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Interacting Fisher–Wright diffusions in a catalytic medium

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Abstract. We study the longtime behaviour of interacting systems in a randomly fluctuating (space–time) medium and focus on models from population genetics. There are two prototypes of spatial models in population genetics: spatial branching processes and interacting Fisher–Wright diffusions. Quite a bit is known on spatial branching processes where the local branching rate is proportional to a random environment (catalytic medium).

Here we introduce a model of interacting Fisher–Wright diffusions where the local resampling rate (or genetic drift) is proportional to a catalytic medium. For a particular choice of the medium, we investigate the longtime behaviour in the case of nearest neighbour migration on the d-dimensional lattice.

While in classical homogeneous systems the longtime behaviour exhibits a dichotomy along the transience/recurrence properties of the migration, now a more complicated behaviour arises. It turns out that resampling models in catalytic media show phenomena that are new even compared with branching in catalytic medium.

1. Introduction

This paper is concerned with evolutions in disordered media where the medium fluctuates both in space and time. We focus on spatial models arising in population genetics.

An object of study in spatial population genetics is a class of stochastic models where on each site of a certain countable set G there is a population of one or more types. The mass and/or relative frequency of the types undergoes a local stochastic evolution that models effects such as genetic drift due to resampling, population growth with limited/unlimited resources, competition of two types for limited resources and so on. In addition spatial models comprise a migration between colonies.

The prototype for a one-type population growth model with unlimited resources is branching random walk as well as its diffusion limit, the Dawson-Watanabe process (see, e.g., Dawson [Daw93]). The most widely studied two-type model with

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a fixed population size (at any site) and a genetic drift due to haploid resampling is that of interacting Fisher–Wright diffusions (see, e.g., Shiga [Shi80], Ethier and Kurtz [EK86], Fleischmann and Greven [FG96]).

In the last years there has been some interest in models where the local diffusion mechanism is influenced by a random medium that is itself a realization of a spatial stochastic process. The best studied model is a spatial branching process where the local branching rate is proportional to the local abundance of a second type which performs an autonomous branching process leading to catalytic branching random walks (see [GKW99]) and catalytic super-Brownian motion (see Dawson and Fleischmann [DF97a] and [DF97b]). Recently also branching models with two types and with a mutual influence have been studied; this is the so called mutually catalytic branching (Dawson and Perkins [DP98]). For an overview on catalytic spatial branching processes we refer to the survey article by one of the authors [Kle00b].

In this paper we introduce a model of interacting Fisher–Wright diffusions where the resampling rate (or strength of the genetic drift) is proportional to a random medium that varies in time and space. The main goal is the investigation of the longtime behaviour. To this end we concentrate on the d-dimensional lattice as site space and on the medium given by the voter model, which is technically better treatable than a medium of interacting diffusions, but closely resembles interacting Fisher–Wright diffusions.

Recall the following properties of the longtime behaviour in the classical case of a space–time homogeneous resampling rate (say for nearest-neighbour migration). Starting in an i.d.d. (or spatially ergodic) random initial state the system of interacting Fisher–Wright diffusions (and similarly the voter model) approaches a non-trivial ergodic equilibrium state as $t \to \infty$ if $d \geq 3$. On the other hand, for $d = 1, 2$ the system approaches a mixture of δ-masses on the traps $\underline{0}$ and $\underline{1}$ of the system. One way of understanding this is the following. The migration has a smoothing effect which drives the local frequencies of the types towards their mean value, whereas the fluctuations caused by the resampling push the components towards the traps 0 and 1. For far reaching interaction (transient migration) the migration wins in this competition, while for short range interaction (recurrent migration) the fluctuations win.

In the random medium things are different. We shall see that in our model in low dimension the new phenomenon occurs that both mechanisms can win with certain probabilities. This is a feature unparallelled by catalytic branching. High dimensions ($d \geq 3$) hide no surprise: as in catalytic branching, the systems behave qualitatively like their classical homogeneous counterparts.

The low dimensions $d = 1$ and especially $d = 2$ are more challenging. In these dimensions we find as longtime limits of the law of the reactant component mixtures of laws concentrated on traps and on spatially constant states different from the traps. We are able to calculate the probabilities of these two possibilities and to give transparent formulas for the limiting joint law of medium and reactant. The situation in $d = 1$ parallels the case of catalytic branching in $d = 2$ (see [GKW99]): the limiting state of the reactant is a process with random intensity where the random factor reflects global properties of the medium and is expressed in terms of the $\sqrt{t}$-rescaling limit of the model.

Finally, the type of behaviour in dimension $d = 2$ is deviant from any known behaviour of catalytic branching. Like for catalytic branching, in the longtime limit we see a random multiple of the counting measure. Howeverm, it is not derived by a rescaling limit of the whole process. It rather arises from a multiple-scale observation of the catalyst process.

Many results of this paper can be carried over easily to a reactant process describing multi-type and infinite-type situations, as interacting multi-dimensional Fisher–Wright diffusions or interacting Fleming-Viot processes.

Some words on the methods used. The starting point is the duality relation connecting the reactant with coalescing random walk in a fluctuating medium. To obtain precise results we exploit the cluster analysis of the voter model. In dimension $d = 1$ Arratia's rescaled voter model ([Arr79], [Arr81]) is the key ingredient for the quantitative description of the longtime behaviour. In $d = 2$ there is *diffusive clustering* of the voter model (see Cox and Griffeath [CG86], and also [Kle96]) and we employ the multiple scale description of this form of cluster formation to derive an intriguing formula for the longtime limit of our model.

This paper is part of a framework to investigate catalytic spatial models where the diffusion coefficient (genetic drift) is of a more general type (diploid resampling, e.g.). The corresponding non-catalytic models could be related to models of the Fisher–Wright type by using a comparison theory (see Cox, Fleischmann and Greven [CFG96]). This theory will be developed for the catalytic models in a forthcoming paper.

1.1. The models and basic tools

We want to define the model in some generality first and concentrate on a special situation later once we come to the longtime behaviour.

Let G be a countable Abelian group and let $\mathscr{B}$ be the generator (q-matrix) of a continuous time random walk on G. Denote by $b_t = \exp(t\mathscr{B})$ is transition probabilities at time $t \geq 0$. We assume that we are given a bounded measurable function

$$\kappa : G \times [0, \infty) \to [0, \infty).$$

This function serves as the *space–time medium* of our model. For the moment it is deterministic but will be chosen to be random later. We need in the sequel the notion of a standard Fisher–Wright diffusion.

Definition 1.1. *A standard Fisher–Wright diffusion is the solution* (*with values in* $[0, 1]$) *of the following SDE*:

$$dY_t = \sqrt{Y_t(1 - Y_t)}\, dW_t. \tag{1.1}$$

The following proposition has been proved by Shiga and Shimizu ([SS80], Remark 2.1 and Theorem 3.2) for $\kappa(g, t)$ not depending on t.

Proposition 1.2. *There exists a unique strong solution* $(\xi_t)_{t\geq 0}$ *of the following system of interacting stochastic differential equations*

$$d\xi_t(g) = (\mathscr{B}\xi_t)(g)\,dt + \sqrt{\kappa(g,t)\xi_t(g)(1-\xi_t(g))}\,dW_t(g), \quad g \in G, \tag{1.2}$$

where $\{(W_t(g))_{t\geq 0},\ g \in G\}$ *is a family of independent standard Brownian motions.*

Proof. The proof of the existence has two components. First we have to show that there exists a solution of this system for a finite index set. Shiga and Shimizu achieve this in their setting by referring to a result of Skorokhod. In the time–inhomogeneous setting, this can be replaced, e.g. by [RW87] Thm. V 23.5, guaranteeing a solution to the corresponding martingale problem and hence also a weak solution to the SDE (cf. [RW87] Thm. V 20.1). Based on this result one can use the technique developed in [SS80] to construct a solution for the system with countably many components. In addition their proof of strong uniqueness based on Gronwall's inequality carries over. Since existence of a weak solution together with strong uniqueness implies the existence of a strong solution, we obtain the existence and uniqueness results for time inhomogeneous κ. $\square$

Definition 1.3 (CIFWD). *We call the solution* $(\xi_t)_{t\geq 0}$ *of* (1.2) *a system of interacting Fisher–Wright diffusions in the catalytic medium* κ *and with migration kernel* $\mathscr{B}$ *(and write CIFWD(*$\mathscr{B},\kappa$*) for short). We also refer to* $(\xi_t)_{t\geq 0}$ *as the reactant.*

The main tool in the analysis of CIFWD is a duality to coalescing random walks with varying rate of coalescence. This kind of duality is well known for the homogeneous model ($\kappa \equiv 1$). We state the duality here and begin with introducing the dual process. We denote by $\mathscr{N}_f(G)$ the set of finite, non-negative integer valued measures on G. For $\mu \in \mathscr{N}_f(G)$ and $x \in [0,\infty)^G$ we define

$$x^\mu := \exp\left(\int \log(x(g))\ \mu(dg)\right), \tag{1.3}$$

where x^μ can be zero.

Definition 1.4 (Coalescing random walk in the catalytic medium). *Fix* $T > 0$ *and let* $(\widetilde{X}_t^T)_{t\in[0,T]}$ *be a system of coalescing random walks with local rate of coalescence* $\kappa_{T-t}(g)$ *at time t at site g. This is,* $\widetilde{X}^T$ *is the Markov process that takes values in* $\mathscr{N}_f(G)$ *with time-dependent infinitesimal generator acting on bounded functions* $F : G \to \mathbb{R}$ *as follows*:

$$\begin{aligned}\mathscr{G}^{T,t}F(x) = &\sum_{g,h\in G} x(g)\mathscr{B}(g,h)F(x+\delta_h-\delta_g)\\ &+\sum_{g\in G}\binom{x(g)}{2}\kappa_{T-t}(g)[F(x-\delta_g)-F(x)].\end{aligned} \tag{1.4}$$

More intuitively, the particles of $\widetilde{X}^T$ *perform independent random walks with generator* $\mathscr{B}$ *and pairs of particles at the same site g at time t coalesce at rate* $\kappa_{T-t}(g)$.

Proposition 1.5 (Duality). *For fixed* κ, $T \geq 0$ *and* $\varphi \in \mathcal{N}_f(G)$

$$\mathbf{E}^{\xi_0}[(\xi_T)^{\varphi}] = \mathbf{E}^{\varphi}[\xi_0^{\widetilde{X}_T}]. \tag{1.5}$$

Proof. For the piecewise (in time) constant medium this follows as in [Shi80], Lemma 2.3. The general case follows by an approximation. □

The construction of the process works for a broad class of media. However the longtime behaviour of the model depends in a subtle way on the medium. In order to obtain precise results we have to make specific choices here. In the sequel we shall consider the special situation where also κ is random and is generated by an autonomous Markovian evolution which we call the catalyst. In this case we consider the process which corresponds to the pair (catalyst, reactant). Precisely

Definition 1.6. *Let* $(\kappa(\cdot, t))_{t\geq 0}$ *be a Markov process and define for given realization of* $(\kappa(\cdot, t))_{t\geq 0}$ *the process* $(\xi_t)_{t\geq 0}$ *by Definition 1.3. The process* $(\kappa(\cdot, t), \xi_t)_{t\geq 0}$ *is called the bivariate process.*

In order to achieve a concise presentation we will specialise in the sequel and make the following

Assumption. $G = \mathbb{Z}^d$ and $\mathscr{B}$ is the q-matrix of nearest neighbour random walk. We also assume that $\kappa = \eta$ is a realization of a (nearest neighbour) voter model on $\mathbb{Z}^d$.

Recall that the voter model $(\eta_t)_{t\geq 0}$ is the Markov process with values in $\{0, 1\}^{\mathbb{Z}^d}$ where in each coordinate $i \in \mathbb{Z}^d$ a flip to $1 - \eta_{t-}(i)$ occurs at a rate proportional to the number of neighbours j with $|j - i| = 1$ such that $\eta_{t-}(i) \neq \eta_{t-}(j)$. For details see [Lig85, Chapter V] or [Dur88, Chapters 2 and 10]. The voter model can be thought of as the limit of (homogeneous) IFWD where the resampling rate tends to infinity. Its duality is the same as the one for IFWD but now the coalescence of two walks is instantly.

Remark 1.7. To distinguish conceptually between the random walks underlying the reactant and our voter medium, we write $\mathscr{A}$ for the q-matrix of nearest neighbour random walk on $\mathbb{Z}^d$ whenever it is associated to the voter model η. We also denote the transition kernels by $a_t = \exp(t\mathscr{A})$.

The main content of this paper is to investigate the longtime behaviour of the IFWD in this medium η. The natural choice for the medium seems to be (classical) interacting Fisher–Wright diffusions. The reason for choosing the voter model instead is that it produces less technical difficulties in places. On the other hand, in many respects the voter model shows a similar behaviour as IFWD. Hence the hope seems justified that we capture essential features by choosing the voter model as the medium.

1.2. Results

We investigate the longtime behaviour of the model and give convergence theorems for the joint distribution of the medium and the reactant. Due to the very different nature of dimension one, two and $d \geq 3$ we give the results in different subsections. We begin with the case closest to the classical situation and then proceed to the new features subsequently.

1.2.1. Dimensions three and more

The longtime behaviour in $d \geq 3$ is very similar to that of space–time homogeneous systems. We consider for the process $(\eta_t, \xi_t)_{t \geq 0}$ the following class of initial laws.

Denote by $\mathscr{P}((\{0,1\} \times [0,1])^{\mathbb{Z}^d})$ the space of probability measures on $(\{0,1\} \times [0,1])^{\mathbb{Z}^d}$ and by $\mathscr{T} \subset \mathscr{P}$ the subset of translation invariant measures. For $(\theta_1, \theta_2) \in [0,1]^2$ define the class of initial states with asymptotic intensity (θ_1, θ_2) by

$$\mathscr{M}_{(\theta_1,\theta_2)} = \Big\{ \mu \in \mathscr{T} : \limsup_{t \to \infty} \int \mu(d(\eta,\xi))(|a_t \eta(i) - \theta_1| + |b_t \xi(i) - \theta_2|) = 0, \\ i \in \mathbb{Z}^d \Big\}. \tag{1.6}$$

Note that in particular translation invariant, spatially ergodic configurations with mean (θ_1, θ_2) are in this class.
The main result in the high dimensional case is:

Theorem 1. *Assume that* $d \geq 3$.

(a) *For every intensity* $(\theta_2, \theta_2) \in [0,1]^2$ *there exists a unique extremal invariant measure* $\nu_{(\theta_1,\theta_2)}$ *of the bivariate process such that*

$$\int \nu_{\theta_1,\theta_2}(d(\eta,\xi)) \begin{pmatrix} \eta(0) \\ \xi(0) \end{pmatrix} = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}. \tag{1.7}$$

The measure ν_{θ_1,θ_2} *has the following properties*

$$\begin{aligned} &- \nu_{\theta_1,\theta_2} \text{ is translation invariant and spatially ergodic.} \\ &- \mathbf{Var}^{\nu_{\theta_1,\theta_2}}[\xi(0)] > 0 \text{ provided } \theta_1 > 0 \text{ and } \theta_2 \in (0,1). \end{aligned} \tag{1.8}$$

(b) *If we choose* $\mu \in \mathscr{M}_{(\theta_1,\theta_2)}$, *and if we denote by* $P_{(\theta_1,\theta_2)}$ *the law of the stationary bivariate process with marginal* ν_{θ_1,θ_2}, *then*

$$\mathscr{L}^{\mu}\left[(\eta_{t+T}, \xi_{t+T})_{t \geq 0}\right] \overset{T \to \infty}{\Longrightarrow} P_{(\theta_1,\theta_2)}. \tag{1.9}$$

The law $P_{(\theta_1,\theta_2)}$ *is space–time mixing.*

Remark 1.8. The statements of Theorem 1 remain true even in a situation of much greater generality. In fact, our proof works without changes for a countable Abelian group G and the case where the (possibly different) migration kernels of catalyst and reactant have transient symmetrisations. One can even replace the catalyst by some translation invariant random medium that follows a Markovian dynamics on $[0,\infty)^G$ and approaches a non-trivial equilibrium. (In particular, we can take interacting Fisher–Wright diffusions as the medium, cf. the remark at the end of

Subsection 1.1) Focusing only on (1.9) we can replace the Markovian process by a space–time mixing stationary process. We chose the formulation of the special case in Theorem 1 only to be consistent with the following theorems for the low-dimensional situation where we *have* to be more specific.

1.2.2. Dimension two

A principal role in our main result for the two-dimensional case is played by the solution $p(\cdot)$ of the following Dirichlet problem (for existence and uniqueness see Lemma 1.13 below):

Definition 1.9. *We define the twice continously differentiable map* $p : [0, 1] \to [0, 1]$ *as the solution of*

$$\frac{d^2}{d\theta^2} p(\theta) = -2 \frac{p(\theta)(1 - p(\theta))}{\theta(1 - \theta)}, \quad \theta \in (0, 1) \tag{1.10}$$

with the boundary conditions

$$p(0) = 0, \ \ p(1) = 1. \tag{1.11}$$

Let Y^1 and Y^2 be independent standard Fisher–Wright diffusions started at θ_1 and θ_2. Denote by λ the counting measure on $\mathbb{Z}^2$. We write π_θ for the product measure on $\{0, 1\}^{\mathbb{Z}^d}$ and $\bar{\pi}_\theta$ for an arbitrary but fixed product measure on $[0, 1]^{\mathbb{Z}^d}$ with intensity $\theta \in [0, 1]$, respectively. Our main result is:

Theorem 2. *Assume* $d = 2$. *Then*

$$\mathscr{L}^{\pi_{\theta_1} \otimes \bar{\pi}_{\theta_2}}[(\eta_t, \xi_t)] \overset{t\to\infty}{\Longrightarrow} \mathscr{L}^{\theta_1,\theta_2}\left[(Y^1_\infty \cdot \lambda,\, Y^2_{\int_0^\infty p(Y^1_s)ds} \cdot \lambda)\right]. \tag{1.12}$$

Remark 1.10. With $m_{\theta_1,\theta_2} = \mathbf{P}^{\theta_1,\theta_2}\left[Y^2_{\int_0^\infty p(Y^1_s)ds} \in \bullet | Y^1_\infty = 0\right]$ the r.h.s. of (1.12) equals

$$\theta_1(\theta_2 \delta_{(\underline{1},\underline{1})} + (1 - \theta_2)\delta_{(\underline{1},\underline{0})}) + (1 - \theta_1) \int \delta_{(\underline{0},\underline{\theta})} m_{\theta_1,\theta_2}(d\theta).$$

This means that for the reactant we see both constant states $\underline{\theta}$ with $\underline{\theta}(g) \equiv \theta \in (0, 1)$ produced by a dominating migration, and constant states $\underline{0}$ or $\underline{1}$ which are traps produced by the dominance of fluctuations.

Remark 1.11. Presumably the initial state $\pi_{\theta_1} \otimes \bar{\pi}_{\theta_2}$ could be replaced by more general elements of $\mathscr{M}_{\theta_1,\theta_2}$ without changing the result; we do not strive for this generalisation here.

In order to understand why the theorem should be true let us give an idea of the proof here. We start by explaining the function $p(\cdot)$. To this end we need to introduce *binary branching Fisher–Wright diffusions* first, which allow a probabilistic representation of $p(\cdot)$. Recall also that every element of the set $\mathscr{N}_f([0, 1])$ of finite, integer valued measures on $[0, 1]$ can be viewed in an obvious way as empirical measure of a collection of particles.

Definition 1.12. *Let* (Z_s) *be rate 1 binary branching Fisher–Wright diffusions. This is,* (Z_s) *is the Markov process with values in* $\mathcal{N}_f([0,1])$ *where each particle undergoes a standard Fisher–Wright diffusion and with rate 1 splits into two particles at the same location undergoing the same (but independent) dynamics.*

The connection of p with Z is given by the following lemma that we prove in Subsection 3.4.

Lemma 1.13. *Equation* (1.10) *with boundary conditions* (1.11) *is uniquely solvable and*

$$p(\theta) = \lim_{t\to\infty} \mathbf{P}^{\delta_\theta}[Z_t(\{1\}) > 0] = \lim_{t\to\infty} \mathbf{P}^{\delta_\theta}[Z_t((0,1]) > 0].$$

The basis for the phenomenon described in the theorem is the following property of the medium (voter model). For the medium $(\eta_t)_{t\geq 0}$ it is well known that the configuration forms big clusters (i.e. connected components) of zeros and ones as $t \to \infty$ and it is even possible to determine how these clusters grow. In fact, they follow a pattern of diffusive clustering, which means that the sizes of clusters of 0's or 1's are of order $t^{\alpha/2}$ with a random exponent α. More precisely, we know that the block averages converge in the following scaling

$$\mathscr{L}^{\pi_\theta}\left[(t^{-\alpha}\eta_t([0,t^{\alpha/2}]^2))_{\alpha\in[0,1]}\right] \underset{\text{fdd}}{\overset{t\to\infty}{\Longrightarrow}} \mathscr{L}^{\theta}\left[(Y^1_{-\log\alpha})_{\alpha\in[0,1]}\right]. \tag{1.13}$$

For given medium the reactant has a dual process and we compute m-th moments of the reactant, first for given medium, via this duality (see Proposition 1.5), and later we average over the medium. Hence we have to start m random walks at time t and let them run backwards in time through the medium. On a logarithmic scale with $\alpha \in [0,1]$ as parameter the times t^α when pairs of the random walks meet form a point process with intensity $\alpha^{-1}d\alpha$ times the number of remaining pairs. We show (Proposition 3.3), that if a pair meets it has a probability $\approx p(\varrho)$ to coalesce, where $\varrho = t^{-\alpha}\eta_{t-t^\alpha}([0,t^{\alpha/2}]^2)$, which is approximated via (1.13) by $Y^1_{-\log\alpha}$. This means at time $t^{\exp(-s)}$, as $t \to \infty$, the total rate of coalescence is $\approx \binom{m_s}{2}p(Y^1_s)$ with m_s being the number of remaining particles. Hence for large t the number of surviving particles is distributed approximately as Kingman's coalescent $(D^m_s)_{s\geq 0}$ started with $D^m_0 = m$ and evaluated at time $\int_0^\infty p(Y^1_s)ds$. (Kingman's coalescent is the pure death process on $\mathbb{N}$ with rates $\binom{m}{2}$ for the transitions $m \mapsto m-1$.) Finally, Kingman's coalescent is connected to the Fisher–Wright diffusion by the following well known duality, leading to the time transformed Y^2 of Proposition 1.15.

Lemma 1.14 (Duality: Fisher–Wright diffusion). *For all* $m \in \mathbb{N}$, $\theta \in [0,1]$ *and* $s \geq 0$,

$$\mathbf{E}^\theta[(Y_s)^m] = \mathbf{E}^m[\theta^{D_s}]. \tag{1.14}$$

This discussion already suggests that the result can be viewed also as a limit result for *coalescing random walk in random medium.* Indeed we will show Theorem 2 by proving the following rescaling result for the joint law of the medium and of the reactant's dual process.

Proposition 1.15. *Fix $m, n \in \mathbb{N}$ and $x_1, \dots, x_m, y_1, \dots, y_n \in \mathbb{Z}^2$. Let $(\widetilde{X}^t_s)_{s\in[0,t]}$ be coalescing random walk in the medium η, started at time t with $\widetilde{X}^t_0 = \delta_{x_1} + \dots + \delta_{x_m}$. For all $z \in \{0,1\}^n$ and $k \le m$,*

$$\lim_{t\to\infty} \mathbf{P}^{\pi_\theta}[\eta_t(y_i) = z^i,\ i = 1, \dots, n;\ \widetilde{X}^t_t(\mathbb{Z}^2) = k]$$
$$= \mathbf{P}\left[Y^1_\infty = z^1;\ D^m_{\int_0^\infty p(Y^1_s)ds} = k\right] \mathbb{1}_{z^1 = \dots = z^n}.$$

Note that together with the duality (Proposition 1.5 and Lemma 1.14) this implies immediately Theorem 2.

1.2.3. Dimension one

A key point for the investigation of our process in dimension $d = 1$ is that it has a natural scaling limit. More precisely, if we scale time by T and space by $T^{1/2}$ we obtain a limiting process $(\eta^\infty_t, \xi^\infty_t)_{t\ge 0}$. The law $\mathscr{L}[(\eta^\infty_1, \xi^\infty_1)]$ will be the ingredient for a quantitative description of the longtime behaviour of the non-rescaled process in analogy to the previous theorems.

Let us start by considering the medium. The interfaces between the zeros and ones perform annihilating random walks. On the Brownian scaling they converge to annihilating Brownian motions. Arratia seems to have been the first who showed this convergence in the sense of an invariance principle. More precisely, there should exist an entrance law (denoted by $\mathscr{L}^{\theta_1}$) for a Markov process $(\eta^\infty_t)_{t>0}$ (note the problem arising at $t = 0$!) where, for each fixed t, η^∞_t is piecewise constant (in space) with values in $\{0, 1\}$, the discontinuities of $(\eta^\infty_t)_{t>0}$ perform annihilating Brownian motions, and

$$\mathscr{L}^{\pi\theta_1}\left[\left(\left(\eta_{Tt}\left(\left\lfloor T^{1/2}\bullet\right\rfloor\right)\right)\right)_{t>0}\right] \overset{T\to\infty}{\Longrightarrow} \mathscr{L}^{\theta_1}\left[\left(\eta^\infty_t(\bullet)\right)_{t>0}\right]. \tag{1.15}$$

Since Arratia's proof is a bit difficult to spot, we give an argument for (1.15) in Proposition 4.1. We will refer to η^∞ as "Arratia medium".

It is reasonable to conjecture that also the rescaled bivariate process converges and that, given η^∞, the limit ξ^∞ is the solution of the following formal SPDE

$$\frac{d}{dt}\xi^\infty_t(x) = \frac{1}{2}\frac{d^2}{dx^2}\xi^\infty_t(x) + \infty \cdot\ \eta^\infty_t(x)\sqrt{\xi^\infty_t(x)(1-\xi^\infty_t(x))}\ \dot{W}(t,x), \tag{1.16}$$

where $\dot{W}$ is space–time white noise. The factor "∞" should be understood in the sense that ξ^∞ is the limit of ξ^K as $K \to \infty$ where the ∞ is replaced by a factor K (in particular $\infty\eta_t(x) = 0$ if $\eta_t(x) = 0$). This would be an SPDE of the Mueller–Tribe type [MT95, Thm. 2] in a catalytic medium. However, the existence of a solution of (1.16) has not been established yet.

What we can show here is for given medium the existence of some process ξ^∞ that is given in terms of its mixed (space–time) moments

$$\mathbf{E}^{(\theta_1,\theta_2)}\left[\prod_{i=1}^m \xi^\infty_{t_i}(x_i)\Big|\eta^\infty\right] = \mathbf{E}^{\theta_1}\left[\theta_2^{|\{\widetilde{W}_1(x_i),\ i=1,\dots,m\}|}\Big|\eta^\infty\right]. \tag{1.17}$$

Here the space–time dual process $((\widetilde{W}_s(x_i))_{s\in[0,1]},\ i = 1, \dots, m)$ is a family of modified coalescing Brownian motions: they are frozen at x_i for $s \le 1 - t_i$ and a pair coalesces at the first instance $s \ge (1-t_i) \vee (1-t_j)$ with $\widetilde{W}_s(x_i) = \widetilde{W}_s(x_j)$ and $\eta^\infty_{1-s}(\widetilde{W}_s(x_i)) = 1$. If (1.16) does make sense then the moments of its solution are given by (1.17). In fact, the existence of a process obeying (1.17) follows easily by the standard Kolmogorov extension theorem, and since the law depends measurably on η, we arrive at:

Remark 1.16. Let $\theta_1, \theta_2 \in [0, 1]$. There exists a bivariate Markov process $(\eta^\infty, \xi^\infty)$ with values in $\{0, 1\}^{\mathbb{R}} \times [0, 1]^{\mathbb{R}}$, where η^∞ is the Arratia medium and where the moments of ξ^∞ given η^∞ are prescribed by (1.17). If x is a continuity point of $\eta^\infty_t(\bullet)$, then the map $x \mapsto \xi^\infty_t(x)$ is continuous at x if $\eta^\infty_t(x) = 0$. If $\eta^\infty_t(x) = 1$ then $\xi_t(x) \in \{0, 1\}$.

Theorem 3. *Let $\theta_1, \theta_2 \in [0, 1]$. Fix $m \in \mathbb{N}$ and $(x^i, t^i) \in \mathbb{R} \times [0, \infty)$, $i = 1, \dots, m$. Assume that we are given sequences $(x^i_T, t^i_T)_{T\ge 0}$ such that $(T^{-1/2}x^i_T, T^{-1}t^i_T) \overset{T\to\infty}{\longrightarrow} (x^i, t^i)$. Then*

$$\begin{aligned}
&\mathscr{L}^{\pi_{\theta_1}\otimes\bar{\pi}_{\theta_2}}\left[\left(\eta_{t^i_T}(x^i_T), \xi_{t^i_T}(x^i_T)\right)_{i=1,\dots,m}\right] \\
&\overset{T\to\infty}{\Longrightarrow} \mathscr{L}^{(\theta_1,\theta_2)}\left[\left(\eta^\infty_{t^i}(x^i), \xi^\infty_{t^i}(x^i)\right)_{i=1,\dots,m}\right].
\end{aligned} \tag{1.18}$$

Remark 1.17. Letting $m_{\theta_1,\theta_2} = \mathscr{L}[\xi^\infty_1(0) \in \bullet | \eta^\infty_1(0) = 0]$ we have

$$\mathscr{L}[(\eta_t, \xi_t)] \overset{t\to\infty}{\Longrightarrow} \theta_1 \left(\theta_2\delta_{(\underline{1},\underline{1})} + (1-\theta_2)\delta_{(\underline{1},\underline{0})}\right) + (1-\theta_1) \int_0^1 \delta_{(\underline{0},\underline{\theta})} m_{\theta_1,\theta_2}(d\theta). \tag{1.19}$$

Remark 1.18. Let us mention that also Theorem 3 could be extended to a more general class of initial states (as described in Remark 1.11).

Remark 1.19. In [GKW99, Thm. 3] it was shown that the reactant of two-dimensional catalytic branching random walk (CBRW) converges to a homogeneous Poisson point process with random intensity. This randomness could be described in terms of the density of catalytic super-Brownian motion (see [FK99, Thm. 1] or [Kle00a, Thm. 1]) which is the scaling limit of CBRW. In this respect the case of *one*-dimensional IFWD is similar to that *two*-dimensional CBRW.

The result of Theorem 3 raises the question whether its statement could be strengthened such that we could view $(\eta_{Tt}(\sqrt{T}\bullet), \xi_{Tt}(\sqrt{T}\bullet))$ as elements of a function space like $D([0, \infty), \{0, 1\}) \times D([0, \infty), [0, 1])$ or whether we can show convergence in path space (in the time variable).

Start with the evolution at a space point in time. The first observation is that the maps

$$t \mapsto \xi_t\left(\left\lfloor x\sqrt{T}\right\rfloor\right), \quad t \mapsto \xi^\infty_t(x) \tag{1.20}$$

do not have the same continuity properties: the first is continuous, the second may have jumps if $\eta_t^\infty(x) = 1$. Hence pathwise convergence cannot hold in the Skorokhod topology. If, however, we consider the measures on $\mathbb{R}$ with density functions $\xi_t(\lfloor\sqrt{T}\bullet\rfloor)$ on $\mathbb{R}$ then we conjecture that one has the pathwise convergence.

The problem with considering the system for fixed time as element of a function space is two-fold. First of all at points where $\eta_t^\infty(\cdot)$ changes values the random variable $\xi_t^\infty(\cdot)$ has no regularity properties and is of complicated nature so that different type function spaces have to be considered. Secondly it is an open problem to verify that a classical one-dimensional system of Fisher–Wright diffusions shows under the $\sqrt{T}$-rescaling as a random function in space the same qualitative limiting behaviour as a voter model.

Thus here are serious open problems which are intimately connected with the question which sense can be given to the equation (1.16).

1.3. Extensions

There are two extensions of our results, one concerning the migration mechanism and the other the state space of the reactant process of a component. We describe both extensions shortly.

In this paper we consider very special migration kernels (symmetric nearest neighbour). One is tempted to believe that the qualitative statements remain true if one assumes only that the kernels have second moments and vanishing drift. However, the technical difficulties in proving such a statement appear to be substantial. At this stage we prefer only to highlight the main features of the longtime behaviour by analysing the important examples.

However, one interesting case that is simple to discuss is that of a kernel *with drift*. More precisely, assume that the migration kernel of the reactant has second moments and a non-zero drift. Clearly, for $d \geq 3$ no qualitative change occurs. In dimension $d = 1$ and $d = 2$ the situation changes drastically. Now also the reactant clusters: For $\mu \in \mathscr{M}_{(\theta_1,\theta_2)}$,

$$\mathscr{L}^\mu\left[(\eta_t, \xi_t)\right] \overset{t\to\infty}{\Longrightarrow} \left(\theta_1\delta_{\underline{1}} + (1-\theta_1)\delta_{\underline{0}}\right) \otimes (\theta_2\delta_{\underline{1}} + (1-\theta_2)\delta_{\underline{0}}).$$

This can be understood easily using the duality given in (1.5). Consider pairs of particles of the coalescing random walk. The difference of two walks is again a random walk, but now without drift, hence it is recurrent. However, due to the drift, the two random walks explore the medium with a linear speed. Since in $d = 1$ the clusters of the voter model are of order $t^{1/2}$ only, two coalescing random walks will finally not only meet but meet also in the presence of the medium and hence coalesce. In $d = 2$ the diameter of clusters is $t^{\alpha/2}$ with random $\alpha \in [0, 1]$ and hence the same argument applies.

Another scope for generalisation is to modify the reactant, in particular its state space. Recall that Fisher–Wright diffusions describe the frequency of one type in a two type population located at the sites of $\mathbb{Z}^d$. Instead we could consider interacting multitype Fisher–Wright diffusions or Fleming–Viot processes describing populations with three or more respectively a continuum of types.

Both these processes have a dual process, which even though it is more complicated, is driven by the coalescing random walk in random medium analysed in this paper (compare Subsection 3(a) in [DGV95]). Thus it is clear that in $d \geq 3$, i.e. in Theorem 1, nothing changes while in $d = 2$, i.e. Theorem 2, we have to replace Y^2 by either a multitype Fisher–Wright diffusion or Fleming–Viot process. Finally, for $d = 1$ in Theorem 3 one has to write (1.17) using the duality relation for the respective one of *these* processes instead for usual FWD.

2. Proof of Theorem 1 ($d \geq 3$)

We give here a proof whose method works for far more general situations than for the special structure of our present model. Indeed, the behaviour in $d \geq 3$ as described here occurs in many other situations. In particular we could replace, in our choice for the medium, the voter model by interacting Fisher–Wright diffusions.

The main idea is to use the fact that the medium evolves autonomously towards an ergodic equilibrium state, which allows to treat it first and then consider the evolution of $(\xi_t)_{t\geq 0}$ for given medium consisting of realizations of the stationary process $(\eta_t)_{t\in(-\infty,\infty)}$. We proceed in steps: after recalling some basic facts about the medium we consider first convergence results on the bivariate law for special initial states and then later for general ones. With these we construct in Step 4 the extremal invariant measures, prove the convergence statement (1.9) and conclude in Step 5 by showing the claimed mixing properties.

Step 1. Observe first that for initial states $\nu \in \mathscr{M}_{(\theta_1,\theta_2)}$ using the projection ν_1 on the medium component as initial state of a voter model leads to the following property ([HL75, Thm. 1.9(c)]) for $d \geq 3$:

$$\mathscr{L}^{\nu_1}[\eta_t] \overset{t\to\infty}{\Longrightarrow} \mu_{\theta_1}, \tag{2.1}$$

where μ_{θ_1} is the unique extremal invariant measure with intensity θ_1. This measure is spatially mixing. Using the Markov and Feller property it is straightforward to prove the following strengthening of the ergodic theorem above:
Denote by $(\widetilde{\eta}_t)_{t\geq 0}$ the stationary process with marginal μ_{θ_1}. Then

$$\mathscr{L}^{\nu_1}\left[(\eta_{t+T})_{t\geq 0}\right] \overset{T\to\infty}{\Longrightarrow} \mathscr{L}\left[(\widetilde{\eta}_t)_{t\geq 0}\right]. \tag{2.2}$$

Step 2. We begin by considering the bivariate process starting in the following special initial state $\nu \in \mathscr{M}_{(\theta_1,\theta_2)}$ with $\nu = \nu_1 \otimes \bar{\pi}_{\theta_2}$ and $\nu_1 := \mu_{\theta_1}$. We start the process in this state at time $-t$ and denote the configuration arising at time 0 by $(\widetilde{\eta}_0^{-t}, \widetilde{\xi}_0^{-t})$. We can construct this process as follows. Realize the stationary process $(\widetilde{\eta}_t)_{t\in(-\infty,\infty)}$. Fix a version of this process and an initial state for the ξ-process, called $\widehat{\xi}$, which is independently sampled from $\bar{\pi}_{\theta_2}$. Then we can (simultaneously for all t) construct the distribution of $\widetilde{\xi}_0^{-t}$ for fixed medium $\widetilde{\eta}$ through a coalescing random walk $(\widetilde{X}_s)_{s\geq 0}$ with coalescence rate $\widetilde{\eta}_{-s}(x)$ at site x at times s. Consider the coalescing random walk $(\widetilde{X}_s)$ starting with k particles placed at the (not necessarily different) sites $x_1, \ldots, x_k \in \mathbb{Z}^d$. Note first that due to the monotonicity of the total number of particles $\widetilde{X}_t(\mathbb{Z}^d)$ converges to a random variable ζ_∞ whose law depends only on k and $\widetilde{\eta}$. Hence (recall (1.3))

$$\begin{aligned}\mathbf{E}\left[\widetilde{\xi}_0^{-t}(x_1)\ldots\widetilde{\xi}_0^{-t}(x_k)|(\widetilde{\eta}_s)_{s\le 0}\right] &= \mathbf{E}^\nu\left[\widehat{\xi}^{\widetilde{X}_t}\middle|(\widetilde{\eta}_s)_{s\le 0}\right] \\ &\overset{t\to\infty}{\longrightarrow} \sum_{j=1}^{k}(\theta_2)^j\mathbf{P}^{\widetilde{\eta},k}\left[\zeta_\infty = j\right].\end{aligned} \tag{2.3}$$

Note that from $\widetilde{\eta}$ only the part $(\widetilde{\eta}_s)_{s\le 0}$ enters in the r.h.s. above, and viewing this part of the process as element of $D((-\infty,0],\{0,1\}^{\mathbb{Z}^d})$ the r.h.s. of (2.3) is a continuous function of $\widetilde{\eta}$, since the difference random walks $(X_t^i - X_t^j)_{t\ge 0}$ starting at x_i and x_j are transient and since the dual process reads the medium backwards. (In fact, think of $\widetilde{X}$ as being constructed from k free random walks $X^1,\ldots,X^k$. If we fix a realization of $X^1,\ldots,X^k$ it is easy to see that the probability of having j free particles in the end is a continuous function of $\widetilde{\eta}$. Furthermore, for $T>0$ and $R>0$ on the event $A_{T,R} := \bigcap_{i\neq j}\{X_t^i \neq X_t^j \text{ if } t>T \text{ or } \|X_t^i\|_2 > R\}$ it is even uniformly (in the realizations of $X^1,\ldots,X^k$) continuous in $\widetilde{\eta}$. However due to the transience of the (difference) walks, $\mathbf{P}[A_{T,R}]\to 1$ as $T,R\to\infty$.)

As a consequence we can define for every fixed $\widetilde{\eta} = (\widetilde{\eta}_s)_{s\le 0}\in D((-\infty,0],\{0,1\}^{\mathbb{Z}^d})$

$$\nu_{\theta_2}(\widetilde{\eta}) = \lim_{t\to\infty}\mathscr{L}^{\widetilde{\eta}}[\widetilde{\xi}_0^{-t}]. \tag{2.4}$$

As shown above, the map

$$\widetilde{\eta}\mapsto \nu_{\theta_2}(\widetilde{\eta}) \tag{2.5}$$

from the path space $D((-\infty,0],\{0,1\}^{\mathbb{Z}^d})$ into $\dot{}([0,1]^{\mathbb{Z}^d})$ is continuous. Put

$$\nu_{\theta_1,\theta_2} := \int\left(\delta_{\widetilde{\eta}_0}\otimes\nu_{\theta_2}(\widetilde{\eta})\right)Q^{\theta_1}(d\widetilde{\eta}),\quad \text{where } Q^{\theta_1} = \mathscr{L}[(\widetilde{\eta}_s)_{s\le 0}]. \tag{2.6}$$

By construction we have for our special choice of ν that

$$\mathscr{L}^\nu\left[(\eta_t,\xi_t)\right] = \int\left(\delta_{\widetilde{\eta}_0}\otimes\mathscr{L}_{\widetilde{\eta}}[\widetilde{\xi}_0^{-t}]\right)Q^{\theta_1}(d\widetilde{\eta}) \overset{t\to\infty}{\Longrightarrow} \nu_{\theta_1,\theta_2}. \tag{2.7}$$

Note that relation (2.3) implies that (with $x_1 = x_2 = 0$)

$$\mathbf{Var}^{\nu_{\theta_1,\theta_2}}[\xi(0)] = \int Q^{\theta_1}(d\widetilde{\eta})\mathbf{P}^{\widetilde{\eta},2}[\zeta_\infty = 1]\theta_2(1-\theta_2) > 0$$

if $\theta_2\in(0,1)$. Hence together with the observation of the previous step we have proved the assertion (1.8) as far as the assertion for the variance goes.

Step 3. We want to show in this step that for *all* initial laws $\nu \in \mathscr{M}_{\theta_1,\theta_2}$ the bivariate process converges as $t \to \infty$ to ν_{θ_1,θ_2}, i.e.,

$$\nu \in \mathscr{M}_{(\theta_1,\theta_2)} \text{ implies } \quad \mathscr{L}^{\nu}[(\eta_t, \xi_t)] \overset{t\to\infty}{\Longrightarrow} \nu_{\theta_1,\theta_2}. \tag{2.8}$$

Denote again by ν_1 the projection of ν on the medium. It suffices according to Step 2 to show that the following pairs of initial distributions (i) $\nu_1 \otimes \bar{\pi}_{\theta_2}$ and $\mu_{\theta_1} \otimes \bar{\pi}_{\theta_2}$ as well as (ii) ν and $\nu_1 \otimes \bar{\pi}_{\theta_2}$ lead to the same bivariate limits in distribution as $t \to \infty$. Next we give these two arguments.

(i) Since $\nu \in \mathscr{M}_{\theta_1,\theta_2}$, the measure ν_1 has the intensity θ_1. Denote by $Q_t^{\nu_1}(d\widetilde{\eta})$ the distribution of the medium started at time $-t$ with distribution ν_1 and set $\equiv 0$ for times earlier than $-t$. With this notation (2.2) becomes $Q_t^{\nu_1} \overset{t\to\infty}{\Longrightarrow} Q^{\theta_1}$. Recalling the discussion following (2.3) it is clear that the convergence in (2.3) is even uniform in $\widetilde{\eta}$. Thus we can interchange limits and replace in (2.7) Q^{θ_1} by $Q_t^{\nu_1}$. In total we get for $\mu = \nu_1 \otimes \bar{\pi}_{\theta_2}$ that:

$$\mathscr{L}^{\mu}[(\eta_t, \xi_t)] \overset{t\to\infty}{\Longrightarrow} \nu_{\theta_1,\theta_2}. \tag{2.9}$$

(ii) Here we have to compare two initial measures which have the same projection on the medium component, so that it suffices to compare the two reactant processes evolving in one given medium. We construct the two processes $(\xi_t^1)_{t\geq 0}$ and $(\xi_t^2)_{t\geq 0}$ on one probability space by using for both the same realization of the medium and the same driving Brownian motions. The initial states are realized by choosing ξ_0^1 according to ν conditioned on the medium and ξ_0^2 according to $\bar{\pi}_{\theta_2}$ independently of everything else. If we can show that

$$f_t(x) = \mathbf{E}[|\xi_t^1(x) - \xi_t^2(x)|] \text{ tends to 0 as } t \to \infty, \tag{2.10}$$

we are done.

Using Itô-calculus we can derive for the collection $\{f_t(x), x \in \mathbb{Z}^d\}$ a system of differential equations (recall that $\mathscr{A}$ is the q-matrix of simple random walk)

$$\begin{aligned} \frac{d}{dt} f_t(x) = \mathscr{A} f_t(x) - 2 \sum_y \mathbf{E}\Big[&|\xi_t^1(y) - \xi_t^2(y)|;\ \operatorname{sign}(\xi_t^1(y) - \xi_t^2(y)) \\ &\neq \operatorname{sign}(\xi_t^1(x) - \xi_t^2(x))\Big] \end{aligned} \tag{2.11}$$

By the translation invariance of the law $\mathscr{L}[\xi_t]$, the first term vanishes. Therefore we see immediately that $f_t(x)$ is monotone decreasing. The system of equations is derived and analysed in [CG94], Subsection 3, a paper dealing with interacting diffusions with time–homogeneous diffusion coefficients. However, we get in our case exactly the same system of equations due to the translation invariance of the distribution of ξ_t averaged over the medium. The reason that f_t converges actually to 0 is due to the irreducibility of the migration and the fact that the diffusion term is mean preserving. The argument is roughly as follows. One shows that equation (2.11) implies that (ξ_t^1, ξ_t^2) must become ordered in the limit $t \to \infty$. If they would, however, become at a site strictly ordered with positive probability, this would contradict the fact that the intensities are preserved in the limit $t \to \infty$ (which follows from a second moment estimate) and are equal for both of the two coupled systems. The details of the argument can be found in the reference mentioned above and the quoted result simply carries over to the inhomogeneous evolution.

Step 4. Now we need to show that ν_{θ_1,θ_2} is an invariant measure of the bivariate process. A straightforward calculation shows that η_t and ξ_t are both mean preserving and since the components are bounded this means

$$\int \eta(x)\nu_{\theta_1,\theta_2}(d(\eta,\xi)) = \theta_1, \qquad \int \xi(x)\nu_{\theta_1,\theta_2}(d(\eta,\xi)) = \theta_2, \quad x \in \mathbb{Z}^d. \tag{2.12}$$

A simple second moment calculation (recall (1.6)), separately for η_t and for ξ_t given $(\eta_s)_{s\geq 0}$ shows that

$$\nu_{\theta_1,\theta_2} \in \mathscr{M}_{(\theta_1,\theta_2)}. \tag{2.13}$$

Since the bivariate process has the *Feller property* we can now argue as usual. Denote by $(S_t)_{t\geq 0}$ the semigroup of the bivariate process. Then

$$(\nu_{\theta_1,\theta_2})S_t = \left(\lim_{u\to\infty}(\pi_{\theta_1}\otimes\bar{\pi}_{\theta_2})S_u\right)S_t = \lim_{u\to\infty}\left(\pi_{\theta_1}\otimes\bar{\pi}_{\theta_2}\right)S_{t+u} = \nu_{\theta_1,\theta_2}. \tag{2.14}$$

This invariant measure ν_{θ_1,θ_2} is obviously translation invariant. Since ν_{θ_1,θ_2} has intensity (θ_1,θ_2) one can use the convergence property given in (2.8) to conclude that ν_{θ_1,θ_2} is an extremal invariant measure.

The Markov property of the bivariate process together with the Feller property allow immediately to conclude from (2.8) the convergence in (1.9).

Step 5. We finally use the duality for ξ and the graphical representation for η to show that ν_{θ_1,θ_2} is spatially mixing (which is stronger than (2.13)). Start with the first assertion. It suffices to show

$$\lim_{|y|\to\infty}\langle\nu_{\theta_1,\theta_2}, f\tau_y g\rangle = \langle\nu_{\theta_1,\theta_2}, f\rangle\langle\nu_{\theta_1,\theta_2}, \tau_y g\rangle \tag{2.15}$$

for functions $f, g : (\{0,1\}\times[0,1])^{\mathbb{Z}^d} \to [0,\infty)$ that are monomials and that depend only the coordinates from a finite set $A \subset \mathbb{Z}^d$. By $\tau_y g$ we denote the function g shifted by $y \in \mathbb{Z}^d$. We use the representation (2.7) for ν_{θ_1,θ_2} that allows us to condition on $\widetilde{\eta}$ first and use the duality for $\widetilde{\xi}^{-t}$. In fact,

$$\mathbf{E}\left[f(\widetilde{\eta}_0,\widetilde{\xi}_0^{-t})\tau_y g(\widetilde{\eta}_0,\widetilde{\xi}_0^{-t})\middle|\widetilde{\eta}\right] - \mathbf{E}\left[f(\widetilde{\eta}_0,\widetilde{\xi}_0^{-t})\middle|\widetilde{\eta}\right]\cdot\mathbf{E}\left[\tau_y g(\widetilde{\eta}_0,\widetilde{\xi}_0^{-t})\middle|\widetilde{\eta}\right]$$

can be bounded in terms of the probability that two random walks, started in A respectively in $y + A$, ever meet. Due to the transience of the difference walk, this probability vanishes as $|y| \to \infty$. Now use that μ_{θ_1} is mixing to conclude (2.15).

We argue similarly for the space–time mixing property of the corresponding stationary process. Here we use space–time observation points $A \subseteq \mathbb{R}\times\mathbb{Z}^d$, $|A| < \infty$ and $(t,y) + A$ and we let $|(t,y)| \to \infty$. We leave the straightforward details to the reader.

3. Proof of Theorem 2 ($d = 2$)

The proof of Theorem 2 is quite involved and uses elaborate techniques such as the multiple scale analysis of the diffusive clustering of the voter model.

In Subsection 3.1 we fix some notation and formulate a version of the well-known multiple scale cluster description of the voter model. With a view to the duality of the reactant to coalescing random walk in the medium η we give in Subsection 3.2 the asymptotics for the probability that *two* random walks coalesce. This is the technical core of the proof of Theorem 2. The generalisation to more than two random walks is carried out in Subsection 3.3 where we proof a statement (Proposition 3.14) that is slightly stronger than Proposition 1.15, which, in turn, implies Theorem 2. In Subsection 3.4 we establish that $p(\cdot)$ is uniquely defined by the boundary value problem (1.10), (1.11).

3.1. Multiple-scale analysis of the voter model

Recall that for η_0 a random initial configuration with intensity θ_1, the process (η_t) satisfies in $d = 2$ the convergence $\mathscr{L}[\eta_t] \overset{t\to\infty}{\Longrightarrow} \theta_1\delta_{\underline{1}} + (1-\theta_1)\delta_{\underline{0}}$. In fact, the order of magnitude of a cluster of 0's or 1's due to one ancestral voter (see [CG86, Thm. 5]) is known: the cluster-size in space is of the order $t^{\alpha/2}$ where α is a random variable with uniform distribution in $[0, 1]$. The age of such a cluster is of order t^{α} (see [FG96, Thm. 7]). We need here a finer analysis, in particular, we will have to investigate the behaviour of the voter-model observed in collections of time–space points which spread at possibly different polynomial scales. Next we make this precise.

Denote by $\mathbb{T}_N$ the set of all sequences e with values in $\{1, 2\}$ of length $\ell(e) \in \{0, \dots, N\}$. $\mathbb{T}_N$ carries the natural tree structure and we use the usual notation $e \wedge f$ for the greatest common ancestor of e and f as well as $\overleftarrow{e}$ for the predecessor of e and $e^{\leftarrow n}$ for the n-th predecessor of e. Finally, we write $\emptyset$ for the root which by convention corresponds to the empty sequence.

Now we analyse the configuration of the voter model in different time–space points which spread with t (age of the system) on various different scales. We specify next the needed time–space configurations.

Fix $\mathbb{T}_N$ and $\{\alpha^e : e \in \mathbb{T}_N\}$, where $\alpha^e \in [0, 1]$. Define $\beta^e = \prod_{f \le e} \alpha^f$. Assume that for these parameters for each $e \in \mathbb{T}_N$ we are given families $(T_t^e)_{t\ge 0}$ of time points $T_t^e \sim t$, and families $(x_t^e)_{t\ge 0}$ of points in $\mathbb{Z}^2$ such that

$$\limsup_{t\to\infty} \frac{\log(|T_t^e - T_t^f|)}{\log t} \le \beta^{e\wedge f}, \qquad e \ne f, \tag{3.1}$$

and

$$\lim_{t\to\infty} \frac{\log(|x_t^e - x_t^f|)}{\log t} = \frac{\beta^{e\wedge f}}{2}, \qquad e \ne f. \tag{3.2}$$

Further choose mass scaling functions (S_t^e) such that

$$\lim_{t\to\infty} \frac{\log S_t^e}{\log t} = \beta^e.$$

Next we need the objects to describe the behaviour of the voter model as $t \to \infty$ viewed on the grid described above. Let $((Y_s^e)_{s \geq 0}, e \in \mathbb{T}_N)$ be a $\mathbb{T}_N$-indexed family of Fisher–Wright diffusions with the following dependence structure

$$\begin{aligned} &Y_s^e = Y_s^f \text{ for } s \leq -\log(\beta^{e \wedge f}) \\ &(Y_s^e)_{s \geq -\log(\beta^{e\wedge f})} \text{ and } (Y_s^f)_{s \geq -\log(\beta^{e\wedge f})} \text{ are independent, given } Y^e_{-\log(\beta^{e\wedge f})}. \end{aligned} \tag{3.3}$$

The following proposition (or rather a slightly weaker version of it) is well known for the hierarchical group instead of $\mathbb{Z}^2$ as the site space (see [FG94, Thm. 3]) and follows easily as in [FG94] from results on random walks in $\mathbb{Z}^2$ which can be found in [CG86, Sec. 5 and Thm. 6].

Proposition 3.1 (Diffusive clustering).

$$\mathscr{L}^{\pi_\theta}\left[((S_t^e)^{-1}\eta_{T_t^e}(x_t^e + (S_t^e)^{1/2}(\bullet))^{e\in\mathbb{T}_N}\right] \overset{t\to\infty}{\Longrightarrow} \mathscr{L}^{\theta}\left[(Y^e_{-\log\beta^e}\cdot\lambda)^{e\in\mathbb{T}_N}\right], \tag{3.4}$$

where λ denotes the two-dimensional Lebesgue measure, and the configurations η_s are also viewed as point measures on $\mathbb{R}^2$.

We focus in the sequel on the situation where $S_t^e = t^{\beta^e}(\log t)^4$ and denote by B_t^e the ball in $\mathbb{Z}^2$ of (Euclidean) radius $\sqrt{S_t^e}$ centred at x_t^e.

The next goal is to strengthen the above statement by showing that asymptotic averages which are 0 or 1 can be replaced by pure configurations. More precisely, we want to show that the cases where the Fisher–Wright diffusion $(Y_t)_{t\geq 0}$ is in its boundary points 0 or 1 reflect in the voter model the cases where the corresponding balls are *completely* empty or filled. Denote

$$\widetilde{Y}_t^e = |B_t^e|^{-1}\eta_{T_t^e}(B_t^e).$$

Proposition 3.2 ("Empty boxes are really empty"). *For every $z \in \{0,1\}^{\mathbb{T}_N}$,*

$$\lim_{t\to\infty} \mathbf{P}^{\pi_\theta}\left[\widetilde{Y}_t^e = z^e,\ e\in\mathbb{T}_N\right] = \mathbf{P}^{\theta}\left[Y^e_{-\log\beta^e} = z^e,\ e\in\mathbb{T}_N\right]. \tag{3.5}$$

Proof. By the previous proposition (and the Portmanteau theorem) it suffices to show that

$$\liminf_{t\to\infty} \mathbf{P}^{\pi_\theta}\left[\widetilde{Y}_t^e = z^e,\ e\in\mathbb{T}_N\right] \geq \mathbf{P}^{\theta}\left[Y^e_{-\log\beta^e} = z^e,\ e\in\mathbb{T}_N\right]. \tag{3.6}$$

To avoid a blow up of notation, we show this only for $N = 0$ and $z = 0$, leaving the obvious generalisations to the reader.

Abbreviate $\alpha = \alpha^{\emptyset}$, $Y := Y^{\emptyset}$ and w.l.o.g. assume that $T_t^{\emptyset} = t$. We have to show that (with B_t the centred ball in $\mathbb{Z}^2$ with radius $t^{\alpha/2}(\log t)^2$)

$$\liminf_{t\to\infty} \mathbf{P}^{\pi_\theta}[\eta_t(B_t) = 0] \geq \mathbf{P}^{\theta}[Y_{-\log\alpha} = 0]. \tag{3.7}$$

To this end fix $\varepsilon > 0$ and $\delta > 0$ such that, $\alpha + \delta < 1$ and

$$\mathbf{P}^{\theta}[Y_{-\log\alpha} = 0] \leq \mathbf{P}^{\theta}[Y_{-\log(\alpha+\delta)} = 0] + \varepsilon. \tag{3.8}$$

Let $(\{(X_s^x)_{s\geq 0},\ \ x\in B_t\},\mathbf{P}^X)$ be a (t-dependent) family of coalescing random walks each of which is started at $X_0^x=x$. By the duality of the voter model with coalescing random walk applied between the time points $t-t^{\alpha+\delta}$ and t:

$$\mathbf{P}^{\pi_\theta}\left[\eta_t(B_t)=0\right]=\mathbf{P}^{\pi_\theta}\otimes\mathbf{P}^X\left[\eta_{t-t^{\alpha+\delta}}\left(X^x_{t^{\alpha+\delta}}\right)=0,\quad\forall\, x\in B_t\right]. \tag{3.9}$$

Now by [BCG86, Theorem 1] (for applications to the voter model see [CG90]), $D^t:=|\{X^x_{t^{\alpha+\delta}}:x\in B_t\}|$ is a random variable which converges in law to $D^\infty_{\log(1+(\delta/\alpha))}$, where $((D_s^\infty)_{s\geq 0},\mathbf{P}^D)$ is Kingman's coalescent started with infinitely many particles at time 0.

Furthermore, given D^t, the D^t remaining components have positions which are, as $t\to\infty$, asymptotically independent and distributed as $a_{t^{\alpha+\delta}}$, where a_t denotes the transition kernel of simple random walk on $\mathbb{Z}^2$.

Return to the proof of (3.7). Express the l.h.s. of this equation using (3.10). Employing the two facts above we obtain with the previous proposition together with the central limit theorem for a_t from (3.8) that

$$\begin{aligned}\liminf_{t\to\infty}\mathbf{P}^{\pi_\theta}[\eta_{t-t^\alpha}(B_t)=0]&=\liminf_{t\to\infty}\mathbf{E}^{\pi_\theta}\mathbf{E}^X\left[(1-(a_{t^{\alpha+\delta}}\eta_{t-t^{\alpha+\delta}})(0))^{D^t}\right]\\&=\mathbf{E}^\theta\mathbf{E}^D\left[(1-Y_{-\log(\alpha+\delta)})^{D^\infty_{\log(1+(\delta/\alpha))}}\right]\\&\geq\mathbf{P}^\theta[Y_{-\log(\alpha+\delta)}=0]\\&\geq\mathbf{P}^\theta[Y_{-\log\alpha}=0]-\varepsilon.\end{aligned} \tag{3.10}$$

Since $\varepsilon>0$ was arbitrary, the claim (3.7) follows. □

3.2. Coalescing random walk in a voter medium: two-particle case

Fix the time horizon t. Let $\bar{p}_t(\theta,x_1,x_2)$ be the probability that the coalescing random walk $\widetilde{X}^T$ in the medium, started with two particles at positions x_1 and x_2, has coalesced by time t. In this subsection we show that $\bar{p}_t(\theta,x_1,x_2)\overset{t\to\infty}{\longrightarrow}p(\theta)$, where $p(\theta)$ is the function introduced in Definition 1.9.

Let $(X_s^i)_{s\geq 0}$, $i=1,2$ be two independent random walks, started at x^1 and x^2, and define the events

$$\begin{aligned}\mathbb{B}_t^{x_1,x_2}&=\left\{\int_0^t\eta_{t-s}(X_s^1)\mathbb{1}_{X_s^1=X_s^2}\,ds>0\right\},\\ \mathbb{B}_t^{x_1,x_2}(K)&=\left\{\int_0^t\eta_{t-s}(X_s^1)\mathbb{1}_{X_s^1=X_s^2}\,ds>K\right\},\quad K>0.\end{aligned} \tag{3.11}$$

Furthermore let

$$p_t(\theta,x^1,x^2):=\mathbf{P}^{\pi_\theta}\left[\mathbb{B}_t^{x^1,x^2}\right]. \tag{3.12}$$

Clearly

$$\inf_{K>0}\liminf_{t\to\infty}\mathbf{P}^{\pi_\theta}[\mathbb{B}_t^{x_1,x_2}(K)]\leq\liminf_{t\to\infty}\bar{p}_t(\theta,x_1,x_2), \tag{3.13}$$

and

$$\limsup_{t\to\infty} p_t(\theta, x_1, x_2) \geq \limsup_{t\to\infty} \bar{p}_t(\theta, x_1, x_2). \tag{3.14}$$

We can show

$$\limsup_{t\to\infty} \mathbf{P}^{\pi_\theta}[\mathbb{B}_t^{x_1,x_2} \setminus \mathbb{B}_t^{x_1,x_2}(K)] = 0, \quad K > 0. \tag{3.15}$$

Indeed, this follows from the recurrence of $X_t^1 - X_t^2$ together with the fact that the medium has the property that for $T < \infty$, $C < \infty$, $x \in \mathbb{Z}^2$:

$$\lim_{s\to\infty} \mathbf{P}^{\pi_\theta}\big[\eta_{s-u}(y) = 1 \quad \forall u \leq T, \quad \forall\, y \text{ such that } ||x-y||_2 \leq C \\ |\eta_s(x) = 1\big] = 1. \tag{3.16}$$

Hence we have

$$\lim_{t\to\infty} (\bar{p}_t(\theta, x_1, x_2) - p_t(\theta, x_1, x_2)) = 0 \tag{3.17}$$

and we can work in the sequel with the function p_t instead of $\bar{p}_t$.

The key technical result of this subsection is:

Proposition 3.3.

$$\lim_{t\to\infty} p_t(\theta, x^1,\ x^2) = p(\theta).$$

Proof. Since the medium covers as $t \to \infty$ arbitrarily large (finite) blocks with either all 0 or all 1, it suffices to consider $p_t(\theta)$ instead of $p_t(\theta, x^1,\ x^2)$ where we used the abbreviation $p_t(\theta) = p_t(\theta, 0, 0)$. We also write $\mathbb{B}_t = \mathbb{B}_t^{0,0}$.

The proof consists of several steps. We have to analyse first the structure of the sets of time points in $[0, t]$ where the two random walks meet, this happens in the first two steps. In the third step we bring the properties of the medium given in Subsection 3.1 into play and in the fourth step we show that the probabilistic representations for $p(\theta)$ given in Lemma 1.13 are asymptotic lower and upper bounds respectively for $p_t(\theta)$.

Step 1. Note that once the two walks meet, there will be many collisions before they separate for a longer time again. This behaviour can be captured best by performing a hierarchically structured multiple-scale analysis.

Recall that we consider the $p_t(\theta) = p_t(\theta, 0, 0)$, thus we let X^1 and X^2 be two independent random walks that are both started in 0. We have to define the last time before t where the random walks meet and put it on a logarithmic scale:

$$A_t := \frac{\log\sup\{s < t:\ X_s^1 = X_s^2\}}{\log t}. \tag{3.18}$$

Note that

$$\mathbf{P}[A_t > \alpha] = \mathbf{P}[X_s^1 = X_s^2 \text{ for some } s \in (t^\alpha, t)].$$

Denote by $\mathscr{U}[0, 1]$ the uniform distribution on $[0, 1]$. The following lemma is one basic ingredient for the proof of Proposition 3.3.

Lemma 3.4.

$$\mathscr{L}[A_t] \overset{t\to\infty}{\Longrightarrow} \mathscr{U}[0,1].$$

Proof. Fix $\alpha \in (0,1)$ and let $\varepsilon > 0$ be arbitrary. We show that $\limsup_{t\to\infty} |\mathbf{P}[A_t > \alpha] - (1-\alpha)| \le \varepsilon$. To this end we localise the most likely relative position of the random walks at time t^α. It follows via the CLT that we can pick $c, C \in (0,\infty)$ such that

$$\liminf_{t\to\infty} \mathbf{P}\left[\|X^1_{t^\alpha} - X^2_{t^\alpha}\|_2 \in (ct^{\alpha/2}, Ct^{\alpha/2})\right] \ge 1 - \frac{\varepsilon}{2}. \tag{3.19}$$

For the relative positions in the above interval we can use the Erdös–Taylor theorem for simple random walk $(X_s)_{s\ge 0}$ in $\mathbb{Z}^2$, started in t-dependent locations $x_t \in \mathbb{Z}^2$ (see [ET60, Eq. (2.16)] or [Kle96, Proposition 2.4]):

$$\lim_{t\to\infty} \mathbf{P}^{x_t}[X_s \text{ hits } 0 \text{ for some } s \in [0, t - t^\alpha]] = 1 - \alpha, \tag{3.20}$$

uniformly in all sequences (x_t) in $\mathbb{Z}^2$ with $\|x_t\|_2 \in (ct^{\alpha/2}, Ct^{\alpha/2})$. Now combine (3.19) and (3.20) to get the conclusion of the lemma. □

Step 2. In the sequel we condition on $A_t = \alpha$ for some $\alpha \in (0,1)$. Having fixed A_t, the difference walk $(X^1_s - X^2_s)_{0\le s\le t^\alpha}$ is a random walk bridge from 0 to 0 with rate 2.

Consider the two random walks with new time horizon $[0, t^\alpha]$ viewed forward respectively viewed backwards from 0 and t^α. Proceeding similarly as above we define for the two endpoints of the bridge from Step 1:

$$\begin{aligned} A^1_t &= \frac{\log \sup\{s\in[0,t^\alpha/\log t]: X^1_s = X^2_s\}}{\alpha \log t}, \\ A^2_t &= \frac{\log \sup\{s\in[0,t^\alpha/\log t]: X^1_{t^\alpha - s} = X^2_{t^\alpha - s}\}}{\alpha \log t}. \end{aligned} \tag{3.21}$$

The key to our construction is the following independence property:

Lemma 3.5.

$$\mathscr{L}\left[\left(A^1_t, A^2_t\right) | A_t = \alpha\right] \overset{t\to\infty}{\Longrightarrow} \mathscr{U}[0,1] \otimes \mathscr{U}[0,1]. \tag{3.22}$$

Proof. The point is that the random variables A^1_t, A^2_t depend on a time span which is small compared to the time span defining the bridge of the difference random walk. We leave the details of the argument to the reader. □

By the procedure preceding (3.21) the bridge has been split into two independent (given $A^1_t = \alpha^1$ and $A^2_t = \alpha^2$ asymptotically as $t \to \infty$) bridges

$$\left(X^1_s - X^2_s\right)_{s\in[0,t^{\alpha\cdot\alpha^1}]} \tag{3.23}$$

and

$$\left(X^1_s - X^2_s\right)_{s\in[t^\alpha - t^{\alpha\cdot\alpha^2}, t^\alpha]}. \tag{3.24}$$

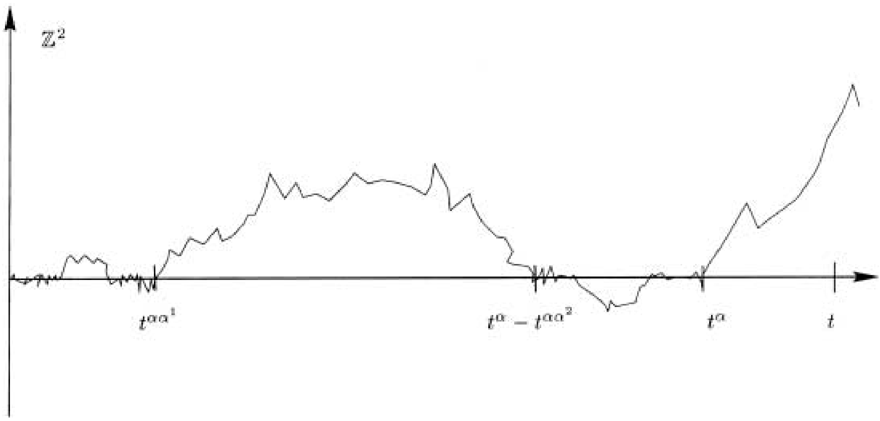

Fig. 1. Sketch of the difference of two random walks.

This construction has the property:

$$\begin{aligned}\limsup_{t\to\infty} \mathbf{P}[X_s^1 = X_s^2 \text{ for some } s \in [t^{\alpha\cdot\alpha^1}, t^{\alpha} - t^{\alpha\cdot\alpha^2}] \big| A_t = \alpha,\\ A_t^1 = \alpha^1, A_t^2 = \alpha^2] = 0. \qquad (3.25)\end{aligned}$$

Indeed by definition $X_s^1 \neq X_s^2$ for $s \in [t^{\alpha\alpha^1}, t^{\alpha}/\log t] \cup [t^{\alpha} - t^{\alpha}/\log t, t - t^{\alpha\alpha^2}]$. Since by Lemma 3.5 the limiting distributions of A_t^1 and A_t^2 (as $t \to \infty$) do not have atoms at 1, we may assume $\alpha^1, \alpha^2 < 1$, thus $(t^{\alpha}/\log t) - t^{\alpha\alpha^i} \sim t^{\alpha}/\log t$, $i = 1, 2$. This implies that

$$\left\| X^1_{t^{\alpha}/\log t} - X^2_{t^{\alpha}/\log t} \right\|_2 \sim (t^{\alpha}/\log t)^{1/2}.$$

The Erdös–Taylor lemma now yields that with high probability also $X_s^1 \neq X_s^2$ for $s \in [t^{\alpha}/\log t, t^{\alpha} - t^{\alpha}/\log t]$.

Next iterate this procedure N times, to obtain thereby 2^N (asymptotically as $t \to \infty$) independent bridges. We index these bridges by $e \in \{1, 2\}^N$. The bridges of "generations" $\leq N$ are then indexed by the tree $\mathbb{T}_N$. The bridge e starts at a time t^e_{start} and ends at a time t^e_{end}. The lifetime t^e_{life} equals $t^e_{\text{end}} - t^e_{\text{start}}$. Analogously to (3.21) we write

$$A_t^e = \frac{\log t^e_{\text{life}}}{\log t^{\overleftarrow{e}}\,\text{life}}.$$

Iterating the argument of Lemma 3.5 we get

Lemma 3.6. $\mathscr{L}[(A_t^e, e \in \mathbb{T}_N)] \overset{t\to\infty}{\Longrightarrow} \mathscr{U}[0,1]^{\otimes \mathbb{T}_N}$.

In the arguments later on we condition on $\{A_t^e = \alpha^e : e \in \mathbb{T}_N\}$ for some numbers $\alpha^e \in (0, 1)$ and we write

$$\beta^e = \prod_{f \leq e} \alpha^f, \quad e \in \mathbb{T}_N. \qquad (3.26)$$

Thus $t^e_{\text{life}} = t^{\beta^e}$, $e \in \mathbb{T}_N$. Note that we have for $e, f \in \mathbb{T}_N$:

$$\left|t^e_{\text{start}} - t^f_{\text{start}}\right| \sim (t^{\beta^{e\wedge f}}), \quad e \wedge f \notin \{e, f\}, \tag{3.27}$$

and

$$\left|X^1_{t^e_{\text{start}}} - X^1_{t^f_{\text{start}}}\right| \sim t^{\beta^{e\wedge f}/2}, \quad e \wedge f \notin \{e, f\}. \tag{3.28}$$

Step 3. In order to handle the influence of the medium on all of the 2^N bridges simultaneously we have to appeal to the corresponding multiple scale analysis of the two-dimensional voter model combined with properties of the range of random walk.

Denote by $\widetilde{B}^e_t$ the ball of radius $\sqrt{t^e_{\text{life}}}\log t$ centred at $X^1_{t^e_{\text{start}}}$. With probability tending to one the bridge e stays in $\widetilde{B}^e_t$ during its lifetime. What we show next is that with high probability the medium is 0 (respectively 1) for *all* time–space points in $[t^e_{\text{start}}, t^e_{\text{end}}] \times \widetilde{B}^e_t$ given that $\eta_{t^e_{\text{start}}}(x) = 0$ for all $x \in B^e_t$ (respectively 1). Recall that B^e_t has radius $\sqrt{t^{\beta^e}}(\log t)^2$.

Lemma 3.7. *For* $z \in \{0, 1\}$,

$$\lim_{t\to\infty} \mathbf{P}^{\pi_\theta}\left[\eta_s(x) \neq z \text{ for some } x \in \widetilde{B}^e_t \text{ and } s \in \left[t^e_{\text{start}}, t^e_{\text{end}}\right] \middle| \eta_{t^e_{\text{start}}} \equiv z \text{ on } B^e_t\right] = 0. \tag{3.29}$$

Proof. W.l.o.g. we may assume $z = 0$. We use the duality of the voter model (see the description below Definition 1.6). In fact we need an extended version of this duality for the time–space process. Consider a collection $\{(X^{x,s}_t)_{t\in[t^e_{\text{start}},s]},\ x \in \widetilde{B}^e_t,\ s \in [t^e_{\text{start}}, t^e_{\text{end}}]\}$ of (instantly) coalescing random walks running backwards in time. The walk $X^{x,s}$ is started at time s in x. The extended duality says in our context that

$$\begin{aligned}
&\mathbf{P}\left[\sup\left\{\eta_s(x), s \in \left[t^e_{\text{start}}, t^e_{\text{end}}\right],\ x \in \widetilde{B}^e_t\right\} \neq 0 \middle| \eta_{t^e_{\text{start}}}\right] \\
&\quad = \mathbf{P}\left[\sup\left\{\eta_{t^e_{\text{start}}}\left(X^{s,x}_{t^e_{\text{start}}}\right),\ s \in \left[t^e_{\text{start}}, t^e_{\text{end}}\right],\ x \in \widetilde{B}^e_t\right\} \neq 0 \middle| \eta_{t^e_{\text{start}}}\right].
\end{aligned} \tag{3.30}$$

See [CG83] for a derivation. The uncountably many random walks coalesce immediately to a finite random number of particles. Furthermore this random number (by scaling) is of order $\mathcal{O}(t^e_{\text{life}}|\widetilde{B}^e_t|) = \mathcal{O}((t^e_{\text{life}}\log t)^2)$. It suffices therefore to show that the probability for one of these particles to be in $(B^e_t)^{\text{c}}$ at time t^e_{start} is uniformly $\mathcal{O}(t^{-3})$.
Note that for simple random walk (X_s) on $\mathbb{Z}$ we have

$$\mathbf{E}\left[e^{\lambda X_s}\right] = \cosh(\lambda)^s, \quad \lambda > 0, s > 0.$$

Choose $\lambda = s^{-1/2}$ and use Chebyshev's inequality to get for $r > 0$:

$$\begin{aligned}
\mathbf{P}[X_s \geq r] &= \mathbf{P}[e^{\lambda X_s} \geq e^{\lambda r}] \\
&\leq e^{-\lambda r}\cosh(\lambda)^s \\
&\leq e^{-1/2}e^{-r/\sqrt{s}}.
\end{aligned} \tag{3.31}$$

Coming back to our problem we see that (choosing $r = \sqrt{S_t^e} = (t^e_{\text{life}})^{1/2}(\log t)^2$ and $s = t^e$life) for any of the random walks the probability to be at time t^e_{start} in $(B^e_t)^c$ is smaller than $4\exp(-(\log t)^2)$. Thus we have established Lemma 3.7. $\square$

Combining Lemma 3.7 with Proposition 3.2 we get the following statement:

Corollary 3.8. *For every* $z \in \{0,1\}^{\mathbb{T}_n}$

$$\liminf_{t\to\infty} \mathbf{P}^{\pi_\theta}\left[\eta_s(x) = z^e \textit{ for all } x \in \widetilde{B}^e_t \textit{ and } s \in \left[t^e_{\text{start}}, t^e_{\text{end}}\right]\right] \geq \mathbf{P}^{\theta}\left[Y^e_{-\log\beta^e} = z^e\right]. \tag{3.32}$$

Since two walks coalesce with very high probability if they meet in a cluster of 1's of the medium that is of size B^e_t we easily derive the following proposition.

Lemma 3.9. *For every fixed* $N \in \mathbb{N}$ *the following estimates hold*:

$$\begin{aligned} &\limsup_{t\to\infty} \mathbf{P}^{\pi_\theta}\left[\mathbb{B}_t \middle| A^e_t = \alpha^e,\ e \in \mathbb{T}_N\right] \\ &\quad \leq 1 - \mathbf{P}^{\theta}\left[Y^e_{-\log\beta^e} = 0 \quad \textit{for all } e \in \{1,2\}^N\right] \\ &\liminf_{t\to\infty} \mathbf{P}^{\pi_\theta}\left[\mathbb{B}_t \middle| A^e_t = \alpha^e,\ e \in \mathbb{T}_N\right] \\ &\quad \geq \mathbf{P}^{\theta}\left[Y^e_{-\log\beta^e} = 1 \textit{ for some } e \in \{1,2\}^N\right]. \end{aligned} \tag{3.33}$$

Step 4. Finally, we derive the formula for $\lim_{t\to\infty} \mathbf{P}^{\pi_\theta}[\mathbb{B}_t]$.

Recall Definition 1.12 of the process Z_t started with one particle at θ. We stop the evolution of each particle once the N-th generation is reached. Denote by Z^N the random population of these 2^N particles. From Lemma 3.9 we get, letting the A^e_t be random again, the following lemma.

Lemma 3.10.

$$\begin{aligned} \mathbf{P}^{\delta_\theta}[Z^N(\{1\}) > 0] &\leq \liminf_{t\to\infty} \mathbf{P}^{\pi_\theta}[\mathbb{B}_t] \\ &\leq \limsup_{t\to\infty} \mathbf{P}^{\pi_\theta}[\mathbb{B}_t] \leq \mathbf{P}^{\delta_\theta}[Z^N((0,1]) > 0]. \end{aligned} \tag{3.34}$$

Proof. Assume that $(A^e,\ e \in \mathbb{T}_N)$ is an i.i.d. family of $\mathscr{U}[0,1]$ random variables. Define

$$B^e = \prod_{f\leq e} A^f. \tag{3.35}$$

Hence $-\log B^e$ is a sum of i.i.d. exp(1) random variables. It follows that (recall 3.3)

$$\mathscr{L}^{\theta}\left[\sum_{e\in\{0,1\}^N} \delta_{Y^e_{B^e}}\right] = \mathscr{L}^{\delta_\theta}[Z_N]. \tag{3.36}$$

Now recall that $(A_t^e,\ e \in \mathbb{T}_N)$ is asymptotically i.i.d. $\mathscr{U}[0,1]$ (see Lemma 3.6). Since the r.h.s. (3.33) depends continuously on $\{\alpha^e,\ e \in \mathbb{T}\}$, we can in (3.33) integrate over $\{A_t^e,\ e \in \mathbb{T}_N\}$ and pass to the limiting distribution of the latter random variables to get (3.34) from (3.33). □

Let us return to the proof of Proposition 3.3. Note that

$$\lim_{N\to\infty} \mathbf{P}^{\delta_\theta}\left[Z^N(\{1\}) > 0\right] = \lim_{t\to\infty} \mathbf{P}^{\delta_\theta}\left[Z_t(\{1\}) > 0\right] \tag{3.37}$$

and

$$\lim_{N\to\infty} \mathbf{P}^{\delta_\theta}\left[Z^N(\{1\}) > 0\right] = \lim_{t\to\infty} \mathbf{P}^{\delta_\theta}[Z_t((0,1]) > 0]. \tag{3.38}$$

Hence by Lemma 1.13 both sides in (3.34) coincide with $p(\theta)$. This finishes the proof of Proposition 3.3. □

Now we can summarise the content of this subsection in the following proposition, which describes the asymptotic of two walks and the relevant statistics of the medium. Recall that $\mathbb{B}_t$ is the event that two random walks meet in the presence of the medium provided they run backwards from time t, evolve independently of the medium and start both at site 0. Consider (Z_t, Y_t) where Y_t is the position in $[0,1]$ of a tagged particle from Z_t. Therefore, (Y_t) is again a Fisher–Wright diffusion. With this object we can describe the block-averages of the medium together with the properties of the coalescent as follows:

Proposition 3.11.

$$\begin{aligned}&\mathscr{L}^{\pi_\theta}\left[\left(\left(\frac{1}{4}t^{-\alpha}\eta_t\left(\left[-t^{\alpha/2}, t^{\alpha/2}\right]^2\right),\ \alpha \in [0,1]\right), \mathbb{1}_{\mathbb{B}_t}\right)\right],\\ &\underset{\text{fdd}}{\overset{t\to\infty}{\Longrightarrow}} \mathscr{L}^{\delta_\theta}\left[\left(((Y_{-\log\alpha}),\ \alpha \in [0,1]), \mathbb{1}_{Z_\infty((0,1])>0)}\right)\right].\end{aligned} \tag{3.39}$$

Proof. Note that the convergence of the second component is an immediate consequence of Lemma 3.9 and 3.10. The convergence of the first component is just a special case of Proposition 3.1. The point is, of course, to show that the dependence structure between the two components is given correctly. To this end go back to Lemma 3.9. From the fact that $\|X^1_{t^\alpha} - X^2_{t^\alpha}\|_2 \sim t^{\alpha/2}$ and Proposition 3.1 we get

$$\begin{aligned}&\mathscr{L}^{\pi_\theta}\left[\left(\left(\frac{1}{4}t^{-\alpha}\eta_t\left(\left[-t^{\alpha/2}, t^{\alpha/2}\right]^2\right),\right.\right.\right.\\ &\qquad\left.\left.\left.\frac{1}{4}t^{-\alpha}\eta_t\left(X^1_{t^\alpha} + \left[-t^{\alpha/2}, t^{\alpha/2}\right]^2\right)\right)\ \alpha \in [0,1]\right)\right]\\ &\qquad\underset{\text{fdd}}{\overset{t\to\infty}{\Longrightarrow}} \mathscr{L}^{\delta_\theta}\left[\left(\left(Y^{\emptyset}_{-\log\alpha}, Y^{\emptyset}_{-\log\alpha}\right),\ \alpha \in [0,1]\right)\right].\end{aligned}$$

Thus we can add on the left hand sides in (3.33) conditions on $\frac{1}{4}t^{-\alpha}\eta_t([-t^{\alpha/2}, t^{\alpha/2}]^2)$ at finitely many points α and have to impose the same conditions at the right hand side of (3.33) – now for $Y^{\emptyset}$ (or any other fixed Y^e). The same is true in (3.34) when we replace $Y^{\emptyset}$ by one tagged particle of Z^N. Now argue as in (3.37)ff in order to replace Z^N first by Z_t and then let $t \to \infty$. □

3.3. Coalescing random walk in a voter medium: m-particle case

Recall that we want to study the asymptotic number of particles of a coalescing random walk in the voter-medium. We did this in the previous subsection for starting with two particles. Now we generalise to m particles. Our strategy is, as in Subsection 3.2, to obtain a result on the times when free particles meet first. Then we use the information on the clusters of the voter model (medium) to compute how many particle really coalesce.

The usual way that coalescing random walk $(\widetilde{X}^t_s)_{s\in[0,t]}$ with $\widetilde{X}^t_0(\mathbb{Z}^2) = m$ is generated from independent random walk $(X^i_s)_{s\geq 0}, i = 1, \dots, m$ is as follows. Two free particles i and j at the same position $X^i_s = X^j_s$ at time s induce a coalescence in $\widetilde{X}^t$ at rate $\eta_{t-s}(X^i_s)$, if the particles have not coalesced before (in $\widetilde{X}^t$).

If we are interested only in the distribution of $\widetilde{X}^t_t$ (instead of in the whole path) then we can reverse the order in which the particles coalesce: i and j coalesce at rate $\eta_{t-s}(X^i_s)\mathbb{1}_{X^i_s = X^j_s}$, where s runs backwards from t to 0. This fact simplifies the investigation considerably since in the limit $t \to \infty$ the times s when two particles meet accumulate at 0, when seen in the logarithmic scale $\alpha_s = \dfrac{\log s}{\log t}$.

3.3.1. Free random walks

Let $(X^i_s)_{s\geq 0}$, $i = 1, \dots, m$, be independent simple random walks on $\mathbb{Z}^2$. We introduce marked coalescence times. Whenever two random walks meet they stay “close” together and keep recolliding for a while until they have again a distance comparable to the mutual distances of the other random walks. These meeting times form asymptotically as $t \to \infty$ a discrete point process of times and labels (which indicate which pair has met). We formalise this idea here and prove a distributional limit result for $t \to \infty$.

Define random variables $\alpha_{n,t}$ and $\beta_{n,t}$, $n \in \mathbb{N}$, $t \geq 0$ by $\alpha_{0,t} = \beta_{0,t} = 1$ and

$$\alpha_{n,t} = \frac{\log \sup\left\{s < t^{\beta_{n-1,t}}/\log t :\ X^i_s = X^j_s \text{ for some } i \neq j\right\}}{\beta_{n-1,t}\log t}, \tag{3.40}$$

where $\beta_{n,t} = \prod_{k\leq n}\alpha_{k,t}$. Denote by $\ell_{n,t}$ the two indices of a pair of particles chosen at random from those that meet at time $t^{\beta_{n,t}}$, so that we have

$$\ell_{n,t} = (\ell_{n,t}(1), \ell_{n,t}(2)) \in \left\{(i,j)|i < j \text{ and } X^i_{t^{\beta_{n,t}}} = X^j_{t^{\beta_{n,t}}}\right\}. \tag{3.41}$$

Finally, let $A_{n,t} = (\alpha_{n,t}, \ell_{n,t})$, $n \in \mathbb{N}$.

The key which allows us to reduce everything to the constructions used in the two particle case is the following:

Lemma 3.12. (i) *Let $(A_n) = (\alpha_n, \ell_n)$, $n \in \mathbb{N}$ an i.i.d. family of random variables with*

$$\mathbf{P}\left[\alpha_n < x, \ell_n = (i,j)\right] = (1-x)^{\binom{m}{2}} \Big/ \binom{m}{2}, \quad x \in [0,1],\ 1 \leq i < j \leq m.$$

Then

$$\mathscr{L}\left[(A_{n,t})_{n\in\mathbb{N}}\right] \overset{t\to\infty}{\Longrightarrow} \mathscr{L}\left[(A_n)_{n\in\mathbb{N}}\right], \tag{3.42}$$

(ii) *For any $n \in \mathbb{N}$ the definition of $\ell_{n,t}$ leads asymptotically (as $t \to \infty$) to a unique pair*:

$$\limsup_{t\to\infty} \mathbf{P}\left[X_s^i = X_s^j \text{ for some } s \in \left[t^{\beta_{n,t}}/\log t, t^{\beta_{n,t}}\right] \text{ and } (i,j) \neq \ell_{n,t}\right] = 0. \tag{3.43}$$

Proof. The first part follows from [CG86, Sec. 5] who showed that asymptotically the different pairs of random walks act independently. Then by Lemma 3.6 together with the exchangeability of the random walks the claim follows.

The second part is obvious since the pairs of particles that do not meet at time $t^{\beta_{n,t}}$ have a distance of order $t^{\beta_{n,t}/2}$ (see [CG86, Lemma 1 on page 363]) and hence do not meet in the time interval of length $t^{\beta_{n,t}}/\log t$ (see (3.20)). □

3.3.2. Coalescence in the medium

We bring the medium back into the picture. Define the event that the pair $\ell_{n,t}$ of walks meeting at time $t^{\beta_{n,t}}$ experiences the catalyst within the time they spend together at this instance. Let

$$\mathbb{B}_{n,t} = \left\{\int_{t^{\beta_{n,t}}/\log t}^{t^{\beta_{n,t}}} \eta_{t-s}\left(X_s^{\ell_{n,t}(1)}\right) \mathbb{1}_{X_s^{\ell_{n,t}(1)}=X_s^{\ell_{n,t}(2)}}\, ds > 0\right\}. \tag{3.44}$$

Lemma 3.13. *Asymptotically as $t \to \infty$: Conditioned on the path of*

$$\bar{Y}_s^t := \frac{1}{4} t^{-e^{-s}} \eta_{t-t^{e^{-s}}}\left(\left[-t^{e^{-s}/2}, t^{e^{-s}/2}\right]^2\right), \quad s \geq 0 \tag{3.45}$$

$(\mathbb{B}_{n,t})_{n\in\mathbb{N}}$ *is an independent sequence and*

$$\lim_{t\to\infty} \mathbf{E}\left[\left|\mathbf{E}\left[\mathbb{B}_{n,t}\middle|\bar{Y}^t\right] - p\left(\bar{Y}^t_{-\log\beta_{n,t}}\right)\right|\right] = 0. \tag{3.46}$$

Proof. The proof is a refinement of the arguments used in showing Proposition 3.3. It suffices to show the statement for fixed $N \in \mathbb{N}$ and $\{B_{n,t},\ n \leq N\}$. As a first step one notes that given the $\beta_{n,t}$ and $(\eta_s)_{s\in[t^{\beta_{n,t}}/\log t, t^{\beta_{n,t}}]}$, $n \leq N$, we have independence of the $B_{n,t}$. In order to compute the conditioned probabilities one proceeds as in Step 2 and 3 in Section 3.2: one splits $(X_s^{\ell_{n,t}(1)} - X_s^{\ell_{n,t}(2)})_{s\in[t^{\beta_{n,t}}/\log t, t^{\beta_{n,t}}]}$ into finitely many bridges and applies Lemma 3.7 to reduce the condition on the path of η to a condition on small spatial windows at finitely many time points. To all of these windows (also for all $n \leq N$) one applies the multiple scale analysis of the clusters of the voter model (Proposition 3.1) to conclude as in Step 4 of Section 3.2 that the conditional probability of $B_{n,t}$ is asymptotically (as $t \to \infty$) our $p(\bar{Y}^t_{-\beta_{n,t}})$. This yields both, asymptotic independence given $\bar{Y}^t$ as well as formula (3.46). We omit the tedious details. □

3.3.3. Rescaling limit of the marked coalescence times

We combine Lemma 3.12 and Lemma 3.13 with the proposition on the diffusive clustering of the voter model (Proposition 3.1) to the following statement which is stronger than Proposition 1.15. This proposition in turn implies Theorem 2. Hence with the next proposition we finish the proof of Theorem 2.

Proposition 3.14.

$$\mathscr{L}^{\pi_\theta}\left[\left(\bar{Y}_s^t\right)_{s\geq 0}, \sum_{n=1}^{\infty} \mathbb{1}_{\mathbb{B}_{n,t}} \delta_{(-\log \beta_{n,t}, \ell_{n,t})}\right] \underset{\text{fdd}}{\overset{t\to\infty}{\Longrightarrow}} \mathscr{L}^{\theta}\left[(Y_s)_{s\geq 0}, \pi^Y\right], \qquad (3.47)$$

where π^Y is a Poisson point process on $\mathbb{R}^+ \times \{(i,j) : \ 1 \leq i < j \leq m\}$ with intensity (λ is the counting measure) $p(Y_s)ds\lambda(d(i,j))$.

Proof of Proposition 1.15. Consider (3.47). The meaning of a point $\delta_{(s,(i,j))}$ in the expression on the l.h.s. of (3.47) is that the particles i and j coalesce by time $t^{e^{-s}}$ if they have not coalesced before (recall (3.11) – (3.17)). In particular, by the exchangeability of the particles, the total rate of coalescence is, in the limit $t \to \infty$, $\binom{m_s}{2} p(Y_s)$ if there are m_s uncoalesced particles at time $t^{e^{-s}}$. Furthermore π^Y depends continuously on Y. This proves the convergence of the coalescent in the voter medium to the time-transformed Kingman coalescent. Since the voter model converges in law to $\theta_1\delta_{\underline{1}} + (1-\theta_1)\delta_{\underline{0}}$ we have proved Proposition 1.15. □

3.4. Harmonic functions of branching Fisher–Wright diffusions

In this subsection we give the
Proof of Lemma 1.13. Existence of a solution of the Dirichlet problem is easy. In fact, for any $\Phi : \mathscr{N}_f([0,1]) \to [0,1]$ which is multiplicative in the sense that $\Phi(z_1+z_2) = \Phi(z_1)\cdot\Phi(z_2)$, $u(t,\theta) := \mathbf{E}^{\delta_\theta}[\Phi(Z_t)]$ solves the backward equation

$$\partial_t u = \frac{1}{2}\theta(1-\theta)u'' + u^2 - u. \qquad (3.48)$$

(Note that the analogue of (3.48) with $\frac{1}{2}u''$ instead of $\frac{1}{2}\theta(1-\theta)u''$ is the well-known KPP equation (cf. [McK75]).)

In this case

$$v(t,\theta) := \mathbf{E}^{\delta_\theta}[1-\Phi(Z_t)] = 1 - u(t,\theta)$$

solves the backward equation

$$\partial_t v = \frac{1}{2}\theta(1-\theta)v'' - v^2 + v. \qquad (3.49)$$

Putting $\Phi(z) := \mathbb{1}_{\{0\}}(z(\{1\}))$ and $\Phi(z) := \mathbb{1}_{\{0\}}(z((0,1]))$, respectively, we see that both probabilistic expressions in Lemma 1.13 are stationary solutions of (3.49) and therefore solve (1.10). Also, they clearly satisfy the boundary conditions (1.11).

We are thus left with showing uniqueness. Assume that p_1 and p_2 are two solutions of (1.10) and (1.11). Note that due to concavity we must have $p_i(x) \geq x$, $x \in [0,1]$, $i = 1, 2$. Let $f = p_1 - p_2$. Then if $x \in (0,1)$ and $f(x) \neq 0$

$$\begin{aligned}\frac{f''(x)}{f(x)} &= \frac{p_1''(x) - p_2''(x)}{f(x)} \\ &= -\frac{2}{x} \cdot \frac{1 - p_1(x) - p_2(x)}{1-x} \\ &> -\frac{2}{x}. \end{aligned} \tag{3.50}$$

Since by assumption $f(0) = f(1) = 0$, Lemma 3.15 below (with $a = 1$) implies $f \equiv 0$. □

Let J_1 the Bessel function of the first kind with parameter 1 and let z_0 its smallest non-trivial zero. It is well known that $z_0 \approx 3.832$, hence $z_0^2/8 \approx 1.836$.

Lemma 3.15. *Let $a \in (0, z_0^2/8)$ and $f : [0,a] \to \mathbb{R}$ be twice continuously differentiable and subject to the differential inequality*

$$\frac{f''(x)}{f(x)} > -\frac{2}{x}, \quad \text{if } x \in (0,a) \text{ and } f(x) \neq 0. \tag{3.51}$$

If $f(0) = f(a) = 0$, then $f \equiv 0$.

Proof. Assume that there exists an $x_0 \in (0,a)$ with $f(x_0) \neq 0$. W.l.o.g. we may assume $f(x_0) > 0$ (otherwise consider $-f$). For $\delta, \gamma > 0$ define the function $H_{\delta,\gamma}$ by

$$H_{\delta,\gamma}(x) = \delta\sqrt{x}\left(\gamma J_1\left(\sqrt{8x}\right) - N_1\left(\sqrt{8x}\right)\right), \tag{3.52}$$

where N_1 is the *Bessel function of the second kind* (or *Neumann function*) with parameter 1. It is well known that $H_{\delta,\gamma}$ is the general solution of the differential equation

$$H_{\delta,\gamma}''(x) = -\frac{2H_{\delta,\gamma}(x)}{x}, \quad x > 0, \tag{3.53}$$

with $H_{\delta,\gamma}(0) = \delta/\pi$.

It is well known that $J_1(0) = 0$ and $J_1(x) > 0$ for all $x \in (0, z_0)$. Now fix a $\gamma_0 > 0$ such that $H_{1,\gamma_0}(x) > 0$ for all $x \in [0,a]$ and define

$$\delta_0 = \inf\left\{\delta > 0 : \; H_{\delta,\gamma_0}(x) \geq f(x), \; x \in [0,a]\right\}. \tag{3.54}$$

By assumption on f, $\delta_0 \in (0,\infty)$. Since $H_{\delta_0,\gamma_0}(0) > f(0)$ and $H_{\delta_0,\gamma_0}(a) > f(a)$, there exists a $t \in (0,a)$ such that $H_{\delta_0,\gamma_0}(t) = f(t)$, hence $H'_{\delta_0,\gamma_0}(t) = f'(t)$. However, by (3.51) and (3.53), $f''(t) > H''_{\delta_0,\gamma_0}(t)$ which contradicts (3.54). □

4. Proof of Theorem 3 ($d = 1$)

The key point here is that medium and reactant have distributions which can be determined via duality in terms of systems of coalescing random walks with and without medium. For such objects we will obtain scaling limits in the classical Brownian scaling. Some technical effort is needed to make the proof rigorous.

In the first subsection we show convergence of the medium in the sense of an invariance principle for the interfaces between the zeros and ones, away from $t = 0$.

In the second subsection we rescale the dual process in the medium. To this end we first replace the dual process by *instantaneously* coalescing random walk in the medium. Then we show that we can forget the small times and that it is enough to have the invariance principle away from $t = 0$. Afterwards we construct instantaneously coalescing random walks in the medium in a deterministic way from objects for which the invariance principle applies directly, and finally we combine the two just mentioned results to obtain the theorem.

4.1. Rescaling the medium

Here we show that the rescaled voter model converges to a process that has the form of an entrance law on $(0, \infty)$ and is given in terms of annihilating Brownian motions. We begin by deriving the limit law for a positive small time.

Let

$$\Delta^t = \{i \in \mathbb{Z} : \ \eta_t(i) \neq \eta_t(i+1)\},$$

and fix $\varepsilon > 0$. It is easy to show (cf. [Dur88], page 242) that $\eta_{\varepsilon T}(\lfloor T^{1/2} \bullet \rfloor)$ converges as $T \to \infty$ to a stationary piecewise constant process $\{\eta_\varepsilon^\infty(x), \ x \in \mathbb{R}\}$ with values in $\{0, 1\}$ and with isolated discontinuities $D_\varepsilon \subset \mathbb{R}$. Hence $T^{-1/2}\Delta^{\varepsilon T}$ converges in distribution to D_ε as $T \to \infty$. Furthermore it is clear that as t evolves, the discontinuities of $\eta_t(\lfloor \bullet \rfloor)$ form annihilating random walks.

Let $((X_t(i))_{t \geq 0}, \ i \in \mathbb{Z})$ be a family of independent random walks from which the annihilating walks are generated. For definiteness we assume that two walks jump to a cemetery state ∂ immediately when they meet. Denote by $((\hat{X}_t^{T,\varepsilon}(i))_{t \geq 0}$, $i \in \Delta^{\varepsilon T})$ the system of annihilating random walks started from $\Delta^{\varepsilon T}$. For $t \geq \varepsilon T$ and $i \in \mathbb{Z}$ define $j_t^{T,\varepsilon}(i)$ by

$$j_t^{T,\varepsilon}(i) = \inf\left\{ j \in \Delta^{\varepsilon T} \ : \ \widehat{X}_{t-\varepsilon T}^{T,\varepsilon}(j) \geq i \right\}. \tag{4.1}$$

It is easily verified that

$$\left(\eta_{\varepsilon T}\left(j_t^{T,\varepsilon}(i)\right), \ i \in \mathbb{Z}\right)_{t \geq \varepsilon T}$$

is a voter model on $\mathbb{Z}$.

Now consider a family of independent Brownian motions $\{(W_t(x))_{t \geq 0}, x \in \mathbb{R}\}$, where $W_0(x) = x$. Clearly $\left(T^{-1/2} X_{tT}(xT^{1/2}), \ x \in T^{-1/2}\Delta^{\varepsilon T}\right)_{t \geq 0}$ converges to $(W_t(x), \ x \in D_\varepsilon)_{t \geq 0}$ in the sense of an invariance principle. The same is true for $\widehat{X}$ and annihilating Brownian motion $(\widehat{W}_t^\varepsilon(x), x \in \Delta^\varepsilon)_{t \geq 0}$.

Now we are ready to define the limiting process η^∞ of the rescaled voter model. Define

$$g_t^\varepsilon(x) = \inf\left\{y \in D_\varepsilon :\ \widehat{W}^\varepsilon_{t-\varepsilon}(y) \geq x\right\}, \tag{4.2}$$

and

$$\eta_t^\infty(x) = \eta_\varepsilon^\infty(g_t^\varepsilon(x)), \quad t \geq \varepsilon,\ x \in \mathbb{R}. \tag{4.3}$$

Then we have proved above the following invariance principle.

Proposition 4.1. *For every $\varepsilon > 0$:*

$$\mathscr{L}\left[\left(\eta_{tT}\left(\left\lfloor T^{1/2}\bullet\right\rfloor\right)\right)_{t\geq\varepsilon}\right] \overset{T\to\infty}{\Longrightarrow} \mathscr{L}\left[\left(\eta_t^\infty\right)_{t\geq\varepsilon}\right] \tag{4.4}$$

in the sense of an invariance principle. By taking now the projective limit $(\varepsilon \to 0)$ *we can define* $(\eta_t^\infty)_{t>0}$ *and can then conclude that*

$$\mathscr{L}\left[\left(\eta_{tT}\left(T^{1/2}\bullet\right)\right)_{t>0}\right] \overset{T\to\infty}{\Longrightarrow} \mathscr{L}\left[\left(\eta_t^\infty\right)_{t>0}\right]. \tag{4.5}$$

4.2. Rescaling the dual process of the reactant

4.2.1. Instantaneous coalescence

Consider the coalescing random walk in voter medium, which was denoted by $\widetilde{X}^T$. The first step is to change from (the usual delayed) coalescing random walk $\widetilde{X}^T$ to instantaneously coalescing random walk $(\bar{X}_t^T)_{t\in[0,T]}$ generated from the same realization of the walks (X_t) but with *instantaneous* coalescence of a pair a, b at the first time they meet in the presence of the catalyst.

Lemma 4.2. *For $A \subset \mathbb{R}$ finite*

$$\lim_{T\to\infty} \mathbf{P}\left[\bar{X}_T^T\left(\left\lfloor T^{1/2}a\right\rfloor\right) \neq \widetilde{X}_T^T\left(\left\lfloor T^{1/2}a\right\rfloor\right) \textit{ for some } a \in A\right] = 0. \tag{4.6}$$

Proof. Fix a finite set $A \subset \mathbb{R}$ and let $m = |A|$. For $a, b \in A$, $a < b$ let

$$\begin{aligned}\tau_T^{(a,b)} = \inf\Big\{t > 0 :\ & X_t\left(\left\lfloor T^{1/2}a\right\rfloor\right) = X_t\left(\left\lfloor T^{1/2}b\right\rfloor\right),\\ & \eta_{T-t}\left(X_t\left(\left\lfloor T^{1/2}a\right\rfloor\right)\right) = 1\Big\}.\end{aligned} \tag{4.7}$$

Let $\tau_1 \leq \cdots \leq \tau_{\binom{m}{2}}$ be the order statistic of $\{\tau_T^{(a,b)} : a, b \in A,\ a < b\}$. Write $a(l)$ and $b(l)$ for the a and b such that $\tau_T^{(a,b)} = \tau_l$.

For $K > 0$ define the event

$$\begin{aligned}E_T(K) = \Bigg\{&\exists l \in \{1, \dots, \binom{m}{2} - 1\} :\ \tau_l < T,\\ &\int_{\tau_l}^{\tau_{l+1}} \eta_{T-s}(X_t(\lfloor T^{1/2}a(l)\rfloor))\mathbb{1}_{X_t(\lfloor T^{1/2}a(l)\rfloor)=X_t(\lfloor T^{1/2}b(l)\rfloor)}dt < K\Bigg\}.\end{aligned} \tag{4.8}$$

Using the recurrence of the difference walk and distributional convergence of η_t to $\theta_1\delta_{\underline{1}} + (1-\theta_1)\delta_{\underline{0}}$, it is easy to check that

$$\lim_{T\to\infty} \mathbf{P}[E_T(K)] = 0, \quad K > 0. \tag{4.9}$$

This concludes the proof of Lemma 4.2. □

4.2.2. Rescaling the coalescent

Now we prepare for using the invariance principle for the medium and the dual process of the reactant. Since the rescaled medium is very irregular for $t \to 0$ we first show that we can neglect the effect of very small times.

For $\varepsilon > 0$ define $(\bar{X}_t^{T,\varepsilon})$ as above but with coalescence allowed only if $t \leq (1-\varepsilon)T$. Hence $(\bar{X}_t^{T,\varepsilon})$ is independent of the medium at times before εT.

Lemma 4.3. *Fix $A \subset \mathbb{R}$ finite.*

$$\lim_{T_0\to\infty} \limsup_{\varepsilon\to 0} \sup_{T\geq T_0} \mathbf{P}\left[\bar{X}_T^{T,\varepsilon}(\lfloor T^{1/2}a\rfloor) \neq \bar{X}_T^{T}(\lfloor T^{1/2}a\rfloor) \textit{ for some } a \in A\right] = 0. \tag{4.10}$$

Proof. The probability on the l.h.s. can be bounded by $\binom{|A|}{2}$ times the probability that a random walk started at 0 is at 0 at some time $t \in [2(1-\varepsilon)T, 2T]$. However this probability converges (uniformly in $T \geq T_0$) to 0 as $\varepsilon \to 0$. □

Let $(\bar{W}_t)_{t\in[0,1]}$ be coalescing Brownian motions in the medium η^∞. Define $(\bar{W}_t^\varepsilon)$ similarly as $(\bar{X}_t^{T,\varepsilon})$ by prohibiting coalescence after time $1-\varepsilon$. Clearly we have the analogous statement to Lemma 4.3.

Lemma 4.4. *Fix $A \subset \mathbb{R}$ finite.*

$$\limsup_{\varepsilon\to 0} \mathbf{P}\left[\bar{W}_1^\varepsilon(a) \neq \bar{W}_1(a) \textit{ for some } a \in A\right] = 0. \tag{4.11}$$

4.2.3. Conclusion

With a view to the proceeding two lemmas and the duality for the reactant, it is enough to show for every $\varepsilon > 0$ and $A \subset \mathbb{R}$ finite that

$$\lim_{T\to\infty} \mathscr{L}\left[\left(\eta_{T\bullet}(\lfloor T^{1/2}\bullet\rfloor), |\bar{X}_T^{T,\varepsilon}(A)|\right)\right] = \mathscr{L}\left[\left(\eta^\infty, |\bar{W}_1^\varepsilon(A)|\right)\right], \tag{4.12}$$

in order to conclude that in the sense of f.d.d.:

$$\mathscr{L}\left[(\eta_{tT}(T^{-1/2}\bullet), \xi_{tT}(T^{-1/2}\bullet))_{t>0}\right] \overset{t\to\infty}{\Longrightarrow} \mathscr{L}\left[(\eta_t^\infty, \xi_t^\infty)_{t>0}\right]. \tag{4.13}$$

We establish this invariance principle (4.12) for the medium and the dual process. Fix $A \subset \mathbb{R}$ finite and $\varepsilon > 0$. Let $((X_t'(i))_{t\geq 0},\ i \in \mathbb{Z})$ be a family of random walks from which the approximate dual process $\left((\bar{X}_t^{T,\varepsilon}(\lfloor T^{1/2}a\rfloor))_{t\in[0,T]},\ a \in A\right)$

is built. Further let $((W'_t(x))_{t\geq 0},\ x \in \mathbb{R})$ be an independent family of Brownian motions from which $((\bar{W}^\varepsilon_t(a))_{t\in[0,1-\varepsilon]},\ a \in A)$ is built.

Note that $|\bar{W}^\varepsilon_1(A)|$ is a deterministic function $F_{A,\varepsilon}$ of $\eta^\infty_\varepsilon(0)$, Δ^ε, W, and W'. Furthermore F is almost everywhere locally constant. In particular, it is almost everywhere continuous.

Note also that the same function F applies to $|\bar{X}^{T,\varepsilon}_T(A)|$:

$$\begin{aligned} |\bar{X}^{T,\varepsilon}_T(A)| = F_{A,\varepsilon}(&\eta_{\varepsilon T}(0), T^{-1/2}\Delta^{\varepsilon T}, T^{-1/2}X_{T\bullet}(\lfloor T^{1/2}\bullet\rfloor), \\ &T^{-1/2}X'_{T\bullet}(\lfloor T^{1/2}\bullet\rfloor)). \end{aligned} \tag{4.14}$$

Note that also the pair $(|\bar{X}^{T,\varepsilon}_T(A)|, \eta_{T\bullet}(\lfloor T^{1/2}\bullet\rfloor))$ is a continuous function of the variables on the right hand side of (4.14). Hence the invariance principle and the continuous mapping theorem yield (4.12). This finishes the proof of Theorem 3. □

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