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F.M. Dekking · P. v.d. Wal

Fractal percolation and branching cellular automata

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Abstract. Branching cellular automata (BCA) are introduced as generalisations of fractal percolation by admitting neighbour dependence. We associate sequences of random sets with BCA's and study their convergence. In case of convergence we derive the Hausdorff dimension of the limit set and of its boundary. To accomplish the latter we proof that the boundary of a set generated by a BCA is again a set generated by a BCA.

1. Introduction

In this paper we study generalisations of a family of random sets introduced by Benoit Mandelbrot in [10]. Mandelbrot coined the name canonical curdling for these sets, but they are commonly known as fractal percolation. Let p be a number with $0 \leq p \leq 1$ and $[0,1]^d$ be the unit cube in $\mathbb{R}^d$. We furthermore choose an integer base $M \geq 2$. Random sets $K_0 = [0,1]^d, K_1, \ldots, K_n$ are generated by a recursive construction. The set K_0 is a union of M^d subcubes with side lengths M^{-1}. Generate K_1 by retaining each of these subcubes with probability p, or discarding it with probability $1-p$, independently of each other. In general K_n is a union of M-adic cubes of order n, i.e., with side lengths M^{-n}, and K_{n+1} is obtained by retaining or discarding each of the order $n+1$ M-adic subcubes of these cubes with probability p respectively $1-p$, independently of each other, and of all the previous choices. The limit set $K = \cap_{n=0}^{\infty} K_n$ is a fractal set with a.s. Hausdorff dimension $\log(pM^d)/\log M$, conditioned on being non-empty (see [2], but also [6], and [5]).

Mandelbrot introduced fractal percolation as an alternative model for turbulence in a fluid in a critique of Kolmogorov's model. However fractal percolation is not more than a metaphor for turbulence. In the paper [15] the authors argue that physically there is dependence on the activity in neighbouring regions in turbulence, and that therefore the independent evolution of the M-adic cubes would be an important deficiency of Mandelbrot's model. They then propose neighbour

F.M. Dekking, P. v.d. Wal: Thomas Stieltjes Institute for Mathematics and Delft University of Technology, Mekelweg 4, 2628 CD Delft, The Netherlands. e-mail: F.M.Dekking@its.tudelft.nl, P.vanderWal@its.tudelft.nl

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interaction to obtain a model which admittedly is still phenomenological. Moreover, they merely study one specific example in the one dimensional case $d = 1$. In this paper we develop a general theory of fractal percolation with neighbour interaction. To assess the success of such a model in the goal of modeling turbulence from a phenomenological point of view, we invite the reader to compare ordinary fractal percolation in Figure 1 with an example involving neighbour interaction in Figure 2. We mention that the interest in fractal percolation goes beyond an attempt to model turbulence. Recently, Yuval Peres has revealed a surprising relationship between fractal percolation (for specific values of p) and the path of Brownian motion [12].

The fractal percolation process can conveniently be defined on the space of M-adic trees, but allowing interaction with the neighbours destroys the tree property. We have chosen to construct these random sets by way of the iteration of random substitutions. We shall call the corresponding process a branching cellular automaton. (See [14] for another approach.) This will be done in Section 2, where we furthermore indicate the importance of multi-type branching processes (with dependent offspring) for the analysis of branching cellular automata (BCA's). In Section 3 we consider the question of extinction of these multi-type branching processes.

Since the sets (K_n) will not necessarily be decreasing anymore in our general model, the question of convergence (in the Hausdorff metric) of the (K_n) arises. This problem is considered in Section 4, where a complete answer is given in Theorem 2. This theorem also gives a structure result: one deduces, for example, directly from this theorem that the limiting set K equals all of $[0, 1]^d$ if all types of the BCA communicate aperiodically. In Section 8 we determine the almost sure Hausdorff dimension of the limit set K, using Lyons' percolation method ([9]). In order to do this we need the notion of a product of two BCA's which is introduced in Section 7. For many BCA's the set K has a non-empty interior, and therefore K is *not* a fractal set (see e.g. the example analyzed in [1]). However, K will often have a fractal boundary. To determine the Hausdorff dimension of this boundary, it is therefore very useful that we show in Section 5 (see Theorem 3) that the boundary itself is again a limit set of a BCA.

2. Definition of branching cellular automata

Let A be a finite set, acting as our alphabet. Its elements are called letters and letters can be concatenated to form words. By A^n we denote the set of words of length n and by $A^{\mathbb{Z}}$ we denote the set of doubly infinite words. Let $M, N \geq 0$ be fixed integers with $M \geq 2$ and $M \geq N + 1$. The last inequality is not essential, but it simplifies some definitions. If u is a word in $A^{\mathbb{Z}}$, then the *N–context* of a letter u_k in u, denoted by $B_N(u, k)$, is defined as

$$B_N(u, k) = u_{k-N} \dots u_{k+N},$$

which is a word in A^{2N+1}.

Define a random substitution $\sigma(\cdot)$, which is a random map from $A^{\mathbb{Z}}$ to itself, as follows. Let $(W_v)_{v \in A^{2N+1}}$ be a collection of random variables, taking values in

A^M. For each $v \in A^{2N+1}$, let $(W_{v,k})_{k\in\mathbb{Z}}$ be a sequence of independent copies of W_v. For each $u \in A^{\mathbb{Z}}$, define $\sigma(u)$ by

$$\sigma(u) = w,$$

where

$$w_{kM} \dots w_{(k+1)M-1} = W_{B_N(u,k),k}$$

for all $k \in \mathbb{Z}$.

Let $\sigma_1, \sigma_2, \dots$ be independent copies of σ. Define the n^{th} iterate of σ, denoted by σ^n, by

$$\begin{aligned}\sigma^0(u) &= u\\ \sigma^n(u) &= \sigma_n(\sigma_{n-1}(\dots\sigma_1(u)\dots))\end{aligned}$$

for each $u \in A^{\mathbb{Z}}$.

Define a Branching Cellular Automaton (BCA) as the quintuple

$$(A, M, N, (W_v)_{v\in A^{2N+1}}, u),$$

where $u \in A^{\mathbb{Z}}$ serves as the starting word for the random substitution σ. If v_k is a letter in the word $v = \sigma^n(u)$, then the *children* of the *parent* v_k are the letters $w_{kM}, \dots, w_{(k+1)M-1}$ in the word $w = \sigma^{n+1}(u)$. The set of types T of a BCA is defined as $T = A^{2N+3}$. A letter v_k of a word $v \in A^{\mathbb{Z}}$ is said to be *of type* t, where t is an element of T, if $B_{N+1}(v,k) = t$. For each realisation of σ^n, the types of all letters $(\sigma^n(u))_k$ with $k = 0, \dots, M^n - 1$ are determined by the type of the letter u_0 in u. Here we used the assumption that $M \geq N+1$.

Let S be a subset of the set of types T and let $t \in T$. Define for $n = 0, 1, \dots$

$$\begin{aligned}J_n(t,S) = \{k :\ & 0 \leq k \leq M^n - 1,\\ & \text{the type of } (\sigma^n(u))_k \text{ is an element of } S\},\end{aligned}$$

where u is a word in $A^{\mathbb{Z}}$, such that the letter u_0 has type t in the word u. Furthermore we define

$$\begin{aligned}Z_n(t,S) &= |J_n(t,S)|\\ K_n(t,S) &= \bigcup_{k\in J_n(t,S)} I_n(k),\end{aligned}$$

where $|\cdot|$ denotes cardinality and

$$I_n(k) = [\frac{k}{M^n}, \frac{k+1}{M^n}]$$

is a level–n M–adic interval.

All definitions can be easily extended to higher dimensions. For example, the N–context of a letter $u_{k,l}$ in a word $u \in A^{\mathbb{Z}^2}$ will be

$$B_N(u,k,l) = \begin{matrix} u_{k-N,l+N} & \dots & u_{k+N,l+N}\\ \vdots & & \vdots\\ u_{k-N,l-N} & \dots & u_{k+N,l-N}\end{matrix}$$

For the sake of notational simplicity we shall mostly deal with the one-dimensional case in the sequel.

The *mean-offspring matrix* $\mathcal{M} = (m_{s,t})_{s,t\in T}$ is defined by

$$\begin{aligned} m_{s,t} &= \text{expected number of children with type } t, \\ &\quad\ \text{generated by a parent of type } s \\ &= \mathbb{E}(Z_1(s,t)), \end{aligned}$$

where $Z_1(s,t)$ is short for $Z_1(s,\{t\})$. Define $\mathcal{M}(n) = (m_{s,t}(n))_{s,t\in T}$ by $m_{st}(n) = \mathbb{E}(Z_n(s,t))$.

Lemma 1. *For* $n = 1, 2, \ldots$

$$\mathcal{M}(n) = \mathcal{M}^n.$$

In the proof of this lemma we use the following random variables. Fix n and define for $s, v, t \in T$ and $k = 1, \ldots, Z_n(s,v)$

$$\begin{aligned} \zeta_k(s,v,t) := &\ \text{number of children with type } t, \text{ generated by} \\ &\ \text{the } k^{\text{th}} \text{ type--}v \text{ letter in } (\sigma^n(u))_0, \ldots, (\sigma^n(u))_{M^n-1}, \end{aligned}$$

where u is a word in $A^{\mathbb{Z}}$, such that the letter u_0 has type s in the word u. To make the $\zeta_k(s,v,t)$'s random variables on the whole probability space, we define $\zeta_k(s,v,t)$ for $k = Z_n(s,v)+1, \ldots, M^n$ as independent copies of $Z_1(v,t)$. Note that $\zeta_1(s,v,t), \ldots, \zeta_{M^n}(s,v,t)$ are identically distributed, that each $\zeta_k(s,v,t)$ is independent of $Z_n(s,v)$ and that

$$Z_{n+1}(s,t) = \sum_{v\in T} \sum_{k=1}^{Z_n(s,v)} \zeta_k(s,v,t).$$

However, the variables $\zeta_1(s,v,t), \ldots, \zeta_{M^n}(s,v,t)$ do not need to be independent.

Proof (of Lemma 1). The proof is by induction. We have

$$\begin{aligned} m_{st}(n+1) &= \mathbb{E}(Z_{n+1}(s,t)) \\ &= \mathbb{E}\Big(\sum_{v\in T} \sum_{k=1}^{Z_n(s,v)} \zeta_k(s,v,t)\Big) \\ &= \sum_{v\in T} \mathbb{E}\Big(\sum_{k=1}^{Z_n(s,v)} \zeta_k(s,v,t)\Big) \\ &= \sum_{v\in T} \mathbb{E}(Z_n(s,v))\, m_{vt} \\ &= \sum_{v\in T} m_{sv}(n)\, m_{vt}. \end{aligned}$$

3. Extinction

Definition 1 (Communicating class). *Let $\mathscr{M} = (m_{st})_{s,t\in T}$ be a non-negative matrix. For $s, t \in T$, we write $s \to t$ if $(\mathscr{M}^r)_{st} > 0$ for some $r \geq 0$. We say that types s and t communicate if $s \to t$ and $t \to s$. The* communicating class $C(t)$ *consists of all types in T that communicate with t.*

Denote the restriction of the matrix $\mathscr{M}$ to the communicating class C by $\mathscr{M}_C = (m_{st})_{s,t\in C}$ and denote the Perron-Frobenius eigenvalue of $\mathscr{M}_C$ by λ_C.

Note that if $t \in C$, then the events $\{Z_n(t, C) > 0 \text{ i.o.}\}$ and $\{Z_n(t, C) > 0 \text{ for all } n\}$ are the same events $\mathbb{P}$–almost surely, where *i.o.* is short for *infinitely often*.

Theorem 1. *Let $t \in C$, with C a communicating class of types.*

(i) If $\lambda_C < 1$, then

$$\mathbb{P}(Z_n(t, C) = 0 \textit{ eventually}) = 1.$$

(ii) If $\lambda_C > 1$, then

$$\mathbb{P}(Z_n(t, C) > 0 \textit{ infinitely often}) > 0.$$

Moreover, for all $\varepsilon > 0$ there are $c_1 = c_1(\varepsilon) > 0$ and $c_2 = c_2(\varepsilon) > 0$ such that

$$\mathbb{P}(c_1\,\lambda_C^n \leq Z_n(t, C) \leq c_2\,\lambda_C^n \textit{ for all } n \mid Z_n(t, C) > 0 \textit{ i.o.}) \geq 1 - \varepsilon.$$

If $\lambda_C = 0$, then $\mathbb{P}(Z_n(t, C) = 0 \text{ for } n = 1, 2, \dots) = 1$, so assume in the following that $\lambda_C > 0$. Furthermore, we assume from now on that our BCA is such, that if $\lambda_C = 1$, then $\mathbb{P}(Z_n(t, C) = 0 \text{ eventually}) = 1$ for $t \in C$.

Write C as $C = \{t_1, \dots, t_r\}$, where r is the cardinality of C. In the remaining part of this section we fix $t \in C$ and assume that $t = t_1$. Furthermore, let $m_{ij} := m_{t_i,t_j}$ and write $\mathscr{M}_C$ as $\mathscr{M}_C = (m_{ij})_{1\leq i,j\leq r}$. Let v_C be a row vector such that its transpose, denoted by v'_C, is a right eigenvector of $\mathscr{M}_C$ corresponding to λ_C, with all entries strictly positive. Define an r–dimensional row-vector $Z_n = (Z_n(1), \dots, Z_n(r))$ by

$$\begin{aligned} Z_n(i) &= Z_n(t, t_i) \\ &= \text{number of type } t_i \text{ letters in } (\sigma^n(u))_0 \dots (\sigma^n(u))_{M^n-1}, \end{aligned}$$

where $u \in A^{\mathbb{Z}}$ is such that u_0 has type t in u. Note that $Z_n(t, C) = Z_n(1) + \dots + Z_n(r)$. Let $(v_C)_1$ denote the first entry of the vector v_C and let $(v_C)_{(1)}$ denote the smallest entry.

Lemma 2. *We have*

$$\mathbb{E}(Z_n(t, C)) \leq \frac{(v_C)_1}{(v_C)_{(1)}} \lambda_C^n.$$

In the sequel we will write v for v_C and λ for λ_C.

Proof (Lemma 2). Let e_1 denote the r–dimensional row vector with a 1 at the first entry and 0's elsewhere. Then for all n

$$\begin{aligned}\mathbb{E}(Z_n(t,C)) &\leq \tfrac{1}{v_{(1)}}\mathbb{E}(Z_n v') \\ &= \tfrac{1}{v_{(1)}} e_1 \mathscr{M}_C^n v' \qquad \text{by Lemma 1} \\ &= \frac{v_1}{v_{(1)}}\lambda^n.\end{aligned}$$

For $n = 0, 1, \ldots$ define

$$\mathscr{F}_n = \text{ the } \sigma\text{–algebra generated by } \sigma^0, \ldots, \sigma^n.$$

Lemma 3. *Assume* $\lambda > 1$. *Then the sequence*

$$\Big(\frac{Z_n v'}{\lambda^n}\Big)_{n\geq 0}$$

is a uniformly integrable martingale with respect to $(\mathscr{F}_n)_{n\geq 0}$.

Fix $n \geq 0$ and define for $i, j = 1, \ldots, r$ and $k = 1, \ldots, M^n$

$$\zeta_k(i,j) = \zeta_k(t, t_i, t_j).$$

Hence, for $k = 1, \ldots, Z_n(i)$

$$\begin{aligned}\zeta_k(i,j) = {} & \text{number of children with type } t_j\text{, generated by} \\ & \text{the } k^{\text{th}} \text{ type–}t_i \text{ letter in } (\sigma^n(u))_0, \ldots, (\sigma^n(u))_{M^n-1},\end{aligned}$$

where $u \in A^{\mathbb{Z}}$ is such that u_0 has type t in u. Note that $\mathbb{P}$–a.s.

$$Z_{n+1}(j) = \sum_{i=1}^{r}\sum_{k=1}^{Z_n(i)} \zeta_k(i,j).$$

Proof (Lemma 3). The fact that the sequence $\big(\frac{Z_n v'}{\lambda^n}\big)_{n\geq 0}$ is a martingale is proved in the same way as in the case that $(Z_n)_{n\geq 0}$ is a multi-type Galton-Watson branching process. To establish uniform integrability, it suffices to show that the sequence $\big(\mathrm{Var}(\frac{Z_n v'}{\lambda^n})\big)_{n\geq 0}$ is uniformly bounded. We have

$$\begin{aligned}\mathrm{Var}(Z_{n+1}v') &= \mathrm{Var}\Big(\sum_{j=1}^{r} v_j \sum_{i=1}^{r}\sum_{k=1}^{Z_n(i)} \zeta_k(i,j)\Big) \\ &= \mathbb{E}\Big(\sum_{j=1}^{r} v_j \sum_{i=1}^{r}\sum_{k=1}^{Z_n(i)} \zeta_k(i,j) - \mathbb{E}\big(\sum_{j=1}^{r} v_j \sum_{i=1}^{r}\sum_{k=1}^{Z_n(i)} \zeta_k(i,j)\big)\Big)^2 \\ &= \mathbb{E}\Big(\sum_{j=1}^{r} v_j \sum_{i=1}^{r}\sum_{k=1}^{Z_n(i)} \big(\zeta_k(i,j) - \mathbb{E}(\zeta_k(i,j))\big) \\ &\quad + \sum_{j=1}^{r} v_j \sum_{i=1}^{r} Z_n(i)\,\mathbb{E}(\zeta_1(i,j)) - \sum_{j=1}^{r} v_j \sum_{i=1}^{r} \mathbb{E}(Z_n(i))\mathbb{E}(\zeta_1(i,j))\Big)^2,\end{aligned}$$

since the $\zeta_1(i,j),\ldots,\zeta_{M^n}(i,j)$ are identically distributed and each one is independent of $Z_n(i)$.

$$\begin{aligned}\mathrm{Var}(Z_{n+1}v') &= \mathbb{E}\Big(\sum_{j=1}^{r} v_j \sum_{i=1}^{r} \sum_{k=1}^{Z_n(i)} \big(\zeta_k(i,j) - \mathbb{E}(\zeta_k(i,j))\big) \\ &\qquad + \sum_{i=1}^{r}\big(Z_n(i) - \mathbb{E}(Z_n(i))\big) \sum_{j=1}^{r} v_j m_{ij}\Big)^2 \\ &= \mathbb{E}\Big(\sum_{j=1}^{r} v_j \sum_{i=1}^{r} \sum_{k=1}^{Z_n(i)} \big(\zeta_k(i,j) - \mathbb{E}(\zeta_k(i,j))\big) \\ &\qquad + \lambda\big(Z_n v' - \mathbb{E}(Z_n v')\big)\Big)^2 \\ &= \mathbb{E}\Big(\sum_{j=1}^{r} v_j \sum_{i=1}^{r} \sum_{k=1}^{Z_n(i)} \big(\zeta_k(i,j) - \mathbb{E}(\zeta_k(i,j))\big)\Big)^2 + \lambda^2 \mathrm{Var}(Z_n v').\end{aligned}$$

The last equality follows by writing out the square. To see that the cross-term cancels, condition on Z_n. We will derive an upper bound for the first term in the last expression.

$$\begin{aligned}A_n &:= \mathbb{E}\Big(\sum_{j=1}^{r} v_j \sum_{i=1}^{r} \sum_{k=1}^{Z_n(i)} \big(\zeta_k(i,j) - \mathbb{E}(\zeta_k(i,j))\big)\Big)^2 \\ &\le \sum_{i=1}^{r}\sum_{j=1}^{r} (r v_j)^2 \mathbb{E}\Big(\sum_{k=1}^{Z_n(i)} \big(\zeta_k(i,j) - \mathbb{E}(\zeta_k(i,j))\big)\Big)^2\end{aligned}$$

The inequality follows by applying $(\sum_{j=1}^{r} x_j)^2 \le r \sum_{j=1}^{r} x_j^2$ twice. Condition on $Z_n(i) = m$ to obtain

$$\begin{aligned}A_n &\le \sum_{i=1}^{r}\sum_{j=1}^{r} (r v_j)^2 \sum_{m=0}^{M^n}\sum_{k=1}^{m}\sum_{l=1}^{m} \mathrm{Cov}\big(\zeta_k(i,j), \zeta_l(i,j)\big)\, \mathbb{P}(Z_n(i) = m) \\ &= \sum_{i=1}^{r}\sum_{j=1}^{r} (r v_j)^2 \sum_{m=0}^{M^n}\sum_{k=1}^{m}\sum_{|k-l|\le 2}^{m} \mathrm{Cov}\big(\zeta_k(i,j), \zeta_l(i,j)\big)\, \mathbb{P}(Z_n(i) = m),\end{aligned}$$

since if $|k-l| > 2$, the k^{th} and the l^{th} type t_i letter in $\sigma^n(u)$ are at least 2 places apart, which implies that the types of the children of the k^{th} type t_i letter and the types of the children of the l^{th} type t_i letter are independent. Bring two summations inside the expectation and apply the inequality $2xy \le x^2 + y^2$ to obtain

$$\begin{aligned}A_n &\le \sum_{i=1}^{r}\sum_{j=1}^{r} (r v_j)^2 \sum_{m=0}^{M^n} \mathbb{E}\Big(5 \sum_{k=1}^{m}\big(\zeta_k(i,j) - \mathbb{E}(\zeta_k(i,j))\big)^2\Big)\, \mathbb{P}(Z_n(i) = m) \\ &= \sum_{i=1}^{r}\sum_{j=1}^{r} 5 (r v_j)^2\, \mathbb{E}(Z_n(i))\, \mathrm{Var}(\zeta_1(i,j))\end{aligned}$$

$$\leq \mathbb{E}(Z_n(t,C))\Big(\sum_{j=1}^{r} 5(rv_j)^2 \max_{1\leq i\leq r}\big(\mathrm{Var}(\zeta_1(i,j))\big)\Big).$$

By Lemma 2, we can bound $\mathbb{E}(Z_n(t,C))$ by $\frac{v_1}{v_{(1)}}\lambda^n$. Writing

$$c = 5r^2\frac{v_1}{v_{(1)}}\sum_{j=1}^{r}(v_j)^2 \max_{1\leq i\leq r}\big(\mathrm{Var}(\zeta_1(i,j))\big),$$

we have found that $A_n \leq c\lambda^n$, and therefore

$$\mathrm{Var}(Z_{n+1}v') \leq c\lambda^n + \lambda^2\,\mathrm{Var}(Z_n v').$$

This recursive inequality implies that

$$\mathrm{Var}(Z_{n+1}v') \leq c\lambda^n\frac{\lambda^{n+1}-1}{\lambda-1}.$$

Hence, for $\lambda > 1$

$$\mathrm{Var}(\frac{Z_{n+1}v'}{\lambda^{n+1}}) \leq c\frac{1}{\lambda-1}$$

and so our martingale sequence is uniformly integrable.

Proof (Theorem 1). The first part of the theorem is easy to prove. For all $n = 0, 1, \ldots$, writing $\lambda = \lambda_C$ and $v = v_C$

$$\begin{aligned}\mathbb{P}(Z_n(t,C) > 0) &\leq \mathbb{E}(Z_n(t,C))\\ &\leq \frac{v_1}{v_{(1)}}\lambda^n\end{aligned}$$

by Lemma 2. If we assume that $\lambda < 1$, then $\mathbb{P}(Z_n(t,C) > 0)$ tends to 0 as $n\to\infty$. This implies that

$$\mathbb{P}(Z_n(t,C) > 0 \text{ for all } n) = 0.$$

For the second part where $\lambda > 1$, we use the fact that the sequence

$$\big(X_n\big)_{n\geq 0} = (\frac{Z_n v'}{\lambda^n})_{n\geq 0}$$

is a uniformly integrable martingale sequence. This implies that the X_n converge to a limit X with $0 < \mathbb{E}(X) = \mathbb{E}(X_1) = v_1 < \infty$. Hence,

$$\mathbb{P}(\text{there is a } c_1 > 0 \text{ such that } Z_n v' \geq c_1\lambda^n \text{ for all } n) > 0$$

and

$$\mathbb{P}(\text{there is a } c_2 \text{ such that } Z_n v' \leq c_2\lambda^n \text{ for all } n) = 1.$$

We will show that

$$\mathbb{P}(\exists c_1 > 0 \text{ such that } Z_n v' \geq c_1\lambda^n \text{ for all } n) > 0$$

implies

$$\mathbb{P}(\exists c_1 > 0 \text{ such that } Z_n v' \geq c_1 \lambda^n \ \forall n \mid Z_m(t, C) > 0 \ \forall m) = 1.$$

To see this, define for each $t_i \in C$ and $n \geq 0$ an r–dimensional row-vector $Y_n(i) := (Y_n(i, 1), \ldots, Y_n(i, r))$ by

$$Y_n(i, j) := Z_n(t_i, t_j).$$

Note that since $t = t_1$, we have $Z_n = Y_n(1)$. Define

$$\rho_i := \mathbb{P}(\exists c_1 > 0 \text{ such that } Y_n(i) v' \geq c_1 \lambda^n \ \forall n)$$
$$\rho := \min\{\rho_i : 1 \leq i \leq r\}.$$

Note that $\rho > 0$. For all m,

$$\mathbb{P}(\exists c_1 > 0 \text{ such that } Z_n v' \geq c_1 \lambda^n \ \forall n \,|\mathscr{F}_m) \geq \rho$$

almost surely on $\{Z_m(t, C) > 0\}$. So on $\{Z_m(t, C) > 0$ for all $m\}$ we have

$$\mathbb{P}(\exists c_1 > 0 \text{ such that } Z_n v' \geq c_1 \lambda^n \ \forall n \,|\mathscr{F}_m) \geq \rho$$

for all m almost surely. By Lévy's 0–1 law,

$$\mathbb{P}(\exists c_1 > 0 : \ Z_n v' \geq c_1 \lambda^n \ \forall n \,|\mathscr{F}_m) \to \mathbf{1}_{\{\exists c_1 > 0:\ Z_n v' \geq c_1 \lambda^n \ \forall n\}} \qquad \text{a.s.}$$

Hence, on $\{Z_m(t, C) > 0$ for all $m\}$ we have $\mathbf{1}_{\{\exists c_1 > 0:\ Z_n v' \geq c_1 \lambda^n \ \forall n\}} = 1$ a.s., which means that

$$\mathbb{P}(\exists c_1 > 0 \text{ such that } Z_n v' \geq c_1 \lambda^n \ \forall n \mid Z_m(t, C) > 0 \ \forall m) = 1.$$

We conclude that

$$\mathbb{P}(\exists c_1, c_2 > 0 \text{ such that } c_1 \lambda^n \leq Z_n v' \leq c_2 \lambda^n \ \forall n \mid Z_m(t, C) > 0 \ \forall m) = 1$$

and since v' has all entries positive

$$\mathbb{P}(\exists c_1, c_2 > 0 \text{ such that } c_1 \lambda^n \leq Z_n(t, C) \leq c_2 \lambda^n \ \forall n \mid Z_m(t, C) > 0 \ \forall m) = 1.$$

The second part of the theorem follows from this.

The last part of the proof is very similar to the technique to show that a branching process (X_n) with $\mathbb{P}(X_1 = 1) < 1$ satisfies

$$\mathbb{P}(\lim_{n \to \infty} X_n = 0 \text{ or } \infty) = 1.$$

4. Convergence

We will consider convergence in the Hausdorff metric m_H, defined for all non-empty compact sets A, B in $[0, 1]^d$ by

$$m_H(A, B) = \inf_{\varepsilon>0}\{A \subseteq B^\varepsilon \text{ and } B \subseteq A^\varepsilon\},$$

where

$$A^\varepsilon = \{x \in [0, 1]^d : \text{ there is an } a \in A \text{ such that } \delta(x, a) \leq \varepsilon\}$$

and δ denotes Euclidean distance. We extend m_H by defining for all non-empty compact A in $[0, 1]^d$

$$m_H(A, \emptyset) = \tfrac{1}{2}\sqrt{d}.$$

Note that compact sets $A_1, A_2, \ldots$ in $[0, 1]^d$ converge with respect to the metric m_H, if and only if for all $\varepsilon > 0$ there is an n_0 such that for all $m, n \geq n_0$ we have $A_n \subseteq A_m^\varepsilon$.

Definition 2 (Period). *Let t be a type in a communicating class C. Define the* period *$d(t)$ of t as the greatest common divisor of integers $r \geq 1$ for which $(\mathscr{M}^r)_{tt} > 0$. It can be shown that the periods of all $t \in C$ are the same. The period of C is defined as the common value d for the periods of the types in C. If $d > 1$, then C is called* periodic *and if $d = 1$, then C is called* aperiodic.

Definition 3 (Cyclic classes and extended cyclic classes). *Let C be a communicating class of T with period d. If t is a type in C, then the* cyclic class *$H(t)$ consists of all types s in C that can be reached from t in a multiple of d steps, i.e. $(\mathscr{M}^{nd})_{ts} > 0$ for some $n = 0, 1, \ldots$. By $H_0, \ldots, H_{d-1}$ we denote the d cyclic classes of C. We assume that the numbering of the classes is such that if $s \in H_i$ and $\mathscr{M}_{st} > 0$, then $t \in H_{(i+1) \bmod d}$.*

If t is a type in C, then the extended cyclic class *$\overline{H}(t)$ consists of all types in T that can be reached from t in a multiple of d steps. By $\overline{H}_0, \ldots, \overline{H}_{d-1}$ we denote the d extended cyclic classes of C.*

By $S^{\geq}$ we denote the set of types in T which can reach an element of S. So

$$S^{\geq} = \{t \in T : t \to S\},$$

where $t \to S$ means that there is an $s \in S$ such that $t \to s$.

Define

$$\mathscr{C}(S) = \{C \subseteq S^{\geq} : C \text{ is a communicating class with } \lambda_C > 1\},$$

where λ_C is the Perron–Frobenius eigenvalue of C.

Theorem 2. *Let $S \subseteq T$. Then $(K_n(t, S))_{n\geq 0}$ converges to $K(t, S)$ $\mathbb{P}$–almost surely for all types $t \in T$ if and only if for each communicating class $C \in \mathscr{C}(S)$*

$$\overline{H}_i \cap S \neq \emptyset$$

for $0 \leq i \leq d-1$, where d is the period of C, and $\overline{H}_0, \ldots, \overline{H}_{d-1}$ are the extended cyclic classes of C. Moreover, in case of convergence

$$K(t, S) = \bigcup_{C \in \mathscr{C}(S)} K(t, C)$$

$\mathbb{P}$–almost surely.

We first make some comments on this result. Let C be a communicating class. Then Theorem 2 implies that the sequence $(K_n(t, C))_{n\geq 0}$ converges for all $t \in T$. If $t \in C$ and $\lambda_C < 1$, then the sequence converges to the empty set with probability 1, by Theorem 1. If $t \in C$ and $\lambda_C > 1$, then the sequence converges with probability one, and it converges to a non-empty set with positive probability. If $t \in C$ and $\lambda_C = M$, i.e., the communicating class C is closed, then the sequence converges with probability one to the unit cube.

For the proof of Theorem 2 we need the following lemma's. Fix a type $t \in T$. If $I = I_m(l)$ is a level–m M–adic interval with $0 \leq l \leq M^m - 1$, define for $n \geq 0$

$$\begin{aligned} J_n(t, S, I) = \{k : l M^n \leq k < (l+1)M^n, \\ \text{the type of } (\sigma^{m+n}(u))_k \text{ is an element of } S\}, \end{aligned}$$

where $u \in A^{\mathbb{Z}}$ is such that u_0 has type t in u, and define

$$Z_n(t, S, I) = |J_n(t, S, I)|.$$

Note that $Z_n(t, S) = Z_n(t, S, [0, 1])$.

Lemma 4. *Let $S_1, S_2 \subseteq T$ and $t \in T$. If for all M–adic intervals I*

$$\mathbb{P}(Z_n(t, S_1, I) > 0 \text{ i.o. and } Z_n(t, S_2, I) = 0 \text{ i.o.}) = 0$$

and

$$\mathbb{P}(Z_n(t, S_1, I) = 0 \text{ i.o. and } Z_n(t, S_2, I) > 0 \text{ i.o.}) = 0,$$

then $\mathbb{P}$–almost surely, $(K_n(t, S_1))_{n\geq 0}$ converges to $K(t, S_1)$, $(K_n(t, S_2))_{n\geq 0}$ converges to $K(t, S_2)$ and $K(t, S_1) = K(t, S_2)$.

Proof. Note that the sequences $(K_n(t, S_1))_{n\geq 0}$ and $(K_n(t, S_2))_{n\geq 0}$ converge to the same limit if and only if for all $\varepsilon > 0$ there is an n_0 such that for all $n, m \geq n_0$ $m_H(K_n(t, S_1), K_m(t, S_2)) < \varepsilon$.

Suppose that with positive probability $(K_n(t,S_1))_{n\ge 0}$ and $(K_n(t,S_2))_{n\ge 0}$ do not converge to the same limit. Then the following event has positive probability:

$$\begin{aligned} E := \{&\exists\varepsilon>0\ \forall n_0\ \exists m,n\ge n_0 \text{ such that } K_n(t,S_1)\not\subseteq (K_m(t,S_2))^{\varepsilon} \\ &\text{or } K_n(t,S_2)\not\subseteq (K_m(t,S_1))^{\varepsilon}\} \\ = \{&\exists\varepsilon>0\ \forall n_0\ \exists m,n\ge n_0 \text{ such that } K_n(t,S_1)\not\subseteq (K_m(t,S_2))^{\varepsilon}\} \\ &\cup\{\exists\varepsilon>0\ \forall n_0\ \exists m,n\ge n_0 \text{ such that } K_n(t,S_2)\not\subseteq (K_m(t,S_1))^{\varepsilon}\} \end{aligned}$$

where $(K_m(t,S_1))^{\varepsilon}$ denotes the set of points closer than ε to $K_m(t,S_1)$. Call the first event in the last expression E_1 and the second event E_2. At least one of the two events has positive probability and we start by assuming that this is E_1. We have

$$\begin{aligned} E_1 = \{&\exists\varepsilon>0\ \forall n_0\ \exists m,n\ge n_0\ \exists x_{n_0}\in K_n(t,S_1) \\ &\text{such that } B_{\varepsilon}(x_{n_0})\cap K_m(t,S_2)=\emptyset\}, \end{aligned}$$

where $B_{\varepsilon}(x_{n_0})$ denotes the ε–ball around the point x_{n_0}.

Fix a realisation $\omega\in E_1$. Then there are sequences $(x_i)_{i\ge 1}$, $(m_i)_{i\ge 1}$ with $m_i\ge i$ and $(n_i)_{i\ge 1}$ with $n_i\ge i$ such that

i) $x_i\in K_{n_i}(t,S_1)=K_{n_i}(t,S_1)(\omega)$ and
ii) $B_{\varepsilon}(x_i)\cap K_{m_i}(t,S_2)=\emptyset$.

By compactness we may assume that the x_i converge to a point $x_0\in[0,1]$. Hence $x_i\in B_{\frac{1}{3}\varepsilon}(x_0)$ for all i large enough. Since $x_i\in K_{n_i}(t,S_1)$, we have $B_{\frac{1}{3}\varepsilon}(x_0)\cap K_{n_i}(t,S_1)\neq\emptyset$ for i large enough. Furthermore, since $B_{\varepsilon}(x_i)\cap K_{m_i}(t,S_2)=\emptyset$, we have $B_{\frac{2}{3}\varepsilon}(x_0)\cap K_{m_i}(t,S_2)=\emptyset$ for i large enough. Hence writing $\eta=\frac{2}{3}\varepsilon$, we have shown that E_1 is a subset of

$$\begin{aligned} \tilde{E} = \{&\exists\eta>0\ \exists x_0\in[0,1] : B_{\frac{1}{2}\eta}(x_0)\cap K_n(t,S_1)\neq\emptyset \text{ i.o.}, \\ &B_{\eta}(x_0)\cap K_n(t,S_2)=\emptyset \text{ i.o.}\} \end{aligned}$$

Fix $\omega\in\tilde{E}$. Choose an integer k such that the diameter of a k^{th}–level M–adic interval is less than $\frac{1}{2}\eta$. This choice for k implies that if a k^{th}–level M–adic interval $I_k(l)$ intersects $B_{\frac{1}{2}\eta}(x_0)$, then it is contained in $B_{\eta}(x_0)$. Consider a covering of $B_{\frac{1}{2}\eta}(x_0)$ $\Lambda=\{I_k(l_1),\dots,I_k(l_r)\}$ of all level–k intervals having a non-empty intersection with $B_{\frac{1}{2}\eta}(x_0)$. Then for all $1\le i\le r$, $I_k(l_i)\subseteq B_{\eta}(x_0)$ and hence $I_k(l_i)\cap K_n(t,S_2)=\emptyset$ for infinitely many n, which implies that $Z_n(t,S_2,I_k(l_i))=0$ for infinitely many n. Furthermore, since $B_{\frac{1}{2}\eta}(x_0)\cap K_n(t,S_1)\neq\emptyset$ i.o., there are M–adic intervals $J_0,J_1,\dots$ and an increasing sequence $(n_i)_{i\ge 0}$ with $n_0\ge k$ such that for all J_i

i) J_i is a level–n_i M-adic interval
ii) $J_i\subseteq K_{n_i}(t,S_1)$
iii) $J_i\cap B_{\frac{1}{2}\eta}(x_0)\neq\emptyset$.

Since the level of each M-adic interval J_i is greater than k, for each J_i there is an $I_k(l) \in \Lambda$ such that $J_i \subseteq I_k(l)$. Since Λ is a finite covering and $B_{\frac{1}{2}\eta}(x_0) \cap K_n(t, S_1) \neq \emptyset$ i.o., there must be an interval $I_k(l) \in \Lambda$ such that $J_i \subseteq I_k(l)$ for infinitely many i. This implies that $Z_n(t, S_1, I_k(l)) > 0$ for infinitely many n. So $\tilde{E}$ is a subset of

$$\hat{E} = \{\exists k, l \; Z_n(t, S_1, I_k(l)) > 0 \text{ i.o.}, \; Z_n(t, S_2, I_k(l)) = 0 \text{ i.o.}\}$$

Define

$$\hat{E}_{k,l} = \{Z_n(t, S_1, I_k(l)) > 0 \text{ i.o.}, \; Z_n(t, S_2, I_k(l)) = 0 \text{ i.o.}\}$$

Then $\hat{E} = \bigcup_{k,l \geq 0} \hat{E}_{k,l}$. So

$$0 < \mathbb{P}(\hat{E}) = \mathbb{P}(\bigcup_{k,l \geq 0} \hat{E}_{k,l}) \leq \sum_{k,l \geq 0} \mathbb{P}(\hat{E}_{k,l}).$$

Hence there are k and l such that $\mathbb{P}(\hat{E}_{k,l}) > 0$. So there is an M–adic interval I such that

$$\mathbb{P}(Z_n(t, S_1, I) > 0 \text{ i.o.}, \; Z_n(t, S_2, I) = 0 \text{ i.o.}) > 0.$$

If we assume that E_2 has positive probability, then we obtain similarly that there is an M-adic interval I such that

$$\mathbb{P}(Z_n(t, S_1, I) = 0 \text{ i.o.}, \; Z_n(t, S_2, I) > 0 \text{ i.o.}) > 0.$$

So if the sequences $(K_n(t, S_1))_{n \geq 0}$ and $(K_n(t, S_2))_{n \geq 0}$ do not converge to the same limit with positive probability, then there is an M-adic interval I such that

$$\mathbb{P}(Z_n(t, S_1, I) > 0 \text{ i.o.}, \; Z_n(t, S_2, I) = 0 \text{ i.o.}) > 0$$

or

$$\mathbb{P}(Z_n(t, S_1, I) = 0 \text{ i.o.}, \; Z_n(t, S_2, I) > 0 \text{ i.o.}) > 0.$$

Lemma 5. *Let $S \subseteq T$ and $t \in T$. The event*

$$\{Z_n(t, S) > 0 \; i.o\}$$

is contained in

$$\{\exists M\text{–adic interval } I \text{ and } \exists \text{communicating class } C \in \mathscr{C}(S) \text{ such that } Z_n(t, C, I) > 0 \text{ for all } n\},$$

$\mathbb{P}$–almost surely.

Proof. Fix $\omega \in \{Z_n(t, S) > 0 \text{ i.o.}\}$. We can find a sequence of M–adic intervals $J_0, J_1, \ldots$ such that

i) J_k is a level–k M–adic interval
ii) $J_0 \supseteq J_1 \supseteq \ldots$
iii) for all intervals J_k we have $Z_n(t, S, J_k)(\omega) > 0$ for infinitely many n.

We can find a subsequence $I_0, I_1, \ldots$ and a type $s \in T$ such that for all $k = 0, 1, \ldots$ we have $Z_0(t, s, I_k)(\omega) = 1$ and $Z_n(t, S, I_0)(\omega) > 0$ for some n. Writing $I = I_0$, it follows that

$$\{Z_n(t, S) > 0 \text{ i.o.}\} \subseteq \{\exists M\text{–adic interval I, and } \exists s \in T \text{ such that } Z_0(t, s, I) = 1,\\ Z_n(t, s, I) > 0 \text{ i.o. and } Z_n(t, S, I) > 0 \text{ for some } n\}.$$

This implies by Theorem 1 that $\mathbb{P}$–a.s.

$$\begin{aligned}\{Z_n(t, S) > 0 \text{ i.o.}\} &\subseteq \{\exists M\text{–adic interval I, and } \exists s \in \bigcup_{C \in \mathscr{C}(S)} C \\ &\qquad \text{such that } Z_0(t, s, I) = 1 \text{ and } Z_n(t, s, I) > 0 \text{ i.o.}\} \\ &\subseteq \{\exists M\text{–adic interval I, and } \exists C \in \mathscr{C}(S) \\ &\qquad \text{such that } Z_0(t, C, I) = 1 \text{ and } Z_n(t, C, I) > 0 \text{ i.o.}\} \\ &\subseteq \{\exists M\text{–adic interval I, and } \exists C \in \mathscr{C}(S) \\ &\qquad \text{such that } Z_n(t, C, I) > 0 \text{ for all n}\}.\end{aligned}$$

Let C be a communicating class with period d and let $H_0, \ldots, H_{d-1}$ be the cyclic classes of C.

Lemma 6. *Assume $s \in H_0$ and $\overline{H}_0 \cap S \neq \emptyset$. The event*

$$\{Z_n(s, C) > 0 \textit{ i.o.}\}$$

is contained in

$$\{Z_{nd}(s, S) > 0 \textit{ eventually}\}$$

$\mathbb{P}$–almost surely.

Proof. Assume $\mathbb{P}(Z_n(s, C) > 0 \text{ i.o.}, Z_{nd}(s, S) = 0 \text{ i.o.}) > 0$. By Theorem 1, for all $\varepsilon > 0$ there is a $c > 0$ such that

$$\begin{aligned}0 &< \mathbb{P}(Z_n(s, C) > 0 \text{ i.o.}, Z_{nd}(s, S) = 0 \text{ i.o.}) \\ &\leq \mathbb{P}(Z_n(s, C) \geq c\,\lambda_C^n \text{ for all } n, Z_{nd}(s, S) = 0 \text{ i.o.}) + \varepsilon \\ &\leq \mathbb{P}(Z_{nd}(s, H_0) \geq c\,\lambda_C^{nd} \text{ for all } n, Z_{nd}(s, S) = 0 \text{ i.o.}) + \varepsilon.\end{aligned}$$

This implies that we can find $v \in H_0$, $w \in S$, $l \in \mathbb{N}$ and $c' > 0$ such that

$$(\mathscr{M}^{ld})_{vw} > 0$$

and

$$\mathbb{P}(E_n \text{ i.o.}) > 0,$$

where

$$E_n = \{Z_{nd}(s, v) \geq c'\,\lambda_C^{nd},\ Z_{(n+l)d}(s, w) = 0\}.$$

Define

$$\rho = \frac{1}{M^{ld}} (\mathscr{M}^{ld})_{vw}.$$

Then

$$\begin{aligned}\mathbb{P}(Z_{ld}(v, w) > 0) &\geq \frac{1}{M^{ld}} \mathbb{E}(Z_{ld}(v, w)) \\ &= \frac{1}{M^{ld}} (\mathscr{M}^{ld})_{vw} \\ &= \rho.\end{aligned}$$

Fix n and define for all $k = 1, \ldots, Z_{nd}(s, v)$

$$\xi_k(s, v, w) := \text{number of descendants with type } w \text{ in } \sigma^{(n+l)d}(u)\text{, generated by the } k^{\text{th}} \text{ type–}v \text{ letter in } (\sigma^{nd}(u))_0, \ldots, (\sigma^{nd}(u))_{M^{nd}-1},$$

where $u \in A^{\mathbb{Z}}$ is such that u_0 has type s in u.

If $\mathbb{P}(Z_{nd}(s, v) \geq c' \lambda_C^{nd}) > 0$, we have

$$\begin{aligned}\mathbb{P}(E_n) &= \mathbb{P}(Z_{nd}(s, v) \geq c' \lambda_C^{nd},\ Z_{(n+l)d}(s, w) = 0) \\ &\leq \mathbb{P}(Z_{(n+l)d}(s, w) = 0 \mid Z_{nd}(s, v) \geq c' \lambda_C^{nd}) \\ &\leq \mathbb{P}(\sum_{k=1}^{Z_{nd}(s,v)} \xi_k(s, v, w) = 0 \mid Z_{nd}(s, v) \geq c' \lambda_C^{nd}) \\ &\leq \mathbb{P}(\sum_{k=1}^{\lceil \frac{1}{3} c' \lambda_C^{nd} \rceil} \xi_{3k-2}(s, v, w) = 0).\end{aligned}$$

Note that if $k \neq l$, the $(3k-2)^{\text{th}}$ and the $(3l-2)^{\text{th}}$ type–v letter in $\sigma^{nd}(u)$ are at least 2 places apart, which implies that the two letters generate the types of their offspring independently of each other. Therefore,

$$\begin{aligned}\mathbb{P}(E_n) &\leq \mathbb{P}(\xi_1(s, v, w) = 0)^{\frac{1}{3} c' \lambda_C^{nd}} \\ &\leq (1 - \rho)^{\frac{1}{3} c' \lambda_C^{nd}}.\end{aligned}$$

By the Borel-Cantelli lemma $\mathbb{P}(E_n \text{ i.o.}) = 0$, which is a contradiction.

Lemma 7. *Let $t \in T$ and $S \subseteq T$. Assume that for each communicating class $C \in \mathscr{C}(S)$*

$$\overline{H}_i \cap S \neq \emptyset$$

for $0 \leq i \leq d-1$, where d is the period of C, and $\overline{H}_0, \ldots, \overline{H}_{d-1}$ are the extended cyclic classes of C. Then for any $C \in \mathscr{C}(S)$, the event

$$\{Z_n(t, C) > 0 \textit{ i.o.}\}$$

is contained in

$$\{Z_n(t, S) > 0 \textit{ eventually}\}$$

$\mathbb{P}$–almost surely.

Proof. By Lemma 5

$$\begin{aligned}\{Z_n(t,C)>0 \text{ i.o.}\} \subseteq \{&\exists M\text{–adic interval I, and } \exists C' \in \mathscr{C}(S)\\ &\text{such that } Z_n(t,C',I)>0 \text{ for all n}\}\\ &= \bigcup_I \bigcup_{C'} \{Z_n(t,C',I)>0 \text{ for all n}\}.\end{aligned}$$

Consider the event $\{Z_n(t,C',I)>0 \text{ for all n}\}$ for some fixed level–m M–adic interval I and $C' \in \mathscr{C}(S)$. Denote the period of C' by d. Let $s_0, s_1, \ldots, s_{d-1}$ be types in C' and let $I_0 = I \supseteq I_1 \supseteq \ldots \supseteq I_{d-1}$ be M–adic intervals such that I_k is a level–$(m+k)$ interval for $k=0,\ldots,d-1$. By Lemma 6 we have

$$\begin{aligned}&\bigcap_{k=0}^{d-1}\{Z_0(t,s_k,I_k)=1,\ Z_n(t,C',I_k)>0 \text{ for all n}\}\\ &\quad\subseteq \bigcap_{k=0}^{d-1}\{Z_{nd}(t,S,I_k)>0 \text{ eventually}\}\\ &\quad= \{Z_n(t,S,I)>0 \text{ eventually}\}.\end{aligned}$$

Since this holds for any choice of $s_0, s_1, \ldots, s_{d-1}$ and $I_0, I_1, \ldots, I_{d-1}$, we obtain

$$\begin{aligned}\{Z_n(t,C',I)>0 \text{ for all n}\} &\subseteq \{Z_n(t,S,I)>0 \text{ eventually}\}\\ &\subseteq \{Z_n(t,S)>0 \text{ eventually}\}.\end{aligned}$$

The lemma follows directly from this.

Proof (Theorem 2). Fix a set $S \subseteq T$. Suppose that for some $t \in T$ the $K_n(t,S)$ do not converge to $\bigcup_{C\in\mathscr{C}(S)} K(t,C)$ with positive probability. Then by Lemma 4, there is an M–adic interval I such that

$$\mathbb{P}(Z_n(t,S,I)>0 \text{ i.o.}, \sum_{C\in\mathscr{C}(S)} Z_n(t,C,I)=0 \text{ i.o.})>0$$

or

$$\mathbb{P}(Z_n(t,S,I)=0 \text{ i.o.}, \sum_{C\in\mathscr{C}(S)} Z_n(t,C,I)>0 \text{ i.o.})>0.$$

However, the first possibility is ruled out by Lemma 5 and the second possibility is ruled out by Lemma 7.

On the other hand, assume without loss of generality that there is a communicating class $C \subseteq S^{\geq}$ with $\lambda_C > 1$ such that $\overline{H}_0 \cap S \neq \emptyset$ and $\overline{H}_1 \cap S = \emptyset$. Let the starting type t be an element of H_0. By Theorem 1 and Lemma 6, $\mathbb{P}(Z_{nd}(t,S)>0 \text{ eventually})>0$. Since $\overline{H}_1 \cap S = \emptyset$, we have $\mathbb{P}(Z_{nd+1}(t,S) = 0 \text{ for all } n) = 1$. So $\mathbb{P}(K_n(t,S)=\emptyset \text{ i.o.}, K_n(t,S)\neq\emptyset \text{ i.o.})>0$, which implies that $K_0(t,S), K_1(t,S), \ldots$ do not converge with positive probability.

5. The boundary of a BCA

Although the results in this section can be derived in general, we consider for reasons of simplicity a 2–dimensional BCA $(A, M, N, (W_v)_{v\in V}, u)$ with $A = \{0, 1\}$ and $V = A^{\{-N,\dots,N\}^2}$. Let $T = A^{\{-(N+1),\dots,N+1\}^2}$ be the set of types.

Hypothesis 1. Let $\mathbf{0}$ denote the type in T consisting of only 0's and let $\mathbf{1}$ denote the type consisting of only 1's. Assume that the sets

$$C_0 = \{\mathbf{0}\}$$
$$C_1 = \{\mathbf{1}\}$$

are closed communicating classes in T and that there are no other closed communicating classes.

It follows from this that $\lambda_{C_0} = \lambda_{C_1} = M^2$. Define

$$D_0 = \{s \in T : s \to C_0, s \nrightarrow C_1\}$$
$$D_1 = \{s \in T : s \to C_1, s \nrightarrow C_0\}$$
$$D_2 = \{s \in T : s \to C_0, s \to C_1\}.$$

Then $\{D_0, D_1, D_2\}$ is a partition of T.

We call the BCA *non-lattice* if for all $u \in A^{\mathbb{Z}^2}$ and $n \in \mathbb{Z}$ the probability is 0 that one of the types of two neighbouring letters in

$$\begin{matrix} (\sigma^n(u))_{0,M^n-1} & \dots & (\sigma^n(u))_{M^n-1,M^n-1} \\ \vdots & & \vdots \\ (\sigma^n(u))_{00} & \dots & (\sigma^n(u))_{M^n-1,0} \end{matrix}$$

is an element of D_0 and the other is an element of D_1.

Hypothesis 2. We will assume that our BCA is non-lattice.

Define the following events:

$$\begin{aligned} G_n = \{ & \text{ the type of } (\sigma^n(u))_{kl} \text{ is an element of } C_0 \text{ for all} \\ & (k,l) \in \{0, M^n - 1\} \times \{0, \dots, M^n - 1\} \text{ and} \\ & (k,l) \in \{0, \dots, M^n - 1\} \times \{0, M^n - 1\}\} \\ G = & \bigcup_{n=0}^{\infty} G_n. \end{aligned}$$

Hypothesis 3. The starting type $t \in T$ is such that the event G has positive probability.

The 'almost sure'–statements in this section will be with respect to the conditional probability measure

$$\mathbb{P}(\,\cdot\,\mid G).$$

This conditioning is added to avoid trivial problems which might arise when the limit set $K(t, C_1)$ would intersect the boundary of the unit square.

Finally, we will denote the boundary of a set $X \subseteq [0,1]^2$ by ∂X, the interior of X by $\mathrm{int}(X)$ and the closure of X by $\mathrm{cl}(X)$.

Theorem 3. *Almost surely, we have*

$$\partial K(t, C_1) = K(t, D_2).$$

Lemma 8. *We have the following almost surely:*

i) $\mathrm{cl}(\mathrm{int}(K(t, C_a))) = K(t, C_a)$ *for* $a = 0, 1$.
ii) $K(t, C_0) \cup K(t, C_1) = [0,1]^2$
iii) $\mathrm{int}(K(t, C_0)) \cap \mathrm{int}(K(t, C_1)) = \emptyset$.

Proof. In this and the following proofs, all statements are almost sure.

i) This follows from letting n tend to infinity in

$$K_n(t, C_a) = \mathrm{cl}(\mathrm{int}(K_n(t, C_a))) \subseteq \mathrm{cl}(\mathrm{int}(K(t, C_a))) \subseteq K(t, C_a)$$

for $a = 0, 1$. Here we used that $(K_n(t, C_a))_{n\geq 0}$ increases to $K(t, C_a)$, denoted by $K_n(t, C_a) \uparrow K(t, C_a)$, since C_a is a closed communicating class.

ii) By Theorem 2,

$$\begin{aligned}[0,1]^2 &= K(t, T)\\ &= \bigcup_{C \in \mathscr{C}(T)} K(t, C).\end{aligned}$$

Since $\mathscr{C}(T) = \mathscr{C}(C_0) \cup \mathscr{C}(C_1)$ by Hypothesis 1 and by another application of Theorem 2 we have

$$\begin{aligned}[0,1]^2 &= \bigcup_{C \in \mathscr{C}(C_0)} K(t, C) \quad \cup \bigcup_{C \in \mathscr{C}(C_1)} K(t, C)\\ &= K(t, C_0) \cup K(t, C_1).\end{aligned}$$

iii) This follows since $(K_n(t, C_0))_{n\geq 0}$ and $(K_n(t, C_1))_{n\geq 0}$ are increasing sequences and for all n

$$\begin{aligned}\mathrm{cl}(\mathrm{int}(K_n(t, C_a))) &= K_n(t, C_a) \qquad \text{for } a = 0, 1\\ \mathrm{int}(K_n(t, C_0)) \cap \mathrm{int}(K_n(t, C_1)) &= \emptyset.\end{aligned}$$

Proposition 1. *Let* $x \in [0,1]^2$, $\varepsilon > 0$ *and* $B_\varepsilon(x)$ *be the open ball of radius* ε *and center* x. *For* $n \geq 0$ *define*

$$\mathscr{I}_n = \{\textit{level-}n\ M\textit{-adic squares contained in } B_\varepsilon(x) \cap [0,1]^2\}.$$

Let $I, J \in \mathscr{I}_n$. *The event*

$$\{I \subseteq K_n(t, D_0),\ J \subseteq K_n(t, D_1)\}$$

is contained in the event

$$\{\textit{there is an } H \in \mathscr{I}_n \textit{ such that } H \subseteq K_n(t, D_2)\}$$

almost surely.

Proof. We can find squares $H_1, \dots, H_k \in \mathscr{I}_n$ such that

i) $H_1 = I$ and $H_k = J$
ii) H_i and H_{i+1} are neighbouring squares.

Since $\{D_0, D_1, D_2\}$ is a partition of T, each H_i is contained in either $K_n(t, D_0)$, $K_n(t, D_1)$ or $K_n(t, D_2)$. Since our BCA is non-lattice and H_i and H_{i+1} are neighbours, it cannot be the case that $H_i \subseteq K_n(t, D_0)$ and $H_{i+1} \subseteq K_n(t, D_1)$. From these two observations it follows that at least one of $H_2, \dots, H_{k-1}$ must be contained in $K_n(t, D_2)$.

Proof (Theorem 3). We will first proof that $K(t, D_2) \subseteq \partial K(t, C_1)$. By Theorem 2 we have $K(t, D_2) = \bigcup_{C \in \mathscr{C}(D_2)} K(t, C)$. Since $\mathscr{C}(D_2) \subseteq \mathscr{C}(C_1)$, we have $K(t, D_2) \subseteq \bigcup_{C \in \mathscr{C}(C_1)} K(t, C) = K(t, C_1)$. Similarly, we have $K(t, D_2) \subseteq K(t, C_0)$. By Lemma 8 part iii), this implies that

$$K(t, D_2) \subseteq \partial K(t, C_0) \cap \partial K(t, C_1) \subseteq \partial K(t, C_1).$$

To prove that $\partial K(t, C_1) \subseteq K(t, D_2)$, assume by contradiction that with positive probability there is an $y \in \partial K(t, C_1)$ such that $y \notin K(t, D_2)$. Since we conditioned on the intersection of $\partial K(t, C_1)$ with the boundary of $[0, 1]^2$ being empty, we can find by Lemma 8 part iii) a $y' \in [0, 1]^2$ with rational coordinates and rational $\eta > 0$ such that with positive probability

$$\begin{aligned} \mathrm{cl}(B_\eta(y')) \cap K(t, D_2) &= \emptyset \\ B_\eta(y') \cap K(t, C_1) &\neq \emptyset \\ B_\eta(y') \cap K(t, C_0) &\neq \emptyset. \end{aligned}$$

Since this last event is a union over rational y' and η, we can find non-random $x \in [0, 1]^2$ and $\varepsilon > 0$ such that with positive probability

$$\begin{aligned} \mathrm{cl}(B_\varepsilon(x)) \cap K(t, D_2) &= \emptyset \\ B_\varepsilon(x) \cap K(t, C_1) &\neq \emptyset \\ B_\varepsilon(x) \cap K(t, C_0) &\neq \emptyset. \end{aligned}$$

By Lemma 8 part i) and ii), it follows that

$$\begin{aligned} B_\varepsilon(x) \cap \mathrm{int}(K(t, C_0)) &\neq \emptyset \\ B_\varepsilon(x) \cap \mathrm{int}(K(t, C_1)) &\neq \emptyset. \end{aligned}$$

Since $K_n(t, C_0) \uparrow K(t, C_0)$ and $K_n(t, C_1) \uparrow K(t, C_1)$, there are n_0 and non-random level-n_0 M-adic squares $I_{n_0}, J_{n_0} \subseteq B_\varepsilon(x)$ such that with positive probability

$$I_{n_0} \subseteq K_{n_0}(t, C_0)$$

and

$$J_{n_0} \subseteq K_{n_0}(t, C_1).$$

Let for $n \geq n_0$ I_n and J_n be level-n M-adic squares such that $I_{n_0} \supseteq I_{n_0+1} \supseteq \ldots$ and $J_{n_0} \supseteq J_{n_0+1} \supseteq \ldots$. Then for all $n \geq n_0$, we have $I_n, J_n \subseteq B_\varepsilon(x)$ and

$$I_n \subseteq K_n(t, C_0)$$
$$J_n \subseteq K_n(t, C_1).$$

By Proposition 1, we can find for all $n \geq n_0$ a level-n M-adic square $H_n \subseteq B_\varepsilon(x)$ such that

$$H_n \subseteq K_n(t, D_2).$$

This implies that that $\mathrm{cl}(B_\varepsilon(x)) \cap K(t, D_2) \neq \emptyset$, which is a contradiction.

Define $S \subseteq T$ as

$$S = \{s \in T : s_{00} = 1\}.$$

We claim that $(K_n(t, S))_{n\geq 0}$ converges almost surely for all starting types t. To see this, let

$$\begin{aligned} S' &= \{s \in T : s_{ij} = 1 \text{ for some } i, j \in \{-(N+1), \ldots, N+1\}\} \\ &= T \backslash \{\mathbf{0}\}. \end{aligned}$$

By Theorem 2, $(K_n(t, S'))_{n\geq 0}$ converges almost surely for all starting types t. Since $S \subseteq S'$ we have for all $n \geq 0$

$$K_n(t, S) \subseteq K_n(t, S')$$

and eventually

$$K_n(t, S') \subseteq (K_n(t, S))^{\sqrt{2}\, M^{-n}(N+1)},$$

where we recall that $(K_n(t, S))^\varepsilon$ denotes the set of points in $[0, 1]^2$ with distance less than ε to the set $K_n(t, S)$. The last inclusion holds almost surely since we conditioned on the event G. Therefore $(K_n(t, S))_{n\geq 0}$ converges almost surely for all starting types t.

Define the set $E \subseteq T$ as

$$E = \{s \in T : s \to C_0, s \notin C_0\}.$$

Corollary 1. *Almost surely, we have*

$$\partial K(t, S) = K(t, E).$$

Proof. Note that $S' = E \cup D_1$ and from Theorem 2 it follows that

$$\begin{aligned} K(t,S) &= K(t,S') \\ &= K(t,E) \cup K(t,D_1) \\ &= K(t,E) \cup K(t,C_1). \end{aligned}$$

Also from Theorem 2 we get that $K(t,E) \subseteq K(t,C_0)$. Using this and Lemma 8 it follows that $K(t,E) \cap \mathrm{int}(K(t,C_1)) = \emptyset$. We claim that $\mathrm{int}(K(t,E)) = \emptyset$. By contradiction, assume that with positive probability $\mathrm{int}(K(t,E)) \neq \emptyset$. Since $\mathrm{int}(K(t,E)) \subseteq K(t,E) \subseteq K(t,C_0)$ and since $K_n(t,C_0) \uparrow K(t,C_0)$, there is an n_0 such that

$$\mathrm{int}(K_{n_0}(t,C_0) \cap K(t,E)) \neq \emptyset.$$

Since $K_n(t,E) \downarrow K(t,E)$, we have

$$\mathrm{int}(K_{n_0}(t,C_0) \cap K_{n_0}(t,E)) \neq \emptyset,$$

which is impossible since $E \cap C_0 = \emptyset$. Hence $\mathrm{int}(K(t,E)) = \emptyset$. Since also $K(t,E) \cap \mathrm{int}(K(t,C_1)) = \emptyset$,

$$\begin{aligned} \partial K(t,S) &= \partial(K(t,E) \cup K(t,C_1)) \\ &= K(t,E) \cup \partial(K(t,C_1)). \end{aligned}$$

Furthermore,

$$\partial K(t,S) = K(t,E) \cup K(t,D_2)$$

by Theorem 3, and since $D_2 \subseteq E$

$$\partial K(t,S) = K(t,E).$$

6. Examples

Example 1. The first example of a BCA is commonly known as ordinary fractal percolation with parameter p in dimension 2, where $p \in [0,1]$. Consider the BCA $(A, M, N, (W_v)_{v \in V}, u)$, with $V = A^{\{-N,\dots,N\}^2}$ and $A = \{0,1\}$. Let $\mathbb{P}_p$ denote the associated probability measure. Fix $v \in A^{\{-N,\dots,N\}^2}$ and write

$$W_v = \begin{matrix} W_{1,M} & \dots & W_{M,M} \\ \vdots & & \vdots \\ W_{1,1} & \dots & W_{M,1} \end{matrix}$$

where the W_{ij} are random variables taking values in A. For fixed p, the probability distribution of W_v is such, that the W_{ij} are independent. So

$$\mathbb{P}_p(W_v = w) = \prod_{i,j \in \{1,\dots,M\}} \mathbb{P}_p(W_{ij} = w_{ij})$$

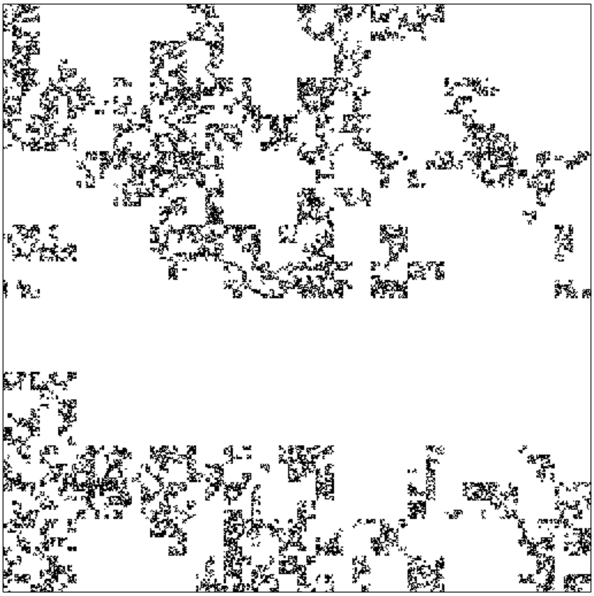

Fig. 1. A realisation of K_9 of ordinary fractal percolation (Example 1) with parameter $p = 0.75$ and $M = 2$.

for all $w \in A^{\{1,...,M\}^2}$. Define the probability distribution of the W_{ij}, $i, j \in \{1, \ldots, M\}$, by

$$\mathbb{P}_p(W_{ij} = 1) = \begin{cases} p & \text{if } v_{00} = 1 \\ 0 & \text{if } v_{00} = 0. \end{cases}$$

Let

$$T = A^{\{-(N+1),...,N+1\}^2}$$

be the set of types. Let $t \in T$ be the type with a 1 in the middle, i.e. $t_{00} = 1$ and 0's elsewhere. Let $C \subseteq T$ be

$$C = \{s \in T : s_{00} = 1\}$$

and define $K_n := K_n(t, C)$. Note that C is a communicating class. By Theorem 2, $K_1, K_2, \ldots$ converge to K $\mathbb{P}_p$–almost surely.

Example 2. To introduce neighbour dependence in fractal percolation, we will follow the same construction as in the previous example. However, the probability measure $\mathbb{P}_p$ will be given by

$$\mathbb{P}_p(W_{ij} = 1) = 1 - (1 - p)^{n(v)},$$

where $n(v) = \sum_{-N \le i, j \le N} v_{ij}$. Again,

$$T = A^{\{-(N+1),...,N+1\}^2}$$

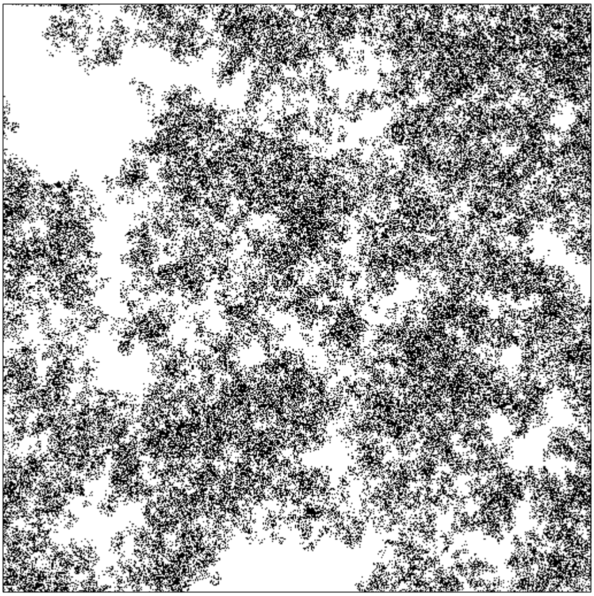

Fig. 2. A realisation of $K_9(t, S)$ of fractal percolation with neighbour dependence (Example 2) with parameter $p = 0.15$, $M = 2$ and $N = 1$.

is the set of types and let $S \subseteq T$ be

$$S = \{s \in T : s_{00} = 1\}.$$

In this example, S is not a communicating class for $N \geq 1$, but

$$C = \{s \in T : s_{ij} = 1 \text{ for some } i, j \in \{-(N+1), \ldots, N+1\}\}$$

is. In fact, $S \subseteq C$ and $S^{\geq} = C$. Since C is aperiodic, we have by Theorem 2 that $K_1(t, S), K_2(t, S), \ldots$ almost surely converge to $K(t, S) = K(t, C)$ for all starting types $t \in T$.

Example 3. Consider a family of 2–dimensional BCA's, parametrised by p with $0 < p < 1$. Each member of the family is a BCA $(A, M, N, (W_v)_{v \in V}, u)$ with $A = \{0, 1\}$, $M = 2$, $N = 1$ and $V = A^{\{-1,0,1\}^2}$. The dependence on p is in the distribution of the random variables W_v. The probability measure which describes the BCA shall be denoted by $\mathbb{P}_p$.

Define for $v \in A^{\{-1,0,1\}^2}$ $n_{1,1}(v)$ to be 2 if both the north–neighbour $v_{0,1}$ of $v_{0,0}$ and the east–neighbour $v_{1,0}$ are equal to 0, to be 1 if exactly one of them is 0 and to be 0 if none of them is 0. Written shortly, $n_{1,1}(v) = 2 - v_{1,0} - v_{0,1}$. In the same spirit, define

$$n_{-1,-1}(v) = 2 - v_{-1,0} - v_{0,-1}$$
$$n_{-1,1}(v) = 2 - v_{-1,0} - v_{0,1}$$
$$n_{1,-1}(v) = 2 - v_{1,0} - v_{0,-1}.$$

Fix $v \in A^{\{-1,0,1\}^2}$ and write

$$W_v = \begin{matrix} W_{-1,1} & W_{1,1} \\ W_{-1,-1} & W_{1,-1} \end{matrix}$$

where the W_{ij} are random variables taking values in A. For fixed p, the probability distribution of W_v is such, that the W_{ij} are independent. So

$$\mathbb{P}_p(W_v = w) = \prod_{i,j \in \{-1,1\}} \mathbb{P}_p(W_{ij} = w_{ij})$$

for all $w \in A^{\{-1,1\}^2}$. Define the probability distribution of the W_{ij}, $i, j \in \{-1, 1\}$, by

$$\mathbb{P}_p(W_{ij} = 1) = \begin{cases} p^{n_{ij}(v)} & \text{if } v_{0,0} = 1 \\ 0 & \text{if } v_{0,0} = 0. \end{cases}$$

For this example, it suffices to take $A^{\{-1,0,1\}^2}$ as set of types T instead of $A^{\{-2,-1,0,1,2\}^2}$, since one is still able to determine the distribution of the types of the offspring, based on the type of the parent.

Let $\mathbf{0}$ denote the type in T consisting of only 0's and let $\mathbf{1}$ denote the type consisting of only 1's. Define the following subsets of $T = A^{\{-1,0,1\}^2}$.

$$\begin{aligned} S &= \{s \in T : s_{00} = 1\}, \qquad C_0 = \{\mathbf{0}\}, \qquad C_1 = \{\mathbf{1}\} \\ D_0 &= \{s \in T : s_{00} = 0\}, \qquad D_1 = C_1 \\ D_2 &= \{s \in T : s_{00} = 1 \text{ and there are } i, j \in \{-1, 0, 1\} \text{ such that } s_{ij} = 0\}. \end{aligned}$$

Note that C_0 and C_1 are closed communicating classes, and that these are the only closed ones. Note that by Theorem 2 we have $K(t, S) = K(t, C_1)$, since $\mathscr{C}(S) \subseteq \mathscr{C}(C_1)$. The sets D_0, D_1 and D_2 play the same role as in the previous section:

$$\begin{aligned} D_0 &= \{s \in T : s \to C_0, s \nrightarrow C_1\} \\ D_1 &= \{s \in T : s \to C_1, s \nrightarrow C_0\} \\ D_2 &= \{s \in T : s \to C_0, s \to C_1\}. \end{aligned}$$

Furthermore, the BCA is non-lattice, since if u_k and u_l are neighbouring letters in $u \in A^{\mathbb{Z}^2}$ and $u_k \in D_0$, then $u_k = 0$ and so $u_l \notin D_1$. Condition on $K(t, S) = K(t, C_1)$ having an empty intersection with $\partial[0, 1]^2$. Then by Theorem 3 we have $\partial K(t, S) = \partial K(t, C_1) = K(t, D_2)$ almost surely.

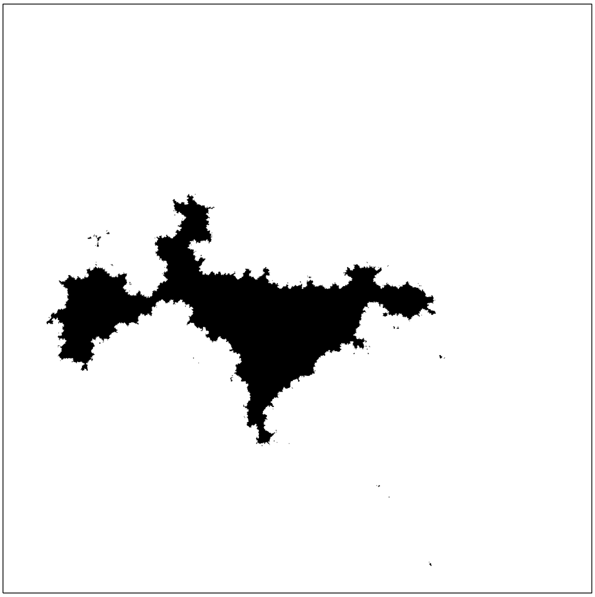

Fig. 3. A realisation of $K_9(t, S)$ of the BCA in Example 3 with $p = \frac{1}{2}$.

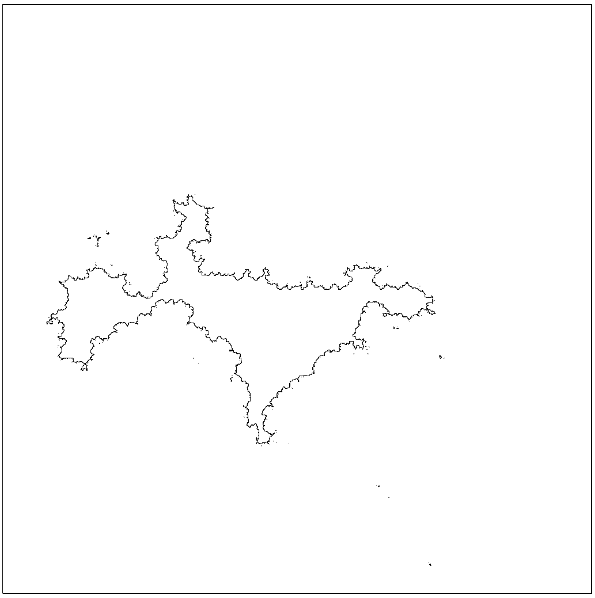

Fig. 4. $K_9(t, D_2)$ of the same realisation as in Figure 3.

7. Product BCA

Let $\mathscr{B} = (A, M, N, (W_v)_{v\in A^{2N+1}}, u)$ and $\mathscr{B}' = (A', M, N, (W'_v)_{v\in(A')^{2N+1}}, u')$ be two BCA's with the same substitution length M and interaction length N. Denote the probability measure associated with $\mathscr{B}$ by $\mathbb{P}$ and the probability measure associated with $\mathscr{B}'$ by $\mathbb{P}'$. We are going to define a product BCA with alphabet $\hat{A} = A \times A'$. For each $n \geq 0$ and for each $v = (v_1, \dots, v_n) \in \hat{A}^n$ we write $v_i = (v_i^{(1)}, v_i^{(2)})$ where $v_i^{(1)} \in A$ and $v_i^{(2)} \in A'$. We denote $(v_1^{(1)}, \dots, v_n^{(1)}) \in A^n$ by $v^{(1)}$ and $(v_1^{(2)}, \dots, v_n^{(2)}) \in (A')^n$ by $v^{(2)}$. Similarly, for $v = (\dots, v_{-1}, v_0, v_1, \dots) \in \hat{A}^{\mathbb{Z}}$ we write $v_i = (v_i^{(1)}, v_i^{(2)})$ and

$$v^{(1)} = (\dots, v_{-1}^{(1)}, v_0^{(1)}, v_1^{(1)}, \dots) \qquad v^{(2)} = (\dots, v_{-1}^{(2)}, v_0^{(2)}, v_1^{(2)}, \dots).$$

Define a sequence $(\hat{W}_v)_{v\in\hat{A}^{2N+1}}$ of random words in $\hat{A}^M$ by

$$\hat{\mathbb{P}}(\hat{W}_v = w) = \mathbb{P}(W_{v^{(1)}} = w^{(1)})\, \mathbb{P}'(W'_{v^{(2)}} = w^{(2)})$$

for all $v \in \hat{A}^{2N+1}$ and $w \in \hat{A}^M$. Furthermore, define $\hat{u} \in \hat{A}^{\mathbb{Z}}$ by

$$\hat{u}^{(1)} = u,$$
$$\hat{u}^{(2)} = u'.$$

Then the BCA $\hat{\mathscr{B}} = (\hat{A}, M, N, (\hat{W}_v)_{v\in\hat{A}^{2N+1}}, \hat{u})$ is called the *product* BCA of $\mathscr{B}$ and $\mathscr{B}'$.

Each random variable and quantity concerning $\mathscr{B}'$, respectively the product BCA $\hat{\mathscr{B}}$ will have a $'$, respectively a $\hat{}$ attached to it.

Let $t \in T$ be the type of u_0 in u, $t' \in T'$ be the type of $(u')_0$ in u' and $\hat{t} \in \hat{T}$ be the type of $(\hat{u})_0$ in $\hat{u}$. Hence $\hat{t}^{(1)} = t$ and $\hat{t}^{(2)} = t'$.

Let $S \subseteq T$, $S' \subseteq T'$ and let $\hat{S} \subseteq \hat{T}$ be defined by

$$\hat{S} = \{s \in \hat{T};\, s^{(1)} \in S, s^{(2)} \in S'\}.$$

The proof of the following rather obvious lemma is left to the reader.

Lemma 9. *Let S, S' and $\hat{S}$ as above. Then*

$$\hat{\mathbb{P}}(\hat{K}_n(\hat{t}, \hat{S}) = \emptyset) = \mathbb{P} \times \mathbb{P}'(K_n(t, S) \cap K'_n(t', S') = \emptyset).$$

8. Dimension

Let B be an arbitrary set in $\mathbb{R}^d$ and let $\{U_i\}_{i\geq 1}$ be a countable collection of sets in $\mathbb{R}^d$. We say that $\{U_i\}_{i\geq 1}$ is a δ–cover of B, if $B \subseteq \bigcup_{i=1}^{\infty} U_i$ and $\mathrm{diam}(U_i) \leq \delta$ for all $i = 1, 2, \dots$, where $\mathrm{diam}(U_i)$ denotes the diameter of the set U_i. Define

$$\mathscr{H}_\delta^\alpha(B) = \inf\{\sum_{i=1}^{\infty}(\mathrm{diam}(U_i))^\alpha : \{U_i\}_{i\geq 1} \text{ is a } \delta\text{–cover of } B\}$$

and let

$$\mathscr{H}^{\alpha}(B) = \lim_{\delta \downarrow 0} \mathscr{H}^{\alpha}_{\delta}(B),$$

which is called the α–dimensional Hausdorff measure of B. Define the Hausdorff dimension of B as

$$\dim_{\mathrm{H}}(B) = \inf\{\alpha : \mathscr{H}^{\alpha}(B) = 0\}.$$

Once more, for the sake of simplicity we will consider in this section a 1-dimensional BCA $(A, M, N, (W_v)_{v \in A^{2N+1}}, u)$, but all results extend easily to the higher dimensional case (including Lemma 11 by Lyons). Let t be the type of the letter u_0 in u and assume that t is an element of a communicating class C. Furthermore assume that the Perron–Frobenius eigenvalue λ_C is strictly larger than 1. In the following, we will write K_n for $K_n(t, C)$ and Z_n for $Z_n(t, C)$. By Theorem 2, the sets $K_0, K_1, \ldots$ converge almost surely to a limit K, which is non–empty with positive probability by Theorem 1. Define the event '*non–extinction*' as $\{Z_n > 0 \text{ i.o.}\}$.

Lemma 10. *Conditioned on non–extinction,* $\dim_{\mathrm{H}} K$ *is a constant* $\mathbb{P}$*–a.s.*

Proof. Let

$$\begin{aligned} D(s) &:= \dim_{\mathrm{H}}(K(s, C)) \\ D &:= D(t) = \dim_{\mathrm{H}}(K(t, C)) \\ d_s &:= \sup\{x : \mathbb{P}(D(s) \le x) < 1\}. \end{aligned}$$

Let $d = \max_{s \in C} d_s$, and let s^* be such that $d_{s^*} = d$. For any $s \in C$ a scaled copy of $K(s^*, C)$ is a subset of $K(s, C)$ with some positive probability, since $s \to s^*$. But then the probability that $D(s) > D(s^*) - \delta$ is positive for all $\delta > 0$. Hence $d_s = d_{s^*} = d$ for all $s \in C$.

Define for $\varepsilon > 0$ and $s \in C$,

$$\rho_s(\varepsilon) := \mathbb{P}(D(s) \ge d - \varepsilon).$$

Note that $\rho_s(\varepsilon) > 0$ for all $\varepsilon > 0$ and for all $s \in C$. So

$$\rho(\varepsilon) := \min\{\rho_s(\varepsilon) : s \in C\}$$

is strictly larger than 0.

Fix $\varepsilon > 0$. Then for all n,

$$\mathbb{P}(D \ge d - \varepsilon | \mathscr{F}_n) \ge \rho(\varepsilon)$$

almost surely on $\{Z_n > 0\}$. To see this, let $\omega \in \{Z_n > 0\}$ and note that $\mathbb{P}(D \ge d - \varepsilon | \mathscr{F}_n)(\omega) = \mathbb{P}(D \ge d - \varepsilon | \sigma^n(u) = v)$, where $v = \sigma^n(u)(\omega)$. Since $\omega \in \{Z_n > 0\}$, at least one of the letters $v_0, \ldots, v_{M^{-n}-1}$ has a type $s \in C$. Conditioned on $\sigma^n(u)$ being equal to v, $K(t, C)$ contains a copy of $K(s, C)$ scaled by a factor M^{-n} and so $\mathbb{P}(D \ge d - \varepsilon | \sigma^n(u) = v) \ge \rho_s(\varepsilon) \ge \rho(\varepsilon)$.

Then also on $\{Z_n > 0 \text{ for all } n\}$ we have

$$\mathbb{P}(D \geq d - \varepsilon | \mathscr{F}_n) \geq \rho(\varepsilon)$$

for all n almost surely. By Lévy's 0–1 law,

$$\mathbb{P}(D \geq d - \varepsilon | \mathscr{F}_n) \to \mathbf{1}_{\{D \geq d - \varepsilon\}} \qquad \text{a.s.}$$

Hence, on $\{Z_n > 0 \text{ for all } n\}$ we have $\mathbf{1}_{\{D \geq d-\varepsilon\}} = 1$ a.s., which means that

$$\mathbb{P}(D \geq d - \varepsilon | Z_n > 0 \text{ for all } n) = 1.$$

Letting $\varepsilon \to 0$, we see that $\mathbb{P}(D = d | Z_n > 0 \text{ for all } n) = 1$.

In this section we will denote by $(A', M, N, (W'_v)_{v \in (A')^{2N+1}}, u')$ ordinary fractal percolation with parameter p (see Example 1). So $A' = \{0, 1\}$ and $u' = (\ldots, 0, 1, 0, \ldots) \in (A')^{\mathbb{Z}}$. All quantities and random variables concerning fractal percolation will be written with a tilde. Recall that $K'_n = K'_n(t', C')$ and $K' = K'(t', C')$, where t' is the type of u'_0 in u' and $C' = \{s \in T' : s_{N+1} = 1\}$.

The following lemma due to Russell Lyons ([9], p. 933) gives a lower bound for the Hausdorff dimension of a non-random closed set B in $[0, 1]$.

Lemma 11 (Lyons). *Let B be a closed set in $[0, 1]$. If $\mathbb{P}'_p(B \cap K' \neq \emptyset) > 0$, then $\dim_{\mathrm{H}} B \geq -\frac{\log p}{\log M}$.*

Consider the product BCA $(\hat{A}, M, N, (\hat{W}_v)_{v \in \hat{A}^{2N+1}}, \hat{u})$ of our initial BCA with fractal percolation with parameter p. So $\hat{A} = A \times A' = A \times \{0, 1\}$ and $\hat{u}^{(1)} = u$, $\hat{u}^{(2)} = u'$. All quantities and random variables concerning the product BCA will be written with a hat. We will write $\hat{K}_n = \hat{K}_n(\hat{t}, \hat{C})$ and $\hat{K} = \hat{K}(\hat{t}, \hat{C})$, where $\hat{t}$ is the type of $\hat{u}_0$ in $\hat{u}$ and $\hat{C}$ is the communicating class in $\hat{T}$ which contains $\hat{t}$.

Define

$$\hat{S} = \{s \in \hat{T} : s^{(1)} \in C, s^{(2)} \in C'\}.$$

Note that $\hat{S}$ does not need to be a communicating class. We have $\hat{C} \subseteq \hat{S}$ and from each $s \in \hat{S}$ we can reach $\hat{t}$. This implies that if s is a type in $\hat{S} \backslash \hat{C}$, then $\hat{t} \nrightarrow s$ and therefore $K_n(\hat{t}, \hat{S}) = K_n(\hat{t}, \hat{C})$ for all $n \geq 0$ almost surely.

Let $\hat{\mathscr{M}}_p = (\hat{m}_{st})_{s,t \in \hat{T}}$ be the mean offspring matrix of the product BCA.

Lemma 12. *For the Perron–Frobenius eigenvalue $\hat{\lambda}_p$ of $\hat{\mathscr{M}}_p$ restricted to $\hat{C}$ we have*

$$\hat{\lambda}_p = p\,\lambda,$$

where $\lambda = \lambda_C$ is the Perron–Frobenius eigenvalue of $\mathscr{M}$ restricted to the communicating class C.

For the proof we need the following proposition due to Furstenberg (cf. [11]).

Proposition 2 (Furstenberg). *Let* $A = (a_{ij})_{1\le i,j\le n}$ *be a non–negative, irreducible* $n \times n$*–matrix and let* $B = (b_{ij})_{1\le i,j\le r}$ *be a non–negative, irreducible* $r \times r$*–matrix, where* $1 \le r \le n$*. Let* $I_1, \dots, I_r$ *be a partition of the index set* $\{1, \dots, n\}$*, where all sets* I_k *are non–empty. Assume that for* A*,* B *and* $\{I_k : k = 1, \dots, r\}$ *the following relation holds. For all* $1 \le i, j \le r$ *we have*

$$\text{for all } l \in I_i \qquad \sum_{k\in I_j} a_{lk} = b_{ij}.$$

Then the Perron–Frobenius eigenvalues of the matrices A *and* B *are the same.*

Proof. Let λ_A and λ_B be the Perron–Frobenius eigenvalues of A and B and let v_A and v_B be associated left eigenvectors with all entries strictly positive. Furthermore, let w_B^{T} be a right eigenvector associated with λ_B having all entries strictly positive. Define

$$R = (r_{ij})_{1\le i\le n,\ 1\le j\le r}$$

by

$$r_{ij} = \begin{cases} 1 & \text{if } i \in I_j \\ 0 & \text{else.} \end{cases}$$

Then $AR = RB$. Define $x \in \mathbb{R}^r$ as $x = v_A R$. Since $\{I_1, \dots, I_r\}$ is a partition, all entries of x are strictly positive. Then

$$xB = v_A RB = v_A AR = \lambda_A v_A R = \lambda_A x,$$

and therefore λ_A is an eigenvalue of B.

Furthermore, we have

$$\lambda_A x w_B^{\mathrm{T}} = xBw_B^{\mathrm{T}} = \lambda_B x w_B^{\mathrm{T}}.$$

Since both x and w_B have all entries strictly positive, we conclude that $\lambda_A = \lambda_B$.

Proof (of Lemma 12). Consider the partition $\{I_s\}_{s\in C}$ of $\hat{C}$, where

$$I_s = \{v \in \hat{C} : v^{(1)} = s\}.$$

Then we claim that for all $s, t \in C$ and for all $v \in I_s$

$$\sum_{w\in I_t} \hat{m}_{vw} = p\, m_{st}.$$

To see this define for $\hat{t} \in \hat{T}$, $S \subseteq T$ and $S' \subseteq T'$

$$\hat{J}(\hat{t}, S, S') = \{k :\ 0 \le k < M, \text{ the type } v \text{ of } (\hat{\sigma}(\hat{u}))_k \\ \text{has } v^{(1)} \in S \text{ and } v^{(2)} \in S'\},$$

where the type of $\hat{u}_0$ in $\hat{u}$ is $\hat{t}$. So we have

$$\hat{J}(\hat{t}, S, S') = \hat{J}(\hat{t}, S, T') \cap \hat{J}(\hat{t}, T, S').$$

Recall that for $t \in T$ and $S \subseteq T$

$$J_n(t, S) = \{k : 0 \leq k \leq M^n - 1, \text{ the type of } (\sigma(u)^n)_k \text{ is an element of } S\},$$

where the type of u_0 in u is t.

Writing $Z(s,t)$ for $Z_1(s, \{t\})$ and $J(s,t)$ for $J_1(s,\{t\})$, we have for all $v \in I_s$

$$\begin{aligned}
\sum_{w \in I_t} \hat{m}_{vw} &= \sum_{w \in I_t} \hat{\mathbb{E}}_p(\hat{Z}(v, w)) \\
&= \hat{\mathbb{E}}_p(\hat{Z}(v, I_t)) \\
&= \hat{\mathbb{E}}_p(\sum_{k=0}^{M-1} \mathbf{1}_{\hat{J}(v,t,C')}(k)) \\
&= \sum_{k=0}^{M-1} \hat{\mathbb{E}}_p(\mathbf{1}_{\hat{J}(v,t,T')}(k)\, \mathbf{1}_{\hat{J}(v,T,C')}(k)) \\
&= \sum_{k=0}^{M-1} \mathbb{E}(\mathbf{1}_{J(v^{(1)},t)}(k))\, \mathbb{E}'_p(\mathbf{1}_{J'(v^{(2)},C')}(k)) \\
&= p\, \mathbb{E}(Z(s,t)) \\
&= p\, m_{st}.
\end{aligned}$$

By Proposition 2, the Perron–Frobenius eigenvalues of $\hat{\mathscr{M}}_p$ restricted to $\hat{C}$ and $p\,\mathscr{M}$ restricted to C are the same. Therefore, $\hat{\lambda}_p = p\,\lambda$.

Lemma 13. *If* $p > \frac{1}{\lambda}$, *then*

$$\mathbb{P} \times \mathbb{P}'_p\,(K \cap K' \neq \emptyset) > 0.$$

Proof. By Lemma 9 we have

$$\begin{aligned}
\hat{\mathbb{P}}_p(\hat{K}_n = \emptyset) &= \hat{\mathbb{P}}_p(\hat{K}_n(\hat{t}, \hat{C}) = \emptyset) \\
&= \hat{\mathbb{P}}_p(\hat{K}_n(\hat{t}, \hat{S}) = \emptyset) \\
&= \mathbb{P} \times \mathbb{P}'_p(K_n(t, C) \cap K'_n(t', C') = \emptyset) \\
&= \mathbb{P} \times \mathbb{P}'_p(K_n \cap K'_n = \emptyset)
\end{aligned}$$

for all $n = 0, 1, \ldots$. Since $\{\hat{K}_n = \emptyset\} \downarrow \{\hat{K} = \emptyset\}$ and similarly $\{K_n \cap K'_n = \emptyset\} \downarrow \{K \cap K' = \emptyset\}$, we have

$$\hat{\mathbb{P}}_p(\hat{K} = \emptyset) = \mathbb{P} \times \mathbb{P}'_p(K \cap K' = \emptyset).$$

If $p > \frac{1}{\lambda}$, then the Perron–Frobenius eigenvalue $\hat{\lambda}_p = p\lambda$ is strictly larger than 1. So by Theorem 1,

$$\hat{\mathbb{P}}_p(\hat{K} \neq \emptyset) = \mathbb{P} \times \mathbb{P}'_p(K \cap K' \neq \emptyset) > 0.$$

Our main theorem specifies the Hausdorff dimension of the limit set K of our BCA.

Theorem 4. *Let $K = K(t, C)$ be a set generated by a BCA with $t \in C$ and C a communicating class with $\lambda_C > 1$. Conditioned on non–extinction, we have*

$$\dim_{\mathrm{H}} K = \frac{\log \lambda}{\log M} \qquad \mathbb{P}\text{–}a.s.$$

Proof. The easy part of the proof is showing that $\dim_{\mathrm{H}} K \le \frac{\log \lambda}{\log M}$ a.s. Fix $\varepsilon > 0$. By Theorem 1 we can find a constant c such that

$$\mathbb{P}(Z_n \le c\,\lambda^n \text{ for all } n) \ge 1 - \varepsilon.$$

Hence

$$\begin{aligned}\mathbb{P}(K_n &\text{ can be covered with less than } c\,\lambda^n \\ &n^{\text{th}}\text{-level } M\text{-adic intervals, for all } n\,) > 1 - \varepsilon.\end{aligned}$$

This implies that

$$\mathbb{P}(\mathscr{H}^{\alpha}(K) < \infty) > 1 - \varepsilon,$$

where $\alpha = \frac{\log \lambda}{\log M}$ and $\mathscr{H}^{\alpha}(K)$ is the α-dimensional Hausdorff measure of K. Since this holds for all ε, we conclude that $\dim_{\mathrm{H}} K \le \frac{\log \lambda}{\log M}$ a.s.

To prove the converse, $\dim_{\mathrm{H}} K \ge \frac{\log \lambda}{\log M}$ a.s., we will use the previous lemma's. Let $\varepsilon > 0$ and $p = p(\varepsilon) = \frac{1}{\lambda} + \varepsilon$. By Lemma 13 we have $\mathbb{P} \times \mathbb{P}'_p\,(K \cap K' \ne \emptyset) > 0$. This implies by Fubini's theorem that the set

$$B = \{\omega \,:\, \mathbb{P}'_p(K(\omega) \cap K' \ne \emptyset) > 0\}$$

has positive $\mathbb{P}$–measure.

By Lemma 11, $\dim_{\mathrm{H}} K \ge -\frac{\log p}{\log M}$ with positive probability. Since conditioned on non-extinction $\dim_{\mathrm{H}} K$ is a constant a.s. (Lemma 10), we have in this case that $\dim_{\mathrm{H}} K \ge -\frac{\log p}{\log M}$ a.s. If we let $\varepsilon \to 0$, then $p(\varepsilon) \to \frac{1}{\lambda}$ and so we have $\dim_{\mathrm{H}} K \ge \frac{\log \lambda}{\log M}$ a.s., conditioned on non-extinction.

Example 4. Consider fractal percolation with parameter p in dimension 2 (Example 1). Recall that $K_n := K_n(t, C)$, where

$$C = \{s \in T : s_{00} = 1\}$$

and t is the type with a 1 in the middle and 0's elsewhere. By Proposition 2, the largest eigenvalue of $\mathscr{M}_p$ is equal to $p\,M^2$. Hence by Theorem 4, we have that conditioned on non–extinction

$$\dim_{\mathrm{H}} K = 2 + \frac{\log p}{\log M} \qquad \text{a.s.}$$

Example 5. Consider the BCA described in Example 3, parametrised by p. Recall that

$$S = \{s \in T : s_{00} = 1\}$$

and

$$D_2 = \{s \in T : s_{00} = 1 \text{ and there are } i, j \in \{-1, 0, 1\} \text{ such that } s_{ij} = 0\}.$$

Conditioned on $K(t, S)$ being non-empty and having an empty intersection with $\partial[0, 1]^2$, we have almost surely

$$\dim_{\mathrm{H}} K(t, S) = 2$$
$$\dim_{\mathrm{H}} \partial K(t, S) = \frac{\log \lambda_p}{\log M},$$

where λ_p is the Perron–Frobenius eigenvalue associated with D_2.

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