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A large-deviation result for the range of random walk and for the Wiener sausage

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Abstract. Let $\{S_n\}$ be a random walk on $\mathbb{Z}^d$ and let R_n be the number of different points among $\mathbf{0}, S_1, \dots, S_{n-1}$. We prove here that if $d \geq 2$, then $\psi(x) := \lim_{n\to\infty}(-1/n)\log P\{R_n \geq nx\}$ exists for $x \geq 0$ and establish some convexity and monotonicity properties of $\psi(x)$. The one-dimensional case will be treated in a separate paper.

We also prove a similar result for the Wiener sausage (with drift). Let $B(t)$ be a d-dimensional Brownian motion with constant drift, and for a bounded set $A \subset \mathbb{R}^d$ let $\Lambda_t = \Lambda_t(A)$ be the d-dimensional Lebesgue measure of the 'sausage' $\bigcup_{0\leq s\leq t}(B(s)+A)$. Then $\phi(x) := \lim_{t\to\infty}(-1/t)\log P\{\Lambda_t \geq tx\}$ exists for $x \geq 0$ and has similar properties as ψ.

1. Introduction

Let $X, X_1, X_2, \dots$ be i.i.d. $\mathbb{Z}^d$-valued random variables such that $P\{X = \mathbf{0}\} < 1$. Let $S_0 = \mathbf{0}$, $S_k = \sum_{i=1}^k X_i$ and let $|A|$ denote the cardinality of the set A. The *range (at time n)* of the random walk $S = \{S_k\}$ is

$$\begin{aligned} R_n &= \big|\{\mathbf{0}, S_1, \dots, S_{n-1}\}\big| \\ &= \text{number of different points among } \mathbf{0}, S_1, \dots, S_{n-1}. \end{aligned} \tag{1.1}$$

(Note that in this definition we take the last point to be S_{n-1} rather than S_n; this gives a somewhat more convenient subadditivity relation for the range.) It was first shown by Spitzer in [S3], pp. 38-40 that

$$\frac{R_n}{n} \to \pi := P\{S_n \neq \mathbf{0} \text{ for all } n \geq 1\} \text{ a.s.} \tag{1.2}$$

(cf. [S2], [De] for later references and improvements). Moreover, since

$$\begin{aligned} R_{n+m} &\leq R_n + \big|\{S_n, S_{n+1}, \dots, S_{n+m-1}\}\big| \\ &= R_n + \big|\{\mathbf{0}, S_{n+1}-S_n, S_{n+2}-S_n, \dots, S_{n+m-1}-S_n\}\big|, \end{aligned} \tag{1.3}$$

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one easily sees that

$$P\{R_{n+m} \le (n+m)x\} \ge P\{R_n \le nx\}P\{R_m \le mx\} \tag{1.4}$$

for $x \ge 0, n, m \ge 1$. It follows from this subadditivity relation (see [PS], problem I.98) that

$$\zeta(x) := \lim_{n\to\infty} \frac{-1}{n} \log P\{R_n \le nx\} \text{ exists.} \tag{1.5}$$

Of course, it follows from (1.2) that $\zeta(x) = 0$ for $x > \pi$. It came as a bit of a surprise when Donsker and Varadhan (see [DV]) proved that for random walks in the domain of normal attraction of a symmetric stable law of index α, and $\lambda \ge 0$,

$$\lim_{n\to\infty} n^{-d/(d+\alpha)} \log Ee^{-\lambda R_n} \text{ exists and is finite.}$$

This shows that the main contributions to $E \exp[-\lambda R_n]$ do not come from values for R_n of order n. Perhaps $P\{R_n \le nx\}$ does not decrease exponentially in n for any $x > 0$. This can indeed be proven by easy lower bounds on $P\{|S_i| \le n^\beta$ for $1 \le i \le n\}$ for $\beta \le 1/d$. Thus, for many random walks $\zeta(x) = 0$ for all $x > 0$. It turns out that a different normalization for $\log P\{R_n \le nx\}$ should be used. Indeed, van den Berg, Bolthausen, den Hollander ([BBH]) recently evaluated the limit

$$I(x) := -\lim_{t\to\infty} t^{-(d-2)/d} \log P\{\Lambda_t \le tx\}, \tag{1.6}$$

where Λ_t denotes the volume of the Wiener sausage, which will be defined more precisely in (1.18) below. They gave a variational characterization of $I(x)$ and found some peculiar dimension dependence for the associated variational problem. One can expect that a similar situation prevails for

$$\widetilde{\zeta}(x) := -\lim_{n\to\infty} n^{-(d-2)/d} \log P\{R_n \le nx\}, \tag{1.7}$$

if the random walk $\{S_k\}$ has mean zero and bounded variance, and $d \ge 3$.

In this paper we consider large deviations for R_n in the *upwards* direction, that is, we study

$$\psi(x) := \lim_{n\to\infty} \frac{-1}{n} \log P\{R_n \ge nx\}. \tag{1.8}$$

Since there is no obvious analogue of (1.4) when $P\{R_n \le nx\}$ is replaced by $P\{R_n \ge nx\}$, it is not clear that the limit in (1.8) exists for $x \ge \pi$. Our first theorem shows that this is indeed the case for essentially all random walks. Throughout we assume that X has a genuinely d-dimensional distribution and that the corresponding random walk is aperiodic, that is, we assume that

$$\text{the group generated by the support of } X \text{ is all of } \mathbb{Z}^d \tag{1.9}$$

(see Section 2 for some discussion of this assumption).

Theorem 1. *Let* S_n, R_n *and* π *be as above and assume that* (1.9) *holds. Then*

$$\psi(x) = \lim_{n\to\infty} \frac{-1}{n} \log P\{R_n \geq nx\} \textit{ exists} \tag{1.10}$$

for all x *(but* $\psi(x)$ *may equal* $+\infty$*).* $\psi(\cdot)$ *has the following properties:*

$$\psi(x) = 0 \textit{ for } x \leq \pi, \tag{1.11}$$

$$0 < \psi(x) < \infty \textit{ for } \pi < x \leq 1, \tag{1.12}$$

$$\psi(x) = \infty \textit{ for } x > 1, \tag{1.13}$$

$$\begin{gathered} x \mapsto \psi(x) \textit{ is continuous on } [0,1], \textit{ and in case } d \geq 2, \\ x \mapsto \psi(x) \textit{ is also convex on } [0,1], \end{gathered} \tag{1.14}$$

and

$$x \mapsto \psi(x) \textit{ is strictly increasing on } [\pi, 1]. \tag{1.15}$$

Remark 1. Our proof also shows convexity of ψ if $d = 1$ and $|X|$ does *not* have an exponentially bounded tail. ψ is also convex when $d = 1$ and $P\{X > 0\}P\{X < 0\} = 0$, but we were unable to prove convexity for all one-dimensional cases. The proof of Theorem 1 for $d = 1$ will be given in a separate paper, though.

The following is a straightforward consequence of Theorem 1. It gives a partial large deviation principle for the range of random walks. The proof of this corollary is given at the end of Section 2.

Corollary 1. *Let* μ_n *be the probability distribution of the random variable* R_n/n. *In the set-up of Theorem 1, we have that*

$$\limsup_{n\to\infty} \frac{1}{n} \log \mu_n(F) \leq -\inf_{x\in F} \psi(x) \tag{1.16}$$

for each closed subset $F \subset [\pi, \infty)$ *and that*

$$\liminf_{n\to\infty} \frac{1}{n} \log \mu_n(G) \geq -\inf_{x\in G} \psi(x) \tag{1.17}$$

for each open subset $G \subset [\pi, \infty)$.

One can consider analogous problems for the cardinality of the set $\{A, S_1 + A, \dots, S_n + A\}$ when A is a fixed finite susbset of $\mathbb{Z}^d$ (compare [S3], Problem 27.14). We shall not do this, but instead shall go directly to continuous space and time and consider the Wiener sausage. Let $\{B(t)\}$ be a d-dimensional Brownian motion with $B(0) = \mathbf{0}$. Contrary to the usual convention we allow the Brownian motion to have a constant drift $\mu \neq \mathbf{0}$. Further, let $A \subset \mathbb{R}^d$ be a bounded, Lebesgue measurable, nonempty set. The set $\bigcup_{0\leq s\leq t}(B(s) + A)$ is called the *Wiener sausage* (associated with A). Its volume is

$$\Lambda_t = \Lambda_t(A) := \lambda_d\Bigg(\bigcup_{0\leq s\leq t} (B(s) + A)\Bigg), \tag{1.18}$$

where $\lambda_d(\cdot)$ denotes the d-dimensional Lebesgue measure. The analogue of (1.2) is now that

$$\theta := \lim_{t\to\infty} \frac{\Lambda_t(A)}{t} \text{ exists a.s.} \tag{1.19}$$

(cf. [S2], [IM, Problem 7.8.4]). When $\mu = \mathbf{0}$

$$\lim_{t\to\infty} \frac{\Lambda_t(A)}{t} = \begin{cases} \text{cap}(A) & \text{if } d \geq 3 \\ 0 & \text{if } d = 1 \text{ or } 2 \end{cases} \tag{1.20}$$

a.s., where

$$\text{cap}(A) = \begin{cases} d\text{-dimensional Newtonian capacity of } A & \text{if } d \geq 3 \\ \text{logarithmic capacity of } A & \text{if } d = 2. \end{cases} \tag{1.21}$$

It can also be shown that θ in (1.19) is strictly positive and finite for either $d \geq 3$, cap $(A) > 0$, or $d = 2$, cap $(A) > 0$, $\mu \neq \mathbf{0}$, or $d = 1$, $A \neq \emptyset, \mu \neq 0$. (We are grateful to an anonymous referee for this last remark.)

Λ_t in the result of [BBH] ((1.6) above) stands for $\Lambda_t(A)$ with A a ball of radius a, and for this choice of A, (1.6) supplements (1.20). We have, however, no nice expression for θ for general drift. Some related results for the case $\mu \neq \mathbf{0}$ are in [EL].

Here we again consider deviations from (1.19) in the upwards direction. Some control of the probabilities $P\{\Lambda_t \geq tx\}$ is provided by [BB] and [BT]. These papers show the existence of

$$\rho(\lambda) := \lim_{t\to\infty} \frac{1}{t} \log E e^{\lambda\Lambda_t} \quad \text{for} \quad \lambda \geq 0, \tag{1.22}$$

again under the assumptions that A is a ball and $\mu = \mathbf{0}$. They further derive some estimates for $\rho(\lambda)$ as $\lambda \downarrow 0$ or $\lambda \uparrow \infty$. We shall indicate here a proof of the following analogue of Theorem 1:

Theorem 2. *If A is a bounded, Lebesgue measurable, nonempty set in $\mathbb{R}^d$, then for any constant (possibly zero) drift μ*

$$\phi(x) := \lim_{t\to\infty} \frac{-1}{t} \log P\{\Lambda_t(A) \geq tx\} \text{ exists} \tag{1.23}$$

for all $x \in \mathbb{R}$. $\phi(\cdot)$ satisfies

$$\phi(x) = 0 \quad \textit{for} \quad x \leq \theta, \tag{1.24}$$

where θ is defined by (1.19). *If $d = 1$, then*

$$\phi(x) = \begin{cases} \frac{x^2}{2} & \textit{if } \mu = 0,\ x \geq 0 \\ 0 & \textit{if } 0 \leq x \leq |\mu| \\ \frac{(x-|\mu|)^2}{2} & \textit{if } 0 < |\mu| \leq x. \end{cases} \tag{1.25}$$

If $d \geq 2$ *and cap*$(A) > 0$, *then* ϕ *has the following properties:*

$$0 < \phi(x) < \infty \text{ for } x > \theta, \tag{1.26}$$

$$x \mapsto \phi(x) \text{ is convex and continuous on } [0, \infty), \tag{1.27}$$

and

$$x \mapsto \phi(x) \text{ is strictly increasing on } [\theta, \infty). \tag{1.28}$$

Remark 2. The case $d \geq 2$ with cap(A)= 0 is not interesting, because in that case

$$P\{B(t) \in A \text{ for some } t > 0\} = 0.$$

For a Brownian motion without drift this was proven by Kakutani in [Ka] for $d = 2$, and the higher dimensional case seems to have been treated first in [Do1]; see also Section 2.IX.5 in [Do2]. This statement remains valid even if the drift is nonzero, because on each finite time interval Brownian motion with drift and without drift are absolutely continuous with respect to each other (see [KS, Section 3.5, especially 3.5C]). But then

$$\begin{aligned} E\Lambda_t(A) &= \int_{\mathbb{R}^d} P\Big\{y \in \bigcup_{s \leq t}(B(s) + A)\Big\} dy \\ &= \int_{\mathbb{R}^d} P\{B(s) \in y - A \text{ for some } s \leq t\} dy = 0. \end{aligned}$$

Thus if cap(A) = 0, then $P\{\Lambda_t(A) = 0\} = 1$ for each fixed t, and then by the monotonicity of $t \mapsto \Lambda_t$, $P\{\Lambda_t(A) = 0 \text{ for all } t\} = 1$.

The following corollary is the analogue of Corollary 1 for the Wiener sausage. Its proof is entirely analogous to that of Corollary 1 and will be left to the reader.

Corollary 2. *Let* A *be a bounded, Lebesgue measureable, nonempty set in* $\mathbb{R}$ *and let* ν_t *be the distribution of* $\Lambda_t(A)/t$. *If* $d \geq 2$ *assume that cap*$(A) > 0$. *The constant* θ *is the same as in* (1.19). *Then we have that*

$$\limsup_{t \to \infty} \frac{1}{t} \log \nu_t(F) \leq - \inf_{x \in F} \phi(x)$$

for each closed set $F \subset [\theta, \infty)$ *and that*

$$\liminf_{t \to \infty} \frac{1}{t} \log \nu_t(G) \geq - \inf_{x \in G} \phi(x)$$

for each open set $G \subset [\theta, \infty)$.

The principal **open question** now is to find a manageable expression or characterization for $\psi(x)$ and $\phi(x)$.

1.1. Outline of the proof of (1.10) *for* d ≥ 2

As we already stated in Remark 1, we only prove Theorem 1 for $d \geq 2$ in this paper. The proof is based on an approximate subaddivity relation. We build a path of length $n+m$ with $R_{n+m} \geq y+z-E(n,m)$ for some error term $E(n,m)$ from two paths, $\mathscr{P}_1$ and $\mathscr{P}_2$, say. $\mathscr{P}_1(\mathscr{P}_2)$ has length $n\,(m)$ and range greater than or equal to $y\,(z$, respectively). The error term comes from the fact that some points are counted in the range of both $\mathscr{P}_1$ and $\mathscr{P}_2$. In order to make this overlap small we do not put the initial point of $\mathscr{P}_2$ at the endpoint of $\mathscr{P}_1$, but at some nearby point. We show that we can place the initial point of $\mathscr{P}_2$ at a distance at most of order $(nm)^{1/(d+1)}$ from the endpoint of $\mathscr{P}_1$ so as to get an overlap of order $(nm)^{1/(d+1)}$. The two paths are then connected at not too large a cost in probability. The result is the relation

$$\begin{aligned}P\{R_{n+m} &\geq y+z-(2d+2)(nm)^{1/(d+1)}\}\\ &\geq \tfrac{1}{2}\zeta^{d(nm)^{1/(d+1)}+d}P\{R_n \geq y\}P\{R_m \geq z\}.\end{aligned} \tag{1.29}$$

for some $\zeta > 0$. (The idea of putting the initial point of $\mathscr{P}_2$ at a point which is near, but not necessarily at, the endpoint of $\mathscr{P}_1$, was also used in [Ke] for estimating numbers of self-avoiding walks. It is of course not surprising that tools for self-avoiding walks are useful here, since the event $\{R_n \geq n\}$ is just the event that the initial piece $S_0, \dots, S_{n-1}$ of the random walk is selfavoiding.)

When $d \geq 2$, then $(nm)^{1/(d+1)}$ is small with respect to $(n \vee m)$. From this one deduces by more or less standard subadditivity arguments that $\lim_{n\to\infty}(-1/n)\log P\{R_n \geq nx\}$ exists at all continuity points $x \in (0,1)$ of $\psi(x) := \liminf_{n\to\infty}(-1/n)\log P\{R_n \geq nx\}$. It is then easy to obtain from (1.29) that the restriction of $\lim_{n\to\infty}(-1/n)\log P\{R_n \geq nx\}$ to the continuity points of ψ in $(0,1)$ is convex (see proof of Lemma 3 for the precise meaning of this statement). This is enough to conclude that ψ is in fact continuous on $(0,1)$, just as in the usual proof of continuity of a convex function. Hence $\lim_{n\to\infty}(-1/n)\log P\{R_n \geq nx\}$ exists for all $x \in (0,1)$. The existence of this limit for $x \notin (0,1)$ is easily shown directly.

We remark that it is also possible to prove Theorem 1 by the methods of Lemmas 3 and 6 of [HK], which are used there for the one-dimensional case. For the higher dimensional case the proof of this paper is more direct.

2. A subadditivity argument; proof of Theorem 1 for $d \geq 2$

Throughout this section $X, X_1, X_2, \dots$ are i.i.d. $\mathbb{Z}^d$-valued random variables and S_n, R_n are as in Section 1.

Before we begin any proofs we point out that assumption (1.9) does not entail any loss of generality. Indeed we can always change coordinates so that (1.9) holds, by Proposition 7.1 in [S3]. More specifically, if $\mathscr{G}$ denotes the subgroup of $\mathbb{Z}^d$ generated by the support of X, then there exists a $d' \leq d$ and d' linearly independent vectors $v_1, v_2, \dots, v_{d'} \in \mathscr{G}$ such that the group generated by $v_1, v_2, \dots, v_{d'}$ is equal to $\mathscr{G}$, and is isomorphic to $\mathbb{Z}^{d'}$. In particular, there exist unique random variables

$X' = (X'(1), \dots, X'(d')),\ X'_j = (X'_j(1), \dots, X'_j(d')) \in \mathbb{Z}^{d'}$ such that

$$X = \sum_{\ell=1}^{d'} X'(\ell) v_\ell, \quad X_j = \sum_{\ell=1}^{d'} X'_j(\ell) v_\ell,$$

and the group generated by the support of X' equals $\mathbb{Z}^{d'}$ Accordingly, we can view $\{S_n\}$ as an aperiodic random walk on $\mathbb{Z}^{d'}$. From now on we therefore drop the primes from our notation and assume that the problem has been set up from the beginning so that (1.9) holds.

We begin with a fundamental subadditivity relation.

Lemma 1. *If* (1.9) *holds, then there exists a constant* $\zeta \in (0, 1)$ *such that for all integers* $n, m \geq 0$ *and* $y, z \in [0, \infty)$, *it holds that*

$$\begin{aligned} &P\{R_{n+m} \geq y + z - (2d+2)(nm)^{1/(d+1)}\} \\ &\quad \geq \frac{1}{2} \zeta^{d(nm)^{1/(d+1)}+d} P\{R_n \geq y\} P\{R_m \geq z\}. \end{aligned} \tag{2.1}$$

Proof. We introduce a number of quantities. The relevance of these quantities will become clear in a little while. We let $\widehat{X}_1, \widehat{X}_2, \dots$ be an independent copy of $X_1, X_2, \dots$. In analogy with our previous notation we define $\widehat{S} = \{\widehat{S}_n\}_{n\geq 0}$ by $\widehat{S}_0 = \mathbf{0}$, $\widehat{S}_n = \sum_{i=1}^{n} \widehat{X}_i$. We define

$$R[a, b] = |\{S_a, \dots, S_b\}| \text{ and } \widehat{R}[a, b] = |\{\widehat{S}_a, \dots, \widehat{S}_b\}|.$$

Note that $R_n = R[0, n-1]$ in this notation. Next we define for $w \in \mathbb{Z}^d$

$$\begin{aligned} N_{n,m}(w) &= N_{n,m}(w, S, \widehat{S}) \\ &= \Big|\Big\{u \in \mathbb{Z}^d : u \in \{S_0, S_1, \dots, S_{n-1}\} \\ &\qquad\qquad \text{and} \quad u \in S_n + w + \{\widehat{S}_0, \widehat{S}_1, \dots, \widehat{S}_{m-1}\}\Big\}\Big|. \end{aligned}$$

Thus $N_{n,m}(w)$ counts the number of points which are visited during the time interval $[0, n-1]$ by the walk S and also visited during $[0, m-1]$ by the walk $\widehat{S}$ shifted by $S_n + w$.

For any fixed integers $p \geq 0, n \geq 0$, consider the random walk defined by

$$T_k = T_k^{(p,n)} = \begin{cases} S_k & \text{for } k \leq n + p \\ S_{n+p} + \widehat{S}_{k-n-p} & \text{for } k > n + p. \end{cases}$$

Of course, $\{T_k\}_{k\geq 0}$ has the same distribution as $\{S_k\}_{k\geq 0}$, and hence also

$$\begin{aligned} P\{R_{n+p+m} \geq \ell\} &= P\{R[0, n+p+m-1] \geq \ell\} \\ &= P\{|\{T_0, \dots, T_{n+p+m-1}\}| \geq \ell\}. \end{aligned} \tag{2.2}$$

We claim that on the event

$$\{S_{n+p} - S_n = w\}, \tag{2.3}$$

it holds that

$$|\{T_0, \dots, T_{n+p+m-1}\}| \geq R_n + \widehat{R}_m - N_{n,m}(w). \tag{2.4}$$

This is immediate from the fact that $R_n = |\{T_0, \dots, T_{n-1}\}|$, and $\widehat{R}_m = |\{T_{n+p}, \dots, T_{n+p+m-1}\}|$, and on the event (2.3),

$$\begin{aligned} N_{n,m}(w) &= N_{n,m}(T_{n+p} - T_n) \\ &= \left|\{T_0, \dots, T_{n-1}\} \cap \{T_{n+p}, \dots, T_{n+p+m-1}\}\right|. \end{aligned}$$

At this stage we remind the reader of (1.9). This allows us to pick d linearly independent vectors $v_1, \dots, v_d \in \mathbb{Z}^d$ for which $P\{X = v_i\} > 0$. We can then choose $0 < \zeta < 1$ such that $P\{X = v_i\} \geq \zeta$ for $i = 1, \dots, d$. We set

$$\Xi_q = \left\{\sum_{i=1}^d k_i v_i : 0 \leq k_i \leq q\right\} \subset \mathbb{Z}^d.$$

For any $w = \sum_{i=1}^d k_i v_i \in \Xi_q$, we then have for $p = p(w) = \sum_{i=1}^d k_i \leq dq$ that

$$P\{S_{n+p} - S_n = w\} = P\{S_p = w\} \geq \zeta^p \geq \zeta^{dq}. \tag{2.5}$$

Moreover,

$$|\Xi_q| = (\text{number of vectors } w \in \Xi_q) = (q+1)^d.$$

We take

$$q = q(n,m) = \lceil (nm)^{1/(d+1)} \rceil, \tag{2.6}$$

where $\lceil a \rceil$ denotes the smallest integer $\geq a$. As a result of (2.2) and (2.4) we have for each $w \in \Xi_q$,

$$\begin{aligned} &P\{R_{n+dq+m} \geq y + z - 2(nm)^{1/(d+1)}\} \\ &\quad \geq P\{R_{n+p(w)+m} \geq y + z - 2(nm)^{1/(d+1)}\} \\ &\quad \geq P\{R_n \geq y, \widehat{R}_m \geq z, S_{n+p(w)} - S_n = w, N_{n,m}(w) \leq 2(nm)^{1/(d+1)}\}. \end{aligned} \tag{2.7}$$

The event (2.3) depends only on the X_i with $n < i \leq n + p$, and is therefore independent of the events $\{R_n \geq y\}$, $\{\widehat{R}_m \geq z\}$ and of the random variable $N_{n,m}(w)$ (for fixed w). Consequently,

$$\begin{aligned} &P\{R_{n+dq+m} \geq y + z - 2(nm)^{1/(d+1)}\} \\ &\quad \geq P\{S_{n+p} - S_n = w\} P\{R_n \geq y, \ \widehat{R}_m \geq z, N_{n,m}(w) \leq 2(nm)^{1/(d+1)}\} \\ &\quad \geq \zeta^{dq} P\{R_n \geq y, \ \widehat{R}_m \geq z, \ N_{n,m}(w) \leq 2(nm)^{1/(d+1)}\}. \end{aligned} \tag{2.8}$$

Since this inequality holds for all $w \in \Xi_q$, we can take its average over Ξ_q to obtain

$$\begin{aligned} &P\{R_{n+dq+m} \ge y + z - 2(nm)^{1/(d+1)}\} \\ &\quad \ge \frac{\zeta^{dq}}{|\Xi_q|} \sum_{w \in \Xi_q} P\{N_{n,m}(w) \le 2(nm)^{1/(d+1)},\ R_n \ge y,\ \widehat{R}_m \ge z\} \\ &\quad \ge \frac{\zeta^{dq}}{|\Xi_q|} E\Big\{ \Big| \{w \in \Xi_q : N_{n,m}(w) \le 2(nm)^{1/(d+1)}\} \Big| \, I[R_n \ge y] I[\widehat{R}_m \ge z] \Big\}. \end{aligned} \tag{2.9}$$

We shall soon show that always

$$\Big| \{w \in \Xi_q \ :\ N_{n,m}(w) \le 2(nm)^{1/(d+1)}\} \Big| \ge \frac{1}{2}(q+1)^d. \tag{2.10}$$

Before we do this we show that this will complete the the proof of the lemma. Indeed, $\{R_n \ge y\}$ and $\{\widehat{R}_m \ge z\}$ are independent because they depend only on the X_i and $\widehat{X}_j$, respectively. Their respective probabilities equal $P\{R_n \ge y\}$ and $P\{R_m \ge z\}$. Thus, if (2.10) is true, then (2.9) yields

$$\begin{aligned} &P\{R_{n+dq+m} \ge y + z - 2(nm)^{1/(d+1)}\} \\ &\quad \ge \frac{\zeta^{dq}}{(q+1)^d} \frac{1}{2}(q+1)^d P\{R_n \ge y\} P\{R_m \ge z\}. \end{aligned} \tag{2.11}$$

Finally, by removing the last dq steps from $S_0 = \mathbf{0}, S_1, \dots, S_{n+dq+m}$, we see that $\{R_{n+dq+m} \ge \ell\}$ implies $\{R_{n+m} \ge \ell - dq\}$. Thus, (2.11) shows that

$$\begin{aligned} &P\{R_{n+m} \ge y + z - 2(nm)^{1/(d+1)} - d\lceil (nm)^{1/(d+1)} \rceil\} \\ &\quad \ge \frac{1}{2}\zeta^{dq} P\{R_n \ge y\} P\{R_m \ge z\}, \end{aligned} \tag{2.12}$$

which will indeed prove the lemma.

We conclude with the promised proof of (2.10). We have

$$\begin{aligned} \sum_{w \in \Xi_q} N_{n,m}(w) &\le \sum_{w \in \mathbb{Z}^d} N_{n,m}(w) \\ &= \sum_{u \in \mathbb{Z}^d} \sum_{w \in \mathbb{Z}^d} I[u \in \{S_0, S_1, \dots, S_{n-1}\}] \\ &\qquad\qquad \times I[u \in S_n + w + \{\widehat{S}_0, \widehat{S}_1, \dots, \widehat{S}_{m-1}\}] \\ &= \sum_{u \in \mathbb{Z}^d} I[u \in \{S_0, S_1, \dots, S_{n-1}\}] \\ &\qquad\qquad \times \Big| \{u - S_n - \widehat{S}_0, u - S_n - \widehat{S}_1, \dots, u - S_n - \widehat{S}_{m-1}\} \Big| \\ &= \sum_{u \in \mathbb{Z}^d} I[u \in \{S_0, S_1, \dots, S_{n-1}\}] \widehat{R}_m \\ &= R_n \widehat{R}_m \le nm. \end{aligned}$$

It follows that

$$\begin{aligned}&\left|\{w \in \Xi_q : N_{n,m}(w) > 2(nm)^{1/(d+1)}\}\right| \\ &\quad \le \frac{nm}{2(nm)^{1/(d+1)}} = \frac{1}{2}(nm)^{d/(d+1)} \le \frac{1}{2}|\Xi_q|,\end{aligned}$$

and hence

$$\left|\{w \in \Xi_q : N_{n,m}(w) \le 2(nm)^{1/(d+1)}\}\right| \ge \frac{1}{2}|\Xi_q|,$$

as desired in (2.10). □

Remark 3. As stated, Lemma 1 is not useful when $d = 1$; the error factor $\zeta^{(nm)^{1/(d+1)}}$ becomes too dominant in this case. However, when $d = 1$ and $\limsup_{k\to\infty}[P\{|X| \ge k\}]^{1/k} = 1$ we can still derive an estimate similar to (2.1), by taking a different set Ξ_q in the proof of Lemma 1. This replacement for (2.1) will not be enough by itself to deduce (1.10), but it will give us an upper bound on $\limsup_{n\to\infty}(-1/n)\log P\{R_n \ge nx\}$.

For $x \in \mathbb{R}$ we now define

$$\psi(x) = \liminf_{n\to\infty} \frac{-1}{n} \log P\{R_n \ge nx\}. \tag{2.13}$$

Observe that $\psi(x)$ is nondecreasing in x. Moreover, it is bounded on $[0, 1]$ because

$$P\{R_n \ge n\} = P\{S_0, \dots, S_{n-1} \text{ are distinct}\} \ge [P\{X(i) > 0\}]^n, \tag{2.14}$$

where $X(i)$ denotes the i-th component of X. Hence, for all $1 \le i \le d$,

$$\psi(1) \le -\log P\{X(i) > 0\}. \tag{2.15}$$

If $P\{X(i) > 0\} = 0$ for all $1 \le i \le d$, then we can replace $X(i) > 0$ by $X(i) < 0$ in this estimate. This always gives a finite upper bound for $\psi(x)$.

We have to prove for (1.10) that the liminf in (2.13) can be replaced by lim. We first show that this is permissible for any $x \in [0, 1)$ at which ψ is continuous from the right.

Proposition 2. *If* (1.9) *holds and* $d \ge 2$ *and if* ψ *is right continuous at a given* $x \in [0, 1)$*, then*

$$\psi(x) = \lim_{n\to\infty} \frac{-1}{n} \log P\{R_n \ge nx\}. \tag{2.16}$$

Proof. The proof here will be based only on (2.11) and the fact that $P\{R_n \ge z\}$ is increasing in n and decreasing in z. It would be somewhat simpler to use (2.1), but a proof which only uses (2.11) has the advantage that it can also be used in the next section for the Wiener sausage.

For simplicity we write $\eta = (d-1)/(d+1)$ and $\xi = 2/(d+1)$. To start with, we define for any integer $N \geq 1$,

$$\sigma(0) = \sigma_N(0) = N,$$
$$\sigma(k+1) = \sigma_N(k+1) = 2\sigma(k) + d\left\lceil [\sigma(k)]^{\xi} \right\rceil, \; k \geq 0.$$

It is immediate from these definitions that

$$\frac{\sigma(i-1)}{\sigma(i)} \leq \frac{1}{2}, \quad \sigma(i) \geq 2^i N, \tag{2.17}$$

and for some constants c_1, $N_0 < \infty$ and $N \geq N_0$

$$\begin{aligned} 1 \; &\leq \frac{\sigma(k)}{2^k N} = \prod_{i=1}^{k} \frac{\sigma(i)}{2\sigma(i-1)} \\ &= \prod_{i=1}^{k} \left[1 + \frac{d\left\lceil [\sigma(i-1)]^{\xi} \right\rceil}{2\sigma(i-1)} \right] \leq \exp\left[\sum_{i=1}^{\infty} d[\sigma(i-1)]^{-\eta} \right] \\ &\leq \exp\left[\sum_{j=0}^{\infty} d(2^j N)^{-\eta} \right] \leq 1 + \frac{c_1}{N^{\eta}} \leq 2. \end{aligned} \tag{2.18}$$

Also, for $c_2 = 2^{\xi}/(1-2^{-\eta})$ and $N \geq N_0$,

$$\begin{aligned} \sum_{i=0}^{k-1} 2^{k-i} [\sigma(i)]^{\xi} &\leq \sum_{i=0}^{k-1} 2^{k-i} 2^{\xi(i+1)} N^{\xi} \\ &\leq c_2 2^k N^{\xi} \leq c_2 N^{-\eta} \sigma(k). \end{aligned} \tag{2.19}$$

Now let $x \in [0,1)$ be such that ψ is right continuous at x and let $\varepsilon > 0$. Take $\delta \in (0,1)$ such that

$$\psi(x+4\delta) \leq \psi(x) + \varepsilon. \tag{2.20}$$

Then take $c_3 = \zeta^d/2 < 1$ and fix $\ell \geq 2$ such that

$$(1 - 2^{-\ell+2})(x+2\delta) \geq x + \delta. \tag{2.21}$$

Finally, fix $N \geq N_0$ so that

$$\begin{aligned} P\{R_N \geq N(x+4\delta)\} &\geq \exp[-N(\psi(x+4\delta)+\varepsilon)] \\ &\geq \exp[-N(\psi(x)+2\varepsilon)], \end{aligned} \tag{2.22}$$

$$1 + \frac{c_1}{N^{\eta}} \leq \frac{x+4\delta}{x+3\delta}, \tag{2.23}$$

$$N^{-\eta} < \min\left\{ \frac{\delta}{c_2}, \frac{-2\varepsilon}{c_2 d \log \zeta}, \frac{1}{2d} \right\}, \tag{2.24}$$

$$2\ell(3d+2)(N^{\xi}+1) < \delta N, \tag{2.25}$$

and

$$\frac{2}{N}|\log c_3| < \varepsilon. \tag{2.26}$$

We shall first consider $P\{R_n \geq nx\}$ for $n \in \{\sigma_N(k)\}_{k\geq 0}$. (2.11) with $m = n = \sigma(k-1)$ and

$$y = z = 2^{k-1}N(x+4\delta) - \sum_{i=0}^{k-2} 2^{k-1-i}[\sigma(i)]^{\xi}$$

now gives for $k \geq 1$

$$P\left\{R_{\sigma(k)} \geq 2^k N(x+4\delta) - \sum_{i=0}^{k-1} 2^{k-i}[\sigma(i)]^{\xi}\right\}$$
$$\geq c_3\zeta^{d[\sigma(k-1)]^{\xi}}\left[P\left\{R_{\sigma(k-1)} \geq 2^{k-1}N(x+4\delta) - \sum_{i=0}^{k-2} 2^{k-1-i}[\sigma(i)]^{\xi}\right\}\right]^2. \tag{2.27}$$

We also have

$$\begin{aligned}
&2^k N(x+4\delta) - \sum_{i=0}^{k-1} 2^{k-i}[\sigma(i)]^{\xi} \\
&\geq \sigma(k)(x+3\delta) - \sum_{i=0}^{k-1} 2^{k-i}[\sigma(i)]^{\xi} \qquad \text{(by (2.18) and (2.23))} \\
&\geq \sigma(k)(x+2\delta) \qquad \text{(by (2.19) and (2.24))}.
\end{aligned}$$

Combining this with (2.27) we obtain

$$\begin{aligned}
&P\{R_{\sigma(k)} \geq \sigma(k)(x+2\delta)\} \\
&\geq P\left\{R_{\sigma(k)} \geq 2^k N(x+4\delta) - \sum_{i=0}^{k-1} 2^{k-i}[\sigma(i)]^{\xi}\right\} \\
&\geq c_3\zeta^{d[\sigma(k-1)]^{\xi}}\left[P\left\{R_{\sigma(k-1)} \geq 2^{k-1}N(x+4\delta) - \sum_{i=0}^{k-2} 2^{k-1-i}[\sigma(i)]^{\xi}\right\}\right]^2 \\
&\geq \cdots \geq \\
&\geq [c_3]^{2^{k+1}} \exp\left[d\sum_{j=1}^{k} 2^{j-1}[\sigma(k-j)]^{\xi}\log\zeta\right]\left[P\{R_{\sigma(0)} \geq N(x+4\delta)\}\right]^{2^k} \\
&\geq [c_3]^{2^{k+1}} \exp\left[c_2 d 2^{k-1}N^{\xi}\log\zeta\right]\exp\left[-2^k N(\psi(x)+2\varepsilon)\right] \\
&\text{(by (2.19) and (2.22))}
\end{aligned}$$

$$= [c_3]^{2^{k+1}} \exp\left[-2^k N\left(\psi(x) + 2\varepsilon - \frac{1}{2}c_2 d N^{-\eta}\log\zeta\right)\right]$$
$$\geq [c_3]^{2^{k+1}} \exp\left[-2^k N(\psi(x) + 3\varepsilon)\right]. \tag{2.28}$$

The last inequality here was obtained from (2.24). With the help of (2.26), the estimate (2.28) implies in particular that

$$\limsup_{k\to\infty} \frac{-1}{\sigma(k)} \log P\{R_{\sigma(k)} \geq \sigma(k)(x + 2\delta)\}$$
$$\leq \psi(x) + 3\varepsilon - \frac{2}{N}\log c_3 \leq \psi(x) + 4\varepsilon,$$

but we shall need the explicit bound of (2.28) for the next step.

In order to deal with general n, we "expand n (approximately) into a linear combination of the $\sigma_N(k)$." More precisely, recall that we fixed ℓ in (2.21). Now let $n \geq \sigma_N(2\ell)$. Take

$$\widehat{n} = n - 2d\ell\lceil n^{\xi}\rceil,$$

and choose k_i, α_i recursively in the following manner ($\sigma = \sigma_N$ again here): k_1 is determined by

$$\sigma(k_1) \leq \widehat{n} < \sigma(k_1 + 1),$$

and

$$\alpha_1 = \begin{cases} 1 & \text{if } \sigma(k_1) \leq \widehat{n} < 2\sigma(k_1) \\ 2 & \text{if } 2\sigma(k_1) \leq \widehat{n} < \sigma(k_1 + 1); \end{cases}$$

then k_r and α_r for $r \geq 2$ are determined by

$$\sigma(k_r) \leq \widehat{n} - \sum_{i=1}^{r-1} \alpha_i \sigma(k_i) < \sigma(k_r + 1) \tag{2.29}$$

and

$$\alpha_r = \begin{cases} 1 & \text{if } \sigma(k_r) \leq \widehat{n} - \sum_{i=1}^{r-1}\alpha_i\sigma(k_i) < 2\sigma(k_r) \\ 2 & \text{if } 2\sigma(k_r) \leq \widehat{n} - \sum_{i=1}^{r-1}\alpha_i\sigma(k_i) < \sigma(k_r + 1). \end{cases}$$

We only choose these k_i, α_i as long as $k_i \geq 0$. Note that $n \geq \sigma(2\ell)$ implies $k_1 \geq 2\ell - 1$. Also, by virtue of (2.24),

$$\frac{\sigma(k+1)}{\sigma(k)} \leq 2 + 2d[\sigma(k)]^{-\eta} \leq 2 + 2dN^{-\eta} < 3,$$

so that we obtain from our choice of k_r, α_r that

$$0 \leq \widehat{n} - \sum_{i=1}^{r} \alpha_i \sigma(k_i) < \sigma(k_r).$$

As a consequence $k_1 > k_2 > \cdots$. Let p be such that $k_p > k_1 - \ell \geq k_{p+1}$. By (2.29), the monotonicity of $\sigma(i)$ in i, and (2.18) we then have that

$$\begin{aligned} 0 \leq \widehat{n} - \sum_{i=1}^{p} \alpha_i \sigma(k_i) &< \sigma(k_{p+1}+1) \leq \sigma(k_1 - \ell + 1) \\ &< 2^{k_1-\ell+2} N \leq 2^{-\ell+2}\sigma(k_1) \leq 2^{-\ell+2} n. \end{aligned} \tag{2.30}$$

Of course we also have $p \leq \ell$.

We set $\beta := \sum_{i=1}^{p} \alpha_i$ and let $n_1 < n_2 < \cdots < n_\beta$ be the numbers of the form $\sum_{i=1}^{j} \alpha_i \sigma(k_i)$ or $\sum_{i=1}^{j} \alpha_i \sigma(k_i) - \sigma(k_j)$; the latter form is included only if $\alpha_j = 2$. If n_γ has the first form for some $1 \leq j \leq p-1$, then $n_{\gamma+1} = n_\gamma + \sigma(k_{j+1})$. If n_γ has the second form for some $1 \leq j \leq p-1$, then $n_{\gamma+1} = n_\gamma + \sigma(k_j)$. We now apply (2.11), with n replaced by $n_\gamma + d\gamma\lceil n^\xi \rceil$ and $m = n_{\gamma+1} - n_\gamma$. We further take $y = n_\gamma(x+2\delta) - 2\gamma n^\xi$ and $z = (n_{\gamma+1} - n_\gamma)(x+2\delta) = m(x+2\delta)$. Note that

$$\beta \leq 2p \leq 2\ell, \tag{2.31}$$

and for $\gamma \leq \beta - 1$,

$$(n_\gamma + d\gamma\lceil n^\xi \rceil)(n_{\gamma+1} - n_\gamma) \leq (\widehat{n} + d\beta\lceil n^\xi \rceil)\widehat{n} \leq n^2.$$

Using the monotonicity properties of $P\{R_r \geq z\}$ we then find

$$\begin{aligned} &P\{R_{n_{\gamma+1}+d(\gamma+1)\lceil n^\xi \rceil} \geq n_{\gamma+1}(x+2\delta) - 2(\gamma+1)n^\xi\} \\ &\geq \frac{1}{2}\zeta^{dn^\xi + d} P\{R_{n_\gamma + d\gamma\lceil n^\xi \rceil} \geq n_\gamma(x+2\delta) - 2\gamma n^\xi\} \\ &\quad \times P\{R_{n_{\gamma+1}-n_\gamma} \geq (n_{\gamma+1} - n_\gamma)(x+2\delta)\}. \end{aligned}$$

We use this inequality successively for $\gamma = \beta-1, \beta-2, \ldots, 1$ (with $n_0 = 0$) to obtain finally

$$\begin{aligned} &P\{R_{n_\beta + d\beta\lceil n^\xi \rceil} \geq n_\beta(x+2\delta) - 2\beta n^\xi\} \\ &\geq [c_3]^\beta \zeta^{\beta d n^\xi} \prod_{\gamma=0}^{\beta-1} P\{R_{n_{\gamma+1}-n_\gamma} \geq (n_{\gamma+1} - n_\gamma)(x+2\delta)\} \\ &\geq [c_3]^{2\ell} \zeta^{2\ell d n^\xi} \prod_{j=1}^{p} \left[P\{R_{\sigma(k_j)} \geq \sigma(k_j)(x+2\delta)\}\right]^{\alpha_j} \\ &\quad \text{(by (2.31) and the definition of } n_\gamma) \\ &\geq [c_3]^{2\ell + \sum_{j=1}^{p} \alpha_j 2^{k_j+1}} \zeta^{2\ell d n^\xi} \exp\Big[-\sum_{j=1}^{p} \alpha_j 2^{k_j} N(\psi(x) + 3\varepsilon)\Big] \\ &\quad \text{(by (2.28)).} \end{aligned} \tag{2.32}$$

Now observe that by (2.30) and (2.31)

$$\begin{aligned} n_\beta(x+2\delta) - 2\beta n^\xi &= \sum_{i=1}^{p} \alpha_i \sigma(k_i)(x+2\delta) - 2\beta n^\xi \\ &\geq (\widehat{n} - 2^{-\ell+2} n)(x+2\delta) - 2\beta n^\xi \\ &\geq (1-2^{-\ell+2}) n(x+2\delta) - 2d\ell(x+2\delta)\lceil n^\xi \rceil - 4\ell n^\xi \\ &\geq n(x+\delta) - 2\ell(3d+2)\lceil n^\xi \rceil \text{ (by (2.21))}. \end{aligned}$$

Then, by (2.25), we have that $n^\beta(x+2\delta) - 2\beta n^\xi \geq nx$ for all $n \geq N$. Note also that

$$n_\beta + d\beta\lceil n^\xi \rceil \leq \widehat{n} + d\beta \lceil n^\xi \rceil \leq n,$$

and that

$$\sum_{j=1}^{p} \alpha_j 2^{k_j} N \leq \sum_{j=1}^{p} \alpha_j \sigma(k_j) \leq \widehat{n} \leq n.$$

Thus, $P\{R_n \geq nx\}$ is at least as large as the left hand side of (2.32), and we have

$$\begin{aligned} &-\frac{1}{n} \log P\{R_n \geq nx\} \\ &\leq -\left(\frac{2\ell}{n} + \frac{2}{N}\right) \log c_3 - \frac{2\ell d}{n^\eta} \log \zeta + \psi(x) + 3\varepsilon \\ &\leq -\frac{2\ell}{n} \log c_3 - \frac{2\ell d}{n^\eta} \log \zeta + \psi(x) + 4\varepsilon. \end{aligned}$$

The proposition follows by taking the limits $n \to \infty$, and then $\varepsilon \downarrow 0$. ◻

Lemma 3. *If* (1.9) *holds and $d \geq 2$, then ψ is convex and continuous on* $(0,1)$.

Proof. We first show that ψ is convex at its continuity points in $[0,1)$, that is, for $x, y, z \in [0,1)$, all of which are continuity points of ψ and such that $x = \alpha y + (1-\alpha)z$ for some $0 < \alpha < 1$ it holds that

$$\psi(x) = \psi(\alpha y + (1-\alpha)z) \leq \alpha\psi(y) + (1-\alpha)\psi(z). \tag{2.33}$$

To see this we use Lemma 1 and the obvious monotonicity in w of $P\{R_n \geq w\}$. Recall that $\xi = 2/(d+1)$ and take $m = m_n = n - d\lceil n^\xi \rceil$ for large n. Since $\lfloor \alpha m \rfloor \cdot \lfloor (1-\alpha)m \rfloor \leq n^2$, and $\lfloor \alpha m \rfloor (y+\varepsilon) + \lfloor (1-\alpha)m \rfloor z - 2\lceil n^\xi \rceil \geq nx$ for large n, we obtain from (2.11) for $\varepsilon \in (0,1)$ and n large that

$$\begin{aligned} &\frac{1}{2}\zeta^{dn^\xi + d} P\{R_{\lfloor \alpha m \rfloor} \geq \lfloor \alpha m \rfloor (y+\varepsilon)\} P\{R_{\lfloor (1-\alpha)m \rfloor} \geq \lfloor (1-\alpha)m \rfloor z\} \\ &\quad \leq P\{R_{\lfloor \alpha m \rfloor + \lfloor (1-\alpha)m \rfloor + d\lceil n^\xi \rceil} \geq \lfloor \alpha m \rfloor (y+\varepsilon) + \lfloor (1-\alpha)m \rfloor z - 2\lceil n^\xi \rceil\} \\ &\quad \leq P\{R_n \geq nx\}, \end{aligned}$$

where $\lfloor a \rfloor$ denotes the largest integer $\leq a$.

Now restrict $\varepsilon > 0$ to such ε for which $y + \varepsilon$ is a continuity point of ψ. As we already observed, ψ is nondecreasing so that this holds for all but at most countably many ε. If we now take the logarithm and divide by n and let $n \to \infty$, then we obtain from Proposition 2 that

$$\psi(x) \le \alpha\psi(y+\varepsilon) + (1-\alpha)\psi(z).$$

The required (2.33) now follows by letting $\varepsilon \downarrow 0$ such that $y + \varepsilon$ runs only through continuity points of ψ.

Now assume, to derive a contradiction, that ψ is not continuous at some $x \in (0, 1)$. Since ψ is nondecreasing, this means that ψ has an upwards jump at x. There therefore exist $x_n \downarrow x$ and $y_n \uparrow x$ such that x_n, y_n are continuity points of ψ and such that

$$\lim_{n\to\infty} \frac{\psi(x_n) - \psi(y_n)}{x_n - y_n} = \infty.$$

However, if $z \in (x, 1)$ is also a continuity point of ψ, then (2.33) implies

$$\frac{\psi(z) - \psi(x_n)}{z - x_n} \ge \frac{\psi(x_n) - \psi(y_n)}{x_n - y_n}$$

as soon as $0 < y_n < x_n < z$. Thus $(\psi(z) - \psi(x_n))/(z - x_n) \to \infty$, which contradicts the boundedness of ψ on [0,1] (see (2.14), (2.15)). This proves the continuity of ψ on $(0, 1)$. □

Proposition 4. *If* (1.9) *holds and* $d \ge 2$*, then* ψ *has the properties* (1.10)–(1.15), *so that Theorem 1 holds when* $d \ge 2$.

Proof. We first note that ψ is continuous at 0. Indeed, we clearly have $P\{R_n \ge nx\} = 1$ and $\psi(x) = 0$ for $x \le 0$, while for $\delta \in (0, 1)$,

$$P\{R_n \ge \delta n\} \ge P\{S_0, \dots, S_{\lfloor \delta n \rfloor} \text{ are distinct}\} \ge [P\{X(i) > 0\}]^{\delta n},$$

exactly as in (2.14). It follows that $\psi(\delta) \le -\delta \log P\{X(i) > 0\}$, as in (2.15). Hence also $\lim_{\delta\downarrow 0} \psi(\delta) = 0$.

This result, together with Lemma 3 and Proposition 2 shows that (1.10) holds for all $x < 1$. Moreover, $\lim_{n\to\infty}(-1/n)P\{R_n \ge n\}$ exists by subadditivity, since $\{R_n \ge n\} = \{S_0, S_1, \dots, S_{n-1}, \text{are distinct}\}$, so that

$$P\{R_{n+m} \ge n + m\} \le P\{R_n \ge n\}P\{R_m \ge m\}.$$

In addition $P\{R_n > n\} = 0$, so that (1.10) holds for all x and also (1.13) holds.

Next we show that $\psi(x)$ is left continuous at $x = 1$. Assume that this is not the case. Since we already know that $\psi(1)$ is finite, there must then exist some $\beta > 0$ such that

$$\psi(1) = \psi(1-) + 2\beta = \lim_{x\uparrow 1} \psi(x) + 2\beta. \tag{2.34}$$

By (1.10) we also know that there exists a constant $c_4 \geq 1$ such that

$$P\{R_n \geq n\} \leq c_4 \exp[-n(\psi(1) - \beta)], \; n \geq 0. \tag{2.35}$$

Now if $R_n \geq nx$, then there are at most $\lfloor (1-x)n \rfloor$ indices $0 \leq k \leq n-1$ for which $S_k = S_i$ for some $0 \leq i < k$ (compare the proof of Theorem 4.1 in [S3]). Let these indices be $0 \leq k_1 < k_2 < \cdots < k_\rho \leq n-1$, (with $\rho \leq (1-x)n$). The number of ways in which these indices can be chosen is for $3/4 \leq x \leq 1$ at most

$$\sum_{\rho=0}^{\lfloor (1-x)n \rfloor} \binom{n}{\rho} \leq c_5 x^{-xn} (1-x)^{-(1-x)n},$$

for some constant c_5, independent of x, n. All times $j \notin \{k_1, \ldots, k_\rho\}$ must be "first visit times", that is, times j with $S_j \neq S_i$ for $0 \leq i < j$, so that $R[k_i, k_{i+1} - 1] = k_{i+1} - k_i, \; 0 \leq i \leq \rho$, where we take $k_0 = 0$ and $k_{\rho+1} = n$. We conclude from this that

$$\begin{aligned} P\{R_n \geq nx\} &\leq \sum_\rho \sum_{k_1,\ldots,k_\rho} \prod_{i=0}^{\rho} P\{R[k_i, k_{i+1} - 1] \geq (k_{i+1} - k_i)\} \\ &\leq \sum_\rho \sum_{k_1,\ldots,k_\rho} [c_4]^{\rho+1} \exp\Bigg[-\sum_{i=0}^{\rho} (k_{i+1} - k_i)(\psi(1) - \beta)\Bigg] \\ &= \sum_\rho \sum_{k_1,\ldots,k_\rho} [c_4]^{\rho+1} \exp[-n(\psi(1) - \beta)] \\ &\leq [c_4]^{(1-x)n+1} c_5 x^{-xn} (1-x)^{-(1-x)n} \exp[-n(\psi(1) - \beta)]. \end{aligned}$$

It follows that

$$\psi(x) \geq -(1-x)\log c_4 + x \log x + (1-x)\log(1-x) + \psi(1) - \beta.$$

By taking the limit $x \uparrow 1$, we see that

$$\lim_{x \uparrow 1} \psi(x) \geq \psi(1) - \beta.$$

This contradicts (2.34) unless $\beta = 0$, and the left continuity of ψ at $x = 1$ follows.

Now that we have continuity of ψ on $[0, 1]$ we obtain the convexity of ψ on $[0, 1]$ from Lemma 3. We also have continuity at π, so that also (1.11) holds.

Next, we prove (1.12). By Markov's inequality,

$$P\{R_n \geq n(\pi + \varepsilon)\} \leq e^{-n(\pi+\varepsilon)\lambda} E e^{\lambda R_n}$$

for any $\lambda \geq 0$. Consequently,

$$\lim_{n\to\infty} \frac{-1}{n} \log P\{R_n \geq n(\pi + \varepsilon)\} \geq (\pi + \varepsilon)\lambda + \lim_{n\to\infty} \frac{-1}{n} \log E e^{\lambda R_n}.$$

The last limit exists by subadditivity, because $E e^{\lambda R_{n+m}} \leq E e^{\lambda R_n} E e^{\lambda R_m}$ for $\lambda \geq 0$. Moreover, it is proven in [H] that the right derivative at $\lambda = 0$ of this limit equals

$-\pi$. (This reference only considers simple random walk, but the proof applies to any random walk on $\mathbb{Z}^d$). Thus for sufficiently small $\lambda > 0$,

$$\lim_{n\to\infty} \frac{-1}{n} \log E e^{\lambda R_n} \geq \lim_{n\to\infty} \frac{-1}{n} \log e^{0\cdot R_n} - \left(\pi + \frac{\varepsilon}{2}\right)\lambda = -\left(\pi + \frac{\varepsilon}{2}\right)\lambda$$

and

$$\lim_{n\to\infty} \frac{-1}{n} \log P\{R_n \geq n(\pi + \varepsilon)\} \geq \frac{\varepsilon}{2}\lambda.$$

This establishes (1.12).

Finally, (1.12) also implies (1.15). Indeed, convexity of ψ implies that for $h > 0, x \geq \pi$ one has $\psi(x+h) - \psi(x) \geq \psi(\pi + h) - \psi(\pi)$. □

The proof of Theorem 1 for $d = 1$ will be given in a separate paper.

We conclude this section with the proof of Corollary 1.

Proof of Corollary 1. The upper bound (1.16) for $\mu_n(F)$ can easily be obtained from a standard large deviation estimate with the help of (1.15) (compare the proof of Theorem 2.2.3 in [DZ]). We therefore concentrate on proving the lower bound (1.17) for $\mu_n(G)$.

Let G be an open set contained in the interval $[\pi, \infty)$. If $\inf G \geq 1$, then the infimum of ψ on G is infinity by (1.13) since $x > 1$ for all $x \in G$. Therefore the lower bound is trivial in this case. In the case that $\inf G < 1$, we can find for each $x \in G \cap [\pi, 1)$ some constant $\delta = \delta(x) > 0$ such that $[x, x + \delta)$ is contained in $(\pi, 1)$. Noting that

$$\mu_n(G) \geq P\{R_n \geq x\} - P\{R_n \geq (x + \delta)n\},$$

and using (1.15), we easily obtain for each $x \in G \cap [\pi, 1)$ that

$$\liminf_{n\to\infty} \frac{1}{n} \log \mu_n(G) \geq -\psi(x).$$

This implies that

$$\liminf_{n\to\infty} \frac{1}{n} \log \mu_n(G) \geq - \inf_{x\in G\cap[\pi,1)} \psi(x).$$

In view of (1.13), the right hand side here is unchanged even if we take the infimum over all of G instead of $G \cap [\pi, 1)$. □

3. The Wiener sausage

We indicate here how to prove Theorem 2. We shall first deal with the one-dimensional case of Theorem 2. Because of the continuity of Brownian motion one can more or less write down the distribution of Λ_t and find $\phi(x)$ explicitly when $d = 1$. For $d \geq 2$ there are only some technical differences with the proof of Theorem 1 and we shall skip some details.

Lemma 5. *If $d = 1$ and $A \subset \mathbb{R}$ is any bounded, Lebesgue measurable, nonempty set, then for $x \geq 0$*

$$\phi(x) = \lim_{t\to\infty} \frac{-1}{t} \log P\{\Lambda_t(A) \geq tx\} = \begin{cases} \frac{x^2}{2} & \text{if } \mu = 0 \\ 0 & \text{if } 0 \leq x \leq \mu \\ \frac{(x-\mu)^2}{2} & \text{if } 0 < \mu \leq x. \end{cases} \tag{3.1}$$

Proof. Since Brownian motion is continuous, it is easy to see that for any $A \subset [-K, K]$

$$\max_{s\leq t} B(s) - \min_{s\leq t} B(s) \leq \Lambda_t(A) \leq 2K + \max_{s\leq t} B(s) - \min_{s\leq t} B(s).$$

For the purposes of (3.1) we may therefore replace $\Lambda_t(A)$ by $\widetilde{\Lambda}_t := \max_{s\leq t} B(s) - \min_{s\leq t} B(s)$. The density of this quantity when $\mu = 0$ is explicitly given in [F1], and this immediately gives the result for $\mu = 0$. When $\mu > 0$ it is still possible to derive an infinite series for the density of $\widetilde{\Lambda}_t$ as in [F1], but it is easier to use the following crude estimates.

$$P\{\widetilde{\Lambda}_t \geq tx\} \geq P\{B(t) \geq tx\} = \frac{1}{\sqrt{2\pi t}} \int_{tx}^{\infty} e^{-(y-t\mu)^2/(2t)} dy.$$

When $x \leq \mu$, the integral in the right hand side is at least 1/2, and when $x > \mu$, then this integral behaves asymptotically like

$$\frac{1}{(x-\mu)\sqrt{2\pi t}} \exp\big[-(x-\mu)^2 t/2\big] \tag{3.2}$$

(see [F2], Lemma VII.1.2). To prove (3.1) we only need an upper bound for $P\{\widetilde{\Lambda}_t \geq tx\}$ when $x > \mu$. For this we can use

$$\begin{aligned} P\{\widetilde{\Lambda}_t \geq tx\} \leq &\sum_{k\leq t} P\{\max_{k\leq s\leq k+1} |B(s) - B(k)| \geq t^{3/4}\} \\ &+ \sum_{k,\ell\leq t} P\{|B(k) - B(\ell)| \geq tx - 2t^{3/4}\}. \end{aligned}$$

The first sum in the right hand side here is by the familiar reflection principle ([IM], Section 1.7) $O\big(t \exp[-(t^{3/4} - \mu)^2/2]\big)$. The summand in the second sum equals

$$\frac{2}{\sqrt{2\pi}} \int_{|k-\ell|^{-1/2}[tx-2t^{3/4}-|k-\ell|\mu]}^{\infty} e^{-y^2/2} dy = O\left(\exp\left[-\frac{(t(x-\mu) - 2t^{3/4})^2}{2t}\right]\right), \tag{3.3}$$

(by virtue of [F2], lemma VII.1.2 again). (3.1) now follows easily. □

For the remainder of this section we take $d \geq 2$ and $A \subset \mathbb{R}^d$ a fixed bounded, Lebesgue measurable set with $\text{cap}(A) > 0$, where $\text{cap}(A)$ is as in (1.21). We usually abbreviate $\Lambda_t(A)$ to Λ_t. The constants c_i which appear in the proof do not necessarily have the same value as in the previous sections. Without loss of generality we assume that A is contained in $\mathscr{S} := \{x \in \mathbb{R}^d : |x| \le 1\}$, the unit ball centered at the origin. Unless otherwise indicated the initial point $B(0)$ equals $\mathbf{0}$. We shall write $B(t;i)$ for the i-th coordinate of $B(t)$. Finally, we define

$$V[a,b] = \bigcup_{a \le s \le b} (B(s) + A).$$

In this notation,

$$\Lambda_t = \lambda_d(V[0,t]).$$

Before we can give the analogue of Lemma 1 we need some crude a priori bounds on the distribution of Λ_t. The following lemma is from [ABP].

Lemma 6. *Let $d \ge 2$ and let A be a Lebesgue measurable set contained in $\mathscr{S}$, and with cap$(A) > 0$. Then*

$$P\{\Lambda_{1/2}(A) \ge x\} > 0 \quad \textit{for all} \quad x \ge 0. \tag{3.4}$$

Proof. This is proven in [ABP] for a Brownian motion without drift. For a Brownian motion with drift this then follows from the fact that on any finite time interval Brownian motions with and without drift are absolutely continuous with respect to each other. □

Lemma 7. *For all $x \ge 0$ there exists an $L_1 = L_1(x, A, d) < \infty$ such that*

$$P\{\Lambda_t \ge tx\} \ge e^{-L_1 t}, \quad t \ge 1. \tag{3.5}$$

Proof. Clearly, for $t \ge 1$

$$P\{\Lambda_t \ge tx\} \ge P\{\Lambda_{\lfloor t \rfloor} \ge 2\lfloor t \rfloor x \ge tx\}.$$

Thus, at the cost of replacing x by $2x$ we may restrict ourselves to t an integer. Now, by Lemma 6, for any $x \ge 0$, $P\{\Lambda_{1/2} \ge x\} > 0$. There then exists a constant c_1 such that even

$$P\Big\{\Lambda_{1/2} \ge x, \sup_{0 \le s \le 1/2} |B(s;1)| \le c_1\Big\} > 0. \tag{3.6}$$

Now let $\mathscr{E}_k = \mathscr{E}_{k,1} \cap \mathscr{E}_{k,2} \cap \mathscr{E}_{k,3}$, where

$$\mathscr{E}_{k,1} := \{B(k + \frac{1}{4}; 1) - B(k;1) > c_1 + 1\};$$

$$\mathscr{E}_{k,2} := \Big\{\lambda_d\big(V[k+\frac{1}{4}, k+\frac{3}{4}]\big) \ge x \quad \text{and} \quad \big|B(k+s;1) - B(k+\frac{1}{4};1)\big| \le c_1 \quad \text{for} \quad \frac{1}{4} \le s \le \frac{3}{4}\Big\};$$

$$\mathscr{E}_{k,3} := \Big\{B(k+1;1) - B(k+\frac{3}{4};1) > 2c_1 + 2\Big\}.$$

Then the events $\mathscr{E}_k$, $k = 0, 1\dots$ are independent and all have the same probability. This probability equals

$$P\Big\{B\Big(\frac{1}{4}; 1\Big) > c_1 + 1\Big\} P\Big\{B\Big(\frac{1}{4}; 1\Big) > 2c_1 + 2\Big\} \\ \times P\{\Lambda_{1/2} \geq x, \sup_{0 \leq s \leq 1/2} |B(s; 1)| \leq c_1\}.$$

Let us denote this probability by α. Then $\alpha > 0$ by virtue of (3.6), and

$$P\Big\{\bigcap_{0 \leq k \leq t-1} \mathscr{E}_k\Big\} \geq \alpha^t. \tag{3.7}$$

Finally, note that if $\mathscr{E}_k$ and $\mathscr{E}_\ell$ occur for some $k < \ell$, then

$$V\big[k + \frac{1}{4}, k + \frac{3}{4}\big] \cap V\big[\ell + \frac{1}{4}, \ell + \frac{3}{4}\big] = \emptyset, \tag{3.8}$$

because the points in $V[k + 1/4, k + 3/4]$ lie within distance 1 of $\{B(s) : k + 1/4 \leq s \leq k + 3/4\}$, and therefore have their first coordinate in

$$\big[B(k + \frac{1}{4}; 1) - c_1 - 1, B(k + \frac{1}{4}; 1) + c_1 + 1\big] \\ \subset \big(B(k; 1), B(k + 1; 1)\big).$$

In addition,

$$\begin{aligned} B(k + 1; 1) &> B(k + \frac{3}{4}) + 2c_1 + 2 \\ &\geq B(k + \frac{1}{4}; 1) + c_1 + 2 > B(k; 1) + 2c_1 + 3, \end{aligned}$$

so that the intervals $(B(k; 1), B(k + 1; 1))$ for different k are disjoint. Thus (3.8) indeed holds, and on the event $\cap_{0 \leq k \leq t-1} E_k$ one has

$$\Lambda_t \geq \sum_{k=0}^{t-1} \lambda_d\Big(V\big[k + \frac{1}{4}, k + \frac{3}{4}\big]\Big) \geq tx.$$

The lemma now follows from (3.7), by taking $L_1 = -\log \alpha$. □

Lemma 8. *For all* $0 \leq L < \infty$, *there exist a constant* $x_1 = x_1(L, A, d) < \infty$ *such that*

$$P\{\Lambda_t \geq tx_1\} \leq e^{-Lt}, \quad t \geq 1. \tag{3.9}$$

Proof. When $\mu = 0$ this is essentially contained in Theorem 1 of [BT]. When $\mu \neq 0$ we cannot use the scaling property of Brownian motion, but we can still get our estimate (even for $\mu = 0$) by following part of the proof in [BT]. We do not attempt to get sharp estimates such as given in [BB]. Probably this is still possible even if $\mu \neq 0$, but crude estimates are good enough for our purposes.

Since Λ_t is increasing in A, it is sufficient to restrict ourselves to $A = \mathscr{S}$. We define

$$\theta_0 = 0,\ \theta_{n+1} = \inf\{s > \theta_n : \|B(s) - B(\theta_n)\| > 1\},$$
$$\tau_n = \theta_n - \theta_{n-1}, \quad n \geq 1,$$

and

$$\nu(t) = \max\{n : \theta_n \leq t\}.$$

These are the definitions of Section 4 of [BT] with $y = 1$. As in (4.5) of [BT], there then exist constants c_2 and h such that $\Lambda_t \leq c_2 + h\nu(t)$. Since the $\tau_n,\ n \geq 1$, are i.i.d. we therefore obtain for $t \geq 1,\ x \geq 2c_2$,

$$\begin{aligned} P\{\Lambda_t \geq tx\} &\leq P\{h\nu(t) \geq \frac{1}{2}tx\} \\ &= P\Big\{\sum_{i=1}^{\lceil tx/(2h)\rceil} \tau_i \leq t\Big\} \\ &\leq e^t\big[E\exp(-\tau_1)\big]^{tx/(2h)}. \end{aligned}$$

Clearly $E\exp(-\tau_1) < 1$, so that (3.9) is satisfied for

$$x_1 \geq \frac{2h(L+1)}{-\log E\exp(-\tau_1)} + 2c_2. \qquad \square$$

We can now prove the following analogue of Lemma 1:

Lemma 9. *For any fixed $x_2 \geq 0$, there exist constants $c_3, c_4 \in (0, \infty)$ such that for $s, t \geq 1$, $y \leq sx_2, z \leq tx_2$, it holds that*

$$\begin{aligned} &P\{\Lambda_{s+t+(st)^{1/(d+1)}} \geq y + z - c_3(st)^{1/(d+1)}\} \\ &\quad \geq c_4\exp[-d(st)^{1/(d+1)}]P\{\Lambda_s \geq y\}P\{\Lambda_t \geq z\}. \end{aligned} \tag{3.10}$$

Proof. To prove this we bring in an additional Brownian motion $\widehat{B} = \{\widehat{B}(t)\}_{t\geq 0}$ which is independent of $B = \{B(t)\}_{t\geq 0}$, but has the same distribution as B. $\widehat{\Lambda}_t$ is defined by (1.18) with B replaced by $\widehat{B}$. We further define for $w \in \mathbb{R}^d$

$$\begin{aligned} M_{s,t}(w) &= M_{s,t}(w, B, \widehat{B}) \\ &= \lambda_d\Big(\bigcup_{r\leq s}(B(r) + A) \cap \bigcup_{r\leq t}(\widehat{B}(r) + B(s) + w + A)\Big), \end{aligned}$$

and for $p, r \geq 0$ we define

$$T(r) = T^{(p,s)}(r) = \begin{cases} B(r) & \text{if } r \leq s + p \\ B(s+p) + \widehat{B}(r - s - p) & \text{if } r > s + p. \end{cases}$$

Finally we define

$$\Xi_q = [-q, q]^d$$

and take

$$q = q(s,t) = (st)^{1/(d+1)}.$$

Analogously to the proof of Lemma 1 we now have for any $c_3, \beta \geq 0$,

$$\begin{aligned}
&P\{\Lambda_{s+t+q} \geq y + z - c_3 q\} \\
&\quad \geq \int_{w\in\Xi_q} \frac{1}{q}\int_{0\leq p\leq q} P\{B(s+p) - B(s) \in dw\}dp \\
&\qquad \times P\{\Lambda_s \geq y, \widehat{\Lambda}_t \geq z,\ M_{s,t}(w, B, \widehat{B}) \leq c_3 q\} \\
&\quad \geq \int_{w\in\Xi_q} \frac{1}{q}\int_{0\leq p\leq q} P\{B(s+p) - B(s) \in dw\}dp \\
&\qquad \times P\{y \leq \Lambda_s \leq s\beta, z \leq \widehat{\Lambda}_t \leq t\beta,\ M_{s,t}(w, B, \widehat{B}) \leq c_3 q\}. \qquad (3.11)
\end{aligned}$$

Of course

$$P\{B(s+p) - B(s) \in dw\} = \frac{1}{(2\pi p)^{d/2}} \exp[-\|w\|^2/(2p)]dw,$$

and for some constant $c_5 > 0$ and all $w \in \Xi_q$,

$$\begin{aligned}
&\frac{1}{q}\int_{0\leq p\leq q} \frac{1}{(2\pi p)^{d/2}} \exp[-\|w\|^2/(2p)]dp \\
&\quad \geq \frac{1}{q(2\pi q)^{d/2}}\int_{q/2}^{q} \exp[-dq^2/q]dp \geq c_5 q^{-d/2}\exp[-dq].
\end{aligned}$$

Thus, the right hand side of (3.11) is at least

$$\begin{aligned}
&c_5 q^{-d/2}\exp[-dq] E\Big\{I[y \leq \Lambda_s \leq s\beta] I[z \leq \widehat{\Lambda}_t \leq t\beta] \\
&\quad \times \lambda_d\big(\{w \in \Xi_q : M_{s,t}(w, B, \widehat{B}) \leq c_3 q\}\big)\Big\}.
\end{aligned} \qquad (3.12)$$

Moreover, for fixed realizations of B and $\widehat{B}$ with $\Lambda_s \leq s\beta$, $\widehat{\Lambda}_t \leq t\beta$ we have

$$\begin{aligned}
&\int_{\Xi_q} M_{s,t}(w, B, \widehat{B})dw \\
&\quad \leq \int_{\mathbb{R}^d} dw \int_{\mathbb{R}^d} db I\Big[b \in \bigcup_{r\leq s}(B(r) + A)\Big] I\Big[b \in \bigcup_{r\leq t}(\widehat{B}(r) + B(s) + w + A)\Big] \\
&\quad = \int_{\mathbb{R}^d} db I\Big[b \in \bigcup_{r\leq s}(B(r) + A)\Big]\widehat{\Lambda}_t \\
&\quad = \Lambda_s \widehat{\Lambda}_t \leq \beta^2 st.
\end{aligned}$$

Thus, if $\Lambda_s \leq s\beta$, $\widehat{\Lambda}_t \leq t\beta$, and if we choose $c_3 \geq \beta^2$ (for a β to be chosen later), then

$$\begin{aligned}
\lambda_d\big(\{w \in \Xi_q : M_{s,t}(w, B, \widehat{B}) \leq c_3 q\}\big) &\geq \lambda_d(\Xi_q) - \frac{\beta^2 st}{c_3 q} \\
&= 2^d q^d - \frac{\beta^2}{c_3} q^d \geq q^d.
\end{aligned}$$

In this situation the expression in (3.12) is at least

$$\begin{aligned} &c_6 q^{-d/2} \exp[-dq] q^d P\{y \le \Lambda_s \le s\beta\} P\{z \le \widehat{\Lambda}_t \le t\beta\} \\ &\ge c_6 \exp[-dq] P\{y \le \Lambda_s \le s\beta\} P\{z \le \widehat{\Lambda}_t \le t\beta\}. \end{aligned}$$

Finally,

$$P\{y \le \Lambda_s \le s\beta\} = P\{\Lambda_s \ge y\} - P\{\Lambda_s > s\beta\}.$$

But if $y \le s x_2$, then

$$P\{\Lambda_s \ge y\} \ge P\{\Lambda_s \ge s x_2\} \ge e^{-L_1 s},$$

for $L_1 = L_1(x_2, A, d)$ as in Lemma 7. By Lemma 8 we can therefore first choose $\beta = c_7(x_2, A, d)$ (and then $c_3 \ge \beta^2$) such that for $s \ge 1$,

$$P\{\Lambda_s > s c_7\} \le e^{-(L_1+1)s} \le \frac{1}{2} P\{\Lambda_s \ge y\},$$

and

$$P\{y \le \Lambda_s \le s c_7\} \ge \frac{1}{2} P\{\Lambda_s \ge y\}.$$

Similarly

$$P\{z \le \widehat{\Lambda}_t \le t c_7\} \ge \frac{1}{2} P\{\Lambda_t \ge z\}.$$

Combining all these estimates we find that for $c_4 = c_6/4$,

$$P\{\Lambda_{s+t+q} \ge y + z - c_3 q\} \ge c_4 \exp[-dq] P\{\Lambda_s \ge y\} P\{\Lambda_t \ge z\}. \qquad \square$$

We now define

$$\phi(x) = \liminf_{n\to\infty} \frac{-1}{n} \log P\{\Lambda_n(A) \ge nx\}.$$

This $\phi(x)$ is finite for all x, by virtue of Lemma 7. We can repeat the proof of Proposition 2 with (3.10) taking the role of (2.11) to obtain that

$$\lim_{n\to\infty} \frac{-1}{n} \log P\{\Lambda_n(A) \ge nx\} = \phi(x)$$

at each $x \ge 0$ at which ϕ is right continuous. This further implies that if $x \pm \varepsilon \ge 0$ are continuity points of ϕ, then

$$\begin{aligned} \limsup_{t\to\infty} \frac{-1}{t} \log P\{\Lambda_t(A) \ge tx\} &\le \limsup_{t\to\infty} \frac{-1}{t} \log P\left\{\Lambda_{\lfloor t\rfloor}(A) \ge \lfloor t\rfloor \frac{t}{\lfloor t\rfloor} x\right\} \\ &\le \lim_{n\to\infty} \frac{-1}{n} \log P\{\Lambda_n(A) \ge n(x+\varepsilon)\} \\ &= \phi(x+\varepsilon), \end{aligned}$$

as well as

$$\liminf_{t\to\infty} \frac{-1}{t} \log P\{\Lambda_t(A) \geq tx\} \geq \liminf_{t\to\infty} \frac{-1}{t} \log P\left\{\Lambda_{\lceil t\rceil}(A) \geq \lceil t\rceil \frac{t}{\lceil t\rceil} x\right\}$$
$$\geq \lim_{n\to\infty} \frac{-1}{n} \log P\{\Lambda_n(A) \geq n(x-\varepsilon)\}$$
$$= \phi(x-\varepsilon).$$

Thus, at any continuity point $x > 0$ of ϕ,

$$\lim_{t\to\infty} \frac{-1}{t} \log P\{\Lambda_t(A) \geq tx\} = \phi(x). \tag{3.13}$$

Further, one can show exactly as in Lemma 3 that

$$\phi(x) \leq \alpha\phi(y) + (1-\alpha)\phi(z),$$

when x, y, z are continuity points of ϕ such that $x, y, z > 0$, $0 < \alpha < 1$, and $x = \alpha y + (1-\alpha)z$. From this convexity property and the finiteness of ϕ we then conclude that ϕ is convex, continuous and finite for all $x > 0$, and that (3.13) holds for all $x > 0$ (compare Lemma 3). Moreover, we trivially have

$$\phi(x) = \lim_{t\to\infty} \frac{-1}{t} \log P\{\Lambda_t(A) \geq tx\}$$
$$= \lim_{t\to\infty} \frac{-1}{t} \log P\{\Lambda_t(A) \geq 0\} = 0 \quad \text{for} \quad x \leq 0.$$

Also, analogously to the proof of Proposition 4, for $0 < \delta < 1$,

$$P\{\Lambda_t(A) \geq t\delta\} \geq P\{\Lambda_{t\delta}(A) \geq t\delta\} \geq e^{-L_1 t\delta}$$

for $L_1 = L_1(1, A, d)$, by Lemma 7. Thus $\phi(\delta) \leq L_1\delta$, and ϕ is also continuous at 0. Thus (3.13) holds for all $x \in \mathbb{R}$. This proves (1.23) and (1.27). Relation (1.24) follows trivially from (1.19). Finally, (1.26) and (1.28) can be proven in exactly the same way as (1.12) and (1.15).

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