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The asymmetric random cluster model and comparison of Ising and Potts models

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Abstract. We introduce the asymmetric random cluster (or ARC) model, which is a graphical representation of the Potts lattice gas, and establish its basic properties. The ARC model allows a rich variety of comparisons (in the FKG sense) between models with different parameter values; we give, for example, values (β, h) for which the 0's configuration in the Potts lattice gas is dominated by the "+" configuration of the (β, h) Ising model. The Potts model, with possibly an external field applied to one of the spins, is a special case of the Potts lattice gas, which allows our comparisons to yield rigorous bounds on the critical temperatures of Potts models. For example, we obtain $0.571 \leq 1 - \exp(-\beta_c) \leq 0.600$ for the 9-state Potts model on the hexagonal lattice. Another comparison bounds the movement of the critical line when a small Potts interaction is added to a lattice gas which otherwise has only interparticle attraction. ARC models can also be compared to related models such as the partial FK model, obtained by deleting a fraction of the nonsingleton clusters from a realization of the Fortuin-Kasteleyn random cluster model. This comparison leads to bounds on the effects of small annealed site dilution on the critical temperature of the Potts model.

1. Introduction

Random cluster models, or graphical representations, have become an increasingly important tool in the study of lattice models. Most prominently, the Fortuin-Kasteleyn random cluster model (or simply, the *FK model*), introduced in [17], [15] and [16], has been used to analyze aspects of the Potts and Ising models, including critical behavior [31], long-range versions [3], mean-field behavior in high dimensions [28], covariance structure [6], mixing properties [4] and efficient simulation [38]. Wiseman and Domany [40] and Pfister and Velenik [35] considered graphical representations of the Ashkin-Teller model, and graphical representations for large classes of models have been considered in the contexts of efficient simulation ([11],[12]) and conditions for Gibbs uniqueness [5].

A principal advantage of random cluster models is that the configuration space, typically $\{0, 1\}^{\mathscr{B}}$ for some set $\mathscr{B}$ of bonds, is partially ordered in a natural way, mean-

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ing that it makes sense to speak of one configuration being "larger than" another, or of one measure on configurations dominating another, in the FKG sense. The standard comparison theorem of [17] (see also [3]) says (in standard notation—see (2.22)) that if

$$1 \le q \le q' \quad \text{and} \quad \frac{p}{q(1-p)} \le \frac{p'}{q'(1-p')}$$

then the FK model with parameters (p', q') dominates the model with parameters (p, q) in the FKG sense (that is, increasing events have larger probabilities at (p', q').) This yields information about the smoothness of the critical line in the (p, q)-parameter space, among other things; see [3]. Another comparison inequality for the FK model appears in [20].

A principal disadvantage of the standard comparison theorem is that it is not very sharp. Rephrased, the theorem says that as one moves in (p, q)-space up any line $p/q(1-p) = c$, the configurations of the FK model get larger. In a sharper result, the corresponding lines would approximately parallel the presumed critical line, given by $p^2/q(1-p)^2 = 1$ for the two-dimensional integer lattice, which is clearly not so for the lines in the standard comparison theorem; the situation in higher dimensions is even worse. A second disadvantage is that an external field in the Potts model cannot be incorporated into the FK model in a very natural way.

In this paper we introduce a new model, the asymmetric random cluster model, (or simply, the *ARC model*), which is a random cluster representation of the Potts lattice gas (that is, the annealed site-diluted Potts model.) As is well known, the q-state Potts lattice gas includes the $(q + 1)$-state Potts model as a special case. We will show that the ARC model allows quite sharp comparison theorems between different parameter values of the Potts lattice gas. This leads to a variety of consequences. We obtain rigorous bounds $\beta_1 \le \beta_c \le \beta_2$ on the critical inverse temperature of the Potts model on various lattices (within about 5%, in many cases, and sometimes much less, of numerical or other nonrigorous estimates in the literature), and establish standard properties of the high-temperature regime, such as exponential decay of correlations and weak mixing, up to the lower bound β_1. By contrast, existing methods for establishing such properties are generally perturbative, working only for very small values of the inverse temperature. (For exceptions, generally involving $q = 2$ or q large, see e.g. [4], [28], [31], [36], [39].) We also obtain bounds on critical line, or critical surface, locations in the parameter space of the Potts lattice gas. We bound the change in the critical temperature, or in the critical line, when certain models are perturbed by adding a small term to the Hamiltonian. One such perturbation is small annealed site dilution added to a standard Potts model. In another perturbation, we begin with a lattice gas with only one species of particle, or essentailly equivalently, a Potts latttice with q species but with no additional energy associated with adjacent particles of mismatched species; we then add a small Potts interaction between the different species.

Numerous aspects of the phase diagram of the Potts lattice gas (though not the ones we consider here) have been studied in [10].

To enrich the set of possible set of comparisons which can be made using the ARC model, we also introduce and analyze what we call the *partial FK model*, which is obtained from the usual FK model by deleting a fraction of the nonsingleton clusters. Comparisons between ARC models and partial FK models are used in our analysis of site dilution.

Like the FK model, the ARC model is useful in constructing couplings between measures under different boundry conditions; we will demonstrate an application of such a construction. Further, we will use the ARC model to examine the question of when the distribution of the set of empty sites of a Potts lattice gas – that is, the distribution of $\{\delta_{[\sigma_x=0]} : x \in \Lambda\}$, where Λ is a subset of the lattice – has the FKG property. This includes, as a special case, the FKG property for the distribution of the set of sites of any one species in the Potts model, which was recently established by L. Chayes [9].

Of course the advantages of the ARC model over the FK model–principally sharper comparison theorems and more natural incorporation of external fields–do not come without a price. For example, there is phase coexistence in the Potts model precisely when there is percolation (under wired boundary conditions) in the corresponding FK model. In the ARC model, by contrast, there are two bond configurations, corresponding to the two pair interactions (Potts interaction and interparticle attraction), and the relation between percolation and phase transition is more complex. Further, correlations in the Potts model are given (under free boundary conditions) by connectivities in the corresponding FK model; the ARC model has no such property. So the ARC model supplements, but does not replace, the FK model.

2. Preliminaries and description of the models

By a *lattice* we mean a periodic graph embedded in Euclidean space. The *degree* of a site (that is, vertex) of a graph is the number of bonds emanating from that site. When the degree is the same for every site of a lattice, this degree is called the *coordination number* of the lattice.

The *q-state Potts lattice gas* on a finite subset Λ of a lattice $\mathbb{L}$ is described by variables $\sigma_x \in \{0, 1, .., q\}$ at each site $x \in \Lambda$; 0 denotes an empty site, and $1, .., q$ are possible spins, or species, for a particle at x. Let $n_x = \delta_{[\sigma_x \in \{1,..,q\}]}$ be the indicator of the presence of a particle at x. We write the Hamiltonian as

$$H(\sigma) = -J \sum_{\langle xy \rangle} n_x n_y \delta_{[\sigma_x=\sigma_y]} - \kappa \sum_{\langle xy \rangle} n_x n_y - \sum_x \mu_x n_x, \tag{2.1}$$

where the first two sums are over adjacent unordered pairs (that is, bonds) $\langle xy \rangle$ with $x, y \in \Lambda$; when there is a boundary condition we include also ajacent pairs with only one of x, y in Λ. We call J the *interaction strength,* κ the *interparticle attraction,* and μ_x the *chemical potential* at x. Note that when $\kappa = 0$, adjacent mismatched particles are energentically equivalent to adjacent empty sites, whereas adjacent matched particles have a lower energy. When $\kappa = 0$ we call the Potts lattice gas *neutral*. Let $\partial\Lambda$ denote the set of sites in Λ^c which are adjacent to Λ; by a *boundary condition* for the Potts lattice gas we mean a configuration $\eta \in \{0, 1, .., q\}^{\partial\Lambda}$. The corresponding Hamiltonian is denoted $H_{\Lambda,\eta}$, and the partition function for the Potts lattice gas at $(\beta, J, \kappa, \{\mu_x\})$ is

$$Z(\Lambda, \eta, \beta, J, \kappa, \{\mu_x\}) = \sum_\sigma e^{-\beta H_{\Lambda,\eta}(\sigma)}.$$

When $\mu_x = \mu$ for all x, the corresponding measure on $\{0, .., q\}^\Lambda$ is denoted $P^{PLG}_{\Lambda,\eta,q,\beta,J,\kappa,\mu}$. There are really only three free numerical parameters (or sets of parameters, if μ depends on x) in the partition function, so the inverse temperature β is a redundant parameter, though at times convenient; we will generally take β to be 1.

A configuration σ together with a boundary condition η on $\partial\Lambda$ (or on Λ^c) yields a combined configuration on $\overline{\Lambda} = \Lambda \cup \partial\Lambda$ (or on $\mathbb{L}$) which we denote $(\sigma\eta)$.

A graph G is designated by a pair $(\Lambda, \mathscr{B})$, where Λ is a set of sites and $\mathscr{B}$ is a set of bonds. The set of sites of G is also denoted $S(G)$, and the set of bonds is also denoted $B(G)$.

Let Λ be a finite set of sites of a lattice and let $\mathscr{B}(\Lambda) = \{\langle xy\rangle : x, y \in \Lambda\}$ and $\overline{\mathscr{B}}(\Lambda) = \{\langle xy\rangle : x \in \Lambda \text{ or } y \in \Lambda\}$. Given a subgraph, either $G = (\overline{\Lambda}, \mathscr{B})$ or $G = (\Lambda, \mathscr{B})$, of $(\overline{\Lambda}, \overline{\mathscr{B}}(\Lambda))$ and given a boundary condition η and a configuration σ on Λ, we define variables $N_{**} = N_{**}(G, (\sigma\eta))$ by

$$\begin{aligned}
N_{00} &= |\{\langle xy\rangle \in \mathscr{B} : (\sigma\eta)_x = (\sigma\eta)_y = 0\}|, \\
N_{ss} &= |\{\langle xy\rangle \in \mathscr{B} : (\sigma\eta)_x = (\sigma\eta)_y \in \{1, .., q\}\}|, \\
N_{ss'} &= |\{\langle xy\rangle \in \mathscr{B} : (\sigma\eta)_x, (\sigma\eta)_y \in \{1, .., q\}, (\sigma\eta)_x \neq (\sigma\eta)_y\}|, \\
N_{0s} &= |\{\langle xy\rangle \in \mathscr{B} : (\sigma\eta)_x \in \{1, .., q\}, (\sigma\eta)_y = 0\}|, \\
N_s &= |\{x \in \Lambda : \sigma_x \in \{1, .., q\}\}|, \\
N_0 &= |\{x \in \Lambda : \sigma_x = 0\}|,
\end{aligned} \tag{2.2}$$

so that

$$H(\sigma) = -(\kappa + J)N_{ss} - \kappa N_{ss'} - \sum_x \mu_x n_x \tag{2.3}$$

and

$$|\mathscr{B}| = N_{00} + N_{ss} + N_{ss'} + N_{0s}. \tag{2.4}$$

For $x \in \Lambda$ let

$$m_x = |\{y : \langle xy\rangle \in \mathscr{B}\}|.$$

In the case of $\mathbb{L}$ with coordination number m and $\mathscr{B} = \overline{\mathscr{B}}(\Lambda)$ we have $m_x = m$ for all x. In general,

$$\sum_x m_x \delta_{[\sigma_x=0]} = 2N_{00} + N_{0s}.$$

Subtracting this from (2.4), multiplying by κ and adding the result to (2.3) gives

$$H(\sigma) = -JN_{ss} - \kappa N_{00} + \sum_x (\mu_x + \kappa m_x)\delta_{[\sigma_x=0]} + c(G) \tag{2.5}$$

where $c(G)$ is a nonrandom constant. If $\kappa = J$ this becomes

$$H(\sigma) = -J(N_{ss} + N_{00} + \sum_x h_x \delta_{[\sigma_x=0]}) + c(G),$$

where h_x is given by

$$Jh_x = -(\mu_x + Jm_x).$$

The Hamiltonian for the $(q+1)$-state Potts model with external field h_x applied to spin 0 at each x is

$$H(\sigma) = -N_{ss} - N_{00} - \sum_x h_x \delta_{[\sigma_x=0]}, \tag{2.6}$$

so the q-state Potts lattice gas at $(1, J, J, \{\mu_x\})$ is the same as the $(q+1)$-state Potts model with inverse temperature β and external fields $\{h_x\}$ given by

$$\beta = J \quad \text{and} \quad \beta h_x = -(\mu_x + Jm_x). \tag{2.7}$$

Note that in the case of fixed coordination number and chemical potential, say $m_x = m$, $\mu_x = \mu$ for all $x \in \Lambda$, we have that $h = h_x$ does not depend on x, and

$$\sum_x h_x \delta_{[\sigma_x=0]} = hN_0.$$

In the further special case of the Ising model ($q + 1 = 2$ states), it is natural (since the external field is applied to spin 0) to relabel 0 as "+"and 1 as "-", and of course $N_{ss'} = 0$. The computation yielding (2.4), done in reverse, is then just the standard lattice-gas transformation of the Ising model:

$$\begin{aligned} H(\sigma) &= -(N_{--} + N_{++} + hN_+) \\ &= -2N_{--} + (m+h)N_- + c(G). \end{aligned} \tag{2.8}$$

Note that in some formulations in the literature, this would be the Hamiltonian corresponding to an external field of $h/2$. From (2.4) and (2.7) we obtain the standard fact that when $q = 1$ and $J = 0$, the Potts lattice gas (which is then called a *binary lattice gas*) is equivalent, under the same relabeling, to an Ising model with parameters (β, h) given by

$$\beta = \frac{\kappa}{2}, \quad \beta h = -\left(\mu + \frac{\kappa m}{2}\right). \tag{2.9}$$

To construct the ARC model, we begin by rewriting the partition function of the Potts lattice gas, as was done in [17] for the Potts model. Let $G = (\Lambda, \mathscr{B})$ be a finite subgraph of a lattice $\mathbb{L}$. For simplicity we first consider free boundary conditions, with $\mu_x = \mu$ for all $x \in \Lambda$. Let $\Omega = \{0, 1\}^{\mathscr{B}}$. A *bond configuration* is an element $\omega \in \Omega$; when convenient we alternatively view ω as a subset of $\mathscr{B}$ or as a subgraph of $(\Lambda, \mathscr{B})$. Bonds e with $\omega_e = 1$ are *open* in ω; those with $\omega_e = 0$ are *closed*. Let $C(\omega)$ denote the number of open clusters in ω, let $\omega \vee \omega'$ and $\omega \wedge \omega'$ denote the coordinatewise maximum and minimum, respectively, and define

$$\begin{aligned}|\omega| &= \{e \in \mathscr{B} : e \text{ is open}\},\\ \mathscr{I}(\omega) &= \{x \in S(G) : x \text{ is an isolated site of the graph } \omega\},\\ I(\omega) &= |\mathscr{I}(\omega)|.\end{aligned}$$

Here an isolated site means a singleton cluster. The partition function corrresponding to the Hamiltonian (2.2), with $\beta = 1$, is

$$\begin{aligned}&Z(\Lambda, J, \kappa, \mu)\\ &\quad = \sum_{\sigma} \exp\left((\kappa + J)N_{ss} + \kappa N_{ss'} + \mu N_s\right)\\ &\quad = e^{\mu|\Lambda|} \sum_{\sigma} e^{-\mu N_0} \prod_{\langle xy\rangle} (1 + (e^{\kappa} - 1)n_x n_y) \prod_{\langle xy\rangle} (1 + (e^{J} - 1)\delta_{[\sigma_x = \sigma_y \neq 0]}).\end{aligned} \tag{2.10}$$

Expanding out the products over bonds yields

$$\begin{aligned}&Z(\Lambda, J, \kappa, \mu)\\ &\quad = e^{\mu|\Lambda|} \sum_{\omega_g \in \Omega} \sum_{\omega_r \in \Omega} \sum_{A \subset \mathscr{I}(\omega_g \vee \omega_r, \Lambda)} e^{-\mu|A|} (e^{\kappa} - 1)^{|\omega_g|} (e^{J} - 1)^{|\omega_r|} q^{C(\omega_r) - |A|}\\ &\quad = e^{\mu|\Lambda|} \sum_{\omega_g \in \Omega} \sum_{\omega_r \in \Omega} \left(1 + \frac{e^{-\mu}}{q}\right)^{I(\omega_g \vee \omega_r)} (e^{\kappa} - 1)^{|\omega_g|} (e^{J} - 1)^{|\omega_r|} q^{C(\omega_r)}.\end{aligned} \tag{2.11}$$

Note that the sets A in the expansion (2.11) correspond to sets $\{x : \sigma_x = 0\}$ in (2.10), the spin values σ_x in the terms of (2.10) are constant on the clusters of ω_r in the corresponding terms in (2.11), and the values n_x are all 1 on each nonsingleton cluster of ω_g. The expression (2.11) motivates us to define the *ARC model on* $(\Lambda, \mathscr{B})$ *with parameters* (p_r, p_g, q, Q) *and free boundary conditions* to be the measure on $\Omega \times \Omega$ given by the weights

$$W(\omega_r, \omega_g) = p_g^{|\omega_g|}(1 - p_g)^{|\mathscr{B}| - |\omega_g|} p_r^{|\omega_r|}(1 - p_r)^{|\mathscr{B}| - |\omega_r|} q^{C(\omega_r)} Q^{I(\omega_r \vee \omega_g)}. \tag{2.12}$$

Here $p_r, p_g \in [0, 1]$, $q > 0$ and $Q \geq 1$.

Edwards and Sokal [13] observed that the Potts and FK model could be constructed on a common probability space. The analog of their result is valid here as well, provided $J, \kappa \geq 0$. Specifically, we relate parameters of the q-state Potts lattice gas and the ARC model by

$$p_r = 1 - e^{-J}, \quad p_g = 1 - e^{-\kappa}, \quad Q = 1 + \frac{e^{-\mu}}{q}; \tag{2.13}$$

the parameter q takes the same value in both models. We call an ARC model and a Potts lattice gas *corresponding* when their parameters are related by (2.13). We view $\Omega \times \Omega$ as a set of configurations on a lattice in which there are two bonds – one green and one red—between each adjacent pair of sites of $(\Lambda, \mathscr{B})$. Given a site configuration σ, we obtain a green-bond configuration ω_g from independent bond percolation at density p_g on $\{\langle xy \rangle : n_x = n_y = 1\}$ and a red-bond configuration ω_r from independent bond percolation at density p_r on $\{\langle xy \rangle : \sigma_x = \sigma_y \in \{1, .., q\}\}$. Conversely, given the bond configurations ω_r and ω_g, we obtain a site configuration σ by choosing a spin from $\{1, .., q\}$ independently and uniformly for each cluster of ω_r which is not an isolated site of $\omega_r \vee \omega_g$; for each isolated site of $\omega_r \vee \omega_g$, we choose a spin independently from $\{0, 1, .., q\}$ with probability proportional to $e^{-\mu}$ for 0 and proportional to 1 for each of $1, .., q$. Thus

$$P(\sigma_x = 0 \mid x \in \mathscr{I}(\omega_r \vee \omega_g)) = \frac{e^{-\mu}}{q + e^{-\mu}} = \frac{Q-1}{Q}, \tag{2.14}$$
$$P(\sigma_x = i \mid x \in \mathscr{I}(\omega_r \vee \omega_g)) = \frac{1}{q + e^{-\mu}} = \frac{1}{qQ} \quad \text{for each } i = 1, .., q.$$

For either construction, the result is a joint distribution of site and bond configurations for which the marginal distribution of the sites in the Potts lattice gas and the marginal of the bonds is the ARC model.

To allow the chemical potential μ_x to vary with x, we can modify the ARC model to allow $Q = Q_x$ to depend on x; we merely replace the term $Q^{I(\omega_r \vee \omega_g)}$ in (2.12) with

$$\prod_{x \in \mathscr{I}(\omega_r \vee \omega_g)} Q_x.$$

Via similar constructions, we can obtain the ARC model with either a site boundary condition or a bond boundary condition, defined as follows. A *bond boundary condition on* $\mathscr{B}^c$ is a configuration $\rho = (\rho_r, \rho_g)$ in $\{0,1\}^{\mathscr{B}^c} \times \{0,1\}^{\mathscr{B}^c}$. Under a bond boundary condition, the *ARC model on* $(\Lambda, \mathscr{B})$ *with parameters* (p_r, p_g, q, Q) *and boundary condition* ρ is again given by the weights $W(\omega_r, \omega_g)$ of (2.12), except that now $C(\omega_r)$ (also written $C(\omega_r \mid \rho_r)$) is defined to be the number of clusters of $(\omega_r \rho_r)$ which intersect Λ. We denote this model by $P^{ARC}_{\Lambda,\rho,p_r,p_g,q,Q}$. When ρ_r is all 1's, the configuration ρ_g is irrelevant (that is, it does not affect the weights $W(\omega_r, \omega_g)$), and we say the resulting ARC model has *red-wired boundary condition*. When also $\mathscr{B} = \overline{\mathscr{B}}(\Lambda)$ we denote the corresponding measure on bond configurations by $P^{ARC}_{\Lambda,rw,p_r,p_g,q,Q}$.

For integer q, a *site boundary condition* is given by a boundary condition for the corresponding Potts lattice gas, that is, an element $\eta \in \{0, 1, .., q\}^{\partial\Lambda}$. Site boundary conditions are defined only when G has form $(\overline{\Lambda}, \overline{\mathscr{B}}(\Lambda))$. Define the events

$$\begin{aligned} D_r(\Lambda, \eta) = \{\omega_r \in \{0,1\}^{\overline{\mathscr{B}}(\Lambda)} : \eta_x = \eta_y \text{ for every } x, y \in \partial\Lambda \text{ for} \\ \text{which } x \leftrightarrow y \text{ in } \omega_r, \text{ and } \{x \in \partial\Lambda : \eta_x = 0\} \subset \mathscr{I}(\omega_r)\}, \end{aligned} \tag{2.15}$$

$$D_g(\Lambda, \eta) = \{\omega_g \in \{0,1\}^{\overline{\mathscr{B}}(\Lambda)} : \{x \in \partial\Lambda : \eta_x = 0\} \subset \mathscr{I}(\omega_g)\},$$
$$D(\Lambda, \eta) = \{(\omega_r, \omega_g) : \omega_r \in D_r(\Lambda, \eta), \omega_g \in D_g(\Lambda, \eta)\}.$$

Here $x \leftrightarrow y$ means there is a path of open bonds connecting x to y. Once again, the *ARC model on* $(\overline{\Lambda}, \overline{\mathscr{B}}(\Lambda))$ *with parameters* (p_r, p_g, q, Q) *and site boundary condition* η, denoted $P^{ARC}_{\Lambda,\eta,p_r,p_g,q,Q}$, is given by the weights in (2.12), with $C(\omega_r)$ now defined to be the number of clusters of ω_r which do not intersect $\partial\Lambda$, and with $I(\omega_r \vee \omega_g)$ now defined to be the number of isolated sites of $\omega_r \vee \omega_g$ in Λ (instead of $\overline{\Lambda}$), except that weight 0 is assigned to configurations not in $D(\Lambda, \eta)$. This is equivalent to the red-wired ARC model conditioned on the event $D(\Lambda, \eta)$. More generally, one can allow the boundary spins η_x to take values in an arbitrary finite set V containing 0, in place of $\{0, 1, .., q\}$, since the definition of $D(\Lambda, \eta)$ caries over to such situations; this will be useful when q is not an integer. We call such a boundary condition a *generalized site boundary condition*.

Since the definitions of $C(\omega)$, $\mathscr{I}(\omega)$ and $I(\omega)$ depend on the boundary condition, when ambiguity is possible we will use the notation $C(\omega \mid \rho)$ for the number of clusters of ω when the bond boundary condition is ρ, $C(\omega, \Lambda)$ for the number of clusters having all sites in Λ, and $\mathscr{I}(\omega, \Lambda)$ and $I(\omega, \Lambda)$ respectively for the set and the number of isolated sites in the set Λ.

When η is all 0's, $D(\Lambda, \eta)$ is the event that every site of $\partial\Lambda$ is isolated; we therefore call site boundary condition η the *isolated boundary condition* and denote the corresponding measure $P^{ARC}_{\Lambda,iso,p_r,p_g,q,Q}$. The ARC model on $(\overline{\Lambda}, \overline{\mathscr{B}}(\Lambda))$ with isolated boundary condition is equivalent to the ARC model on $(\Lambda, \mathscr{B}(\Lambda))$ with free boundary condition. (But see Remark 2.1 below.)

In the ARC model, a green bond and a red bond connect each adjacent pair of sites. It is convenient to add a third bond, colored black, which we define to be open precisely when either the red or the green bond is open. Thus the corresponding configuration of black bonds is $\omega_b = \omega_r \vee \omega_g$. It is easy to see that only the black and red bonds (not the green) are needed when one constructs a Potts lattice gas configuration by labeling the clusters of an ARC model configuration. To each ARC model there thus corresponds what we call a *red/black ARC model with parameters* (p_b, p_{rb}, q, Q) given by the weights

$$W(\omega_b, \omega_r) = p_b^{|\omega_b|}(1 - p_b)^{|\mathscr{B}|-|\omega_b|} p_{rb}^{|\omega_r|}(1 - p_{rb})^{|\omega_b|-|\omega_r|} q^{C(\omega_r)} Q^{I(\omega_b)} \quad \text{for all } (\omega_b, \omega_r) \text{ with } \omega_r \subset \omega_b, \tag{2.16}$$

where p_b and p_{rb} are given by

$$1 - p_b = (1 - p_r)(1 - p_g), \qquad p_{rb} = p_r/p_b. \tag{2.17}$$

The weights (2.16) are obtained by first rewriting the "independent bonds" weight:

$$\begin{aligned} &p_g^{|\omega_g|}(1 - p_g)^{|\mathscr{B}|-|\omega_g|} p_r^{|\omega_r|}(1 - p_r)^{|\mathscr{B}|-|\omega_r|} \\ &= p_b^{|\omega_b|}(1 - p_b)^{|\mathscr{B}|-|\omega_b|} p_{rb}^{|\omega_r|}(1 - p_{rb})^{|\omega_b|-|\omega_r|} p_g^{|\omega_g \wedge \omega_r|}(1 - p_g)^{|\omega_r|-|\omega_g \wedge \omega_r|}, \end{aligned} \tag{2.18}$$

then summing over all choices of $\omega_g \wedge \omega_r$ for a given (ω_b, ω_r). Equality (2.18) reflects the fact that one can choose red and green configurations by first choosing a black configuration, then a red configuration which is a subset of the black one, then a green configuration which is a subset of the red one, then adding this green configuration an open green bond wherever there is an open black bond but a closed red bond.

There are three important special cases of the ARC model. The first is the *neutral ARC model*, in which $p_g = 0$. From (2.13) a neutral ARC model corresponds precisely to a neutral Potts lattice gas, that is, one with $\kappa = 0$. There are no green bonds so the weights (2.12) become

$$W(\omega_r) = p_r^{|\omega_r|}(1 - p_r)^{|\mathscr{B}|-|\omega_r|} q^{C(\omega_r)} Q^{I(\omega_r)}.$$

The second special case is the *Potts ARC model*, in which $p_g = p_r$. Recall that the q-state Potts lattice gas at $(1, J, J, \{\mu_x\})$ is the same as the $(q + 1)$-state Potts model at $(\beta, \{h_x\})$ with β and h_x given by (2.7). From (2.13), the condition $\kappa = J$ is equivalent to $p_g = p_r$, so for integer q a Potts ARC model corresponds to a $(q+1)$-state Potts model. More precisely, the $(q+1)$-state Potts model at $(\beta, \{h_x\})$ corresponds to a Potts ARC model with parameters $(p, p, q, \{Q_x\})$ where

$$p = 1 - e^{-\beta} \quad \text{and} \quad Q_x = 1 + \frac{e^{\beta(m_x + h_x)}}{q}. \tag{2.19}$$

In the absense of an extenal field, the Potts model is of course symmetric in the spin variables $0, 1, .., q$, except for boundary conditions. By contrast, in constructing the Potts ARC model from the $(q + 1)$-state Potts model by independent percolation (open red bonds with probability p_r on matching pairs with spins $1, .., q$; open green bonds with probability p_g on general pairs with spins $1, .., q$), the spin values are clearly treated asymmetrically, with 0 given special treatment. This asymmetric treatment of a symmetric model is a key part of what makes the Potts ARC model a useful tool.

Remark 2.1. It was mentioned above that the ARC model on $(\overline{\Lambda}, \overline{\mathscr{B}}(\Lambda))$ with isolated boundary condition is equivalent to the ARC model on $(\Lambda, \mathscr{B}(\Lambda))$ with free boundary condition. In the case of the Potts ARC model, it should be noted that the values of m_x are different for these two graphs, which affects the translation between the external fields h_x in the Potts model and the parameters Q_x in the Potts ARC model. Consider for example a Potts model on a finite subset Λ of a lattice of coordination number m, with constant external field h. A free boundary condition on this model corresponds to a Potts ARC model with parameters $\{Q_x\}$ that are different for sites x adjacent to $\partial\Lambda$. The constant-Q Potts ARC model corresponds instead to the Potts model with 0's boundary condition.

The third special case is the *Ising ARC model*, in which $q = 1$. This is really a special case of the black-bond configuration in the red/black ARC model (2.16). The red bonds are removed, or summed out, because they are irrelevant when $q = 1$; the open red bonds are just obtained from independent percolation on the open black bonds. The *Ising ARC model with parameters* (p, Q) is given by the weights

$$W(\omega) = p^{|\omega|}(1 - p)^{|\mathscr{B}|-|\omega|} Q^{I(\omega)}.$$

Values Q_x depending on x are allowed as before. For a lattice with coordination number m, the Ising model on Λ at (β, h) with boundary condition η corresponds to an Ising ARC model model on $(\overline{\Lambda}, \overline{\mathscr{B}}(\Lambda))$ with site boundary condition η and parameters

$$p = 1 - e^{-2\beta}, \quad Q = 1 + e^{\beta(m+h)}, \tag{2.20}$$

provided we relabel 0 as "+"and 1 as "-". An Ising ARC model configuration can be obtained by independent percolation at density p on the "- -" bonds of an Ising configuration. Conversely an Ising configuration can be obtained by labeling each isolated site independently, according to the following analog of (2.14):

$$\begin{aligned} P(\sigma_x = + \mid x \in \mathscr{I}(\omega_r \vee \omega_g)) &= \tfrac{Q-1}{Q}, \\ P(\sigma_x = - \mid x \in \mathscr{I}(\omega_r \vee \omega_g)) &= \tfrac{1}{Q}. \end{aligned} \tag{2.21}$$

Note that $1/Q$ is precisely the probability that a site is "-" given that all its neighbors are "+".

By contrast, the FK model (with $q = 2$) is obtained from independent percolation on both "++" and "- -" bonds of an ising configuration, at the lower density $p = 1 - e^{-\beta}$. The density is higher for the Ising ARC model because the Ising ARC model configuration is essentially the union of the red and green configurations, each of which is obtained by independent percolation at density $1 - e^{-\beta}$ on "- -" bonds.

Of course, one could equally well construct a joint Ising/Ising ARC model configuration using independent percolation on the "++" bonds of an Ising configuration, though as we have defined things, the Ising ARC model would then have the opposite site boundary condition from the Ising model. We will refer to this as the *reversed polarity* construction of the Ising ARC model.

Remark 2.2. Even when $q > 0$ is not an integer, one can still construct a joint site-bond configuration with site variables $n_x \in \{0, 1\}$, by using the first half of (2.14) to label the isolated sites of an ARC model configuration; all sites not labeled 0 are labeled 1. If the ARC model has parameters $(p_r, p_g, q, \{Q_x\})$ with q not an integer, and $J, \kappa, \{\mu_x\}$ are given by (2.13), we call the resulting site-bond model the *q-state Potts lattice gas with parameters* $(1, J, \kappa, \{\mu_x\})$, thereby extending the definition to noninteger q. A bond or generalized site boundary condition can be applied in the natural way. If the ARC model is a Potts ARC model, and $\beta, \{h_x\}$ are given by (2.19), we similarly call the site-bond model the *$(q + 1)$-state Potts model with parameters* $(\beta, \{h_x\})$. To distinguish things when necessary, we will refer to the standard site-variables-only Potts model or Potts lattice gas with integer q as the *usual* model, and refer to the joint site-bond model just defined for general q as the *particle/bond* model. We call the random variable $\{\delta_{[n_x=0]} : x \in \Lambda\}$ (or its distribution, in a harmless abuse of terminology) the *0's configuration* of the (usual or particle/bond) Potts lattice gas or (usual or particle/bond) Potts model. When appropriate, the *j's configuration* is defined similarly for $j \neq 0$.

Recently and independently, L. Chayes and J. Machta ([11], [12]) introduced particle/bond random cluster models for a wide class of lattice gases, in the context of efficient simulation. Our particle/bond Potts lattice gas is one example of this class.

We turn next to our other new models, the partial and bicolored FK models. The FK model on $(\Lambda, \mathscr{B})$ with parameters (p, q) assigns weights

$$W(\omega) = p^{|\omega|}(1-p)^{|\mathscr{B}|-|\omega|}q^{C(\omega)} \tag{2.22}$$

to bond configurations. As shown in [13], for β given by $p = 1-e^{-\beta}$, a configuration of the usual q-state Potts model at inverse temperature β can be obtained from a configuration ω of the FK model at (p, q), by choosing a label for each cluster of ω independently and uniformly from $\{0, 1, .., q-1\}$; this construction yields a joint site-bond configuration for which the sites are a Potts model and the bonds are an FK model. Fix an integer $0 < t < q-1$ and suppose that we color yellow all open bonds in such a joint configuration with (necessarily matching) endpoints labeled $0, .., t-1$, and color white all open bonds with endpoints labeled $t, .., q-1$. The weight of a given yellow/white bond configuration is then

$$\begin{aligned} W(\omega_y, \omega_w) &= p^{|\omega_y|+|\omega_w|}(1-p)^{|\mathscr{B}|-|\omega_y|-|\omega_w|}q^{C(\omega_y\vee\omega_w)}\left(\frac{t}{q}\right)^{C(\omega_y)-I(\omega_y)} \\ &\quad\cdot\left(1-\frac{t}{q}\right)^{C(\omega_w)-I(\omega_w)} \qquad \delta_E\big((\omega_y, \omega_w)\big) \\ &= p^{|\omega_y|}(1-p)^{|\mathscr{B}|-|\omega_y|}t^{C(\omega_y)}t^{-I(\omega_y)}\left(\frac{p}{1-p}\right)^{|\omega_w|} \\ &\quad (q-t)^{C(\omega_w)+I(\omega_y\vee\omega_w)-I(\omega_w)}\cdot\left(\frac{q}{q-t}\right)^{I(\omega_y\vee\omega_w)} \qquad \delta_E\big((\omega_y, \omega_w)\big) \end{aligned} \tag{2.23}$$

where

$$E = E(\Lambda, \mathscr{B}) = \{(\omega_y, \omega_w) : \text{no site is an endpoint of both an open yellow bond and an open white bond}\}.$$

One can obtain such a yellow/white configuration directly from an FK configuration, without the intermediate step of the joint Potts/FK configuration, by independently coloring each FK cluster (including singletons) yellow with probability t/q, and white with probability $1-t/q$. Thus we neeed not restrict t or q to be an integer; any $0 < t < q$ will do. We call the distribution of the yellow/white site-bond configuration the *bicolored FK model on* $(\Lambda, \mathscr{B})$ *with parameters* (p, q, t) (and free boundary.) When $(\Lambda, \mathscr{B})$ is a subgraph of a lattice, a bond boundary condition for the bicolored FK model can be imposed by specifying a bond boundary condition ρ for the uncolored FK model, then specifying a color for each cluster of ρ. Alternately, as a special case of generalized site boundary conditions one can specify a color, yellow or white, for each site of $\partial\Lambda$. Under such a bicolored site boundary condition η, the event E in (2.23) should be replaced by

$$A(\Lambda,\eta) = E(\overline{\Lambda}, \overline{\mathscr{B}}(\Lambda)) \cap \{(\omega_y,\omega_w) : \eta_x = \eta_y \text{ for every } x, y \in \partial\Lambda \text{ for which } x \leftrightarrow y \text{ in } \omega_y \vee \omega_w\}.$$

We use the notation $C(\omega,\Lambda)$, and $C(\omega \mid \rho)$ for (bicolored) bond boundary conditions ρ, as we do for the ARC model. Summing (2.23) over ω_w for a given ω_y yields the weight of the yellow configuration ω_y, under bicolored site boundary condition $\eta \in \{yellow, white\}^{\partial\Lambda}$:

$$W(\omega_y) = p^{|\omega_y|}(1-p)^{|\mathscr{B}|-|\omega_y|} t^{C(\omega_y,\Lambda)} t^{-I(\omega_y,\Lambda)} F(\omega_y), \tag{2.24}$$

where

$$F(\omega_y) = \sum_{\omega_w \in \{0,1\}^{\overline{\mathscr{B}}(\mathscr{I}(\omega_y,\Lambda))}} \left(\frac{p}{1-p}\right)^{|\omega_w|} (q-t)^{C(\omega_w,\Lambda)+I(\omega_y\vee\omega_w,\Lambda)-I(\omega_w,\Lambda)} \cdot \left(\frac{q}{q-t}\right)^{I(\omega_y\vee\omega_w,\Lambda)} \delta_{A(\Lambda,\eta)}\big((\omega_y,\omega_w)\big). \tag{2.25}$$

We call the model given by the weights (2.24) the *partial FK model on* $(\overline{\Lambda}, \overline{\mathscr{B}}(\Lambda))$ *with parameters* (p,q,t) *and bicolored site boundary condition* η. Note that the exponent $C(\omega_w,\Lambda)+I(\omega_y\vee\omega_w,\Lambda)-I(\omega_w,\Lambda)$ in (2.25) is the number of clusters of ω_w not intersecting $\partial\Lambda$ which have (all) sites in $I(\omega_y,\Lambda)$; from this observation we see that $F(\omega_y)$ is precisely the partition function of the neutral ARC model on $(\mathscr{I}(\omega_y,\Lambda), \overline{\mathscr{B}}(\mathscr{I}(\omega_y,\Lambda))$ with parameters $(p, 0, q-t, \frac{q}{q-t})$, boundary condition η on white sites in $\partial\Lambda$, and 0's (or free) boundary condition on $\Lambda\backslash\mathscr{I}(\omega_y,\Lambda)$ and on yellow sites in $\partial\Lambda$. This, together with (2.14), proves the following result.

Proposition 2.3. *Let* $p \in [0,1]$, $q \geq 1$ *and* $0 < t < q$. *Let* Λ *be a finite subset of the sites of a lattice* $\mathbb{L}$ *and let* η *be a bicolored generalized site boundary condition for the bicolored FK model on* $(\overline{\Lambda}, \overline{\mathscr{B}}(\Lambda))$. *Conditionally on the yellow-bond configuration* ω_y *of the bicolored FK model with parameters* (p,q,t),

(i) the white bonds form a neutral ARC model on $(\mathscr{I}(\omega_y,\Lambda), \overline{\mathscr{B}}(\mathscr{I}(\omega_y,\Lambda)))$ *with parameters* $(p, 0, q-t, \frac{q}{q-t})$ *and boundary condition* η *on the white sites in* $\partial\Lambda$*, and 0's (or free) boundary condition on both* $\Lambda\backslash\mathscr{I}(\omega_y,\Lambda)$ *and the yellow sites in* $\partial\Lambda$*;*

(ii) the yellow sites in $\mathscr{I}(\omega_y,\Lambda)$ *have the distribution of the 0's configuration of a* $(q-t)$*-state neutral Potts lattice gas on* $\mathscr{I}(\omega_y)$ *with the same boundary condition as in (i), with parameters* $(1, J, 0, \mu)$ *given by*

$$p = 1 - e^{-J}, \quad t = e^{-\mu}.$$

We call the neutral Potts lattice gas of Proposition 2.3(ii) the *conditional neutral Potts lattice gas* of the bicolored FK model.

Remark 2.4. The case $t = q - 1$ in Proposition 2.3 is of particular interest, for F is then the partition function of an Ising ARC model. In this case, for integer q, part (ii) says that for the joint Potts/FK configuration, conditionally on the bonds with (matching) endpoints in $\{1, .., q - 1\}$, the 0 and non-0 sites left isolated by these bonds form an Ising model. Further, for β given by $p = 1 - e^{-\beta}$, let $Q = 1 + e^{\beta m}/(q - 1)$ (corresponding to a Potts model with no external field – cf. (2.19).) Then the yellow bonds of the bicolored FK model with parameters $(p, q, q - 1)$ have the same distribution as the red bonds of the Potts ARC model with parameters $(p, p, q - 1, Q)$, as both configurations are obtained from a q-state Potts model configuration by independent percolation at density p on $\{\langle xy\rangle \in \mathscr{B} : \sigma_x = \sigma_y \neq 0\}$.

One type of bicolored generalized site boundary condition specifies only a color for each site; equivalently, all white boundary sites are 0's and all yellow boundary sites have a second spin, say 1. Thus we may, for example, have a bicolored FK model with all-white or all-yellow site boundary condition.

3. Statement of main results

In this section we describe our main results; proofs appear in later sections.

Let us use "$\leq$" to denote the natural partial ordering on $\{0, 1\}^{\mathscr{B}}$. An event A is called *increasing* if $\omega \in A$, $\omega \leq \omega'$ imply $\omega' \in A$, and *decreasing* if its complement is increasing. A probability measure P on $\{0, 1\}^{\mathscr{B}}$ is said to have the *FKG property* if

$$P(A \cap B) \geq P(A)P(B) \quad \text{for all increasing events } A, B.$$

P is said to satisfy the *FKG lattice condition* if

$$P(\omega \vee \omega')P(\omega \wedge \omega') \geq P(\omega)P(\omega') \quad \text{for all} \quad \omega, \omega'. \tag{3.1}$$

As proved in [18], the FKG lattice condition implies the FKG property. For P_1 and P_2 probability measures on $\{0, 1\}^{\mathscr{B}}$, we say P_1 *dominates* P_2 (in the FKG sense) if $P_1(A) \geq P_2(A)$ for all increasing events A. This is equivalent to the statement that there exists a coupling $\tilde{P}$ of $\{0, 1\}^{\mathscr{B}} \times \{0, 1\}^{\mathscr{B}}$ with marginals P_1 and P_2 for which $\tilde{P}(\{(\omega, \omega') : \omega \geq \omega'\}) = 1$. As is well-known, if P_1 and P_2 are determined by weights W_1 and W_2 respectively, P_1 or P_2 has the FKG property, and W_1/W_2 is an increasing function on $\{0, 1\}^{\mathscr{B}}$, then P_1 dominates P_2.

Let η^i denote the all-i's boundary condition; if $i \neq 0$ we call η^i a *constant-species boundary condition*. More generally, we say that a generalized site boundary condition η has a *single particle species* if there exists $i \neq 0$ such that $\eta_x = 0$ or i for all $x \in \partial\Lambda$. Note that for the Ising ARC model ($q = 1$), every (nongeneralized) site boundary condition has just a single particle species.

L. Chayes [9] proved that for the usual Potts model without external field, the 0's configuration satisfies the FKG lattice condition and thus has the FKG property. Of course, 0 can be replaced by any other spin. In two dimensions, Chayes obtained as a consequence that Gibbs nonuniqueness is characterized by the percolation of spin i under boundary condition η^i. Chayes' proof of the FKG property allows for boundary conditions specified in terms of the variables $\delta_{[\sigma_x=0]}$, $x \in \partial\Lambda$, but does not cover general site boundary conditions. There is good reason for this as the following example shows.

Example 3.1. Consider a 3-state Potts model with spin space $\{0, 1, 2\}$ at inverse temperature β without external field, on $\Lambda = \{x, y, z\} \subset \mathbb{Z}^3$, where $x = (1, 0, 0)$, $y = (2, 0, 0)$, $z = (3, 0, 0)$. The boundary condition is as follows: of the five boundary sites adjacent to x, three have spin 1 and two have spin 2; of the four boundary sites adjacent to y, two have spin 1 and two have spin 2; and of the five boundary sites adjacent to z, two have spin 1 and three have spin 2. We write ijk for the configuration σ with $\sigma_x = i, \sigma_y = j, \sigma_z = k$. For large β, most of the probability is concentrated on the energy-minimizing configurations 111, 112, 122, 222 so the partition function Z corresponding to the Hamiltonian (2.6) satisfies $Z \sim 4e^{9\beta}$ as $\beta \to \infty$. Hence as $\beta \to \infty$ we have

$$P(\sigma_x = \sigma_z = 0) = \frac{3e^{2\beta}}{Z},$$

$$P(\sigma_z = 0) = P(\sigma_x = 0) \sim P(022) = \frac{e^{6\beta}}{Z} \sim \frac{e^{-3\beta}}{4}$$

and hence

$$P(\sigma_z = 0 \mid \sigma_x = 0) \sim 3e^{-4\beta}.$$

Thus for large β, the events $[\sigma_z = 0]$ and $[\sigma_x = 0]$ are negatively correlated, so the FKG property fails for the 0's configuration.

This brings up the more general question of just when the FKG property holds for the 0's configuration of a Potts lattice gas. A sufficient condition is given by the following result; Example 3.1 demonstrates the need to restrict to single-particle-species boundary conditions.

Theorem 3.2. *The 0's configuration of any particle/bond Potts lattice gas on a finite subset Λ of the sites of a lattice, with $J, \kappa \geq 0$, under free boundary conditions or under any site boundary condition η which has a single particle species, satisfies the FKG lattice condition.*

We say that *percolation of spin i* occurs under a measure P if with probability 1, there exists an infinite self-avoiding lattice path on which all sites have spin i. Using Theorem 3.2 we will establish the following.

Corollary 3.3. *Consider a q-state Potts lattice gas (q an integer) with parameters (β, J, κ, μ), with $J, \kappa \geq 0$, on a planar lattice $\mathbb{L}$. If percolation of 0's occurs under boundary condition η^i for some $i \in \{1, .., q\}$, then there is a unique Gibbs distribution at (β, J, κ, μ).*

We turn next to conditions for Gibbs uniqueness and weak mixing, and to the bounds on critical points that can be obtained by establishing such properties throughout most of the high-temperature regime. A bond or site model with specified parameters (but unspecified boundary condition) is said to have the *weak mixing property* if there exist C, λ as follows. Given finite sets $\Delta \subset \Lambda$ and any two boundary conditions (bond or generalized site) η_1 and η_2 for the model on $(\overline{\Lambda}, \overline{\mathscr{B}}(\Lambda))$, the corresponding distributions P_1 and P_2 of the configuration on $(\Delta, \mathscr{B}(\Delta))$ satisfy

$$\mathrm{Var}(P_1, P_2) \leq C \sum_{x \in \Delta, y \notin \Lambda} e^{-\lambda |y-x|},$$

where $\mathrm{Var}(\cdot, \cdot)$ denotes total variation distance. Loosely this says that the maximum influence, on a fixed region, of the boundary condition decays exponentially to 0 as the boundary recedes to infinity. Turning to the FK model, fix p, q and for each finite subset Λ of the sites of a lattice $\mathbb{L}$, let $P^{FK}_{\Lambda,w}$ denote the model at (p, q) on $(\overline{\Lambda}, \overline{\mathscr{B}}(\Lambda))$ with wired boundary condition. The infinite-volume limit, denoted P^{FK}_{w}, is said to have *exponential decay of local wired-boundary connectivities* if there exist $C, \lambda > 0$ such that for every finite $\Lambda \ni 0$,

$$P^{FK}_{\Lambda,w}(0 \leftrightarrow \partial\Lambda \text{ by a path of open bonds}) \leq C e^{-\lambda r(\Lambda)},$$

where $r(\lambda) = \min\{|x| :\in \partial\Lambda\}$. Note this is stronger than the usual notion of exponential decay of connectivities (for the infinite-volume limit), as studied e.g. in [23], though for the FK model on planar lattices the two notions have been proven equivalent [4]. It is not hard to show (see [4]) that if the FK model at (p, q) has exponential decay of local wired-boundary connectivities, then it has the weak mixing property, as does the corresponding Potts model if q is an integer. Weak mixing for the Potts model has a variety of useful consequences, particularly in two dimensions; see [32].

A planar lattice $\mathbb{L}$ divides the plane into polygonal faces. The *dual lattice* $\mathbb{L}^*$ is constructed by placing a *dual site* at the center of each such face, and then a *dual bond* between each pair of dual sites for which the corresponding faces have a bond (that is, an edge) in common. For example, the dual of the triangular lattice is the hexagonal lattice, and vice versa. When necessary for clarity, bonds of $\mathbb{L}$ are called *regular bonds*. To each regular bond e there is associated a unique dual bond e^* connecting the centers of the two faces of which e is an edge. The dual bond e^* is defined to be open precisely when e is closed, so that for each bond configuration ω on $\mathbb{L}$, there is unique dual configuration ω^* on $\mathbb{L}^*$. For each $q > 0$, for $p \in [0, 1]$ the value p^* *dual to* p *at level* q is given by

$$\frac{p}{q(1-p)} = \frac{1-p^*}{p^*}.$$

If the regular bonds are distributed as the infinite-volume FK model at (p, q) on $\mathbb{L}$ with wired boundary condition, then the dual bonds form the infinite-volume FK model at (p^*, q) on $\mathbb{L}^*$ with free boundary condition (see [21].) If p is the *self-dual point*

$$p_{sd}(q) = \frac{\sqrt{q}}{1+\sqrt{q}},$$

then $p = p^*$. For the FK model on the square lattice it is conjectured that the percolation critical point is the self-dual point for all q; this is known for $q \geq 25.72$, from [31].

For the Potts model, the value β^* dual to a given β can be obtained using FK duality and the correspondence $p = 1 - e^{-\beta}$.

Let $p_c^{FK}(q, \mathbb{L})$ denote the percolation critical point of the FK model on a lattice $\mathbb{L}$, and let $\beta_c^{Potts}(q, \mathbb{L})$ denote the critical inverse temperature of the q-state Potts model on $\mathbb{L}$, so that

$$p_c^{FK}(q, \mathbb{L}) = 1 - e^{-\beta_c^{Potts}(q, \mathbb{L})}.$$

If q is not an integer, we take this as the definition of $\beta_c^{Potts}(q, \mathbb{L})$. For $q = 2$ we alternately write $\beta_c^{Ising}(\mathbb{L})$.

It is believed that weak mixing should hold for the q-state Potts model for all $q \geq 1$ and all subcritical β, but for the most part it has only been established perturbatively, for β near 0. Exceptions include the Ising model [24], and $q \geq 25.72$ on the square lattice [31]. Here we will establish weak mixing for the Potts model on planar lattices throughout most of the subcritical region, that is, nearly up to β_c^{Potts}. As a byproduct we obtain rigorous bounds on the Potts and FK critical points on such lattices. The specifics are as follows.

Theorem 3.4. *Let $\mathbb{L}$ be a planar lattice of coordination number m, and suppose the dual lattice $\mathbb{L}^*$ has coordination number m^*. Let $q > 1$ and define $\beta_1 = \beta_1(q+1, m)$ and $p_1 = p_1(q+1, m)$ by*

$$e^{\beta_1} = \frac{q-1}{q^{(m-2)/m} - 1}, \quad p_1 = 1 - e^{-\beta_1} = \frac{1 - q^{-2/m}}{1 - q^{-1}}. \tag{3.2}$$

(i) The FK model on $\mathbb{L}$ at $(p, q+1)$ has exponential decay of local wired-boundary connectivities, and has the weak mixing propety, for all $p < p_1(q+1, m)$. Its critical point $p_c = p_c^{FK}(q+1, \mathbb{L})$ satisfies $p_1 \leq p_c \leq p_2$, where $p_2 = p_1(q+1, m^*)^*$ is the value dual to $p_1(q+1, m^*)$ at level $q+1$.

(ii) If q is an integer, the $(q+1)$-state Potts model at $(\beta, 0)$ on $\mathbb{L}$ has the weak mixing property for all $\beta < \beta_1(q+1, m)$. Its critical point $\beta_c = \beta_c^{Potts}(q+1, \mathbb{L})$ satisfies $\beta_1 \leq \beta_c \leq \beta_2$, where $\beta_2 = \beta_1(q+1, m^*)^*$ is the value dual to $\beta_1(q+1, m^*)$ at level $q+1$.

(iii) The lower bounds p_1 and β_1 remain valid even if there is no m^* (that is, not all sites of $\mathbb{L}^*$ have the same degree.)

We now apply Theorem 3.4 to some examples.

Example 3.5. For the square lattice with certain values of q, it is known [31] that the FK critical point is the self-dual point:

$$p_c^{FK}(q, \mathbb{Z}^2) = \frac{\sqrt{q}}{1 + \sqrt{q}} \quad \text{for} \quad q = 1, 2 \quad \text{and all} \quad q \geq 25.72,$$

and this is believed to hold for all $q \geq 1$; it is known (see [21]) that $p_c^{FK}(q, \mathbb{Z}^2) \geq p_{sd}(q)$ for all $q \geq 1$. By contrast, the lower bound given by Theorem 3.4 is

$$p_1(q,4) = \frac{\sqrt{q-1}}{1+\sqrt{q-1}} = p_{sd}(q-1) = p_{sd}(q) - \frac{1}{2q^{3/2}} - O\left(\frac{1}{q^2}\right) \quad \text{as } q \to \infty,$$

and the upper bound is

$$\frac{\sqrt{q}}{\sqrt{\frac{q-1}{q}} + \sqrt{q}} = p_{sd}(q) + \frac{1}{2q^{3/2}} + O\left(\frac{1}{q^2}\right).$$

For $q = 10$, for example, we obtain $.760 \leq p_c \leq .769$, compared to the conjectured value $p_{sd} = .760$, and we establish exponential decay for all $p < .750$.

Example 3.6. For the triangular and hexagonal lattices, there are computations of the Potts critical point in the physics literature using the star-triangle transformation and variants thereof ([26],[41]), but it is not clear whether these can be made rigorous. Since the triangular and hexagonal lattices are dual to each other, rigorous upper bounds on each of these lattices come from lower bounds on the other lattice. For example, the values .513 and .740 are dual at level $q = 3$, as are .413 and .810 (see Table 1). For many other planar lattices, there are only estimates obtained by series expansion methods, renormalization group methods and/or simulation; see [42] for a summary and references. For the Kagomé lattice lower bounds are computed in [29], but again, the level of rigor is unclear. Our rigorous bounds, and corresponding nonrigorous values from the physics literature, are summarized in Table 1. The nonrigorous values for the triangular and hexagonal lattices are from the presumably exact general formula in [26]. The nonrigorous values for the Kagomé lattice are from [29] for $q = 4$ and from the conjectured general formula in [41] for $q = 9, 30$; it should be noted that this general formula was found in [14] to be incorrect. The lower bound from [29] for the Kagomé lattice with $q = 4$ is .672, better than our rigorous bound .634. The accuracy of the rigorous bounds becomes quite high for larger values of q. Nonetheless, this table should perhaps be seen less as a source of new information about critical points and more as a numerical quantification of the sharpness of the comparison result, Proposition 4.13.

The idea behind the proof of Theorem 3.4 is as follows. The main step is to prove that there is exponential decay of local wired-boundary connectivities for $p < p_1$; this establishes weak mixing which shows that $p_1 \leq p_c^{FK}$, and then, using duality, that $p_c \leq p_2$. The bounds on β_c^{Potts} follow from the Potts/FK correspondence using (2.19). The wired-boundary FK model on a finite Λ can be obtained from percolation at density p on the matching bonds in the Potts model with a constant, say all-0, boundary condition. To obtain the exponential decay of local wired-boundary connectivities, then, we need only consider percolation on "00" bonds. It is enough to show that the probability of a path in the Potts model from the origin to $\partial\Lambda$ on which all spins are 0's decays exponentially in $r(\Lambda)$. For this, it is enough to show that the 0's configuration of the Potts model is dominated by the configuration of some other species in some other model where this exponential decay property is

Table 1. Nonrigorous values and rigorous bounds for the critical point of the FK model on planar lattices.

Lattice	m	q	Rigorous bounds on $p_c^{FK}(q, \mathbb{L})$	Nonrigorous value
Triangular	6	3	$0.413 \le p_c \le 0.513$	0.468
Triangular	6	4	$0.460 \le p_c \le 0.532$	0.500
Triangular	6	9	$0.571 \le p_c \le 0.600$	0.588
Triangular	6	30	$0.699 \le p_c \le 0.706$	0.703
Hexagonal	3	3	$0.740 \le p_c \le 0.810$	0.773
Hexagonal	3	4	$0.779 \le p_c \le 0.824$	0.800
Hexagonal	3	9	$0.857 \le p_c \le 0.871$	0.863
Hexagonal	3	30	$0.926 \le p_c \le 0.928$	0.927
Kagomé	4	4	$0.634 \le p_c$	0.686
Kagomé	4	9	$0.739 \le p_c$	0.761
Kagomé	4	30	$0.843 \le p_c$	0.851

known to hold. The latter role is played by the "+" configuration of an Ising model with negative external field. The ARC model, and in particular the fact that the 0's configuration is obtained by independent site percolation on the isolated sites of the ARC configuration, is used to facilitate the comparison of the Potts 0's to the Ising "+" configuration.

We will prove a somewhat weaker analog of Theorem 3.4 for higher dimensions. It requires an assumption on the Ising model to which the Potts 0's configuration is compared, as follows. Given an inverse temperature β, a value of q and a lattice with coordination number m, define β'' by

$$e^{\beta''} = \frac{q - 1 + e^{\beta}}{q^{(m-1)/m}} \tag{3.3}$$

and let β_1 be as in 3.2. We will need the following condition:

independent percolation at density $1 - e^{-\beta}$ on the "++" bonds of the minus phase of the Ising model at $(\beta'', 0)$ produces no infinite cluster, a.s. (3.4)

Unfortunately we have no way to verify this for $d > 2$, except when we are in the Peierls regime $e^{\beta''} > 3$, where it is known that "+" spins do not percolate in the minus phase. From (3.3), a sufficient condition for $e^{\beta''} > 3$ is that $q > 3^{2d}$. This leads to the following two results.

Theorem 3.7. *For $q > 1$, consider the FK model at $(p, q + 1)$ on $\mathbb{Z}^d$, and, for integer q, the corresponding $(q+1)$-state Potts model at $(\beta, 0)$, with $p = 1 - e^{-\beta}$. If (3.4) holds, with β'' given by (3.3), and*

$$p < \frac{1 - q^{-1/d}}{1 - q^{-1}} \tag{3.5}$$

or equivalently

$$e^{\beta} < \frac{q-1}{q^{(d-1)/d} - 1}, \tag{3.6}$$

then there is no percolation in the FK model, and (for integer q) the Potts model has a unique Gibbs distribution. If (3.4) holds for all β in a neighborhood of $\beta_1(q+1, 2d)$, then

$$p_c^{FK}(q+1, \mathbb{Z}^d) < \frac{1 - q^{-1/d}}{1 - q^{-1}}.$$

Corollary 3.8. *For all $d \geq 2$ and $q > 3^{2d}$, the FK model on $\mathbb{Z}^d$ with parameters $(p, q+1)$ has no percolation provided (3.5) holds. If $q > 3^{2d}$ is an integer, the $(q+1)$-state Potts model on $\mathbb{Z}^d$ at inverse temperature β has a unique Gibbs distribution provided (3.6) holds.*

Remark 3.9. The equivalent conditions (3.5) and (3.6) are apparently quite sharp for $\mathbb{Z}^d$, even for small q. For example, for the 4-state Potts model ($q = 3$) in dimension 3, we have

$$\frac{1 - q^{-1/d}}{1 - q^{-1}} = .460$$

while nonrigorous estimates of p_c in the physics literature range from .468 to .477 (see [42].) It is not hard to show that $\beta/2 \leq \beta'' \leq \beta$ for all $\beta \leq \beta_1$. Since the FK model has at most one infinite cluster a.s., we know that independent percolation at density $1 - e^{-\beta''}$ on the "++" bonds of the minus phase of the Ising model at $(\beta'', 0)$ produces no infinite cluster a.s.; in (3.4) we replace $1 - e^{-\beta''}$ with the larger percolation density $1 - e^{-\beta}$. Percolation at the still-larger density $1 - e^{-2\beta''}$ on the "++" bonds of the minus phase of the Ising model at $(\beta'', 0)$ produces the Ising ARC model (with reversed polarity), so to establish (3.4) it is enough to show that

there is no percolation in the Ising ARC model, corresponding to an Ising model at $(\beta'', 0)$, with isolated boundary condition. (3.7)

Remark 3.10. We consider (3.4) and (3.7) in the mean field limit, $m = 2d \to \infty$, for the integer lattice. Let $\beta_1''(q+1, m)$ be the value of β'' when $\beta = \beta_1(q+1, m)$, that is,

$$e^{\beta_1''(q+1,m)} = \frac{q-1}{q^{1/m}(q^{(m-2)/m} - 1)} = \frac{1}{q^{1/m}} e^{\beta_1(q+1,m)} \tag{3.8}$$

(cf. (3.3).) We have from (3.2) that

$$\lim_{m\to\infty} m\beta_1(q+1, m) = \frac{2q \log q}{q-1},$$

which is the "right" mean field limit, in that it is believed that

$$\lim_{d\to\infty} 2d\beta_c(q+1, \mathbb{Z}^d) = \frac{2q \log q}{q-1} \tag{3.9}$$

and it is proved in [28] that

$$\limsup_{d\to\infty} 2d\beta_c(q+1,\mathbb{Z}^d) \le \frac{2q\log q}{q-1}.$$

The limit (3.9) is known for the Ising model, $q+1=2$, with the RHS interpreted as 2. Thus if for all sufficiently large d one could establish (3.4) for all β in a neighborhood of β_1, it would prove (3.9). From (3.8),

$$\lim_{m\to\infty} m\beta_1''(q+1,m) = \frac{(q+1)\log q}{q-1},$$

and the latter is an increasing function of $q>1$, so for all $q>1$,

$$\beta_1''(q+1,2d) > \beta_c^{Potts}(2,\mathbb{Z}^d) \quad \text{for all sufficiently large } d.$$

Thus for (3.9) it would be enough to prove (3.4) or (3.7) when β'' is above the Ising critical point $\beta_c^{Potts}(2,\mathbb{Z}^d)$. Define the percolation threshhold

$$\begin{aligned} p_c^{Ising}(\beta'',\mathbb{L}) = \inf\{p\in[0,1]\colon &\text{ independent percolation at density } p \text{ on the “++”}\\ &\text{bonds of the minus phase of the Ising model at } (\beta'',0) \text{ on } \mathbb{L}\\ &\text{produces an infinite cluster a.s.}\}. \end{aligned}$$

From Remark 3.9 we know $p_c^{Ising}(\beta'',\mathbb{L}) \ge 1-e^{-\beta''}$ and for (3.4) it suffices that

$$\begin{aligned} p_c^{Ising}(\beta'',\mathbb{L}) > 1-e^{-\beta}, \quad &\text{for all } \beta \text{ in a neighborhood of form } (\beta_1-\epsilon,\beta_1)\\ &\text{and for } \beta'' \text{ as in (3.3),} \end{aligned} \tag{3.10}$$

where $\epsilon>0$. We are unable to verify (3.10) for the integer lattice but we can verify the analog for the Cayley tree $\mathbb{T}_m$ with large coordination number m as follows. It is easily checked that

$$p_c^{Ising}(\beta'',\mathbb{T}_m) \sim (mP^{Ising}_{\mathbb{T}_m,-,\beta_m'',0}(\sigma_x=+))^{-1} \quad \text{as} \quad m\to\infty, \tag{3.11}$$

where x is arbitrary and $P^{Ising}_{\mathbb{T}_m,-,\beta_m'',0}$ denotes the infinite-volume Ising model on $\mathbb{T}_m$ at $(\beta_m'',0)$ with minus boundary condition. This is just the branching-process approximation which says that the critical percolation density is such that the mean number of sites x adjacent to 0 for which both $\sigma_x=+$ and $\langle 0x\rangle$ is open is approximately 1. Suppose we have a sequence $\{\beta_m\}$ with $m\beta_m\to c$ for some c “sufficiently close” to $2q(\log q)/(q-1)$; then from (3.3), $m\beta_m''\to a(c)=\frac{c}{q}+\log q$, while from the mean field limit for the magnetization on a tree (which is straightforward),

$$P^{Ising}_{\mathbb{T}_m,-,\beta_m'',0}(\sigma_x=+) \to \frac{1-M_{a(c)}}{2} \tag{3.12}$$

where M_a is the positive solution of $M=\tanh(aM/2)$. We claim that

$$\frac{2}{1-M_{a(c)}} > c \quad \text{for all } c \text{ in a neighborhood of } \frac{2q\log q}{q-1}, \tag{3.13}$$

which with (3.11) and (3.12) establishes (3.10) for $\mathbb{T}_m$ with m large. Thus it seems plausible that (3.10) may hold for the integer lattice when $q > 1$, at least in high dimensions. It is sufficient to prove (3.13) for $c = 2q(\log q)/(q-1)$, so $c > 2$ and $a(c) = (q+1)(\log q)/(q-1)$. Now (3.13) for this c is equivalent to $M_{a(c)} > 1 - \frac{2}{c}$ or

$$\tanh\left(\frac{a(c)}{2}\left(1-\frac{2}{c}\right)\right) > 1 - \frac{2}{c}$$

or

$$\tanh\left(\frac{q+1}{2}\left(\frac{\log q}{q-1} - \frac{1}{q}\right)\right) > 1 - \frac{q-1}{q\log q}, \tag{3.14}$$

and (3.14) can be verified for all $q > 1$ by a tedious but straightforward calculation, using $e^x > 1 + x + x^2/2$ for $x > 0$. Thus (3.13) and hence (3.10) hold for m large, so for each $q > 1$, for $\mathbb{T}_m$ with m large, (3.4) holds for all β in a neighborhood of β_1, as required.

The obvious problem with Theorem 3.7 is the difficulty of verifying (3.4). We give next an alternate theorem which has more readily verifiable hypotheses (at least in certain limits.) The price paid for this is that the resulting bound on p_c^{FK} or β_c^{Potts} is weaker, particularly for small q.

The *magnetization* $M(\beta, h) = M(\beta, h, \mathbb{L})$ of the Ising model at (β, h) on the lattice $\mathbb{L}$ is the mean value of σ_0 in the infinite-volume plus phase. For $h \geq 0$, the *susceptibility* at (β, h) is the quantity

$$\chi(\beta, h) = \frac{1}{\beta}\frac{\partial}{\partial h}M(\beta, h).$$

Theorem 3.11. *For $q > 1$ consider the FK model at $(p, q+1)$ on $\mathbb{Z}^d$, and, for integer q, the corresponding $(q+1)$-state Potts model at $(\beta, 0)$, with $p = 1 - e^{-\beta}$. Suppose*

$$2\chi\left(\frac{\beta}{2}, 0\right) < 1 + M\left(\frac{\beta}{2}, 0\right) \tag{3.15}$$

and suppose

$$p < 1 - \frac{1}{q^{1/d}} \tag{3.16}$$

or equivalently

$$e^\beta < q^{1/d}.$$

Then there is no percolation in the FK model, and (for integer q) the Potts model has a unique Gibbs distribution at inverse temperature β. If (3.15) holds for all β in a neighborhood of $(\log q)/d$, then

$$p_c(q+1, \mathbb{Z}^d) \geq 1 - \frac{1}{q^{1/d}}.$$

Condition (3.16) is clearly less sharp than (3.5), especially for small q; in particular one could not hope to obtain the mean-field limit (3.9) using (3.16). For example, for $q+1=4$ and $d=3$, (3.16) allows $p<.306$ while (3.5) allows $p<.460$ (see Remark 3.9.) For $q+1=30$ and $d=10$, there is less difference, as (3.16) allows $p<.286$ while (3.5) allows $p<.296$.

We can verify (3.15) in certain limits, which leads to the following.

Corollary 3.12. *Suppose $q+1\geq 10.56$. For all sufficiently large d, the FK model satisfies*

$$p_c^{FK}(q+1,\mathbb{Z}^d)\geq 1-\frac{1}{q^{1/d}}.$$

Remark 3.13. For the Potts model, Kesten and Schonmann [28] established a lower bound for $q+1\geq 2$ of the form

$$\beta_c^{Potts}(q+1,\mathbb{Z}^d)\geq \frac{c(q)}{d}\quad \text{for all } d.$$

Their $c(q)$ is very close to the mean-field value $(q\log q)/(q-1)$ for small q, which is better than what Corollary 3.12 gives, but for large q their $c(q)$ is only about half the mean-field value, so Corollary 3.12 is better for large q.

Remark 3.14. Since $\chi(\beta)\to 0$ as $\beta\to\infty$, (3.15) holds for all sufficiently large β. Therefore for fixed d, Theorem 3.11 shows $p_c^{FK}(q+1,\mathbb{Z}^d)\geq 1-q^{-1/d}$ for all sufficiently large q. But this is of less interest that large d, because a series expansion for $p_c^{FK}(q+1,\mathbb{Z}^d)$ for large q is known [30].

We turn next to the analysis of certain perturbations of the (zero-field) Potts and Ising models. The main question of interest to us is how these perturbations affect the critical inverse temperature of the model. We consider first annealed site dilution, that is, a Potts lattice gas in which 0's are rare. It should be pointed out that the coresponding problem for annealed bond dilution is much simpler, because bond dilution is essentially just a change of temperature (see [37].)

We define the *dilution parameter* θ of a q-state Potts lattice gas at $(1,J,\kappa,\mu)$ on a lattice with coordination number m by

$$\theta=e^{-(\mu+\kappa m)}.$$

For the corresponding ARC model, this becomes

$$\theta=q(Q-1)(1-p_g)^m. \tag{3.17}$$

Heuristically, and to an extent quantifiably (see Lemma 4.18), θ gives the order of magnitude of the typical fraction of 0's in the system. Define the ARC model critical point for red bonds by

$$\begin{aligned}p_c^{ARC}(p_g,q,Q,\mathbb{L})=\inf\{p_r:\ &\text{percolation of red bonds occurs in the}\\ &\text{in the red-wired ARC model at } (p_r,p_g,q,Q) \text{ on } \mathbb{L}\}\end{aligned}$$

and the corresponding Potts lattice gas critical point by

$$J_c^{PLG}(q, \kappa, \mu, \mathbb{L}) = \inf\{J : \text{there is symmetry breaking in the } q-\text{state Potts lattice gas at } (1, J, \kappa, \mu) \text{ on } \mathbb{L}\}$$

As is easy to show (see Proposition 4.7), if (p_g, Q) and (κ, μ) are related as in (2.13), we have

$$p_c^{ARC}(p_g, q, Q, \mathbb{L}) = 1 - e^{-J_c^{PLG}(q,\kappa,\mu,\mathbb{L})}. \tag{3.18}$$

In the next theorem, the underlying heuristic is that when small annealed site dilution is added to the Potts model, the change in critical temperature, the fraction of empty sites, and the dilution parameter θ are all of the same order of magnitude. What we actually prove are one-sided bounds consistent with this picture. These bounds essentially compare the effect of the dilution to the effect of change much simpler to analyze: replacing the parameter q with $q + \theta$ in the FK model.

Theorem 3.15. *Let $\mathbb{L}$ be a lattice of coordination number m.*

(i) If $q, Q \geq 1$, $p_g \in [0, 1]$, $(Q - 1)(1 - p_g)^{m/2} \leq 1$ and the dilution parameter θ is sufficiently small, then

$$p_c^{FK}(q, \mathbb{L}) \leq p_c^{ARC}(p_g, q, Q, \mathbb{L}) \leq p_c^{FK}(q + \theta, \mathbb{L}).$$

(ii) If $\kappa \geq 0$, $\mu \in \mathbb{R}$, $q \geq 1$, $\mu + \frac{1}{2}\kappa m + \log q \geq 0$ and the dilution parameter $\theta = e^{-(\mu+\kappa m)}$ is sufficiently small, then

$$\beta_c^{Potts}(q, \mathbb{L}) \leq J_c^{PLG}(q, \kappa, \mu, \mathbb{L}) \leq \beta_c^{Potts}(q + \theta, \mathbb{L}).$$

Remark 3.16. Let us compare Theorem 3.15 to what can be obtained by much simpler techniques, similar to the standard comparison theorem discussed in the introduction. As we will show (see Lemma 4.4), since $Q^{2C(\omega)-I(\omega)}$ is a decreasing function of ω, these simpler techniques show that the ARC model, call it P, at (p_r, p_g, q, Q) dominates the ARC model, call it P', at $(p_r, p_g, qQ^2, 1)$. Hence the red bond configuration under P', which forms the FK model at (p_r, qQ^2), is dominated by the red bond configuration under P. This shows that $p_c^{ARC}(p_g, q, Q, \mathbb{L}) \leq p_c^{FK}(qQ^2, \mathbb{L})$. But $qQ^2 \geq q + 2q(Q - 1) \geq q + 2\theta$, so this result is worse than Theorem 3.15 by a factor of at least 2 in the correction θ, and by a much larger factor if p_g is near 1, or Q is large.

For the square lattice, and $q \geq 25.72$, the FK critical point is known exactly [31]. This tells us the exact change in p_c^{FK} when q is replaced by $q + \theta$, so we can get more detailed information from Theorem 3.15 about the change in critical point induced by the dilution. We summarize this in the following corollary.

Corollary 3.17. *Suppose $\kappa \geq 0$, $\mu \in \mathbb{R}$, $q \geq 25.72$, and $\mu + \frac{1}{2}\kappa m + \log q \geq 0$. Then for the square lattice, for $\theta = e^{-(\mu+\kappa m)}$, as $\theta \to 0$,*

$$\beta_c^{Potts}(q, \mathbb{Z}^2) \leq J_c^{PLG}(q, \kappa, \mu, \mathbb{Z}^2) \leq \beta_c^{Potts}(q, \mathbb{Z}^2) + \frac{1}{2}\sqrt{q}(1 + \sqrt{q})\theta + o(\theta). \tag{3.19}$$

We do not expect that the factor $\frac{1}{2}\sqrt{q}(1+\sqrt{q})$ multiplying θ in (3.19) is sharp, but we do expect, though we cannot prove, that the true correction term is of order θ.

A special case of “dilution” of the q-state Potts model is the $(q+1)$-state Potts model with a large negative external field applied to one of the spins. This is the subject of our next result.

Theorem 3.18. *Let* $\mathbb{L}$ *be a lattice and* $q \geq 1$*, and for* $h \geq 0$*, let*

$$\theta(h) = \exp\bigl(-\beta_c^{Potts}(q,\mathbb{L})h\bigr).$$

Then

$$\beta_c^{Potts}(q,\mathbb{L}) \leq \beta_c^{Potts}(q+1,-h,\mathbb{L}) \leq \beta_c^{Potts}(q+\theta(h),\mathbb{L}). \tag{3.20}$$

Note that (3.20) is an equality both for $h=0$ and in the limit $h \to \infty$.

We turn now to a different perturbation: the Potts lattice gas with small J. When $J=0$ the particles form a binary lattice gas with the Gibbs weight multiplied by an entropy factor q^{N_s}, where N_s is the number of particles; this factor just adds $\log q$ to the chemical potential. Thus small J may be considered a perturbation of a binary lattice gas, or equivalently, of an Ising model. More precisely, presuming we relabel particles as “-”and empty sites as “+”, the $J=0$ Potts lattice gas becomes an Ising model with parameters (β_0, h_0) given by (cf. (2.9))

$$\beta_0 = \frac{\kappa}{2}, \quad \beta_0 h_0 = -(\mu + \frac{\kappa m}{2} + \log q). \tag{3.21}$$

We call h_0 the *effective external field* of the $J=0$ Potts lattice gas. Thus for $J=0$, the phase diagram in (κ,μ)-space is known – there is a critical line $\mu = -\frac{1}{2}\kappa m - \log q,\ \kappa > 2\beta_c^{Ising}(\mathbb{L})$ where there is phase coexistence, and Gibbs uniqueness holds everywhere outside the closure of this critical line. For fixed $\kappa > 2\beta_c^{Ising}(\mathbb{L})$, as μ increases, there is a first-order aggregation transition at $\mu = -\frac{1}{2}\kappa m - \log q$ from an empty-dominated regime to a particle-dominated regime. If the lattice is planar these regimes are characterized by the percolation of empty sites and of particles, respectively. For small J one expects this phase diagram to be perturbed only slightly. It is outside the scope of this work to make this phrase “only slightly” into a rigorous statement – this nontrivial problem would involve showing that for small J the transition remains sharp, meaning there is no interval of μ values in which there is an intermediate phase, and showing that the minimum value of κ for which the transition is first-order remains near $2\beta_c^{Ising}(\mathbb{L})$ for small J. (See [10] for more on the phase diagram for positive J.) Instead we will establish a one-sided bound—adding a positive J reduces the critical μ by at least a certain function of J. Define

$$\mu_c^{PLG}(q,J,\kappa,\mathbb{L}) = \sup\{\mu \in \mathbb{R} : \text{ there is percolation of 0's in the infinite-volume } q-\text{state Potts lattice gas at } (1,J,\kappa,\mu) \text{ on } \mathbb{L} \text{ with 0's boundary condition}\}$$

so that for planar $\mathbb{L}$, from the preceding discussion,

$$\mu_c^{PLG}(q, 0, \kappa, \mathbb{L}) = -\frac{\kappa m}{2} - \log q \quad \text{for all } \kappa > 2\beta_c^{Ising}(\mathbb{L}).$$

Theorem 3.19. *Let $\mathbb{L}$ be a planar lattice of coordination number m. For $q \geq 1$, $J \geq 0$ and $\kappa > 2\beta_c^{Ising}(\mathbb{L})$,*

$$\mu_c^{PLG}(q,J,\kappa,\mathbb{L}) \leq \mu_c^{PLG}(q, 0, \kappa, \mathbb{L}) - \frac{m}{2} \log\left(\frac{1}{q}e^J + \frac{q-1}{q}\right) \tag{3.22}$$

$$= \mu_c^{PLG}(q, 0, \kappa, \mathbb{L}) - \frac{m}{2q}J - \frac{m(q-1)}{4q^2}J^2 + O(J^3) \quad \text{as } J \to 0.$$

We expect, but do not prove, that the order-J term in the RHS of (3.22) is the true first-order correction, and that the order-J^2 terms is correct as well for certain lattices, including the square and hexagonal lattices, which have the property that for a bond $\langle xy\rangle$, the length of the shortest path from x to y outside $\langle xy\rangle$ is more than 2. To see why, note first that the Potts lattice gas with small J has particles of species which are approximately independent and uniform in $\{1, .., q\}$, so that for any bond, the endpoints match species with probability approximately $1/q$. Conditionally on the particle locations, there is approximate pairwise independence, but not mutual independence, among the variables $\delta_{[\sigma_x=\sigma_y]}$ as $\langle xy\rangle$ varies over bonds with particles at both ends. If these variables $\delta_{[\sigma_x=\sigma_y]}$ *were* mutually independent, then each bond with particles at both ends would make a contribution to the Gibbs weight $e^{-H_{\Lambda,\eta}(\sigma)}$ of e^κ from the interparticle attraction and of

$$\frac{1}{q}e^J + \frac{q-1}{q}$$

from the Potts interaction. Defining κ_0 by

$$e^{\kappa_0} = e^\kappa\left(\frac{1}{q}e^J + \frac{q-1}{q}\right)$$

we see that the effect of positive J under mutual independence would be merely to change the interparticle attraction from κ to κ_0; we would still have effectively just a binary lattice gas. What actually happens is a slight variation of this, as follows. The contribution to the Gibbs weight from a bond $\langle xy\rangle$ with particles at both ends is

$$e^\kappa\left(\lambda e^J + (1-\lambda)\right),$$

where the random value $\lambda = \lambda(x, y, \{n_z\})$ is the probability that $\sigma_x = \sigma_y$ for a Potts model at inverse temperature J on the particles, but with the interaction between x and y "turned off," and $\lambda e^J + (1-\lambda)$ is the ratio of two partition functions for the Potts model on the particles, one with the $\langle xy\rangle$ Potts interaction "turned on" and one with it "turned off." Assuming "nice" boundary conditions, e.g. free or having a single particle species, we have $\lambda \geq 1/q$. We therefore call κ_0 (which corresponds to $\lambda = 1/q$) the *minimum effective interparticle atttraction* of the q-state Potts lattice gas at $(1, J, \kappa, \mu)$. The heuristic content of Theorem 3.19 is that since $\mu_c^{PLG}(q, 0, \kappa, \mathbb{L})$ is a decreasing function of κ, and the small-J system is roughly like a $J = 0$ system with interparticle attraction κ_0 or greater, we should expect $\mu_c^{PLG}(q, J, \kappa, \mathbb{L}) \leq \mu_c^{PLG}(q, 0, \kappa_0, \mathbb{L})$. This is fairly sharp so long as λ is typically close to $1/q$. The deviation $\lambda - 1/q$ should be of order of the probability of a path of open bonds in the FK model from x to y outside $\langle xy \rangle$, which is of order J^n for small J when the shortest possible path has length n. Thus the inequality in (3.22) should be accurate to within order J^n.

The $J = 0$ Potts lattice gas with interparticle attraction κ_0, which we are effectively using as a bound for the positive-J system, is equivalent (when we ignore particle species) to an Ising model with parameters (β', h') given by

$$\beta' = \frac{\kappa_0}{2}, \quad \beta' h' = -\left(\mu + \frac{1}{2}\kappa_0 m + \log q\right) \tag{3.23}$$

or equivalently

$$e^{2\beta'} = e^{\kappa}\left(\frac{1}{q}e^J + \frac{q-1}{q}\right), \quad e^{\beta'(m+h')} = \frac{e^{-\mu}}{q}. \tag{3.24}$$

We call h' the *maximum effective external field,* and β' the *minimum effective inverse temperature,* of the q-state Potts lattice gas at $(1, J, \kappa, \mu)$.

Our final topic is couplings. Couplings have been a useful tool in studying how the boundary condition influences probabilities under a Gibbs distribution (see [33],[5].) For a Potts lattice gas configuration σ on a finite set Λ with boundary condition η on $\partial\Lambda$, we define the *boundary particle cluster*

$$\begin{aligned} C(\partial\Lambda, \sigma) = \{x \in \Lambda : x \text{ is connected to } \partial\Lambda \text{ by a path in which} \\ \text{all sites } x \in \Lambda \text{ have } n_x = 1\}. \end{aligned}$$

We say that the Potts lattice gas at $(1, J, \kappa, \mu)$ on a lattice $\mathbb{L}$ has the *boundary coupling property with respect to particles* if for every finite Λ and every boundary condition η on $\partial\Lambda$, there exists a coupling $\tilde{P}$ of the measures with boundary conditions η and η^1 satisfying

$$\tilde{P}(\{(\sigma, \sigma') : \sigma \text{ and } \sigma' \text{ agree on } \Lambda \backslash C(\partial\Lambda, \sigma')\}) = 1.$$

This property is instrumental in [4] in establishing the following property for the Potts model on a planar lattice with nonnegative external field applied to spin 0: exponential decay of the (infinite-volume) probability of connecting two sites by a path with no 0 spins implies weak mixing.

Theorem 3.20. *For every lattice $\mathbb{L}$ and all $J, \kappa \geq 0, q \geq 1$ and $\mu \in \mathbb{R}$, the q-state Potts lattice gas at $(1, J, \kappa, \mu)$ on $\mathbb{L}$ has the boundary coupling property with respect to particles.*

4. Basic properties of the ARC model and the partial FK model

We begin with the FKG property for the ARC model. Note that an ARC model is a probability distribution on pairs of configurations. Hence if $\omega = (\omega_r, \omega_g)$ and $\omega' = (\omega_r', \omega_g')$, then $\omega \geq \omega'$ means precisely that both $\omega_r \geq \omega_r'$ and $\omega_g \geq \omega_g'$. In particular, there is never a comparison of a red bond to a green one, and one should not view red and green as two possible states of a single bond $\langle xy \rangle$, but rather as two parallel bonds between x and y.

Lemma 4.1. *Let P be an ARC model on a graph $(\Lambda, \mathscr{B})$, possibly with a boundary condition, given by the weights (2.12) for all $\omega = (\omega_r, \omega_g) \in \{0, 1\}^{\mathscr{B}} \times \{0, 1\}^{\mathscr{B}}$, with $q, Q \geq 1$. Then P satisfies the FKG lattice condition (3.1). Consequently, P has the FKG property.*

Proof. As in the analogous proof for the FK model (see [22]), we have

$$C(\omega_r \vee \omega_r') - C(\omega_r) \geq C(\omega_r') - C(\omega_r \wedge \omega_r').$$

Similarly, setting $K(\omega) = I(\omega_r \vee \omega_g)$,

$$K(\omega \vee \omega') - K(\omega) \geq K(\omega') - K(\omega \wedge \omega'),$$

and (3.1) follows easily. That (3.1) implies the FKG property is a result of [18]. □

Remark 4.2. Lemma 4.1 applies to the ARC model under any bond boundary condition, but does not apply to the ARC model with site boundary conditions in general, because the weights (2.12) then only apply to a restricted set of configurations ω. Constant-species boundary conditions are covered by Lemma 4.1 since the event $D(\Lambda, \eta)^c$ of (2.15) is empty in such cases. More generally, suppose we have a generalized site boundary condition η which has a single particle species. Let $F' = \{x \in \partial\Lambda : \eta_x = i\}$ and $E' = \{x \in \partial\Lambda : \eta_x = 0\}$, and let $(\Lambda', \mathscr{B}')$ be the graph obtained by deleting E' and all bonds with an endpoint in E' from $(\overline{\Lambda}, \overline{\mathscr{B}}(\Lambda))$. Then the ARC model on $(\overline{\Lambda}, \overline{\mathscr{B}}(\Lambda))$ with boundary condition η on $\partial\Lambda$ is equivalent to the ARC model on $(\Lambda', \mathscr{B}')$ with all-i's boundary condition on F'. Thus Lemma 4.1 applies under boundary conditions having a single particle species as well. Here we are using the fact that when the graph is $(\overline{\Lambda}, \overline{\mathscr{B}}(\Lambda))$, the proof of Lemma 4.1 is not changed if $I(\cdot)$ means $I(\cdot, \Lambda)$ and not $I(\cdot, \overline{\Lambda})$.

Let us call an ARC model on a graph $(\Lambda, \mathscr{B})$ *unconditioned* if the weights (2.12) apply to all configurations in $\{0, 1\}^{\mathscr{B}} \times \{0, 1\}^{\mathscr{B}}$. Thus ARC models with free or bond boundary conditions are unconditioned, and an ARC model with site boundary condition having a single particle species is equivalent to an unconditioned ARC model on an appropriate subgraph, as in Remark 4.2.

Exactly as for the FK model (see [21],Theorem 3.1), we obtain the following using Lemma 4.1.

Corollary 4.3. *For fixed $p_r \in [0,1]$, $p_g \in [0,1]$, $q \geq 1$ and $Q \geq 1$, the measures on $\{0,1\}^{\mathscr{B}(\mathbb{L})} \times \{0,1\}^{\mathscr{B}(\mathbb{L})}$ given by the weak limits*

$$P^{ARC}_{rw,p_r,p_g,q,Q} = \lim_{\Lambda \nearrow S(\mathbb{L})} P^{ARC}_{\Lambda,rw,p_r,p_g,q,Q} \quad \textit{and}$$
$$P^{ARC}_{iso,p_r,p_g,q,Q} = \lim_{\Lambda \nearrow S(\mathbb{L})} P^{ARC}_{\Lambda,iso,p_r,p_g,q,Q}$$

exist, and are translation-invariant and ergodic.

We continue with an analog for the ARC model of the standard comparison theorem of [17] (see also [3]) for the FK model.

Lemma 4.4. *Consider an unconditioned ARC model on a finite graph (possibly with a boundary condition). Let $p_r, p_g, p'_r, p'_g \in [0,1]$ and let $q, Q, Q' \geq 1$ and $q' > 0$. The model at (p'_r, p'_g, q', Q') dominates the model at (p_r, p_g, q, Q) under any of the following conditions:*

(i) $p_r \leq p'_r$, $p_g \leq p'_g$, $q = q'$ *and* $Q = Q'$;
(ii) $p_r = p'_r$, $p_g = p'_g$, $q \geq q'$ *and* $Q \geq Q'$;
(iii) $p_r \leq p'_r$, $p_g = p'_g$, $Q = Q'$ *and*

$$\frac{p_r}{q(1-p_r)} \leq \frac{p'_r}{q'(1-p'_r)};$$

(iv) $p_r \leq p'_r$, $p_g \leq p'_g$, $q = q'$ *and*

$$\frac{p_r}{Q^2(1-p_r)} \leq \frac{p'_r}{(Q')^2(1-p'_r)}.$$

Proof. Let $W'(\omega_r, \omega_g)$ and $W(\omega_r, \omega_g)$ be the weight functions as in (2.12), for the two parameter choices. It is easy to see that $|\omega_r|$, $|\omega_g|$, $|\omega_r| + C(\omega_r)$, $2|\omega_r| + I(\omega_r \vee \omega_g)$ and $2|\omega_g| + I(\omega_r \vee \omega_g)$ are increasing functions of (ω_r, ω_g), while $C(\omega_r)$ and $I(\omega_r \vee \omega_g)$ are decreasing functions. It follows easily that in all four cases, W'/W is an increasing function. □

Suppose we have two models, P_A and P_B, for configurations on a finite Λ, each with boundary conditions, and we have species i appearing under P_A and j appearing under P_B. As a shorthand terminology, we say that the i's configuration under P_A dominates the j's configuration under P_B if $P_A(\{\delta_{[\sigma_x=i]} : x \in \Lambda\} \in \cdot)$ dominates $P_B(\{\delta_{[\sigma_x=j]} : x \in \Lambda\} \in \cdot)$. Note these are measures on $\{0,1\}^{\Lambda}$.

For a Potts lattice gas on a finite set Λ, define

$$X_0 = X_0(\Lambda, \sigma) = \{x \in \Lambda : \sigma_x = 0\}.$$

Lemma 4.5. *Consider a q-state Potts lattice gas and a q'-state Potts lattice gas on a finite set Λ under respective site boundary conditions η and η', with parameter values $(1, J, \kappa, \mu)$ and $(1, J', \kappa', \mu')$ respectively, satisfying $J, \kappa, J', \kappa' \geq 0$. Let (p_r, p_g, q, Q) and (p'_r, p'_g, q', Q') be the parameters of the corresponding ARC models. Suppose $Q \geq Q'$ and the black configuration $\omega_b = \omega_r \vee$*

ω_g of the ARC model under $P^{ARC}_{\Lambda,\eta',p'_r,p'_g,q',Q'}$ dominates the black configuration under $P^{ARC}_{\Lambda,\eta,p_r,p_g,q,Q}$. Then the 0's configuration of the Potts lattice gas under $P^{PLG}_{\Lambda,\eta,q,1,J,\kappa,\mu}$ dominates the 0's configuration under $P^{PLG}_{\Lambda,\eta',q',1,J',\kappa',\mu'}$.

Proof. Let P and P' denote the ARC models at (p_r, p_g, q, Q) and at (p'_r, p'_g, q', Q'), respectively. There exists a coupling $\tilde{P}$ of P and P' for which $\tilde{P}(\{(\omega_b, \omega'_b) : \omega_b \leq \omega'_b\}) = 1$, and hence $\mathscr{I}(\omega_b, \Lambda) \supset \mathscr{I}(\omega'_b, \Lambda)$ a.s. From (2.14), since $Q \geq Q'$, the ARC configurations ω_b and ω'_b can therefore be labeled to produce lattice-gas configurations σ and σ' satisfying $X_0(\Lambda, \sigma) \supset X_0(\Lambda, \sigma')$.

Applying Lemmas 4.4 and 4.5 to the Ising model and Ising ARC model yields the following result, obtained by Schonmann and Shlosman ([36], Lemma 1) using different methods.

Lemma 4.6. *([36]) Consider the Ising model on a finite subset Λ of a lattice with coordination number m, with boundary condition η. Suppose that*

$$\beta'(m - h') \geq \beta(m - h); \tag{4.1}$$

$$\beta'(m + h') \leq \beta(m + h). \tag{4.2}$$

Then the "+" configuration on Λ at (β, h) dominates the "+" configuration at (β', h').

Note that if $-m \leq h' \leq m$, then (4.1) and (4.2) imply $h \geq h'$.

Proof (Proof of Lemma 4.6). The comparison is made by way of the model with a third set of parameters, (β'', h''). Define these by

$$\beta'(m + h') = \beta''(m + h''), \quad \beta(m - h) = \beta''(m - h'').$$

It is easy to check that $\beta'' \leq \min(\beta, \beta')$. From Lemma 4.5 and Lemma 4.4(i) we have the following two conclusions:

(i) the "+" configuration at (β'', h'') dominates the "+" configuration at (β', h');
(ii) the "+" configuration at $(\beta'', -h'')$ dominates the "+" configuration at $(\beta, -h)$.

We can restate (ii) as:

(iii) the "-" configuration at $(\beta, -h)$ dominates the "-" configuration at $(\beta'', -h'')$.

Since (iii) is valid under arbitrary boundary condition, we can interchange the roles of "+"and "-" in (iii) to obtain:

(iv) the "+" configuration at (β, h) dominates the "+" configuration at (β'', h'').

Now (i) and (iv) prove the lemma.

For the lattice $\mathbb{Z}^d$, a *plaquette* is a face of a unit hypercube centered at a lattice site. Each plaquette is the perpendicular bisector of a unique bond. A *dual surface* (consisting of plaquettes) is the outer boundary of a connected set which is the union of a finite collection of such hypercubes.

Proposition 4.7. *Consider the red-wired ARC model on a lattice $\mathbb{L}$ with parameters (p_r, p_g, q, Q), with $q \in \mathbb{Z}$, and the (usual) q-state Potts lattice gas with corresponding parameters $(1, J, \kappa, \mu)$, given by (2.13).*

(i) If the ARC model with red-wired boundary condition has no percolation in the black configuration $\omega_b = \omega_r \vee \omega_g$, then the Potts lattice gas has a unique Gibbs distribution.

(ii) The red bonds of the ARC model with red-wired boundary condition percolate if and only if the Potts lattice gas exhibits symmetry breaking, that is, there is a Gibbs distribution not symmetric in $\{1, .., q\}$.

Proof. We give the proof for the integer lattice only; for other lattices one need only extend the notion of an "outermost dual surface" in the appropriate way. By Lemma 4.1, on a finite $(\overline{\Lambda}, \overline{\mathscr{B}}(\Lambda))$ the ARC model with red-wired boundary dominates the ARC model with any other generalized site boundary condition η. If $\Delta \subset \Lambda$ and in some configuration ω_b there is no path of open bonds from $\partial\Lambda$ to Δ, then there is a unique outermost dual surface $\Gamma = \Gamma(\omega_b)$ surrounding Δ which is crossed by no open black bond. Let P_1 and P_2 denote the red-wired measure (that is, the measure under boundary condition η^1) and the measure under η, respectively, for the ARC model on $(\overline{\Lambda}, \overline{\mathscr{B}}(\Lambda))$. As is well-known in the context of the FK model (see e.g. [33]), the coupling $\tilde{P}$ of P_1 and P_2 can be chosen so that

$$\tilde{P}\big(\{(\omega_b, \omega_b') : \omega_b \text{ and } \omega_b' \text{ agree inside } \Gamma(\omega_b)\} \mid \partial\Lambda \not\leftrightarrow \Delta \text{ in } \omega_b\big) = 1.$$

When ω_b and ω_b' agree inside $\Gamma(\omega_b)$, clusters of ω_b and ω_b' can be labeled identically to create Potts configurations, under boundary conditions η^1 and η, which also agree inside $\Gamma(\omega_b)$. Letting $\Lambda \nearrow S(\mathbb{L})$ we have $\tilde{P}(\partial\Lambda \not\leftrightarrow \Delta \text{ in } \omega_b) \to 1$ and (i) follows. The proof of (ii) is similar to the the proof for the FK model (see [3].) □

The next result is an analog of Proposition 2.3.

Lemma 4.8. *Let Λ be a finite subset of a lattice $\mathbb{L}$ with coordination number m. Consider a q-state particle/bond Potts lattice gas on $(\overline{\Lambda}, \overline{\mathscr{B}}(\Lambda))$ with a generalized site boundary condition η, with parameters $(1, J, \kappa, \mu)$ satisfying $\kappa, J \geq 0$. Then conditionally on ω_r, the 0's of the Potts lattice gas form the "+" configuration of an Ising model on $\mathscr{I}(\omega_r, \Lambda)$ with parameters $(\tilde{\beta}, \tilde{h})$ given by*

$$\tilde{\beta} = \frac{\kappa}{2}, \quad e^{\tilde{\beta}(m+\tilde{h})} = \frac{e^{-\mu}}{q}$$

and with boundary condition as follows: "-" on $\Lambda \backslash \mathscr{I}(\omega_r, \Lambda)$ and on $\{x \in \partial\Lambda : \eta_x \neq 0\}$, "+" on $\{x \in \partial\Lambda : \eta_x = 0\}$. In particular, for the $(q+1)$-state particle/bond Potts model at $(\beta, 0)$, we have

$$\tilde{\beta} = \frac{\beta}{2}, \quad e^{\beta\tilde{h}/2} = \frac{e^{\beta m/2}}{q}. \tag{4.3}$$

Proof. It is immediate from (2.12) that conditionally on ω_r, the green bonds of the ARC model in $\overline{\mathscr{B}}(\Gamma)$ form an Ising ARC model on $(\overline{\Gamma}, \overline{\mathscr{B}}(\Gamma))$, where $\Gamma = \mathscr{I}(\omega_r, \Lambda)$, with "-", or equivalently wired, boundary condition. Applying (2.19) yields the result. □

We call the Ising model of Lemma 4.8 the *conditional Ising model* of the (particle/bond) Potts lattice gas or of the equivalent ARC model. Note that conditionally on ω_r, in addition to the Ising ARC model formed by the green bonds in $\overline{\mathscr{B}}(\Gamma)$, the remaining green bonds of the ARC model – those with neither endpoint isolated in ω_r – are independently open with probability p_g.

For $\mathbb{L} = \mathbb{Z}^d$ we have from (4.3) that $\tilde{h} < 0$ if and only if $e^\beta < q^{1/d}$. This is very close to the condition that β is subcritical, at least for large q [31]. Thus, except perhaps near the critical point, 0's are favored relative to particles on $\mathscr{I}(\omega_r, \Lambda)$ when β is supercritical, and particles are favored when β is subcritical.

Given a bond configuration ω and a bond e, we let $\omega \vee e$ denote the configuation obtained by adding the bond e to ω (that is, by declaring e to be open.) The ratio $U(\omega)/V(\omega)$ of two functions is an increasing function if and only if

$$\frac{U(\omega \vee e)}{U(\omega)} \geq \frac{V(\omega \vee e)}{V(\omega)} \quad \text{for all } \omega \text{ and } e. \tag{4.4}$$

In some situations of interest, the ratios appearing in (4.4) can be interpreted as probabilities, as the next three lemmas show. Let $P^{Ising}_{\Lambda,\eta,\beta,h}$ denote the distribution of the Ising model with parameters (β, h) on a finite set Λ with boundary condition η on $\partial\Lambda$, and let $P^{Ising}_{+,\beta,h}$ and $P^{Ising}_{-,\beta,h}$ denote the infinite volume limits under "+" and "-" boundary conditions, respectively, on the full lattice $\mathbb{L}$.

Lemma 4.9. *Let $p \in (0,1)$ and $Q \geq 1$, and let Λ be a finite set of sites of a lattice $\mathbb{L}$ with coordination number m. For $\omega \in \{0,1\}^{\overline{\mathscr{B}}(\Lambda)}$ define*

$$G(\omega) = \sum_{\omega_r \in \{0,1\}^{\mathscr{B}(\mathscr{I}(\omega,\Lambda))}} \left(\frac{p}{1-p}\right)^{|\omega_r|} Q^{I(\omega_r \vee \omega, \Lambda)}.$$

Then for $e = \langle xy \rangle \in \overline{\mathscr{B}}(\Lambda)$,

$$\frac{G(\omega \vee e)}{G(\omega)} = (Q-1)^{I(\omega \vee e, \Lambda) - I(\omega,\Lambda)} \, P^{Ising}_{\mathscr{I}(\omega,\Lambda),-,\beta/2,h}(\sigma_x = \sigma_y = -), \tag{4.5}$$

where β and h are given by

$$p = 1 - e^{-\beta}, \quad Q = 1 + e^{\beta(m-h)/2}. \tag{4.6}$$

If $x \notin \Lambda$ then $P^{Ising}_{\mathscr{I}(\omega,\Lambda),-,\beta/2,h}(\sigma_x = \sigma_y = -)$ should be interpreted as $P^{Ising}_{\mathscr{I}(\omega,\Lambda),-,\beta/2,h}(\sigma_y = -)$. Similarly if both $x, y \notin \Lambda$ then $P^{Ising}_{\mathscr{I}(\omega,\Lambda),-,\beta/2,h}(\sigma_x = \sigma_y = -)$ should be interpreted as 1. Also, in the event that $\mathscr{B}(\mathscr{I}(\omega, \Lambda))$ is empty, we define $G(\omega)$ to be $Q^{I(\omega,\Lambda)}$.

Given an ARC-model red-bond configuration ω, there is a conditional Ising model on $\mathscr{I}(\omega, \Lambda)$, and a corresponding reversed-polarity Ising ARC model. $G(\omega)$ is a version of the partition function for this reversed-polarity Ising ARC model. In the case that the original ARC model is a Potts ARC model, we may view ω as the FK portion of a joint FK/Potts configuration, with the "00" bonds deleted. The configurations ω_r summed to obtain $G(\omega)$ are just the possible choices for the set of "00" bonds to make $\omega \vee \omega_r$ a full FK configuration. This is made precise in Lemma 4.12.

Proof (Proof of Lemma 4.9). Fix ω and $e = \langle xy \rangle$. Let $\Gamma = \mathscr{I}(\omega, \Lambda)$ and $\Delta = I(\omega, \Lambda) - I(\omega \vee e, \Lambda) = |\{x, y\} \cap \Gamma|$. Consider an Ising ARC model, which we denote P_1, on $(\overline{\Gamma}, \overline{\mathscr{B}}(\Gamma))$ with parameters (p, Q) and "+", or equivalently isolated, boundary conditions. This is equivalent (see Remark 2.1) to an Ising ARC model P_2 on $(\Gamma, \mathscr{B}(\Gamma))$ with the same parameters (p, Q) but with free boundary. The weights for P_2 are given by the terms of the sum $G(\omega)$:

$$W_2(\omega_r) = \left(\frac{p}{1-p}\right)^{|\omega_r|} Q^{I(\omega_r, \Gamma)}, \quad \omega_r \in \{0, 1\}^{\mathscr{B}(\Gamma)}.$$

Let P_3 be the Ising ARC model on $(\Gamma \backslash \{x, y\}, \mathscr{B}(\Gamma \backslash \{x, y\}))$ with parameters (p, Q) and free boundary, and let $\mathscr{A} = \mathscr{B}(\Gamma) \backslash \mathscr{B}(\Gamma \backslash \{x, y\})$. Note $e \in \mathscr{A}$. Each ω_r for which all bonds of $\mathscr{A}$ are closed (that is, $\{x, y\} \cap \Gamma \subset \mathscr{I}(\omega_r, \Gamma)$) corresponds to a unique configuration α_r which is the restriction of ω_r to $\mathscr{B}(\Gamma \backslash \{x, y\})$, and conversely. The corresponding weight is

$$W_3(\alpha_r) = \left(\frac{p}{1-p}\right)^{|\alpha_r|} Q^{I(\alpha_r, \Gamma \backslash \{x,y\})} = W_2(\omega_r) Q^{-\Delta}. \tag{4.7}$$

Summing $W_3(\alpha_r)$ over all α_r yields $G(\omega \vee e)$, so summing (4.7) and dividing by $G(\omega)$ yields

$$\frac{G(\omega \vee e)}{G(\omega)} = P_2\big(\{x, y\} \cap \Gamma \subset \mathscr{I}(\omega_r, \Gamma))\big) Q^{-\Delta} = P_1\big(\{x, y\} \cap \Gamma \subset \mathscr{I}(\omega_r, \Gamma)\big) Q^{-\Delta}. \tag{4.8}$$

But from (2.14),

$$\begin{aligned} P^{Ising}_{\Gamma,-,\beta/2,h}(\sigma_x = \sigma_y = -) &= P^{Ising}_{\Gamma,+,\beta/2,-h}(\sigma_x = \sigma_y = +) \\ &= P_1\big(\{x, y\} \cap \Gamma \subset \mathscr{I}(\omega_r, \Gamma)\big) \left(\frac{Q-1}{Q}\right)^{\Delta}, \end{aligned}$$

and (4.5) follows. □

We have viewed $G(\omega)$ as the partition function of a reversed-polarity Ising ARC model. We could do the same without reversing the polarity. This yields a different partition function $T(\omega)$, expressed below as a sum over configurations ω_g. These configurations ω_g are precisely those of the Ising ARC model which appeared in the proof of Lemma 4.8.

Lemma 4.10. *Let $p \in (0, 1)$ and $Q \geq 1$, and let Λ be a finite subset of a lattice $\mathbb{L}$. For $\omega \in \{0, 1\}^{\overline{\mathscr{B}}(\Lambda)}$ define $b(\omega) = |\overline{\mathscr{B}}(\mathscr{I}(\omega, \Lambda))|$ and*

$$T(\omega) = \sum_{\omega_g \in \{0,1\}^{\overline{\mathscr{B}}(\mathscr{I}(\omega,\Lambda))}} p^{|\omega_g|}(1-p)^{b(\omega)-|\omega_g|} Q^{I(\omega \vee \omega_g, \Lambda)}.$$

Then for $e = \langle xy \rangle \in \overline{\mathscr{B}}(\Lambda)$,

$$\frac{T(\omega \vee e)}{T(\omega)} = P^{Ising}_{\mathscr{I}(\omega,\Lambda),-,\beta/2,h}(\sigma_x = \sigma_y = -), \tag{4.9}$$

where β *and* h *are given by*

$$p = 1 - e^{-\beta}, \quad Q = 1 + e^{\beta(m+h)/2}. \tag{4.10}$$

Note that the configurations ω_r in Lemma 4.9 are on $\mathscr{B}(\mathscr{I}(\omega, \Lambda))$, while the configurations ω_g in Lemma 4.10 are on $\overline{\mathscr{B}}(\mathscr{I}(\omega, \Lambda))$.

Proof (Proof of Lemma 4.10). As in the proof of Lemma 4.9, fix ω and $e = \langle xy \rangle$ and let $\Gamma = \mathscr{I}(\omega, \Lambda)$, $\Delta = I(\omega, \Lambda) - I(\omega \vee e, \Lambda) = |\{x, y\} \cap \Lambda|$ and $b(\omega) = |\overline{\mathscr{B}}(\Gamma)|$. This time consider an Ising ARC model P_1 on $(\overline{\Gamma}, \overline{\mathscr{B}}(\Gamma))$ with parameters (p, Q) and with "-", or equivalently wired, boundary condition. The weights for P_1 are given by the terms of the sum $T(\omega)$:

$$W_1(\omega_g) = p^{|\omega_g|}(1-p)^{b(\omega)-|\omega_g|} Q^{I(\omega \vee \omega_g, \Lambda)}, \quad \omega_g \in \{0, 1\}^{\overline{\mathscr{B}}(\Gamma)}.$$

Let P_3 be the Ising ARC model on $(\Gamma \backslash \{x, y\}, \overline{\mathscr{B}}(\Gamma \backslash \{x, y\}))$ with parameters (p, Q) and "-" boundary condition, and let $\mathscr{C} = \overline{\mathscr{B}}(\Gamma) \backslash \overline{\mathscr{B}}(\Gamma \backslash \{x, y\})$. Note $e \in \mathscr{C}$. Each ω_g for which all bonds of $\mathscr{C}$ are open corresponds to a unique configuration ζ_g which is the restriction of ω_g to $\overline{\mathscr{B}}(\Gamma \backslash \{x, y\})$, and conversely. The corresponding weight is

$$W_3(\zeta_g) = p^{|\zeta_g|}(1-p)^{b(\omega)-|\mathscr{C}|-|\zeta_g|} Q^{I(\zeta_g, \Gamma \backslash \{x,y\})} = W_1(\omega_g) p^{-|\mathscr{C}|}. \tag{4.11}$$

Summing $W_3(\zeta_g)$ over all ζ_g yields $T(\omega \vee e)$, so summing (4.11) and dividing by $T(\omega)$ yields

$$\frac{T(\omega \vee e)}{T(\omega)} = P_1(\text{all bonds of } \mathscr{C} \text{ are open}) p^{-|\mathscr{C}|}.$$

But from (2.14),

$$P_1(\text{all bonds of } \mathscr{C} \text{ are open}) = P^{Ising}_{\Gamma, -, \beta/2, h}(\sigma_x = \sigma_y = -) p^{|\mathscr{C}|}$$

and (4.9) follows. □

Here is a related result for the FK model. Write $\omega \backslash \{e\}$ for the configuration obtained by closing the bond $e \in \omega$.

Lemma 4.11. *Let* $p \in (0, 1)$ *and* $q \geq 1$, *let* Λ *be a finite set of sites in a lattice* $\mathbb{L}$, *and let* η *be a generalized site boundary condition. For* $\omega \in \{0, 1\}^{\overline{\mathscr{B}}(\Lambda)} \cap D_g(\Lambda, \eta)$ *define*

$$R(\omega) = \sum_{\omega_r \subset \omega} p^{|\omega_r|}(1-p)^{|\omega|-|\omega_r|} q^{C(\omega_r, \Lambda)} \delta_{D_r(\Lambda, \eta)}(\omega_r).$$

Then for $e \in \omega$,

$$\frac{R(\omega \backslash \{e\})}{R(\omega)} = \frac{1}{1-p} P^{FK}_{\omega}(e \text{ is closed}), \tag{4.12}$$

where P^{FK}_{ω} *denotes probability for the FK model on the graph* $(\overline{\Lambda}, \omega)$ *with parameters* (p, q) *and site boundary condition* η. *Further,*

$$\frac{R(\omega)}{(\frac{p}{q} + 1 - p)^{|\omega|}} \text{ is an increasing function.} \tag{4.13}$$

Proof. The proof of (4.12) is similar to those of Lemmas 4.9 and 4.10 so we omit it. From [3] we have

$$P_\omega^{FK}(e \text{ is closed}) \le \frac{q(1-p)}{p+q(1-p)}$$

so (4.13) follows from (4.12) and the criterion (4.4). □

We have used red/green coloring for the ARC model and its cousin the particle/bond Potts model, and yellow/white coloring for the bicolored FK model, to help avoid confusion between the models. In the next lemma we want to add bonds to the particle/bond Potts model red configuration to obtain a bicolored FK model. To maintain our color scheme, this requires thinking of these red bonds as instead being yellow. Thus we refer to the particle/bond Potts model "with the red bonds recolored yellow."

Lemma 4.12. *Let Λ be a finite subset of a lattice $\mathbb{L}$, and consider a $(q+1)$-state particle/bond Potts model on $(\overline{\Lambda}, \overline{\mathscr{B}}(\Lambda))$ at $(\beta, 0)$ with 0's boundary condition, with the red bonds recolored yellow. The yellow bonds of this model, supplemented by independent percolation of white bonds at density $p = 1 - e^{-\beta}$ on the "00" bonds, form a bicolored FK model on $(\overline{\Lambda}, \overline{\mathscr{B}}(\Lambda))$ with parameters $(p, q+1, q)$ and all-white boundary condition.*

Proof. From Remark 2.4, the yellow bonds of the particle/bond Potts model form a partial FK model on $(\overline{\Lambda}, \overline{\mathscr{B}}(\Lambda))$ with parameters $(p, q+1, q)$ and all-white boundary condition. From Lemma 4.8, conditionally on ω_y (or equivalently ω_r), the 0's (relabeled "-") and the particles (relabeled "+") form an Ising model on $\mathscr{I}(\omega_y, \Lambda)$ at $(\tilde{\beta}, \tilde{h})$, where $\tilde{\beta}$ and $\tilde{h}$ are given by (4.3), with boundary condition "+" on $\Lambda \backslash \mathscr{I}(\omega_y, \Lambda)$ and "-" on $\partial\Lambda$. Hence, still conditionally on ω_y, the white bonds from the independent percolation form an Ising ARC model at (p, Q) with this same boundary condition, where $Q = 1 + e^{\tilde{\beta}(m+\tilde{h})} = q+1$. However, by Proposition 2.3(i), for the bicolored FK model with parameters $(p, q+1, q)$, the white bonds have exactly this same Ising ARC model as their conditional distribution given ω_y. The result follows. □

The next proposition is the key to the proof of Theorem 3.4.

Proposition 4.13. *Let Λ be a finite set of sites of a lattice $\mathbb{L}$ with coordination number m, let $J, \kappa \ge 0$ and $q \ge 1$ and consider a q-state Potts lattice gas on Λ at $(1, J, \kappa, \mu)$ with site boundary condition η. Let β' be the minimum effective inverse temperature, and h' the maximum effective external field, of this Potts lattice gas, as given in (3.24), and define an Ising-model boundary condition η' by $\eta'_x = +$ if $\eta_x = 0$, $\eta'_x = -$ if $\eta_x \ne 0$. Then the following hold:*

- *(i) The 0's configuration of this Potts lattice gas is dominated by the "+" configuration of an Ising model on Λ at (β', h') with boundary condition η'.*
- *(ii) If $h' < 0$, this "+" configuration is further dominated by the "+" configuration of an Ising model on Λ at $(\beta'', 0)$ with boundary condition η', where*

$$\beta'' = \beta' \frac{m - h'}{m},$$

or equivalently,

$$e^{\beta''} = (qe^{\mu})^{1/m} e^{\kappa} \left(\frac{1}{q} e^J + \frac{q-1}{q} \right).$$

In particular:

(iii) The 0's configuration of a $(q+1)$*-state Potts model at* (β, h) *with boundary condition* η *is dominated by the "+" configuration of an Ising model on* Λ *at* (β', h') *(and at* $(\beta'', 0)$*, if* $h' < 0$*) with boundary condition* η'*, where* (β', h') *is given by*

$$e^{2\beta'} = e^{\beta} \left(\frac{1}{q} e^{\beta} + \frac{q-1}{q} \right), \quad e^{\beta'(m+h')} = \frac{e^{\beta(m+h)}}{q} \tag{4.14}$$

and $\beta'' = \beta'(m-h')/m$*, or equivalently*

$$e^{\beta''} = \frac{q-1+e^{\beta}}{q^{(m-1)/m}} e^{\beta h/m}. \tag{4.15}$$

Proof. Corresponding to the Potts lattice gas there is a red/black ARC model on $(\overline{\Lambda}, \overline{\mathscr{B}}(\Lambda))$ with site boundary condition η and parameters given by (2.13) and (2.17):

$$p_b = 1 - e^{-(\kappa+J)}, \quad p_{rb} = \frac{1-e^{-J}}{1-e^{-(\kappa+J)}}, \quad Q = 1 + \frac{e^{-\mu}}{q}.$$

Summing (2.16) over ω_r, we see that the black configuration has weights given by

$$W(\omega_b) = p_b^{|\omega_b|} (1-p_b)^{|\overline{\mathscr{B}}(\Lambda)| - |\omega_b|} Q^{I(\omega_b, \Lambda)} R(\omega_b) \delta_{D_g(\Lambda,\eta)}(\omega_b)$$

where

$$R(\omega_b) = \sum_{\omega_r \subset \omega_b} p_{rb}^{|\omega_r|} (1-p_{rb})^{|\omega_b| - |\omega_r|} q^{C(\omega_r, \Lambda)} \delta_{D_r(\Lambda,\eta)}(\omega_r).$$

Since the Ising ARC model with arbitrary (nongeneralized) site boundary condition has the FKG property, by (4.13) in Lemma 4.11 this black configuration dominates an Ising ARC model with parameters (p', Q) and site boundary condition η', where p' is given by

$$\frac{p'}{1-p'} = \left(\frac{p_{rb}}{q} + 1 - p_{rb} \right) \frac{p_b}{1-p_b} = 1 + e^{\kappa} \left(\frac{1}{q} e^J + \frac{q-1}{q} \right).$$

But then $p' = 1 - e^{-2\beta'}$, so from (2.20) this Ising ARC model corresponds to an Ising model at (β', h') with boundary condition η'. Now (i) follows from Lemma 4.5, and then (ii) from Lemma 4.6; (iii) is a special case of (i) and (ii). □

Remark 4.14. Proposition 4.13 is particularly useful when $h' < 0$, for then the dominating Ising "+" configuration is a minority spin. Particularly for the $(q+1)$-state Potts model in two dimensions, we will see that the comparison can be used to transfer known properties of the Ising model to the Potts model. This is useful because a number of properties are easier to prove for the Ising model, where one has tools such as symmetry inequalities which are not available for the Potts model in general. From (3.24) we have

$$h' < 0 \quad \text{if and only if} \quad \frac{e^{-\mu}}{q} < e^{\beta' m},$$

or equivalently

$$h' < 0 \quad \text{if and only if} \quad \frac{e^{-(\mu+\frac{1}{2}\kappa m)}}{q} < \left(\frac{1}{q}e^{J} + \frac{q-1}{q}\right)^{m/2}. \tag{4.16}$$

Lemma 4.15. *In any infinite-volume limit of the Potts ARC model corresponding to a Potts model at (β, h) with $h > 0$, red bonds a.s. do not percolate.*

Proof. As is well-known (see [4]), by use of a "ghost site" one can construct an FK model corresponding to a (usual) Potts model with a positive external field applied to species 0. This model has the finite energy property (see [34] or [7] for the definition) so a configuration a.s. has at most one infinite cluster [7]. It follows easily that in the joint Potts/FK configuration, there is no percolation of open bonds whose endpoints x, y have species $\sigma_x = \sigma_y \neq 0$. These are precisely the red bonds of the ARC model. □

The next result will be used in the proof of Theorem 3.15, when we compare the ARC model corresponding to a q-state Potts lattice gas with dilution parameter θ to a partial FK model at $(p, q+\theta, q)$. The red bonds of this "q-state" ARC model may be viewed loosely as an FK model with the same q, diluted by the addition of some 0 sites, just as the original Potts lattice gas is a diluted Potts model. Similarly, the bicolored FK model at $(p, q+\theta, q)$ is, again loosely, an FK model at (p, q), with bonds colored yellow, diluted by the addition of some white bonds. It is thus reasonable to try to compare these two types of models, particularly when the dilution is small. The main question to be answered is, given an ARC model, what is the comparable value of θ in the partial FK model? An answer, or more precisely a one-sided bound on an answer, comes from the following.

Proposition 4.16. *Let Λ be a finite set of sites of a lattice $\mathbb{L}$ with coordination number m and consider an ARC model P on $(\overline{\Lambda}, \overline{\mathscr{B}}(\Lambda))$ at (p_r, p_g, q, Q), with $q > 1$, and with single-species site boundary condition η. Let $\theta = q(Q-1)(1-p_g)^m$ be the dilution parameter.*

(i) If

$$p_g \leq p_r \quad \text{and} \quad \frac{p_g}{1-p_g} \leq \frac{p_r}{\theta(1-p_r)}, \tag{4.17}$$

then the red-bond configuration under P dominates the partial FK model at $(p_r, q+\theta, q)$ on $(\overline{\Lambda}, \overline{\mathscr{B}}(\Lambda))$ with site boundary condition η' given by $\eta'_x =$ "white" if $\eta_x = 0$, $\eta'_x =$ "yellow" if $\eta_x \neq 0$.

(ii) Let $\beta/2$ and h be the parameters of the conditional Ising model of the ARC model P, given by

$$p_g = 1 - e^{-\beta}, \quad \frac{q}{\theta} = e^{\beta(m-h)/2}, \tag{4.18}$$

and suppose that, in the infinite volume limit, for some $0 < \delta < 1/2$,

$$P^{Ising}_{-,\beta/2,h}(\sigma_x = + \text{ for some } x \text{ adjacent to } 0 \mid \sigma_0 = +) < \delta. \tag{4.19}$$

Suppose also that η is a constant-species (equivalently, red-wired) boundary condition. Then the red-bond configuration under P dominates the partial FK model at $(p_r, q+\theta', q)$ on $(\overline{\Lambda}, \overline{\mathscr{B}}(\Lambda))$ with all-yellow boundary condition, where

$$\theta' = \frac{1-\delta}{1-2\delta}\theta.$$

Since the Ising model in (4.19) is (except for boundary condition) the conditional Ising model of the ARC model P, the "+"spins in (4.19) correspond approximately to the 0's configuration of the ARC model P. Loosely, (4.19) holds when 0's are so rare that most 0's are isolated from any other 0's, and this will be true when θ is small. Thus (4.19) is a substitute for (4.17) when the dilution is very small. The values of greatest interest in Proposition 4.16 are small θ and p_r near the FK critical point $p_c^{FK}(q, \mathbb{L})$. Since $\theta' > \theta$, the conclusion in (ii) is weaker than (i), but for small δ the difference is small, and for the aforementioned values of greatest interest, we only expect our θ to be sharp up to a constant depending on q anyway; see Remark 4.17.

Proof (Proof of Proposition 4.16). The basic technique is roughly to compare the conditional Ising model of the ARC model P to the conditional neutral Potts lattice gas of the partial FK model. For clarity of exposition we give the proof of (i) only when η is the all-1's, or equivalently red-wired, boundary condition (so that η' is all-yellow); the general case is quite similar. The weights for the partial FK model are given by (2.24) and (2.25):

$$W^{PFK}(\omega_y) = p_r^{|\omega_y|}(1-p_r)^{|\overline{\mathscr{B}}(\Lambda)|-|\omega_y|} q^{C(\omega_y,\Lambda)} q^{-I(\omega_y,\Lambda)} F(\omega_y), \quad \omega_y \in \{0,1\}^{\overline{\mathscr{B}}(\Lambda)},$$

where

$$F(\omega_y) = \sum_{\omega_w \in \{0,1\}^{\mathscr{B}(\mathscr{I}(\omega_y,\Lambda))}} \left(\frac{p_r}{1-p_r}\right)^{|\omega_w|} \theta^{C(\omega_w, \mathscr{I}(\omega_y,\Lambda))} \left(\frac{q+\theta}{\theta}\right)^{I(\omega_y \vee \omega_w, \Lambda)}.$$

Here we use the fact that under the all-yellow boundary condition, we have $(\omega_y, \omega_w) \in A(\Lambda, \eta)$ if and only if $(\omega_w)_e = 0$ for all $e \notin \mathscr{B}(\mathscr{I}(\omega_y, \Lambda))$, meaning that effectively $\omega_w \in \{0,1\}^{\mathscr{B}(\mathscr{I}(\omega_y,\Lambda))}$. The weights for the ARC-model red bonds are given by summing (2.12):

$$W^{ARC}(\omega_r) = p_r^{|\omega_r|}(1-p_r)^{|\overline{\mathscr{B}}(\Lambda)|-|\omega_r|}q^{C(\omega_r,\Lambda)}T(\omega_r), \quad \omega_r \in \{0,1\}^{\overline{\mathscr{B}}(\Lambda)}, \tag{4.20}$$

where

$$T(\omega_r) = \sum_{\omega_g \in \{0,1\}^{\overline{\mathscr{B}}(\mathscr{I}(\omega_r,\Lambda))}} p_g^{|\omega_g|}(1-p_g)^{b(\omega_r)-|\omega_g|}Q^{I(\omega_r \vee \omega_g,\Lambda)} \tag{4.21}$$

with $b(\omega_r) = |\overline{\mathscr{B}}(\mathscr{I}(\omega_r,\Lambda))|$. Note that green bonds not in $\overline{\mathscr{B}}(\mathscr{I}(\omega_r,\Lambda))$ have been summed out; this is possible because the states of such bonds (open or closed) do not affect the factor $Q^{I(\omega_r \vee \omega_g,\Lambda)}$. Define

$$Q' = 1 + \frac{1}{(Q-1)(1-p_g)^m} = \frac{q+\theta}{\theta},$$

so that the values β, h obtained from p_g and Q via (4.6) are the same as the values β, h obtained from p_g and Q' via (4.10). Define

$$G(\omega) = \sum_{\omega_w \in \{0,1\}^{\mathscr{B}(\mathscr{I}(\omega,\Lambda))}} \left(\frac{p_g}{1-p_g}\right)^{|\omega_w|}(Q')^{I(\omega_w \vee \omega,\Lambda)}.$$

By Lemmas 4.9 and 4.10,

$$\frac{T(\omega \vee e)}{T(\omega)} = (Q'-1)^{I(\omega,\Lambda)-I(\omega\vee e,\Lambda)}\frac{G(\omega \vee e)}{G(\omega)} \quad \text{for all } \omega \text{ and } e. \tag{4.22}$$

Let $P^{IA}_{\mathscr{I}(\omega,\Lambda)}$ be the Ising ARC model on $(\mathscr{I}(\omega,\Lambda), \mathscr{B}(\mathscr{I}(\omega,\Lambda)))$ at (p_g, Q') with free boundary, and let $P^{NA}_{\mathscr{I}(\omega,\Lambda)}$ be the neutral ARC model on $(\mathscr{I}(\omega,\Lambda), \mathscr{B}(\mathscr{I}(\omega,\Lambda)))$ at $(p_r, 0, \theta, \frac{q+\theta}{\theta})$ with free boundary. Note that $P^{IA}_{\mathscr{I}(\omega,\Lambda)}$ is also the ARC model at $(p_g, 0, 1, Q')$. By (4.8) in the proof of Lemma 4.9,

$$\frac{G(\omega \vee e)}{G(\omega)} = P^{IA}_{\mathscr{I}(\omega,\Lambda)}\big(\{\omega_w : \{x,y\} \cap \mathscr{I}(\omega,\Lambda) \subset \mathscr{I}(\omega_w,\Lambda)\}\big)\left(\frac{1}{Q'}\right)^{I(\omega,\Lambda)-I(\omega\vee e,\Lambda)}. \tag{4.23}$$

By an argument similar to the proof of (4.8) we have

$$\frac{F(\omega \vee e)}{F(\omega)} = P^{NA}_{\mathscr{I}(\omega,\Lambda)}\big(\{\omega_w : \{x,y\} \cap \mathscr{I}(\omega,\Lambda) \subset \mathscr{I}(\omega_w,\Lambda)\}\big)\left(\frac{1}{q+\theta}\right)^{I(\omega,\Lambda)-I(\omega\vee e,\Lambda)}. \tag{4.24}$$

Under (4.17), by Lemma 4.4, $P^{NA}_{\mathscr{I}(\omega,\Lambda)}$ dominates $P^{IA}_{\mathscr{I}(\omega,\Lambda)}$. Hence (4.24), (4.22) and (4.23) show that

$$\frac{F(\omega \vee e)}{F(\omega)} \le \left(\frac{Q'}{q+\theta}\right)^{I(\omega,\Lambda)-I(\omega\vee e,\Lambda)} \frac{G(\omega \vee e)}{G(\omega)} = \left(\frac{1}{q}\right)^{I(\omega,\Lambda)-I(\omega\vee e,\Lambda)} \frac{T(\omega \vee e)}{T(\omega)}. \tag{4.25}$$

Using (4.4) this shows that W^{ARC}/W^{PFK} is an increasing function. Since from Lemma 4.1 the ARC model (with boundary condition η) has the FKG property, so does the red-bond configuration alone, and (i) follows.

Now suppose (4.19) holds. The value of $I(\omega,\Lambda) - I(\omega \vee e, \Lambda) = |\{x, y\} \cap \mathscr{I}(\omega,\Lambda)|$ is either 0, 1 or 2; the case of 0 is trivial because then (4.22), (4.23) and (4.24) are all equal to 1. Let us assume $I(\omega,\Lambda) - I(\omega \vee e, \Lambda) = 2$; the case of 1 is similar. Let U denote the event that $\sigma_z = +$ for some z adjacent to 0. We have

$$\begin{aligned}
&P^{IA}_{\mathscr{I}(\omega,\Lambda)}\big(\{\omega_w : x \notin \mathscr{I}(\omega \vee \omega_w, \Lambda)\}\big)\\
&\quad\le P^{Ising}_{\mathscr{I}(\omega,\Lambda),-,\beta/2,h}(\sigma_x = +, \sigma_z = + \text{ for some } z \text{ adjacent to } x)\\
&\quad\le P^{Ising}_{-,\beta/2,h}([\sigma_0 = +] \cap U)\\
&\quad< \frac{P^{Ising}_{-,\beta/2,h}([\sigma_0 = +] \cap U)}{P^{Ising}_{-,\beta/2,h}(U^c)}\\
&\quad= \frac{P^{Ising}_{-,\beta/2,h}(\sigma_0 = + \mid U^c)\,P^{Ising}_{-,\beta/2,h}(U \mid \sigma_0 = +)}{P^{Ising}_{-,\beta/2,h}(U^c \mid \sigma_0 = +)}\\
&\quad\le \frac{\delta}{Q'(1-\delta)}
\end{aligned}$$

and similarly for y, so, using the FKG property,

$$P^{IA}_{\mathscr{I}(\omega,\Lambda)}\big(\{\omega_w : \{x, y\} \subset \mathscr{I}(\omega \vee \omega_w, \Lambda)\}\big) \ge \left(1 - \frac{\delta}{Q'(1-\delta)}\right)^{I(\omega,\Lambda)-I(\omega\vee e,\Lambda)}. \tag{4.26}$$

Let

$$\tau = 1 - \frac{\delta}{Q'(1-\delta)}.$$

Combining (4.26) with (4.22) and (4.23) yields

$$\frac{T(\omega \vee e)}{T(\omega)} = (Q'-1)^{I(\omega,\Lambda)-I(\omega\vee e,\Lambda)} \frac{G(\omega \vee e)}{G(\omega)} \ge \left(\frac{(Q'-1)\tau}{Q'}\right)^{I(\omega,\Lambda)-I(\omega\vee e,\Lambda)}.$$

It follows using the criterion (4.4) that the red-bond configuration of the ARC model P dominates the red-bond configuration of the neutral ARC model at $(p_r, 0, q, Q'')$ on $(\overline{\Lambda}, \overline{\mathscr{B}}(\Lambda))$ with boundary condition η, where

$$Q'' = \frac{Q'}{(Q'-1)\tau} = \frac{q+\theta}{q\tau}.$$

But this neutral ARC model satisfies the hypotheses of (i), so its red-bond configuration dominates the partial FK model at $(p_r, q+\theta'', q)$ on $(\overline{\Lambda}, \overline{\mathscr{B}}(\Lambda))$ with all-yellow (equivalently wired) boundary condition, where

$$\theta'' = q(Q''-1) = \frac{q+\theta}{\tau} - q.$$

A short calculation shows $\theta'' < \theta'$, and (ii) follows. □

Remark 4.17. It is apparent from the proof of Proposition 4.16 that if we could find a smaller value of θ (so that $Q' < 1 + q/\theta$) for which the inequality between the first and last terms of (4.25) were reversed, the domination in (i) and (ii) would then also be reversed. Now from (2.14), if we extend the neutral ARC model to its particle/bond form, we have

$$\begin{aligned} &P^{NA}_{\mathscr{I}(\omega,\Lambda)}\big(\{\omega_w : \{x,y\} \subset \mathscr{I}(\omega \vee \omega_w, \Lambda)\}\big)\left(\frac{q}{q+\theta}\right)^{I(\omega,\Lambda)-I(\omega\vee e,\Lambda)} \\ &\quad = P^{NA}_{\mathscr{I}(\omega,\Lambda)}(n_x = n_y = 0); \end{aligned}$$

note that this particle/bond neutral ARC model has 0's boundary condition on $\partial\mathscr{I}(\omega,\Lambda)$. Thus to establish the reverse of (4.25), it is enough to choose θ so that for the particle/bond neutral ARC model at $(p_r, 0, \theta, \frac{q+\theta}{\theta})$ we have (in the notation of the last proof)

$$P^{NA}_{\mathscr{I}(\omega,\Lambda)}(n_x = n_y = 0) \geq \left(\frac{Q'-1}{Q'}\right)^{I(\omega,\Lambda)-I(\omega\vee e,\Lambda)}. \tag{4.27}$$

The bonds of this neutral ARC model are precisely the white bonds of the bicolored FK model from which it was obtained, and the 0's of the neutral ARC model are precisely the isolated yellow sites of this bicolored FK model. Thus we might expect that for fixed ω, when $x \in \mathscr{I}(\omega,\Lambda)$,

$$\begin{aligned} P^{NA}_{\mathscr{I}(\omega,\Lambda)}(n_x = 1) &= P^{BFK}(x \text{ is white} \mid \omega_y = \omega) \\ &\approx P^{BFK}(x \text{ is white } \mid x \in \mathscr{I}(\omega_y,\Lambda)) \\ &= \frac{P^{BFK}(x \text{ is white})}{P^{BFK}(x \text{ is white or isolated yellow})} \end{aligned} \tag{4.28}$$

where P^{BFK} denotes the bicolored FK model at $(p_r, q+\theta, q)$ on $(\overline{\Lambda}, \overline{\mathscr{B}}(\Lambda))$ with boundary condition η'. (It is only the approximation that is nonrigorous here.) Since

$$P^{BFK}(x \text{ is white}) \leq \frac{\theta}{q+\theta} \quad \text{and} \quad P^{BFK}(x \text{ is yellow } \mid x \text{ is isolated}) = \frac{q}{q+\theta},$$

the right side of (4.28) is at most

$$\frac{\theta}{\theta + qP^{FK}(x \text{ is isolated})}, \tag{4.29}$$

where P^{FK} denotes the FK model at $(p_r, q+\theta)$ on $(\overline{\Lambda}, \overline{\mathscr{B}}(\Lambda))$ with generalized site boundary condition η'. We might expect (4.27) to hold when (4.29) is approximately $1/Q'$, that is,

$$\theta \approx \frac{qP^{FK}(x \text{ is isolated})}{Q'},$$

which for small θ is within approximately a factor of $P^{FK}(x$ is isolated) of the value $\theta = q/(Q'-1)$ of Proposition 4.16. Thus we expect that the value of θ in the Proposition is sharp "up to a constant," but we are unable to prove this; the main obstacle is the absense of the FKG property for the neutral ARC model at $(p_r, 0, \theta, \frac{q+\theta}{\theta})$ when $\theta < 1$.

Our next lemma shows that 0's are rare in the Potts lattice gas roughly when the dilution parameter θ is small and the effective external field (on empty sites–see (3.21)) is negative for the binary lattice gas obtained by replacing parameters J, κ with $0, J+\kappa$.

Lemma 4.18. *Suppose $\kappa_k \geq 0$ and $\mu_k \in \mathbb{R}$ for each $k \geq 1$, and $q \geq 1$, $J \geq 0$. Let $\mathbb{L}$ be a lattice of coordination number m, and let P_k be the infinite-volume q-state Potts lattice gas at $(1, J, \kappa_k, \mu_k)$ with all-1's boundary condition. If*

$$\mu_k + \kappa_k m \to \infty \tag{4.30}$$

and

$$\mu_k + \frac{1}{2}\kappa_k m + \log q \geq 0 \quad \textit{for all sufficiently large } k, \tag{4.31}$$

then $P_k(n_0 = 0) \to 0$ as $k \to \infty$. Conversely if $P_k(n_0 = 0) \to 0$ then (4.30) holds and

$$\liminf_{k\to\infty} \frac{\mu_k + \frac{1}{2}\kappa_k m + \log q}{\kappa_k} \geq 0. \tag{4.32}$$

(The quantity $(\mu_k + \frac{1}{2}\kappa_k m + \log q)/\kappa_k$ should be interpreted as $+\infty$ if $\kappa_k = 0$.)

Proof. We may assume our Potts lattice gas is in particle/bond form. Let $(p_r, (p_g)_k, q, Q_k)$ be the parameters of the ARC model corresponding to P_k (see (2.13). The conditional Ising model of P_k has parameters $(\beta_k/2, h_k)$ given by Lemma 4.8:

$$\beta_k = \kappa_k, \quad -\frac{1}{2}\beta_k h_k = \mu_k + \frac{1}{2}\kappa_k m + \log q,$$

so

$$\frac{1}{2}\beta_k(m - h_k) = \mu_k + \kappa_k m + \log q.$$

Thus (4.30) and (4.31) are equivalent respectively to

$$\beta_k(m - h_k) \to \infty \quad \text{and} \quad h_k \le 0, \tag{4.33}$$

while (4.30) and (4.32) are equivalent respectively to

$$\beta_k(m - h_k) \to \infty, \quad \text{and} \quad \limsup_{k\to\infty} h_k \le 0. \tag{4.34}$$

Now the ARC-model red-bond configuration on the full lattice under P_k is dominated by the FK model at (p_r, q) so $P_k(\Lambda \subset \mathscr{I}(\omega_r, S(\mathbb{L})))$ stays bounded away from 0 as $k \to \infty$ for each finite set of sites Λ. Hence in particular

$$P_k(n_0 = 0) \to 0 \quad \text{if and only if} \quad P_k\big(n_0 = 0 \mid 0 \in \mathscr{I}(\omega_r, S(\mathbb{L}))\big) \to 0. \tag{4.35}$$

But this latter probability is just the probability of a "+" at 0 for the conditional Ising model on $\mathscr{I}(\omega_r, S(\mathbb{L}))$ which has "-" boundary condition, and this probability is bounded above by the same probability for the infinite-volume minus phase of the Ising model on $\mathbb{L}$. That is,

$$P_k\big(n_0 = 0 \mid 0 \in \mathscr{I}(\omega_r, S(\mathbb{L}))\big) \le P_k^{Ising}(\sigma_0 = +), \tag{4.36}$$

where P_k^{Ising} denotes the infinite-volume minus phase of the Ising model at $(\beta_k/2, h_k)$.

If (4.30) and (4.31) hold, or equivalently (4.33) holds, then for large k either $-h_k$ is large or both β_k is large and $h_k \le 0$; either way we obtain $P_k^{Ising}(\sigma_0 = +) \to 0$ and hence from (4.35) and (4.36), $P_k(n_0 = 0) \to 0$.

Conversely suppose $P_k(n_0 = 0) \to 0$. Analogously to (4.36) we have

$$\begin{aligned} P_k\big(n_0 = 0 \mid 0 \in \mathscr{I}(\omega_r, S(\mathbb{L}))\big) &\ge P_k^{Ising}(\sigma_0 = + \mid \sigma_x = - \text{ for all } x \text{ adjacent to } 0) \\ &= \frac{1}{1 + e^{\beta(m-h_k)/2}}, \end{aligned}$$

so the first half of (4.34) follows from (4.35).

To prove the second half of (4.34), suppose the first half holds but $\limsup h_k > \epsilon$ for some $\epsilon > 0$; we may assume $h_k \ge \epsilon$ for all k, so $\beta_k \to \infty$. If Λ is sufficiently large then for all large k,

$$P_k\big(n_0 = 0 \mid \Lambda \subset \mathscr{I}(\omega_r, S(\mathbb{L}))\big) \ge P_k^{Ising}(\sigma_0 = + \mid \sigma_x = - \text{ for all } x \in \partial\Lambda) > \frac{1}{2}.$$

Since $P_k(\Lambda \subset \mathscr{I}(\omega_r, S(\mathbb{L})))$ is bounded away from 0, it follows that $P_k(n_0 = 0)$ is bounded away from 0.

In terms of ARC model parameters, (4.31) can be restated as

$$(Q_k - 1)(1 - (p_g)_k)^{m/2} \le 1. \tag{4.37}$$

Remark 4.19. The proof of Lemma 4.18 contains the fact that (4.33) implies $P_k^{Ising}(\sigma_0 = +) \to 0$. One can actually obtain the stronger conculsion from (4.33) that

$$P_k^{Ising}(\sigma_x = + \text{ for some } x \text{ adjacent to } 0 \mid \sigma_0 = +) \to 0.$$

Indeed, (4.33) implies that for large k, either $-h_k$ is very large or both $h_k \leq 0$ and β_k is large enough that a Peierls-type argument shows that a cluster of two or more "+" spins is much less likely than a single isolated "+" spin.

5. Proofs of the main theorems

Proof (Proof of Theorem 3.2). The Hamiltonian for the q-state Potts lattice gas is as given by (2.1); for simplicity we assume $\mu_x = \mu$ for all x and assume there is a site boundary condition which has a single particle species i. Let $F' = \{x \in \partial\Lambda : \eta_x = i\}$. As in Remark 4.2, the corresponding ARC model on a subgraph $(\Lambda', \mathscr{B}')$ of $(\overline{\Lambda}, \overline{\mathscr{B}}(\Lambda))$ is unconditioned. We use P^{PLG} to denote the Potts lattice gas and P^{ARC} to denote this ARC model on $(\Lambda', \mathscr{B}')$, which has weights $W(\omega_r, \omega_g)$ given by (2.12), parameters given by (2.13) and site boundary condition η equivalent to a red-wired boundary condition on F'. Let $W^{FK}(\omega_r)$ and P^{FK} denote the weights and probabilities, respectively, for an FK model on $(\Lambda', \mathscr{B}')$ at (p_r, q) with wired boundary condition on F'. Let P^{ind} denote probability corresponding to independent percolation at density p_g on $\mathscr{B}'$. For $A, B \subset \Lambda$ we have using (2.14):

$$\begin{aligned} P^{PLG}(X_0 = A) &= \sum_{K \supset A} P^{ARC}(\mathscr{I}(\omega_b, \Lambda) = K) \left(\frac{Q-1}{Q}\right)^{|A|} \left(\frac{1}{Q}\right)^{|K|-|A|} \\ &= (Q-1)^{|A|} Z_{ARC}^{-1} \sum_{(\omega_r, \omega_g): \mathscr{I}(\omega_r \vee \omega_g, \Lambda) \supset A} \frac{W(\omega_r, \omega_g)}{Q^{I(\omega_r \vee \omega_g)}} \\ &= (Q-1)^{|A|} Z_{ARC}^{-1} \sum_{\omega_r : \mathscr{I}(\omega_r, \Lambda) \supset A} W^{FK}(\omega_r) \sum_{\omega_g : \mathscr{I}(\omega_g, \Lambda) \supset A} P^{ind}(\omega_g) \\ &= (Q-1)^{|A|} Z_{ARC}^{-1} Z_{FK} P^{FK}(\overline{\mathscr{B}}(A) \text{ all closed}) P^{ind}(\overline{\mathscr{B}}(A) \text{ all closed}), \end{aligned} \tag{5.38}$$

where Z_{ARC} and Z_{FK} are the partition functions of the ARC and FK models, respectively. The FKG property of the measure $P^{FK}(\cdot \mid \overline{\mathscr{B}}(A \cap B) \text{ all closed})$ yields that

$$\frac{P^{FK}(\overline{\mathscr{B}}(A \cup B) \text{ all closed})}{P^{FK}(\overline{\mathscr{B}}(A) \text{ all closed})} \geq \frac{P^{FK}(\overline{\mathscr{B}}(B) \text{ all closed})}{P^{FK}(\overline{\mathscr{B}}(A \cap B) \text{ all closed})},$$

and similarly for P^{ind}. This and (5.38) readily yield

$$P^{PLG}(X_0 = A \cup B) P^{PLG}(X_0 = A \cap B) \geq P^{PLG}(X_0 = A) P^{PLG}(X_0 = B),$$

which is the FKG lattice condition. The proof under free boundary condition, or for nonconstant μ_x, is essentially similar. □

Proof (Proof of Corollary 3.3). For a joint ARC/Potts lattice gas configuration on $\mathbb{L}$, black bonds cannot percolate if each site is surrounded by a circuit on which every site x has $\sigma_x = 0$. As in [9], with this observation the corollary is a direct consequence of Proposition 4.7, Theorem 3.2 and the main result of [19]. □

Proof (Proof of Theorem 3.4). Consider first the FK model with $p < p_1$; define β by $p = 1 - e^{-\beta}$. Let β' be the minimum effective inverse temperature, and h' the maximum effective external field, of the corresponding $(q+1)$-state Potts model at $(\beta, 0)$, as given by (4.14). It is easily checked that

$$h' < 0 \quad \text{if and only if} \quad \beta < \beta_1.$$

Let β'' be as in (4.15) and let Λ be a finite subset of the sites of $\mathbb{L}$, with $0 \in \Lambda$. By Proposition 4.13(iii), the 0's configuration of the q-state Potts model at $(\beta, 0)$ with 0's boundary condition on $\partial\Lambda$ is dominated by the "+" configuration of the Ising model at (β', h') with "+" boundary condition on $\partial\Lambda$. Therefore by Lemma 4.12, the white-bond configuration of the bicolored FK model at $(p, q+1, q)$ on $(\overline{\Lambda}, \overline{\mathscr{B}}(\Lambda))$ with all-white boundary condition is dominated by independent percolation at density p on the "++"bonds of this Ising model. Since $h' < 0$, results from [24] and [36] say that the Ising model has the weak mixing property, and results from [25] and [8] say that it has exponential decay of "+" connectivity. (These proofs are only written for the square lattice, but everything works for general planar lattices; the key fact used in [8] is the result of [2] that the Ising model has exponential decay of correlations for all $\beta < \beta_c^{Ising}(\mathbb{L})$, and this proof works on general periodic lattices with minor modifications [1].) Therefore there exist constants $C, \lambda > 0$, not depending on Λ, such that

$$\begin{aligned} P^{Ising}_{\Lambda,+,\beta',h'}(0 \leftrightarrow \partial\Lambda \text{ by a lattice path on which all sites } x \text{ have} \\ \sigma_x = +) \le Ce^{-\lambda r(\Lambda)}. \end{aligned} \tag{5.39}$$

Define probability measures as follows: $P^{Ising,++}_{\Lambda,+,\beta',h',p}$ for the distribution of the bond-site configuration produced by independent percolation at density p on "++" bonds, in $(\overline{\Lambda}, \overline{\mathscr{B}}(\Lambda))$, of the Ising model at (β', h') on Λ with "+" boundary condition; $P^{BFK}_{\Lambda,wh}$ for the bicolored FK model on $(\overline{\Lambda}, \overline{\mathscr{B}}(\Lambda))$ with all-white boundary condition; and $P^{FK}_{\Lambda,w}$ for the FK model on $(\overline{\Lambda}, \overline{\mathscr{B}}(\Lambda))$ with wired boundary condition. From (5.39),

$$P^{Ising,++}_{\Lambda,+,\beta',h',p}(0 \leftrightarrow \partial\Lambda \text{ by a path of open bonds}) \le Ce^{-\lambda r(\Lambda)}. \tag{5.40}$$

Hence by the domination,

$$P^{BFK}_{\Lambda,wh}(0 \leftrightarrow \partial\Lambda \text{ by a path of open white bonds}) \le Ce^{-\lambda r(\Lambda)},$$

which is equivalent to

$$P^{FK}_{\Lambda,w}(0 \leftrightarrow \partial\Lambda \text{ by a path of open bonds}) \le Ce^{-\lambda r(\Lambda)}; \tag{5.41}$$

this proves exponential decay of local wired-boundary connectivities, and thus also weak mixing, for the FK model. This proves (iii). Applying this result to the dual lattice $\mathbb{L}^*$, we see that when $p > p_2$, that is, $p^* < p_1(q+1, m^*)$, the dual configuration has exponential decay of local wired-boundary connectivities, and thus has the weak mixing property. But weak mixing for the dual configuration is equivalent to weak mixing for the regular configuration, and (i) follows. (ii) is an immediate consequence of (i). □

Proof (Proof of Theorem 3.7). For simplicity we restrict attention to the integer lattice. Consider the FK model in d dimensions at $(p, q+1)$ with $p < p_1(q+1, 2d)$. Let β', h', β'' be as in the proof of Theorem 3.4. In order to show that there is no percolation in the wired-boundary infinite-volume limit, one must establish the following analog of (5.41):

$$\lim_{\Lambda_0 \nearrow \mathbb{Z}^d} \lim_{\Lambda \nearrow \mathbb{Z}^d} P^{FK}_{\Lambda,w}(0 \leftrightarrow \partial\Lambda_0 \text{ by a path of open bonds}) = 0.$$

For this it is enough to establish the following analog of (5.40):

$$\lim_{\Lambda_0 \nearrow \mathbb{Z}^d} \lim_{\Lambda \nearrow \mathbb{Z}^d} P^{Ising,++}_{\Lambda,+,\beta',h',p}(0 \leftrightarrow \partial\Lambda_0 \text{ by a path of open bonds}) = 0. \tag{5.42}$$

Since $h' < 0$, the Ising model at (β', h') has a unique Gibbs distribution, so one can replace the "+" boundary condition with "-" in (5.42). From Proposition 4.13(ii), one can then also replace (β', h') with $(\beta'', 0)$ in (5.42). Further, since the FK model is monotone in p, one need only consider β close to β_1. Thus the proof of Gibbs uniqueness in Theorem 3.4 goes through since we have assumed (3.4) for appropriate β. □

Proof (Proof of Theorem 3.11). The idea is to show that the corresponding Potts ARC model is dominated by another Potts ARC model corresponding to a Potts model with a positive external field on 0's; then Lemma 4.15 can be applied.

We claim that for some $\epsilon > 0$, we have

$$\frac{2}{\beta}\frac{\partial}{\partial h} P^{Ising}_{\Lambda,+,\beta/2,h}(\sigma_x = +) \le \left(\frac{1}{2} - \epsilon\right) P^{Ising}_{\Lambda,+,\beta/2,h}(\sigma_x = +) \quad \text{for all } h > 0 \text{ and all finite } \Lambda \text{ and } x \in \Lambda. \tag{5.43}$$

Indeed, as is standard, from symmetry inequalities the left side of (5.43) is a decreasing function of h and an increasing function of Λ, while from the FKG property the right side is a decreasing function of Λ. So it is enough to verify (5.43) in the limit as $h \searrow 0$ and $\Lambda \nearrow \mathbb{Z}^d$, but this is exactly (3.15). Thus (5.43) is proved. By symmetry, (5.43) is equivalent to

$$-\frac{2}{\beta}\frac{\partial}{\partial h} P^{Ising}_{\Lambda,-,\beta/2,h}(\sigma_x = -) \le \left(\frac{1}{2} - \epsilon\right) P^{Ising}_{\Lambda,-,\beta/2,h}(\sigma_x = -) \quad \text{for all } h < 0 \text{ and all finite } \Lambda \text{ and } x \in \Lambda. \tag{5.44}$$

Let $Q = 1 + e^{2d\beta}/q$, so the Potts ARC model corresponding to the Potts model at $(\beta, 0)$ has parameters (p, p, q, Q). It is sufficient to show that this Potts ARC model, with red-wired boundary condition, has no percolation of red bonds in the infinite-volume limit. By Lemma 4.15, for this it is enough to find q' and $h' > 0$ such that, letting (p, p, q', Q') be the parameters of the Potts ARC model corresponding to a q'-state Potts model at (β, h'), this Potts ARC model dominates the Potts ARC model at (p, p, q, Q), both models having red-wired boundary condition on some $(\overline{\Lambda}, \overline{\mathscr{B}}(\Lambda))$. The external field h_I of the conditional Ising model of the latter Potts ARC model is given by

$$Q = 1 + e^{\beta(2d+h_I)/2},$$

so $e^{\beta h_I/2} = e^{d\beta}/q < 1$ and thus $h_I < 0$. Hence we can choose $0 > h'_I > h_I$, then choose $q' < q$ and $0 < h' \le \epsilon(h'_I - h_I)$ satisfying

$$e^{\beta(2d+h'_I)/2} = \frac{e^{\beta(2d+h')}}{q'},$$

and set

$$Q' = 1 + \frac{e^{\beta(2d+h')}}{q'}.$$

We now show that under red-wired boundary condition, the red-bond configuration of the Potts ARC model at (p, p, q', Q') dominates the red-bond configuration of the Potts ARC model at (p, p, q, Q). The weights for the ARC model red-bond configuration at (p, p, q, Q) on $(\overline{\Lambda}, \overline{\mathscr{B}}(\Lambda))$ with red-wired boundary condition, and the definition of $T(\omega_r)$, are given by (4.20) and (4.21). We let $W'(\omega_r)$ and $T'(\omega_r)$ denote the corresponding quantities for the model at (p, p, q', Q'). To establish the desired domination, by (4.4) it is sufficient to show that

$$\left(\frac{q}{q'}\right)^{C(\omega,\Lambda)-C(\omega\vee e,\Lambda)} \frac{T'(\omega \vee e)}{T'(\omega)} \ge \frac{T(\omega \vee e)}{T(\omega)} \quad \text{for all } \omega \text{ and all } e = \langle xy \rangle. \tag{5.45}$$

If neither x nor y is in $\mathscr{I}(\omega, \Lambda)$, then both sides of (5.45) are 1. Let us assume both x and y are in $\mathscr{I}(\omega, \Lambda)$, so $C(\omega, \Lambda) - C(\omega \vee e, \Lambda) = 1$; the case in which exactly one is in $\mathscr{I}(\omega, \Lambda)$ is similar. By Lemma 4.10,

$$\begin{aligned} \frac{T(\omega \vee e)}{T(\omega)} &= P^{Ising}_{\mathscr{I}(\omega,\Lambda),-,\beta/2,h_I}(\sigma_x = \sigma_y = -) \\ &= P^{Ising}_{\mathscr{I}(\omega,\Lambda),-,\beta/2,h_I}(\sigma_x = -) P^{Ising}_{\mathscr{I}(\omega,\Lambda)\cup\{x\},-,\beta/2,h_I}(\sigma_y = -), \end{aligned} \tag{5.46}$$

and similarly for T', with h_I replaced by h'_I. Also,

$$\frac{q}{q'} = e^{-\beta(h_I-h'_I)/2} e^{-\beta h'} \ge e^{(\frac{1}{2}-\epsilon)\beta(h'_I-h_I)}. \tag{5.47}$$

Integrating (5.44) gives

$$\log P^{Ising}_{\mathscr{I}(\omega,\Lambda),-,\beta/2,h'_I}(\sigma_x = -) - \log P^{Ising}_{\mathscr{I}(\omega,\Lambda),-,\beta/2,h_I}(\sigma_x = -) \geq -\left(\frac{1}{2} - \epsilon\right)\frac{\beta(h'_I - h_I)}{2}$$

and

$$\log P^{Ising}_{\mathscr{I}(\omega,\Lambda)\cup\{x\},-,\beta/2,h'_I}(\sigma_y = -) - \log P^{Ising}_{\mathscr{I}(\omega,\Lambda)\cup\{x\},-,\beta/2,h_I}(\sigma_y = -) \geq -\left(\frac{1}{2} - \epsilon\right)\frac{\beta(h'_I - h_I)}{2},$$

which with (5.46) and (5.47) proves (5.45).

Proof (Proof of Corollary 3.12). We will show that for large d, (3.15) holds for all β in a neighborhood of $(\log q)/d$. Consider the mean-field magnetization $M_0(\beta_0, h_0)$, which for $\beta_0 > 2$ and $h_0 \geq 0$ is the positive solution of

$$M_0 = \tanh\left(\frac{\beta_0(M_0 + h_0)}{2}\right). \tag{5.48}$$

We claim that for $\beta_0 > 2.257$ we have

$$\frac{1}{\beta_0}\frac{\partial M_0}{\partial h_0}(\beta_0, 0) < 1 + M_0(\beta_0, 0). \tag{5.49}$$

In fact, differentiating (5.48) yields

$$\frac{2}{\beta_0}\frac{\partial M_0}{\partial h_0} = \frac{1 - M_0^2}{1 - \frac{1}{2}\beta_0(1 - M_0^2)}$$

so (5.49) is equivalent to

$$\frac{1 - M_0}{2} < 1 - \frac{1}{2}\beta_0(1 - M_0^2)$$

at $(\beta_0, 0)$. For this it suffices that

$$M_0 > 1 - \frac{1}{\beta_0},$$

which is equivalent to

$$\tanh\left(\frac{\beta_0 - 1}{2}\right) > 1 - \frac{1}{\beta_0}. \tag{5.50}$$

A routine calculation verifies that (5.50) holds for $\beta_0 > 2.257$.

Now suppose $\{\beta_d\}$ is a sequence with $d\beta_d \to \log q$. Since $q + 1 > 10.56$ we have $\log q > 2.257$. Also, from convergence to mean-field limits [28],

$$M\left(\frac{\beta_d}{2}, 0\right) \to M_0(\log q, 0)$$

and

$$\chi\left(\frac{\beta_d}{2},0\right) \to \frac{1}{\log q}\frac{\partial M_0}{\partial h_0}(\log q,0).$$

With (5.49) this proves (3.15) for all β in a neighborhood of $(\log q)/d$, for large d. □

Proof (Proof of Theorem 3.15). Fix $\delta > 0$ and let $\theta' = \theta(1+\delta)/(1+2\delta)$. Fix $p_r > p_c^{FK}(q+\theta',\mathbb{L})$. From Lemma 4.18, Remark 4.19 and (3.17), if θ is sufficiently small then (4.19) holds for β, h as in (4.18). Hence by Proposition 4.16(ii) the red bonds percolate in the infinite-volume ARC model on $\mathbb{L}$ at (p_r, p_g, q, Q) with red-wired boundary condition. Thus $p_c^{ARC}(p_g, q, Q, \mathbb{L}) \le p_c^{FK}(q+\theta',\mathbb{L})$. Since δ is arbitrary, (i) follows. Part (ii) is just a restatement of (i) in the context of the Potts lattice gas, using (3.18). □

Proof (Proof of Corollary 3.17). This is immediate from Theorem 3.15(ii) and the result from [31] that $\exp(\beta_c^{Potts}(q,\mathbb{Z}^2)) = 1+\sqrt{q}$ for all $q \ge 25.72$. □

Proof (Proof of Theorem 3.18). Let $\beta > \beta_c^{Potts}(q+\theta(h),\mathbb{L})$. Since $\beta_c^{Potts}(q+t,\mathbb{L})$ is an increasing function of t, we have $\beta > \beta_c^{Potts}(q+e^{-\beta h},\mathbb{L})$. Let (p_r, p_g, q, Q) be the parameters of the Potts ARC model corresponding to the Potts model at $(\beta, -h)$. The dilution parameter of this Potts ARC model is $e^{-\beta h} \le 1$, and $p_r = p_g = 1-e^{-\beta}$, so Proposition 4.16 applies and shows that the red bond configuration of this ARC model, with red-wired boundary condition, dominates the partial FK model at $(p_r, q+e^{-\beta h}, q)$. Since $\beta > \beta_c^{Potts}(q+e^{-\beta h},\mathbb{L})$, there is percolation in the infinite-volume wired-boundary FK model at $(p_r, q+e^{-\beta h})$, so there is also percolation a.s. in the yellow-boundary partial FK model at $(p_r, q+e^{-\beta h}, q)$. Therefore the red bonds of our Potts ARC model percolate, meaning $\beta \ge \beta_c^{Potts}(q+1,-h,\mathbb{L})$, and the theorem follows. □

Proof (Proof of Theorem 3.19). Since the Ising model has the FKG property, on a planar lattice there is no percolation of "+" spins when the external field is negative (see [24], [25].) Hence the theorem is an immediate consequence of Proposition 4.13 and (4.16). □

Proof (Proof of Theorem 3.20). Fix a finite Λ and a boundary condition η. The corresponding ARC model on $(\overline{\Lambda}, \overline{\mathscr{B}}(\Lambda))$ with boundary condition η^1 (equivalently, red-wired) dominates the ARC model with boundary condition η, so as in the proof of Proposition 4.7, the Potts lattice gases under the two boundary conditions can be coupled, creating a pair of configurations $((\omega_r,\omega_g),(\omega_r',\omega_g'))$ which agree everywhere inside each dual surface which is crossed by no open bond (red or green) of the red-wired ARC model configuration (ω_r,ω_g). But every site not inside such a dual surface is necessarily in the boundary particle cluster, and the result follows. □

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