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On the cohomology of flows of stochastic and random differential equations

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Abstract. We consider the flow of a stochastic differential equation on d-dimensional Euclidean space. We show that if the Lie algebra generated by its diffusion vector fields is finite dimensional and solvable, then the flow is conjugate to the flow of a non-autonomous random differential equation, i.e. one can be transformed into the other via a random diffeomorphism of d-dimensional Euclidean space. Viewing a stochastic differential equation in this form which appears closer to the setting of ergodic theory, can be an advantage when dealing with asymptotic properties of the system. To illustrate this, we give sufficient criteria for the existence of global random attractors in terms of the random differential equation, which are applied in the case of the Duffing-van der Pol oscillator with two independent sources of noise.

Introduction

In this paper we try to answer the following basic question: when is a stochastic differential equation cohomologous to a non-autonomous random ordinary differential equation? In other words: under which conditions can one find a random coordinate change on the state space which transforms the flow generated by the stochastic differential equation into the flow of a non-autonomous random one?

Let us first state this problem in a little more precise terms, for systems on Wiener space $(\Omega, \mathscr{F}, P)$ with an m-dimensional Wiener process W and the canonical shift θ_t on Ω by time t which is P-ergodic for $t \neq 0$. Suppose $f_0, \cdots, f_m$ are smooth vector fields in $\mathbf{R}^d$ and let $\phi = (\phi_t)_{t \in \mathbf{R}}$ denote the (possibly only local) flow of the stochastic differential equation

$$dx_t = f_0(x_t)\,dt + \sum_{i=1}^{m} f_i(x_t) \circ dW_t^i \tag{1}$$

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A non-autonomous random differential equation is given by a smooth random vector field $g(\cdot, x)$, $x \in \mathbf{R}^d$, through

$$dx_t = g(\theta_t \cdot, x_t)\, dt. \tag{2}$$

Then our question asks for a random diffeomorphism Φ of the state space $\mathbf{R}^d$ such that ϕ and the (local) flow χ generated by (2) are related by the *conjugation* equation

$$\Phi \circ \theta_t \ \chi_t \ \Phi^{-1} = \phi_t, \quad t \in \mathbf{R}. \tag{3}$$

Why could one be interested in having a relationship such as (3)? For our initial aims, the reason was this. While the framework of the treatment of (1) is stochastic analysis, (2), describing a motion along a stationary vector field, fits better into the methodology of ergodic theory. Fortunately, the cohomology relation (3) preserves asymptotic invariants such as Lyapunov exponents, rotation numbers, or invariant subspaces such as Oseledets spaces, invariant manifolds, or random attractors (see Arnold [Arn98]). So, if aspects of ergodic theory are involved in the study of asymptotic properties of (1), it could be much simpler to look at (2) instead, and then let Φ do the rest of the work. In a simpler setting, this concept has already been used in [Imk98] to study the existence of global random attractors of systems like the randomly perturbed Duffing-van der Pol oscillator, or the Lorenz equation. We shall give another illustration of this idea in section 4 below, where we study the existence of global random attractors, and consider the Duffing-van der Pol oscillator with two different sources of noise as another example.

The answer we shall give in this paper to the conjugation problem is this: we show that if the Lie algebra $\mathscr{L}$ generated by $f_1 \cdots, f_m$ is solvable and finite dimensional, then there is a random diffeomorphism Φ solving (3). This reminds somewhat the generalization of the well known Doss-Sussmann method of solving a stochastic differential equation through an associated ordinary differential equation, given by Yamato [Yam79], Kunita [Kun80] and Krener and Lobry [Kre81]. In fact, some algebraic aspects of the algorithm to be described, are similar to the ones used in the literature. This algorithm, which is our principal tool to derive the main results, reduces gradually the algebraic complexity of the Lie algebra $\mathscr{L}$, and this way creates a chain of random diffeomorphisms the composition of which yields Φ. The algorithm had to be taylor made for the central purposes of ergodic theory, however. They can be expressed by requiring that the noise terms of the "remainder" stochastic differential equations updated in each step, have to be made stationary. To achieve this goal, we use the following simple observation. If X, Y are smooth stationary semimartingales, then the generally non-stationary process $X \circ dY$ can be made stationary by passing to its moving average process

$$e^{-t} \int_{-\infty}^{t} e^s\, X_s \circ dY_s, \quad t \in \mathbf{R}.$$

The paper is organized as follows.

In section 1 we collect some auxiliary results concerning stationary semimartingales, and prove some algebraic identities to be used crucially in the reduction algorithm. The latter is first described in the relatively simple framework of nilpotent

Lie algebra in section 2, and leads to the conjugation theorem (Theorem 2.1). In section 3 we pass to the case of a solvable Lie algebra $\mathscr{L}$. In this framework the reduction algorithm turns out to be even formally simpler. Yet, the stochastic differential equations to be solved in each step become gradually more involved, and are much less explicit and transparent. In section 4, we first discuss some general sufficient conditions for the existence of global random attractors in the situation of the preceding sections. We finally consider a concrete problem, the Duffing-van der Pol oscillator with two independent linear sources of noise: multiplicative noise on both position and velocity. In this simple case the Lie algebra of the linear diffusion vector fields is solvable, but not nilpotent.

We expect that more asymptotic properties of random differential equations given by stochastic differential equations become treatable via conjugation. One example promises to be local linearization of random dynamical systems, as described in the theorem by Hartman-Grobman: for non-autonomous random differential equations Wanner [Wan93] derived local linearization results, which are preserved by conjugation.

We restrict our attention to the case of nilpotent resp. linear solvable Lie algebras in order to see explicitly the dependence of the conjugation in ω, which would be not so clear in the case of nonlinear solvable Lie algebras. We also remark that it seems possible to obtain a conjugation result using an implicit technique without any assumption on the Lie algebra generated by the diffusion vector fields. At the moment however this implicit technique does not provide enough information about the nature of the conjugation (e.q. temperedness) to be used for proving existence of attractors.

Notations and preliminaries

Our basic probability space is the $m-$dimensional canonical Wiener space $(\Omega, \mathbf{F}, P)$, enlarged such as to carry an $m-$dimensional *Wiener process* indexed by $\mathbf{R}$. More precisely, $\Omega = C(\mathbf{R}, \mathbf{R}^m)$ is the set of continuous functions on $\mathbf{R}$ with values in $\mathbf{R}^m$, $\mathbf{F}$ the $\sigma-$algebra of Borel sets with respect to uniform convergence on compacts of $\mathbf{R}$, P the probability measure on $\mathbf{F}$ for which the *canonical Wiener process* $W_t = (W_t^1, ..., W_t^m)$, $t \in \mathbf{R}$, satisfies that both $(W_t)_{t\geq 0}$ and $(W_{-t})_{t\geq 0}$ are usual $m-$dimensional Brownian motions. The natural filtration $\{\mathbf{F}_s^t = \sigma(W_u - W_v : s \leq u, v \leq t) : \mathbf{R} \ni s \leq t \in \mathbf{R}\}$ of W is assumed to be completed by the $P-$completion of $\mathbf{F}$. For $t \in \mathbf{R}$, let $\theta_t : \Omega \to \Omega, \omega \mapsto \omega(t + \cdot) - \omega(t)$, the *shift* on Ω by t. It is well known that θ_t preserves Wiener measure P for any $t \in \mathbf{R}$ and is even ergodic for $t \neq 0$. Hence $(\Omega, \mathbf{F}, P, (\theta_t)_{t\in\mathbf{R}})$ is an ergodic *metric dynamical system* (see Arnold [Arn98]). As usual, we use a "$\circ$" to denote Stratonovich integrals with respect to Wiener process.

For a random vector X, we denote by P_X the law of X with respect to P. ∇ is used as a symbol for the gradient of vector fields on $\mathbf{R}^d$. Lie brackets between vector fields will be denoted by the usual symbol $[\cdot, \cdot]$, scalar products in $\mathbf{R}^m$ by $\langle ., . \rangle$.

1. Stationary stochastic integrals and some algebra

It is well known that the Wiener process can be made stationary by just adding a suitable drift. This way one obtains the stationary Ornstein-Uhlenbeck process. To be more precise, the sde

$$dz_t = dW_t - z_t\, dt$$

has the stationary solution

$$z_t = e^{-t} \int_{-\infty}^{t} e^s dW_s.$$

Now suppose that X and Y are stationary semimartingales of the Brownian filtration. Then the stochastic integral $Y \circ dX$ need not be stationary, just as the Wiener process. By passing to the same moving average process as above, we may add a drift to the stochastic integral to make it stationary. We will briefly elaborate on this, and then consider particular cases of semimartingales of this type, generated by multiple integrals of the Ornstein-Uhlenbeck process.

For $p > 1$, $x \geq 0$ let $F_p(x) = [\ln(x+1)]^p$. Then F_p is a moderate function (see for example Revuz, Yor [Rev99]). Denote by $\mathscr{S}$ the set of continuous semimartingales X of the Brownian filtration with decomposition $dX_t = \langle a_t, dW_t\rangle + b_t dt$ such that X, a, b are stationary, and such that

$$E(F_p(\sup_{0\leq t\leq 1} |a_t|)) < \infty, \quad E(F_p(\sup_{0\leq t\leq 1} |b_t|)) < \infty, \quad E(F_p(\sup_{0\leq t\leq 1} |X_t|)) < \infty$$

for all $p > 1$. Clearly, an Ornstein-Uhlenbeck process belongs to the class $\mathscr{S}$.

Lemma 1.1. *Let $X, Y \in \mathscr{S}$ with canonical (forward) decomposition $dX = \langle a, dW\rangle + b\, dt$, $dY = \langle c, dW\rangle + d\, dt$,*

$$Z_t = e^{-t} \int_{-\infty}^{t} e^s\, Y_s \circ dX_s,$$

$t \in \mathbf{R}$. Then $Z \in \mathscr{S}$, satisfies

$$E(F_p(\sup_{0\leq t\leq 1} |Z_t|)) < \infty, \quad p > 1,$$

the sde

$$dZ_t = Y_t \circ dX_t - Z_t dt,$$

and has the (forward) decomposition

$$dZ_t = \langle Y_t a_t, dW_t\rangle + (\frac{1}{2}\langle a_t, c_t\rangle + Y_t b_t - Z_t)\, dt.$$

Proof. In the following C will denote a constant varying from line to line and depending only on p unless stated explicitely.

Once we know that Z_0 is well defined, stationarity of Z is automatic from the following equation, which is a consequence of the stationarity of X and Y

$$Z_0 \circ \theta_t = \int_{-\infty}^0 e^s Y_{s+t} \circ dX_{s+t} = e^{-t} \int_{-\infty}^t e^s Y_s \circ dX_s = Z_t.$$

Let us first show that

$$E(F_p(|\int_0^1 e^s\, Y_s \circ dX_s|)) < \infty. \tag{4}$$

First of all, we have the decomposition

$$\int_0^1 e^s\, Y_s \circ dX_s = \int_0^1 e^s\, Y_s \langle a_s, dW_s\rangle + \int_0^1 e^s [Y_s b_s + \frac{1}{2}\langle a_s, c_s\rangle] ds.$$

Hence, using the inequality of Burkholder, Davis and Gundy for the moderate function F_p (see Revuz, Yor [Rev99], p. 170) we have

$$\begin{aligned}
&E(F_p(|\int_0^1 e^s\, Y_s \circ dX_s|)) \\
&\quad \leq C(E(F_p(|\int_0^1 e^s\, Y_s \langle a_s, dW_s\rangle|)) + E(F_p(|\int_0^1 e^s [Y_s b_s + \frac{1}{2}\langle a_s, c_s\rangle] ds|))) \\
&\quad \leq C\,(E(F_p([\int_0^1 |Y_s|^2 |a_s|^2 ds]^{\frac{1}{2}})) + E(F_p(|\int_0^1 e^s [Y_s b_s + \frac{1}{2}\langle a_s, c_s\rangle] ds|))) \\
&\quad \leq C(E(F_p(\sup_{0\leq t\leq 1} |Y_t|)) + E(F_p(\sup_{0\leq t\leq 1} |a_t|)) + E(F_p(\sup_{0\leq t\leq 1} |b_t|)) + E(F_p(\sup_{0\leq t\leq 1} |c_t|))).
\end{aligned}$$

Now we are able to prove (4). To see next that Z_0 is well defined, note that

$$|Z_0| \leq \sum_{n=1}^\infty e^{-n}\, |\int_0^1 e^s\, Y_s \circ dX_s| \circ \theta_{-(n+1)}.$$

So by the lemma of Borel-Cantelli, Z_0 will be well defined, if we can show that

$$\sum_{n=1}^\infty P(|\int_0^1 e^s\, Y_s \circ dX_s| > e^{\alpha n}) < \infty$$

for some $0 < \alpha < 1$. By definition of F_p, this amounts to show

$$\sum_{n=1}^\infty P(F_p(|\int_0^1 e^s\, Y_s \circ dX_s|) > n) < \infty.$$

This in turn is an obvious consequence of (4). Now the SDE valid for Z as well as the semimartingale decomposition of Z are obvious. Moreover, since $F_p(xy) \leq 2^p \max(F_p(x), F_p(y))$ for $x, y \geq 0$, we may restrict to the verification of

$$E(F_p(\sup_{0\leq t\leq 1} |Z_t|)) < \infty, \qquad p > 1. \tag{5}$$

Indeed, we have

$$E(F_p(\sup_{0\le t\le 1}|Z_t|)) \tag{6}$$

$$\le C\left\{E(F_p(Z_0)) + E(F_p(\sup_{0\le t\le 1}|\int_{-\infty}^t e^s\, Y_s\langle a_s, dW_s\rangle|))\right.$$

$$\left. + E(F_p(\int_{-\infty}^1 e^s|[Y_s b_s + \frac{1}{2}\langle a_s, c_s\rangle]|ds))\right\} \tag{7}$$

$$\le C\{E(F_p(Z_0)) + E(F_p([\int_{-\infty}^1 e^{2s}\,|Y_s|^2|a_s|^2 ds]^{\frac{1}{2}}))$$

$$+ E(F_p(\int_{-\infty}^1 e^s|[Y_s b_s + \frac{1}{2}\langle a_s, c_s\rangle]|ds))\Big\}. \tag{8}$$

We proceed to estimate the first term in the last line of (6), the second one being treated similarly. We have with some $0 < \alpha < 1$, a constant $C(\alpha)$ varying from line to line and $q > p$

$$E(F_p([\int_{-\infty}^1 e^{2s}\,|Y_s|^2|a_s|^2 ds]^{\frac{1}{2}}))$$

$$= E(F_p(\sum_{n=0}^{\infty} e^{-n}\,[\int_0^1 e^{2s}\,|Y_s|^2|a_s|^2 ds]^{\frac{1}{2}}\circ\theta_{-(n+1)}))$$

$$\le 1 + \sum_{l=1}^{\infty} P(F_p(\sum_{n=0}^{\infty} e^{-n}\,[\int_0^1 e^{2s}\,|Y_s|^2|a_s|^2 ds]^{\frac{1}{2}}\circ\theta_{-(n+1)}) > l)$$

$$\le 1 + \sum_{l=1}^{\infty} P(\sum_{n=0}^{\infty} e^{-n}\,[\int_0^1 e^{2s}\,|Y_s|^2|a_s|^2 ds]^{\frac{1}{2}}\circ\theta_{-(n+1)} > e^{l^{\frac{1}{p}}} - 1)$$

$$\le 1 + C(\alpha)\,\sum_{l=1}^{\infty}\sum_{n=0}^{\infty} P([\int_0^1 e^{2s}\,|Y_s|^2|a_s|^2 ds]^{\frac{1}{2}} > C\,e^{\alpha n + l^{\frac{1}{p}}})$$

$$\le 1 + C(\alpha)\,\sum_{l=1}^{\infty}\sum_{n=0}^{\infty}\frac{1}{(\alpha n + l^{\frac{1}{p}})^q} E(F_q([\int_0^1 e^{2s}\,|Y_s|^2|a_s|^2 ds]^{\frac{1}{2}}))$$

$$\le 1 + C(\alpha)\,E(F_q([\int_0^1 e^{2s}\,|Y_s|^2|a_s|^2 ds]^{\frac{1}{2}})).$$

Now, finally, the same arguments as used for (4) yield (5). This finishes the proof. □

In the construction of stationary diffeomorphisms which describe the conjugation of the flows in the following section, a recursively defined family of processes of the above type will play an essential role. To describe them, we use the following notation. Let

$$\Lambda_0 = \{1,\cdots,m\},$$

and recursively for $n \in \mathbf{N}$

$$\Lambda_n = \bigcup_{k \geq 2} \Lambda_{n-1}^k.$$

Let then for $i \in \Lambda_0$, $t \in \mathbf{R}$

$$z_i^0(t) = e^{-t} \int_{-\infty}^t e^s \, dW_s^i.$$

To set up the recursion, suppose for $i \in \Lambda_n$ stationary processes z_i^n have been defined. Let $i = (i_1, \cdots, i_k) \in \Lambda_{n+1}$ be given, with $i_1, \cdots, i_k \in \Lambda_n$. In this case we also denote $|i| = k$. Then we define for $t \in \mathbf{R}$

$$z_i^{n+1}(t) = e^{-t} \int_{-\infty}^t e^s \prod_{j=1}^{k-1} z_{i_j}^n(s) \circ dz_{i_k}^n(s).$$

For obvious reasons we call these integrals *stationary multiple Ornstein-Uhlenbeck integrals*. According to Lemma 1.1 and by induction on n, they are elements of the class $\mathscr{S}$.

For the algorithm of reduction of algebraic complexity to be discussed in the following section, we shall need some identities concerning the Lie algebra generated by the diffusion vector fields of a stochastic system. It is the aim of the following considerations to provide them.

To this end we have to introduce some more notation. A vector field $A : \mathbf{R}^d \to \mathbf{R}^d$ is called *complete* if it generates a global flow $(\Phi_t^A)_{t \in \mathbf{R}}$. In this case we set

$$\exp(tA) \; := \; \Phi_t^A.$$

(This notation is consistent since $\Phi_1^{tA} = \Phi_t^A$.) For smooth complete vector fields A, B the Lie bracket $[A, B]$ is defined by

$$[A, B](x) \; = \; \frac{d}{dt}\bigg|_{t=0} \Big(\exp(tA)'(x)\Big)^{-1} \; B \circ \exp(tA)(x).$$

We define recursively the n-fold Lie brackets $[A, B]_n$ by

$$[A, B]_n = \begin{cases} B, & n = 0, \\ [A, [A, B]_{n-1}], & n \in \mathbf{N}. \end{cases}$$

Let us briefly recall the notion of nilpotence. If $(\mathscr{L}, [\cdot, \cdot])$ is a Lie algebra, let

$$\mathscr{L}^1 = [\mathscr{L}, \mathscr{L}] = \{[A, B] : A, B \in \mathscr{L}\},$$
$$\mathscr{L}^{n+1} = [\mathscr{L}, \mathscr{L}^n],$$

$n \in \mathbf{N}$. Then $\mathscr{L}$ is called *nilpotent* if there exists n such that $\mathscr{L}^n = 0$.

Suppose now that $\mathscr{L}$ is generated by finitely many smooth complete vector fields $A_1, \ldots, A_m$. If $\mathscr{L}$ is nilpotent then $\mathscr{L}$ is obviously finite dimensional. In this case it is well known that every element of $\mathscr{L}$ is complete (see e.g. [Pal57]).

Lemma 1.2. *Let $A, B : \mathbf{R}^d \to \mathbf{R}^d$ be smooth complete vector fields and assume that the Lie algebra generated by A and B is nilpotent. Let $(\Phi_t)_{t\in\mathbf{R}}$ denote the flow generated by A. Then we have*

$$\Phi_t'^{-1}\ B\circ\Phi_t \ = \ \sum_{n=0}^{\infty}\frac{t^n}{n!}\,[A,B]_n, \quad t\in\mathbf{R}.$$

Proof. Let

$$U(t,x) = \Phi_t'^{-1}(x)\ B\Big(\Phi_t(x)\Big),$$
$$V(t,x) = \sum_{n=0}^{\infty}\frac{t^n}{n!}[A,B]_n(x) \qquad (t\in\mathbf{R}, x\in\mathbf{R}^d).$$

Then we have $U(0,\cdot) = B = V(0,\cdot)$ and furthermore

$$\begin{aligned}\frac{\partial}{\partial t}U(t,x) &= \frac{\partial}{\partial s}\Big|_{s=0}\ \Phi_{t+s}'(x)^{-1}\ B(\Phi_{t+s}(x))\\ &= \frac{\partial}{\partial s}\Big|_{s=0}\ \Phi_s'(x)^{-1}\ \Phi_t'(\Phi_s(x))^{-1}B\,(\Phi_t(\Phi_s(x)))\\ &= \frac{\partial}{\partial s}\Big|_{s=0}\ \phi_s'(x)^{-1}\ U(t,\Phi_s(x))\\ &= [A, U(t,\cdot)](x)\end{aligned}$$

and

$$\frac{\partial}{\partial t}V(t,x) \ = \ \sum_{n=0}^{\infty}\frac{t^n}{n!}\,[A,B]_{n+1}(x) \ = \ [A, V(t,\cdot)](x).$$

This implies $U = V$. □

Lemma 1.3. *Let $A, B : \mathbf{R}^d \to \mathbf{R}^d$ be smooth complete vector fields and assume that the Lie algebra generated by A, B is nilpotent. Then we have*

$$\frac{\partial}{\partial\lambda}\exp(A+\lambda B) \ = \ B\circ\exp(A+\lambda B) + \exp(A+\lambda B)'\sum_{n=1}^{\infty}c_n\,[A,B]_n$$

where $c_n = \frac{1}{(n+1)!} - \frac{1}{n!}$, $n\in\mathbf{N}$.

Proof. For $\lambda\in\mathbf{R}$ let Φ^λ denote the flow of $\dot x = A(x)+\lambda B(x)$. Fix $x\in\mathbf{R}^d$. Then

$$\begin{aligned}\frac{d}{dt}\frac{\partial}{\partial\lambda}\Phi_t^\lambda(x) &= \frac{\partial}{\partial\lambda}\big(A(\Phi_t^\lambda(x))+\lambda B(\Phi_t^\lambda(x))\big)\\ &= \big(A'(\Phi_t^\lambda(x))+\lambda B'(\Phi_t^\lambda(x))\big)\frac{\partial}{\partial\lambda}\Phi_t^\lambda(x) + B(\Phi_t^\lambda(x))\end{aligned}$$

and $\frac{\partial}{\partial\lambda}\Phi_t^\lambda(x)\big|_{t=0} = 0$.

Since the fundamental solution of

$$\dot{Z} = \left(A'(\Phi_t^\lambda(x)) + \lambda B'(\Phi_t^\lambda(x))\right) Z$$

is given by $t \mapsto \Phi_t^{\lambda\,\prime}(x)$, the variations of constants formula and the above lemma give us

$$\begin{aligned} \frac{\partial}{\partial\lambda}\Phi_t^\lambda(x) &= \Phi_t^{\lambda\,\prime}(x) \int_0^t \Phi_s^{\lambda\,\prime}(x)^{-1} B(\Phi_s^\lambda(x))\, ds \\ &= \Phi_t^{\lambda\,\prime}(x) \int_0^t \sum_{n=0}^\infty \frac{s^n}{n!}\, [A, B]_n(x)\, ds. \end{aligned}$$

According to the above lemma we have furthermore

$$B(\Phi_t^\lambda(x)) = \Phi_t^{\lambda\,\prime}(x) B(x) + \Phi_t^{\lambda\,\prime}(x) \sum_{n=1}^\infty \frac{t^n}{n!}\, [A, B]_n(x).$$

Hence

$$\begin{aligned} &\frac{\partial}{\partial\lambda} \exp(A+\lambda B)(x) \\ &= \frac{\partial}{\partial\lambda}\Phi_1^\lambda(x) \\ &= \Phi_1^{\lambda\,\prime}(x)B(x) + \Phi_1^{\lambda\,\prime}(x) \sum_{n=1}^\infty \frac{1}{(n+1)!}\, [A, B]_n(x) \\ &= B(\Phi_1^\lambda(x)) - \Phi_1^{\lambda\,\prime}(x) \sum_{n=1}^\infty \frac{1}{n!}\, [A, B]_n(x) + \Phi_1^{\lambda\,\prime}(x) \sum_{n=1}^\infty \frac{1}{(n+1)!}\, [A, B]_n(x) \\ &= B\left(\exp(A+\lambda B)(x)\right) + \exp(A+\lambda B)'(x) \sum_{n=1}^\infty \left(\frac{1}{(n+1)!} - \frac{1}{n!}\right) [A, B]_n(x). \end{aligned}$$

□

2. The case of diffusion vector fields with nilpotent Lie algebra

Let $A_0, \ldots, A_m$ be smooth vector fields on $\mathbf{R}^d$. We consider the stochastic differential equation

$$dx_t = A_0(x_t)\, dt + \sum_{i=1}^m A_i(x_t) \circ dW_t^i. \tag{9}$$

The aim of the following considerations is to construct a random diffeomorphism Φ and a random vector field $g : \Omega \times \mathbf{R}^d \to \mathbf{R}^d$ such that the flow $(\chi_t)_{t\in\mathbf{R}}$ of the random differential equation

$$dy_t = g(\theta_t\cdot, y_t)\, dt \tag{10}$$

is *conjugate* to the flow $(\phi_t)_{t\in\mathbf{R}}$ of (9) by virtue of Φ, i.e. we have for $t \in \mathbf{R}$

$$\phi_t \;=\; \Phi(\theta_t, \cdot) \circ \chi_t \circ \Phi^{-1} \tag{11}$$

In this section we suppose in addition that the diffusion vector fields $A_1, \ldots, A_m$ are complete and that the Lie algebra generated by these vector fields is nilpotent. This will allow for a rather explicit construction of a natural finite family of random diffeomorphisms which compose to yield the desired Φ. In the next section we shall work under the weaker assumption of solvability of the Lie algebra. Our algorithm will then yield Φ as a finite composition of diffeomorphisms as well, but with less transparent structure.

We shall now describe an algorithm by which the complexity of the Lie algebra of the diffusion vector fields is gradually reduced. In this algorithm we shall encounter the stationary multiple Ornstein-Uhlenbeck integrals of the preceding section, and an analogous family of Lie brackets of the diffusion vector fields.

To define it, we use the notation of the preceding section. For $i \in \Lambda_0$, $A_i^0 = A_i$ is just the diffusion vector field appearing in the i^{th} diffusion term of (9). If for $i \in \Lambda_n$, A_i^n is defined, and we pick $i = (i_1, \cdots, i_k) \in \Lambda_{n+1}$, we let

$$A_i^{n+1} = [A_{i_1}^n, \ldots, [A_{i_{k-1}}, A_{i_k}] \ldots].$$

We begin by rewriting (9) with stationary drift and driving processes. Let z_i^0 be the stationary Ornstein-Uhlenbeck processes corresponding to W^i, $1 \le i \le m$. Defining

$$B_t^0 = A_0 + \sum_{i=1}^m A_i\, z_i^0(t) \qquad t \in \mathbf{R}, \tag{12}$$

we may write (9) as

$$dx_t \;=\; B_t^0(x_t)\, dt + \sum_{i\in\Lambda_0} A_i^0(x_t) \;\circ dz_i^0(t). \tag{13}$$

Note that the vector field B^0 is stationary in t.

Now let

$$\Phi^0 \;=\; \exp(\sum_{i\in\Lambda_0} z_i^0(0)\, A_i^0).$$

Then $\Phi_t^0 \;=\; \Phi \circ \theta_t, t \in \mathbf{R}$ is a stationary process of random diffeomorphisms of $\mathbf{R}^d$. Lemma 1.3 allows us to compute the generator of the flow Φ^0. The formally infinite sums we shall write in the sequel are in fact finite, due to the hypothesis of nilpotence for the Lie algebra of the diffusion vector fields. We have for $x \in \mathbf{R}^d$

$$d\Phi_t^0(x) \tag{14}$$

$$= \sum_{i\in\Lambda_0} \frac{\partial}{\partial t_i} \exp\Big(\sum_{j\in\Lambda_0} t_j A_j^0\Big)(x)\Bigg|_{t_j = z_j^0(t),\ j\in\Lambda_0} \circ\, dz_i^0(t)$$

$$= \sum_{i\in\Lambda_0} \Big\{ A_i^0 \circ \exp\Big(\sum_{j\in\Lambda_0} z_j^0(t) A_j^0 \Big)(x) + \exp\Big(\sum_{j\in\Lambda_0} z_j^0(t) A_j^0 \Big)'(x) \sum_{n=1}^{\infty} c_n \, [\sum_{j\in\Lambda_0} z_j^0(t) A_j^0, A_i^0]_n(x) \Big\} \circ dz_i^0(t)$$

$$= \sum_{i\in\Lambda_0} A_i^0(\Phi_t^0(x)) \circ dz_i^0(t) + (\Phi_t^0)'(x) \sum_{i\in\Lambda_0} \sum_{n=1}^{\infty} c_n \, [\sum_{j\in\Lambda_0} z_j^0(t) A_j^0, A_i^0]_n(x) \circ dz_i^0(t).$$

Now recall the definition of iterated Lie brackets, and note that they match the notation Λ_1. Recall also the Definition of the stationary processes z_j^1, $j \in \Lambda_1$, and note that

$$z_{j_1}^0(t) \dots z_{j_{k-1}}^0(t) \circ dz_{j_k}^0(t) = \circ\, dz_j^1(t) + z_j^1(t) dt, \qquad (j_1, \dots, j_k \in \Lambda_0,\ j = (j_1, \dots, j_k) \in \Lambda_1).$$

Hence we have for $n \geq 1$

$$\sum_{i\in\Lambda_0} [\sum_{j\in\Lambda_0} z_j^0(t) A_j^0, A_i^0]_n(x) \circ dz_i^0(t)$$
$$= \sum_{i\in\Lambda_0} \sum_{j_1,\dots,j_n\in\Lambda_0} [A_{j_1}^0, \dots, [A_{j_n}^0, A_i^0] \dots](x)\, z_{j_1}^0(t) \dots z_{j_n}^0(t) \circ dz_i^0(t)$$
$$= \sum_{j\in\Lambda_1, |j|=n+1} A_j^1(x) \circ (dz_j^1(t) + z_j^1(t) dt).$$

So we can rewrite (14) as

$$d\Phi_t^0(x) = \sum_{i\in\Lambda_0} A_i^0(\Phi_t^0(x)) \circ dz_i^0(t) + (\Phi_t^0)'(x) \sum_{i\in\Lambda_1} c_{|i|}\, A_i^1(x) \circ (dz_i^1(t) + z_i^1(t) dt). \tag{15}$$

In order to solve (13) we use the ansatz

$$x_t \ = \ \Phi_t^0(y_t),$$

where y is an unknown (forward) semimartingale. This yields

$$\begin{aligned} dx_t &= (\circ\, d\Phi_t^0)(y_t) + (\Phi_t^0)'(y_t) \circ dy_t \\ &= \sum_{i\in\Lambda_0} A_i^0(\Phi_t^0(y_t)) \circ dz_i^0(t) \\ &\quad + (\Phi_t^0)'(y_t) \sum_{i\in\Lambda_1} c_{|i|}\, A_i^1(y_t) \circ (dz_i^1(t) + z_i^1(t) dt) + (\Phi_t^0)'(y_t) \circ dy_t \end{aligned} \tag{16}$$

$$= \sum_{i \in \Lambda_0} A_i^0(x_t) \circ dz_i^0(t)$$
$$+ (\Phi_t^0)'(y_t) \Big\{ \sum_{i \in \Lambda_1} c_{|i|} \; A_i^1(y_t) \circ (dz_i^1(t) + z_i^1(t)dt) + dy_t \Big\}.$$

Comparing this equation with (13) we see that x is a solution of (13) iff y is a solution of

$$B_t^0 \circ \Phi_t^0(y_t) \, dt \;=\; (\Phi_t^0)'(y_t) \Big\{ \sum_{i \in \Lambda_1} c_{|i|} \; A_i^1(y_t) \circ (dz_i^1(t) + z_i^1(t)dt) + dy_t \Big\}. \quad (17)$$

This equation is equivalent with

$$dy_t \;=\; B_t^1(y_t)dt - \sum_{i \in \Lambda_1} c_{|i|} A_i^1(y_t) \circ dz_i^1(t), \quad (18)$$

where the stationary vector field B^1 is defined by

$$B_t^1(y) \;=\; (\Phi_t^0)'(y)^{-1} B_t^0 \circ \Phi_t^0(y) - \sum_{i \in \Lambda_1} c_{|i|} A_i^1(y) z_i^1(t).$$

Now let ϕ^0 be the flow of (13) and denote the flow of (18) by ϕ^1. Suppose that $x_0 = \xi \in \mathbf{R}^d$. Then we have

$$\phi_t^0(\xi) \;=\; x_t \;=\; \Phi_t^0(y_t) \;=\; \Phi_t^0 \circ \phi_t^1(y_0) \;=\; \Phi_t^0 \circ \phi_t^1 \circ (\Phi_0^0)^{-1}(\xi)$$

Since Φ^0 is stationary this means that ϕ^1 is *conjugate* to ϕ^0 via Φ_0^0, i.e.

$$\phi_t^0 \;=\; (\Phi^0 \circ \theta_t) \;\circ\; \phi_t^1 \circ (\Phi^0)^{-1}, \quad t \in \mathbf{R}. \quad (19)$$

Note that equation (18) has the same structure as the original equation (13). In particular the Lie algebra generated by A_i^1, $i \in \Lambda_1$, is again nilpotent since the vector fields A_i^1 lie in $\mathscr{L}^1 = [\mathscr{L}, \mathscr{L}]$, where $\mathscr{L}$ denotes the Lie algebra generated by A_i^0, $i \in \Lambda_0$.

So, if $\mathscr{L}$ is nilpotent of degree n (i.e. $\mathscr{L}^{n+1} = \{0\}$) it is clear that we have just to repeat this reduction algorithm n times to arrive at a stationary random differential equation whose flow is conjugate to the flow ϕ^0 of the original sde (13).

For the sake of completeness we summarize the general reduction step. In the nth step we consider a flow ϕ^n which has the generator

$$d\phi_t^n(x) \;=\; B_t^n[\phi_t^n(x)] \, dt - \sum_{i \in \Lambda_n} a_i^n A_i^n[\phi_t^n(x)] \circ dz_i^n(t) \quad (20)$$

where B^n is a stationary vector field and a_i^n, $i \in \Lambda_n$, are given real numbers (for $n = 0$ we have $a_i^0 = 1$ and B^0 is given by (12)). Define the $(n+1)^{st}$ diffeomorphism Φ^n of $\mathbf{R}^d$ by the formula

$$\Phi^n \;=\; \exp\Big(\sum_{i \in \Lambda_n} a_i^n A_i^n z_i^n(0)\Big).$$

Define the real numbers a_i^{n+1}, $i \in \Lambda_{n+1}$, by

$$a_i^{n+1} = -c_{|i|} \prod_{j=1}^{|i|} a_{i_j}^n, \quad i = (i_1, \ldots, i_{|i|}) \in \Lambda_{n+1}$$

and the stationary vector field B^{n+1} by

$$B_t^{n+1}(x) = (\Phi^n)'(\theta_t \cdot, x)^{-1} B_t^n \circ \Phi^n(\theta_t \cdot, x) \sum_{i \in \Lambda_1} a_i^{n+1} A_i^{n+1}(x) z_i^{n+1}(t).$$

Then the flow ϕ^{n+1} of

$$d\phi_t^{n+1}(x) = B_t^{n+1}[\phi_t^{n+1}(x)]\, dt + \sum_{i \in \Lambda_{n+1}} a_i^{n+1} A_i^{n+1}[\phi_t^{n+1}(x)] \circ dz_i^{n+1}(t) \quad (21)$$

is conjugate to ϕ^n by virtue of the diffeomorphism Φ^n, i.e. we have

$$\phi_t^n = (\Phi^n \circ \theta_t) \circ \phi_t^{n+1} \circ (\Phi^n)^{-1}, \quad t \in \mathbf{R}.$$

If the Lie algebra generated by the vector fields A_i^0, $i \in \Lambda_0$, is nilpotent of degree n then the vector fields A_i^{n+1} in (21) vanish, i.e. (21) is a stationary random differential equation. Our main result is therefore proved.

Theorem 2.1. *Let $A_0, \ldots, A_m$ be smooth vector fields on $\mathbf{R}^d$. Let $(\phi_t)_{t \in \mathbf{R}}$ be the (possibly local) flow associated with the stochastic differential equation*

$$dx_t = A_0(x_t)\, dt + \sum_{i=1}^m A_i\, x_t \circ dW_t^i$$

Assume that the diffusion vector fields $A_1, \ldots, A_m$ are complete and that the Lie algebra generated by these vector fields is nilpotent. Then there is a random vector field g and a random diffeomorphism Φ such that ϕ and the flow $(\chi_t)_{t \in \mathbf{R}}$ of the random differential equation

$$dy_t = g(\theta_t \cdot, y_t)\, dt$$

are conjugate, i.e. for any $t \in \mathbf{R}$ we have

$$\phi_t = \Phi(\theta_t, \cdot) \circ \chi_t \circ \Phi^{-1}.$$

Proof. Just take n as in the remark above, let $g = B_0^{n+1}$, and $\Phi = \Phi^0 \circ \ldots \circ \Phi^n$. □

3. The case of linear diffusion vector fields with solvable Lie algebra

In this section we consider the stochastic differential equation

$$dx_t \;=\; f_0(x_t)dt + \sum_{j=1}^{m} A_j x_t \circ dW_t^j \tag{22}$$

where f_0 is a smooth vector field on $\mathbf{R}^d$ and $A_1, \ldots, A_m \in \mathbf{R}^{d\times d}$. So we restrict ourselves to the case of linear diffusion vector fields. In return the condition we impose on the Lie algebra generated by the diffusion vector fields is weakened: we shall suppose that it is solvable. We briefly recall this notion. For a Lie algebra $(\mathscr{L}, [\cdot,\cdot])$, we let

$$\mathscr{L}_1 = [\mathscr{L}, \mathscr{L}],$$
$$\mathscr{L}_{n+1} = [\mathscr{L}_n, \mathscr{L}_n],$$

$n \in \mathbf{N}$. Then $\mathscr{L}$ is called *solvable* if there exists $n \in \mathbf{N}$ such that $\mathscr{L}_n = 0$. Note that nilpotence implies solvability. Note also that the Lie algebra generated by two vector fields is always solvable.

In our case, solvability has more consequences, which help us to regain the framework of nilpotent Lie algebras after a modification of our reduction algorithm. In fact, our Lie algebra $\mathscr{L}$ generated by the diffusion vector fields $A_1, \cdots, A_m$ is finite dimensional. Hence, due to the theorem of Lie there exists a basis $B_1, \cdots, B_n$ of $\mathscr{L}$ such that, if

$$\mathscr{K}_i = \text{span}\{B_i, \cdots, B_n\},$$

then $[\mathscr{L}, \mathscr{K}_i] \subset \mathscr{K}_{i+1}$, $1 \le i \le n$. Since $\mathscr{L} = \mathscr{K}_1$, we have $[\mathscr{L}, \mathscr{L}] \subset \mathscr{K}_2$, and hence $[\mathscr{L}, \mathscr{L}]$ is nilpotent.

We shall now modify the reduction algorithm presented in the previous section, so that it fits the given situation. We will in fact see that it is ready made for the solvable case.

We start again by introducing stationary drifts and driving noises in (22), so that our flow ϕ^0 is generated by the sde

$$dx_t \;=\; B_t^0(x_t)dt + \sum_{i\in\Lambda_0} A_i^0 x_t \circ dz_i^0(t), \tag{23}$$

where the stationary vector field B^0 is defined by

$$B_t^0(x) \;=\; f_0(x) + \sum_{j=1}^{m} A_i^0 x\; z_j^0(t). \tag{24}$$

Define the linear stationary vector field

$$C_t^0 \;=\; \sum_{i\in\Lambda_0} z_i^0(t)\, A_i^0, \quad t \in \mathbf{R}. \tag{25}$$

Let

$$\Phi_t^0 \;=\; \exp(-C_t^0)$$

and define the flow ϕ^1 by

$$\phi_t^1 \;=\; \Phi_t^0\, \phi_t^0\, (\Phi_0^0)^{-1}. \tag{26}$$

In order to compute the generator of the flow ϕ^1 we need a modification of lemma 1.3. In the sequel $[A, B]$ denotes the commutator of the matrices A, B instead of the Lie brackets which will cause a change of the signs at some places (note that $[A, B]_{\text{commutator}} = -[A, B]_{\text{Lie bracket}}$).

Lemma 3.1. *Suppose $A : \mathbf{R} \to \mathbf{R}^{d\times d}$ is continuously differentiable. Then we have*

$$\begin{aligned}\frac{d}{d\lambda} e^{A(\lambda)} &= e^{A(\lambda)} A'(\lambda) - \int_0^1 s\, e^{sA(\lambda)}[A(\lambda), A'(\lambda)]\, e^{-sA(\lambda)}\, ds\; e^{A(\lambda)} \\ &= e^{A(\lambda)} A'(\lambda) + \sum_{n=1}^{\infty} c_n\; [A(\lambda), A'(\lambda)]_n\, e^{A(\lambda)}.\end{aligned}$$

where (as before) $c_n = \frac{1}{(n+1)!} - \frac{1}{n!}$.

Proof. The variation of constants formula for $\frac{d}{d\lambda} e^{tA(\lambda)}$ gives us

$$\frac{d}{d\lambda} e^{A(\lambda)} \;=\; \int_0^1 B(s)\, ds$$

where

$$B(s) \;=\; e^{(1-s)A(\lambda)} A'(\lambda) e^{sA(\lambda)}.$$

Since

$$B'(s) \;=\; -e^{(1-s)A(\lambda)}[A(\lambda), A'(\lambda)]e^{sA(\lambda)}$$

we have

$$\begin{aligned}\frac{d}{d\lambda} e^{A(\lambda)} &= \int_0^1 B(s)\, ds \;=\; \int_0^1 \left(B(0) + \int_0^s B'(u)\, du \right) ds \\ &= e^{A(\lambda)}\, A'(\lambda) - \int_0^1 \int_0^s e^{(1-u)A(\lambda)}[A(\lambda), A'(\lambda)]e^{uA(\lambda)}\, du\, ds \\ &= e^{A(\lambda)} A'(\lambda) - \int_0^1 (1-u)\, e^{(1-u)A(\lambda)}[A(\lambda), A'(\lambda)]e^{uA(\lambda)}\, du \\ &= e^{A(\lambda)} A'(\lambda) - \int_0^1 s\, e^{sA(\lambda)}[A(\lambda), A'(\lambda)]\, e^{-sA(\lambda)}\, ds\; e^{A(\lambda)}.\end{aligned}$$

The proof of the second equation is similar to the proof of lemma 1.3. □

Using the first equation of lemma 3.1 and noting that we can rewrite (23) as

$$d\phi_t^0(x) \;=\; B_t^0[\phi_t^0(x)]dt + \circ dC_t^0\, \phi_t^0(x)$$

we get for $x \in \mathbf{R}^d$

$$\begin{aligned} d\phi_t^1(x) &= \circ d\Phi_t^0\, \phi_t^0\, (\Phi_0^0)^{-1}(x) + \Phi_t^0\, \circ d\phi_t^0\, (\Phi_0^0)^{-1}(x) \\ &= \left[-\Phi_t^0 \circ dC_t^0 - \int_0^1 s\, e^{-sC_t^0}\, [C_t^0, \circ dC_t^0] e^{sC_t^0}\, ds\, \Phi_t^0 \right] \phi_t^0\, (\Phi_0^0)^{-1}(x) \\ &\quad + \Phi_t^0 \left[B_t^0 \phi_t^0\, dt + \circ dC_t^0\, \phi_t^0 \right] (\Phi_0^0)^{-1}(x) \\ &= -\int_0^1 s\, e^{-sC_t^0}\, [C_t^0, \circ dC_t^0] e^{sC_t^0}\, ds\, \phi_t^1 + \Phi_t^0 B_t^0 \phi_t^0 (\Phi_0^0)^{-1}\, dt \\ &= -\int_0^1 s\, e^{-sC_t^0}\, [C_t^0, \circ dC_t^0] e^{sC_t^0}\, ds\, \phi_t^1 + \Phi_t^0 B_t^0 (\Phi_t^0)^{-1}\, \phi_t^1\, dt. \end{aligned} \tag{27}$$

Next define the process Γ^1 by the following Stratonovich differential

$$\circ d\Gamma_t^1 = -\int_0^1 s\, e^{-sC_t^0}\, [C_t^0, \circ dC_t^0] e^{sC_t^0}\, ds \tag{28}$$

Note that by definition the semimartingale Γ^1 possesses stationary characteristics, but need not be stationary itself. In order to see that we may apply Lemma 1.1 to pass to a stationary moving average of Γ^1, let us fix $p > 1$ and compute the characteristics of Γ^1. Indeed, we have

$$\circ d\Gamma_t^1 = -\sum_{j,k=1,\, j<k}^{m} \int_0^1 s\, e^{-sC_t^0} [A_j^0, A_k^0]\, e^{sC_t^0}\, ds\, da_{jk}(t),$$

where

$$da_{jk}(t) = z_j^0(t)\, dz_k^0(t) - z_k^0(t)\, dz_j^0(t) = z_j^0(t) \circ dz_k^0(t) - z_k^0(t) \circ dz_j^0(t) = \circ da_{jk}(t),$$

is the differential of the *Ornstein-Uhlenbeck area process*. Note that we even have

$$da_{jk}(t) = z_j^0(t)\, dW_t^k - z_k^0(t)\, dW_t^j, \quad 1 \le j < k \le m.$$

Hence we have to prove that

$$E(F_p(\sup_{0 \le t \le 1} |v_t|)) < \infty \tag{29}$$

for $p > 1$, where for $t \in [0, 1]$, $1 \le l \le m$

$$v_t^l = -\sum_{j,k=1,\, j<k}^{m} \int_0^1 s\, e^{-sC_t^0} [A_j^0, A_k^0]\, e^{sC_t^0}\, ds\, [z_j^0(t)\, 1_{\{l\}}(k) - z_k^0(t)\, 1_{\{l\}}(j)].$$

The property

$$F_p(x + y) \le c_p\, [F_p(x) + F_p(y)],$$

$x, y \geq 0$, shows that (29) will be a consequence of

$$E(F_p(\sup_{0\leq t\leq 1} |\int_0^1 s\, e^{-sC_t^0}[A_j, A_k]\, e^{sC_t^0}\, ds\, z_j^0(t)|)) < \infty \tag{30}$$

for $j < k$ fixed. But, again due to the properties of the function F_p, this is a consequence of the inequalities

$$E(F_p(\sup_{0\leq t\leq 1} |z_j^0(t)|)) < \infty, \quad E(\sup_{0\leq t\leq 1} |C_t^0|^p) < \infty$$

for any $p > 1$, which follow easily from the properties of the Ornstein-Uhlenbeck process. So Lemma 1.1 applies, and we my define

$$C_t^1 = e^{-t} \int_{-\infty}^t e^s \circ d\Gamma_s^1, \quad t \in \mathbf{R}, \tag{31}$$

which is a stationary semimartingale belonging to the class $\mathscr{S}$. The equation

$$\circ d\Gamma_t^1 = \circ dC_t^1 + C_t^1\, dt,$$

which is valid due to Lemma 1.1, allows us then to rewrite (27) in the following form

$$\begin{aligned} d\phi_t^1(x) &= \circ dC_t^1\, \phi_t^1 + C_t^1 \phi_t^1\, dt \\ &\quad + \Phi_t^0\, B_t^0\, (\Phi_t^0)^{-1}\, \phi_t^1(x)\, dt. \end{aligned} \tag{32}$$

We next define the stationary vector field

$$B_t^1 = \Phi_t^0\, B_t^0\, (\Phi_t^0)^{-1} + C_t^1,$$

$t \in \mathbf{R}$. So we may rewrite (32) in the final form

$$d\phi_t^1(x) = B_t^1\, \phi_t^1(x)\, dt + \circ dC_t^1\, \phi_t^1(x). \tag{33}$$

We now use the fact stated above that $[\mathscr{L}, \mathscr{L}]$ is nilpotent, to start a modification of the algorithm of the preceding section. Let

$$\Phi^1 = \exp(-C_0^1), \quad \Phi_t^1 = \Phi^1 \circ \theta_t, \quad t \in \mathbf{R},$$

and set

$$\phi_t^2 = \Phi_t^1\, \phi_t^1\, (\Phi^1)^{-1}, \quad t \in \mathbf{R}. \tag{34}$$

We now argue with the second part of Lemma 3.1 to find the generator of ϕ^2. For $x \in \mathbf{R}^d$ we can write

$$\begin{aligned} d\phi_t^2(x) &= \circ d\Phi_t^1\, \phi_t^1 (\Phi^1)^{-1}(x) + \Phi_t^1 \circ d\phi_t^1\, (\Phi^1)^{-1}(x) \\ &= \Big[-\Phi_t^1 \circ dC_t^1 + \sum_{n=1}^{\infty} (-1)^{n+1} c_n [C_t^1, \circ dC_t^1]_n\ \Phi_t^1 \Big]\, \phi_t^1\, (\Phi^1)^{-1}(x) \\ &\quad + \Phi_t^1 \circ dC_t^1 \phi_t^1\, (\Phi^1)^{-1}(x) + \Phi_t^1\, B_t^1\, \phi_t^1\, (\Phi^1)^{-1}(x)\, dt \\ &= \sum_{n=1}^{\infty} (-1)^{n+1} c_n [C_t^1, \circ dC_t^1]_n\ \phi_t^2(x) + \Phi_t^1\, B_t^1\, (\Phi_t^1)^{-1}\, \phi_t^2(x)\, dt. \end{aligned} \tag{35}$$

Since C^1 is a process in $[\mathscr{L}, \mathscr{L}]$, the sum in (35) is in fact finite. Let

$$C_t^2 = e^{-t} \sum_{n=1}^{\infty} (-1)^{n+1} c_n \int_{-\infty}^{t} e^s \, [C_s^1, \circ dC_s^1]_n, \quad t \in \mathbf{R}.$$

Then, Lemma 1.1 applies again to show that $C^2 \in \mathscr{S}$. So, if we set

$$B_t^2 = \Phi_t^1 \, B_t^1 (\Phi^1)^{-1} + C_t^2,$$

we may write

$$d\phi_t^2(x) = \circ dC_t^2 \ \phi_t^2(x) + B_t^2 \phi_t^2(x) \, dt. \tag{36}$$

We turn to the recursion step of our algorithm. Let $k \geq 2$, and suppose that a flow ϕ^k on $\mathbf{R}^d$, a stationary vector field B^k, and a stationary vector field $C^k \in \mathscr{S}$ are given such that

$$d\phi_t^k(x) = \circ dC_t^k \ \phi_t^k(x) + B_t^k \ \phi_t^k(x) \, dt. \tag{37}$$

We define the diffeomorphism of step k by

$$\Phi^k = \exp(-C_t^k), \tag{38}$$

and let $\Phi_t^k = \Phi^k \circ \theta_t$, $t \in \mathbf{R}$. Then we may set

$$\phi_t^{k+1} = \Phi_t^k \ \phi_t^k \ (\Phi^k)^{-1}, \quad t \in \mathbf{R}. \tag{39}$$

Then the same computation as above gives us the generator of ϕ^{k+1}. For $x \in \mathbf{R}^d$ we have

$$d\phi_t^{k+1}(x) \ = \ \sum_{n=1}^{\infty} (-1)^{n+1} c_n [C_t^k, \circ dC_t^k]_n \ \phi_t^{k+1}(x) + \Phi_t^k \ B_t^k \ (\Phi_t^k)^{-1} \ \phi_t^{k+1}(x) \, dt, \tag{40}$$

where again the sum is finite. To obtain a stationary SDE we again define

$$C_{k+1}(t) = e^{-t} \sum_{n=1}^{\infty} (-1)^{n+1} c_n \int_{-\infty}^{t} e^s [C_s^k, \circ dC_s^k]_n, \quad t \in \mathbf{R}. \tag{41}$$

According to lemma 1.1, $C^{k+1} \in \mathscr{S}$. So, finally setting

$$B_t^{k+1} = \Phi_t^k \ B_t^k (\Phi^k)^{-1} + C_t^{k+1},$$

$t \in \mathbf{R}$, we obtain the asserted stochastic differential equation

$$d\phi_t^{k+1}(x) = \circ dC_t^{k+1} \phi_t^{k+1}(x) + B_t^{k+1} \phi_t^{k+1}(x). \tag{42}$$

This completes the recursion step.

Since $[\mathscr{L}, L]$ is nilpotent, we know that our algorithm stops after finitely many steps. So we obtain our second main result

Theorem 3.1. *Let f_0 be a smooth vector field on $\mathbf{R}^d$, $A_1, \cdots, A_m \in \mathbf{R}^{d\times d}$. Let $\phi = (\phi_t)_{t\in\mathbf{R}}$ denote the (possibly local) flow associated with the stochastic differential equation*

$$dx_t \;=\; f_0(x_t)\,dt + \sum_{i=1}^{m} A_i\, x_t \circ dW_t^i.$$

Assume that the Lie algebra generated by $A_1, \cdots, A_m$ is solvable. Then the assertions of Theorem 2.1 hold true.

4. An application: the existence of random attractors

Let us first recall the notion of a random attractor. For more details consult Crauel, Debussche, Flandoli [Cra97] or Keller, Schmalfuss [Kel98]. Note first that under the smoothness conditions assumed from section 1 on for the vector fields, the completion result of Arnold, Scheutzow [Arn95] implies that the flows of diffeomorphisms generated by our stochastic differential equations in fact generate *random dynamical systems* (see Arnold [Arn98]). More precisely, the flow $(\phi_t)_{t\geq 0}$ of diffeomorphisms on $\mathbf{R}^d$ generated by a stochastic differential equation is called *random dynamical system* on the metric dynamical system $(\Omega, \mathbf{F}, P, (\theta_t)_{t\in\mathbf{R}})$ if the following *cocycle property* is satisfied:

$$\phi_{s+t}(\omega) = \phi_t(\theta_s\omega) \circ \phi_s(\omega), \quad \phi_0(\omega) = id_{\mathbf{R}^d},$$

for $\omega \in \Omega$, $s, t \geq 0$. An obvious modification gives the notion of a random dynamical system for flows with parameter space $\mathbf{R}$ instead of $\mathbf{R}_+$. Whenever we speak of a flow, we shall, as our hypotheses on the vector fields allow, tacitly assume that it is a random dynamical system.

A family $(A(\omega), \omega \in \Omega)$ of closed subsets of $\mathbf{R}^d$ is called *measurable* if for any $x \in \mathbf{R}^d$ the function $\omega \mapsto d(A(\omega), x) = \inf\{|x-y| : y \in A(\omega)\}$ is measurable. Motivated by the needs of section 4.2, we shall define random attractors for more general systems of attracted sets. Let $\mathcal{D}$ be a system of measurable closed and nonempty sets $\omega \mapsto D(\omega)$. In addition we suppose that D fulfills the following filtering property: if D' is a measurable set with closed and nonempty images and $D'(\omega) \subset D(\omega)$ for $\omega \in \Omega$ and $D \in \mathcal{D}$ then $D' \in \mathcal{D}$. Such a system is briefly named *universe*. We hasten to emphasize that the system of compact random sets uniformly bounded in ω is a universe, the one the reader may imagine if we speak of universes. We call it *universe of compact sets*. As we shall see in section 4.2, it is however not the only one which matters for us.

For a given universe $\mathcal{D}$ a measurable set $A \in \mathcal{D}$ with compact images is called a *random attractor* for the random dynamical system $(\phi_t)_{t\geq 0}$ if A is *ϕ-invariant*, i.e. for $\omega \in \Omega$ we have

$$\phi_t(\omega)A(\omega) = A(\theta_t\omega),$$

and absorbs sets from $\mathcal{D}$, i.e.

$$\lim_{t\to\infty} \mathrm{dist}(\phi(\theta_{-t}\omega)D(\theta_{-t}\omega), A(\omega)) = 0$$

for any $D \in \mathscr{D}$, see Flandoli, Schmalfuss [Fla96], where *dist* denotes the semi-Hausdorff distance

$$\text{dist}(A, B) = \sup_{x \in A} \inf_{y \in B} |x - y|.$$

Note that a random attractor is unique. We remark that the more intuitive relationship

$$\lim_{t \to \infty} \text{dist}(\phi_t(\omega) B, A(\theta_t(\omega)) = 0$$

holds only for convergence in probability.

The following theorem is a version of Crauel, Flandoli [Cra94], Flandoli, Schmalfuss [Fla96] or Schmalfuss [Sch97]:

Theorem 4.1. *Let $\mathscr{D}$ be a universe of measurable sets. Suppose that $x \to \phi_t(\omega)x$ is continuous. In addition we suppose that there exists a compact measurable set $B \in \mathscr{D}$ such that*

$$\phi_t(\theta_{-t}\omega)D(\theta_{-t}\omega) \subset B(\omega)$$

for $t \geq t(\omega, D)$ and any $D \in \mathscr{D}$. Then there exists a random attractor with respect to $\mathscr{D}$.

The other important example of universes is given by the *tempered* random sets. A random variable $R > 0$ is tempered if

$$\lim_{t \to \pm\infty} \frac{1}{|t|} \log^+ R(\theta_t \omega) = 0 \tag{43}$$

for $\omega \in \Omega$, see Arnold [Arn98], p.164. Note that (43) is equivalent to

$$\lim_{t \to \pm\infty} e^{-c|t|} R(\theta_t \omega) = 0 \quad \text{for any} \quad c > 0.$$

A measurable set D is called *tempered* if $D(\omega)$ is contained in a ball with center zero and tempered radius $R(\omega)$, $\omega \in \Omega$. Then the system of measurable sets with compact and nonempty tempered images forms the universe of *tempered sets*. The universe which matters in section 4.2 consists of tempered sets with a simple additional condition and will be described precisely later on.

Of course, the universe of compact sets is contained in the tempered one. Let us briefly point out that the difference is not very big from the point of view of random dynamical systems, however. Temperedness of R may be paraphrased by stating that the Lyapunov exponent of the stationary process $t \mapsto R(\theta_t \omega)$ is zero. But if it is not zero, then we automatically have

$$\limsup_{t \to \pm\infty} \frac{1}{|t|} \log^+ R(\theta_t \omega) = +\infty.$$

4.1. *Sufficient criteria for existence*

In this section we shall give an application of the main result of the previous section to the problem of existence of global attractors for flows generated by stochastic differential equations. Let f_0 be a C^∞ vector field on $\mathbf{R}^d$. We consider a stochastic perturbation of the dynamical system described by the differential equation

$$dx_t = f_0(x_t)\, dt. \tag{44}$$

More precisely, let $A_1, \cdots, A_m \in \mathbf{R}^{d\times d}$ and consider the sde

$$dx_t = f_0(x_t)\, dt + \sum_{i=1}^{m} A_i\, x_t \circ dW_t^i. \tag{45}$$

We assume that the flow ϕ generated by (45) is forward complete. Following the ideas of our prototypical approach of random attractors in [Imk98], we assume that (44) has a Lyapunov function V, so that the system has an attractor. The problem of existence of a random attractor for (45) will be approached in the following form. We ask whether V is still a Lyapunov function for the perturbed system. As such it will then provide a random attractor.

Assume from now on that the Lie algebra generated by $A_1, \cdots, A_m$ is solvable. Fix $n \in \mathbf{N}$ such that $C^{n+1} = 0$, let $C^0, \cdots, C^n$, $\Phi^0, \cdots, \Phi^n$ be defined as in the preceding section, and let

$$\begin{aligned} g(., y) = {} & \Phi^n \circ \cdots \circ \Phi^0\ f_0((\Phi^n \circ \cdots \circ \Phi^0)^{-1} y) \\ & + \sum_{k=0}^{n} \Phi^n \circ \cdots \circ \Phi^k\, C^k\, (\Phi^n \circ \cdots \circ \Phi^k)^{-1} y. \end{aligned}$$

Then the main result of the preceding section states that the flow χ generated by the random differential equation

$$dy_t = g(\theta_t \cdot, y_t)\, dt \tag{46}$$

and ϕ are conjugate. It is very easy to deduce from the definition of Φ^i and Lemma 1.1 that the random diffeomorphisms are tempered (i.e. that the random variables $\|\Phi^i\|$ are tempered). The following Theorem describes the natural relation between random attractors in the different coordinates.

Theorem 4.2. *Let $\Phi = \Phi^n \circ \cdots \circ \Phi^0$. Then there is a one-to-one correspondence between random attractors of ϕ and χ. If $(A(\omega), \omega \in \Omega)$ is a random attractor of χ, then $(\Phi(\omega)\, A(\omega), \omega \in \Omega)$ is a random attractor of ϕ attracting tempered sets. If $(B(\omega), \omega \in \Omega)$ is a random attractor of ϕ, then $(\Phi^{-1}(\omega)\, B(\omega), \omega \in \Omega)$ is a random attractor of χ attracting tempered sets.*

Proof. We know that ϕ and χ are conjugate by the tempered diffeomorphism Φ. Hence the proof is the same as in [Imk98]. □

We next introduce the function

$$h(c^0, \cdots, c^n, y) = e^{-c^n} \circ \cdots \circ e^{-c^0} \; f_0((e^{-c^n} \circ \cdots \circ e^{-c^0})^{-1} y) + \sum_{k=0}^{n} e^{-c^n} \circ \cdots \circ e^{-c^k} \, c^k \, (e^{-c^n} \circ \cdots \circ e^{-c^k})^{-1} y,$$

$c^0, \cdots, c^n \in \mathbf{R}^{d\times d}$, $y \in \mathbf{R}^d$. Note that h and g are related by the formula

$$g(., y) = h(C_0^0, \cdots, C_0^n, y), \quad y \in \mathbf{R}^d.$$

We shall subsequently use the abbreviations c for a vector $(c^0, \cdots, c^n)$ and C for the vector of processes $(C^0, \cdots, C^n)$. The following result is basic for the existence of attractors and generalizes Theorem 2.2 of [Imk98] to our situation.

Theorem 4.3. *Let $U : \mathbf{R}^d \to \mathbf{R}_+$ be a C^1-function such that $\lim_{|x|\to\infty} U(x) = \infty$. Suppose that for any $M > 0$*

$$l_M(c) = \sup_{|y|\leq M} |h(c, y)|, \quad c \in (\mathbf{R}^{d\times d})^{n+1}$$

we have

$$E_{P_C}(l_M) < \infty \tag{47}$$

where P_C is the law of C. Suppose there exists a measurable function $k : (\mathbf{R}^{d\times d})^{n+1} \to \mathbf{R}$ such that we have

$$E(\sup_{0\leq s\leq 1} |k(C \circ \theta_s)|) < \infty, \tag{48}$$

$$\limsup_{|y|\to\infty} \sup_{c\in \text{supp}(P_C)} \frac{\langle \nabla \ln U(y), h(c, y)\rangle}{k(c)} \leq 1, \tag{49}$$

$$E_{P_C}(k) < 0. \tag{50}$$

Then χ has a random attractor which attracts compact sets. If U preserves temperedness, then χ has a random attractor for tempered sets.

Proof. The proof is essentially the same as for Theorem 2.2 in [Imk98]. The hypothesis of subexponential growth there is replaced by the two hypotheses (47) for l_M and (48) for k. (48) is needed to justify the application of Birkhoff's ergodic theorem, (47) for verifying the finiteness of the random variable

$$Y = \int_{-\infty}^{0} \exp(\int_{v}^{0} k(C_0 \circ \theta_u)\, du)\, l_M(C_0 \circ \theta_v)\, dv$$

from which a random attractor is constructed. □

Also the perturbation result of [Imk98] generalizes to our setting of solvable Lie algebras for the diffusion vector fields.

Theorem 4.4. *Let V be a Lyapunov function of*

$$dy_t = f_0(y_t)\,dt,$$

i.e. suppose there exists $\alpha > 0$ such that

$$\limsup_{|y|\to\infty} \langle \nabla \ln V(y),\, f_0(y)\rangle \leq -\alpha.$$

Suppose that there exists a measurable function $k : (\mathbf{R}^{d\times d})^{n+1} \to \mathbf{R}_+$ such that (47) and (48) are fulfilled, as well as

$$\limsup_{|y|\to\infty} \sup_{c\in \mathrm{supp}(P_C)} \frac{|\langle \nabla \ln V(y),\, f_0(y) - h(c, y)\rangle|}{|\langle \nabla \ln V(y),\, f_0(y)\rangle|\, k(c)} \leq 1, \tag{51}$$

$$E_{P_C}(k) < \alpha. \tag{52}$$

Then χ has a global random attractor for the compact sets. If V preserves temperedness, then χ has a random attractor for the tempered sets.

Proof. By a suitable modification of k, (51) and (52) are seen to imply (49) and (50). □

4.2. An example

We shall now give an example for a situation in which the Lie algebra generated by the diffusion vector fields is solvable, but not nilpotent. This situation occurs if we consider the well known noisy Duffing-van der Pol oscillator with independent noise sources coupled to the position and velocity components. Formally, the system is given by the second order sde

$$\ddot{y}_t - \beta\, \dot{y}_t + y_t^3 + y_t^2\, \dot{y}_t + y_t - \sigma\, y_t \circ \dot{W}_t^1 - \rho\, \dot{y}_t \circ \dot{W}_t^2 = 0,$$

where $\beta \in \mathbf{R}$, $\sigma, \rho \neq 0$.We pass in the usual way to a two-dimensional system of first order equations by putting $y_1 = y$, $y_2 = \dot{y}$. With the matrices

$$A_1 = \begin{bmatrix} 0 & 0 \\ \sigma & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 \\ 0 & \rho \end{bmatrix},$$

and the nonlinear vector field

$$g_0(y) = \begin{bmatrix} y_2 \\ -y_1 + \beta y_2 - y_1^3 - y_1^2 y_2 \end{bmatrix}, \quad y \in \mathbf{R},$$

we obtain the following sde

$$dy_t = g_0(y_t)\,dt + A_1\, y_t \circ dW_t^1 + A_2\, y_t \circ dW_t^2.$$

Since

$$[A_2, A_1]_n = \begin{bmatrix} 0 & 0 \\ \sigma\,\rho^n & 0 \end{bmatrix} = \rho^n\, A_1,$$

we have for the Lie algebra $\mathscr{L}$ generated by A_1, A_2

$$\mathscr{L} = \text{span}\{A_1, A_2\}, \quad \mathscr{L}_1 = [\mathscr{L}, \mathscr{L}] = \text{span}\{A_1\}, \quad \mathscr{L}_2 = 0.$$

So $\mathscr{L}$ is solvable, but not nilpotent.

Let us compute the conjugation diffeomorphisms, starting with the processes C^0, C^1. We have

$$C_t^0 = z_1^0(t)\, A_1 + z_2^0(t)\, A_2, \quad t \in \mathbf{R},$$

and therefore by an elementary computation

$$\Phi^0 = e^{-C_0^0} = \begin{bmatrix} 1 & 0 \\ v_2 & v_1 \end{bmatrix},$$

where

$$v_1 = e^{-\rho\, z_2^0(0)}, \quad v_2 = -\frac{\sigma\, z_1^0(0)}{\rho\, z_2^0(0)}(1 - e^{-\rho\, z_2^0(0)}).$$

To compute C^1, let us start with the following special case of (28) and (31)

$$\begin{aligned} C_t^1 &= -e^{-t} \int_{-\infty}^t e^u \int_0^1 s\, e^{-sC_u^0}\, [C_u^0, \circ dC_u^0] e^{sC_u^0}\, ds \qquad (53) \\ &= -\sum_{j,k=1, j<k}^m e^{-t} \int_{-\infty}^t e^u \int_0^1 s\, e^{-s \sum_{l=1}^m z_l^0(u)\, A_l}\, [A_j, A_k] e^{s \sum_{l=1}^m z_l^0(u)\, A_l}\, ds \\ &\qquad \circ\, da_{jk}(u), \end{aligned}$$

where

$$\circ da_{jk}(u) = z_j^0(u) \circ dz_k^0(u) - z_k^0(u) \circ dz_j^0(u) = z_j^0(u) dz_k^0(u) - z_k^0(u) dz_j^0(u)$$

is the differential of the *Ornstein-Uhlenbeck area* process corresponding to z_j^0 and z_k^0, for $1 \le j < k \le m$. Note that in this differential, Itô and Stratonovich integration yields identical results. Returning to our special case, we obtain

$$\begin{aligned} &\int_0^1 s\, e^{-s \sum_{l=1}^2 z_l^0(u)\, A_l}\, [A_1, A_2] e^{s \sum_{l=1}^2 z_l^0(u)\, A_l}\, ds \\ &\quad = -\rho \int_0^1 s \begin{bmatrix} 1 & 0 \\ v_2(s) & v_1(s) \end{bmatrix} A_1 \begin{bmatrix} 1 & 0 \\ -\frac{v_2(s)}{v_1(s)} & \frac{1}{v_1(s)} \end{bmatrix} ds \\ &\quad = -\frac{1}{\rho} \frac{1}{z_2^0(u)^2} [1 - e^{-\rho\, z_2^0(u)} - z_2^0(u)\, e^{-\rho\, z_2^0(u)}]\, A_1. \end{aligned}$$

Here $v_1(s) = e^{-s\,\rho\, z_2^0(u)}$, $v_2(s) = \frac{\sigma\, z_1^0(u)}{\rho\, z_2^0(u)}(1 - e^{-s\,\rho\, z_2^0(0)})$. Consequently, setting

$$v_3 = -\frac{\sigma}{\rho} \int_{-\infty}^0 e^u \frac{1}{z_2^0(u)^2} [1 - e^{-\rho\, z_2^0(u)} - \rho\, z_2^0(u)\, e^{-\rho\, z_2^0(u)}] \circ da_{12}(u),$$

we arrive at the equations

$$C_0^1 = \begin{bmatrix} 0 & 0 \\ -v_3 & 0 \end{bmatrix}, \quad \Phi^1 = e^{-C_0^1} = \begin{bmatrix} 1 & 0 \\ v_3 & 1 \end{bmatrix}.$$

So the conjugation diffeomorphism is given by

$$\Phi = \Phi^1 \circ \Phi^0 = \begin{bmatrix} 1 & 0 \\ v_2 + v_3 & v_1 \end{bmatrix}.$$

Let

$$h(c, y) = e^{-c^1} \circ e^{-c^0} g_0(e^{c^0} \circ e^{c^1} y) + C^1 y + e^{-c^1} c^0 e^{c^1} y,$$

$c = (c^0, c^1) \in (\mathbf{R}^{2\times 2})^2$, $y \in \mathbf{R}^2$. To prove that for small enough coupling constants σ, ρ the system has a global random attractor, our main task will consist in verifying the conditions of Theorem 4.3. Now we know from [Imk98] that the Lyapunov function is most easily given in the *Lienard coordinates*. We denote them by the symbol x. The transformations are given by

$$t(y) = \begin{bmatrix} y_1 \\ y_2 - \beta\, y_1 + \frac{1}{3} y_1^3 \end{bmatrix}, \quad t^{-1}(x) = \begin{bmatrix} x_1 \\ x_2 + \beta\, x_1 - \frac{1}{3} x_1^3 \end{bmatrix},$$

$x, y \in \mathbf{R}^2$. The Lyapunov function in the Lienard coordinates is then given by the formula

$$V(x) = \frac{7}{24} x_1^4 + \frac{1}{2} x_1^2 + \frac{1}{4} x_2^2 + \frac{1}{2} (x_1 - x_2)^2,$$

$x \in \mathbf{R}^2$, the Lyapunov function in the $y-$coordinates by

$$U(y) = V(t(y)), \quad y \in \mathbf{R}^2.$$

First note that due to the integrability properties of the Ornstein-Uhlenbeck processes z_1^0, z_2^0, and the fact that the quadratic variation of the area process a_{12} is bounded by

$$\int_0^t [(z_1^0(s))^2 + (z_2^0(s))^2] ds, \quad t \geq 0,$$

we have

$$v_1, \frac{1}{v_1}, v_2, v_3 \in L^p \quad \text{for any} \quad p \geq 1. \tag{54}$$

This in particular implies that Φ^0, Φ^1 are tempered. As another consequence of (54) and the fact that g_0 has polynomial components, we can note that (47) is valid.

For the remainder of the hypotheses of Theorem 4.3, we may concentrate on treating the nonlinear part of $g_0 - h$, which comes from the nonlinear part of g_0. The arguments for the linear part are simple (see [Imk98]. Recall first the formula for g_0 in Lienard coordinates. We have

$$f_0(x) = Dt(t^{-1}(x))\, g_0(t^{-1}(x)) = \begin{bmatrix} x_2 + \beta\, x_1 - \frac{1}{3} x_1^3 \\ -x_1 - x_1^3 \end{bmatrix}.$$

Let us denote its nonlinear part by

$$n(x) = \begin{bmatrix} -\frac{1}{3}x_1^3 \\ -x_1^3 \end{bmatrix},$$

$x \in \mathbf{R}^2$. Then we have on $\mathrm{supp}(P_C)$, writing

$$e^{-c^0} = \begin{bmatrix} 1 & 0 \\ u_2 & u_1 \end{bmatrix}, \quad e^{-c^1} = \begin{bmatrix} 1 & 0 \\ u_3 & 1 \end{bmatrix},$$

$$n(x) - e^{-c^1} \circ e^{-c^0} n(e^{c^0} \circ e^{c^1} x) = \begin{bmatrix} 0 \\ [1 - \frac{1}{3}(u_2 + u_3) - u_1]\, x_1^3 \end{bmatrix}.$$

This in turn implies the equation

$$\langle \nabla_x V(x), n(x) - e^{-c^1} \circ e^{-c^0} n(e^{c^0} \circ e^{c^1} x)\rangle = (\frac{3}{2}x_2\, x_1^3 - x_1^4)\,[1 - \frac{1}{3}(u_2 + u_3) - u_1].$$

We next use the simple estimate

$$|x_2 x_1^3| \le \frac{1}{2}(x_1^6 + x_2^2) = \kappa(x), \quad x \in \mathbf{R}^2,$$

and remember (see [Imk98]) that κ and $|\langle \nabla_x V, f_0\rangle|$ are asymptotically equivalent, define k on the above mentioned linear subspace of $(\mathbf{R}^{2\times 2})^2$ by

$$k(c^0, c^1) = [1 - \frac{1}{3}(u_2 + u_3) - u_1]$$

and by 0 outside, to arrive at the estimate

$$\limsup_{|x|\to\infty} \sup_{c \in \mathrm{supp}(P_C)} \frac{|\langle \nabla \ln V(x), n(x) - e^{-c^1} \circ e^{-c^0} n(e^{c^0} \circ e^{c^1} x)\rangle|}{|\langle \nabla \ln V(x), f_0(x)\rangle|\, k(c)} \le \gamma, \tag{55}$$

for some constant γ, which may be normalized by multiplying it to k. This gives (51) for the nonlinear part. It remains to verify (48) and (52) for k. Now (48) is a consequence of the standard martingale inequalities and the integrability properties (54). Next, note that as $c \to 0$ on the linear subspace on which k does not vanish, we have $u_2, u_3 \to 0$, whereas $u_1 \to 1$, hence $k(c) \to 0$. But the random variables v_i have just this asymptotic behavior as $\rho \to 0$. Indeed, $v_1 \to 1$, and both $v_2, v_3 \to 0$ as $\rho \to 0$, for any $\sigma \neq 0$. So dominated convergence implies that, if $\sigma \neq 0$ is fixed, by choosing ρ small enough, we may get $E_{P_C}(k) < \alpha$. Hence our system possesses a global random attractor for any $\sigma \neq 0$, and small enough ρ.

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