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Large deviations for the Ginzburg–Landau $\nabla\phi$ interface model

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Abstract. Hydrodynamic large scale limit for the Ginzburg-Landau $\nabla\phi$ interface model was established in [6]. As its next stage this paper studies the corresponding large deviation problem. The dynamic rate functional is given by

$$I(h) = \frac{1}{4}\int_0^T dt \int_{\mathbb{T}^d} \{\partial h/\partial t - \mathrm{div}(\nabla\sigma(\nabla h))\}^2 \, d\theta$$

for $h = h(t,\theta)$, $t \in [0,T]$, $\theta \in \mathbb{T}^d$, where $\sigma = \sigma(u)$ is the surface tension for mean tilt $u \in \mathbb{R}^d$. Our main tool is H^{-1}-method exploited by Landim and Yau [9]. The relationship to the rate functional obtained under the static situation by Deuschel et al. [3] is also discussed.

1. Introduction

The Ginzburg-Landau $\nabla\phi$ interface model determines a stochastic dynamics for a discretized hypersurface embedded in the $d+1$ dimensional space. Such hypersurface is interpreted as an interface separating two distinct phases. The position of hypersurface is described by height variables $\phi = \{\phi(x), x \in \Gamma_N\}$ measured from a fixed hyperplane Γ_N. We shall always work on a periodic cubic lattice so that $\Gamma_N = (\mathbb{Z}/N\mathbb{Z})^d = \{1, 2, \dots, N\}^d$. Its side length N is large and eventually goes to infinity.

The dynamics of the interface ϕ is governed by the stochastic differential equations (SDEs)

$$d\phi_t(x) = -\sum_{y\in\Gamma_N:|x-y|=1} V'(\phi_t(x) - \phi_t(y))dt + \sqrt{2}\, dw_t(x), \quad x \in \Gamma_N, \tag{1.1}$$

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where $\{w_t(x), x \in \Gamma_N\}$ is a family of independent one dimensional standard Brownian motions. The potential V satisfies the following conditions:

(i) $V \in C^2(\mathbb{R})$,
(ii) (symmetry) $V(-\eta) = V(\eta), \quad \eta \in \mathbb{R}$,
(iii) (strict convexity) $c_- \le V''(\eta) \le c_+, \quad \eta \in \mathbb{R}, \quad$ for some $c_-, c_+ > 0$.
(1.2)

In [6], the macroscopic behavior of the microscopically defined interface dynamics $\phi_t = \{\phi_t(x), x \in \Gamma_N\}$ is investigated. It is shown that, after taking the limit $N \to \infty$, the interface dynamics viewed at the macroscopic level is governed by the motion by mean curvature except for some anisotropy effects. To formulate more precisely, let us define macroscopic height variables for the interface as a step function on the torus $\mathbb{T}^d = (\mathbb{R}/\mathbb{Z})^d = [0,1)^d$:

$$h^N(t,\theta) := N^{-1}\phi_{N^2 t}(x), \quad \theta \in B(x/N, 1/N), \tag{1.3}$$

where $B(\theta, a) = \prod_{i=1}^d [\theta_i - a/2, \theta_i + a/2)$ denotes a box in $\mathbb{T}^d$ with center $\theta = (\theta_i)_{i=1}^d$ and side length $a > 0$. Note that (1.3) introduces a diffusive scaling for ϕ_t and both x- and ϕ-axes are rescaled by a factor $1/N$. This is because the ϕ-field represents a hypersurface embedded in $d+1$ dimensional space. The function $h^N(t,\theta)$ is sometimes simply denoted by $h^N(t)$.

One of the main results of [6] can now be stated. Assume that initial random configuration ϕ_0 of the SDEs (1.1) converges to some non-random $h_0 \in L^2(\mathbb{T}^d)$ in the sense that

$$\lim_{N\to\infty} E[||h^N(0) - h_0||^2] = 0, \tag{1.4}$$

where $||\cdot||$ denotes the usual L^2-norm of the space $L^2(\mathbb{T}^d)$. Then, for every $t > 0$

$$\lim_{N\to\infty} E[||h^N(t) - h(t)||^2] = 0 \tag{1.5}$$

holds and $h(t) = h(t,\theta)$ is a unique solution of the partial differential equation (PDE)

$$\frac{\partial}{\partial t}h(t,\theta) = \sum_{i=1}^d \frac{\partial}{\partial \theta_i}\left\{\frac{\partial \sigma}{\partial u_i}(\nabla h(t,\theta))\right\}, \quad \theta \in \mathbb{T}^d, \tag{1.6}$$

having initial data h_0, where $\nabla h = (\partial h/\partial \theta_i)_{i=1}^d$. The function $\sigma = \sigma(u)$ is the so-called surface tension determined by the statistical property of random interfaces with mean tilt $u = (u_i)_{i=1}^d \in \mathbb{R}^d$, see [6]. Since the limit $h(t,\theta)$ of random field $h^N(t,\theta)$ is non-random, this result can be thought as a kind of law of large numbers. The equation (1.6) describes the motion by mean curvature, except for some anisotropy due to the underlying lattice structure. Further physical motivation can be found in [12].

The aim of this paper is to study the corresponding large deviation problem as a natural next stage. An extension of the H^{-1}-method first used by Chang and

Yau [2] effectively works to show the hydrodynamic limit, see [6]. This is again the case for the large deviation problem. We shall in fact apply the method employed by Landim and Yau [9], and also the approach used by [4], [8].

We remark that the stochastic dynamics defined by (1.1) is reversible under the finite-volume Gibbs measures associated with an energy of the interface ϕ

$$H_N(\phi) = \sum_{b\in\Gamma_N^*} V(\nabla\phi(b)). \tag{1.7}$$

Here Γ_N^* stands for the family of all directed bonds $b = (x, y)$, $x, y \in \Gamma_N$, $|x-y| = 1$ in Γ_N, and $\nabla\phi(b) := \phi(x) - \phi(y)$ for $b = (x, y)$. Each bond $b = (x, y)$ is directed from y to x. We write $x_b = x$ and $y_b = y$ for $b = (x, y)$. Reversely directed bond $-b$ of b is defined by $-b := (y_b, x_b)$. Note that each undirected bond appears twice in Γ_N^*. The family of all directed bonds in $\mathbb{Z}^d$ is similarly denoted by $(\mathbb{Z}^d)^*$.

The main result is stated in Section 2, see Theorem 2.1. Superexponential estimate which is essential for the proof of the main result is also formulated there, see Theorem 2.2. To prove such estimate, superexponential one-block and two-blocks estimates are required. These estimates are given in Sections 4 and 5, respectively. Section 3 prepares several a priori exponential estimates. Large deviation lower and upper bounds are proved in Sections 6 and 7, respectively. Finally Section 8 discusses the relationship between our dynamic approach and the static result obtained by [3].

2. Main result

Before stating the main theorem, we prepare several notation. For a microscopic height variable $\phi \equiv \phi^N = \{\phi(x), x \in \Gamma_N\} \in \mathbb{R}^{\Gamma_N}$, $h^N \equiv h^N(\theta) := N^{-1}\sum_{x\in\Gamma_N}\phi(x)1_{B(x/N,1/N)}(\theta)$ denotes a macroscopic height variable. Similarly, for a microscopic height process $\phi_t \equiv \{\phi_t(x), x \in \Gamma_N\}$, we denote by $\phi_t^N \equiv \{\phi_t^N(x) := \phi_{N^2t}(x), x \in \Gamma_N\}$ a height process which is macroscopic in time and microscopic in space and by $h^N(t,\theta) := N^{-1}\sum_{x\in\Gamma_N}\phi_t^N(x)1_{B(x/N,1/N)}(\theta)$ a macroscopic height process, respectively; $h^N(t,\theta)$ is the same as that defined in (1.3). For each directed bond $b = (x, y)$, we set $\eta_t(b) \equiv \nabla\phi_t(b) := \phi_t(x) - \phi_t(y)$ and $\eta_t^N(b) \equiv \nabla\phi_t^N(b) := \phi_t^N(x) - \phi_t^N(y)$. Note that, from the SDEs (1.1) for ϕ_t, the process ϕ_t^N satisfies the SDEs

$$d\phi_t^N(x) = -N^2 \sum_{b\in\Gamma_N^*:x_b=x} V'(\nabla\phi_t^N(b))dt + \sqrt{2}N dw_t(x), \quad x \in \Gamma_N, \tag{2.1}$$

in law sense. Namely, (2.1) holds with $N^{-1}w_{N^2t}(x)$ in place of $w_t(x)$ but the laws are certainly the same. The generator of the process ϕ_t^N is given by

$$L^N = N^2 \sum_{x\in\Gamma_N} \left\{ \frac{\partial^2}{\partial\phi(x)^2} - \sum_{b:x_b=x} V'(\nabla\phi(b)) \frac{\partial}{\partial\phi(x)} \right\}. \tag{2.2}$$

Throughout the paper, we assume the initial configurations $\phi^N = \phi_0^N$ of the SDEs (2.1) are deterministic and satisfy

$$\sup_N \left\{ |\phi^N(0)| + N^{-d} \sum_{b\in\Gamma_N^*} (\nabla\phi^N(b))^2 \right\} < \infty. \tag{2.3}$$

Note that this condition implies $\sup_N ||h^N|| < \infty$ for the corresponding macroscopic height variables. We also assume the condition (1.4) without taking expectations for the corresponding $h^N(0)$ and some $h_0 \in L^2(\mathbb{T}^d)$. The distribution of the process ϕ_t^N, $t \in [0,T]$ with initial configuration ϕ^N is denoted by P_{ϕ^N}; $T > 0$ is fixed. We denote the space $L^2(\mathbb{T}^d)$ equipped with the weak topology by $L^2_w(\mathbb{T}^d)$ and the class of all continuous functions $h : [0,T] \to L^2_w(\mathbb{T}^d)$ by $C([0,T], L^2_w(\mathbb{T}^d))$. The space $H^1(\mathbb{T}^d)$ stands for the Sobolev space on $\mathbb{T}^d$ so that $L^2([0,T], H^1(\mathbb{T}^d))$ is the family of all $h = h(t,\theta) \in L^2([0,T]\times\mathbb{T}^d)$ satisfying that $h(t) \in H^1(\mathbb{T}^d)$ for a.e.$t \in [0,T]$ and $\int_0^T dt \int_{\mathbb{T}^d} |\nabla h(t,\theta)|^2 \, d\theta < \infty$.

For each $h = h(t,\theta)$ which is differentiable in (t,θ), let

$$I(h) \equiv I_T(h) := \frac{1}{4}\int_0^T dt \int_{\mathbb{T}^d} \left\{ \frac{\partial h}{\partial t}(t,\theta) - \operatorname{div}(\nabla\sigma(\nabla h(t,\theta))) \right\}^2 d\theta, \tag{2.4}$$

if $h(0) = h_0$ and $I(h) = +\infty$ if $h(0) \neq h_0$; see Proposition 6.3 below. More precisely saying, $I(h) = +\infty$ unless $h \in C([0,T], L^2_w(\mathbb{T}^d))$ satisfies $h \in L^2([0,T], H^1(\mathbb{T}^d))$, and for such h

$$I(h) = \sup_{J=J(t,\theta)\in C^1([0,T]\times\mathbb{T}^d)} I(h; J), \tag{2.5}$$

where

$$\begin{aligned} I(h;J) = & \int_{\mathbb{T}^d} J(T,\theta)h(T,\theta)\,d\theta - \int_{\mathbb{T}^d} J(0,\theta)h_0(\theta)\,d\theta \\ & - \int_0^T dt \int_{\mathbb{T}^d} \frac{\partial J}{\partial t}(t,\theta)h(t,\theta)\,d\theta \\ & + \int_0^T dt \int_{\mathbb{T}^d} \nabla J(t,\theta)\cdot\nabla\sigma(\nabla h(t,\theta))\,d\theta - \int_0^T dt \int_{\mathbb{T}^d} J^2(t,\theta)\,d\theta. \end{aligned} \tag{2.6}$$

Recall that the function σ enjoys the bound $|\nabla\sigma(u)| \le C(1+|u|)$, see Theorem 3.4 (v) of [6]. Our main result asserts that the large deviation principle holds for $h^N = h^N(t,\theta)$ with the rate functional $I(h)$:

Theorem 2.1. *For every closed subset $\mathcal{C}$ and open subset $\mathcal{O}$ of $C([0,T], L^2_w(\mathbb{T}^d))$, we have*

$$\limsup_{N\to\infty} N^{-d} \log P_{\phi^N}(h^N \in \mathcal{C}) \le -\inf_{h\in\mathcal{C}} I(h), \tag{2.7}$$

$$\liminf_{N\to\infty} N^{-d} \log P_{\phi^N}(h^N \in \mathcal{O}) \ge -\inf_{h\in\mathcal{O}} I(h). \tag{2.8}$$

Remark 2.1. (i) Deuschel et al. [3] recently studied the large deviations under the static situation. They proved that the large deviation principle holds for a sequence of finite volume Gibbs measures $\{\mu^N\}$ and the rate functional is given by the total surface tension $\mathbb{F}(h)$ for $h = h(\theta)$. The details will be discussed in Section 8.
(ii) For $d = 1$, the $\nabla\phi$ interface model is identical to the Ginzburg-Landau model treated by Donsker and Varadhan [4]. In fact, [4] established the large deviation principle for the process η_t^N in our setting.

The proof of Theorem 2.1 relies on a superexponential estimate, see Theorem 2.2 below. To formulate it, we recall some notation from [6] and introduce further notation. The space $\mathcal{X}$ stands for the set of all $\eta \in \mathbb{R}^{(\mathbb{Z}^d)^*}$ satisfying the plaquette condition and μ_u denotes the unique shift invariant, ergodic tempered Gibbs measure on $\mathcal{X}$ with mean $u \in \mathbb{R}^d$, see [6] for details. For $\Lambda \subset \mathbb{Z}^d$, we define Λ^* by $\Lambda^* = \{b \in (\mathbb{Z}^d)^*; x_b, y_b \in \Lambda\}$. Let $C_{\mathrm{loc},b}(\mathcal{X})$ be the family of all functions F on $\mathcal{X}$ of the forms $F(\eta) = \bar{F}\left(\{\eta(b)\}_{b\in\left(\check{\Lambda}\right)^*}\right)$ for some finite $\check{\Lambda} \subset \mathbb{Z}^d$ and $\bar{F} \in C_b\left(\mathbb{R}^{\left(\check{\Lambda}\right)^*}\right)$. The class of all $F \in C_{\mathrm{loc},b}(\mathcal{X})$ defined with $\bar{F} \in C_0\left(\mathbb{R}^{\left(\check{\Lambda}\right)^*}\right)$ is denoted by $C_{\mathrm{loc},0}(\mathcal{X})$. The (minimal) set $\check{\Lambda}$ is called the support of F for $F \in C_{\mathrm{loc},b}(\mathcal{X})$. We denote the expectation of $F \in C_{\mathrm{loc},b}(\mathcal{X})$ with respect to μ_u by $\tilde{F}(u)$:

$$\tilde{F}(u) = E^{\mu_u}[F], \quad u \in \mathbb{R}^d. \tag{2.9}$$

For a positive integer l and $x \in \mathbb{Z}^d$, denote the empirical mean of the configuration η on a box $\Lambda_l + x$ with side length $2l + 1$ centered at x by $\bar{\eta}_x^l$:

$$\bar{\eta}_x^l = (2l+1)^{-d} \sum_{y\in\Lambda_l+x} \sum_{i=1}^{d} \eta(e_i + y)e_i \in \mathbb{R}^d,$$

where $\Lambda_l = \{y \in \mathbb{Z}^d; \max_{1\le i\le d} |y_i| \le l\}$ is a box centered at 0 and $e_i \in \mathbb{Z}^d$ is the i-th unit vector given by $(e_i)_j = \delta_{ij}$; note that $e_i + y$ sometimes represents the directed bond $(e_i + y, y)$. Finally, for a positive integer l and a function $F \in C_{\mathrm{loc},b}(\mathcal{X})$, set

$$\mathrm{Av}_{\Lambda_l+x}\, F(\eta) = (2l+1)^{-d} \sum_{y\in\Lambda_l+x} \tau_y F(\eta),$$

and

$$W_l^F(\eta) = \mathrm{Av}_{\Lambda_l}\, F(\eta) - \tilde{F}(\bar{\eta}_0^l),$$

where τ_y denotes the shift by y. Then, the superexponential estimate for the replacement of sample mean by its average under equilibrium measures is formulated as follows:

Theorem 2.2. *For every* $F \in C_{\mathrm{loc},b}(\mathcal{X})$ *and* $\alpha > 0$, *we have*

$$\limsup_{\epsilon\downarrow 0}\limsup_{N\to\infty} N^{-d}\log E_{\phi^N}\left[\exp\left\{\alpha\int_0^T \sum_{x\in\Gamma_N}\left|\tau_x W^F_{N\epsilon}(\nabla\phi^N_t)\right| dt\right\}\right] \le 0. \tag{2.10}$$

In particular, for every $\delta > 0$

$$\limsup_{\epsilon\downarrow 0}\limsup_{N\to\infty} N^{-d}\log P_{\phi^N}\left\{N^{-d}\int_0^T \sum_{x\in\Gamma_N}\left|\tau_x W^F_{N\epsilon}(\nabla\phi^N_t)\right| dt > \delta\right\} = -\infty. \tag{2.11}$$

3. A priori exponential estimates

In this section, we prove several exponential estimates for $h^N(t)$ and $\nabla\phi^N_t$. Proposition 3.1 and its Corollary 3.2 will play key roles in the subsequent sections, while Propositions 3.3 and 3.4 will be used in Section 7 to show the superexponential tightness of $h^N(t)$.

Proposition 3.1. *For every* $T > 0, 0 < \beta \le 1, N \ge 1$ *and for every initial configuration* ϕ^N *with finite* L^2 *norm for the corresponding* h^N *(i.e.* $\|h^N\| < \infty$*)*

$$\begin{aligned} &E_{\phi^N}\left[\exp\left\{\beta e^{-5T}\int_0^T\left(\sum_{b\in\Gamma_N^*} V'(\nabla\phi^N_t(b))\nabla\phi^N_t(b) + N^d\|h^N(t)\|^2\right) dt\right\}\right] \\ &\quad\le \exp\{\beta N^d\|h^N\|^2 + 2\beta N^d/5\}. \end{aligned} \tag{3.1}$$

The following corollary is readily shown by applying this proposition and recalling that $V'(\eta)\eta \ge c_-\eta^2$.

Corollary 3.2. *Let* $(\phi^N)_{N\ge 1}$ *be a sequence satisfying* $\sup_N \|h^N\| < \infty$. *Then,*

$$\begin{aligned} &\limsup_{\beta\downarrow 0}\limsup_{N\to\infty} N^{-d} \\ &\quad\times\log E_{\phi^N}\left[\exp\left\{\beta\int_0^T\left(\sum_{b\in\Gamma_N^*}\left(\nabla\phi^N_t(b)\right)^2 + N^d\|h^N(t)\|^2\right) dt\right\}\right] \le 0. \end{aligned}$$

Proof of Proposition 3.1. For every $\kappa \ge 0$, we introduce a martingale

$$\begin{aligned} M^\kappa_t &= \|h^N(t)\|^2 e^{-\kappa t} - \|h^N(0)\|^2 \\ &\quad - \int_0^t e^{-\kappa s}\left(L^N\left\|h^N(s)\right\|^2 - \kappa\|h^N(s)\|^2\right) ds, \quad t \ge 0. \end{aligned}$$

From $\|h^N\|^2 = N^{-d-2}\sum_{x\in\Gamma_N}\phi(x)^2$, we have

$$\begin{aligned} L^N \left\|h^N\right\|^2 &= N^{-d}\sum_{x\in\Gamma_N}\left\{2-2\sum_{b:x_b=x}V'(\nabla\phi(b))\phi(x)\right\} \\ &= N^{-d}\sum_{x\in\Gamma_N}\left\{2-\sum_{b:x_b=x}V'(\nabla\phi(b))\phi(x_b)+\sum_{b:y_b=x}V'(\nabla\phi(b))\phi(y_b)\right\} \\ &= 2-N^{-d}\sum_{b\in\Gamma_N^*}V'(\nabla\phi(b))\nabla\phi(b). \end{aligned} \tag{3.2}$$

The second equality uses the symmetry of V which implies $V'(\nabla\phi(b)) = -V'(\nabla\phi(-b))$. From the representation

$$M_t^\kappa = 2\sqrt{2}N^{-d-1}\sum_{x\in\Gamma_N}\int_0^t e^{-\kappa s}\phi_s^N(x)\,dw_s(x),$$

we see that its quadratic variation is given by

$$\langle M^\kappa\rangle_t = 8N^{-d}\int_0^t e^{-2\kappa s}\|h^N(s)\|^2\,ds. \tag{3.3}$$

Since M_t^κ is a continuous martingale vanishing at time 0, for every initial configuration ϕ^N with finite L^2 norm for the corresponding h^N,

$$\exp\left\{\beta N^d M_t^\kappa - \frac{\beta^2 N^{2d}}{2}\langle M^\kappa\rangle_t\right\}$$

is a positive local martingale and therefore a supermartingale. Since it is equal to 1 at $t=0$,

$$E_{\phi^N}\left[\exp\left\{\beta N^d M_T^\kappa - \frac{\beta^2 N^{2d}}{2}\langle M^\kappa\rangle_T\right\}\right] \le 1.$$

Hence, by the definition of M_t^κ and (3.2), (3.3),

$$\begin{aligned} E_{\phi^N}\Bigg[\exp\Bigg\{ &-\beta N^d\|h^N(0)\|^2 \\ &-\beta N^d\int_0^T e^{-\kappa t}\left(2-N^{-d}\sum_{b\in\Gamma_N^*}V'(\nabla\phi_t^N(b))\nabla\phi_t^N(b)-\kappa\|h^N(t)\|^2\right)dt \\ &-4\beta^2N^d\int_0^T e^{-2\kappa t}\|h^N(t)\|^2\,dt\Bigg\}\Bigg] \le 1. \end{aligned}$$

Note that we have dropped the non-negative term $\beta N^d \|h^N(T)\|^2 e^{-\kappa T}$ appearing in the exponential of the left hand side. Therefore, if $\kappa - 4\beta e^{-\kappa s} \geq 1$ holds for $s \in [0, T]$,

$$E_{\phi^N}\Bigg[\exp\Bigg\{\beta \int_0^T e^{-\kappa t} \sum_{b\in\Gamma_N^*} V'(\nabla\phi_t^N(b))\nabla\phi_t^N(b)\, dt + \beta N^d \int_0^T e^{-\kappa t} \|h^N(t)\|^2\, dt\Bigg\}\Bigg] \leq \exp\Bigg\{\beta N^d \|h^N(0)\|^2 + 2\beta N^d \int_0^T e^{-\kappa t}\, dt\Bigg\}.$$

Choose now $\kappa = 5$. Then $\kappa - 4\beta e^{-\kappa s} \geq 1$ holds for every $0 < \beta \leq 1$ and $s \in [0, T]$, and thus we obtain the desired estimate (3.1). □

The next proposition improves the exponential estimate for the integral $\int_0^T \|h^N(t)\|^2\, dt$ given in Corollary 3.2 into that for $\sup_{0\leq t\leq T} \|h^N(t)\|^2$.

Proposition 3.3. *Let* $(\phi^N)_{N\geq 1}$ *be a sequence satisfying* (2.3). *Then,*

$$\limsup_{\beta\downarrow 0}\limsup_{N\to\infty} N^{-d} \log E_{\phi^N}\left[\exp\left\{\beta N^d \sup_{0\leq t\leq T} \|h^N(t)\|^2\right\}\right] \leq 0. \tag{3.4}$$

Proof. Set

$$A_t = \|h^N(0)\|^2 + \int_0^t L^N \left\|h^N(s)\right\|^2 ds.$$

Then $\|h^N(t)\|^2 = M_t + A_t$ holds with the martingale $M_t \equiv M_t^0$ (i.e. $\kappa = 0$) given in the proof of Proposition 3.1. Therefore, by Schwarz's inequality

$$E_{\phi^N}\left[\exp\left\{\beta N^d \sup_{0\leq t\leq T} \|h^N(t)\|^2\right\}\right] \leq \left(I_{1,\beta}^N\right)^{1/2} \cdot \left(I_{2,\beta}^N\right)^{1/2}, \tag{3.5}$$

where

$$I_{1,\beta}^N = E_{\phi^N}\left[\exp\left\{2 \sup_{0\leq t\leq T}\left(\beta N^d M_t - \frac{\beta^2 N^{2d}}{2}\langle M\rangle_t\right)\right\}\right],$$

$$I_{2,\beta}^N = E_{\phi^N}\left[\exp\left\{2 \sup_{0\leq t\leq T}\left(\beta N^d A_t + \frac{\beta^2 N^{2d}}{2}\langle M\rangle_t\right)\right\}\right].$$

Let us estimate the term $I_{1,\beta}^N$. Since

$$M_{1,t} := \exp\left\{\beta N^d M_t - \frac{\beta^2 N^{2d}}{2}\langle M\rangle_t\right\}$$

is a local martingale, there exists an increasing sequence of stopping times $\{t_n\}$ such that $\{M_{1,t\wedge t_n}\}$ are martingales for all n. Thus, by Doob's inequality, the term $I_{1,\beta}^{N,n}$

which is defined by replacing M_t and $\langle M\rangle_t$ with $M_{t\wedge t_n}$ and $\langle M\rangle_{t\wedge t_n}$ respectively in $I_{1,\beta}^N$ can be estimated as

$$\begin{aligned} I_{1,\beta}^{N,n} &= E_{\phi^N}\left[\sup_{0\le t\le T}\left(M_{1,t\wedge t_n}\right)^2\right] \\ &\le 4E_{\phi^N}\left[\exp\left\{2\beta N^d M_{T\wedge t_n} - \beta^2 N^{2d}\langle M\rangle_{T\wedge t_n}\right\}\right] \\ &\le 4E_{\phi^N}\left[\exp\left\{6\beta^2 N^{2d}\langle M\rangle_{T\wedge t_n}\right\}\right]^{1/2}. \end{aligned}$$

The last inequality follows by Schwarz's inequality, since $\exp\left\{4\beta N^d M_{t\wedge t_n} -8\beta^2 N^{2d}\langle M\rangle_{t\wedge t_n}\right\}$ is a supermartingale which is equal to 1 at $t = 0$. Letting $n\to\infty$, from Fatou's lemma and monotone convergence theorem, we get

$$I_{1,\beta}^N \le 4E_{\phi^N}\left[\exp\left\{6\beta^2 N^{2d}\langle M\rangle_T\right\}\right]^{1/2}.$$

By Corollary 3.2 and noting $\langle M\rangle_T = 8N^{-d}\int_0^T \|h^N(s)\|^2\,ds$, we have

$$\limsup_{\beta\downarrow 0}\limsup_{N\to\infty} N^{-d}\log\left(I_{1,\beta}^N\right)^{1/2} \le 0.$$

Next, we estimate the term $I_{2,\beta}^N$ in (3.5). Since $L^N\|h^N\|^2 \le 2$, we have

$$I_{2,\beta}^N \le \exp\left(2\beta N^d\|h^N(0)\|^2 + 4\beta N^d T\right) E_{\phi^N}\left[\exp\left\{8\beta^2 N^d\int_0^T \|h^N(t)\|^2\,dt\right\}\right],$$

and therefore, from Corollary 3.2 we have

$$\limsup_{\beta\downarrow 0}\limsup_{N\to\infty} N^{-d}\log\left(I_{2,\beta}^N\right)^{1/2} \le 0.$$

This completes the proof. □

Finally, we show superexponential weak equicontinuity estimate for $h^N(t)$. For $J = J(\theta)\in C(\mathbb{T}^d)$, we introduce the process $h_t^N(J)$ as

$$h_t^N(J) := \int_{\mathbb{T}^d} J(\theta)h^N(t,\theta)\,d\theta = N^{-(d+1)}\sum_{x\in\Gamma_N}\phi_t^N(x)J^N(x),$$

where $J^N(x)$ is the function defined by

$$J^N(x) = N^d\int_{B(x/N,1/N)} J(\theta)\,d\theta, \quad x\in\Gamma_N.$$

Proposition 3.4. *Let* $(\phi^N)_{N\ge 1}$ *be a sequence satisfying* (2.3). *For all* $\epsilon > 0$ *and* $J \in C^1(\mathbb{T}^d)$,

$$\limsup_{\delta\downarrow 0}\limsup_{N\to\infty} N^{-d}\log P_{\phi^N}\left(\sup_{\substack{0\le t_1<t_2\le T,\\ t_2-t_1<\delta}}\left|h_{t_1}^N(J)-h_{t_2}^N(J)\right|>\epsilon\right)=-\infty. \tag{3.6}$$

Proof. Using Itô's formula, we get

$$h_t^N(J)=h_0^N(J)-\xi_t^N(J)+w_t^N(J),$$

where

$$\xi_t^N(J)=\frac{1}{2}N^{-d+1}\int_0^t\sum_{b\in\Gamma_N^*}\nabla J^N(b)V'(\nabla\phi_s^N(b))\,ds,$$
$$w_t^N(J)=\sqrt{2}N^{-d}\sum_{x\in\Gamma_N}J^N(x)w_t(x).$$

Therefore, for completing the proof, it is sufficient to show the superexponential estimate (3.6) for $\xi_t^N(J)$ and $w_t^N(J)$ in place of $h_t^N(J)$, respectively. First we consider for $\xi_t^N\equiv\xi_t^N(J)$. Since we see

$$P_{\phi^N}\left(\sup_{\substack{0\le t_1<t_2\le T,\\ t_2-t_1<\delta}}\left|\xi_{t_1}^N-\xi_{t_2}^N\right|>\epsilon\right)\le\sum_{k=0}^{[T/\delta]}P_{\phi^N}\left(\sup_{k\delta\le t<(k+1)\delta}\left|\xi_t^N-\xi_{k\delta}^N\right|>\frac{\epsilon}{4}\right)$$

and

$$\left|\xi_{t_1}^N-\xi_{t_2}^N\right|\le\frac{1}{2}c_+N^{-d}\|\nabla J\|_\infty\int_{t_1}^{t_2}\sum_{b\in\Gamma_N^*}\left|\nabla\phi_s^N(b)\right|\,ds$$

for $0\le t_1<t_2\le T$, we get

$$\begin{aligned}&P_{\phi^N}\left(\sup_{\substack{0\le t_1<t_2\le T,\\ t_2-t_1<\delta}}\left|\xi_{t_1}^N-\xi_{t_2}^N\right|>\epsilon\right)\\&\quad\le\sum_{k=0}^{[T/\delta]}P_{\phi^N}\left(N^{-d}\int_{k\delta}^{(k+1)\delta\wedge T}\sum_{b\in\Gamma_N^*}|\nabla\phi_s^N(b)|\,ds>\frac{\epsilon}{c}\right),\end{aligned} \tag{3.7}$$

where $c=2c_+\|\nabla J\|_\infty$; we may assume J is not a constant function. However, for every $\beta>0$, dividing the integral into the region of $s\in[k\delta,(k+1)\delta\wedge T]$ satisfying $|\nabla\phi_s^N(b)|\le\beta^{-1}$ and its complement, we have

$$\int_{k\delta}^{(k+1)\delta\wedge T}\sum_{b\in\Gamma_N^*}\left|\nabla\phi_s^N(b)\right|\,ds\le 2dN^d\beta^{-1}\delta+\beta\int_0^T\sum_{b\in\Gamma_N^*}\left|\nabla\phi_s^N(b)\right|^2\,ds.$$

Thus, by Chebyshev's inequality, for every $\alpha > 0$

$$P_{\phi^N}\left(N^{-d}\int_{k\delta}^{(k+1)\delta\wedge T}\sum_{b\in\Gamma_N^*}|\nabla\phi_s^N(b)|\,ds > \frac{\epsilon}{c}\right)$$
$$\le \exp\left\{\alpha\left(2d\beta^{-1}\delta-\frac{\epsilon}{c}\right)N^d\right\}E_{\phi^N}\left[\exp\left\{\alpha\beta\int_0^T\sum_{b\in\Gamma_N^*}\left|\nabla\phi_t^N(b)\right|^2\,dt\right\}\right].$$

Therefore, we get

$$\limsup_{\delta\downarrow 0}\limsup_{N\to\infty}N^{-d}\log P_{\phi^N}\left(\sup_{\substack{0\le t_1<t_2\le T,\\ t_2-t_1<\delta}}\left|\xi_{t_1}^N-\xi_{t_2}^N\right|>\epsilon\right)$$
$$\le \limsup_{N\to\infty}N^{-d}\log E_{\phi^N}\left[\exp\left\{\alpha\beta\int_0^T\sum_{b\in\Gamma_N^*}\left|\nabla\phi_t^N(b)\right|^2\,dt\right\}\right]-\frac{\alpha\epsilon}{c}.$$

Take the limit $\beta\downarrow 0$ in the right hand side. Then, by Corollary 3.2, the first term vanishes. Since $\alpha > 0$ is arbitrary, the estimate (3.6) is shown for $\xi_t^N = \xi_t^N(J)$ instead of $h_t^N(J)$. The estimate (3.6) for $w_t^N(J)$ is easy, since

$$\left\{2N^{-2d}\sum_{x\in\Gamma_N}(J^N(x))^2\right\}^{-1/2} w_t^N(J)$$

are Brownian motions under P_{ϕ^N}. The proof of Proposition 3.4 is concluded. □

4. One-block estimate

The goal of the present section is to show the following theorem from which the superexponential one-block estimate follows immediately. This theorem will be used to prove Theorem 2.2 combining with the superexponential two-blocks estimate which will be shown in the next section.

Theorem 4.1. *Let $(\phi^N)_{N\ge1}$ be a sequence of initial configurations satisfying $\sup_N\|h^N\| < \infty$ for the associated $(h^N)_{N\ge1}$. Then, for every $F\in C_{\mathrm{loc},b}(\mathcal{X})$ and $\alpha > 0$,*

$$\limsup_{l\to\infty}\limsup_{N\to\infty}N^{-d}\log E_{\phi^N}\left[\exp\left\{\alpha\int_0^T\sum_{x\in\Gamma_N}\left|\tau_x W_l^F(\nabla\phi_t^N)\right|\,dt\right\}\right]\le 0.$$

To conclude the proof of the theorem we have to show that for every $\alpha > 0$,

$$\limsup_{l\to\infty}\limsup_{N\to\infty}N^{-d}\log E_{\phi^N}\left[\exp\left\{\alpha N^d\int_0^T U_{N,l}^F(\nabla\phi_t^N)\,dt\right\}\right]\le 0, \tag{4.1}$$

where

$$U^F_{N,l}(\eta) = N^{-d} \sum_{x\in\Gamma_N} \left| \mathrm{Av}_{\Lambda_l+x}\, F(\eta) - \tilde{F}(\bar{\eta}^l_x) \right|.$$

The proof of the superexponential one-block estimate (4.1) will be reduced to the usual one-block estimate (see (4.3) below) by means of Portenko's argument, [11, p. 117]. Such idea was already taken by [9]; however, since they treated one dimensional model, it was sufficient to divide the time interval [0,T] into small pieces for the integral appearing in the exponential in (4.1). To study higher dimensional model, it turns out to be necessary for the lattice space simultaneously divided into suitably small domains, namely, we divide Γ_N into boxes with side length $N^{1/d}$ (we assume $N^{1/d}$ is an integer for simplicity); see Remark 4.1 below. In other words, we have

$$\Gamma_N = \bigcup_{a\in\tilde{\Gamma}_N} (\Lambda^{(N)} + a),$$

with $\tilde{\Gamma}_N := N^{1/d}(\mathbb{Z}/N^{(1-1/d)}\mathbb{Z})^d = \{N^{1/d}, 2N^{1/d}, \dots, N\}^d$ and $\Lambda^{(N)} := \Lambda_{N^{1/d}/2}$ (as we have remarked, such partition in space is unnecessary in one dimension). Then, for $A > 0$, set

$$U^{A,F}_{N,l}(\eta) := N^{-d} \sum_{a\in\tilde{\Gamma}_N} U^{A,F}_{a,N,l}(\eta),$$

$$U^{A,F}_{a,N,l}(\eta) := 1_{\{\mathrm{Av}_{(\Lambda^{(N)}+a)^*}\,\eta^2\le A\}} \sum_{x\in\tilde{\Lambda}^{(N)}+a} |\,\mathrm{Av}_{\Lambda_l+x}\, F - \tilde{F}(\bar{\eta}^l_x)|,$$

where

$$\mathrm{Av}_{(\Lambda^{(N)}+a)^*}\,\eta^2 := \left|(\Lambda^{(N)}+a)^*\right|^{-1} \sum_{b\in(\Lambda^{(N)}+a)^*} \eta(b)^2,$$

$$\tilde{\Lambda}^{(N)} := \Lambda_{(N^{1/d}-N^{1/2d})/2}.$$

Lemma 4.2. *To prove Theorem* 4.1, *it is enough to show* (4.1) *for every* $\alpha, A > 0$ *with* $U^{A,F}_{N,l}$ *replacing* $U^F_{N,l}$*:*

$$\limsup_{l\to\infty} \limsup_{N\to\infty} N^{-d} \log E_{\phi^N}\left[\exp\left\{\alpha N^d \int_0^T U^{A,F}_{N,l}(\nabla\phi^N_t)\,dt\right\}\right] \le 0. \tag{4.2}$$

Proof. For $A > 0$,

$$\begin{aligned}&\left|U^F_{N,l}(\eta) - U^{A,F}_{N,l}(\eta)\right| \\ &\quad\le 2\|F\|_\infty N^{-d}\left\{\sum_{a\in\tilde{\Gamma}_N} 1_{\{\mathrm{Av}_{(\Lambda^{(N)}+a)^*}\,\eta^2 > A\}} |\tilde{\Lambda}^{(N)}| + |\tilde{\Gamma}_N|\,|\Lambda^{(N)} \setminus \tilde{\Lambda}^{(N)}|\right\}\end{aligned}$$

$$\leq 2\|F\|_\infty \left\{ N^{-d+1} A^{-1} \sum_{a\in\tilde{\Gamma}_N} \mathrm{Av}_{(\Lambda^{(N)}+a)^*}\, \eta^2 + N^{-1/2d} \right\}$$
$$= 2\|F\|_\infty \left\{ N^{-d} (4dA)^{-1} \sum_{b\in\Gamma_N^*} \eta(b)^2 + N^{-1/2d} \right\}.$$

Using Corollary 3.2, (4.1) follows from (4.2) by taking A sufficiently large. □

We now reduce the proof of (4.2) to the usual one-block estimate:

Lemma 4.3. *The inequality* (4.2) *follows from*

$$\lim_{l\to\infty} \limsup_{N\to\infty} \sup_{\phi,\phi_t} E_{\phi,\phi_t}\left[\int_0^{T_0/N} U_{0,N,l}^{A,F}(\nabla\phi_s^N)\, ds \right] = 0, \tag{4.3}$$

where $T_0 = T/c_1$, $c_1 = [2\alpha\|F\|_\infty T] + 1$ *and the supremum is taken over all initial configurations* $\phi = \{\phi(x), x \in \Lambda^{(N)}\} \in \mathbb{R}^{\Lambda^{(N)}}$ *and all moving boundary conditions* $\phi_t = \{\phi_t(x), x \in \partial\Lambda^{(N)}\} \in C\left([0, T_0/N], \mathbb{R}^{\partial\Lambda^{(N)}}\right)$, *where* $\partial\Lambda^{(N)} = \{x \in \Lambda^{(N)};\ \mathrm{dist}(x, (\Lambda^{(N)})^c) = 1\}$. *The dynamics* $\phi_t^N = \{\phi_t^N(x), x \in \Lambda^{(N)}\}$ *is defined by the SDEs* (2.1) *for* $x \in \Lambda^{(N)}$ *having initial data* ϕ *on* $\Lambda^{(N)}$ *and boundary condition* ϕ_t *at* $\partial\Lambda^{(N)}$.

Proof. The expectation in the left hand side of (4.2) is rewritten and then bounded as follows:

$$E_{\phi^N}\left[\prod_{k=1}^{c_1 N} \prod_{a\in\tilde{\Gamma}_N} \exp\left\{ \alpha \int_{(k-1)T_0/N}^{kT_0/N} U_{a,N,l}^{A,F}(\nabla\phi_s^N)\, ds \right\} \right]$$
$$\leq \left\{ \sup_{\phi,\phi_t} E_{\phi,\phi_t}\left[\exp\left\{ \alpha \int_0^{T_0/N} U_{0,N,l}^{A,F}(\nabla\phi_s^N)\, ds \right\} \right] \right\}^{c_1 N^d}. \tag{4.4}$$

The second line is obtained by noting the (space-time) Markov property and the shift invariance of the dynamics. Therefore, (4.2) is shown once we can prove

$$\limsup_{l\to\infty} \limsup_{N\to\infty} \sup_{\phi,\phi_t} E_{\phi,\phi_t}\left[\exp\left\{ \alpha \int_0^{T_0/N} U_{0,N,l}^{A,F}(\nabla\phi_s^N)\, ds \right\} \right] \leq 1. \tag{4.5}$$

Now, expanding the exponential as a sum of powers, using identity

$$E_{\phi,\phi_t}\left[\left(\int_0^t W(\nabla\phi_s^N)\, ds \right)^k \right]$$
$$= k!\, E_{\phi,\phi_t}\left[\int \cdots \int_{0\leq s_1\leq\cdots\leq s_k\leq t} \prod_{j=1}^k W(\nabla\phi_{s_j}^N)\, ds_1 \ldots ds_k \right]$$

and repeating the same trick in time on Markov process presented above, we obtain that the expected value appearing in (4.5) is bounded from above by

$$\sum_{k=0}^{\infty}\left(\sup_{\phi,\phi_t} E_{\phi,\phi_t}\left[\int_0^{T_0/N} \alpha U_{0,N,l}^{A,F}(\nabla\phi_s^N)\,ds\right]\right)^k$$
$$=\left(1-\sup_{\phi,\phi_t} E_{\phi,\phi_t}\left[\int_0^{T_0/N} \alpha U_{0,N,l}^{A,F}(\nabla\phi_s^N)\,ds\right]\right)^{-1}.$$

Note that T_0 is chosen in such a manner that the expected value in the last formula is strictly less than 1. Therefore, the proof of (4.2) can be reduced to that of (4.3). □

Remark 4.1. Note that $|\tilde{\Gamma}_N| = N^{d-1}$ so that, once we divide the time interval $[0,T]$ into $c_1 N$ pieces, the whole space-time region $\Gamma_N \times [0,T]$ is divided into $N^{d-1} \times c_1 N = c_1 N^d$ pieces. This compensates with the factor N^d appearing in the exponential of (4.1). If we try to introduce the division only in time, we need to divide it into N^d pieces. Then, the time length TN^{-d} of each piece, which corresponds to TN^{2-d} in microscopic time scale, is too short for the system to reach the equilibrium states if $d \geq 2$.

The goal is now to prove (4.3). Let us define probability measures μ_{ϕ,ϕ_t}^N on $\mathcal{X}_{(\Lambda^{(N)})^*}$, the configuration space on $\left(\Lambda^{(N)}\right)^*$, by the space-time average of the distribution of $\nabla\phi_t^N$, i.e.,

$$E^{\mu_{\phi,\phi_t}^N}[F(\eta)] := \frac{1}{T_0}\sum_{x\in\Lambda^{(N)}} E_{\phi,\phi_t}\left[\int_0^{T_0/N} F(\tau_x^{(N)}\nabla\phi_s^N)\,ds\right], \tag{4.6}$$

for every $F \in C_{\mathrm{loc},b}(\mathcal{X})$ with support inside $\Lambda^{(N)}$, where $\tau_x^{(N)}$ is the shift operator on $\Lambda^{(N)}$; we regard $\Lambda^{(N)}$ as a periodic lattice by putting bonds to connect corresponding boundary points of opposite side and define $\eta(b) := 0$ for such bonds. Extending the configurations periodically, μ_{ϕ,ϕ_t}^N can be regarded as probability measures on $\mathcal{X}$. Define the function $\mathrm{Av}\,\eta^2$ on $\mathcal{X}$ by

$$\mathrm{Av}\,\eta^2 := \limsup_{K\to\infty} \mathrm{Av}_{(\Lambda_K)^*}\,\eta^2 \in [0,\infty].$$

The operator $L = \sum_{x\in\mathbb{Z}^d} L_x$ denotes the generator of the process $\eta_t = \{\eta_t(b) \equiv \nabla\phi_t(b), b \in (\mathbb{Z}^d)^*\}$, where $\phi_t = \{\phi_t(x), x \in \mathbb{Z}^d\}$ is the solution of the SDEs (1.1) defined for $x \in \mathbb{Z}^d$. See [6], Sect. 4.2 for L_x.

Lemma 4.4. *Every vague limit μ of $\{\mu_{\phi,\phi_t}^N\}_{N,\phi,\phi_t}$ as $N \to \infty$ is L-stationary and shift invariant. Here, vague convergence means those for all finite dimensional marginal distributions as well as for the distributions of* $\mathrm{Av}\,\eta^2$.

Proof. We recall that arbitrary family of probability measures on $\mathbb{R}^{\Lambda^*}$ for finite $\Lambda \subset \mathbb{Z}^d$ is vaguely tight. Assume that μ is a vague limit of $\mu^{N'} := \mu^{N'}_{\phi^{N'},\phi_t^{N'}}$ along a subsequence $\{N' \nearrow \infty\}$ of $\{N\}$ and set $L_N = \sum_{x\in\Lambda^{(N)}} L_x$. Then, since

$$\left|E^{\mu^{N'}}[LF]\right| = \left|E^{\mu^{N'}}[L_{N'}F]\right| \le \left\{2(N')^{-2} + |\Lambda| \cdot |\partial\Lambda^{(N')}|/|\Lambda^{(N')}|\right\} \|F\|_\infty$$

holds for every $F \in C_{\mathrm{loc},0}(\mathcal{X})$ if its support Λ satisfies $\Lambda \subset \Lambda^{(N')}$, we have $E^\mu[LF] = 0$ by letting $N' \to \infty$. Accordingly, μ is L-stationary. Shift invariance of μ follows by definition. □

The next lemma establishes the uniform law of large numbers for the Gibbs measures $\{\mu_u, |u| \le A\}$ for every $A > 0$, where μ_u is the unique shift invariant ergodic tempered Gibbs measure on $\mathcal{X}$ with mean $u \in \mathbb{R}^d$. A probability measure μ on $\mathcal{X}$ is called tempered if $E^\mu[\eta(b)^2] < \infty$ for each $b \in (\mathbb{Z}^d)^*$, see [6].

Lemma 4.5. *For every $F \in C_{loc,b}(\mathcal{X})$ and $A > 0$,*

$$\lim_{l\to\infty} \sup_{|u|\le A} E^{\mu_u}\left[\left|\mathrm{Av}_{\Lambda_l} F(\eta) - \tilde{F}(\bar{\eta}_0^l)\right|\right] = 0. \tag{4.7}$$

Proof. For every $u, v \in \mathbb{R}^d$, Proposition 2.1 in [6] gives a shift invariant coupled probability measure P on $\mathcal{X} \times \mathcal{X}$ with μ_u and μ_v as its first and second marginals, respectively, in such a way that P satisfies

$$E^P[||\eta^1 - \eta^2||_e^2] \le C_1|u-v|^2 \tag{4.8}$$

for $C_1 > 0$, where $(\eta^1, \eta^2) \in \mathcal{X}\times\mathcal{X}$ and $||\eta||_e^2 = \sum_{i=1}^d |\eta(e_i)|^2$ for $\eta = \{\eta(b)\} \in \mathcal{X}$. Set the expectation in (4.7) $g^l(u)$ and estimate the difference

$$\begin{aligned} |g^l(u) - g^l(v)| \le\ & E^P\left[\left|\mathrm{Av}_{\Lambda_l} F(\eta^1) - \mathrm{Av}_{\Lambda_l} F(\eta^2)\right|\right] \\ & + E^P\left[\left|\tilde{F}\left(\overline{(\eta^1)}_0^l\right) - \tilde{F}\left(\overline{(\eta^2)}_0^l\right)\right|\right]. \end{aligned} \tag{4.9}$$

We first assume that the function $F \in C_{loc,b}(\mathcal{X})$ is Lipschitz; namely, F has a form $F(\eta) = \bar{F}(\{\eta(b)\}_{b\in(\check{\Lambda})^*})$ for some finite $\check{\Lambda} \subset \mathbb{Z}^d$ and Lipschitz continuous function $\bar{F}$ on $\mathbb{R}^{(\check{\Lambda})^*}$. Then, since

$$\left|\mathrm{Av}_{\Lambda_l} F(\eta^1) - \mathrm{Av}_{\Lambda_l} F(\eta^2)\right| \le C_2|\Lambda_l|^{-1} \sum_{x\in\Lambda_l} ||\tau_x\eta^1 - \tau_x\eta^2||_{\check{\Lambda}}$$

where $||\eta||_{\check{\Lambda}}^2 = \sum_{b\in(\check{\Lambda})^*} |\eta(b)|^2$, using (4.8) and recalling the shift invariance of P, the first term in the right hand side of (4.9) can be bounded by $C_3|u-v|$. The second term also has a similar bound, since we see

$$|\tilde{F}(a) - \tilde{F}(b)| \le E^P[|F(\eta^1) - F(\eta^2)|] \le C_4|a-b|, \qquad a, b \in \mathbb{R}^d.$$

Accordingly, the uniform Lipschitz continuity of $g^l(u)$ in l is shown. However, $\lim_{l\to\infty} g^l(u) = 0$ is easy for each $u \in \mathbb{R}^d$, and therefore the uniform convergence (4.7) is established at least for Lipschitz continuous F. Since functions in $C_{loc,b}(\mathcal{X})$ can be uniformly approximated by Lipschitz continuous bounded local functions, the lemma is proved. □

Let us continue the proof of (4.3). We first assume $F \in C_{loc,0}(\mathcal{X})$ with support $\check{\Lambda}$. Take a function $\varphi_K \in C_0(\mathbb{R})$, $K > 0$ for cut-off in such a way that $0 \le \varphi_K \le 1$, $\varphi_K(z) = 1$ for $|z| \le K$ and $\varphi_K(z) = 0$ for $|z| \ge K + 1$. Then, the expectation in (4.3) divided by T_0 can be estimated by the sum

$$E^{\mu^N_{\phi,\phi_t}}\left[1_{\{\mathrm{Av}\,\eta^2 \le A\}}\left|\mathrm{Av}_{\Lambda_l}\, F(\eta) - \tilde{F}(\bar{\eta}^l_0)\varphi_K(\bar{\eta}^l_0)\right|\right] + E^{\mu^N_{\phi,\phi_t}}\left[\left|\tilde{F}(\bar{\eta}^l_0)(\varphi_K(\bar{\eta}^l_0) - 1)\right|\right]. \tag{4.10}$$

Note that $\mathrm{Av}\,\eta^2 = \mathrm{Av}_{(\Lambda^{(N)})^*}\,\eta^2$ under μ^N_{ϕ,ϕ_t}. Since F has a bound $|F(\eta)| \le ||F||_\infty 1_{\{||\eta||_{\check{\Lambda}} \le R\}}(\eta)$ for some $R > 0$, for every $\epsilon > 0$ one can find $K > 0$ such that

$$|\tilde{F}(u)| = |E^{\mu_u}[F]| \le ||F||_\infty \mu_u(||\eta||_{\check{\Lambda}} \le R) \le \epsilon$$

if $|u| \ge K$. The last inequality follows from $\sup_{u \in \mathbb{R}^d, 1 \le i \le d} E^{\mu_u}[|\eta(e_i) - u_i|^2] < \infty$, see (3.6) in [6]. In particular, the second term of (4.10) is uniformly bounded by ϵ. On the other hand for the first term of (4.10), since $F(\eta)$ and $\tilde{F} \cdot \varphi_K\left(\bar{\eta}^l_0\right) \in C_{loc,0}(\mathcal{X})$, from Lemma 4.4

$$\begin{aligned}\limsup_{N\to\infty}\sup_{\phi,\phi_t} E^{\mu^N_{\phi,\phi_t}}&\left[1_{\{\mathrm{Av}\,\eta^2 \le A\}}\left|\mathrm{Av}_{\Lambda_l}\, F(\eta) - \tilde{F}(\bar{\eta}^l_0)\varphi_K(\bar{\eta}^l_0)\right|\right]\\ &\le \sup_{\mu \in \mathcal{S}} E^{\mu}\left[1_{\{\mathrm{Av}\,\eta^2 \le A\}}\left|\mathrm{Av}_{\Lambda_l}\, F(\eta) - \tilde{F}(\bar{\eta}^l_0)\varphi_K(\bar{\eta}^l_0)\right|\right],\end{aligned}$$

where $\mathcal{S}$ is the class of all L-stationary and shift invariant subprobability measures μ on $\mathcal{X}$. However, each $\mu \in \mathcal{S}$ can be represented as a mixture of extremal probability measures $\bar{\mu}$ for which $\mathrm{Av}\,\eta^2 = E^{\bar{\mu}}[||\eta||_e^2]$ hold. Therefore the above supremum is unchanged when it is restricted over all $\mu \in \mathcal{S}$ being tempered. Such μ is a mixture of $\{\mu_u, u \in \mathbb{R}^d\}$ (see [6]) and accordingly the right hand side is bounded by

$$\sup_{u \in \mathbb{R}^d : E^{\mu_u}[||\eta||_e^2] \le A} E^{\mu_u}\left[\left|\mathrm{Av}_{\Lambda_l}\, F(\eta) - \tilde{F}(\bar{\eta}^l_0)\right|\right] + \epsilon,$$

if K is taken as above. This proves (4.3) for $F \in C_{loc,0}(\mathcal{X})$ from Lemma 4.5.

Next we consider for $F \in C_{loc,b}(\mathcal{X})$ with support $\check{\Lambda}$. Take another cut-off function $\psi = \psi_K(\eta_{\check{\Lambda}}) \in C_{loc,0}(\mathcal{X})$, $K > 0$ such that $0 \le \psi \le 1$, $\psi(\eta) = 1$ for $||\eta||_{\check{\Lambda}} \le K$ and $\psi(\eta) = 0$ for $||\eta||_{\check{\Lambda}} \ge K + 1$, and decompose F into the sum $F = F \cdot \psi + F(1 - \psi)$. Since $F \cdot \psi \in C_{loc,0}(\mathcal{X})$, one can apply the result obtained above. On the other hand, for the term $F(1 - \psi)$

$$\frac{1}{N} U^{A,F(1-\psi)}_{0,N,l}(\eta) \le \frac{||F||_\infty}{N} 1_{\{\mathrm{Av}_{(\Lambda(N))^*}\,\eta^2 \le A\}} \sum_{x \in \tilde{\Lambda}^{(N)}} \left\{\mathrm{Av}_{\Lambda_{l+x}}\,\Psi_K(\eta) + \tilde{\Psi}_K(\bar{\eta}^l_x)\right\},$$

where $\Psi_K(\eta) := 1_{\{||\eta||_{\check\Lambda} \geq K\}}$. Under the condition $\mathrm{Av}_{(\Lambda^{(N)})^*}\, \eta^2 \leq A$ for $\eta \in \mathcal{X}$, if N is larger enough than l (l is indeed fixed here),

$$\frac{1}{N}\sum_{x\in\tilde\Lambda^{(N)}} \mathrm{Av}_{\Lambda_l+x}\, \Psi_K(\eta) \leq \frac{1}{N}\sum_{x\in\Lambda_{(N^{1/d}-N^{1/2d})/2+l}} \tau_x \Psi_K(\eta)$$

$$\leq \frac{1}{NK}\sum_{x\in\Lambda_{(N^{1/d}-N^{1/2d})/2+l}} ||\eta||_{\check\Lambda+x} \leq \frac{|\check\Lambda|}{K}\sqrt{A}.$$

We similarly have under the same condition for $\eta \in \mathcal{X}$

$$\frac{1}{N}\sum_{x\in\tilde\Lambda^{(N)}} \tilde\Psi_K(\bar\eta_x^l) \leq \frac{C_1|\check\Lambda|}{K}\left\{1+\frac{1}{N}\sum_{x\in\tilde\Lambda^{(N)}} |\bar\eta_x^l|\right\} \leq \frac{C_1|\check\Lambda|}{K}(1+\sqrt{A}),$$

since

$$\tilde\Psi_K(u) \leq \frac{1}{K}E^{\mu_u}[||\eta||_{\check\Lambda}] \leq \frac{C_1|\check\Lambda|}{K}(1+|u|),$$

use the estimate $E^{\mu_u}[|\eta(b)|^2] \leq C_2(1+|u|^2)$ which follows from Theorem 3.4-(iv), (v) of [6]. We have therefore shown

$$\frac{1}{N}U_{0,N,l}^{A,F(1-\psi)}(\eta) \leq \frac{C_3|\check\Lambda|}{K}(1+\sqrt{A})$$

which can be made small enough for large $K > 0$. Thus the proof of (4.3) and therefore that of Theorem 4.1 is concluded.

5. Two-Blocks Estimate

For $f \in C(\mathbb{R}^d)$, set

$$\bar W_{l,\epsilon}^{f,N}(\eta) := \mathrm{Av}_{\Lambda_{N\epsilon}}\left\{f\left(\bar\eta_0^l\right)\right\} - f(\bar\eta_0^{N\epsilon}).$$

In this section, we shall prove the following theorem. Theorem 2.2 is readily shown from this theorem and Theorem 4.1; see the end of this section.

Theorem 5.1. *Let* $(\phi^N)_{N\geq 1}$ *be a sequence satisfying the condition* (2.3). *Then, for every bounded and globally Lipschitz continuous function* f *on* $\mathbb{R}^d$ *and every* $\alpha > 0$,

$$\limsup_{l\to\infty}\limsup_{\epsilon\downarrow 0}\limsup_{N\to\infty} N^{-d} \times \log E_{\phi^N}\left[\exp\left\{\alpha\int_0^T \sum_{x\in\Gamma_N}\left|\tau_x \bar W_{l,\epsilon}^{f,N}(\nabla\phi_t^N)\right| dt\right\}\right] \leq 0. \qquad (5.1)$$

The proof of Theorem 5.1 is given based on the next theorem.

Theorem 5.2. *Under the condition* (2.3) *on* $(\phi^N)_{N\ge 1}$*, for every* $\alpha > 0$*,*

$$\limsup_{\epsilon\downarrow 0}\limsup_{N\to\infty} N^{-d}\log E_{\phi^N}\left[\exp\left\{\alpha(2N\epsilon+1)^{-d}\right.\right.$$
$$\left.\left.\times\sum_{y\in\Lambda_{N\epsilon}}\int_0^T\left(\sum_{b\in\Gamma_N^*}\Psi_y(b;\nabla\phi_t^N)-4N^d\right)dt\right\}\right]\le 0,$$

where

$$\Psi_y(b;\eta) := \{V'(\eta(b+y)) - V'(\eta(b))\}\{\eta(b+y)-\eta(b)\}$$

and the bond $b+y$ *is defined by* $b+y := (x_b+y, y_b+y)$*.*

Proof of Theorem 5.1. We assume Theorem 5.2 is already shown. For a fixed positive integer l and $x, y\in\mathbb{Z}^d$ such that $(x+\Lambda_l)\cap(y+\Lambda_l)=\emptyset$, define $U_{x,y}$ and $W^l_{x,y}$ as

$$U_{x,y}(\eta) = 2\sum_{i=1}^d\left\{V'(\eta(e_i+x)) - V'(\eta(e_i+y))\right\}\{\eta(e_i+x)-\eta(e_i+y)\} - 4, \tag{5.2}$$

$$W^l_{x,y}(\eta) = (2l+1)^{-2d}\sum_{\substack{z_1\in\Lambda_l+x\\ z_2\in\Lambda_l+y}} U_{z_1,z_2}(\eta). \tag{5.3}$$

From Proposition 3.1 and Theorem 5.2, we get the following estimate for every l and $\alpha > 0$:

$$\limsup_{\epsilon\downarrow 0}\limsup_{N\to\infty} N^{-d}\log E_{\phi^N}\left[\exp\left\{\alpha\int_0^T(2N\epsilon+1)^{-d}\right.\right.$$
$$\left.\left.\times\sum_{y\in\Lambda_{N\epsilon}\setminus\Lambda_{2l}}\sum_{x\in\Gamma_N} W^l_{x,x+y}(\nabla\phi_t^N)\,dt\right\}\right]\le 0. \tag{5.4}$$

We now introduce cut-off. For $K>0$, define functions V'_K and I_K on $\mathbb{R}$ by

$$V'_K(\eta) = \big(V'(\eta)\vee V'(-K)\big)\wedge V'(K),$$
$$I_K(\eta) = (\eta\vee(-K))\wedge K,$$

for $\eta\in\mathbb{R}$ and $U^K_{x,y}$ and $W^{l,K}_{x,y}$ on $\mathcal{X}$ by

$$U^K_{x,y}(\eta) = 2\sum_{i=1}^d\left\{V'_K(\eta(e_i+x)) - V'_K(\eta(e_i+y))\right\}$$
$$\times\{I_K(\eta(e_i+x)) - I_K(\eta(e_i+y))\} - 4,$$
$$W^{l,K}_{x,y}(\eta) = (2l+1)^{-2d}\sum_{\substack{z_1\in\Lambda_l+x\\ z_2\in\Lambda_l+y}} U^K_{z_1,z_2}(\eta),$$

for $\eta \in \mathcal{X}$, respectively. Since V is strictly convex, $W^l_{x,y} \geq -4$ and $W^l_{x,y} \geq W^{l,K}_{x,y}$ hold, and therefore we get

$$W^l_{x,x+y}(\eta) \geq W^{l,K}_{x,x+y}(\eta) 1_{|\bar\eta^l_x| \vee |\bar\eta^l_{x+y}| \leq A} - 4 \cdot 1_{|\bar\eta^l_x| \vee |\bar\eta^l_{x+y}| > A} \tag{5.5}$$

for every $A, K > 0$ and $\eta \in \mathcal{X}$. However, $W^{l,K}_{x,y}$ has the representation

$$\begin{aligned} W^{l,K}_{x,y} = 2 \Bigg\{ & \mathrm{Av}_{\Lambda_l + x}\, g_K + \mathrm{Av}_{\Lambda_l + y}\, g_K \\ & - \sum_{i=1}^d \left(\mathrm{Av}_{\Lambda_l + x}\, V'_{K,i} \cdot \mathrm{Av}_{\Lambda_l + y}\, I_{K,i} + \mathrm{Av}_{\Lambda_l + y}\, V'_{K,i} \cdot \mathrm{Av}_{\Lambda_l + x}\, I_{K,i} \right) \Bigg\} - 4, \end{aligned} \tag{5.6}$$

where g_K, $V'_{K,i}$ and $I_{K,i}$, $1 \leq i \leq d$, are functions on $\mathcal{X}$ defined respectively by

$$\begin{aligned} g_K(\eta) &= \sum_{i=1}^d V'_K(\eta(e_i)) I_K(\eta(e_i)), \\ V'_{K,i}(\eta) &= V'_K(\eta(e_i)), \\ I_{K,i}(\eta) &= I_K(\eta(e_i)). \end{aligned}$$

For every $\alpha > 0$, using one-block estimate (Theorem 4.1), we get from (5.4), (5.5) and (5.6)

$$\begin{aligned} & \limsup_{l\to\infty} \limsup_{\epsilon\downarrow 0} \limsup_{N\to\infty} N^{-d} \log E_{\phi^N} \Bigg[\exp \Bigg\{ \alpha \int_0^T (2N\epsilon + 1)^{-d} \\ & \times \sum_{y \in \Lambda_{N\epsilon} \setminus \Lambda_{2l}} \sum_{x \in \Gamma_N} \Bigg\{ 2 \Bigg(\tilde g_K \left(\bar\eta^l_x(t) \right) + \tilde g_K \left(\bar\eta^l_{x+y}(t) \right) \\ & \quad - \sum_{i=1}^d \tilde V'_{K,i} \left(\bar\eta^l_x(t) \right) \tilde I_{K,i} \left(\bar\eta^l_{x+y}(t) \right) \\ & \quad - \sum_{i=1}^d \tilde V'_{K,i} \left(\bar\eta^l_{x+y}(t) \right) \tilde I_{K,i} \left(\bar\eta^l_x(t) \right) - 2 \Bigg) 1_{|\bar\eta^l_x(t)| \vee |\bar\eta^l_{x+y}(t)| \leq A} \\ & \quad - 4 \times 1_{|\bar\eta^l_x(t)| \vee |\bar\eta^l_{x+y}(t)| > A} \Bigg\} dt \Bigg\} \Bigg] \leq 0, \end{aligned} \tag{5.7}$$

where the functions $\tilde g_K(u)$, $\tilde V'_{K,i}(u)$ and $\tilde I_{K,i}(u)$ on $\mathbb{R}^d$ are defined by the expected values under μ_u of g_K, $V'_{K,i}$ and $I_{K,i}$, respectively; recall the formula (2.9). We have simply denoted $\eta^N_t \equiv \nabla\phi^N_t$ by $\eta(t)$. From Theorem 3.4-(iv) of [6] and Brascamp-Lieb uniform exponential bound on μ_u (see (3.6) in [6]), the sequence $\tilde g_K$ converges as $K \to \infty$ to

$$\tilde{g}(u) := \sum_{i=1}^{d} E^{\mu_u}[\eta(e_i)V'(\eta(e_i))] = u \cdot \nabla\sigma(u) + 1$$

uniformly on every compact set of $\mathbb{R}^d$. In the same manner, $\tilde{V'}_{K,i}$ and $\tilde{I}_{K,i}$ converge uniformly on every compact set to $\partial\sigma/\partial u_i$ and u_i, respectively. Therefore,

$$\limsup_{l\to\infty}\limsup_{\epsilon\downarrow 0}\limsup_{N\to\infty} N^{-d}\log E_{\phi^N}\left[\exp\left\{\alpha\int_0^T 2(2N\epsilon+1)^{-d}\right.\right.$$
$$\left.\left.\times\sum_{y\in\Lambda_{N\epsilon}\setminus\Lambda_{2l}}\sum_{x\in\Gamma_N}\left\{\tilde{U}^{l,A}_{x,x+y}(\eta(t)) - 2\times 1_{|\bar{\eta}^l_x(t)|\vee|\bar{\eta}^l_{x+y}(t)|>A}\right\}dt\right\}\right] \le 0, \tag{5.8}$$

where $\tilde{U}^{l,A}_{x,y}$ is defined by

$$\tilde{U}^{l,A}_{x,y}(\eta) = (\nabla\sigma(\bar{\eta}^l_x) - \nabla\sigma(\bar{\eta}^l_y))\cdot(\bar{\eta}^l_x - \bar{\eta}^l_y)1_{|\bar{\eta}^l_x|\vee|\bar{\eta}^l_y|\le A}.$$

Using Schwarz's inequality and Proposition 3.1, for every $\alpha > 0$,

$$\limsup_{A\to\infty}\limsup_{l\to\infty}\limsup_{\epsilon\downarrow 0}\limsup_{N\to\infty} N^{-d}\log E_{\phi^N}\left[\exp\left\{\alpha\int_0^T (2N\epsilon+1)^{-d}\right.\right.$$
$$\left.\left.\times\sum_{y\in\Lambda_{N\epsilon}\setminus\Lambda_{2l}}\sum_{x\in\Gamma_N}\tilde{U}^{l,A}_{x,x+y}(\nabla\phi^N_t)\,dt\right\}\right] \le 0. \tag{5.9}$$

From the convexity of σ, the expected value in (5.9) is increasing in A. Therefore, for every $A > 0$,

$$\limsup_{l\to\infty}\limsup_{\epsilon\downarrow 0}\limsup_{N\to\infty} N^{-d}\log E_{\phi^N}\left[\exp\left\{\alpha\int_0^T (2N\epsilon+1)^{-d}\right.\right.$$
$$\left.\left.\times\sum_{y\in\Lambda_{N\epsilon}\setminus\Lambda_{2l}}\sum_{x\in\Gamma_N}\tilde{U}^{l,A}_{x,x+y}(\nabla\phi^N_t)\,dt\right\}\right] \le 0. \tag{5.10}$$

Since f is globally Lipschitz and σ is strictly convex (see [3] or [7]), the proof of Theorem 5.1 is concluded. □

We turn to the proof of Theorem 5.2, which requires some notation. For $\theta' \in \mathbb{T}^d$, $\bar{\tau}_{\theta'}$ represents the macroscopic space shift by θ'. In this way, for a function f on $\mathbb{T}^d$,

$$(\bar{\tau}_{\theta'}f)(\theta) = f(\theta'+\theta), \quad \theta\in\mathbb{T}^d.$$

The proof of Theorem 5.2 relies on three lemmas. We simply denote ϕ^N by ϕ.

Lemma 5.3. *For every $y \in \Lambda_{N\epsilon}$,*

$$\left\| \bar{\tau}_{y/N} h^N - h^N \right\|^2 \le \epsilon^2 N^{-d} \sum_{b \in \Gamma_N^*} (\nabla\phi(b))^2. \tag{5.11}$$

Proof. By the definition of h^N,

$$\begin{aligned}\left\| \bar{\tau}_{y/N} h^N - h^N \right\|^2 &= N^{-d-2} \sum_{x \in \Gamma_N} |\phi(x+y) - \phi(x)|^2 \\ &= N^{-d-2} \sum_{x \in \Gamma_N} \left| \sum_{b \in C_{0,y}} \nabla\phi(b+x) \right|^2.\end{aligned}$$

Here $C_{0,y}$ denotes a sequence of bonds b connecting two sites 0 and y. One can take in such a manner that $|C_{0,y}| = |y|$. Noting that the right hand side is an average over all $x \in \Gamma_N$, we get the conclusion by Schwarz's inequality. □

The next lemma corresponds to Lemma 4.4 of [9] and its proof is based on the symmetry of V and similar to that of Lemma 2.3 of [6].

Lemma 5.4.

$$L^N \left\| \bar{\tau}_{y/N} h^N - h^N \right\|^2 = 4 - N^{-d} \sum_{b \in \Gamma_N^*} \Psi_y(b; \nabla\phi). \tag{5.12}$$

Proof. We may assume $y \ne 0$. Since $\left\| \bar{\tau}_{y/N} h^N - h^N \right\|^2$ has a representation given in the proof of Lemma 5.3,

$$L^N \left\| \bar{\tau}_{y/N} h^N - h^N \right\|^2 = N^{-d} \sum_{x \in \Gamma_N} \left\{ 4 - 2 \sum_{b: x_b = x} \Phi_y(b)(\phi(x+y) - \phi(x)) \right\},$$

where we set

$$\Phi_y(b) := V'(\nabla\phi(b+y)) - V'(\nabla\phi(b)).$$

However, $\sum_{b: x_b = x} \Phi_y(b) = -\sum_{b: y_b = x} \Phi_y(b)$ by symmetry of V and therefore

$$\begin{aligned}L^N \left\| \bar{\tau}_{y/N} h^N - h^N \right\|^2 &= N^{-d} \sum_{x \in \Gamma_N} \left\{ 4 - \sum_{b: x_b = x} \Phi_y(b)(\phi(x_b+y) - \phi(x_b)) \right. \\ &\qquad \left. + \sum_{b: y_b = x} \Phi_y(b)(\phi(y_b+y) - \phi(y_b)) \right\} \\ &= 4 - N^{-d} \sum_{b \in \Gamma_N^*} \Phi_y(b)(\nabla\phi(b+y) - \nabla\phi(b)),\end{aligned}$$

which shows (5.12). □

For a fixed $y \in \Gamma_N$, we consider the martingale $M_t^N(y)$ defined by

$$M_t^N(y) := N^d \left\| \bar{\tau}_{y/N} h^N(t) - h^N(t) \right\|^2 - N^d \left\| \bar{\tau}_{y/N} h^N(0) - h^N(0) \right\|^2 - N^d \int_0^t L^N \left\| \bar{\tau}_{y/N} h^N(s) - h^N(s) \right\|^2 ds.$$

Define $M_t^{N,\epsilon}$, $\epsilon > 0$ as a martingale which is the average of the martingales $M_t^N(y)$ in $y \in \Lambda_{N\epsilon}$:

$$M_t^{N,\epsilon} = (2N\epsilon + 1)^{-d} \sum_{y \in \Lambda_{N\epsilon}} M_t^N(y).$$

To keep notation simple, we denote the martingale $M_t^{N,\epsilon}$ by M_t^N.

Lemma 5.5. *The quadratic variation of M_t^N has a bound*

$$\langle M^N \rangle_t \le 32\epsilon^2 \int_0^t \sum_{b \in \Gamma_N^*} (\nabla \phi_s^N(b))^2 \, ds.$$

Proof. Since ϕ_t^N is a solution of the SDEs (2.1), the martingale $M_t^N(y)$ has the following representation:

$$M_t^N(y) = 2\sqrt{2} N^{-1} \int_0^t \sum_{x \in \Gamma_N} (2\phi_s^N(x) - \phi_s^N(x-y) - \phi_s^N(x+y)) \, dw_s(x).$$

Therefore, for any $y, z \in \Lambda_{N\epsilon}$, computing cross-variation of $M_t^N(y)$ and $M_t^N(z)$,

$$\begin{aligned} &\langle M^N(y), M^N(z) \rangle_t \\ &\quad = 8N^{-2} \int_0^t \sum_{x \in \Gamma_N} (2\phi_s^N(x) - \phi_s^N(x-y) - \phi_s^N(x+y)) \\ &\qquad\qquad \times (2\phi_s^N(x) - \phi_s^N(x-z) - \phi_s^N(x+z)) \, ds \\ &\quad \le 16N^{-2} \int_0^t \sum_{x \in \Gamma_N} \{(\phi_s^N(x) - \phi_s^N(x-y))^2 + (\phi_s^N(x) - \phi_s^N(x-z))^2\} \, ds \\ &\quad \le 32\epsilon^2 \int_0^t \sum_{b \in \Gamma_N^*} (\nabla \phi_s^N(b))^2 \, ds. \end{aligned}$$

The last inequality is shown similarly to that in the proof of Lemma 5.3. The conclusion follows by recalling the definition of M_t^N. □

Proof of Theorem 5.2. Since we have

$$E_{\phi^N}[\exp\{2\alpha M_T^N - 2\alpha^2 \langle M^N \rangle_T\}] \le 1,$$

for every $\alpha > 0$, recalling the definition of M_t^N and using Lemmas 5.3-5.5,

$$E_{\phi^N}\left[\exp\left\{-2\alpha\epsilon^2\sum_{b\in\Gamma_N^*}(\nabla\phi_0^N(b))^2 - 2\alpha(2N\epsilon+1)^{-d}\right.\right.$$
$$\times\sum_{y\in\Lambda_{N\epsilon}}\int_0^T\left(4N^d - \sum_{b\in\Gamma_N^*}\Psi_y(b;\nabla\phi_t^N)\right)dt$$
$$\left.\left.-64\alpha^2\epsilon^2\int_0^T\sum_{b\in\Gamma_N^*}(\nabla\phi_t^N(b))^2\,dt\right\}\right]\leq 1.$$

An application of Schwarz's inequality implies

$$E_{\phi^N}\left[\exp\left\{\alpha(2N\epsilon+1)^{-d}\sum_{y\in\Lambda_{N\epsilon}}\int_0^T\left(\sum_{b\in\Gamma_N^*}\Psi_y(b;\nabla\phi_t^N)-4N^d\right)dt\right\}\right]$$
$$\leq E_{\phi^N}\left[\exp\left\{2\alpha\epsilon^2\sum_{b\in\Gamma_N^*}(\nabla\phi_0^N(b))^2+64\alpha^2\epsilon^2\int_0^T\sum_{b\in\Gamma_N^*}(\nabla\phi_t^N(b))^2\,dt\right\}\right]^{1/2}.$$

Thus the proof of Theorem 5.2 is concluded by applying Corollary 3.2. □

Proof of Theorem 2.2. For every $l \geq 1$, we estimate

$$\begin{aligned}\sum_{x\in\Gamma_N}\left|\tau_x W_{N\epsilon}^F(\eta)\right| \leq &\sum_{x\in\Gamma_N}\left|\mathrm{Av}_{\Lambda_{N\epsilon}+x}F(\eta)-\mathrm{Av}_{\Lambda_{N\epsilon}+x}\mathrm{Av}_{\Lambda_l}F(\eta)\right|\\ &+\sum_{x\in\Gamma_N}\left|\mathrm{Av}_{\Lambda_{N\epsilon}+x}\mathrm{Av}_{\Lambda_l}F(\eta)-\mathrm{Av}_{\Lambda_{N\epsilon}+x}\left\{\tilde F\left(\bar\eta_0^l\right)\right\}\right|\\ &+\sum_{x\in\Gamma_N}\left|\mathrm{Av}_{\Lambda_{N\epsilon}+x}\left\{\tilde F\left(\bar\eta_0^l\right)\right\}-\tilde F(\bar\eta_x^{N\epsilon})\right|\\ =:&\, I_1^{N,l,\epsilon}(\eta)+I_2^{N,l,\epsilon}(\eta)+I_3^{N,l,\epsilon}(\eta).\end{aligned}\tag{5.13}$$

The first term has a trivial bound: $I_1^{N,l,\epsilon}(\eta)\leq Cl(N\epsilon)^{d-1}\|F\|_\infty$. We therefore obtain (2.10) by applying Theorem 4.1 for $I_2^{N,l,\epsilon}$ noting that $I_2^{N,l,\epsilon}(\eta) \leq \sum_{x\in\Gamma_N}|\tau_x W_l^F(\eta)|$ and Theorem 5.1 for $I_3^{N,l,\epsilon}$ taking $f(u)=\tilde F(u)$, respectively. The equality (2.11) is an immediate consequence of (2.10).

6. Lower bound

Before starting the proof of the large deviation principle, we show the following proposition which guarantees that the rate functional $I(h)$ of the large deviation principle is defined as $+\infty$ unless $h\in L^2([0,T],H^1(\mathbb{T}^d))$.

Proposition 6.1. *Let $h \in C([0,T], L^2_w(\mathbb{T}^d))$ be given. If there exists $\rho > 0$ such that*

$$\limsup_{N\to\infty} N^{-d} \log P_{\phi^N}(h^N \in \mathcal{O}) > -\rho \tag{6.1}$$

holds for every open set $\mathcal{O}$ in $C([0,T], L^2_w(\mathbb{T}^d))$ which contains h, then $h \in L^2([0,T], H^1(\mathbb{T}^d))$.

Proof. We define the operator $\mathcal{A}^\epsilon = \left(\mathcal{A}^\epsilon_i\right)_{1\le i\le d}$, $\epsilon > 0$ by

$$\left(\mathcal{A}^\epsilon_i h\right)(t,\theta) = (2\epsilon)^{-d-1} \int_{B(\theta,\epsilon)} (h(t,\theta'+\epsilon e_i) - h(t,\theta'-\epsilon e_i))\, d\theta',$$
$$(t,\theta) \in [0,T]\times\mathbb{T}^d,$$

for $h \in L^2([0,T]\times\mathbb{T}^d)$. Then, by Lemma 5.3,

$$\|\mathcal{A}^\epsilon_i h^N(t,\cdot)\|^2_{L^2(\mathbb{T}^d)} \le N^{-d} \sum_{b\in\Gamma^*_N} (\nabla\phi^N_t(b))^2.$$

Combining with Corollary 3.2, we get

$$\limsup_{\beta\downarrow 0} \limsup_{N\to\infty} \sup_{\epsilon} N^{-d} \log E_{\phi^N}\left[\exp\left(\beta N^d \left\|\mathcal{A}^\epsilon_i h^N\right\|^2_{L^2([0,T]\times\mathbb{T}^d)}\right)\right] \le 0$$

for every $1 \le i \le d$. On the other hand, using Chebyshev's inequality,

$$\begin{aligned} N^{-d} \log P_{\phi^N}(h^N \in \mathcal{O}) + \beta \sup_{\epsilon} \inf_{h\in\mathcal{O}} \|\mathcal{A}^\epsilon_i h\|^2_{L^2([0,T]\times\mathbb{T}^d)} \\ \le \sup_{\epsilon} N^{-d} \log E_{\phi^N}\left[\exp\left(\beta N^d \left\|\mathcal{A}^\epsilon_i h^N\right\|^2_{L^2([0,T]\times\mathbb{T}^d)}\right)\right]. \end{aligned}$$

Therefore, letting $N\to\infty$ and taking β sufficiently small, we get

$$\beta \sup_{\epsilon} \inf_{h\in\mathcal{O}} \|\mathcal{A}^\epsilon_i h\|^2_{L^2([0,T]\times\mathbb{T}^d)} < \rho.$$

Now, we note that for every $\epsilon > 0$, $\mathcal{A}^\epsilon_i h_n$ converges to $\mathcal{A}^\epsilon_i h$ pointwise on $[0,T]\times\mathbb{T}^d$ if h_n converges to h in $C([0,T], L^2_w(\mathbb{T}^d))$. Therefore we can conclude

$$\sup_{\epsilon} \|\mathcal{A}^\epsilon_i h\|^2_{L^2([0,T]\times\mathbb{T}^d)} < \frac{\rho}{\beta},$$

that is, the family $\{\mathcal{A}^\epsilon_i h;\ \epsilon > 0\}$ is precompact in $L^2([0,T]\times\mathbb{T}^d)$ under the weak topology. Thus, one can find a limit point $\check{h}_i \in L^2([0,T]\times\mathbb{T}^d)$ of this sequence as $\epsilon \downarrow 0$, and can easily see $\check{h}_i = \partial h/\partial\theta_i$, $1 \le i \le d$. This shows $h \in L^2([0,T], H^1(\mathbb{T}^d))$. □

We proceed to the proof of the lower bound (2.8) of the large deviation principle. To this end, let us consider the following weakly perturbed SDEs:

$$d\phi_t^{N,G}(x) = -N^2 \sum_{b\in\Gamma_N^*:x_b=x} V'(\nabla\phi_t^{N,G}(b))\,dt + NG(t,N^{-1}x)\,dt$$
$$+\sqrt{2}N\,dw_t(x), \quad x\in\Gamma_N, \tag{6.2}$$

where $G(t,\theta)$ is an arbitrarily given smooth function in t and θ. The generator of $\phi_t^{N,G}$ is denoted by $L^{N,G}$:

$$L^{N,G} = N^2 \sum_{x\in\Gamma_N} \left\{ \frac{\partial^2}{\partial\phi(x)^2} - \sum_{b:x_b=x} V'(\nabla\phi(b)) \frac{\partial}{\partial\phi(x)} \right.$$
$$\left. + N^{-1}G(t,N^{-1}x)\frac{\partial}{\partial\phi(x)} \right\}.$$

The distributions on the space $C([0,T],\mathbb{R}^{\Gamma_N})$ of weakly perturbed process $\phi_t^{N,G}$ and original unperturbed process ϕ_t^N are denoted by $P^{N,G}$ and P^N, respectively. We assume the initial configurations are common: $\phi_0^{N,G} = \phi_0^N$. The macroscopic height process corresponding to $\phi_t^{N,G}$ is denoted by $h^{N,G}(t)$; namely, $h^{N,G}(t,\theta) = N^{-1}\sum_{x\in\Gamma_N}\phi_t^{N,G}(x)1_{B(x/N,1/N)}(\theta)$ for $\theta\in\mathbb{T}^d$.

Theorem 6.2. *We assume that there exists a function $h_0\in L^2(\mathbb{T}^d)$ such that the initial configurations $h^{N,G}(0)\equiv h^N(0)$ satisfy*

$$\lim_{N\to\infty}\|h^{N,G}(0)-h_0\| = 0.$$

Then, for all $J\in C^\infty([0,T]\times\mathbb{T}^d)$,

$$\lim_{N\to\infty}\int_0^T dt\int_{\mathbb{T}^d} J(t,\theta)h^{N,G}(t,\theta)\,d\theta = \int_0^T dt\int_{\mathbb{T}^d} J(t,\theta)h^G(t,\theta)\,d\theta$$

in probability on $P^{N,G}$, where $h^G(t)$ is the unique weak solution of

$$\frac{\partial h^G}{\partial t} = \operatorname{div}\left(\nabla\sigma(\nabla h^G)\right) + G \tag{6.3}$$

with the initial data h_0.

Proof. Since we have

$$P^{N,G}(A) \le P^N(A)^{1/2} E^{P^N}\left[\left(\frac{dP^{N,G}}{dP^N}\right)^2\right]^{1/2}$$

for all events A, once we can prove

$$\limsup_{N\to\infty} N^{-d} \log E^{P^N}\left[\left(\frac{dP^{N,G}}{dP^N}\right)^2\right] < \infty, \tag{6.4}$$

the superexponential estimate (2.11) holds for $P^{N,G}$ in place of $P^N \equiv P_{\phi^N}$. Then, it is standard in the theory of hydrodynamic limit to show the conclusion. To prove (6.4) we use Girsanov's formula and Itô's formula, and obtain

$$\begin{aligned}
\left(\frac{dP^{N,G}}{dP^N}\right)^2 &= \exp\Bigg(\sqrt{2}\sum_{x\in\Gamma_N}\int_0^T G(t,N^{-1}x)\,dw_t(x) \\
&\qquad -\frac{1}{2}\sum_{x\in\Gamma_N}\int_0^T G^2(t,N^{-1}x)\,dt\Bigg) \\
&= \exp\Bigg(N^{-1}\sum_{x\in\Gamma_N} G(T,N^{-1}x)\phi_T^N(x) \\
&\qquad -N^{-1}\sum_{x\in\Gamma_N} G(0,N^{-1}x)\phi_0^N(x) \\
&\qquad -N^{-1}\sum_{x\in\Gamma_N}\int_0^T \frac{\partial G}{\partial t}(t,N^{-1}x)\phi_t^N(x)\,dt \\
&\qquad -N\sum_{x\in\Gamma_N}\int_0^T G(t,N^{-1}x)\sum_{b:x_b=x} V'(\nabla\phi_t^N(b))\,dt \\
&\qquad -\frac{1}{2}\sum_{x\in\Gamma_N}\int_0^T G^2(t,N^{-1}x)\,dt\Bigg).
\end{aligned} \tag{6.5}$$

Therefore, there exists a constant $C > 0$ which depends on $\|G\|_\infty$, $\|\partial G/\partial t\|_\infty$ and $\|\nabla G\|_\infty$ such that

$$\begin{aligned}
\left(\frac{dP^{N,G}}{dP^N}\right)^2 \le \exp\Bigg(& C\beta^{-1}N^d + C\beta N^d \sup_{0\le t\le T}\|h^N(t)\|^2 \\
&+ C\beta\sum_{b\in\Gamma_N^*}\int_0^T |\nabla\phi_t^N(b)|^2\,dt\Bigg)
\end{aligned} \tag{6.6}$$

for every $\beta > 0$. From Corollary 3.2 and Proposition 3.3, we get (6.4) taking β sufficiently small. □

The next proposition is routine so that the proof is omitted, see Lemma 2.4 in [4].

Proposition 6.3. *If the rate functional of h is finite (i.e. $I(h) < \infty$), then $G_h := \partial h/\partial t - \mathrm{div}(\nabla\sigma(\nabla h)) \in L^2([0,T]\times\mathbb{T}^d)$. Moreover, in this case, $I(h)$ has the following representation:*

$$I(h) = \frac{1}{4}\int_0^T dt \int_{\mathbb{T}^d} G_h(t,\theta)^2\, d\theta. \tag{6.7}$$

An energy estimate gives the continuity of the solution h^G of the PDE (6.3) in G:

Proposition 6.4. *There exists a constant $K > 0$ such that for every $G, \bar{G} \in L^2([0,T], \mathbb{V}^*)$*

$$\begin{aligned}\sup_{0\le t\le T} \|h^G(t) - h^{\bar{G}}(t)\|^2 + \int_0^T \|h^G(t) - h^{\bar{G}}(t)\|_{\mathbb{V}}^2\, dt \\ \le K\int_0^T \|G(t) - \bar{G}(t)\|_{\mathbb{V}^*}^2\, dt\end{aligned} \tag{6.8}$$

holds if we assume $h^G(0) = h^{\bar{G}}(0)$, where $\mathbb{V} = H^1(\mathbb{T}^d)$ and $\mathbb{V}^* = H^{-1}(\mathbb{T}^d)$ is the dual space of $\mathbb{V}$.

Proof. From the PDE (6.3), we get

$$\begin{aligned}\frac{\partial}{\partial t}\left\|h^G(t) - h^{\bar{G}}(t)\right\|^2 = 2\,{}_{\mathbb{V}}\Big\langle h^G(t) - h^{\bar{G}}(t), \left\{A(h^G(t)) - A(h^{\bar{G}}(t))\right\} \\ + \left\{G(t) - \bar{G}(t)\right\}\Big\rangle_{\mathbb{V}^*},\end{aligned} \tag{6.9}$$

where $A(h) = \mathrm{div}\,(\nabla\sigma(\nabla h))$, see Appendix I of [6] for details. Using Lemma 3.6 of [3], there is a constant c such that

$${}_{\mathbb{V}}\Big\langle h^G(t) - h^{\bar{G}}(t), A(h^G(t)) - A(h^{\bar{G}}(t))\Big\rangle_{\mathbb{V}^*} \le -c\|h^G(t) - h^{\bar{G}}(t)\|_{\mathbb{V}}^2.$$

Therefore,

$$\begin{aligned}\frac{\partial}{\partial t}\left\|h^G(t) - h^{\bar{G}}(t)\right\|^2 &\le -2c\|h^G(t) - h^{\bar{G}}(t)\|_{\mathbb{V}}^2 \\ &\quad +2\,{}_{\mathbb{V}}\Big\langle h^G(t) - h^{\bar{G}}(t), G(t) - \bar{G}(t)\Big\rangle_{\mathbb{V}^*} \\ &\le -c\left\|h^G(t) - h^{\bar{G}}(t)\right\|_{\mathbb{V}}^2 + \frac{1}{c}\left\|G(t) - \bar{G}(t)\right\|_{\mathbb{V}^*}^2\end{aligned} \tag{6.10}$$

holds. Integrating both sides, the conclusion is shown. □

Proof of (2.8). By Girsanov's formula,

$$\frac{dP^N}{dP^{N,G}} = \exp\Bigg(-\frac{1}{\sqrt{2}}\sum_{x\in\Gamma_N}\int_0^T G(t,N^{-1}x)\,dw_t(x) \\ -\frac{1}{4}\sum_{x\in\Gamma_N}\int_0^T G^2(t,N^{-1}x)\,dt\Bigg),$$

where $\{w_t(x), x \in \Gamma_N\}$ is a family of independent one dimensional standard Brownian motions under $P^{N,G}$. By similar calculation to (6.5), we can rewrite $dP^N/dP^{N,G}$ and get

$$\begin{aligned}\lim_{N\to\infty} N^{-d}\log\frac{dP^N}{dP^{N,G}} &= -\frac{1}{2}\int_{\mathbb{T}^d} G(T,\theta)h^G(T,\theta)\,d\theta + \frac{1}{2}\int_{\mathbb{T}^d} G(0,\theta)h_0(\theta)\,d\theta \\ &\quad + \frac{1}{2}\int_0^T dt\int_{\mathbb{T}^d}\frac{\partial G}{\partial t}(t,\theta)h^G(t,\theta)\,d\theta \\ &\quad + \frac{1}{2}\int_0^T dt\int_{\mathbb{T}^d} G(t,\theta)\,\mathrm{div}(\nabla\sigma(\nabla h^G))\,d\theta \\ &\quad + \frac{1}{4}\int_0^T dt\int_{\mathbb{T}^d} G^2(t,\theta)\,d\theta\end{aligned}$$

in the sense of convergence in probability under $P^{N,G}$. Note that the sum of first three terms coincides with

$$-\frac{1}{2}\int_0^T dt\int_{\mathbb{T}^d} G(t,\theta)\frac{\partial h^G}{\partial t}(t,\theta)\,d\theta.$$

Hence, recalling the definition of h^G,

$$\begin{aligned}\lim_{N\to\infty} N^{-d}\log\frac{dP^N}{dP^{N,G}} &= -\frac{1}{4}\int_0^T dt\int_{\mathbb{T}^d}\left(\frac{\partial h^G}{\partial t}(t,\theta) - \mathrm{div}(\nabla\sigma(\nabla h^G))\right)^2 d\theta \\ &= -I(h^G),\end{aligned}$$

in probability. Now, from the entropy inequality

$$\log\frac{P^N(\mathcal{O})}{P^{N,G}(\mathcal{O})} \geq -\frac{\mathbb{H}(P^{N,G}|P^N) + e^{-1}}{P^{N,G}(\mathcal{O})}$$

and Theorem 6.2, we obtain

$$\liminf_{N\to\infty} N^{-d}\log P^N(\mathcal{O}) \geq -I(h^G), \tag{6.11}$$

for every $G \in C^\infty([0,T] \times \mathbb{T}^d)$ such that $h^G \in \mathcal{O}$, where $\mathbb{H}(P^{N,G}|P^N) = E^{P^{N,G}}\left[\log \frac{dP^{N,G}}{dP^N}\right]$ is the relative entropy. Now, by Propositions 6.3 and 6.4, we get

$$\inf_{\substack{G\in C^\infty([0,T]\times\mathbb{T}^d) \\ \text{s.t. } h^G\in\mathcal{O}}} I(h^G) = \inf_{h\in\mathcal{O}} I(h). \tag{6.12}$$

Combining (6.11) and (6.12), the conclusion is shown. □

7. Upper bound

In this section, we prove the the upper bound (2.7). To do this, we require some notation. For $f = f(t,\theta)$, operators ∇_i^ϵ and $\tilde{\mathcal{A}}^\epsilon = (\tilde{\mathcal{A}}_i^\epsilon)_{1\le i\le d}$, $\epsilon > 0$ are defined by

$$\nabla_i^\epsilon f(t,\theta) := (2\epsilon)^{-1}\left(f(t,\theta+\epsilon e_i) - f(t,\theta-\epsilon e_i)\right),$$
$$\tilde{\mathcal{A}}_i^\epsilon f(t,\theta) := (2\epsilon)^{-d}\int_{B(\theta,\epsilon)} \nabla_i^{\epsilon^2} f(t,\theta+\theta')\,d\theta'.$$

For every smooth function $J(t,\theta)$ on $[0,T]\times\mathbb{T}^d$, we denote

$$J^N(t,\theta) := N^d \int_{B(\theta,1/N)} J(t,\theta')\,d\theta', \quad \theta\in\mathbb{T}^d,$$

and define $K_t^{N,\epsilon}$ and K_t^N by

$$K_t^{N,\epsilon} = S_t^{N,1} + S_t^{N,\epsilon,2}, \tag{7.1}$$
$$K_t^N = S_t^{N,1} + S_t^{N,2}, \tag{7.2}$$

where $S_t^{N,1}$, $S_t^{N,\epsilon,2}$ and $S_t^{N,2}$ are defined respectively by

$$S_t^{N,1} = N^d \int_{\mathbb{T}^d} h^N(t,\theta)J(t,\theta)\,d\theta - N^d\int_0^t ds\int_{\mathbb{T}^d} h^N(s,\theta)\frac{\partial J}{\partial s}(s,\theta)\,d\theta$$
$$S_t^{N,\epsilon,2} = N^d\int_0^t ds\int_{\mathbb{T}^d}\sum_{i=1}^d \frac{\partial J}{\partial\theta_i}(s,\theta)\frac{\partial\sigma}{\partial u_i}\left(\tilde{\mathcal{A}}^\epsilon h^N(s,\theta)\right)d\theta,$$
$$\begin{aligned} S_t^{N,2} &= N\int_0^t \sum_{x\in\Gamma_N}\sum_{b:x_b=x} V'(\nabla\phi_s^N(b))J^N(s,x/N)\,ds \\ &= \int_0^t\sum_{i=1}^d\sum_{x\in\Gamma_N} N\left(J^N(s,x/N+e_i/N) - J^N(s,x/N)\right) \\ &\qquad \times V'(\nabla\phi_s^N(e_i+x))\,ds. \end{aligned}$$

Note that the symmetry of V implies the last equality for $S_t^{N,2}$.

Our first goal is to derive an exponential bound on $K_T^{N,\epsilon}$:

$$\limsup_{\epsilon\downarrow 0}\limsup_{N\to\infty} N^{-d}\log E[\exp K_T^{N,\epsilon}] \le \int_{\mathbb{T}^d} J(0,\theta)h_0(\theta)\,d\theta + \int_0^T dt\int_{\mathbb{T}^d} J^2(t,\theta)\,d\theta, \tag{7.3}$$

where the expectation under P_{ϕ^N} is simply denoted by $E[\cdot]$. Using Itô's formula, one can observe that $\exp\left(K_t^N - \int_0^t \sum_{x\in\Gamma_N}\left(J^N(s,y/N)\right)^2 ds\right)$ is a martingale relative to measure P_{ϕ^N}. Therefore, the expectation of $\exp(K_T^N)$ is given by

$$E[\exp(K_T^N)] = \exp\left(N^d\int_{\mathbb{T}^d} h^N(0,\theta)J(0,\theta)\,d\theta + \int_0^T \sum_{x\in\Gamma_N}\left(J^N(t,y/N)\right)^2 dt\right). \tag{7.4}$$

On the other hand, we can write

$$K_T^{N,\epsilon} - K_T^N = S_T^{N,\epsilon,2} - S_T^{N,2} = I_1 + I_2 + I_3,$$

where

$$I_1 = \int_0^T \sum_{i=1}^d \sum_{x\in\Gamma_N}\left\{\left(\frac{\partial J}{\partial\theta_i}\right)^N(t,x/N) - N\left(J^N(t,x/N+e_i/N) - J^N(t,x/N)\right)\right\}V'(\nabla\phi_t^N(e_i+x))\,dt,$$

$$I_2 = \int_0^T \sum_{i=1}^d \sum_{x\in\Gamma_N}\left(\frac{\partial J}{\partial\theta_i}\right)^N(t,x/N) \times\left\{\frac{\partial\sigma}{\partial u_i}\left(\overline{(\nabla\phi_t^N)_x}^{N\epsilon}\right) - V'(\nabla\phi_t^N(e_i+x))\right\}dt,$$

$$I_3 = N^d\int_0^T dt\int_{\mathbb{T}^d}\sum_{i=1}^d \frac{\partial J}{\partial\theta_i}(t,\theta)\frac{\partial\sigma}{\partial u_i}\left(\tilde{\mathcal{A}}^\epsilon h^N(t,\theta)\right)d\theta - \int_0^T \sum_{i=1}^d\sum_{x\in\Gamma_N}\left(\frac{\partial J}{\partial\theta_i}\right)^N(t,x/N)\frac{\partial\sigma}{\partial u_i}\left(\overline{(\nabla\phi_t^N)_x}^{N\epsilon}\right)dt$$

for every $\epsilon > 0$. Using Hölder's inequality,

$$\begin{aligned} E[\exp K_T^{N,\epsilon}] &= E[\exp (K_T^N + I_1 + I_2 + I_3)] \\ &\leq E[\exp (pK_T^N)]^{1/p} E[\exp q(I_1 + I_2 + I_3)]^{1/q} \\ &= \exp \Bigg(N^d \int_{\mathbb{T}^d} h^N(0,\theta) J(0,\theta)\, d\theta \\ &\qquad + p \int_0^T \sum_{x\in\Gamma_N} \left(J^N(t, y/N)\right)^2 dt \Bigg) \\ &\qquad \times E[\exp q(I_1 + I_2 + I_3)]^{1/q}, \end{aligned} \tag{7.5}$$

where $p, q > 1$ satisfy $1/p + 1/q = 1$. We have used (7.4) with pJ in place of J.

We can estimate the second term in the right hand side of (7.5) as

$$E[\exp q(I_1 + I_2 + I_3)]^{1/q} \leq E[\exp 3qI_1]^{1/3q} E[\exp 3qI_2]^{1/3q} E[\exp 3qI_3]^{1/3q}. \tag{7.6}$$

Since J is continuously differentiable in θ, there is a constant $c_1(N)$ that goes to 0 as $N \to \infty$ such that

$$|I_1| \leq c_1(N) \int_0^T \sum_{b\in\Gamma_N^*} |V'(\nabla\phi_t^N(b))|\, dt. \tag{7.7}$$

Therefore, by Corollary 3.2, we have

$$\limsup_{N\to\infty} N^{-d} \log E[\exp 3qI_1] = 0. \tag{7.8}$$

For I_2, by boundedness of ∇J and Theorem 2.2,

$$\limsup_{\epsilon\downarrow 0} \limsup_{N\to\infty} N^{-d} \log E[\exp 3qI_2] = 0. \tag{7.9}$$

Finally, we estimate the third term in the right hand side of (7.6). By simple calculation, there exists a constant $c_2 > 0$ such that

$$\begin{aligned} &\sum_{x\in\Gamma_N} \left| \overline{(\nabla\phi_t^N)}_x^{N\epsilon} - (2N\epsilon + 1)^{-d} \sum_{y\in\Lambda_{N\epsilon}} \nabla_i^{\epsilon'} h^N(t, y/N + x/N) \right| \\ &\leq c_2 (2N\epsilon + 1)^{-1} (2N\epsilon' + 1) \left(\sum_{b\in\Gamma_N} \left(\nabla\phi_t^N(b)\right)^2 + N^d \right). \end{aligned}$$

Choosing $\epsilon' = \epsilon^2$ and combining with Proposition 3.1, we get

$$\limsup_{N\to\infty} N^{-d} \log E[\exp 3qI_3] = 0. \tag{7.10}$$

By (7.5), (7.6) and (7.8)–(7.10), letting $p \downarrow 1$, we obtain (7.3).

Now we are at the position to show the large deviation upper bound (2.7). Let us define

$$\begin{aligned}\bar{I}^{\epsilon}(h; J) &\equiv \bar{I}^{\epsilon}_T(h(\cdot); J) \\ &:= \int_{\mathbb{T}^d} J(T,\theta)\, h(T,\theta)\, d\theta - \int_0^T dt \int_{\mathbb{T}^d} \frac{\partial J}{\partial t}(t,\theta) h(t,\theta)\, d\theta \\ &\quad + \int_0^T dt \int_{\mathbb{T}^d} \sum_{i=1}^d \frac{\partial J}{\partial \theta_i}(t,\theta) \frac{\partial \sigma}{\partial u_i}\left(\tilde{\mathcal{A}}^{\epsilon} h(t,\theta)\right) d\theta. \end{aligned} \tag{7.11}$$

Then, we have the following relation:

$$E[\exp K_T^{N,\epsilon}] = E[\exp(N^d \bar{I}^{\epsilon}(h^N; J))]. \tag{7.12}$$

Using Chebyshev's inequality, for every $J = J(t,\theta)$ and $\epsilon > 0$

$$\limsup_{N\to\infty} N^{-d} \log P_{\phi^N}(h^N \in \mathcal{C}) \le \limsup_{N\to\infty} N^{-d} \log E[\exp K_T^{N,\epsilon}] - \inf_{h\in\mathcal{C}} \bar{I}^{\epsilon}_T(h; J). \tag{7.13}$$

Therefore, letting $\epsilon \downarrow 0$ in the right hand side, we conclude from (7.3)

$$\limsup_{N\to\infty} N^{-d} \log P_{\phi^N}(h^N \in \mathcal{C}) \le - \inf_{h\in\mathcal{C}} I(h; J).$$

Taking infimum in J, since $\sup_J$ and $\inf_{h\in\mathcal{C}}$ can be interchanged if $\mathcal{C}$ is compact, (2.7) is shown for compact $\mathcal{C}$. Now, Propositions 3.3 and 3.4 mean superexponential tightness of $h^N(t)$. Therefore, we can generalize the result for closed set $\mathcal{C}$.

8. Discussion

Deuschel et al. [3] recently investigated the large deviations for a sequence of finite volume Gibbs measures $\{\mu^N\}_{N\ge 1}$ for ϕ-field defined on $D \cap N^{-1}\mathbb{Z}^d$ having Dirichlet boundary conditions 0, where D is a bounded domain in $\mathbb{R}^d$ with Lipschitz boundary; the Gaussian case was studied by [1]. They have shown that the rate functional is given by the total surface tension

$$\mathbb{F}_D(\bar{h}) := \int_D \sigma(\nabla \bar{h}(\theta))\, d\theta, \qquad \bar{h} = \bar{h}(\theta) \in H_0^1(D). \tag{8.1}$$

The dependence of random variables $\{\phi(x)\}_x$ under μ^N extended over long distances caused by massless character of the model makes the proof non-trivial. In fact, their method relies on the technique exploited by Naddaf and Spencer [10], especially representations of covariances in terms of random walks in random environments fluctuating in time, and also PDE technique to treat boundary effect.

This paper, on the other hand, studies the corresponding dynamical problem. Since the distribution of $\nabla h^N(T,\cdot)$ weakly converges as $T \to \infty$ to the finite volume Gibbs measures μ^N on Γ_N for $\nabla\phi$-field, the natural question is whether one can recover the rate functional $\mathbb{F}_{\mathbb{T}^d}(\bar{h})$ for $\{\mu^N\}_N$ defined by the integral over $\mathbb{T}^d$ in place of D in (8.1) from the dynamical rate functional $I(h) = I_T(h)$ as $T \to \infty$. To answer such question, we denote the distribution of $h^N(T,\cdot)$ under P_{ϕ^N} by μ_T^N. Then, the contraction principle implies that the large deviation principle holds for $\{\mu_T^N\}_N$ with rate functional

$$S_T(\bar{h}) := \inf_{h=h(t,\theta) \text{ s.t. } h(T,\theta)=\bar{h}(\theta)} I_T(h), \qquad \bar{h} = \bar{h}(\theta) \in H^1(\mathbb{T}^d).$$

The relationship between $S_T(\bar{h})$ and the total surface tension $\mathbb{F}_{\mathbb{T}^d}(\bar{h})$ is stated in the following proposition:

Proposition 8.1.

$$\lim_{T\to\infty} S_T(\bar{h}) = \mathbb{F}_{\mathbb{T}^d}(\bar{h}). \tag{8.2}$$

Proof. Since the Fréchet derivative of the functional $\mathbb{F}_{\mathbb{T}^d}$ is given by

$$\frac{\delta \mathbb{F}_{\mathbb{T}^d}}{\delta h(\theta)}(h) = -\operatorname{div}(\nabla\sigma(\nabla h))(\theta),$$

the hydrodynamic equation which characterizes the minimal point $h = h(t,\theta)$ of I, i.e. h satisfying $I(h) = 0$, is simply a gradient flow determined by the potential energy $\mathbb{F}_{\mathbb{T}^d}$, see [12]. Another remark is that the critical points of $\mathbb{F}_{\mathbb{T}^d}$ are the horizontal surfaces: In fact, if $\delta\mathbb{F}_{\mathbb{T}^d}/\delta h(\theta) = 0$,

$$\begin{aligned} 0 &= \langle h(\cdot+\theta') - h(\cdot), \operatorname{div}(\nabla\sigma(\nabla h))(\cdot+\theta') - \operatorname{div}(\nabla\sigma(\nabla h))(\cdot)\rangle \\ &= -\langle \nabla h(\cdot+\theta') - \nabla h(\cdot), \nabla\sigma(\nabla h)(\cdot+\theta') - \nabla\sigma(\nabla h)(\cdot)\rangle \\ &\le -C\|\nabla h(\cdot+\theta') - \nabla h(\cdot)\|^2, \end{aligned}$$

for every $\theta' \in \mathbb{T}^d$ and for some $C > 0$. We have used the strict convexity of σ for the last inequality, see [3]. This shows $\|\nabla h(\cdot+\theta') - \nabla h(\cdot)\| = 0$ and consequently $\nabla h \equiv \text{const}$ which implies $\nabla h \equiv 0$ since $\int_{\mathbb{T}^d} \nabla h(\theta)d\theta = 0$. Therefore h is a horizontal surface: $h \equiv \text{const}$.

The infimum of $I_T(h)$ is attained by the time-reversed classical trajectory. Indeed, let $\tilde{h}(t,\theta), 0 \le t \le T$, be the solution of the hydrodynamic equation $\partial\tilde{h}/\partial t = \operatorname{div}(\nabla\sigma(\nabla\tilde{h}))$ with initial data $\tilde{h}(0,\theta) = \bar{h}(\theta)$, and define $h(t,\theta) := \tilde{h}(T-t,\theta), t \in [0,T]$. Then, we have $I_T(h) = \mathbb{F}_{\mathbb{T}^d}(\bar{h}) - \mathbb{F}_{\mathbb{T}^d}(\tilde{h}(T,\cdot))$, since $\|\partial h/\partial t - \operatorname{div}(\nabla\sigma(\nabla h))\|^2 = 4d\mathbb{F}_{\mathbb{T}^d}(h(t))/dt$. However, since the critical points of $\mathbb{F}_{\mathbb{T}^d}$ are horizontal surfaces, we have $\lim_{T\to\infty}\mathbb{F}_{\mathbb{T}^d}(\tilde{h}(T,\cdot)) = 0$ and this completes the proof. □

Remark 8.1. (i) The left hand side of (8.2) is called a quasi potential. It is known that, if the classical dynamics which is the minimizer of I is a gradient flow for a certain potential $\mathbb{F}$ and if all stable equilibrium points are global minimal points of $\mathbb{F}$, the quasi potential actually coincides with the potential $\mathbb{F}$ itself, see Theorem 3.1, p. 118 of [5] in a finite dimensional setting.

(ii) Proposition 8.1 shows that one can at least recover the static rate functional from the dynamic one. This does not imply that the static large deviations themselves can be recovered from the dynamic large deviations. What we did here is to take the limits first $N \to \infty$ and then $T \to \infty$. We need to interchange the order of the limits, for which the uniformity in T for the dynamic large deviations is required. This is left as a future problem.

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