t, x_{f,.}^t) \circ dw_t, \quad 0 < t < T, \\ x_{f,.}^0 = \gamma_{f,.}^0, \end{cases} \tag{I.f.1}$$

then an application of Itô's formula yields the following expression for the Stratonovich correction term:

$$\begin{cases} \Delta F_f^1(t, x_{f,.}^t) = \dfrac{1}{2}\displaystyle\sum_{i=1}^{k}\Big\{(p_2 \circ \hat{Z})(x_{c,t}, \tau_{-\delta,t}(x_{c,.}))F_f^1(t, x_{f,.}^t)(e_i)X_2(x_{c,t})(e_i) \\ \qquad\qquad + \tau_{-\delta,t}(x_{c,.})DX_2(x_{c,t})\tau_{t,-\delta}(x_{c,.})F_f^1(t, x_{f,.}^t)(e_i)\Big\}. \end{cases} \tag{2.11}$$

The above relation together with (2.8) immediately implies that F_f^i, $i = 1, 2$, satisfy Hypothesis (H)(i). This is because the vector fields X_i, $i = 1, 2$, are smooth, M is compact and stochastic parallel transport is a rotation on frames.

It remains to check that F_f^i, $i = 1, 2$, in (2.8) satisfy H(ii). For this we need to examine the Lipschitz dependence of the solution of (2.9) on the characteristics (A, B) of the path $\gamma_{c,.}$ In (2.9), we will indicate by $\hat{x}(A, B)$ the dependence of the solution on the characteristics (A, B) of $\gamma_{c,.}$. In the proof of the next lemma and the rest of the paper, we will denote by C_i, $i = 1, 2, 3, \cdots$, generic deterministic positive constants.

Lemma II.1. *In the sde* (2.9)*, suppose* (A, B), (A', B') *are such that there is a positive deterministic constant* R *where* $\|A_t\| + |B_t| + \|A_t'\| + |B_t'| \le R$ *almost surely for all* $t \in [-\delta, T]$. *Then there exists a positive constant* $K := K(R)$ *such that*

$$E[\sup_{-\delta\le s\le t} |\hat{x}_s(A, B) - \hat{x}_s(A', B')|^2]$$

$$\le K E[\int_{-\delta}^{t} (\|A_s - A_s'\|^2 + |B_s - B_s'|^2)ds] \tag{2.12}$$

for all $t \in [-\delta, T]$.

Proof. Let the characteristics (A, B), (A', B') of $\gamma_{c,.}$, $\gamma_{c,.}'$ satisfy the hypotheses of the lemma. Then by (2.10) we have

$$
\begin{aligned}
&d\,\hat{x}_t(A,B) - d\hat{x}_t(A',B') \\
&\quad = \hat{Z}(\hat{x}_t(A,B))(A_t - A'_t)dw_t + (\hat{Z}(\hat{x}_t(A,B)) \\
&\quad - \hat{Z}(\hat{x}_t(A',B')))A'_t dw_t + \text{trace}[\hat{Y}(\hat{x}_t(A,B))\{A_t A_t^* - A'_t(A'_t)^*\}]dt \\
&\quad + \text{trace}[\{\hat{Y}(\hat{x}_t(A,B)) - \hat{Y}(\hat{x}_t(A',B'))\}A'_t(A'_t)^*]dt \\
&\quad + \hat{Z}(\hat{x}_t(A,B))(B_t - B'_t)dt + (\hat{Z}(\hat{x}_t(A,B)) - \hat{Z}(\hat{x}_t(A',B')))B'_t\,dt
\end{aligned} \tag{2.13}
$$

for all $t \in [-\delta, T]$. Now by compactness of M and the orthogonality of stochastic parallel transport, it follows that there is a positive deterministic constant $C_1 := C_1(R)$ (independent of (A, B)) such that whenever $\|A_t\| + |B_t| \le R$ a.s. for all $t \in [-\delta, T]$, then

$$
|\hat{x}_t(A,B)| + |\hat{Z}(\hat{x}_t(A,B))| + |\hat{Y}(\hat{x}_t(A,B))| \le C_1
$$

for all $t \in [-\delta, T]$. Since $\|A_t\| + \|A'_t\|$ is a.s. uniformly bounded in $t \in [0, T]$ by R, then $\|A_t A_t^* - A'_t(A'_t)^*\| \le R\|A_t - A'_t\|$ a.s. for all $t \in [0, T]$. Using (2.12), Burkholder's inequality, the uniform boundedness of $\|A_.\|$, $\|A'_.\|$, $|B'_.|$ and the fact that $\hat{Y}$, $\hat{Z}$ are Lipschitz on bounded sets, it is not hard to see that

$$
\begin{aligned}
&E[\sup_{-\delta \le s \le t} |\hat{x}_s(A,B) - \hat{x}_s(A',B')|^2] \\
&\quad \le C_2 E[\int_{-\delta}^{t} (\|A_s - A'_s\|^2 + |B_s - B'_s|^2)ds] \\
&\qquad + C_3 \int_{-\delta}^{t} E[\sup_{-\delta \le s \le u} |\hat{x}_s(A,B) - \hat{x}_s(A',B')|^2]\,du
\end{aligned}
$$

for all $t \in [-\delta, T]$. The conclusion of the lemma now follows from the above inequality and Gronwall's lemma. □

We now complete the proof of the local Lipschitz property (H)(ii) for F_f^i, $i = 1, 2$. We give the proof only for F_f^1; the corresponding argument for F_f^2 is similar and is left to the reader. Let $\gamma_{f,.}, \gamma'_{f,.} \in S_{2,f}^T$ have characteristics (A, B), (A', B') a.s. bounded by a deterministic constant R. Then $\Delta F_f^1(t, \gamma_{f,.}^t)$ is given by an expression similar to the right-hand-side of (2.11) with x_c, x_f replaced by γ_c, γ_f. Now by the Lipschitz property of X_2 and Lemma II.1, one gets

$$
\begin{aligned}
E|X_2(\gamma_{c,t}) - X_2(\gamma'_{c,t})|^2 &\le C_4 E[\int_{-\delta}^{t} (\|A_s - A'_s\|^2 + |B_s - B'_s|^2)ds] \\
&= C_4 \|\gamma_{f,.}^t - (\gamma')_{f,.}^t\|_2^2
\end{aligned} \tag{2.14}
$$

and

$$
E|\tau_{-\delta,t}(\gamma_{c,.}) - \tau_{-\delta,t}(\gamma'_{c,.})|^2 \le C_5 \|\gamma_{f,.}^t - (\gamma')_{f,.}^t\|_2^2 \tag{2.15}
$$

for all $t \in [-\delta, T]$. Using the boundedness of X_i, $i = 1, 2$, $\tau_{-\delta,s}(\gamma_{c,.})$, (2.14) and (2.15), it follows from (2.8) that

$$
E|F_f^1(t, \gamma_{f,.}^t) - F_f^1(t, (\gamma')_{f,.}^t)|^2 \le C_6 \|\gamma_{f,.}^t - (\gamma')_{f,.}^t\|_2^2 \tag{2.16}
$$

for all $t \in [-\delta, T]$. Finally, use the representation (2.11) coupled with the Lipschitz properties (2.14), (2.15) and (2.16) in order to obtain

$$E|\Delta F_f^1(t, \gamma_{f,.}^t) - \Delta F_f^1(t, (\gamma')_{f,.}^t)|^2 \leq C_7 \|\gamma_{f,.}^t - (\gamma')_{f,.}^t\|_2^2 \tag{2.17}$$

for all $t \in [0, T]$. The last inequality and (2.16) imply that F_f^1 satisfies H(ii). This shows that our geometrical examples $(I.g_1)$, $(I.g_2)$ satisfy Hypotheses (H).

We now state the main theorem of this section.

Theorem II.2 *Assume that the sfde* $(I.c)$ *satisfies Hypotheses (H). Suppose that* $\gamma_{c,.}^0 \in \mathscr{S}_{2,c}^0$ *has characteristics* (A_t, B_t), $t \in [-\delta, 0]$, *which are adapted and almost surely bounded by a deterministic constant* $C > 0$. *Then the sfde* $(I.c)$ *has a unique global solution* $x_{c,.}$ *such that* $x_{c,.}|[-\delta, T] \in \mathscr{S}_{2,c}^T$ *for every* $T > 0$.

Proof. It is sufficient to prove existence and uniqueness of the solution to the flat Itô sfde $(I.if)$. To do this, we use successive approximations. Define the sequence $\{x_{f,.}^n\}_{n=1}^\infty \subset \mathscr{S}_{2,f}^T$ inductively by setting $x_{f,.}^1 := \gamma_{f,.}^0$, and

$$\begin{cases} dx_{f,t}^{n+1} := F_f(t, x_{f,.}^{t,n})dw_t + \Delta F_f(t, x_{f,.}^{t,n})\,dt, \quad 0 < t < T, \\ x_{f,.}^{0,n+1} := \gamma_{f,.}^0 \end{cases} \tag{2.18}$$

for all $n \geq 2$. By Hypothesis (H)(i), the characteristics of each $x_{f,.}^n$ are a.s. bounded by a deterministic constant independent of $t \in [-\delta, T]$ and n. From (2.18) and Hypothesis (H)(ii), it is easy to see that

$$\|x_{f,.}^{t,n+1} - x_{f,.}^{t,n}\|_2^2 \leq C_8 \int_0^t \|x_{f,.}^{s,n} - x_{f,.}^{s,n-1}\|_2^2 ds, \quad t \in [0, T], \; n \geq 1. \tag{2.19}$$

Therefore, by induction on n, we obtain

$$\|x_{f,.}^{t,n+1} - x_{f,.}^{t,n}\|_2^2 \leq \frac{C_8^n t^n}{n!} \tag{2.20}$$

for all $n \geq 1$ and all $t \in [0, T]$. This shows that $\{x_{f,.}^n\}_{n=1}^\infty$ is a Cauchy sequence in $\mathscr{S}_{2,f}^T$ which coverges to a solution $x_{f,.}$ of (2.5). By the Itô map, this gives a solution of $(I.c)$ in $\mathscr{S}_{2,c}^T$.

It remains to show uniqueness. Suppose that there are two solutions $x_{f,.}^1$ and $x_{f,.}^2$ of $(I.f)$. By Hypothesis (H)(i), the characteristics of these two solutions are almost surely bounded. Therefore the relations

$$\begin{cases} dx_{f,t}^1 = F_f(t, x_{f,.}^{1,t})dw_t + \Delta F_f(t, x_{f,.}^{1,t})\,dt, \quad 0 < t < T, \\ dx_{f,t}^2 = F_f(t, x_{f,.}^{2,t})dw_t + \Delta F_f(t, x_{f,.}^{2,t})\,dt, \quad 0 < t < T, \\ x_{f,0}^1 = x_{f,0}^2 = \gamma_f^0, \end{cases} \tag{2.21}$$

together with (H)(ii) imply that

$$\|x_{f,.}^{1,t} - x_{f,.}^{2,t}\|_2^2 \leq C_9 \int_0^t \|x_{f,.}^{1,s} - x_{f,.}^{2,s}\|_2^2 ds, \quad 0 < t < T. \tag{2.22}$$

This shows that $\|x_{f,.}^{1,t} - x_{f,.}^{2,t}\|_2^2 = 0$ for all $t \in [0, T]$, and uniqueness follows. □

Remark. For a sfde F_c satisfying the delay condition (2.6), one can prove existence of a solution to (I.c) by using forward steps of length δ. In particular, given $\gamma^0_{c,.} \in S^T_{2,c}$, we write

$$\begin{aligned} x_{f,t} =& \gamma^0_{f,t}, \quad -\delta \le t \le 0, \\ x_{f,t} =& \gamma^0_{f,0} + \int_0^t F^1_f(u, \gamma^{0,u-\delta}_{f,.})\, dw_u, \quad 0 < t \le \delta, \\ x_{f,t} =& x_{f,\delta} + \int_\delta^t F^1_f(u, x^{u-\delta}_{f,.})\, dw_u, \quad \delta < t \le 2\delta, \end{aligned}$$

and similarly for the delay periods $[2\delta, 3\delta], [3\delta, 4\delta], \cdots$. Note that this procedure automatically guarantees uniqueness of the solution to the sfde (I.c) *without* the Lipschitz condition (H)(ii).

The following result shows that the solution of $(I.c)$ (or $(I.f)$) depends in a Lipschitz manner on sets of initial paths whose characteristics are almost surely bounded by a deterministic constant.

Theorem II.3. *Assume Hypotheses (H). Let $\gamma^0_{f,.}, (\gamma')^0_{f,.} \in \mathscr{S}^T_{2,f}$ have characteristics $(A, B), (A', B')$ that are a.s. uniformly bounded on $[-\delta, 0]$ by a positive deterministic constant R. Denote by $x_{f,.}(\gamma^0_{f,.}),\ x_{f,.}((\gamma')^0_{f,.})$ the solutions of the sfde $(I.f)$ with initial states $\gamma^0_{f,.}$ and $(\gamma')^0_{f,.}$ respectively. Then there is a positive constant $C := C(R)$ such that*

$$\|x_{f,.}(\gamma^0_{f,.}) - x_{f,.}((\gamma')^0_{f,.})\|^2_2 \le C\|\gamma^0_{f,.} - (\gamma')^0_{f,.}\|^2_2 \tag{2.23}$$

Proof. Using $(I.if)$, Burkholder's inequality and property (H)(ii), we easily see that

$$\begin{aligned} \|x^t_{f,.}(\gamma^0_{f,.}) - x^t_{f,.}((\gamma')^0_{f,.})\|^2_2 \le \|\gamma^0_{f,.} - (\gamma')^0_{f,.}\|^2_2 \\ +C_{10}\int_0^t \|x^s_{f,.}(\gamma^0_{f,.}) - x^s_{f,.}((\gamma')^0_{f,.})\|^2_2 ds \end{aligned} \tag{2.24}$$

for all $t \in [0, T]$. The conclusion of the theorem now follows from the above inequality and Gronwall's lemma. □

We will conclude this section by a discussion of a type of Markov property for solutions of the geometrical example $(I.g_1)$. To do this, we will first parametrize the flat sfde $(I.f)$ with the initial point $z \in M$; that is, consider a family of flat sfde's $F_f(\cdot, \cdot, z) : [0, T] \times \mathscr{S}^T_{2,f}(z) \to L^0(\Omega, T_z(M)),\ z \in M$, where $\mathscr{S}^T_{2,f}(z)$ denotes the set of all semimartingales $\gamma_{f,.}(z)$ in $T_z(M)$ satisfying $\gamma_{f,-\delta}(z) = 0$ (or $\gamma_{c,-\delta}(z) = z$) and (2.3). Now "randomize" z by introducing a random variable $Z \in L^0(\Omega, M)$ independent of $w_t, t \ge -\delta$. Then consider the equation

$$\begin{cases} dx_{f,t}(Z) = F_f(t, x^t_{f,.}(Z), Z) \circ dw_t, \quad t \ge 0 \\ x^0_{f,.}(Z) = \gamma^0_{f,.}(Z) \in \mathscr{S}^T_{2,f}(Z). \end{cases} \tag{2.25}$$

Note the starting condition $x_{c,-\delta}(Z) = Z$. Assume that $F_f(\cdot, \cdot, z)$ satisfies Hypotheses (H)(i)(ii) uniformly in $z \in M$. If we fix $z \in M$, we get a unique solution $x_{f,.}(z)$ of the sfde (2.25) when Z is replaced by z. Since Z is independent of $w_t, t \geq -\delta$, one may obtain a unique solution $x_{f,.}(Z)$ of (2.25) starting from 0 in $T_Z(M)$. This folllows from a Picard iteration argument on $T_Z(M)$, which is a linear space. By the Itô map, the corresponding solution $x_{c,.}(Z)$ on M starts from Z instead of the deterministic point x.

Let us now turn to the geometrical sfde

$$\begin{cases} dx_{c,t} = \left\{ \int_{t-\delta}^{t} \tau_{t,s}(x_{c,.}) X_1(x_{c,s})\, ds + X_2(x_{c,t}) \right\} \circ dw_t, \quad 0 < t < T, \\ x^0_{c,.} = \gamma^0_{c,.}. \end{cases} \qquad (I.g_1)$$

If $\gamma_{c,.} : [-\delta, T] \times \Omega \to M$ is a semimartingale, we will denote by $\gamma_{c,.}(t)$ its restriction to the time interval $[t - \delta, t]$ for each $t \in [0, T]$.

Fix $t_0 > 0$. Then, for $t \in (t_0, T)$, $x_{c,t}$ is the unique solution of the sfde

$$\begin{cases} dx'_{c,t} = \left\{ \int_{t-\delta}^{t} \tau_{t,s}(x'_{c,.}) X_1(x'_{c,s})\, ds + X_2(x'_{c,t}) \right\} \circ dw_t, \quad t_0 < t < T, \\ x'_{c,.}(t_0) = x_{c,.}(t_0). \end{cases} \qquad (2.26)$$

Now $x_{c,t_0-\delta}$ is independent of dw_t, $t \geq t_0 - \delta$, and $(I.g_1)$ has a unique solution. Therefore,

$$x'_{c,t} = x_{c,t}, \quad t \geq t_0, \qquad (2.27)$$

because the parallel transport in $(I.g_1)$ depends only on the path between $t - \delta$ and t. The above identity constitutes a type of Markov property. Indeed, let $x_.(\gamma^0_{c,.})(w_.)$ denote the solution of the geometrical sfde $(I.g_1)$ with initial condition $\gamma^0_{c,.}$. Then the following equality holds almost surely

$$x_t(\gamma^0_{c,.})(w.) = x_{t-t'}(x_{c,.}(t')(\gamma^0_{c,.}))(w_{t'+.}), \quad t > t', \qquad (2.28)$$

where $w_{t'+.}$ is the Brownian shift $w_{t'+.} : s \mapsto w_{t'+s} - w_{t'}$.

Remark. Relation (2.28) also holds for the geometrical delay equation $(I.g_2)$. This follows by a similar argument to the above.

III. Differentiability in the Chen–Souriau sense

In this part, we consider the following parametrized version of the geometrical sdde $(I.g_2)$:

$$\begin{cases} dx_{c,t}(u) = \tau_{t,t-\delta}(x_{c,.}(u)) X_1(x_{c,t-\delta}(u)) \circ dw_t, \quad 0 < t < T, \\ x^0_{c,.}(u) = \gamma^0_{c,.}(u), \end{cases} \qquad (3.1)$$

with $u \in U$, a bounded open subset of R^n, X_1 a smooth section of the k-frame bundle $L(\underline{\mathbf{R}}^k, T(M)) \to M$, and initial conditions $\gamma^0_{c,.}(u)$.

We would like to study the sample-path differentiability of $x_{c,t}(u)$ in the parameter u. It is sufficient to examine the flat version of the sdde (3.1):

$$\begin{cases} dx_{f,t}(u) = \tau_{-\delta,t-\delta}(x_{c,\cdot}(u))X_1(x_{c,t-\delta}(u)) \circ dw_t, \quad 0 < t < T, \\ x^0_{f,\cdot}(u) = \gamma^0_{f,\cdot}(u). \end{cases} \tag{3.2}$$

The fact that the parameter u is finite-dimensional will allow us to use traditional tools such as Kolmogorov's lemma, Sobolev's embedding theorem, etc... In order to facilitate this, we will first examine the a.s. dependence on u of the stochastic parallel transport term $\tau_{-\delta,t-\delta}(x_{c,\cdot}(u))$ in (3.2). Introduce the following notation. Let $\mathscr{S}^T_{\infty,f}$ denote the Fréchet space of all semimartingales $\gamma_{f,\cdot} \in \mathscr{S}([-\delta,T], T_x(M); -\delta, 0)$ such that

$$\gamma_{f,t} = \int_{-\delta}^t A_s\, dw_s + \int_{-\delta}^t B_s\, ds, \quad 0 \le t \le T,$$

and

$$\|\gamma_{f,\cdot}\|_p^p := \int_{-\delta}^T E\|A_s\|^p\, ds + \int_{-\delta}^T E|B_s|^p\, ds < \infty, \tag{3.3}$$

for all integers $p \ge 1$. As before, $A : [-\delta, T] \times \Omega \to L(\mathbf{R}^k, T_x(M))$ and $B : [-\delta, T] \times \Omega \to T_x(M)$ are adapted, previsible processes. We will denote by $\mathscr{S}^T_{\infty,c}$ the image of $\mathscr{S}^T_{\infty,f}$ under the Itô map with the induced topology. (See section II). Let $\|\cdot\|_{p,t}$ denote the corresponding norms when T is replaced by t in (3.3). Suppose $\alpha := (\alpha_1, \cdots \alpha_p)$ is a multi-index of order $|\alpha| := \sum_{i=1}^n \alpha_i$. The partial derivatives of order $|\alpha|$ with respect to $u := (u_1, u_2, \cdots, u_n)$ are denoted by $D^\alpha := \dfrac{\partial^{|\alpha|}}{\partial u_1^{\alpha_1} \ldots \partial u_n^{\alpha_n}}$. Consider the following differentiability hypotheses on the characteristics $(A_\cdot(u), B_\cdot(u))$ of a parametrized family $\gamma_{f,\cdot}(u) \in \mathscr{S}([-\delta,T], T_x(M); -\delta, 0)$.

Hypotheses (D).

(i) There exists a deterministic constant R (independent of $u \in U$) such that $\|A_t(u)\| + |B_t(u)| \le R$ almost surely for all $t \in [-\delta, T]$ and all $u \in U$.

(ii) $(A_\cdot(u), B_\cdot(u))$ have modifications which are a.s. smooth in u, with derivatives $(D^\alpha A_\cdot(u), D^\alpha B_\cdot(u))$, and the mappings

$$U \ni u \mapsto D^\alpha A_\cdot(u) \in L^p([-\delta, T] \times \Omega, L(\mathbf{R}^k, T_x(M)))$$
$$U \ni u \mapsto D^\alpha B_\cdot(u) \in L^p([-\delta, T] \times \Omega, T_x(M))$$

are continuous (in the underlying L^p-norms (3.3)) for every positive integer p.

Lemma III.1. *Let the manifold M be embedded (isometrically) in $\mathbf{R}^{d'}$ for some $d' > d$, and denote all embedded entities by the same symbols. Assume that the family $\gamma_{c,\cdot}(u) \in \mathscr{S}([-\delta, T], M; -\delta, x)$, $u \in U$, satisfies Hypotheses (D). Then the pair*

$$\hat{x}_t(u) := (\gamma_{c,t}(u), \tau^{-1}_{t,-\delta}(\gamma_{c,\cdot}(u)))$$

has a modification with almost all sample functions smooth in u. *Furthermore, for any multi-index* α *and any positive integer* p, *there exist positive deterministic constants* $K_i := K_i(p,\alpha)$, $i = 1, 2$, *independent of* $u \in U$, $t \in [-\delta, T]$, *such that*

$$\sup_{u\in U} E \sup_{s\in[-\delta,t]} \|D^\alpha \hat{x}_s(u)\|^p \le K_1 e^{K_2 t} \tag{3.4}$$

for all $t \in [-\delta, T]$.

Proof. Using (2.10), the couple $(\gamma_{c,t}(u), \tau^{-1}_{t,-\delta}(\gamma_{c,\cdot}(u))) := \hat{x}_t(u)$ satisfies the Itô stochastic differential equation

$$\begin{cases} d\hat{x}_t(u) = \hat{Z}(\hat{x}_t(u))A_t(u)\,dw_t + \mathrm{trace}(\hat{Y}(\hat{x}_t(u))A_t(u)A_t^*(u))\,dt \\ \qquad\qquad + \hat{Z}(\hat{x}_t(u))B_t(u)\,dt, \quad -\delta < t < T, \\ \hat{x}_{-\delta}(u) = (x, Id_{\mathbf{R}^d}), \end{cases} \tag{3.5}$$

where $\hat{Z}$, $\hat{Y}$ are as in (2.9). Since the characteristics $(A_\cdot(u), B_\cdot(u))$ have a.s. smooth modifications in u, it follows from ([Kun], Theorem 4.6.5, p. 173) that $\hat{x}_t(u)$ has a modification which is a.s. smooth in u. In order to prove (3.4), we pick such a modification of $\hat{x}_t(u)$ and show first that (3.4) holds for $|\alpha| = 1$. The derivative $D\hat{x}_t(u)$ of $\hat{x}_t(u)$ with respect to u satisfies the following stochastic differential equation which is obtained by formally differentiating (3.5) with respect to u:

$$\begin{cases} dD\hat{x}_t(u) = D\hat{Z}(\hat{x}_t(u))D\hat{x}_t(u)A_t(u)\,dw_t \\ \qquad + \hat{Z}(\hat{x}_t(u))DA_t(u)\,dw_t + \mathrm{trace}\Big\{D\hat{Y}(\hat{x}_t(u))D\hat{x}_t(u)A_t(u)A_t^*(u) \\ \qquad + \hat{Y}(\hat{x}_t(u))DA_t(u)A_t^*(u) + \hat{Y}(\hat{x}_t(u))A_t(u)DA_t^*(u)\Big\}\,dt \\ \qquad + D\hat{Z}(\hat{x}_t(u))D\hat{x}_t(u)B_t(u)\,dt + \hat{Z}(\hat{x}_t(u))DB_t(u)\,dt, \\ \qquad -\delta < t < T, \\ D\hat{x}_{-\delta}(u) = (0,0). \end{cases} \tag{3.6}$$

Note that in the above sde, the process $\hat{x}_t(u)$ lives in a compact (non-random) set on which $\hat{Y}$ and $\hat{Z}$ are bounded together with all their derivatives. Therefore we can take p-th moments in (3.6), use Burkholder's inequality and Hypotheses (D) to obtain

$$\alpha_t \le C_7 + C_8 \int_{-\delta}^t \alpha_s\,ds, \quad -\delta \le t \le T, \tag{3.7}$$

where $\alpha_t := \sup_{u\in U} E \sup_{s\in[-\delta,t]} \|D\hat{x}_s(u)\|^p$ for $-\delta \le t \le T$, and the constants C_7, C_8 are independent of $u \in U$. Applying Gronwall's lemma to (3.7) gives

$$\sup_{u\in U} E \sup_{s\in[-\delta,t]} \|D\hat{x}_s(u)\|^p \le C_7 e^{C_8 t}$$

for all $t \in [-\delta, T]$. This shows that (3.4) holds for $\alpha = 1$. We complete the proof by induction on $|\alpha|$. Suppose the estimate (3.4) holds for all multi-indices $\alpha = \alpha_0$

with $|\alpha_0| < |\alpha| + 1$. Let α' be a multi-index such that $|\alpha'| = |\alpha| + 1$. By repeated differentiation of (3.5) with respect u, it is not hard to see that there are polynomials Q_i, $i = 1, 2$, such that

$$\begin{cases} dD^{\alpha'}\hat{x}_t(u) = D\hat{Z}(\hat{x}_t(u))D^{\alpha'}\hat{x}_t(u)A_t(u)\,dw_t \\ \qquad + D\hat{Z}(\hat{x}_t(u))D^{\alpha'}\hat{x}_t(u)B_t(u)\,dt \\ \qquad + \operatorname{trace}\{D\hat{Y}(\hat{x}_t(u))D^{\alpha'}\hat{x}_t(u)A_t(u)A_t^*(u)\}\,dt \\ \qquad + \sum_{\alpha^i:\sum_{i=1}^5 |\alpha^i|<|\alpha'|} Q_1(D^{\alpha^1}\hat{Z}(\hat{x}_t(u)), D^{\alpha^2}\hat{x}_t(u), D^{\alpha^3}A_t(u), \\ \qquad\qquad D^{\alpha^4}A_t^*(u), D^{\alpha^5}B_t(u))\,dw_t \\ \qquad + \sum_{\beta^i:\sum_{i=1}^6 |\beta^i|<|\alpha'|} Q_2(D^{\beta^1}\hat{Y}(\hat{x}_t(u)), D^{\beta^2}\hat{Z}(\hat{x}_t(u)), \\ \qquad\qquad D^{\beta^3}\hat{x}_t(u), D^{\beta^4}A_t(u), D^{\beta^5}A_t^*(u), D^{\beta^6}B_t(u))\,dt, \\ \qquad\qquad -\delta < t < T, \\ D^{\alpha'}\hat{x}_{-\delta}(u) = (0, 0). \end{cases} \tag{3.8}$$

Note that in the above equation, the term $D^{\alpha'}\hat{x}_t(u)$ appears *linearly*, while, by the inductive hypothesis, the lower order derivatives $D^{\alpha_0}\hat{x}_t(u)$ satisfy the inequality (3.4) for $|\alpha_0| < |\alpha'|$. Using this fact, Hypotheses (D) and Burkholder's inequality, it follows from (3.8) that there are positive constants C_i, $i = 9, 10$, independent of u such that

$$\beta_t \le C_9 + C_{10}\int_{-\delta}^t \beta_s\,ds, \quad -\delta \le t \le T, \tag{3.9}$$

where $\beta_t := \sup_{u\in U} E \sup_{t\in[-\delta,T]} \|D^{\alpha'}\hat{x}_t(u)\|^p$ for $-\delta \le t \le T$. The conclusion of the lemma now follows from (3.9) by Gronwall's lemma and induction on $|\alpha|$. □

In the sequel, the abbreviation "$l.o.$" will denote lower order terms (e.g. the last two terms on the right hand side of (3.8)) whose moments are readily computed and estimated by induction on α.

Theorem III.2 *Assume that the characteristics $(A^0_\cdot(u), B^0_\cdot(u))$ of $\gamma^0_{c,\cdot}(u)$ in (3.1) satisfy Hypotheses (D). Then the solution $x_{c,t}(u)$ of (3.1) has a modification a.s. smooth in u. Furthermore, the solution $x_{f,\cdot}(u)$ of the flat equation (3.2) satisfies the inequality*

$$\sup_{u\in U} E \sup_{s\in[-\delta,t]} \|D^\alpha x_{f,s}(u)\|^p \le K_3 e^{K_4 t} \tag{3.10}$$

for all $t \in [-\delta, T]$, and for some positive constants $K_3 := K_3(p,\alpha)$, $K_4 := K_4(p,\alpha)$, independent of $u \in U$.

Proof. To prove the first assertion of the theorem it is sufficient to show that for each multindex α, $x_{f,t}(u)$ admits a version with continuous partial derivatives of order $|\alpha|$ in u. Embed M (isometrically) in $\mathbf{R}^{d'}$ for some $d' > d$. We proceed by induction on α. Let $g(y, z) := zX(y)$, where z represents stochastic parallel transport and is therefore an orthogonal matrix, and y belongs to M. Then g is bounded and has bounded derivatives of all orders. Now rewrite (3.2) in the form

$$\begin{cases} dx_{f,t}(u) = g(\hat{x}_{c,t-\delta}(u))dw_t, & t > 0, \\ x^0_{f,.}(u) = \gamma^0_{f,.}(u). \end{cases} \tag{3.11}$$

where $\hat{x}_{c,t} := (x_{c,t}, \tau^{-1}_{t,-\delta}(x_{c,.}))$.

In (3.11), the initial condition $\gamma^0_{c,.}(u)$ is given by $\gamma^0_{f,t}(u) = \int_{-\delta}^t A^0_s(u)\,dw_s + \int_{-\delta}^t B^0_s(u)\,ds$ for $-\delta \le t \le 0$, where $A^0_.(u)$ and $B^0_.(u)$ satisfy Hypotheses (D). These imply that $\gamma^0_{f,t}(u)$ has a modification which is a.s. smooth in u (and H'older continuous in $t \in [-\delta, 0]$ with exponent $< \frac{1}{2}$) ([Kun], Theorem 3.3.3, pp. 94–95).

We will prove the differentiability of $x_{f,t}(u)$, $t \in [0, T]$, in u using forward steps of length δ. On $[0, \delta]$, the identity

$$\begin{cases} x_{f,t}(u) = \gamma^0_{f,0}(u) + \displaystyle\int_0^t g(\gamma^0_{c,s-\delta}(u), \tau^{-1}_{s-\delta,-\delta}(\gamma^0_{c,.}(u)))\,dw_s, & t \in [0, \delta] \\ x^0_{f,.}(u) = \gamma^0_{f,.}(u), \quad u \in U, \end{cases} \tag{3.12}$$

and ([Kun], Theorem 3.3.3, pp. 94–95) imply that $x_{f,t}(u)$ has an a.s. smooth modification in u. Indeed, $D^\alpha x_{f,t}$ satisfies the equation obtained by taking partial derivatives of order $|\alpha|$ under the stochastic integral sign in (3.12), viz.

$$\begin{cases} D^\alpha x_{f,t}(u) = D^\alpha \gamma^0_{f,0}(u) + \displaystyle\int_0^t Dg(\gamma^0_{c,s-\delta}(u), \tau^{-1}_{s-\delta,-\delta}(\gamma^0_{c,.}(u)))(D^\alpha \gamma^0_{c,s-\delta}(u), \\ \qquad D^\alpha \tau^{-1}_{s-\delta,-\delta}(\gamma^0_{c,.}(u)))\,dw_s + l.o., \quad t \in [0, \delta], \\ D^\alpha x^0_{f,.}(u) = D^\alpha \gamma^0_{f,.}(u). \end{cases} \tag{3.13}$$

Using Burkholder's inequality, Hypotheses (D) and Lemma III.1, it follows from (3.13) that the estimate (3.10) holds for all $t \in [-\delta, \delta]$.

A similar argument to the above works for the forward intervals $[\delta, 2\delta]$, $[2\delta, 3\delta]$, $\cdots$, and hence by induction for all $t \in [-\delta, T]$. This completes the proof of the lemma. □

Remark. Consider the following generalization of (3.1):

$$\begin{cases} dx_{c,t}(u) = \tau_{t,t-\delta}(x_{c,.}(u))X_1(x_{c,t-\delta})(\circ A_t(u)dw_t + B_t(u)dt), & t > 0, \\ x^0_{c,.}(u) = \gamma^0_{c,.}(u), \quad u \in U, \end{cases} \tag{3.14}$$

where X_1 is a smooth section of the k-frame bundle $L(\underline{\mathbf{R}^k}, T(M)) \to M$, and $A_t(u) \in L(\mathbf{R}^k)$, $B_t(u) \in \mathbf{R}^k$ for $t > 0, u \in U$. Suppose that the characteristics $(A^0_.(u), B^0_.(u))$ of $\gamma^0_{c,.}(u)$ and $(A_.(u), B_.(u))$ all satisfy Hypotheses (D). By a similar argument to the one used in the proof of Lemma III.2, the solution $x_{c,.}(u)$ of (3.14) admits a smooth version in u.

We conclude this section by expressing the result in the above remark in terms of the stochastic calculus of Chen-Souriau ([Le$_2$], [Le$_3$]).

Definitions III.3. A *stochastic plot* on the space $S_{\infty,c}([-\delta,0],M;-\delta,x)\times S_\infty([0,T],\mathbf{R}^k;0,0)$ is a triplet $(U,\phi,\mathbf{R}^n)$ consisting of an open subset U of some Euclidean space $\mathbf{R}^n$ and a mapping $U\ni u\mapsto \phi_{.}(u):=(\gamma_{.}(u),z_{.}(u))\in S_{\infty,c}([-\delta,0],M;-\delta,x)\times S_\infty([0,T],\mathbf{R}^k;0,0)$ such that the characteristics of $\gamma_{.}(u)$ and $z_{.}(u)$ satisfy Hypotheses (D).

Let $(U,\phi,\mathbf{R}^n)$ be a stochastic plot on $S_{\infty,c}([-\delta,0],M;-\delta,x)\times S_\infty([0,T],\mathbf{R}^k;0,0)$, and let $j:U_1\to U$ be a smooth deterministic map where U_1 is an open subset of $\mathbf{R}^{n_1}$. Define $\phi^1_{.}(u_1):=\phi_{.}(j\circ u_1)$ for all $u_1\in U_1$. It is easy to check that $(U_1,\phi^1,\mathbf{R}^{n_1})$ is a stochastic plot, called the *composite* plot.

Next we consider the effect of a measure-space isomorphism on a stochastic plot. More specifically, let $(U,\phi,\mathbf{R}^n)$ be a stochastic plot on $S_{\infty,c}([-\delta,0],M;-\delta,x)\times S_\infty([0,T],\mathbf{R}^k;0,0)$. Suppose $\Psi:(\Omega,\mathscr{F})\to(\Omega,\mathscr{F})$ is a P-preserving measurable bijection. Assume that the spaces $S_{\infty,c}([-\delta,0],M;-\delta,x)$ and $S_\infty([0,T],\mathbf{R}^k;0,0)$ consist of semimartingales based on a Brownian motion w_t, $-\delta\le t\le T$ on a filtered probability space $(\Omega,\mathscr{F},(\mathscr{F}_t)_{t\in[-\delta,T]},P)$. For any $\gamma_{.}\in S_{\infty,c}([-\delta,0],M;-\delta,x)$ and $z_{.}\in S_\infty([0,T],\mathbf{R}^k;0,0)$ define the processes

$$\gamma^\Psi_{.}(\omega):=\gamma_{.}(\Psi(\omega)),\ \ z^\Psi_{.}(\omega):=z_{.}(\Psi(\omega))$$

for all $\omega\in\Omega$. Then $\gamma^\Psi_{.}$ and $z^\Psi_{.}$ are semimartingales on the filtered probability space $(\Omega,\mathscr{F},(\Psi^{-1}(\mathscr{F}_t)_{t\in[-\delta,T]},P)$ based on the Brownian motion $w^\Psi_t(\omega):=w_t(\Psi(\omega))$, $\omega\in\Omega$. If $\gamma_{.}$ has characteristics $(A_{.},B_{.})$ (with respect w), then $\gamma^\Psi_{.}$ has characteristics $(A^\Psi_{.},B^\psi_{.})$ (with respect w^Ψ) where $A^\Psi_{.}(\omega):=A_{.}(\Psi(\omega))$, $B^\Psi_{.}(\omega):=B_{.}(\Psi(\omega))$ for all $\omega\in\Omega$. Let $S^\Psi_{\infty,c}([-\delta,0],M;-\delta,x)$ denote the set of all $\gamma^\Psi_{.}$ whose characteristics satisfy a relation analogous to (3.3) for all integers $p\ge 1$. Define $\phi^\Psi_{.}(u)(\omega):=\phi_{.}(u)(\Psi(\omega))$ for all $u\in U$ and $\omega\in\Omega$. Then $(U,\phi^\Psi,\mathbf{R}^n)$ is a stochastic plot on $S^\Psi_{\infty,c}([-\delta,0],M;-\delta,x)\times S^\psi_\infty([0,T],\mathbf{R}^k;0,0)$. It is clear that Ψ induces an isometry between $S_{\infty,f}([-\delta,0],T_x(M);-\delta,0)$ and $S^\Psi_{\infty,f}([-\delta,0],T_x(M);-\delta,0)$. A similar relationship holds for $S_\infty([0,T],\mathbf{R}^k;0,0)$ and $S^\psi_\infty([0,T],\mathbf{R}^k;0,0)$. In what follows we shall identify these spaces and sfde's defined on them. In particular, we will drop the superscript Ψ from all entities and processes induced by Ψ.

We next introduce the following definition of a *smooth functional*

$$S_{\infty,c}([-\delta,0],M;-\delta,x)\times S_\infty([0,T],\mathbf{R}^k;0,0)\to L^0(\Omega,M)$$

in the Chen-Souriau sense:

Definition III.4. A functional $\Lambda:S_{\infty,c}([-\delta,0],M;-\delta,x)\times S_\infty([0,T],\mathbf{R}^k;0,0)\to L^0(\Omega,M)$ is said to be *smooth in the Chen-Souriau sense* if it satisfies the following requirements:

(i) To each stochastic plot $(U,\phi,\mathbf{R}^n)$, the composite process $\Lambda(\phi_{.}(u))$ has an a.s. smooth version in $u\in U$.

(ii) Let $j : U_1 \to U_2$ be a smooth deterministic map from an open subset U_1 of R^{n_1} into an open subset U_2 of R^{n_2}. Let $(U_2, \phi^2, \mathbf{R}^{n_2})$ be a stochastic plot, and denote by $(U_1, \phi^1, \mathbf{R}^{n_1})$ the composite plot $\phi^1_.(u_1) := \phi^2_.(j \circ u_1)$ for all $u_1 \in U_1$. Then there is a sure event $\Omega_{\phi_1,\phi_2} \subseteq \Omega$ such that

$$\Lambda(\phi^1_.(u_1))(\omega) = \Lambda(\phi^2_.(j \circ u_1))(\omega) \tag{3.15}$$

for all $\omega \in \Omega_{\phi_1,\phi_2}$ and all $u_1 \in U_1$.

(iii) Let $(U, \phi^2_., \mathbf{R}^{n_2})$ be a stochastic plot. Let $\Psi : (\Omega, \mathscr{F}) \to (\Omega, \mathscr{F})$ be a P-preserving measurable transformation. Define the stochastic plot $(U, \phi^1_., \mathbf{R}^{n_2})$ by $\phi^1_.(u)(\omega) := \phi^2_.(u)(\Psi(\omega))$ for almost all $\omega \in \Omega$. Then

$$\Lambda(\phi^1_.(u))(\omega) = \Lambda(\phi^2_.(u))(\Psi(\omega)) \tag{3.16}$$

for almost all $\omega \in \Omega$.

Remark. Using Kolmogorov's lemma, we may, in part (iii) of Definition III.4, assume that our plot $\phi^1_t(u)(\omega)$ has a smooth version in u for the L^p topology (and not the semimartingale topology).

We now state the main result of this part of the article.

Theorem III.5. *Consider the solution $x_{c,.}(\gamma^0_{c,.}, z_.)$ of the geometrical sdde $(I.g_2)$ starting from $\gamma^0_{c,.}$ in $S_{\infty,c}([-\delta, 0], M; -\delta, x)$ and driven by a semimartingale path $z_.$ in $S_\infty([0, T], \mathbf{R}^k; 0, 0)$. Then the map $(\gamma^0_{c,.}, z_.) \mapsto x_{c,.}(\gamma^0_{c,.}, z_.)$ is smooth in the Chen-Souriau sense.*

Proof. The requirements (i)- (iii) in Definition III.4 follows from the fact that they are easily satisfied on $[-\delta, \delta]$ by the Itô integral in (3.12), and hence on the whole interval $[-\delta, T]$ by using forward steps of length δ. □

Remark. Using a (lengthy) Peano approximation argument, it can be shown that the solution of the geometrical sfde $(I.g_1)$ is smooth in the Chen-Souriau sense. Note that the method of forward steps does not apply for the sfde $(I.g_1)$.

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