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Aging of spherical spin glasses

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Abstract. Sompolinski and Zippelius (1981) propose the study of dynamical systems whose invariant measures are the Gibbs measures for (hard to analyze) statistical physics models of interest. In the course of doing so, physicists often report of an "aging" phenomenon. For example, aging is expected to happen for the Sherrington-Kirkpatrick model, a disordered mean-field model with a very complex phase transition in equilibrium at low temperature. We shall study the Langevin dynamics for a simplified spherical version of this model. The induced rotational symmetry of the spherical model reduces the dynamics in question to an N-dimensional coupled system of Ornstein-Uhlenbeck processes whose random drift parameters are the eigenvalues of certain random matrices. We obtain the limiting dynamics for N approaching infinity and by analyzing its long time behavior, explain what is aging (mathematically speaking), what causes this phenomenon, and what is its relationship with the phase transition of the corresponding equilibrium invariant measures.

1. Introduction

Spin glasses are expected to show a very complex phase transition in equilibrium at low temperature (the so-called spin glass phase), at least in the mean field model (that is the Sherrington-Kirkpatrick model, hereafter **SK**). This prediction of a spin glass phase is due to Parisi (see [23] for a survey); the mathematical understanding is yet far from complete (see [1, 9, 11–13, 32]) in spite of recent progress mainly due to Talagrand ([27–29]) and Bolthausen-Sznitman [7].

Studying spin glass dynamics might seem premature, since statics are not yet fully understood. Nevertheless, following Sompolinski and Zippelius [26], a mathematical study of the Langevin dynamics has been undertaken by two of the authors in the recent years (see [2, 3, 18]). The output of this line of research has been to prove convergence and large deviation results for the empirical measure on path

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space as well as averaged and quenched propagation of chaos. The same problem has been solved by M. Grunwald for discrete spins and Glauber dynamics [9, 17]. The law of the limiting dynamics (the self consistent single spin dynamics) is characterized in various equivalent ways, from a variational problem to a non-Markovian implicit stochastic differential equation, none of which being yet amenable, for the moment, to a serious understanding. The task at hand is to understand the behavior of these dynamics for large times, and in particular to check the prediction that they exhibit aging regime, i.e. that the correlation between the spin at times s and t really depends on both s and t in a complex way for low temperature (see [8] for a very interesting survey on this subject).

This paper deals with a considerably simpler model than SK, that is "soft" spherical SK (denoted **SSSK** in the sequel). The most serious difficulty in SK dynamics is that the law of the coupling matrix and of the thermal noise have rotational symmetry whereas the state space is a hypercube (or its vertices for the discrete case). The spherical model has a sphere as its state space (or a spherical constraint) and thus is far simpler (see [12, 20] for former studies of states of SSK). More precisely, we shall consider the following stochastic differential system (denoted hereafter **SDS**)

$$du_t^i = \sum_{j=1}^N J_{ij} u_t^j dt - f'\left(\frac{1}{N}\sum_{j=1}^N (u_t^j)^2\right) u_t^i dt + \beta^{-1/2} dW_t^i \tag{1.1}$$

where β is a positive constant, f' is a uniformly Lipschitz, bounded below function on $\mathbb{R}^+$ such that $f(x)/x \to \infty$ as $x \to \infty$ and $(W^i)_{1\le i\le N}$ is an N-dimensional Brownian motion, independent of $\{J_{i,j}\}$ and of the initial data $\{u_0^i\}$.

The term containing f is a Lagrange multiplier used to implement a "soft" spherical constraint. The system (1.1) is the Langevin dynamics for

$$\nu_{\mathbf{J}}^N(d\boldsymbol{u}) = \frac{1}{Z_{\mathbf{J}}^N} \exp\left\{\beta\left(\sum_{i,j=1}^N J_{ij}u^j u^i - Nf\Big(\frac{1}{N}\sum_{j=1}^N (u^j)^2\Big)\right)\right\}\prod_{i=1}^N du^i, \tag{1.2}$$

with $\nu_{\mathbf{J}}^N(\cdot)$ thus being the equilibrium (invariant) measure of (1.1). For standard SSK model, J_{ij} is a symmetric matrix of centered Gaussian random variables such that

$$\mathbb{E}[J_{ij}^2] = \frac{1}{N} \qquad \mathbb{E}[J_{ii}^2] = \frac{2}{N}.$$

However, we shall also consider more general entries in the sequel. Namely, we can take any coupling $\mathbf{J}$ of the form

$$\mathbf{J} = \boldsymbol{G}^* \boldsymbol{D}\boldsymbol{G}$$

where the orthogonal matrix $\boldsymbol{G}$ follows the uniform law on the sphere, $\boldsymbol{D}$ is a diagonal matrix and also $\boldsymbol{G}$ and $\boldsymbol{D}$ are independent.

In this article, we study the convergence for solutions of (1.1) as $N \to \infty$ and prove the Large Deviation Principle (denoted hereafter **LDP**) with Good Rate Functions (denoted **GRF**) for related objects of interest.

We describe the limits of the empirical covariance for the solution of (1.1) (see Theorem 2.3). Moreover, we study their time evolutions and show that they exhibit a dynamical phase transition which is a weak type of aging (see Section 3).

Under a spherical constraint, the equation for the time evolution of the correlations has already been proposed and its consequences derived in the physics literature [14]. However, the treatment of [14], even though correct on the level of intuition, does not contain detailed proofs which induced the length of the present paper; the convergence of the empirical covariance is not completely proven (see indication at the end of Section 2 to make it rigorous) and the dynamical phase transition study, relying on fine complex analysis (existing Tauberian type of theorems failing), is only stated (see Section 7).

The methods used here to prove convergence of the dynamics are different from the ones used in [2, 3, 17, 18] which were based on a perturbation argument using Girsanov theorem. Here, we can even tackle the case of zero temperature, i.e. $\beta = \infty$ (and thus the dynamics is deterministic except for the randomness of the coupling and of the initial conditions), and we can choose rather general random coupling $\mathbf{J}$.

The important simplification offered by the spherical model is that the system (1.1) is invariant with respect to rotations. In particular, if we write $\mathbf{J} = \boldsymbol{G}^*\boldsymbol{D}\boldsymbol{G}$ where $\boldsymbol{G}$ is an orthogonal matrix and $\boldsymbol{D}$ is the diagonal matrix of the eigenvalues $(\lambda_1, .., \lambda_N)$ of $\mathbf{J}$, then $\boldsymbol{v}_t := \boldsymbol{G}\boldsymbol{u}_t$ satisfies the simpler SDS

$$d\boldsymbol{v}_t = \boldsymbol{D}\boldsymbol{v}_t dt - f'\left(\frac{1}{N}|\boldsymbol{v}_t|^2\right)\boldsymbol{v}_t dt + \beta^{-1/2} d\boldsymbol{B}_t \tag{1.3}$$

where $\boldsymbol{B} := \boldsymbol{G}\boldsymbol{W}$ is an N-dimensional Brownian motion and $|\boldsymbol{v}_t|$ is the Euclidean norm of the vector $\boldsymbol{v}_t$.

This diagonalized system is interesting in its own right; it is a system of random modes Ornstein-Uhlenbeck processes coupled by a function f' of the instantaneous (in t) empirical variance.

The main merit of the SSSK model we treat in this paper is that it is simple enough to allow for rigorous proofs and a first mathematical understanding of the aging phenomenon, which is a form of dynamical phase transition, and enable us to isolate the main features that are crucial for this phenomenon to appear.

Aging is supposedly common in disordered systems, and even in some non disordered ones. Let us first mention a few of these examples, from the physics literature on the subject. The survey [8] contains a very interesting list of possible examples of aging. The simplest being certainly low temperatures dynamics for disordered ferromagnets (see also the recent mathematically rigorous papers [24, 25]). Aging is also expected in the Random Field Ising Model (see [10]). In the context of spin glasses, the debate among physicists seems still unresolved for local models (like Edwards-Anderson model), see the discussion in [8]. For mean-field spin glasses a very complex picture of aging is advocated for the SK model, but even with the help of the whole apparatus of the replica symmetry breaking methods, it is not yet fully understood in the physics literature, and needless to say in the mathematics literature. One model seems to be well understood by physicists, and that is the p-spin model (with $p \geq 3$). This model is still beyond reach of any mathematical understanding, even though some progress has been made recently in [28] for the statics of the p-spin model. The much simpler SSSK model is amenable to a mathematical analysis, and shows an interesting dynamical phase transition. It should be noticed that we also show here a static phase transition, and that the static and dynamic critical temperatures are equal, which is not always true if one believes [8].

So what is aging? It is a manifestation of the complexity of an energy landscape, for time scales which are long but much shorter than for metastability or equilibration to happen. It is very dependent on **initial conditions**: it happens typically for low temperature dynamics started from i.i.d initial conditions (or after a "deep quench", i.e from the equilibrium measure at very high temperatures). This is a very different context than for metastability questions or for equilibrium dynamics questions, which are concerned by much longer time scales, and different initial conditions: typically pure-state initial conditions for metastability and equilibrium measures for spectral gap questions and equilibrium dynamics. Aging is present when a system needs a long time to forget its state at time t, when t is large (usually at low temperatures only). More precisely when the time needed to forget the state at time t depends on t. This is usually measured by the time correlation function $K(s,t)$. Aging implies that the time translation invariance is lost for this function, and that as [8] puts it: one must think in the two-times plane where $K(s,t)$ does not decay to zero when s (the age of the system) and $t-s$ (the duration of the experiment) both tend to ∞ but when $t-s$ is not large enough compared to s.

One should emphasize here that the order of the limits is crucial. One first takes the thermodynamic limit ($N \to \infty$, N being here the size of the system) and then studies the long time behavior of the limiting dynamics. This order of the limit operations precludes any possibility of metastability transitions (at least when the barriers between wells grow with N); the system is not given enough time to go from one deep well to another one. This will explain what happens for initial conditions (IC3) below, where the SSSK model starts from a zero-temperature ground state, i.e., the top eigenvector of the random matrix. The question of aging is much more about how the systems lingers "on the boundary" of basins of attractions than how it gets from one to another as for metastability or how it evolves when started from mixtures of such pure states as in equilibrium questions (see Section 3.4). Another point must be emphasized about the order of the limits: for aging we look at the behavior of the limiting dynamics when **both** s and t go to ∞. Clearly when s goes first to ∞ alone, then the equilibrium picture should prevail. We will see this is really the case.

The aging mechanism we show in this work might seem surprising at first sight, precisely because of the short time scales involved. We emphasize though that this is precisely the time scales involved in the works on the RFIM [10], or on Ising model [24]. Another, completely different, mechanism of aging could be sought, for much longer time scales. If the time scale were dependent on the size N of the system, one could envision the possibility for randomly coupled systems to age through a mechanism of visiting in various time scales, different wells of depths in various scales. The question then is closer to the picture of a complex energy landscape advocated for the static picture of most spin glass models: this other aging mechanism would result from a competition between the long time needed to find randomly located wells with depth of a given scale and then the time to get out of those wells. This mechanism is not possible in the shorter time scales we are considering in this work.

Finally, the model is flexible enough to enable us to analyze the role of the randomness of the coupling in the appearance of aging. We work with rather general coupling randomness (see **(H0)**, **(H1)** and **(H2)** below), and isolate the natural hypothesis for aging. The most important conclusion about the role of this randomness is that aging does not appear unless the levels just below the top of the spectrum are populated enough, i.e., the limiting spectral measure should give enough weight to the high modes. In fact, aging appears because the energy levels of our Hamiltonian next to their minimum value, which are related with the eigenvalues of the matrix **J**, differ only by an energy of order of the inverse of the number N of particles. Consequently, the system may visit all the states corresponding to these energies in finite time (e.g. independently of N), creating a specific long time behavior of the limiting system determined by the distribution of this cloud of energy levels. In fact, aging could also appear in inhomogeneous (possibly non random) environment models, provided the energy levels present such a distribution.

With $\hat{\sigma}^N = \frac{1}{N}\sum_{i=1}^N \delta_{\lambda_i}$ denoting the empirical measure of the eigenvalues of **J** the following assumption is made throughout this article.

(H0) *There exists a (non-random) compactly supported probability measure σ with at most finitely many atoms such that $\hat{\sigma}^N \to \sigma$ in $\mathscr{P}(\mathbb{R})$ for a.e. $\boldsymbol{\lambda}$.*

Let $[\lambda_-^*, \lambda^*]$ denote the smallest interval which supports σ. Since the diagonalized system (1.3) is invariant to the transformation $\lambda_i \leftarrow \lambda_i + k$, $f'(x) \leftarrow f'(x) + k$, for any $k \in \mathbb{R}$, we may and shall choose the constant k hereafter to be such that $\lambda_-^* = -\lambda^*$.

We also assume throughout that the extreme modes almost surely converge to within the support of σ. That is,

(H1)

$$\mathbb{P}(\limsup_{N\to\infty} \max_{i=1}^{N} |\lambda_i| \leq \lambda^*) = 1.$$

For our results about the dynamics we further assume the following strict positivity of σ

(H2) $\sigma(G) > 0$ *for every open set $G \subset [-\lambda^*, \lambda^*]$.*

Although we assume the support of σ convex (as for Wigner's semi-circular law), our work could be generalized to other compactly supported measures if we assume that for N large enough all the eigenvalues remain in the support of the limiting measure σ, as insured here by **(H1)** and **(H2)**. Without such an assumption, the GRF governing our LDPs might no longer be determined completely by the limiting measure σ (as shown for example in the somewhat different context of [5]). In particular, the free energy for the model may well be different. However, we could significantly relax **(H1)** and **(H2)** if we were to prove only the convergence of the empirical dynamical covariance when the system starts for instance from i.i.d. initial conditions.

The organization of the paper is as follows. In Section 2 we state our a.s. convergence results for the diagonalized system (1.3) and its invariant measure as $N \to \infty$. These a.s. statements are derived here from the quenched LDPs we obtain for the dynamic (1.3) and its invariant measure (given by (2.1)), in Sections 5 and 4, respectively. The *quenched* LDPs are about the large deviations due to the randomness in the Brownian motion $\boldsymbol{B}$ and the initial conditions $\boldsymbol{v}_0$, per given infinite realization $\boldsymbol{\lambda}$ of the eigenvalues of **J**. Each of our quenched LDPs holds for a.e. such $\boldsymbol{\lambda}$, with the non-random GRF independent of $\boldsymbol{\lambda}$, and we obtain the corresponding a.s. convergence result by finding the unique minimizer of the GRF.

This study of (1.3) is done under four different initial conditions, the first consisting of i.i.d random variables, the second of rotated i.i.d (that is, u_0^i of (1.1) are i.i.d. random variables), the third indicating that we are beginning from the eigenvector of **J** corresponding to the maximal eigenvalue, and the last corresponding to the stationary initial conditions given by (2.1).

Under the somewhat stronger assumptions **(H0a)** and **(H1a)** about the distribution of the eigenvalues of **J** we derive in Section 5 also *annealed* LDPs for (1.3). The latter LDPs involve expectations with respect to $\boldsymbol{\lambda}$, thus taking into account also the possible large deviations in $\boldsymbol{\lambda}$ (but are not needed in so far as a.s. convergence is concerned). Section 6 collects new tools and results, needed mostly for the derivation of our LDPs, which are of a more general scope and of some independent interest.

As illustrated at the end of Section 2, large deviation techniques are not needed for the a.s. convergence results for the diagonalized system starting from i.i.d initial conditions. However, these techniques are crucial for the study of the static phase transition as well as when considering other initial conditions, for example, stationary. Moreover, they provide a more complete picture which may help in the study of other models.

In Section 3 we present the analysis of the a.s. limit laws of (1.3) for $N \to \infty$ in the special case of $f'(x) = cx$, deferring the proofs of most key technical steps to Section 7. (To simplify matters we do not study the phase transition phenomenon for non-quadratic f. However, the derivation of the limiting equations is exactly the same for any super linear f, so we keep it at this level of generality.) We find that the dynamical phase transition with respect to the parameter β matches the static phase transition for the invariant measure of (2.1) and is characterized by the onset of aging above criticality, at least for i.i.d. and rotated i.i.d. initial conditions.

We conclude the introduction by illustrating our main results as they apply for a specific model of SSSK, the coupling **J** of which corresponds to the Gaussian Orthogonal Ensemble (denoted hereafter **GOE**), where the entries $(J_{ij})_{1\le i\le j\le N}$ are independent centered Gaussian variables with covariance $\mathbb{E}[J_{ij}^2] = (1+\delta_{i=j})/N$.

To give explicit formulae, we choose $f(x) = cx^2/2$. The GOE satisfies **(H0)** with σ the so-called semi-circular law (see [4] and Section 6.1), which is positive in the sense of **(H2)**. The fact that the GOE satisfies **(H1)** follows for example from the LDP of Theorem 6.2 for the maximal eigenvalue, which is of some independent interest. (However, since the eigenvalues for the GOE are not uniformly bounded, **(H1a)** does not hold then, rendering the annealed LDPs of Section 5 irrelevant for this choice of **J**.)

Starting with the static properties of this SSSK model, our first result is the almost sure convergence of the free energy for $\beta > 0$

$$\begin{aligned} F_\beta &= \lim_{N\to\infty} \frac{1}{N} \log Z_{\mathbf{J}}^N \qquad \mathbf{J} \quad a.s. \\ &= \frac{1}{2} \inf_{s\geq 2} \left\{ \frac{\beta}{c} s^2 - \frac{1}{2\pi} \int_{|\lambda|\leq 2} \log(s-\lambda)\sqrt{4-\lambda^2} d\lambda \right\} + \frac{1}{2} \log(\pi\beta^{-1}) \end{aligned}$$

Alternatively, we obtain the more explicit formula

$$F_\beta = \frac{1}{4} + \frac{1}{2}\log(\pi\beta^{-1}) + \begin{cases} \frac{x_\beta}{2(1-x_\beta^2)} + \int_1^{(1-x_\beta^2)^{-1/2}} \sqrt{w^2-1}\,dw & \text{if} \quad \beta < \beta_c \\ \frac{1}{2} x_\beta & \text{otherwise,} \end{cases} \tag{1.4}$$

where $x_\beta = \beta/\beta_c - 1$ and $\beta_c = c/4$ at which $\partial_\beta^3 F_\beta$ is not continuous (see Section 3.1).

The static phase transition can also be characterized by the diagonalized Gibbs measure $\mu_{\boldsymbol{\lambda}}^N$ defined in (2.1). We show in Theorem 2.1 that the empirical measure $\hat{\nu}_0^N = N^{-1} \sum_{i=1}^N \delta_{v_i}$ converges under $\mu_{\boldsymbol{\lambda}}^N$, for almost every **J**, towards a non-random probability measure $\nu_0^* \in \mathscr{P}(\mathbb{R})$. Below criticality ν_0^* has a sub-Gaussian tail whereas ν_0^* has infinite fourth moment above criticality. We note in passing that the empirical measure under $\nu_{\mathbf{J}}^N$ always converges towards a Gaussian law with a finite variance $u_\beta > 0$ (this follows from Theorem 2.1 and the ideas of Section 5.3).

About the dynamics for this special model, we get a quenched LDP for the non centered empirical covariance

$$K^N(t,s) = \frac{1}{N} \sum_{i=1}^N u_t^i u_s^i. \tag{1.5}$$

of the solution of (1.1) starting either from i.i.d initial conditions, spherical initial conditions (including the Gibbs invariant measure) or initial conditions depending on the matrix **J**, namely the eigenvector of **J** with maximum eigenvalue. Studying the GRFs of the LDP shows that K^N converges almost surely towards the limiting covariance $K(t,s)$ of Theorem 2.3. In fact, the centered covariance

$$K_N(t,s) = K^N(t,s) - \left(\frac{1}{N} \sum_{i=1}^N u_t^i \right) \left(\frac{1}{N} \sum_{i=1}^N u_s^i \right) \tag{1.6}$$

has the same limit behavior as K^N when N goes to infinity for all of the above initial conditions, except when $(u_0^i)_{1\le i\le N}$ are i.i.d. with a non-centered law (See Proposition 3.7). Even in the latter case the limits of K^N and K_N have the same long time behaviors (See Section 3.6). Hence, in the following, we shall focus on K^N and, to simplify the exposition, call K^N the empirical covariance.

In Section 3.2, we analyze the limiting covariances $K(t,s)$ and show that, starting from i.i.d initial conditions, they exhibit a dynamical phase transition at the critical temperature $(1/\beta_c)$ and an aging phenomenon for low temperature (see Proposition 3.2). More precisely, when starting from i.i.d initial conditions:

- If $\beta<\beta_c$, then $K(t,s)\le C_\beta\exp(-\delta_\beta|t-s|)$ for some $\delta_\beta>0$, $C_\beta<\infty$ and all (t,s).
- If $\beta=\beta_c$, then $K(t,s)\to 0$ as $t-s\to\infty$. If t/s is bounded, then the polynomial decay is of power $(t-s)^{-1/2}$, and otherwise it behaves like $s^{1/2}t^{-1}$.
- If $\beta>\beta_c$ and $t\gg s\gg 1$, then $K(t,s)(t/s)^{3/4}$ is bounded away from zero and infinity. In particular, the convergence of $K(t,s)$ to zero occurs if and only if $t/s\to\infty$.

In contrast, starting from the top eigenvector, the aging phenomenon does not appear. That is, $K(t,s)\to c_{EA}\in(0,\infty)$ for any $\beta>\beta_c$, regardless of how $t-s$ and s approach infinity (see Theorem 3.4). The Edwards-Anderson parameter c_{EA} is also the limit of $K(t,s)=K(|t-s|)$ when starting from the invariant measure $\nu_{\mathbf{J}}^N$ of (1.2), as shown in Proposition 3.5.

The convergence of the empirical covariance K^N follows from the study of the diagonalized system (1.3) which has the same empirical covariance. We will in fact prove a quenched LDP for the couple of the empirical covariance and the empirical measure of this diagonalized system (see Theorem 2.4). As a consequence, we shall get the almost sure convergence of K^N and of the empirical measure for this diagonalized system. The limit law for the empirical measure is shown to be the mixture $\int\mu_\lambda d\sigma(\lambda)$, where μ_λ is the weak solution of the Ornstein-Uhlenbeck processes (2.15) with an appropriate initial data.

Whenever we state a LDP or convergence result for random probability measures, such LDP or convergence are in the space $\mathscr{P}(\Sigma)$ of Borel probability measures on a Polish alphabet space Σ, endowed with the corresponding $C_b(\Sigma)$-topology. The space $\mathscr{P}(\Sigma)$ is also often considered a subset of the vector space $C_b(\Sigma)'$, the algebraic dual of $C_b(\Sigma)$, endowed with the $C_b(\Sigma)$-topology. Likewise, a LDP or convergence for $\mathbb{R}^d$-valued random functions whose domain is such Σ, is to be understood in the space $C_b(\Sigma)$ of continuous, bounded functions endowed with the uniform (supremum-norm) topology, whereas such statements for $\mathbb{R}^d$-valued random vectors are with respect to the usual (Euclidean-norm) topology. Statements about product objects the coordinates of which are random measures, random functions and/or finite-dimensional random vectors are to be understood with respect to the product of the above mentioned topologies. The σ-fields involved in such statements are always the Borel σ-fields for the relevant topology, completed with respect to the collection of null sets common to the sequence of laws considered.

LDP or convergence for dynamical systems will hold until a finite time $T > 0$, hereafter fixed but as large as required. The GRF of our LDPs often involve the relative entropy function, denoted $I(\mu|\nu)$. Another notation we frequently use is γ_u for the law of a centered Gaussian variable with variance $(1/u)$ for $u > 0$.

2. Study of the diagonalized system

In this section, we assume that we are given the law σ^N of the eigenvalues $(\lambda_1, .., \lambda_N)$ of $\mathbf{J}$ and consider the mean field Gibbs measure

$$\mu^N_{\boldsymbol{\lambda}}(d\boldsymbol{v}) = \frac{1}{Z^N_{\boldsymbol{\lambda}}} \exp\left\{\beta\left(\sum_{i=1}^N \lambda_i v_i^2 - Nf\Big(\frac{1}{N}\sum_{i=1}^N v_i^2\Big)\right)\right\}\prod_{i=1}^N dv_i \tag{2.1}$$

with

$$Z^N_{\boldsymbol{\lambda}} = \int \exp\left\{\beta\Big(\sum_{i=1}^N \lambda_i v_i^2 - Nf(N^{-1}\sum_{i=1}^N v_i^2)\Big)\right\}\prod_{i=1}^N dv_i .$$

We also consider the associated Langevin dynamics

$$dv_t^i = \lambda_i v_t^i dt - f'\left(\frac{1}{N}\sum_{j=1}^N (v_t^j)^2\right) v_t^i dt + \beta^{-1/2} dB_t^i, \tag{2.2}$$

including the case of $\beta = \infty$. As shown in Section 6.4, our hypotheses that f' is uniformly Lipschitz and bounded below on $\mathbb{R}^+$ imply the existence of a unique strong solution of (2.2) in $\mathscr{C}([0,T],\mathbb{R}^N)$ for any finite time T and any initial condition $\boldsymbol{v}_0$ which is independent of $\{B_t^i\}$.

We want to study the asymptotic behavior of the empirical measure

$$\hat{\nu}_0^N := \frac{1}{N}\sum_{i=1}^N \delta_{v_i}$$

under $\mu^N_{\boldsymbol{\lambda}}$ as well as that of the empirical measure on path space

$$\hat{\nu}_T^N := \frac{1}{N}\sum_{i=1}^N \delta_{v^i_{[0,T]}} \tag{2.3}$$

and the (empirical) covariance term

$$K^N(s,t) = \frac{1}{N}\sum_{i=1}^N v_s^i v_t^i \ , \tag{2.4}$$

when the spectral measures $\hat{\sigma}^N$ of $\mathbf{J}$ converge a.s. to σ as in **(H0)**. To state our convergence result for the statics, let us introduce

$$h(u,v) := \frac{u}{2} - \beta f(u) + \beta v + \frac{1}{2}\log(2\pi) \tag{2.5}$$

and

$$k(u,v) := \sup_{\alpha,\rho} \{\alpha v + \rho u - L(\rho,\alpha)\}, \tag{2.6}$$

for

$$L(\rho,\alpha) := \begin{cases} -\frac{1}{2}\int \log(1-2(\alpha\lambda+\rho))d\sigma(\lambda) & \text{if} \quad |\alpha|\lambda^* + \rho \le 1/2 \\ \infty & \text{otherwise}. \end{cases} \tag{2.7}$$

Then,

Theorem 2.1. *Assume* **(H0)**, **(H1)** *and f is strictly convex. Then, for almost all $\boldsymbol{\lambda}$,*
a) The free energy converges and

$$F_\beta := \lim_{N\to\infty} \frac{1}{N} \log Z^N_{\boldsymbol{\lambda}} = -\inf_{u,v}\{k(u,v) - h(u,v)\}. \tag{2.8}$$

b) For almost all $\boldsymbol{\lambda}$, $\hat{\nu}^N_0$ converges almost surely under $\mu^N_{\boldsymbol{\lambda}}$ in $\mathscr{P}(\mathbb{R})$ towards

$$\nu^*_0 := \int \gamma_{1-2(\alpha_\beta\lambda+\rho_\beta)} d\sigma(\lambda), \tag{2.9}$$

with $N^{-1}\sum_{i=1}^N v_i^2$ converging almost surely to $u_\beta > 0$. The couple (u_β, v_β) is the unique minimizer of (2.8). For $\lambda^* > 0$, the couple $(\rho_\beta, \alpha_\beta)$ is where the supremum in $k(u_\beta, v_\beta)$ is uniquely achieved. In contrast, $v_\beta = 0$ for $\lambda^* = 0$, rendering α_β irrelevant and resulting with $\nu^*_0 = \gamma_{1/u_\beta}$.

Remark. $Z^N_{\mathbf{J}} = Z^N_{\boldsymbol{\lambda}}$, so (2.8) gives also the free energy limit for the invariant measure (1.2).

We shall see in Section 3 that the phase transition can be described by the couple $(\rho_\beta, \alpha_\beta)$ in case $\lambda^* > 0$. Below criticality, $|\alpha_\beta|\lambda^* + \rho_\beta < 1/2$ so that ν^*_0 has a sub-Gaussian tail. Above criticality, $|\alpha_\beta|\lambda^* + \rho_\beta = 1/2$ and in fact ν^*_0 may have only a finite number of moments (for the semi-circular law, ν^*_0 has finite second but not fourth moments). In contrast, for constant modes, or more generally whenever $\lambda^* = 0$, there is no phase transition as ν^*_0 is always Gaussian.

Theorem 2.1 is a direct consequence of the following quenched LDP result, proved in Section 4.

Theorem 2.2. *Under* **(H0)** *and* **(H1)**, *for $\beta \in (0,\infty)$ and almost all $\boldsymbol{\lambda}$, the free energy converges to F_β of (2.8) and the random variables*

$$Y^N_0 = \left(\frac{1}{N}\sum_{i=1}^N v_i^2, \frac{1}{N}\sum_{i=1}^N \lambda_i v_i^2, \frac{1}{N}\sum_{i=1}^N \delta_{\lambda_i, v_i} \right)$$

satisfy the LDP in $\mathbb{R}^2 \times \mathscr{P}(\mathbb{R}^2)$ under $\mu^N_{\boldsymbol{\lambda}}$ with the GRF

$$\mathscr{H}^0(u,v,\pi) = \mathscr{K}(u,v,\pi) - h(u,v) + F_\beta \tag{2.10}$$

where $h(u,v)$ *is given by (2.5),*

$$\mathscr{K}(u,v,\pi)=\begin{cases} I(\pi|\sigma\otimes\gamma)+\frac{1}{2}\left(u-\int w^2 d\pi(\lambda,w)\right) & \text{if } \quad \pi\in\mathscr{A}(u,v)\\ \infty & \text{otherwise,}\end{cases} \tag{2.11}$$

for $I(\cdot|\cdot)$ *the relative entropy function, and (with* $0/0:=0$*),*

$$\mathscr{A}(u,v)=\Big\{\pi\in\mathscr{P}(\mathbb{R}^2):\pi_1=\sigma,\\ u\geq\int w^2 d\pi(\lambda,w)+|v-\int\lambda w^2 d\pi(\lambda,w)|/\lambda^*\Big\}. \tag{2.12}$$

When f *is strictly convex, the unique minimizer of* $\mathscr{H}^0$ *is* (u_β,v_β,π^*) *for* $d\pi^*(\lambda,w)$ $=\gamma_{1-2(\alpha_\beta\lambda+\rho_\beta)}(dw)d\sigma(\lambda)$ *corresponding to (2.9).*

For dynamics, we shall prove a convergence result when starting from the following initial conditions

(IC1). *The independent initial conditions:* $(v_0^i)_{1\leq i\leq N}$ are i.i.d random variables, independent of $\boldsymbol{\lambda}$ and of $\boldsymbol{B}$. The marginal law ν_0 of each v_0^i is such that

$$\eta\mapsto\Lambda_0(\eta):=\log\int e^{\eta v^2}d\nu_0(v)\,, \tag{2.13}$$

is continuous and $\Lambda_0(\eta_0)<\infty$ for some $\eta_0>0$.

(IC2). *The rotated independent initial conditions:* $\boldsymbol{v}_0=\boldsymbol{G}\boldsymbol{u}_0$ where $\boldsymbol{G}$, independent of $\boldsymbol{\lambda}$ and $\boldsymbol{B}$, follows the normalized Haar measure H_N on SO(N), that is $\boldsymbol{G}$ follows the uniform law on the random orthogonal matrices and $\boldsymbol{u}_0$, independent of $\boldsymbol{G}$, of $\boldsymbol{\lambda}$ and of $\boldsymbol{B}$, is such that $\{N^{-1}|\boldsymbol{u}_0|^2\}$ satisfy the LDP with some GRF $\kappa(\cdot)$ that has a unique, strictly positive, minimizer.

(IC3). *The top eigenvector initial conditions:* Set $\lambda_1=\lambda_N^*:=\max\{\lambda_1,..,\lambda_N\}$ and $\boldsymbol{v}_0=(\sqrt{N},0,..,0)$, so that $\boldsymbol{u}_0$ is the eigenvector of $\mathbf{J}$ corresponding to the maximum eigenvalue λ_N^* normalized to be on the sphere of radius $\sqrt{N}$.

(IC4). *The stationary initial conditions:* f is strictly convex and $\boldsymbol{v}_0$ follows the Gibbs invariant measure $\mu^N_{\boldsymbol{\lambda}}$, independently of $\boldsymbol{B}$, resulting with the stationary solution $\boldsymbol{v}_t$ of (2.2).

The more general case where the law of $\boldsymbol{v}_0$ is spherical, can also be considered. That is,

$$d\mathbb{P}\left((v_0^i)_{1\leq i\leq N}\right)=\frac{1}{Z_N}\exp\{-Ng(\frac{1}{N}\sum_{i=1}^N(v_0^i)^2)+Nh(\frac{1}{N}\sum_{i=1}^N\lambda_i(v_0^i)^2)\}\prod_{i=1}^N dv_0^i \tag{2.14}$$

for g and h continuous such that g is super-linear, non-decreasing and $h((\lambda^*+1)x)/g(|x|)\to 0$ when $|x|\to\infty$ (g and h may well depend upon β).

The first case is the most standard choice for the vector $\boldsymbol{v}_0$ independently of the original system, where the v_0^i's are i.i.d. The second case corresponds for example to i.i.d. initial conditions for the original system (1.1). The last two cases study what is happening when the initial conditions depend heavily on **J**; we shall see in Section 3 that it very much affects the dynamical phase transition.

We shall prove

Theorem 2.3. *Under* **(H0)**, **(H1)** *and* **(H2)**, *for almost all* $\boldsymbol{\lambda}$ *the empirical measure* $\hat{\nu}_T^N$ *converges in* $\mathscr{P}(C_b([0,T]))$ *towards* $\nu(w) = \int \nu_\lambda(w) d\sigma(\lambda)$ *and the empirical covariance* K^N *converges in* $C_b([0,T]^2)$ *towards* $K(t,s)$. *Here* ν_λ *is the law of the Ornstein-Uhlenbeck process*

$$\begin{cases} dv_t = \lambda v_t dt - f'(K_d(t)) v_t dt + \beta^{-1/2} dB_t \\ Law\ of\ (v_0) = \nu_0\,, \end{cases} \tag{2.15}$$

$K(t,s)$ *is the unique solution of the non-linear equation*

$$\begin{aligned} K(t,s) = \exp\{&-\int_0^t f'(K_d(u))du - \int_0^s f'(K_d(u))du\}\mathscr{L}_0(t+s) \\ &+\beta^{-1}\int_0^{t\wedge s} \exp\{-\int_v^t f'(K_d(u))du \\ &-\int_v^s f'(K_d(u))du\}\mathscr{L}(t+s-2v)dv \end{aligned} \tag{2.16}$$

with $K_d(t) = K(t,t)$ *and*

$$\mathscr{L}(\theta) := \int e^{\theta\lambda} d\sigma(\lambda).$$

The law ν_0 *and the function* $\mathscr{L}_0(\cdot)$ *are determined by the initial conditions as follows.*

1) For (IC1), ν_0 *is the given law of* v_0^i *and* $\mathscr{L}_0(\theta) = \mathscr{L}(\theta)\int v^2 d\nu_0(v)$.

2) For (IC2), $\nu_0 = \gamma_{1/u^*}$ *with variance* $u^* > 0$ *such that* $\kappa(u^*) = 0$ *and again* $\mathscr{L}_0(\theta) = \mathscr{L}(\theta)\int v^2 d\nu_0(v)$.

3) For (IC3), $\nu_0 = \delta_0$ *and* $\mathscr{L}_0(\theta) = \exp(\theta\lambda^*)$.

4) For (IC4), $\nu_0 = \gamma_{1-2(\alpha_\beta\lambda+\rho_\beta)}$ *and* $\mathscr{L}_0(\theta) = \int v^2 e^{\lambda\theta} d\pi^*(\lambda, v) + (u_\beta - \int v^2 d\pi^*(\lambda, v))e^{\lambda^*\theta}$.

Moreover, except for the case of (IC4), all of the above applies also when $\beta = \infty$.

It is easy to check that $K(t,s) = \mathbb{E}[v_t v_s]$ is the covariance of the solution of (2.15) for the initial conditions (IC1) and (IC2), whereas in general this is not the case for (IC3) and (IC4).

We will see in Section 3 that for (IC1) and (IC2) the solution of (2.16) undergoes a dynamical phase transition; above criticality, the covariance $K(t,s)$ goes to zero as $|t-s|$ goes to infinity only if it goes to infinity faster than $s \wedge t$. In contrast, for both (IC3) and (IC4) the solution of (2.16) does not exhibit such an aging phenomenon.

Recall that K^N is also the empirical covariance (1.5) of the SDS (1.1), so our LDPs and convergence results apply for the latter as well. In particular, the analysis of K^N for (IC4) applies as well to the (unique) stationary solution of the SDS (1.1).

We shall deduce Theorem 2.3 from the following quenched LDP results, proved in Section 5 (see Theorem 5.1 and Corollary 5.2).

Theorem 2.4. *Under* **(H0)**, **(H1)** *and* **(H2)**, *for almost every* $\boldsymbol{\lambda}$ *and initial conditions of type (IC1), (IC2), (IC3), or (IC4),* $(K^N, \frac{1}{N}\sum_{i=1}^N \delta_{\lambda_i, v^i_{[0,T]}})$ *satisfies the LDP with a non-random GRF that achieves its minimum value uniquely at the corresponding* $(K, \sigma(\lambda)\otimes\nu_\lambda)$ *of Theorem 2.3.*

We next illustrate how one may derive Theorem 2.3 without LDP techniques, at least for initial conditions (IC1). For this, it suffices to assume in addition to **(H0)** that the following weak form of **(H1)** holds:

$$(\mathbf{H1})' \quad \limsup_{N\to\infty} \frac{1}{N}\sum_{i=1}^N e^{\eta\lambda_i} < \infty \qquad \forall \eta \in \mathbb{R}\,.$$

The main idea of the proof is to assume that the empirical covariance $K_d^N(t) = K^N(t,t)$ converges in $C_b([0,T])$ towards some non-random $K_d(t)$. Then, one can verify that K^N converges towards the solution of (2.16). Indeed, we have that

$$v_t^i = e^{\lambda_i t - \int_0^t f'(K_d^N(u))du} v_0^i + \beta^{-1/2} e^{\lambda_i t - \int_0^t f'(K_d^N(u))du} \int_0^t e^{-\lambda_i v + \int_0^v f'(K_d^N(u))du} dB_v^i$$

for all $i \in \{1, \dots, N\}$, yielding

$$\begin{aligned} K^N(t,s) = \exp\{&-\int_0^t f'({K_d}^N(u))du - \int_0^s f'({K_d}^N(u))du\}\mathscr{L}_0^N(t+s) \\ &+\beta^{-1}\int_0^{t\wedge s} \exp\{-\int_v^t f'({K_d}^N(u))du - \int_v^s f'({K_d}^N(u))du\} \\ &\times \mathscr{L}^N(t+s-2v)dv + R_N \end{aligned}$$

with R_N a (stochastic) reminder term of order $N^{-1/2}$ under **(H1)**$'$, and where we have set

$$\mathscr{L}_0^N(\theta) = \frac{1}{N}\sum_{i=1}^N e^{\lambda_i\theta}(v_0^i)^2 \quad \text{and} \quad \mathscr{L}^N(\theta) = \frac{1}{N}\sum_{i=1}^N e^{\lambda_i\theta}.$$

Since $\boldsymbol{v}_0$ is independent of $\boldsymbol{\lambda}$, it follows that $N^{-1}\sum_{i=1}^N \delta_{\lambda_i, v_0^i}$ converges towards $\sigma(\lambda)\otimes\nu_0(v)$ and, with (2.13), we see that under **(H1)**$'$, $\mathscr{L}_0^N$ and $\mathscr{L}^N$ converge towards $\mathscr{L}_0$ and $\mathscr{L}$ of Theorem 2.3, respectively.

Thus, if K_d^N converges towards K_d, then K^N converges towards the solution of (2.16) under (IC1).

Further, as far as the empirical measures are concerned, one can approximate (2.2) by the system of *independent* Ornstein-Uhlenbeck processes with law P_{λ_i, v_0^i}, $1 \le i \le N$, given as the weak solution of

$$d\widetilde{v}_t^i = \lambda_i \widetilde{v}_t^i dt - f'(K_d(t))\widetilde{v}_t^i dt + \beta^{-1/2} dB_t^i,$$

with the same initial data $\boldsymbol{v}_0$. It is not hard to see that, for any fixed $T < \infty$, the empirical measures $\widetilde{\pi}_T^N = \frac{1}{N}\sum_{i=1}^N \delta_{\lambda_i, \widetilde{v}^i_{[0,T]}}$ converge towards the law $\pi(\lambda, v) =$

$\sigma(\lambda)\otimes\nu_\lambda(v)$, where ν_λ is given by (2.15). Indeed, it is easy to check that the sequence of random measures $\{\tilde{\pi}_T^N\}_N$ is tight (for example, using an approximation scheme as in the proof of Theorem 5.4). Therefore, we only need to prove that this sequence has a unique limit point π. To this end, note that for any bounded continuous function V on $\mathbb{R}\times C_b([0,T])$ such that $\int V(\lambda,v)dP_{\lambda,v_0}d\sigma(\lambda)d\nu_0(v_0)=0$, and any $\epsilon>0$, $L>0$, one has by Chebyshev's inequality that

$$\begin{aligned}
&\mathbb{P}\left(\int V(\lambda,v)d\tilde{\pi}_T^N(\lambda,v)\geq\epsilon\right)\\
&\leq e^{-L\epsilon}\prod_{i=1}^N\int e^{\frac{L}{N}V(\lambda_i,v)}dP_{\lambda_i,v_0^i}(v)\\
&=e^{-L\epsilon}\exp\left\{\frac{L}{N}\sum_{i=1}^N\int V(\lambda_i,v)dP_{\lambda_i,v_0^i}(v)+O\left(\frac{L^2}{N}\right)\right\}
\end{aligned}$$

The convergence of $N^{-1}\sum_{i=1}^N\delta_{\lambda_i,v_0^i}$ towards $\sigma(\lambda)\otimes\nu_0(v)$ and the continuity of $(\lambda,v)\mapsto P_{\lambda,v}$ thus result with

$$\mathbb{P}\left(\int V(\lambda,v)d\tilde{\pi}_T^N(\lambda,v)\geq\epsilon\right)\leq e^{-L(\epsilon-o(1))+O(\frac{L^2}{N})}.$$

Hence, choosing L of order $o(\sqrt{N})$, we can deduce the almost sure convergence of $\int V(\lambda,v)d\tilde{\pi}_T^N(\lambda,v)$ towards zero by Borel-Cantelli's lemma. Since this applies whenever $\int V(\lambda,v)dP_{\lambda,v_0}d\sigma(\lambda)d\nu_0(v_0)=0$, we conclude that π_T^N converges to π as needed. To make the above heuristic sketch into a complete rigorous proof, one should mainly prove the convergence of K_d^N to a non-random function K_d. This may be done for instance by using Theorem 5.3 which shows that K^N is a continuous function of a random variable $\mathscr{C}^N$ that is easily seen to converge.

3. Phase transition and aging

In this section, we shall study both statics and dynamics phase transition. Throughout we assume that $f(\cdot)$ is strictly convex, super-linear, with f' uniformly Lipschitz on $\mathbb{R}^+$ and in case of $\beta=\infty$ also that $f'(0)\leq 0$, whereas σ is a probability measure with at most finitely many atoms, supported on $[-\lambda^*,\lambda^*]$ and positive in the sense of **(H2)**. To simplify the analysis, we also assume in Sections 3.1, 3.2.3, 3.3, 3.4 and 3.5 that

$$f(x)=\frac{c}{2}x^2$$

for some $c>0$ and that σ is symmetric, that is

$$\sigma(\lambda\in.)=\sigma(-\lambda\in.).$$

3.1. Static phase transition

For $\lambda^* = 0$ it follows from Theorem 2.1 that the free energy $F_\beta = \frac{1}{4}\log(2e\pi^2/(\beta c))$ is a smooth function of $\beta \in (0,\infty)$ as is the variance $u_\beta = 1/\sqrt{2\beta c}$ of the Gaussian limit of $\hat{\nu}_0^N$. In particular, there is then no static phase transition.

Turning to the (more interesting) case of $\lambda^* > 0$, let $\mathbf{L}$ denote the Stieljes transform of σ,

$$\mathbf{L}(s) := \int \frac{1}{s-\lambda} d\sigma(\lambda), \tag{3.1}$$

with $\beta_c \in (0,+\infty]$ corresponding to the critical *temperature* such that

$$\beta_c = \frac{c}{2\lambda^*}\mathbf{L}(\lambda^*) \ , \tag{3.2}$$

and

$$p(s,\beta) := \frac{2\beta}{c}s - \mathbf{L}(s) \ . \tag{3.3}$$

The next theorem summarizes the static phase transition at $\beta = \beta_c$ in case $\beta_c < \infty$.

Theorem 3.1. *For any $\beta \in (0,\beta_c)$ there exists a unique solution of $p(s,\beta) = 0$ on (λ^*,∞), denoted s_β. With $s_\beta := \lambda^*$ when $\beta \geq \beta_c$, the free energy of (2.8) is for any $\beta > 0$*

$$F_\beta = \frac{\beta s_\beta^2}{2c} - \frac{1}{2}\int \log(s_\beta - \lambda) d\sigma(\lambda) + \frac{1}{2}\log(\pi\beta^{-1}). \tag{3.4}$$

The function F_β is thus non-analytic at $\beta = \beta_c$, a characterization of the phase transition. Corresponding to F_β are the parameters $u_\beta = s_\beta/c$, $v_\beta = u_\beta s_\beta - 1/(2\beta)$ and the limit law ν_0^* of (2.9), where $\alpha_\beta = \beta$ and $\rho_\beta = \frac{1}{2} - \beta s_\beta$. For any $\beta < \beta_c$ the law ν_0^* has a sub-Gaussian tail, whereas if $|\mathbf{L}^{(k)}(s)| \to \infty$ as $s \downarrow \lambda^*$ then the $2(k+1)$-st moment of ν_0^* is infinite for all $\beta \geq \beta_c$.

To illustrate Theorem 3.1, consider σ which is the semicircular law of Theorem 6.1 corresponding to the GOE $\mathbf{J}$ of the SSSK model. Then, for $s \geq \lambda^* = 2$,

$$\mathbf{L}(s) = \frac{s}{2} - \sqrt{\frac{s^2}{4} - 1}$$

(c.f. [4, Proof of Lemma 2.7]). The critical temperature corresponds to $\beta_c = c/4$ (see (3.2)), with $s_\beta = 2(1-(1-\beta/\beta_c)^2)^{-1/2}$ when $\beta < \beta_c$. The formula (1.4) for the free energy then follows from (3.4) (see also [4, Lemma 2.7]). It is easy to check that $\partial_\beta^3 F_\beta$ is not continuous at $\beta = \beta_c$ and that the fourth moment of ν_0^* is infinite for any $\beta \geq \beta_c$.

3.2. Dynamical phase transition starting from (IC1) or (IC2) initial conditions

Our goal here is to analyze the asymptotic behavior of the limiting covariance function $K(t,s)$ of the Langevin dynamics (1.1) (or its diagonalized form (2.2)), for large values of t and s. Up to the value of $K(0,0) = \int v^2 d\nu_0(v) \in \mathbb{R}^+$, the same solution $K(t,s)$ of (2.16) corresponds to both (IC1) and (IC2) initial conditions. For the sake of definiteness, we set hereafter $t \geq s$ and $K(0,0) = 1$, in which case $K(t,s)$ is the unique solution of

$$K(t,s) = \frac{1}{\sqrt{R(t)R(s)}}\Big(\mathscr{L}(t+s) + \beta^{-1}\int_0^s R(\tau)\mathscr{L}(t+s-2\tau)d\tau\Big)\,, \quad (3.5)$$

where $\mathscr{L}(\theta) := \int e^{\lambda\theta} d\sigma(\lambda)$ and

$$R(t) := \exp\{2\int_0^t f'(K_d(u))du\} \quad (3.6)$$

for $K_d(u) = K(u,u)$. In particular, we provide a characterization of the phase transition in terms of the asymptotics of $K(t,s)$ and exhibit a primitive form of the so-called *aging regime* introduced in [8]. We begin by describing the case of non-random modes and find, as we should, an absence of aging regime, that is, at any positive temperature, the covariance $K(t,s)$ approaches zero regardless of the way the time parameters t and s go to infinity. Then, we study the instructive and simple zero temperature model where we show that an aging phenomenon occurs. Finally, we consider positive and random modes models where we show that a dynamical phase transition occurs; for $\beta < \beta_c$, the covariance decreases exponentially in $|t-s|$ regardless of the way the time parameters go to infinity, whereas for $\beta > \beta_c$, an aging phenomenon shows up.

3.2.1. Constant modes

We start with the case of constant (non-random) modes, or more generally, that of $\sigma = \delta_0$. In this case, $\mathscr{L}(\theta) = 1$ and (3.5) leads to the following equation for $K_d(\cdot)$,

$$K_d(t) = 1 + \int_0^t \phi(K_d(\tau))d\tau\ ,$$

where $\phi(x) := \beta^{-1} - 2xf'(x)$. With f strictly convex and super-linear it is easy to see that for any $\beta < \infty$ there exists a unique positive point K_∞ such that $\phi(K_\infty) = 0$. Furthermore, $\phi(x) < 0$ for $x > K_\infty$ and $\phi(x) > 0$ for $x < K_\infty$, implying that $K_d(t) \to K_\infty$ monotonically as $t \to \infty$. In particular, $K_d(\cdot)$ is bounded above by $\max(K_\infty, 1)$. The same applies for $\beta = \infty$ when $f'(0) \leq 0$ since then there exists a non negative K_∞ such that $K_\infty f'(K_\infty) = 0$ (where $K_\infty = 0$ when $f'(0) = 0$).

If $\beta = \infty$ then $K(t,s) = \sqrt{K_d(t)K_d(s)}$ converges to K_∞ independently of the way (t,s) approach infinity (we do not expect convergence to zero since here all the randomness is in the initial conditions). Considering $\beta < \infty$, there exists $s_0 = s_0(\beta) < \infty$ large enough such that

$$\inf_{s \geq s_0} f'(K_d(s)) \geq (4\beta K_\infty)^{-1} := \delta_\beta > 0 \,,$$

and then for all $t \geq s \geq s_0$

$$K(t,s) = K_d(s) \frac{\sqrt{R(s)}}{\sqrt{R(t)}} \leq \max(K_\infty, 1) e^{-\delta_\beta (t-s)} \,.$$

Thus, we see no phase transition in the dynamics, with the covariance $K(t,s)$ approaching zero exponentially in $|t-s|$ for t, s large.

3.2.2. Random modes at zero temperature

In this subsection, we discuss the case where $\beta = \infty$ and $\lambda^* > 0$ (so σ is any probability measure with at most finitely many atoms, supported on $[-\lambda^*, \lambda^*]$ and positive in the sense of **(H2)**). Then, by (3.5), $K_d(t) = \mathscr{L}(2t)/R(t)$ is continuously differentiable with

$$K_d'(t) = \phi_t(K_d(t)) \tag{3.7}$$

for

$$\phi_t(x) := 2x\Big(\frac{\mathscr{L}'(2t)}{\mathscr{L}(2t)} - f'(x)\Big).$$

The positive solution K_∞ of $f'(x) = \lambda^*$ is unique (for f strictly convex, superlinear such that $f'(0) \leq 0$). Moreover, standard Laplace method yields

$$\lim_{t\to\infty} \frac{\mathscr{L}'(2t)}{\mathscr{L}(2t)} = \lambda^* \,,$$

so for any $\delta > 0$ and t large enough, $\phi_t(x) > 0$ for $x < K_\infty - \delta$ and $\phi_t(x) < 0$ for $x > K_\infty + \delta$. Thus, from (3.7) we see that $K_d(t) \to K_\infty$ as $t \to \infty$. Recall that by (3.5), for $\beta = \infty$,

$$K(t,s) = \sqrt{K_d(t)K_d(s)} \frac{\mathscr{L}(t+s)}{\sqrt{\mathscr{L}(2t)\mathscr{L}(2s)}}$$

Therefore, the asymptotic behavior of $K(t,s)$ depends on the asymptotic behavior of $\mathscr{L}(\cdot)$. Suppose for example that $\mathscr{L}(\theta) \sim b\theta^{-q} e^{\lambda^*\theta}$ for some $q \geq 0$, finite $b > 0$ and large θ, where hereafter the notation $f(\cdot) \sim g(\cdot)$ means that $f/g \to 1$. Let also $f \approx g$ denote the case where f/g is bounded and bounded away from zero, and $t \gg s$ mean that $t - s \to \infty$. Then, we see that

$$K(t,s) \sim K_\infty$$

for large t, s such that $|t-s| = o(|t|)$ whereas for $t - s \gg s \gg 1$,

$$K(t,s) \approx K_\infty \left| \frac{t-s}{s} \right|^{-\frac{q}{2}} \,,$$

exhibiting an aging phenomenon when $q > 0$.

3.2.3. Random modes at positive temperature

In the previous zero temperature model, we observed that the asymptotic behavior of the Laplace transform $\mathscr{L}(\theta)$ was at the heart of the simple aging phenomenon encountered in SSSK. This is still the case when we consider positive temperature, e.g. $\beta < \infty$, with random modes $\lambda^* > 0$ and $f(x) = cx^2/2$. We shall also assume here that the symmetric probability measure σ (of at most finitely many atoms, supported on $[-\lambda^*, \lambda^*]$ and positive in the sense of **(H2)**), is such that for some $q > 1$ and finite $b_1 > 0$

$$\mathscr{L}(\theta) \sim_{\theta\uparrow\infty} b_1 \theta^{-q} e^{\lambda^*\theta} . \tag{3.8}$$

Observe that this assumption is equivalent to (c.f. [6, Theorem 1.7.1'])

$$\sigma([\lambda^* - x, \lambda^*]) \sim_{x\downarrow 0} \frac{b_1 x^q}{\Gamma(q+1)} \tag{3.9}$$

in view of the integration by parts formula

$$e^{-\lambda^*\theta}\mathscr{L}(\theta) = \theta \int_0^{2\lambda^*} e^{-x\theta}\sigma([\lambda^* - x, \lambda^*])dx + e^{-2\lambda^*\theta}.$$

The assumption that $q > 1$ is needed to insure that $\mathbf{L}(\lambda^*) < \infty$, resulting by Theorem 3.1 with a static phase transition at $\beta_c \in (0, \infty)$ of (3.2). For example, the semicircular law that corresponds to the SSSK model with independent Gaussian coupling satisfies (3.8) (alternatively, (3.9)), for $q = 3/2$.

Our assumption (3.8) yields the following asymptotic behavior of $K(t, s)$ for $t \geq s \gg 1$, in which we clearly see the appearance of one aging regime at $\beta > \beta_c$.

Proposition 3.2. *For any $\beta \in (0, \beta_c)$ and $\delta_\beta < (s_\beta - \lambda^*)$, there exists $C = C_\beta < \infty$ such that for all (t, s),*

$$K(t, s) \leq Ce^{-\delta_\beta|t-s|} . \tag{3.10}$$

In contrast, for $\beta = \beta_c$, $q \neq 2$, and $t \gg s \gg 1$, we have the polynomial decay

$$K(t, s) \approx \begin{cases} (t-s)^{1-q} & \text{for } t/s \text{ bounded} \\ \frac{s^{1-\psi_q/2}}{t^{q-\psi_q/2}} & \text{otherwise ,} \end{cases} \tag{3.11}$$

where $\psi_q = \max(2 - q, 0)$. When $\beta > \beta_c$ we get that

$$K(t, s) \approx (s/t)^{q/2} , \tag{3.12}$$

so $K(t, s) \to 0$ if and only if $t/s \to \infty$.

Proposition 3.2 is a direct consequence of the next lemma,

Lemma 3.3. *Let $\psi = 0$ for $\beta < \beta_c$, $\psi = \psi_q$ for $\beta = \beta_c$, $q \neq 2$, and $\psi = q$ for $\beta > \beta_c$. Then, for s_β of Theorem 3.1 and some $C_{q,\beta} \in (0, \infty)$,*

$$R(x) \sim_{x\uparrow\infty} C_{q,\beta} x^{-\psi} e^{2s_\beta x} . \tag{3.13}$$

Indeed, plugging our assumption (3.8) and the estimate (3.13) into (3.5), yields after some computations the stated bounds (3.10)–(3.12) of Proposition 3.2.

The case of $\beta = \beta_c$ and $q = 2$ can be similarly handled, at the cost of cumbersome notations and proof, resulting with x^{ψ} replaced by $\log x$ in (3.11) and (3.13). For simplicity, we shall not do so here.

The proof of Lemma 3.3 follows from the observation that (3.5) gives an equation for R which leads to an explicit formula for the Laplace or the Fourier transform of R. From the formula for the Laplace transform of R, one can deduce by Tauberian theorems the first order term in the asymptotics of R given in equation (3.13) but not the polynomial second order term which is needed to conclude. Hence, one needs to use the formula for the Fourier transform and proceed by use of complex analysis. This proof is detailed in Section 7.

3.3. *Absence of aging regime when starting from the top eigenvector*

In case of (IC3) initial conditions, the limiting covariance function $K(t,s)$ of the Langevin dynamics (1.1) (or its diagonalized form (2.2)), is the unique solution of

$$K(t,s) = \frac{1}{\sqrt{R(t)R(s)}}\left(e^{\lambda^*(t+s)} + \beta^{-1}\int_0^s R(\tau)\mathscr{L}(t+s-2\tau)d\tau\right), \qquad (3.14)$$

where $t \geq s$ and $\mathscr{L}(\cdot)$, $R(\cdot)$, $K_d(\cdot)$ are as introduced in (3.5).

In case $\lambda^* = 0$ we recover the same solution as for (s3.5) so the analysis of Section 3.2.1 applies here as well. In case $\beta = \infty$ the analysis of Section 3.2.2 applies here as well, but for $q = 0$, resulting with the absence of the aging regime.

Turning to deal with $\lambda^* > 0$ and $\beta < \infty$, we make the same assumptions as in Section 3.2.3, in particular assuming (3.8) holds for some $q > 1$. Adapting the analysis to the setting of (3.14), we show in Section 7 that

Theorem 3.4. *Let β_c be as in (3.2). Then, (3.10) holds for any $\beta \in (0, \beta_c)$ (with same choice of δ_β). In case of $\beta = \beta_c$, $q \neq 2$ and $t \gg s \gg 1$, we have a polynomial decay of $K(t,s)$ to zero, albeit with a power that is no longer that of (3.11). For any fixed $\beta > \beta_c$, regardless of the way in which $t - s$ and s approach infinity, $K(t,s) \to c_{EA} := p(\lambda^*, \beta)/(2\beta) > 0$. There is thus no aging regime for the initial condition (IC3).*

3.4. *Dynamic phase transition for stationary initial conditions*

We shall examine the dynamic phase transition for the diagonalized SDS (2.2) in case of $f(x) = cx^2/2$ and starting from the stationary initial conditions determined by the mean field Gibbs measure (2.1). We assume as in Section 3.2.3 that $\beta < \infty$, $\lambda^* > 0$, and that the symmetric probability measure σ satisfies (3.8) for some $q > 1$ and $b_1 \in (0, \infty)$. Our main result is then,

Proposition 3.5. *Under* **(H0)**, **(H1)** *and* **(H2)**, *for almost all* $\boldsymbol{\lambda}$ *the empirical measure $\hat{\nu}_T^N$ converges in $\mathscr{P}(C_b([0,T]))$ towards $\nu(w) = \int \nu_\lambda(w)d\sigma(\lambda)$ where ν_λ is* *the law of the Ornstein-Uhlenbeck process*

$$\begin{cases} dv_t = (\lambda - s_\beta)v_t dt + \beta^{-1/2} dB_t \\ Law\ of\ (v_0) = \gamma_{2\beta(s_\beta-\lambda)}\ , \end{cases} \tag{3.15}$$

and the empirical covariance K^N converges in $C_b([0,T]^2)$ towards $K_{inv}(|t-s|)$, where

$$K_{inv}(\tau) = \frac{1}{2\beta}\left[\int (s_\beta - \lambda)^{-1} e^{-(s_\beta-\lambda)\tau} d\sigma(\lambda) + p(s_\beta, \beta)\right]\ . \tag{3.16}$$

Recall that $p(s_\beta, \beta) = 0$ when $\beta \le \beta_c$ and $p(s_\beta, \beta) > 0$ otherwise (see (3.3)). The solution of (3.16) undergoes a dynamical phase transition; below criticality, $c_{EA} = 0$ and the covariance $K(t,s)$ goes to zero exponentially fast as $|t-s| \to \infty$, whereas above criticality it converges to $c_{EA} = p(\lambda^*, \beta)/(2\beta) > 0$. This is very similar to (IC3) in which case also $K(t,s) \to c_{EA}$ for $s \to \infty$ and $(t-s) \to \infty$. Here, as in the case of (IC3), there is no aging phenomenon.

Proof. By Theorems 2.2 and 3.1 we have that $f'(u_\beta) = cu_\beta = s_\beta$, $\alpha_\beta = \beta$ and $\rho_\beta = \frac{1}{2} - \beta s_\beta$, with $\pi^*(\lambda, v) = \gamma_{2\beta(s_\beta-\lambda)}(v) \otimes \sigma(\lambda)$. It is thus easy to check that

$$c_{EA} := u_\beta - \int v^2 d\pi^*(\lambda, v) = \frac{s_\beta}{c} - \frac{1}{2\beta}\int (s_\beta - \lambda)^{-1} d\sigma(\lambda) = \frac{1}{2\beta} p(s_\beta, \beta),$$

and in view of (3.9),

$$\begin{aligned} \int v^2 e^{-2(s_\beta-\lambda)t} d\pi^*(\lambda, v) &= \frac{1}{2\beta}\int e^{-2(s_\beta-\lambda)t}(s_\beta-\lambda)^{-1} d\sigma(\lambda) \\ &= \beta^{-1}\int_t^\infty e^{-2s_\beta\theta}\mathscr{L}(2\theta)d\theta\ . \end{aligned}$$

Since $p(s_\beta, \beta) = 0$ whenever $s_\beta \neq \lambda^*$, also

$$e^{-2s_\beta t}\mathscr{L}_0(2t) = \beta^{-1}\left[\int_t^\infty e^{-2s_\beta\theta}\mathscr{L}(2\theta)d\theta + \frac{1}{2}p(s_\beta, \beta)\right] = K_{inv}(2t)\ , \tag{3.17}$$

resulting with

$$e^{-2s_\beta t}\mathscr{L}_0(2t) + \beta^{-1}\int_0^t e^{-2s_\beta\theta}\mathscr{L}(2\theta)d\theta = \mathscr{L}_0(0) \tag{3.18}$$

independently of t. Note that in Theorem 2.3, $K_d(0) = K(0,0) = \mathscr{L}_0(0) = u_\beta$ for (IC4), with the expression of (3.18) being the value of $K(t,t)$ obtained upon setting $f'(K_d(t)) = f'(u_\beta) = s_\beta$ in (2.16). Thus, the unique solution of (2.16) under (IC4) corresponds to $K_d(t) = u_\beta$ for all t. The dynamics (3.15) for ν_λ is then merely (2.15) of Theorem 2.3. Moreover, setting $t = s + \tau > s$ and $K_d(t) = u_\beta$ in (2.16), we have by (3.18) that

$$\begin{aligned} K(s+\tau, s) &= e^{-2s_\beta(s+\tau/2)}\mathscr{L}_0(2(s+\tau/2)) + \beta^{-1}\int_{\tau/2}^{s+\tau/2} e^{-2s_\beta\theta}\mathscr{L}(2\theta)d\theta \\ &= e^{-s_\beta\tau}\mathscr{L}_0(\tau) = K_{inv}(\tau) \end{aligned}$$

thus yielding (3.16). □

3.5. *The convergence of $K(s+\tau, s)$ to $K_{inv}(\tau)$*

We shall examine next the convergence of $K(s+\tau, s)$ of (2.16) to $K_{inv}(\tau)$ of (3.16) as $s \to \infty$.

Proposition 3.6. *Suppose $f(x) = cx^2/2$, $\beta < \infty$, $\lambda^* > 0$ and the symmetric probability measure σ satisfies (3.8) for some $q > 1$ and $b_1 \in (0, \infty)$. Then, the unique solution $K(t, s)$ of (2.16) for (IC1), (IC2) or (IC3) is such that for all $\tau > 0$,*

$$\lim_{s \to \infty} K(s+\tau, s) = K_{inv}(\tau) \tag{3.19}$$

Proof. Note that for all three initial conditions, it follows from (2.16) that

$$K_d(s+\frac{\tau}{2})R(s+\frac{\tau}{2}) - K(s+\tau, s)\sqrt{R(s)R(s+\tau)}$$
$$= \beta^{-1} \int_0^{\tau/2} R(s+\frac{\tau}{2}-\theta)\mathscr{L}(2\theta)d\theta$$

for $R(\cdot)$ of (3.6). With $r_1(t,\theta) := \sqrt{R(t)R(t+\theta)}/R(t+\frac{\theta}{2})$ and $r_2(t,\theta) := R(t-\theta)/R(t)$ we thus have that

$$K(s+\tau, s) = \frac{1}{r_1(s,\tau)} \left[K_d(s+\frac{\tau}{2}) - \beta^{-1} \int_0^{\tau/2} r_2(s+\frac{\tau}{2}, \theta)\mathscr{L}(2\theta)d\theta \right] .$$

Fixing $\tau \geq 0$, it follows from (3.13) and (7.37) that $r_1(s,\tau) \to 1$ and $r_2(s+\frac{\tau}{2}, \theta) \to e^{-2s_\beta\theta}$ as $s \to \infty$, uniformly in $\theta \in [0, \tau/2]$. Moreover, an analysis similar to that done when proving Lemma 3.3 and Theorem 3.4 shows that $K_d(x) = R'(x)/(2cR(x)) \to u_\beta$ as $x \to \infty$. Therefore, by (3.17) and (3.18)

$$\lim_{s \to \infty} K(s+\tau, s) = u_\beta - \beta^{-1} \int_0^{\tau/2} e^{-2s_\beta\theta}\mathscr{L}(2\theta)d\theta = K_{inv}(\tau) ,$$

for all three initial conditions, as stated. □

3.6. *Limiting behavior of the centered covariance K_N*

Until now, we have studied the asymptotic behavior of

$$K^N(t,s) = \frac{1}{N}\sum_{i=1}^N v_t^i v_s^i = \frac{1}{N}\sum_{i=1}^N u_t^i u_s^i,$$

with $(u^i)_{1 \leq i \leq N}$ being the solution of (1.1), and called it empirical covariance even though this covariance should rather be given by

$$K_N(t,s) = K^N(t,s) - m_N(t)m_N(s), \quad m_N(t) := \frac{1}{N}\sum_{i=1}^N u_t^i.$$

The next proposition shows that starting from (IC1)–(IC4), the centered covariance K_N converges and its limit has the same long time behavior as that of K of Theorem 2.3.

Proposition 3.7. *Assume* **(H0)**, **(H1)** *and* **(H2)**. *Starting the SDS (1.1) with either (IC1), (IC3) or (IC4), for almost all* $\boldsymbol{\lambda}$ *the centered covariance* $K_N(t,s)$ *of (1.6) converges almost surely to the corresponding* $K(t,s)$ *of (2.16), uniformly in* $(t,s) \in [0,T]^2$. *Further, in case of (IC2), for i.i.d. initial conditions, that is, where* $\boldsymbol{u}_0$ *has law* $\mu^{\otimes N}$, *we have that a.s. and uniformly in* $(t,s) \in [0,T]^2$,

$$\lim_{N\to\infty} K_N(t,s) = K(t,s) - (\int x d\mu(x))^2 \frac{\mathscr{L}(t)\mathscr{L}(s)}{\sqrt{R(t)R(s)}} . \tag{3.20}$$

Remarks. Recall that our assumption (3.8) implies the convergence of $\mathscr{L}(t)\mathscr{L}(s)/\mathscr{L}(t+s)$ to zero when $t \wedge s \to \infty$. Thus, $K(t,s)$ of (3.5) dominates the right hand side of (3.20) for large $t \wedge s$, or in other words, $\lim_{N\to\infty} K_N(t,s)$ then behaves as $K(t,s) = \lim_{N\to\infty} K^N(t,s)$.

Proof. Let $\mathbb{1}$ denote the N-dimensional vector $(1,\dots,1)$ and $\boldsymbol{s} := N^{-1/2}\boldsymbol{G}\mathbb{1}$ which follows the uniform law on the unit sphere in $\mathbb{R}^N$. Recall that $\boldsymbol{u}_t = \boldsymbol{G}^*\boldsymbol{v}_t$ for the solution $\boldsymbol{v}_t$ of the diagonalized SDS (1.3), hence $m_N(t) = N^{-1/2}\langle \boldsymbol{s}, \boldsymbol{v}_t\rangle$. Note that under our assumptions, $\boldsymbol{B}_\cdot$ and $\boldsymbol{D}$ of (1.3) are independent of $\boldsymbol{G}$, as is $\boldsymbol{v}_0$, at least for (IC1), (IC3) and (IC4). Thus, for these initial conditions and each fixed $t \in [0,T]$, the random vectors $\boldsymbol{s}$ and $\boldsymbol{v}_t$ are independent. Consequently, the law of $m_N(t)$ is then the same as that of $N^{-1/2}g_1|\boldsymbol{v}_t|/|\boldsymbol{g}|$ where $\boldsymbol{g}$ is independent of $\boldsymbol{v}_t$ and follows the standard centered Gaussian law $\gamma^{\otimes N}$. With $K^N(t,t) = N^{-1}|\boldsymbol{v}_t|^2$ it follows that for every $\delta > 0$, and $r < \infty$,

$$\mathbb{P}(|m_N(t)| > \delta) \le \mathbb{P}(K^N(t,t) > r) + \mathbb{P}(g_1^2 > \delta^2 r^{-1} \sum_{i=2}^N g_i^2) .$$

Note that $\mathbb{P}(K^N(t,t) > r) \to 0$ exponentially in N whenever $r > K(t,t)$, by the LDPs of Theorem 2.4. Since the same applies for $\mathbb{P}(g_1^2 > \delta^2 r^{-1} \sum_{i=2}^N g_i^2)$, the a.s. convergence of $m_N(t)$ to zero follows by the Borel-Cantelli lemma. We thus see that for (IC1), (IC3), (IC4) and each $(t,s) \in [0,T]^2$, almost surely,

$$\lim_{N\to\infty} K_N(t,s) = \lim_{N\to\infty} K^N(t,s) = K(t,s) . \tag{3.21}$$

We turn to deal with (IC2) in the special case where $\boldsymbol{u}_0$ has law $\mu^{\otimes N}$ (and $\int \exp(\eta x^2) d\mu(x) < \infty$ for some $\eta > 0$). Solving (1.1) it is easy to check that,

$$m_N(t) = \frac{1}{N\sqrt{R_N(t)}}\langle e^{t\boldsymbol{J}}\mathbb{1}, \boldsymbol{u}_0\rangle + \frac{1}{N\sqrt{\beta R_N(t)}}\int_0^t \sqrt{R_N(s)}\langle e^{(t-s)\boldsymbol{J}}\mathbb{1}, d\boldsymbol{W}_s\rangle \tag{3.22}$$

where $R_N(s) := \exp(2\int_0^s f'(K^N(u,u))du)$. We next show that the second term in the right hand side of (3.22) converges a.s. to zero, that is, for each $t \in [0,T]$,

$$|m_N(t) - R_N(t)^{-\frac{1}{2}} V_N(t)| \to 0 \qquad a.s. \tag{3.23}$$

where $V_N(t) := N^{-1}\langle e^{t\boldsymbol{J}}\mathbb{1}, \boldsymbol{u}_0\rangle$. To this end, consider the martingales

$$u \mapsto Y_N(u) := N^{-1}\int_0^u \sqrt{R_N(s)}\langle e^{(t-s)\boldsymbol{J}}\mathbb{1}, d\boldsymbol{W}_s\rangle$$

at the stopping times

$$\tau_N^M = \inf\{s \ge 0 \,:\, R_N(s) \ge e^{Ms}\} \wedge T$$

for some $M > 2\sup_{t\in[0,T]} f'(K(t,t))$. By the Burkholder-Davies-Gundy inequality (c.f. [19, Theorem 3.3.28]), for some $A < \infty$ and all $\delta > 0$,

$$\mathbb{P}\left(\sup_{u\in[0,\tau_N^M]} |Y_N(u)| > \delta\right) \le \frac{2}{\delta^4}\mathbb{E}[(\sup_{u\in[0,\tau_N^M]} Y_N(u))^4] \le \frac{A}{\delta^4}\mathbb{E}[(\langle Y_N\rangle_{\tau_N^M})^2]$$
$$\le \frac{AT}{\delta^4 N^2}e^{2MT+4T\lambda_N^*}\,.$$

Recall that almost surely $R_N(s) \to R(s)$ uniformly in $s \in [0,T]$ (by the LDPs of Theorem 2.4 and the Lipschitz continuity of f'). Hence, by the choice of M, almost surely $\tau_N^M \ge t$ for all N large enough, yielding by the Borel-Cantelli lemma that

$$\limsup_{N\to\infty} |Y_N(t)| \le \limsup_{N\to\infty} \sup_{u\in[0,\tau_N^M]} |Y_N(u)| = 0. \tag{3.24}$$

With $R_N(t) \to R(t) > 0$, we thus get (3.23) by combining (3.22) and (3.24).

Continuing from (3.23) and fixing $\boldsymbol{J}$, note that

$$\mathbb{E}(V_N(t)) = \int x d\mu(x) N^{-1}\langle \mathbb{1}, e^{t\mathbf{J}}\mathbb{1}\rangle = \int x d\mu(x)\sum_{i=1}^N s_i^2 e^{t\lambda_i}$$

where $\boldsymbol{s} = (s_1, \dots, s_N)$ follows the uniform law on the unit sphere of $\mathbb{R}^N$, independently of the eigenvalues $(\lambda_1, \dots, \lambda_N)$ of $\mathbf{J}$. Applying Theorem 5.4 for $\nu_0 = \gamma$, it follows by the contraction principle that almost surely,

$$\limsup_{N\to\infty}\sup_{t\in[0,T]} |N^{-1}\sum_{i=1}^N g_i^2 e^{t\lambda_i} - \mathscr{L}(t)| = 0\,,$$

where $\boldsymbol{g} = (g_1, \dots, g_N)$ follows the standard centered Gaussian law $\gamma^{\otimes N}$, independently of the eigenvalues of $\mathbf{J}$. Hence, the representation $\boldsymbol{s} = \boldsymbol{g}/|\boldsymbol{g}|$ results with

$$\limsup_{N\to\infty}\sup_{t\in[0,T]} |\mathbb{E}(V_N(t)) - \int x d\mu(x)\mathscr{L}(t)| = 0\,, \tag{3.25}$$

for almost every $\mathbf{J}$. Considering $V_N(t) - \mathbb{E}(V_N(t))$ amounts to centering μ, that is, assuming $\int x d\mu(x) = 0$. In this case, the independence of $(u_0^i)_{1\le i\le N}$ and **(H1)** imply that for some $B < \infty$ and any fixed $t \in [0,T]$ and $\boldsymbol{J}$,

$$\mathbb{E}[(N^{-1}\langle e^{t\mathbf{J}}\mathbb{1}, \boldsymbol{u}_0\rangle)^4] \le \frac{B}{N^2}\int x^4 d\mu(x) e^{4t\lambda_N^*}\,,$$

hence by the Borel-Cantelli lemma, for each $t \in [0, T]$, almost surely, $N^{-1}\langle e^{t\boldsymbol{J}}\mathbb{1}, \boldsymbol{u}_0\rangle \to 0$.

Combining this, the convergence results $R_N(t) \to R(t)$ and (3.23), (3.25), we see that for each $t \in [0, T]$, almost surely,

$$\lim_{N\to\infty} m_N(t) = \int x d\mu(x)\mathscr{L}(t)/\sqrt{R(t)}\,.$$

Clearly it then follows that (3.20) holds for each $(t, s) \in [0, T]^2$.

Finally, recall that by Theorem 2.4, up to a null set $\mathscr{N}_0$ of $\boldsymbol{J}$ values, the functions K^N are exponentially tight on $C([0, T]^2, \mathbb{R})$. Since $|m_N(t) - m_N(s)|^2 \leq K^N(t, t)+K^N(s, s)-2K^N(t, s)$ and $m_N(t)^2 \leq K^N(t, t)$, the Arzela-Ascoli's theorem results with the exponential tightness of m_N on $C([0, T], \mathbb{R})$, for all $\boldsymbol{J} \notin \mathscr{N}_0$. Fix $\boldsymbol{J} \notin \mathscr{N}_0 \cup \mathscr{N}_1$, where $\mathscr{N}_1$ denotes the null set of $\boldsymbol{J}$ such that $m_N(t) \to m(t)$ for all $\boldsymbol{J} \notin \mathscr{N}_1$ and each rational $t \in [0, 1]$. Then, the exponential tightness of m_N implies that almost surely it has a limit point $m'_{\boldsymbol{J}}(\cdot) \in C([0, T], \mathbb{R})$ with respect to the uniform convergence. Moreover, $m'_{\boldsymbol{J}}(t) = m(t)$ for each rational $t \in [0, 1]$, hence $m(\cdot)$ is the only possible limit point of $m_N(\cdot)$. Consequently, the a.s. convergence results (3.20) and (3.21) hold uniformly in $(t, s) \in [0, T]^2$. □

4. Proof of Theorem 2.2

The spherical constraint induced by the super-linear f insures that the covariance $N^{-1}\sum_{i=1}^{N} v_i^2 = \int x^2 d\hat{\nu}_0^N(x)$ is well controlled, but it can not insure a quasi-continuity of the map $\mu \mapsto \int x^2 d\mu(x)$ under $\mu_{\boldsymbol{\lambda}}^N$. For this reason, it is not possible to derive Theorem 2.2 as a contraction from the much simpler quenched LDP for the empirical measure.

The next lemma, providing the LDP for Y_0^N under the Gaussian measure $\gamma^{\otimes N}$ is the key to the proof of Theorem 2.2.

Lemma 4.1. *Under* **(H0)** *and* **(H1)**, *for almost all* $\boldsymbol{\lambda}$, *the law of* Y_0^N *under* $\gamma^{\otimes N}$ *satisfies a LDP with GRF* $\mathscr{K}(u, v, \mu)$, *whereas the law of* $(N^{-1}\sum_{i=1}^{N} v_i^2, N^{-1}\sum_{i=1}^{N} \lambda_i v_i^2)$ *under* $\gamma^{\otimes N}$ *satisfies the LDP with GRF* $k(u, v)$.

We wish to apply Gärtner-Ellis theorem to prove Lemma 4.1. However, because this theorem requires the essential smoothness of certain limiting logarithmic moment generating functions, we first consider the case where $\hat{\sigma}^N$ are supported on the same finite, non-random set (and hence so is σ), then relax this assumption on the joint law of $\boldsymbol{\lambda}$ by the use of exponentially good approximations. The following simple lemma is key to the success of this program, which we follow in all the LDPs proved in this article.

Lemma 4.2. *Let* $\mathscr{J}$ *denote the finite set of jump discontinuity points of a given piecewise constant function* $\phi : \mathbb{R} \to \mathbb{R}$. *If* **(H0)** *holds for* $\hat{\sigma}^N$ *and* σ *such that* $\sigma(\mathscr{J}) = 0$, *then it holds for* $\hat{\sigma}^N \circ \phi^{-1}$ *and* $\sigma \circ \phi^{-1}$.

Proof. Suppose that $\mu_n \to \sigma$ in $\mathscr{P}(\mathbb{R})$. Then, by the Portmanteau theorem $\mu_n(I) \to \sigma(I)$ for any interval I such that $\sigma(\partial I) = 0$. Since $\sigma(\mathscr{J}) = 0$ this applies to the finitely many intervals $\{I_j\}$ in the partition ϕ^{-1} of $\mathbb{R}$, resulting with

$$\mu_n \circ \phi^{-1} = \sum_j \mu_n(I_j)\delta_{\phi(I_j)} \to \sum_j \sigma(I_j)\delta_{\phi(I_j)} = \sigma \circ \phi^{-1} . \tag{4.1}$$

Thus, $\hat{\sigma}^N \to \sigma$ a.s. implies $\hat{\sigma}^N \circ \phi^{-1} \to \sigma \circ \phi^{-1}$ a.s. □

Proof of Lemma 4.1. First, we assume that $\hat{\sigma}^N$ are supported on the same finite, non-random set for all N, and by **(H1)** so is their limit σ. Since $\sigma(\{\lambda^*\}) > 0$ and $\sigma(\{-\lambda^*\}) > 0$, under **(H0)** and **(H1)**, we may and shall take the sequence $\boldsymbol{\lambda} = (\lambda_i)_{1\le i\le N}$ in a subset Ω of full probability such that $\hat{\sigma}^N \to \sigma$ and $\lambda^* = \max_{i=1}^N \lambda_i = -\min_{i=1}^N \lambda_i$ for all N large enough. For $V \in C_b(\mathbb{R}^2)$ and $(\rho, \alpha) \in \mathbb{R}^2$ let

$$\psi(\lambda) := \log \int e^{V(\lambda,w)+(\alpha\lambda+\rho)w^2} d\gamma(w)$$

and consider the Laplace transforms

$$\Lambda_N(\rho, \alpha, V) := \frac{1}{N} \log \left[\int \exp\Big(\sum_{i=1}^N V(\lambda_i, v_i) + \alpha \sum_{i=1}^N \lambda_i v_i^2 + \rho \sum_{i=1}^N v_i^2 \Big) \times \prod_{i=1}^N d\gamma(v_i) \right] = \frac{1}{N} \sum_{i=1}^N \psi(\lambda_i) = \int \psi(\lambda) d\hat{\sigma}^N(\lambda) .$$

Since $\psi(\lambda) = \infty$ iff $\alpha\lambda + \rho \ge 1/2$, it follows that on Ω and all N large enough, $\Lambda_N(\rho, \alpha, V)$ is finite iff $|\alpha|\lambda^* + \rho < 1/2$. For such (ρ, α), by dominated convergence we see that $\psi \in C_b([-\lambda^*, \lambda^*])$, hence

$$\Lambda(\rho, \alpha, V) := \lim_{N\to\infty} \Lambda_N(\rho, \alpha, V) = \lim_{N\to\infty} \int_{-\lambda^*}^{\lambda^*} \psi(\lambda) d\hat{\sigma}^N(\lambda) = \int \psi(\lambda) d\sigma(\lambda) \tag{4.2}$$

exists and is finite. Obviously, $\Lambda(\rho, \alpha, V) := \int \psi(\lambda) d\sigma(\lambda)$ is infinite whenever $|\alpha|\lambda^* + \rho \ge 1/2$ (recall that $\sigma(\{\lambda^*\}) > 0$ and $\sigma(\{-\lambda^*\}) > 0$). Using dominated convergence it is easy to check that the function $(\rho, \alpha, \mathbf{t}) \mapsto \Lambda(\rho, \alpha, \sum_i t_i V_i)$ is continuous everywhere and differentiable wherever it is finite, thus being essentially smooth for any fixed $V_i \in C_b(\mathbb{R}^2)$. Combining the Gärtner-Ellis theorem with projective limits for $\mathscr{Y} = \mathbb{R}^2 \times C_b(\mathbb{R}^2)$ (see [15, Corollary 4.6.11(a)]), we deduce that Y_0^N satisfies the LDP in $\mathscr{Y}'$ with the convex GRF

$$\mathscr{K}(u, v, \tau) = \sup_{(\rho,\alpha,V)\in\mathscr{Y}} \{\langle V, \tau\rangle + \rho u + \alpha v - \Lambda(\rho, \alpha, V)\} . \tag{4.3}$$

Note that for any $V \in C_b(\mathbb{R}^2)$ and any $\sigma \in \mathscr{P}(\mathbb{R})$

$$\Lambda(0,0,V) := \int \Big(\log \int e^{V(\lambda,w)} d\gamma(w)\Big) d\sigma(\lambda)$$
$$= \sup_{\{\mu \in \mathscr{P}(\mathbb{R}^2):\mu_1=\sigma\}} \Big[\int V d\mu - I(\mu|\sigma \otimes \gamma)\Big] \tag{4.4}$$

(see [15, Lemma 6.2.13] for a similar computation). Let $\mathscr{I}_0(\mu) = I(\mu|\sigma \otimes \gamma)$ when $\mu \in \mathscr{P}(\mathbb{R}^2)$ is such that $\mu_1 = \sigma$ and $\mathscr{I}_0(\tau) = \infty$ for any other $\tau \in C_b(\mathbb{R}^2)'$. Then, $\mathscr{I}_0(\cdot)$ is a convex GRF on $C_b(\mathbb{R}^2)'$ (for example, this follows from [15, Lemma 6.2.12]). Moreover, by (4.3), (4.4) and Fenchel duality

$$\mathscr{K}(u,v,\tau) \geq \sup_{V \in C_b(\mathbb{R}^2)} \big\{\langle V,\tau\rangle - \Lambda(0,0,V)\big\} = \mathscr{I}_0(\tau)$$

(see [15, Lemma 4.5.8]), is infinite whenever $\tau \notin \mathscr{P}(\mathbb{R}^2)$, allowing us to consider hereafter all LDPs in the space $\mathscr{X} := \mathbb{R}^2 \times \mathscr{P}(\mathbb{R}^2)$. The space $\mathscr{X}$ is metrizable by the metric $d(\cdot,\cdot)$ such that for any $(x,y) \in \mathscr{X}^2$

$$d(x,y) = |x_1 - y_1| + |x_2 - y_2| + D(x_3, y_3) \tag{4.5}$$

where $|\cdot|$ is the standard Euclidean distance on $\mathbb{R}$ and

$$D(p,q) := \sup\{|\int \phi dp - \int \phi dq| : \sup_z |\phi(z)| \leq 1,$$
$$\sup_{y \neq z} |\phi(z) - \phi(y)|/|z-y| \leq 1\}$$

To remove the assumption that $\hat{\sigma}^N$ and σ are supported on the same non-random finite set, we proceed by constructing exponentially good approximations of Y_0^N in $(\mathscr{X}, d)$, as follows. Let $\phi_m : \mathbb{R} \to \mathbb{R}$, $m \in \mathbb{N}$, be piecewise constant, monotone non-decreasing functions and denote $\mathscr{J}_m$ the finite set of jump discontinuity points of $\phi_m : \mathbb{R} \to \mathbb{R}$, $m \in \mathbb{N}$. Since σ has finitely many atoms, we may and shall consider such a sequence ϕ_m for which $\phi_m(x) = -\phi_m(-x)$, $\phi_m(\lambda^*) \leq \lambda^*$, $\lambda^* \notin \mathscr{J}_m$, $-\lambda^* \notin \mathscr{J}_m$, $\sigma(\mathscr{J}_m) = 0$ and

$$\sup_{|x| \leq \lambda^*+1} |\phi_m(x) - x| \leq m^{-1}\ , \tag{4.6}$$

for all m. Letting $\lambda_i^m = \phi_m(\lambda^i)$ and $\lambda_m^* = \phi_m(\lambda^*)$ the condition **(H1)** then obviously holds for $\boldsymbol{\lambda}^m$, whereas by Lemma 4.2, the empirical measures $\hat{\sigma}^N \circ \phi_m^{-1}$ of $\{\lambda_i^m\}_{i=1}^N$ converge a.s. to $\sigma \circ \phi_m^{-1}$ as $N \to \infty$. Thus, by the above proof we deduce that for any m, the sequence

$$Y_0^{N,m} := \Big(\frac{1}{N}\sum_{i=1}^N v_i^2, \frac{1}{N}\sum_{i=1}^N \lambda_i^m v_i^2, \frac{1}{N}\sum_{i=1}^N \delta_{\lambda_i^m, v_i}\Big)$$

satisfies the LDP in $\mathscr{X}$ with some convex GRF $\mathscr{K}^m(u, v, \mu)$. Note that for a.e. $\boldsymbol{\lambda}$ and all N large enough, by **(H1)** and (4.6),

$$d(Y_0^{N,m}, Y_0^N) \le N^{-1} \sum_{i=1}^{N} |\lambda_i^m - \lambda_i|(1 + v_i^2) \le m^{-1} N^{-1} \sum_{i=1}^{N} (1 + v_i^2)$$

where d is chosen as in (4.5). Therefore, for all $\delta > 0$, $r < 1/2$ and a.e. $\boldsymbol{\lambda}$

$$\limsup_{N\to\infty} \frac{1}{N} \log \mathbb{P}(d(Y_0^{N,m}, Y_0^N) \ge \delta) \\ \le r + \log \int e^{rw^2} d\gamma(w) - mr\delta \to_{m\to\infty} -\infty ,$$

implying that $Y_0^{N,m}$ are exponentially good approximations of Y_0^N.

Recall that (4.4) implies that the empirical measures $N^{-1} \sum_{i=1}^N \delta_{\lambda_i, v_i}$ satisfy the LDP in the Polish space $\mathscr{P}(\mathbb{R}^2)$ with the GRF $\mathscr{I}_0(\mu)$ (even when σ is not supported on a finite discrete set). Therefore, these $\mathscr{P}(\mathbb{R}^2)$-valued random variables are exponentially tight (see for example [21, Lemma 2.6]). The two covariance terms $N^{-1} \sum v_i^2$ and $N^{-1} \sum \lambda_i v_i^2$ have finite exponential moments, hence it follows that Y_0^N is also exponentially tight in $\mathscr{X}$. From this and [15, Theorem 4.2.16(a)] it follows that Y_0^N satisfies the LDP in $(\mathscr{X}, d)$ with the GRF

$$\mathscr{K}(x) = \sup_{\eta} \lim_{m\uparrow\infty} \inf_{\{y:d(x,y)<\eta\}} \mathscr{K}^m(y) . \tag{4.7}$$

Since $\mathscr{K}^m(\cdot)$ are convex for all m and $(x, y) \mapsto d(x, y) : \mathscr{X}^2 \to \mathbb{R}$ is convex (see (4.5)), it is easy to check that necessarily $\mathscr{K}(\cdot)$ is also convex on $\mathscr{X}$. To identify this convex GRF we shall apply Proposition 6.4 for $c : \mathbb{R}^2 \to \mathbb{R}^2$ such that $c(\lambda, v) = (v^2, \lambda v^2)$. To this end, by Lemma 6.5, it suffices to check that $\psi(\lambda)$ is bounded on $[-\lambda^* - \delta, \lambda^* + \delta]$ for some $\delta = \delta(p, \rho, \alpha) > 0$ whenever $V = 0$ and $\Lambda(p\rho, p\alpha, 0) < \infty$ for some $p > 1$. The latter condition is equivalent to $|\alpha|\lambda^* + \rho < 1/2$, in which case $\psi(\lambda) = -0.5 \log(1 - 2(\alpha\lambda + \rho))$ is indeed bounded on $[-\lambda^* - \delta, \lambda^* + \delta]$ for some $\delta > 0$. Hence, the convex GRF of the LDP for Y_0^N is

$$\mathscr{K}(u, v, \mu) = \mathscr{I}_0(\mu) + F(u - \int w^2 d\mu(\lambda, w), v - \int \lambda w^2 d\mu(\lambda, w))$$

for μ such that $\mathscr{I}_0(\mu) < \infty$ and infinite otherwise, where

$$F(a, b) = \sup_{\{\rho,\alpha:|\alpha|\lambda^*+\rho<1/2\}} \{\rho a + \alpha b\} = \sup_{\{\rho,\alpha:|\alpha|\lambda^*+\rho\le 1/2\}} \{\rho a + \alpha b\} . \tag{4.8}$$

It is easy to check that $F(a, b) = a/2$ when $a \ge |b|/\lambda^*$ (with $0/0 = 0$), and $F(a, b) = \infty$ otherwise, leading to the formula (2.11) for $\mathscr{K}$.

Note that $(N^{-1} \sum_{i=1}^N v_i^2, N^{-1} \sum_{i=1}^N \lambda_i v_i^2)$, satisfies the LDP in $\mathbb{R}^2$ with the convex GRF of (6.17), by the above application of Lemma 6.5. For $|\alpha|\lambda^* + \rho < 1/2$, the function $L(\rho, \alpha)$ of (2.7) equals $\Lambda(\rho, \alpha, 0)$, hence the continuity of $t \mapsto$

$L(t\rho, t\alpha)$ as $t \uparrow 1/(2|\alpha|\lambda^* + 2\rho) > 0$ results with this GRF being given by $k(u, v)$ of (2.6). □

Proof of Theorem 2.2. By **(H1)**, the LDP of Lemma 4.1 may and shall be considered without loss of generality to hold in the closed subset $\widetilde{\mathscr{X}} = \{(u, v, \mu) : |v| \leq (\lambda^* + 1)u, \mu \in \mathscr{P}(\mathbb{R})\}$ of $\mathscr{X}$ in which Y_0^N is for a.e. $\boldsymbol{\lambda}$ and all N large enough (see [15, Lemma 4.1.5(b)]). Since f is super-linear, $h(u, v, \mu) = h(u, v)$ of (2.5) is continuous and bounded above on $\widetilde{\mathscr{X}}$. So, with $Z_{\boldsymbol{\lambda}}^N = \int \exp(Nh(Y_0^N(\boldsymbol{\lambda}, \boldsymbol{v})))\gamma^{\otimes N}(d\boldsymbol{v})$, by Varadhan's lemma for a.e. $\boldsymbol{\lambda}$

$$F_\beta = \lim_{N\to\infty} \frac{1}{N} \log Z_{\boldsymbol{\lambda}}^N = \sup_{(u,v,\mu)\in\widetilde{\mathscr{X}}} \{h(u, v) - \mathscr{K}(u, v, \mu)\} \tag{4.9}$$

exists, is finite, and by Lemma 4.1 can also be written as (2.8). Moreover, for all $\Phi \in C_b(\widetilde{\mathscr{X}})$,

$$\lim_{N\to\infty} \frac{1}{N} \log \int e^{N\Phi(Y_0^N(\boldsymbol{\lambda},\boldsymbol{v}))} \mu_{\boldsymbol{\lambda}}^N(d\boldsymbol{v}) = \sup_{(u,v,\mu)\in\widetilde{\mathscr{X}}} \{\Phi(u, v, \mu) - \mathscr{H}^0(u, v, \mu)\} , \tag{4.10}$$

with $\mathscr{H}^0(u, v, \mu)$ given by (2.10) (see [15, Theorem 4.3.1]). Since $\mathscr{H}^0 = \mathscr{K} - h + F_\beta$ is a GRF, the LDP for Y_0^N under $\mu_{\boldsymbol{\lambda}}^N$ with this GRF thus follows (see [15, Theorem 4.4.13]).

Assume hereafter that f is strictly convex. Then, the GRF $\mathscr{H}^0$ is strictly convex in (u, μ) wherever it is finite (by the strict convexity of $I(\cdot|\sigma \otimes \gamma)$ and $f(\cdot)$). In particular, the supremum in (4.9) is finite and is obtained at a unique (u_β, π^*). The set $\mathscr{S} := \{v : \pi^* \in \mathscr{A}(u_\beta, v)\}$ is a non-empty, bounded, closed interval on which $\mathscr{K}(u_\beta, \cdot, \pi^*)$ is a finite constant, while $\mathscr{K}(u_\beta, v, \pi^*) = \infty$ for $v \notin \mathscr{S}$. With $v \mapsto h(u_\beta, v)$ strictly increasing, the unique solution of $\mathscr{H}^0(\cdot) = 0$ is at $(u_\beta, v_\beta, \pi^*)$ for $v_\beta = \sup\{v : v \in \mathscr{S}\}$. Since $k(u, v) = \inf_\mu \mathscr{K}(u, v, \mu)$ it follows that (u_β, v_β) is also the unique minimizer in (2.8) and π^* is the unique probability measure for which $\mathscr{K}(u_\beta, v_\beta, \pi^*) = k(u_\beta, v_\beta)$.

For $\lambda^* > 0$, by (2.11), if $\mu \in \mathscr{A}(u, v)$ for $u \leq |v|/\lambda^*$, then $\mu(\{w = 0\}) > 0$ for which $I(\mu|\sigma \otimes \gamma) = \infty$, or else, either $\mu(\{\lambda = \lambda^*\}) = 1$ or $\mu(\{\lambda = -\lambda^*\}) = 1$, in contradiction with the definition of $\lambda^* > 0$ such that $\sigma([-a, b]) < 1$ whenever $a \wedge b < \lambda^*$. Consequently, $|v_\beta| < \lambda^* u_\beta$. Then, for $|\alpha|\lambda^* + \rho \leq 1/2$

$$\begin{aligned} &\alpha v_\beta + \rho u_\beta - L(\rho, \alpha) \\ &\leq \frac{|v_\beta|}{2\lambda^*} + \rho(u_\beta - |v_\beta|/\lambda^*) + \frac{1}{2}\log(2 - 4\rho) \to_{\rho\to-\infty} -\infty \end{aligned}$$

and by lower semi-continuity of $L(\cdot)$, there exist conjugate exponents $(\rho_\beta, \alpha_\beta)$ achieving the supremum in $k(u_\beta, v_\beta)$ (and such exponents are unique by the strict convexity of $L(\cdot)$ wherever finite). Note that $\rho_\beta = 1/2 - |\alpha_\beta|\lambda^*$ is at all possible only if $\sigma(\{\lambda^*\}) = 0$ and $\alpha_\beta > 0$ or $\sigma(\{-\lambda^*\}) = 0$ and $\alpha_\beta < 0$. By Fubini's theorem, for such σ and (ρ, α), as well as whenever $|\alpha|\lambda^* + \rho < 1/2$,

$$\begin{aligned} d\mu^{\rho,\alpha}(\lambda, w) &:= (2\pi)^{-1/2}(1 - 2(\alpha\lambda + \rho))^{1/2} e^{-(1-2(\alpha\lambda+\rho))w^2/2} dw d\sigma(\lambda) \\ &= \gamma_{1-2(\alpha\lambda+\rho)}(dw) d\sigma(\lambda) , \end{aligned} \tag{4.11}$$

is in $\mathscr{P}(\mathbb{R}^2)$ with $\mu_1^{\rho,\alpha} = \sigma$. For such (ρ, α), if $I(\mu|\sigma \otimes \gamma) < \infty$, $\mu_1 = \sigma$ and $\mu \neq \mu^{\rho,\alpha}$, then

$$I(\mu|\sigma \otimes \gamma) - \int (\alpha\lambda + \rho) w^2 d\mu + L(\rho, \alpha) = I(\mu|\mu^{\rho,\alpha}) > 0 \,. \tag{4.12}$$

Consequently, if $\mu_1 = \sigma$ and $\mu \neq \mu^{\rho_\beta,\alpha_\beta}$ then (see (4.8)),

$$\begin{aligned}\mathscr{K}(u_\beta, v_\beta, \mu) &= I(\mu|\sigma \otimes \gamma) + \sup_{\{\rho,\alpha:|\alpha|\lambda^*+\rho\le 1/2\}} \alpha v_\beta + \rho u_\beta - \int (\alpha\lambda + \rho) w^2 d\mu \\ &> \alpha_\beta v_\beta + \rho_\beta u_\beta - L(\rho_\beta, \alpha_\beta) = k(u_\beta, v_\beta) \,.\end{aligned}$$

Since $\mathscr{K}(u, v, \mu) = \infty$ when $\mu_1 \neq \sigma$, it follows that $\pi^* = \mu^{\rho_\beta,\alpha_\beta}$ as stated.

If $\lambda^* = 0$ then $\sigma = \delta_0$. By (2.11), $\mathscr{H}^0(u, v, \mu) = \infty$ except when $v = 0$, $u > 0$ and $\mu_1 = \sigma$, in which case $\mathscr{H}^0(u, v, \mu) = \int H_u(\mu_{2|1}) d\sigma + \kappa_g(u)$ for the GRFs $H_u(\cdot)$ of (6.19) and $\kappa_g(u) := 0.5(u - 1 - \log u)$. It is easy to check that $\gamma_{1/u}$ is the unique minimizer of $H_u(\cdot)$ for any $u > 0$, leading to $k(u, 0) = \kappa_g(u)$. The unique minimizer of $\kappa_g(u) - h(u, 0)$ is $u_\beta > 0$, the unique solution of $uf'(u) = 1/(2\beta)$. □

5. LDPs for the dynamics (2.2)

In this section, we study the Langevin dynamics of (2.2) starting from (IC1), (IC2), (IC3) or (IC4). Hereafter, we use $\mathscr{W}$ to denote the law of a standard Brownian motion $B_{[0,T]}$ and allow for any $\beta \in (0, \infty]$. The main difficulty in our program is to deal with the non-continuity of the map $\mu \mapsto \int x^2 d\mu(x)$ in the C_b-topology. Consequently, it is not enough to prove the LDP only for the path empirical measure $\hat{\nu}_T^N$ of (2.3). We are forced to prove instead the LDPs for the couple $(K^N, \hat{\nu}_T^N)$, where the (empirical) covariance term K^N of (2.4) is considered an element of $C_b([0, T]^2)$. More precisely, letting

$$\hat{\pi}_T^N = \frac{1}{N} \sum_{i=1}^N \delta_{\lambda_i, v^i_{[0,T]}}, \tag{5.1}$$

we shall prove in this section that:

Theorem 5.1. *Under* **(H0)**, **(H1)** *and* **(H2)**, *starting from* $(ICi)_{1\le i\le 4}$ *initial conditions, the couple* $(K^N, \hat{\pi}_T^N)$ *satisfies an a.s. quenched LDP in* $C_b([0, T]^2) \times \mathscr{P}(\mathbb{R} \times C_b([0, T]))$ *with GRF*

$$\mathscr{H}_T^i(K, \pi) := \inf \left\{ I_T^i(\mathscr{C}, \mu) \;:\; K = FP_T^1(\mathscr{C}),\ \pi = FP_T^2(\mathscr{C}, \mu) \right\}.$$

The functions (FP_T^1, FP_T^2) *are described in Theorem 5.3. The GRFs* I_T^1, I_T^2, I_T^3 *and* I_T^4 *are described in Theorems 5.4, 5.7, 5.8 and 5.9, respectively.*

From this theorem, we deduce the following convergence result (in particular, implying Theorem 2.4).

Corollary 5.2. *Under* **(H0)**, **(H1)** *and* **(H2)**, *starting from* $(ICi)_{1\le i\le 4}$ *initial conditions,* $(K^N, \hat{\pi}_T^N)$ *converges almost surely towards the corresponding* (K, π) *with* $\pi(\lambda, w) = \nu_\lambda(w) \otimes \sigma(\lambda)$ *and* ν_λ, K *described in Theorem 2.3.*

Proof. By Theorem 5.1 it suffices to check that each of the GRF $\mathscr{H}_T^i$, $i = 1, 2, 3, 4$, admits a unique minimizer corresponding to (K, π) described above. By part (c) of Theorem 5.3 this amounts to checking that $(\mathscr{C}^*, \mu_T^*)$ are the unique minimizers of the GRF I_T^i, $i = 1, 2, 3, 4$, for $\mathscr{C}^*$ of (5.5) and $\mu_T^* = \sigma \otimes \nu_0 \otimes \mathscr{W}$. Indeed, by Theorem 5.4 the unique minimizer of the GRF I_T^1 is $(\int c_T d\mu_T^*, \mu_T^*)$ and it is easy to check that $\mathscr{C}^* = \int c_T d\mu_T^*$ in case of (IC1). The GRF I_T^2 and I_T^3 are obtained as contractions in Theorems 5.7 and 5.8, respectively, with I_T^4 obtained in Theorem 5.9 via Varadhan's lemma. The unique minimizers are identified there to be the couples $(\mathscr{C}^*, \mu_T^*)$ corresponding to (IC2), (IC3) and (IC4), respectively. □

We also show in this section that for all four initial conditions, $(K^N, \hat{\pi}_T^N)$ satisfy the *annealed* LDP on $C_b([0, T]^2) \times \mathscr{P}(\mathbb{R} \times C_b([0, T]))$ with the same GRF as in Theorem 5.1, provided **(H0)** and **(H1)** are replaced by the following stronger assumptions.

(H0a) *The sequence* $\hat{\sigma}^N$ *satisfies the LDP in* $\mathscr{P}(\mathbb{R})$ *with speed N and trivial GRF which is zero at the compactly supported* σ *with finitely many atoms and* $+\infty$ *elsewhere.*

(H1a) *For some finite integer* N_0

$$\mathbb{P}(\sup_{N\ge N_0} \max_{i=1}^{N} |\lambda_i| \le \lambda^*) = 1.$$

The proof of Theorem 5.1 is based on the contraction principle. Namely, we shall first prove in Section 5.1 that $(K^N, \hat{\pi}_T^N)$ is a continuous function of the simpler object $(\mathscr{C}^N, \hat{\mu}_T^N)$ (typically, corresponding to independent variables). We then derive LDPs for $(\mathscr{C}^N, \hat{\mu}_T^N)$ under different initial conditions, considering (IC1) in Section 5.2, (IC2) in Section 5.3, (IC3) in Section 5.4, and finally, (IC4) in Section 5.5.

5.1. The contraction

To state our contraction result, let us first introduce some notations. Given $T > 0$, let $\mathbf{\Sigma}_T := \mathbb{R}^2 \times C_b([0, T])$ and $\mathscr{X}_T := C_b([0, T]^3, \mathbb{R}^9)$, both endowed with the uniform (supremum-norm) topology, $\mathscr{Q}_T := \mathscr{P}(\mathbf{\Sigma}_T)$ endowed with the C_b-topology and $\mathscr{Y}_T = \mathscr{X}_T \times \mathscr{Q}_T$ endowed with the corresponding product topology. Note that $c_T : \mathbf{\Sigma}_T \to \mathscr{X}_T$ defined for $(u, v, w) \in [0, T]^3$ as

$$c_T(\lambda, v_0, B_\cdot)(u, v, w) := \Big(v_0^2 \lambda^k e^{w\lambda}, \beta^{-1/2} B_u v_0 \lambda^k e^{w\lambda}, \beta^{-1} B_u B_v \lambda^k e^{w\lambda}, \\ k = 0, 1, 2\Big), \tag{5.2}$$

is a continuous mapping. With the free path empirical measure

$$\hat{\mu}_T^N := \frac{1}{N}\sum_{i=1}^N \delta_{\lambda_i, v_0^i, B^i_{[0,T]}} \tag{5.3}$$

in $\mathscr{Q}_T$, it follows that the corresponding empirical covariance

$$\mathscr{C}^N := \int c_T(\lambda, v_0, B_\cdot) d\hat{\mu}_T^N(\lambda, v_0, B_\cdot) = \frac{1}{N}\sum_{i=1}^N c_T(\lambda_i, v_0^i, B_\cdot^i)\ , \tag{5.4}$$

is in $\mathscr{X}_T$ for almost all $\boldsymbol{v}_0$, $\boldsymbol{\lambda}$ and the realization of $\boldsymbol{B}$. Consequently, $(\mathscr{C}^N, \hat{\mu}_T^N) \in \mathscr{Y}_T$.

We are now in position to state the main result of this section.

Theorem 5.3. *For f' uniformly Lipschitz, bounded below function on $\mathbb{R}^+$ and any $T < \infty$:*
a) There exists a continuous map $FP_T^1 : \mathscr{X}_T \to C_b([0,T]^2)$, such that K^N of (2.4) can be expressed as

$$K^N(s,t) = FP_T^1\left(\mathscr{C}^N\right)(s,t).$$

b) There exists a continuous map $FP_T^2 : \mathscr{Y}_T \to \mathscr{P}(\mathbb{R}\times C_b([0,T]))$ such that $\hat{\pi}_T^N$ of (5.1) can be expressed as

$$\hat{\pi}_T^N = FP_T^2\left(\mathscr{C}^N, \hat{\mu}_T^N\right).$$

c) Define $\mu_T^* := \sigma\otimes\nu_0\otimes\mathscr{W}$ for ν_0 of Theorem 2.3. Then, $K = FP_T^1(\mathscr{C}^*)$ is the unique solution of (2.16) when taking

$$\mathscr{C}^* := (\mathscr{L}_0^{(k)}(w), 0, \beta^{-1}(u\wedge v)\mathscr{L}^{(k)}(w),\ k = 0,1,2) \tag{5.5}$$

and $FP_T^2(\mathscr{C}^*, \mu_T^*)$ is the law $\pi(\lambda, w)$ defined in Corollary 5.2.

Proof. a). Let us denote $K_d(t) := K(t,t)$ for $K \in C_b([0,T]^2)$ and

$$F_t(K,\lambda) := f'(K_d(t)) - \lambda.$$

Note that (2.2) is equivalent to

$$\begin{aligned} v_t^i = &\ \exp\{-\int_0^t F_s(K^N,\lambda_i)ds\}v_0^i \\ &+\beta^{-1/2}\exp\{-\int_0^t F_s(K^N,\lambda_i)ds\}\int_0^t \exp\{\int_0^u F_s(K^N,\lambda_i)ds\}dB_u^i \end{aligned}$$

holding for $i = 1, \dots, N$, where the stochastic integral is against a previsible process. Applying Ito's formula gives

$$
\begin{aligned}
v_t^i = & \exp\{-\int_0^t F_s(K^N, \lambda_i)ds\}v_0^i + \beta^{-1/2}B_t^i \\
& -\beta^{-1/2}\int_0^t B_u^i F_u(K^N, \lambda_i)\exp\{-\int_u^t F_s(K^N, \lambda_i)ds\}du. \qquad (5.6)
\end{aligned}
$$

It follows from (5.6) that there exists a non-trivial map $\Phi_T : \mathscr{X}_T \times C_b([0, T]^2) \to C_b([0, T]^2)$ such that for any $(s, t) \in [0, T]^2$

$$
K^N(s, t) = \Phi_T\left(\mathscr{C}^N, K^N\right)(s, t). \qquad (5.7)
$$

To explicitly describe Φ_T we denote the coordinates of $\mathscr{C} \in \mathscr{X}_T$ in accordance with the coordinates of c of (5.2) as

$$
\mathscr{C} := \left(\mathscr{C}_{1,k}(w), \mathscr{C}_{2,k}(u, w), \mathscr{C}_{3,k}(u, v, w), k = 0, 1, 2\right)
$$

and also use the notations $S^2 = \{(1, 2), (2, 1)\}$,

$$
H_\tau^\theta(K) := \exp(-\int_\tau^\theta f'(K_d(\xi))d\xi)
$$

and

$$
DH_\tau^\theta(K) = \frac{d}{d\tau}H_\tau^\theta(K) = f'(K_d(\tau))\exp(-\int_\tau^\theta f'(K_d(\xi))d\xi)\,.
$$

In this context, $D^0 H = H$ by definition. It is not difficult to check that (5.7) holds for Φ_T such that for $(s_1, s_2) \in [0, T]^2$, $\mathscr{C} \in \mathscr{X}_T$ and $K \in C_b([0, T]^2)$,

$$
\begin{aligned}
\Phi_T(\mathscr{C}, K)(s_1, s_2) = & \mathscr{C}_{1,0}(s_1 + s_2)H_0^{s_1}(K)H_0^{s_2}(K) + \mathscr{C}_{3,0}(s_1, s_2, 0) \\
& + \sum_{\sigma \in S^2} H_0^{s_{\sigma(2)}}(K)\mathscr{C}_{2,0}(s_{\sigma(2)}, s_{\sigma(1)}) \\
& + \int_0^{s_1}\int_0^{s_2}\sum_{j,k=0,1}(-1)^{j+k}\mathscr{C}_{3,2-j-k}(u, v, s_1 + s_2 - u - v) \\
& \qquad D^j H_u^{s_1}(K) D^k H_v^{s_2}(K)dudv \\
& + \sum_{\sigma \in S^2} H_0^{s_{\sigma(2)}}(K)\int_0^{s_{\sigma(1)}}\sum_{k=0,1}(-1)^{k+1}\mathscr{C}_{2,k}(u, s_1 + s_2 - u) \\
& \qquad D^{1-k} H_u^{s_{\sigma(1)}}(K)du \\
& + \sum_{\sigma \in S^2}\int_0^{s_{\sigma(1)}}\sum_{k=0,1}(-1)^{k+1}\mathscr{C}_{3,k}(u, s_{\sigma(2)}, s_{\sigma(1)} - u) \\
& \qquad D^{1-k} H_u^{s_{\sigma(1)}}(K)du. \qquad (5.8)
\end{aligned}
$$

For the purpose of studying the continuity properties of Φ_T we may and shall assume that $f' \geq 0$ without loss of generality. In this case, for any $0 \leq u \leq t \leq T$ and $K \in C_b([0,T]^2)$,

$$0 \leq H_u^t(K) \leq 1, \qquad \text{and} \qquad \int_0^t |DH_u^t(K)| du \leq 1 .$$

Moreover, because f' is Lipshitz and $|e^{-x} - e^{-y}| \leq |x-y| \wedge 1$ for $x, y \geq 0$, it follows that

$$\sup_{\tau \leq \theta} |H_\tau^\theta(K) - H_\tau^\theta(\widetilde{K})| \leq \|f'\|_{\mathcal{L}} \int_0^\theta |K_d(s) - \widetilde{K_d}(s)| ds \ ,$$

where $||f'||_{\mathcal{L}}$ denotes the Lipschitz norm of f'.
Furthermore, since $|ze^{-x} - we^{-y}| \leq |z-w| + |x-y|(ze^{-x} + we^{-y})$ for any $x, y, z, w \geq 0$, it follows that

$$|DH_\tau^\theta(K) - DH_\tau^\theta(\widetilde{K})| \leq \|f'\|_{\mathcal{L}} \Bigg[|K_d(\tau) - \widetilde{K_d}(\tau)| + (DH_\tau^\theta(K) + DH_\tau^\theta(\widetilde{K})) \\ \times \int_0^\theta |K_d(s) - \widetilde{K_d}(s)| ds \Bigg].$$

It is now an easy matter to deduce from (5.8) that there exists a finite constant $k(T, ||f'||_{\mathcal{L}})$ so that, for any $\mathcal{C} \in \mathcal{X}_T$, any $K, \widetilde{K} \in C_b([0,T]^2)$, and any $S \leq T$,

$$||\Phi_T(\mathcal{C}, K) - \Phi_T(\mathcal{C}, \widetilde{K})||_\infty^S \leq k(T, ||f'||_{\mathcal{L}}) ||\mathcal{C}||_\infty^T \int_0^S ||K - \widetilde{K}||_\infty^v dv. \quad (5.9)$$

In the above inequality, $|| \cdot ||_\infty^S$ denotes for a (vector valued) function on $[0,T]^r$, $r = 1, 2, 3$, the supremum norm over $[0, S]^r$. It follows from (5.9) that the sequence $K_{n+1} = \Phi_T(\mathcal{C}, K_n)$, $n \geq 0$, is such that

$$||K_{n+1} - K_n||_\infty^S \leq \frac{(k(T, ||f'||_{\mathcal{L}})||\mathcal{C}||_\infty^T S)^n}{n!} ||K_1 - K_0||_\infty^S \ .$$

Hence, $K_n \to K_\infty \in C_b([0,T]^2)$. By (5.9), the mapping $K \mapsto \Phi_T(\mathcal{C}, K)$ is continuous on $C_b([0,T]^2)$ so $K_\infty = \Phi_T(\mathcal{C}, K_\infty)$ and by Gronwall's lemma this is the unique fixed point of $\Phi_T(\mathcal{C}, \cdot)$ in $C_b([0,T]^2)$. Denoting $K_\infty := FP_T^1(\mathcal{C})$ if follows that $K^N = FP_T^1(\mathcal{C}^N)$. Let us tackle next the continuity of $FP_T^1(\cdot)$. Considering again (5.8), one can find a finite constant $C(T)$ such that for any $(\mathcal{C}, \widetilde{\mathcal{C}}) \in \mathcal{X}_T$, any $\widetilde{K} \in C_b([0,T]^2)$ and any $S \leq T$,

$$||\Phi_T(\mathcal{C}, \widetilde{K}) - \Phi_T(\widetilde{\mathcal{C}}, \widetilde{K})||_\infty^S \leq C(T) ||\mathcal{C} - \widetilde{\mathcal{C}}||_\infty^S. \qquad (5.10)$$

Applying (5.9) and (5.10) for $K = FP_T^1(\mathcal{C})$ and $\widetilde{K} = FP_T^1(\widetilde{\mathcal{C}})$, Gronwall's lemma yields

$$||FP_T^1(\mathcal{C}) - FP_T^1(\widetilde{\mathcal{C}})||_\infty^T \leq C(T) e^{k(T, ||f'||_{\mathcal{L}}) T ||\mathcal{C}||_\infty^T} ||\mathcal{C} - \widetilde{\mathcal{C}}||_\infty^T$$

and therefore the continuity of $FP_T^1(\cdot)$.

b). Recall that (5.6) implies that

$$v_t^i = \phi(K^N, \lambda_i, v_0^i, B_\cdot^i)(t)$$

for $\phi : C_b([0,T]^2) \times \mathbf{\Sigma}_T \to C_b([0,T])$ such that

$$\begin{aligned}\phi(K,\lambda,v_0,B)(t) := \; & \beta^{-1/2}B_t + e^{-\int_0^t F_s(K,\lambda)ds}v_0 \\ & -\beta^{-1/2}\int_0^t B_u F_u(K,\lambda)e^{-\int_u^t F_s(K,\lambda)ds}du.\end{aligned}$$

Consequently,

$$\hat{\pi}_T^N = \psi(K^N, \hat{\mu}_T^N)\,,$$

where, by definition, for any $h : \mathbb{R} \times C_b([0,T]) \to [-1,1]$ of uniform Lipshitz constant at most 1,

$$\begin{aligned}&\int h(\lambda, v_t,\, t \le T)d\psi(K,\mu)(\lambda, v) \\ &:= \int h(\lambda, \phi(K,\lambda,v_0,B)(t),\, t\le T) \\ &\qquad d\mu(\lambda, v_0, B).\end{aligned}$$

Fixing h as above, $K_n \to K$ in $C_b([0,T]^2)$ and $\mu_n \to \mu$ in $\mathcal{Q}_T$, let $g(K,x) := h(\lambda, \phi(K,x))$ with $x := (\lambda, v_0, B) \in \mathbf{\Sigma}_T$. Since $x \mapsto \phi(K,x)$ is continuous, $g(K,\cdot) \in C_b(\mathbf{\Sigma}_T)$ implying that $\int g(K,x)d\mu_n(x) \to \int g(K,x)d\mu(x)$. By tightness of $\{\mu_n\}$, there exist compacts $\Gamma_\eta \subset \mathbf{\Sigma}_T$ such that $\mu_n(\Gamma_\eta) \ge 1-\eta$ for all $n \ge 1$. For any $\eta > 0$, the continuous mapping $K \mapsto \phi(K,x)$ is uniformly continuous on Γ_η, so

$$\begin{aligned}&\limsup_{n\to\infty} \left| \int g(K_n,x)d\mu_n(x) - \int g(K,x)d\mu_n(x)\right| \\ &\quad \le 2\eta + \limsup_{n\to\infty}\sup_{x\in\Gamma_\eta} \|\phi(K_n,x) - \phi(K,x)\| = 2\eta.\end{aligned}$$

Considering $\eta \downarrow 0$, we see that $\int h(\lambda,\phi(K_n,x))d\mu_n(x) \to \int h(\lambda,\phi(K,x))d\mu(x)$ for any h bounded and uniformly Lipschitz. Consequently, $\psi(K_n,\mu_n) \to \psi(K,\mu)$ in $\mathcal{P}(\mathbb{R}\times C_b([0,T]))$. Thus, $\psi : C_b([0,T]^2) \times \mathcal{Q}_T \to \mathcal{P}(\mathbb{R}\times C_b([0,T]))$ is continuous and $\hat{\pi}_T^N = FP_T^2(\mathscr{C}^N, \hat{\mu}_T^N)$ where

$$FP_T^2(\mathscr{C},\mu) := \psi\left(FP_T^1(\mathscr{C}), \mu\right)$$

is continuous.

c). Computing $\Phi_T(\mathscr{C}^*, K)$ of (5.8) for $\mathscr{C}^*$ of (5.5) results after integration by parts with the right-side of (2.16). Accordingly, $FP_T^1(\mathscr{C}^*)$ equals K of (2.16). To complete the proof, note that $\psi(K,\mu_T^*)$ is exactly the law $\pi(\lambda, w)$ of Corollary 5.2 per given K and v_0. □

5.2. LDP of the free measure for initial conditions (IC1)

The goal of this part is to prove a LDP for the couple $(\mathscr{C}^N, \hat{\mu}_T^N)$ and the initial conditions (IC1). To do so, we follow the program taken when proving Theorem 2.2. That is, we combine the Gärtner-Ellis theorem and projective limits for σ composed of finitely many atoms, using exponentially good approximations to establish the LDP with a convex GRF in case of a general σ. For an explicit representation of this GRF we then apply Proposition 6.4 relying upon the fact that $\hat{\mu}_T^N$ is made of the i.i.d. random variables $(v_0^i, B_\cdot^i)$ and the modes $(\lambda_i)_{1\le i\le N}$ whose empirical measures converge to σ (super-exponentially in N, for the annealed case). We thus derive the following result.

Theorem 5.4. *Assuming* **(H0)–(H1)** *(or* **(H0a)–(H1a)***) and* **(H2)***, the couple* $(\mathscr{C}^N, \hat{\mu}_T^N)$ *satisfies a quenched (respectively, annealed) LDP in* $\mathscr{Y}_T$ *with respect to the law* σ^N *of the modes, under the product law* $\nu_0^{\otimes N} \otimes \mathscr{W}^{\otimes N}$ *for* $(\boldsymbol{v}, \boldsymbol{B})$*. The GRF* $I_T^1(\cdot)$ *of this LDP is finite only for* $\mu_1 = \sigma$ *and* $I(\mu|\sigma \otimes \nu_0 \otimes \mathscr{W}) < \infty$ *in which case it is given by*

$$I_T^1(\mathscr{C}, \mu) = I(\mu|\sigma \otimes \nu_0 \otimes \mathscr{W}) + F(\mathscr{C} - \int c_T d\mu)$$

where F *is some convex non-negative function such that* $F(x) > 0$ *for all* $x \neq 0$ *and* c_T *is the continuous mapping of (5.2).*

To deal with the spherical law of (2.14), let $\hat{g}(u) = g(u) - u/2$ noting that $h(v) - \hat{g}(u)$ is continuous and bounded above on $\{(u, v) : |v| \le (\lambda^* + 1)u\}$. Hence, considering Theorem 5.4 for $\nu_0 = \gamma$ and applying Varadhan's lemma for $\Phi(\mathscr{C}, \mu) - \hat{g}(\mathscr{C}_{1,0}(0)) + h(\mathscr{C}_{1,1}(0))$, with $\Phi \in C_b(\mathscr{Y}_T)$ we get the following corollary.

Corollary 5.5. *If the law of* $\boldsymbol{v}_0$ *is given by (2.14) then assuming* **(H0)–(H1)** *(or* **(H0a)–(H1a)***) and* **(H2)***, the couple* $(\mathscr{C}^N, \hat{\mu}_T^N)$ *satisfies a quenched (respectively, annealed) LDP in* $\mathscr{Y}_T$ *with respect to the law* σ^N *of the modes. The GRF for this LDP is*

$$\begin{aligned} I_T^{sp}(\mathscr{C}, \mu) &:= I_T^1(\mathscr{C}, \mu) + \hat{g}(\mathscr{C}_{1,0}(0)) - h(\mathscr{C}_{1,1}(0)) \\ &\quad - \inf_{(\widetilde{\mathscr{C}}, \widetilde{\mu}) \in \mathscr{Y}_T} \{I_T^1(\widetilde{\mathscr{C}}, \widetilde{\mu}) + \hat{g}(\widetilde{\mathscr{C}}_{1,0}(0)) - h(\widetilde{\mathscr{C}}_{1,1}(0))\} \end{aligned}$$

where in $I_T^1(\cdot)$ *we set* $\nu_0 = \gamma$.

For the proof of Theorem 5.4 we need the following simple adaptation of Lemma 4.2.

Lemma 5.6. *Let* $\mathscr{J}$ *denote the finite set of jump discontinuity points of a given piecewise constant function* $\phi : \mathbb{R} \to \mathbb{R}$*. If* **(H0a)** *holds for* $\hat{\sigma}^N$ *and* σ *such that* $\sigma(\mathscr{J}) = 0$*, then it holds for* $\hat{\sigma}^N \circ \phi^{-1}$ *and* $\sigma \circ \phi^{-1}$.

Proof. The proof of Lemma 4.2 shows that the mapping $\mu \mapsto \mu \circ \phi^{-1} : \mathscr{P}(\mathbb{R}) \to \mathscr{P}(\mathbb{R})$ is continuous at σ. By **(H0a)** σ is the only point of finite rate for the LDP of $\hat{\sigma}^N$. The contraction principle thus implies that **(H0a)** applies to $\hat{\sigma}^N \circ \phi^{-1}$ and $\sigma \circ \phi^{-1}$ (for example, see [15, Remark (c), page 127]). □

Proof of Theorem 5.4. Assuming **(H0a)–(H1a)** and **(H2)** we establish the annealed LDP by:
1) Proving the LDP for $\hat{\mu}_T^N$ in $\mathscr{Q}_T$ with the GRF $\mathscr{I}_0(\mu) = I(\mu|\sigma \otimes \nu_0 \otimes \mathscr{W})$ when $\mu_1 = \sigma$ and $\mathscr{I}_0(\mu) = \infty$ otherwise.
2) Proving that $\mathscr{C}^N$ is exponentially tight in $\mathscr{X}_T$, hence the couple $(\mathscr{C}^N, \hat{\mu}_T^N)$ is exponentially tight in $\mathscr{Y}_T$.
3) Combining the Gärtner-Ellis theorem and projective limits we show that the couple $(\mathscr{C}^N, \hat{\mu}_T^N)$ satisfies the LDP in $\mathscr{Y}_T$ with a convex GRF whenever $\hat{\sigma}^N$, $N \geq 1$ are supported on the same finite, non-random set, on which σ is strictly positive.
4) With $D(\cdot, \cdot)$ denoting the Wasserstein distance in $\mathscr{Q}_T$ and

$$\|\mathscr{C}\| := \sup_{u,v,w\in[0,T]} \|\mathscr{C}(u, v, w)\| = \sup_{u,v,w\in[0,T]} \sup_{j,k} |\mathscr{C}_{j,k}(u, v, w)|$$

for $\mathscr{C} \in \mathscr{X}_T$, it is easy to check that

$$d((\mathscr{C}, \mu), (\widetilde{\mathscr{C}}, \widetilde{\mu})) := \|\mathscr{C} - \widetilde{\mathscr{C}}\| + D(\mu, \widetilde{\mu}) \tag{5.11}$$

is a complete metric for the (separable) topology of $\mathscr{Y}_T$. Invoking **(H2)** for the first time in the proof and using exponentially good approximations in $(\mathscr{Y}_T, d)$ we conclude that the LDP for $(\mathscr{C}^N, \hat{\mu}_T^N)$ holds with a convex GRF regardless of the support of $\hat{\sigma}^N$.
5) Relying again on **(H2)**, we apply Proposition 6.4 and Lemma 6.5 in order to identify the GRF of the LDP as having the form of $I_T^1(\mathscr{C}, \mu)$.

Let us now detail these five steps of the proof.

<u>*Step 1*</u> By **(H1a)** we may and shall assume hereafter that σ and $\hat{\sigma}^N$, $N \geq 1$ are supported on $[-\lambda^*, \lambda^*]$. For $W \in C(\mathbf{\Sigma}_T)$ let

$$\psi_W(\lambda) := \log \int e^{W(\lambda,v,B)} d\nu_0(v) d\mathscr{W}(B) \ . \tag{5.12}$$

Noting that $\psi_V \in C_b([-\lambda^*, \lambda^*])$ for all $V \in C_b(\mathbf{\Sigma}_T)$, it then follows by **(H0a)** and Varadhan's lemma that

$$\begin{aligned} \Lambda(V) &:= \lim_{N\to\infty} \frac{1}{N} \log \left[\int e^{N\int V d\hat{\mu}_T^N} \, d\nu_0^{\otimes N}(\boldsymbol{v}) d\mathscr{W}^{\otimes N}(\boldsymbol{B}) d\sigma^N(\boldsymbol{\lambda}) \right] \\ &= \lim_{N\to\infty} \frac{1}{N} \log \int e^{N\int \psi_V d\hat{\sigma}^N} \, d\sigma^N(\boldsymbol{\lambda}) = \int \psi_V(\lambda) d\sigma(\lambda) \ , \end{aligned} \tag{5.13}$$

exists and is finite. Moreover, by dominated convergence, $\boldsymbol{\theta} \mapsto \Lambda(\sum_i \theta_i V_i)$ is continuous and differentiable everywhere for any fixed $V_i \in C_b(\mathbf{\Sigma}_T)$. Taking projective limits we deduce the LDP for $\hat{\mu}_T^N$ in $C_b(\mathbf{\Sigma}_T)'$ (endowed with the $C_b(\mathbf{\Sigma}_T)$-topology and its Borel σ-field), where the convex GRF of the LDP is the Fenchel-Legendre dual of $\Lambda(\cdot)$ (see [15, Corollary 4.6.11(a)]). As in (4.4) it is not hard to verify that for any $V \in C_b(\mathbf{\Sigma}_T)$

$$\Lambda(V) := \int \Big(\log \int e^{V(\lambda,v,B)} d\nu_0(v) d\mathscr{W}(B) \Big) d\sigma(\lambda) = \sup_{\{\mu \in \mathscr{P}(\mathbf{\Sigma}_T) : \mu_1 = \sigma\}} \Big[\int V d\mu - I(\mu | \sigma \otimes \nu_0 \otimes \mathscr{W}) \Big] \tag{5.14}$$

(see [15, Lemma 6.2.13] for a similar computation). With $\mathscr{I}_0(\tau) := \infty$ for any $\tau \notin \mathscr{P}(\mathbf{\Sigma}_T)$, we see that $\mathscr{I}_0(\cdot)$ is a convex GRF on $C_b(\mathbf{\Sigma}_T)'$ (c.f. [15, Lemma 6.2.12]). By (5.14) and Fenchel duality (see [15, Lemma 4.5.8]), the convex GRF for the LDP of $\hat{\mu}_T^N$ is necessarily $\mathscr{I}_0(\cdot)$ which is infinite outside $\mathscr{Q}_T = \mathscr{P}(\mathbf{\Sigma}_T)$, allowing us to consider hereafter all LDPs in the space $\mathscr{Q}_T$ (or $\mathscr{Y}_T = \mathscr{X}_T \times \mathscr{Q}_T$).

Step 2 By Step 1, $\hat{\mu}_T^N$ satisfies the LDP with GRF in the Polish space $\mathscr{Q}_T$, hence it is exponentially tight (see [21, Lemma 2.6]). We thus turn to prove the exponential tightness of $\mathscr{C}^N$ in $\mathscr{X}_T$. To this end, let $\Gamma_M = \cap_{n \geq 0} \Gamma_{M,n}$, where

$$\Gamma_{M,0} := \{(\boldsymbol{\lambda}, \boldsymbol{v}_0, \boldsymbol{B}) : \frac{1}{N} \sum_{i=1}^N (v_0^i)^2 \leq M, \frac{1}{N} \sum_{i=1}^N \sup_{t \in [0,T]} (B_t^i)^2 \leq M\} ,$$

and for all $n \geq 1$,

$$\Gamma_{M,n} := \{(\boldsymbol{\lambda}, \boldsymbol{v}_0, \boldsymbol{B}) : \sup_{\substack{s,t \in [0,T] \\ |s-t| \leq n^{-3}}} \frac{1}{N} \sum_{i=1}^N |B_t^i - B_s^i|^2 \leq n^{-2} M\} .$$

We claim that $\mathscr{A}_M := \{\mathscr{C}^N(\boldsymbol{\lambda}, \boldsymbol{v}_0, \boldsymbol{B})(\cdot) : (\boldsymbol{\lambda}, \boldsymbol{v}_0, \boldsymbol{B}) \in \Gamma_M\}$ is pre-compact in $\mathscr{X}_T$ for any fixed $M < \infty$. Indeed, with $\rho = \max(\lambda^*, 1)^2 e^{\lambda^* T} < \infty$ it is easy to check that $\|c_T(x)\| \leq \rho(v_0^2 + \beta^{-1} \|B_\cdot\|^2)$ for $x = (\lambda, v_0, B_\cdot) \in \mathbf{\Sigma}_T$ implying that the functions in $\mathscr{A}_M$ are uniformly bounded by $\rho M(1 + \beta^{-1})$. Moreover,

$$\sup_{u,v \in [0,T]} \|c_T(x)(u,v,w) - c_T(x)(u,v,w')\| \leq \lambda^* |w - w'| \rho(v_0^2 + \beta^{-1} \|B_\cdot\|^2) ,$$

implying that any $\mathscr{C}^N \in \mathscr{A}_M$ is of uniform Lipschitz norm of at most $\lambda^* \rho M(1 + \beta^{-1})$ with respect to w. Consequently, $\mathscr{A}_M$ is equicontinuous with respect to w. Similarly,

$$\sup_{v,w \in [0,T]} \|c_T(x)(u,v,w) - c_T(x)(u',v,w)\| \leq \rho \beta^{-1/2} (|v_0| + \beta^{-1/2} \|B_\cdot\|) |B_u - B_{u'}| ,$$

so applying the Cauchy-Schwartz inequality, it follows that

$$\sup_{\substack{v,w,u,u' \in [0,T] \\ |u-u'| \leq n^{-3}}} \|\mathscr{C}^N(u,v,w) - \mathscr{C}^N(u',v,w)\| \leq n^{-1} \rho M(\beta^{-1/2} + \beta^{-1})$$

for any $\mathscr{C}^N \in \mathscr{A}_M$ and all $n \geq 1$, resulting with the equicontinuity of $\mathscr{A}_M$ with respect to u. The latter bounds also apply with respect to v, so $\mathscr{A}_M$ being bounded and equicontinuous, is by Arzela-Ascoli, a pre-compact subset of $\mathscr{X}_T$.

By the finiteness of $\Lambda_0(\eta_0)$ of (2.13) and the independence of v_0^i, for any $L > 0$ there exists $M_L < \infty$ so that

$$\mathbb{P}(\frac{1}{N}\sum_{i=1}^{N}(v_0^i)^2 > M_L) \leq \exp\{-LN\}. \tag{5.15}$$

By Désirè André reflection principle, for $\eta < 1/(2T)$,

$$\mathbb{E}(\exp(\eta \sup_{t\in[0,T]} B_t^2)) \leq 2\mathbb{E}(\exp(\eta B_T^2)) < \infty\,, \tag{5.16}$$

so increasing M_L as needed, also

$$\mathbb{P}(\frac{1}{N}\sum_{i=1}^{N}\sup_{t\in[0,T]}(B_t^i)^2 > M_L) \leq \exp\{-LN\} \tag{5.17}$$

Partitioning $[0, T]$ into intervals of length n^{-3} each, by the stationarity of the increments of the Wiener law, we get for some $r < \infty$ and all n, M, N,

$$\begin{aligned}
&\mathbb{P}\Bigg(\sup_{\substack{s,t\in[0,T]\\ |s-t|\leq n^{-3}}} \frac{1}{N}\sum_{i=1}^{N}|B_t^i - B_s^i|^2 > \frac{M}{n^2}\Bigg)\\
&\leq n^3 T\,\mathbb{P}\Bigg(\sup_{u\in[0,n^{-3}]}\frac{1}{N}\sum_{i=1}^{N}(B_u^i)^2 \geq \frac{M}{9n^2}\Bigg)\\
&\leq n^3 T\,\mathbb{P}\Bigg(\sum_{i=1}^{N}\sup_{u\in[0,1]}(B_u^i)^2 \geq \frac{n}{9}NM\Bigg) \leq n^3 T r^N e^{-nNM/27}
\end{aligned} \tag{5.18}$$

Combining (5.15), (5.17) and (5.18) we have that for any L there exists $M = M_L$ finite such that for all N,

$$\mathbb{P}(\mathscr{C}^N \notin \mathscr{A}_M) \leq 2e^{-LN} + Tr^N\sum_{n=1}^{\infty} n^3 e^{-nNM/27} \leq 3e^{-LN}$$

and the exponential tightness of $\mathscr{C}^N$ follows.

Step 3 Assume that $\hat{\sigma}^N$, $N \geq 1$ are supported on a finite, non-random set $\mathscr{S} = \{s_1, \ldots, s_m\}$ such that $p_r = \sigma(\{s_r\}) > 0$ for $r = 1, \ldots, m$. Without loss of generality we may and shall take $\mathbf{\Sigma}_T = \mathscr{S}\times\mathbb{R}\times C_b([0, T])$ throughout Step 3. Then, the identity (5.13) applies for any $W \in C(\mathbf{\Sigma}_T)$ such that $\max_{\lambda\in\mathscr{S}} \psi_W(\lambda) < \infty$. Moreover, by **(H0a)**, $\sigma^N(\{\hat{\sigma}^N(s_r) = 0\}) < 1$ for all r and N large enough. Hence, $\Lambda(W) = \infty$ whenever $\max_{\lambda\in\mathscr{S}} \psi_W(\lambda) = \infty$, with the identity (5.13) applicable for all $W \in C(\mathbf{\Sigma}_T)$. Let $\mathscr{M}_a$ denote the vector space of all $\mathbb{R}^9$-weighted finite sums of atomic (dirac) measures on $[0, T]^3$, and $\mathscr{E} = \{(\alpha, V) : V \in C_b(\mathbf{\Sigma}_T), \alpha \in \mathscr{M}_a\}$. Any $(\alpha_l, V_l) \in \mathscr{E}, l = 1, \dots, d,$ defines a projection

$$\pi_{\boldsymbol{\alpha},\boldsymbol{V}}(\mathscr{C}, \mu) := (\langle \alpha_l, \mathscr{C} \rangle, \int V_l d\mu,\ l = 1, \dots, d)$$

from $\mathscr{Y}_T$ to $\mathbb{R}^{2d}$. Fixing such d, and (α_l, V_l) we thus have that

$$h(\boldsymbol{\theta}) := \Lambda(\sum_{l=1}^{d} \theta_l \langle \alpha_l, c_T \rangle + \theta_{l+d} V_l) = \sum_{r=1}^{m} p_r h_r(\boldsymbol{\theta}) ,$$

for $h_r(\boldsymbol{\theta}) := \log \mathbb{E}(\exp(\langle \boldsymbol{\theta}, \mathbf{Y}_r \rangle))$ and the $\mathbb{R}^{2d}$-valued random vectors

$$\mathbf{Y}_r = \mathbf{Y}_r(v_0, B) = \pi_{\boldsymbol{\alpha},\boldsymbol{V}}(c_T(s_r, v_0, B), \delta_{s_r, v_0, B}) .$$

By Fatou's lemma and dominated convergence, the functions $h_r : \mathbb{R}^{2d} \to (-\infty, \infty]$ are lower semi-continuous with $h_r(\cdot)$ differentiable in the interior of $\mathscr{D}_r := \{\boldsymbol{\theta} : h_r(\boldsymbol{\theta}) < \infty\}$. Moreover, $|\mathbf{Y}_r| \le \rho_r(1 + v_0^2 + \beta^{-1}\|B.\|^2)$ for some non-random constants $\rho_r < \infty$, hence $\mathbf{0}$ is in the interior of $\mathscr{D}_r$ by (5.16) and the finiteness of $\Lambda_0(\eta_0)$ of (2.13). Consequently, if $\boldsymbol{\theta} \mapsto h(\boldsymbol{\theta})$ is a steep function, then by Gärtner-Ellis theorem $\pi_{\boldsymbol{\alpha},\boldsymbol{V}}(\mathscr{C}^N, \hat{\mu}_T^N)$ satisfy the LDP in $\mathbb{R}^{2d}$ with a convex GRF (see [15, Theorem 2.3.6]). We show below the steepness of $h(\cdot)$ for all $d < \infty$ and $(\alpha_l, V_l) \in \mathscr{E}$. Considering the projective limits as in [15, Theorem 4.6.9], we get that $(\mathscr{C}^N, \hat{\mu}_T^N)$ satisfies the LDP with some convex GRF in the algebraic dual $\mathscr{E}'$ of $\mathscr{E}$, endowed with the $\mathscr{E}$-topology. The $\mathscr{E}$-topology induces a Hausdorff topology on $\mathscr{Y}_T \subset \mathscr{E}'$ corresponding to the pointwise convergence of functions in $\mathscr{X}_T$, hence the identity map is a continuous injection from $(\mathscr{Y}_T, d)$ to $\mathscr{E}'$. Applying the inverse contraction principle for this injection, the exponential tightness of $(\mathscr{C}^N, \hat{\mu}_T^N)$ in $(\mathscr{Y}_T, d)$ as established in Step 2, results with the LDP with same convex GRF holding for $(\mathscr{C}^N, \hat{\mu}_T^N)$ in $(\mathscr{Y}_T, d)$ (see [15, Theorem 4.2.4]). Recall that the functions $h_r(\cdot)$ are bounded below on compacts, hence it suffices for the steepness of $h(\cdot)$ to show for $r = 1, \dots, m$ that $h_r(\boldsymbol{\theta}_n) \to \infty$ whenever $\boldsymbol{\theta}_n \in \mathscr{D}_r^o$ converges to $\boldsymbol{\theta}_\infty \in \partial\mathscr{D}_r$. Observing that $\sum_{l=1}^d \theta_{l+d} V_l$ is bounded for $\boldsymbol{\theta}$ in any compact set, we may and shall assume without loss of generality that $V_l = 0, l = 1, \dots, d$ and embed all functions of $\lambda = s_r$ into $\alpha_l \in \mathscr{M}_a$. It is then easy to check that

$$\langle \boldsymbol{\theta}, \mathbf{Y}_r \rangle = a(\boldsymbol{\theta}) v_0^2 + v_0 \langle \boldsymbol{b}(\boldsymbol{\theta}), \boldsymbol{g} \rangle + \frac{1}{2} \langle \boldsymbol{g}, \boldsymbol{A}(\boldsymbol{\theta}) \boldsymbol{g} \rangle ,$$

where $\boldsymbol{g} = (B_{t_1}, \dots, B_{t_n})$ for some $0 < t_1 < t_2 < \cdots < t_n \le T$, and $a(\boldsymbol{\theta}) \in \mathbb{R}$, $\boldsymbol{b}(\boldsymbol{\theta}) \in \mathbb{R}^n$ and the symmetric n-dimensional matrix $\boldsymbol{A}(\boldsymbol{\theta})$ are non-random and linear in $\boldsymbol{\theta} \in \mathbb{R}^d$. With $\boldsymbol{K}_g$ denoting the strictly positive definite covariance matrix of $\boldsymbol{g}$, it is not hard to check that $h_r(\boldsymbol{\theta}) = \infty$ unless $\boldsymbol{K}_g^{-1} - \boldsymbol{A}(\boldsymbol{\theta})$ is positive definite in which case

$$h_r(\boldsymbol{\theta}) = -\frac{1}{2} \log \det(\mathbf{I} - \boldsymbol{K}_g \boldsymbol{A}(\boldsymbol{\theta})) + \Lambda_0(\eta(\boldsymbol{\theta})) \tag{5.19}$$

with $\eta(\boldsymbol{\theta}) = a(\boldsymbol{\theta}) + 0.5\langle \boldsymbol{b}, (\boldsymbol{K}_g^{-1} - \boldsymbol{A})^{-1}\boldsymbol{b}\rangle(\boldsymbol{\theta})$. Hence, for $\boldsymbol{\theta}_n \to \boldsymbol{\theta}_\infty \in \partial\mathscr{D}_r$ either $\lambda_{\min}(\boldsymbol{K}_g^{-1} - \boldsymbol{A}(\boldsymbol{\theta}_n)) \downarrow 0$ in which case $h_r(\boldsymbol{\theta}_n) \to \infty$ or else $\boldsymbol{\theta} \mapsto \eta(\boldsymbol{\theta})$ is continuous at $\boldsymbol{\theta}_\infty$ with $\eta(\boldsymbol{\theta}_\infty)$ necessarily at the boundary of $\{\eta : \Lambda_0(\eta) < \infty\}$. The assumed continuity of $\Lambda_0(\cdot)$ then results with $h_r(\boldsymbol{\theta}_n) \to \infty$, completing Step 3 of our proof.

Step 4 For the general case we introduce exponentially good approximations based on the quantizations $\phi_m(\cdot)$ of the modes $\boldsymbol{\lambda}$ as in the proof of Lemma 4.1. Indeed, letting $\lambda_i^m = \phi_m(\lambda^i)$ and $\lambda_m^* = \phi_m(\lambda^*)$ the condition **(H1a)** then obviously holds for $\boldsymbol{\lambda}^m$, whereas by Lemma 5.6, the empirical measures $\hat{\sigma}^N \circ \phi_m^{-1}$ satisfy the LDP in $\mathscr{P}([-\lambda_m^*, \lambda_m^*])$ with GRF that is zero at $\sigma \circ \phi_m^{-1}$ and $+\infty$ otherwise. Moreover, the support of $\hat{\sigma}^N \circ \phi_m^{-1}$, $N \geq 1$ is a finite non-random set on which $\sigma \circ \phi_m^{-1}$ is strictly positive (by the positivity of σ as in **(H2)**). Denote in analogy with (5.3) and (5.4),

$$\hat{\mu}_T^{N,m} := \frac{1}{N}\sum_{i=1}^N \delta_{\lambda_i^m, v_0^i, B^i_{[0,T]}} , \qquad \mathscr{C}^{N,m} := \frac{1}{N}\sum_{i=1}^N c_T(\lambda_i^m, v_0^i, B^i_\cdot) .$$

Then, for any m, by Step 3 the sequence $(\mathscr{C}^{N,m}, \hat{\mu}_T^{N,m})$ satisfies the LDP in $\mathscr{Y}_T$ with some convex GRF. Observing that $\|c_T(\lambda', v_0, B_\cdot) - c_T(\lambda, v_0, B_\cdot)\| \leq |\lambda - \lambda'|\rho'(v_0^2 + \beta^{-1}\|B_\cdot\|^2)$, for some $\rho' < \infty$ and all $\lambda, \lambda' \in [-\lambda^*, \lambda^*]$, it follows from (5.4) and (4.6) that

$$\begin{aligned} \|\mathscr{C}^{N,m} - \mathscr{C}^N\| &\leq \rho' \frac{1}{N}\sum_{i=1}^N |\lambda_i - \lambda_i^m|((v_0^i)^2 + \beta^{-1}\|B^i_\cdot\|^2) \\ &\leq m^{-1}\rho'\left(\frac{1}{N}\sum_{i=1}^N (v_0^i)^2 + \frac{1}{\beta N}\sum_{i=1}^N \|B^i_\cdot\|^2\right) . \end{aligned}$$

With

$$D(\hat{\mu}_T^{N,m}, \hat{\mu}_T^N) \leq \frac{1}{N}\sum_{i=1}^N |\lambda_i^m - \lambda_i| \leq m^{-1} ,$$

we thus conclude by (5.15) and (5.17) that $(\mathscr{C}^{N,m}, \hat{\mu}_T^{N,m})$ are exponentially good approximations of $(\mathscr{C}^N, \hat{\mu}_T^N)$ in $(\mathscr{Y}_T, d)$. From this, the LDP with convex GRF satisfied by $(\mathscr{C}^{N,m}, \hat{\mu}_T^{N,m})$ for all m, and the exponential tightness of $(\mathscr{C}^N, \hat{\mu}_T^N)$, it follows by [15, Theorem 4.2.16(a)] that $(\mathscr{C}^N, \hat{\mu}_T^N)$ also satisfies the LDP in $\mathscr{Y}_T$ with some convex GRF. For the last conclusion we rely on the convexity of $(x, y) \mapsto d(x, y) : \mathscr{Y}_T^2 \to \mathbb{R}$ of (5.11), as done for example in case of (4.7).

Step 5 Having proved in Step 4 the LDP for $(\mathscr{C}^N, \hat{\mu}_T^N)$ in $\mathscr{X}_T \times \mathscr{P}(\boldsymbol{\Sigma}_T)$ with a convex GRF, we next identify this GRF to be $I_T^1(\cdot)$ by applying Proposition 6.4 for $c = c_T$ of (5.2) and the vector space $\mathscr{M}_a$ that separates points in $\mathscr{X}_T$ (recall the formula we got in Step 1 for the GRF $\mathscr{I}_0(\mu)$ of the LDP of $\hat{\mu}_T^N$). Indeed, we have seen in Step 2 that $\|c_T\| \le \rho(v_0^2 + \beta^{-1}\|B_\cdot\|^2)$, hence by (5.16) and the finiteness of $\Lambda_0(\eta_0)$ of (2.13), for some $\eta > 0$

$$\Lambda_\infty(\eta\|c_T\|) := \limsup_{N\to\infty} \frac{1}{N} \log \mathbb{E}\Big[e^{\eta N \int \|c_T\| d\hat{\mu}_T^N}\Big] \le \log \mathbb{E}\Big[e^{\eta\rho(v_0^2+\beta^{-1}\|B_\cdot\|^2)}\Big] < \infty \,.$$

By Lemma 6.5 it remains only to verify that $\psi_{r\langle\alpha,c_T\rangle}(\cdot)$ of (5.12) is bounded on $[-\lambda^*, \lambda^*]$ for any $r \in [0,1)$ and $\alpha \in \mathscr{M}_a$ for which $\int \psi_{\langle\alpha,c_T\rangle}(\lambda)d\sigma(\lambda) < \infty$. Fixing such $\alpha \in \mathscr{M}_a$, a computation similar to the one leading to (5.19), yields for some strictly positive definite matrix $\boldsymbol{K}_g$, σ-a.e. λ and all $r \in [0,1]$,

$$\psi_{r\langle\alpha,c_T\rangle}(\lambda) = -\frac{1}{2}\log\det(\mathbf{I} - r\boldsymbol{K}_g\boldsymbol{A}(\lambda)) + \Lambda_0(r\eta_r(\lambda)) \;,$$

where $\eta_r(\lambda) := a(\lambda) + 0.5\langle \boldsymbol{b}, (r^{-1}\boldsymbol{K}_g^{-1} - \boldsymbol{A})^{-1}\boldsymbol{b}\rangle(\lambda)$ for some $a(\lambda) \in \mathbb{R}$, $\boldsymbol{b}(\lambda) \in \mathbb{R}^n$ and symmetric n-dimensional matrix $\boldsymbol{A}(\lambda)$ that are non-random and continuous in λ, such that $\boldsymbol{K}_g^{-1} - r\boldsymbol{A}(\lambda)$ is strictly positive definite for $r = 1$ and σ-a.e. λ. By **(H2)** and continuity of $\boldsymbol{A}(\cdot)$, the eigenvalues of $(r^{-1}\boldsymbol{K}_g^{-1} - \boldsymbol{A})$ are bounded below by $(r^{-1}-1)\lambda_{min}(\boldsymbol{K}_g^{-1}) > 0$ for all $r \in [0,1)$ and all $\lambda \in [-\lambda^*, \lambda^*]$. Therefore, since $\boldsymbol{A}(\cdot)$ is continuous, $\lambda \in [-\lambda^*, \lambda^*] \mapsto -\frac{1}{2}\log\det(\mathbf{I} - r\boldsymbol{K}_g\boldsymbol{A}(\lambda))$ is bounded continuous. For the same reasons, η_r is bounded continuous on $[-\lambda^*, \lambda^*]$ for any $r \in [0,1)$. Now, let $\bar{\eta} = \sup\{\eta : \Lambda_0(\eta) < \infty\} \in (0,\infty]$. Since $\int \Lambda_0(\eta_1(\lambda))d\sigma(\lambda) < \infty$, $\Lambda_0(\eta_1(\lambda))$ is finite for σ-a.e. λ, resulting with $\eta_1(\lambda) < \bar{\eta}$ for σ-a.e. λ, and by monotonicity of $r \mapsto \eta_r(\lambda)$, $\eta_r(\lambda) < \bar{\eta}$ for σ-a.e. λ and all $r \in [0,1)$. By continuity of η_r, we deduce that $\eta_r(\lambda) \le \bar{\eta}$ for all $\lambda \in [-\lambda^*, \lambda^*]$ and any $r \in [0,1)$. Hence, $r\sup_{|\lambda|\le\lambda^*}\eta_r(\lambda) < \bar{\eta}$ resulting with $\sup_{|\lambda|\le\lambda^*}\Lambda_0(r\eta_r(\lambda)) < \infty$ as needed to complete Step 5 and with it the proof of the annealed LDP.

Assuming now that **(H0)–(H1)** and **(H2)** hold, fix $\boldsymbol{\lambda}$ such that $\hat{\sigma}^N \to \sigma$ and $\limsup_{N\to\infty}\max_{i=1}^N|\lambda_i| \le \lambda^*$. Fix $\delta' \in (0,1)$ and $N_0(\boldsymbol{\lambda}) < \infty$ such that $\max_{i=1}^N|\lambda_i^N| \le \lambda^* + \delta'$ for all $N \ge N_0$. Then, the degenerate laws $\sigma^N = \delta_{(\lambda_1^N,\dots,\lambda_N^N)}$ on $[-\lambda^*-\delta', \lambda^*+\delta']^N$ satisfy **(H0a)–(H1a)**, upon replacing λ^* by $\lambda^*+\delta'$ in **(H1a)**. The corresponding annealed LDP is actually the stated $\boldsymbol{\lambda}$-quenched LDP. It is easy to check that Steps 1 and 2 of the preceding proof remain valid upon changing λ^* to $\lambda^*+\delta'$ and considering $N \ge N_0(\boldsymbol{\lambda})$. Moreover, Step 3 applies even when $\hat{\sigma}^N$, $N \ge 1$ are supported on some finite set $\mathscr{S}'$ on which $\sigma(\cdot)$ is not strictly positive, provided that eventually $\hat{\sigma}^N(\{s_r\}) = 0$ for all $s_r \notin \mathscr{S}$. By **(H1)** and **(H2)** this indeed applies for $\hat{\sigma}^N \circ \phi_m^{-1}$ of Step 4, regardless of m. In the identification of the GRF in Step 5 we now need to establish the boundedness of $\psi_{r\langle\alpha,c_T\rangle}(\cdot)$ of (5.12) on $[-\lambda^*-\delta, \lambda^*+\delta]$ for some $\delta = \delta(\alpha,r) > 0$. This is done as in the preceding proof, noting that by continuity of $\boldsymbol{A}(\cdot)$, per $r < 1$ the strict positive definiteness of $\boldsymbol{K}_g^{-1} - r\boldsymbol{A}(\lambda)$ extends to $[-\lambda^*-\delta, \lambda^*+\delta]$ for some $\delta = \delta(\alpha,r) > 0$. Changing $\delta > 0$ as needed, the same applies for the continuity and boundedness of $-\frac{1}{2}\log\det(\mathbf{I} - r\boldsymbol{K}_g\boldsymbol{A}(\cdot))$, $\eta_r(\cdot)$ and $\Lambda_0(r\eta_r(\cdot))$. The resulting "annealed" GRF $I_T^1(\cdot)$ depends on the (degenerate) laws σ^N only through σ of **(H0a)**, hence is independent of the particular $\boldsymbol{\lambda}$ chosen, as claimed. □

5.3. LDP of the free measure for initial conditions (IC2)

The case of rotated initial conditions (IC2) is slightly different than that of (IC1) since the coordinates of $\boldsymbol{v}_0$ are no longer independent. However, we shall see that

Theorem 5.7. *Assuming* **(H0)–(H1)** *(or* **(H0a)–(H1a)***) and* **(H2)***, the couple* $(\mathscr{C}^N, \hat{\mu}_T^N)$ *satisfies a quenched (respectively, annealed) LDP in* $\mathscr{Y}_T$ *with respect to the law* σ^N *of the modes, under the law induced on* $(\boldsymbol{v}_0, \boldsymbol{B})$ *by (IC2) and* $\mathscr{W}^{\otimes N}$*. The GRF of this LDP is given by*

$$I_T^2(\mathscr{C}, \mu) = \inf\{I_T^1(\widetilde{\mathscr{C}}, \widetilde{\mu}) + \kappa(u) \; : (\mathscr{C}, \mu) = S(u, \widetilde{\mathscr{C}}, \widetilde{\mu})\}$$

where I_T^1 *is the GRF of Theorem 5.4 for* $\nu_0 = \gamma$ *and* $S : \mathbb{R}^+ \times \mathscr{Y}_T \to \mathscr{X}_T \times \mathscr{Q}_T$ *is such that*

$$S_1(u, \widetilde{\mathscr{C}}, \widetilde{\mu}) = S_1(u, \widetilde{\mathscr{C}}) = (L^2\widetilde{\mathscr{C}}_{1,k}, L\widetilde{\mathscr{C}}_{2,k}, \widetilde{\mathscr{C}}_{3,k}, \quad k = 0, 1, 2)$$

for $L := \sqrt{u/\widetilde{\mathscr{C}}_{1,0}(0)} 1_{\widetilde{\mathscr{C}}_{1,0}(0)>0}$ *and*

$$\int h(\lambda, v, B) dS_2(u, \widetilde{\mathscr{C}}, \widetilde{\mu})(\lambda, v, B) := \int h(\lambda, Lg, B) d\widetilde{\mu}(\lambda, g, B) \; ,$$

for all $h : \boldsymbol{\Sigma}_T \to [-1, 1]$ *uniformly Lipshitz continuous.*

Remarks.
(a). Considering the unique minimizer of the GRF $\kappa(\cdot) + I_T^1(\cdot)$ appearing above, that is $(u^*, \int c_T dp_1, p_1)$ for $p_u = \sigma \otimes \gamma_{1/u} \otimes \mathscr{W}$, it is easy to check that the unique minimizer of the GRF $I_T^2(\cdot)$ is

$$S(u^*, \int c_T dp_1, p_1) = (\int c_T dp_{u^*}, p_{u^*}) \; .$$

This is exactly the couple $(\mathscr{C}^*, \mu_T^*)$ of (5.5) corresponding to (IC2).
(b). The quenched LDP of Theorem 5.7 holds $\boldsymbol{\lambda}$-a.e. In the context of the SDS (1.1) it is natural to ask for a quenched LDP that holds for almost every $\mathbf{J} = (\boldsymbol{G}, \boldsymbol{\lambda})$. Indeed, Theorem 5.7 provides such a result whenever the law of $\boldsymbol{u}_0$ is invariant to rotations (so fixing $\boldsymbol{G}$ is the same as averaging over it). For other laws of $\boldsymbol{u}_0$ the GRF of a $\mathbf{J}$-quenched LDP is possibly quite different from $I_T^2(\cdot)$. Nevertheless, our conclusions from the quenched LDP of Theorem 5.7 which are about convergence almost surely in $(\boldsymbol{\lambda}, \boldsymbol{G}, \boldsymbol{u}_0, \boldsymbol{B})$ readily apply to the SDS (1.1).
Proof. Observe first that fixing $\boldsymbol{u}_0 \in \mathbb{R}^N$, the Haar measure H_N induces on $\boldsymbol{v}_0 = \boldsymbol{G}\boldsymbol{u}_0$ the uniform law on the sphere S^{N-1} with radius $|\boldsymbol{u}_0|$. It is well known that this law can be represented as the law of $\frac{|\boldsymbol{u}_0|}{|\boldsymbol{g}|}\boldsymbol{g}$ where $\boldsymbol{g}$ is independent of $\boldsymbol{u}_0$ and follows the standard centered Gaussian law $\gamma^{\otimes N}$. By the assumed independence of $\boldsymbol{G}$ and $\boldsymbol{u}_0$, we can describe $\boldsymbol{v}_0$ of (IC2) as $\frac{|\boldsymbol{u}_0|}{|\boldsymbol{g}|}\boldsymbol{g}$, where now $\boldsymbol{u}_0$ independent of $\boldsymbol{g}$ is also random. Taking

$$\hat{\mu}_T^N = \frac{1}{N} \sum_{i=1}^N \delta_{\lambda_i, g_i, B^i_{[0,T]}} \; ,$$

by Theorem 5.4 the GRF $\kappa(u) + I_T^1(\cdot)$ applies to the $\boldsymbol{\lambda}$-quenched (annealed) LDP of $(N^{-1}|\boldsymbol{u}_0|^2, \int c_T d\hat{\mu}_T^N, \hat{\mu}_T^N)$ in $\mathbb{R}^+ \times \mathscr{Y}_T$. Since $N^{-1}|\boldsymbol{g}|^2 = \int (c_T)_{1,0}(0) d\hat{\mu}_T^N > 0$, the transformation S corresponds to replacing g_i by $v_0^i = \frac{|\boldsymbol{u}_0|}{|\boldsymbol{g}|} g_i$ and then recomputing $(\mathscr{C}^N, \hat{\mu}_T^N)$. By the contraction principle, the LDP then holds with the stated GRF $I_T^2(\cdot)$ provided S is continuous at every $(u, \widetilde{\mathscr{C}}, \widetilde{\mu})$ for which $\kappa(u) + I_T^1(\widetilde{\mathscr{C}}, \widetilde{\mu}) < \infty$. By Cramèr's theorem, $N^{-1}|\boldsymbol{g}|^2$ satisfies the LDP in $\mathbb{R}^+$ with the GRF $\kappa_g(r) = 0.5(r - 1 - \log r)$. Recall that the GRF $\kappa(u) + I_T^1(\widetilde{\mathscr{C}}, \widetilde{\mu})$ is at least $\kappa_g(0) = +\infty$ whenever $\widetilde{\mathscr{C}}_{1,0}(0) = 0$. It is easy to check that S is continuous except at points for which $\widetilde{\mathscr{C}}_{1,0}(0) = 0$. □

5.4. LDP for the free measure starting from the eigenvector with maximum eigenvalue

In this section, we consider (IC3) where $\boldsymbol{u}_0$ is the eigenvector of $\mathbf{J}$ corresponding to the maximum eigenvalue $\lambda_N^* = \max_{i=1}^N \lambda_i$, set without loss of generality to be λ_1, so that $\boldsymbol{v}_0 = (\sqrt{N}, 0, \ldots, 0)$. For $\hat{\mu}_T^N$ such initial conditions are approximately equivalent to zero initial conditions, whereas $\mathscr{C}^N$ of (5.4) is then

$$\mathscr{C}^N(u, v, w) = \Big((\lambda_N^*)^k e^{w\lambda_N^*}, \frac{1}{\sqrt{\beta N}} B_u^1 (\lambda_N^*)^k e^{w\lambda_N^*}, \frac{1}{\beta N} \sum_{i=1}^N B_u^i B_v^i (\lambda_i)^k e^{w\lambda_i} \quad k = 0, 1, 2 \Big)$$

Thus, an annealed LDP for $(\mathscr{C}^N, \hat{\mu}_T^N)$ requires an LDP for $\{\lambda_N^*\}$. For example, such an LDP with a non-trivial GRF is proved in Section 6.1 for the Wigner semi-circular law. However, the latter does not satisfy **(H1a)**. It turns out that **(H0a)**, **(H1a)** and **(H2)** imply the LDP for $\{\lambda_N^*\}$ with the (trivial) GRF $\kappa_*(\lambda^*) := 0$ and $\kappa_*(r) := +\infty$ for all $r \neq \lambda^*$, making the following annealed LDP possible,

Theorem 5.8. *Assuming* **(H0)–(H1)***(or* **(H0a)–(H1a)***) and* **(H2)***, the couple* $(\mathscr{C}^N, \hat{\mu}_T^N)$ *satisfies a quenched (respectively, annealed) LDP in* $\mathscr{Y}_T$ *with respect to the law* σ^N *of the modes, under the law induces on* $(\boldsymbol{v}_0, \boldsymbol{B})$ *by (IC3) and* $\mathscr{W}^{\otimes N}$*. The GRF of this LDP is given by*

$$I_T^3(\mathscr{C}, \mu) = \inf\{I_T^1(\widetilde{\mathscr{C}}, \mu) + \mathscr{J}(\phi) + \kappa_*(r) : \mathscr{C} = \bar{S}(r, \phi, \widetilde{\mathscr{C}}),\ \phi \in H_{0,T}^1,\ |r| \leq \lambda^*\},$$

where I_T^1 *is the GRF of Theorem 5.4 for* $\nu_0 = \delta_0$,

$$\mathscr{J}(\phi) := \frac{1}{2} \int_0^T |\dot{\phi}(s)|^2 ds$$

is a GRF on $H_{0,T}^1 := \{\phi : \phi(0) = 0, \mathscr{J}(\phi) < \infty\}$*, and the continuous mapping* $\bar{S} : \mathbb{R} \times H_{0,T}^1 \times \mathscr{X}_T \to \mathscr{X}_T$ *is such that*

$$\bar{S}(r, \phi, \widetilde{\mathscr{C}}) := \Big(r^k e^{rw}, \beta^{-1/2} r^k e^{rw} \phi(u), \widetilde{\mathscr{C}}_{3,k}(u, v, w) + \beta^{-1} r^k e^{rw} \phi(u)\phi(v) \quad k = 0, 1, 2 \Big)$$

Remark. Considering the unique minimizer of the GRF $I_T^1(\cdot) + \mathscr{J}(\phi) + \kappa_*(r)$ appearing above, that is $(\lambda^*, 0, \int c_T d\mu_T^*, \mu_T^*)$ for $\mu_T^* = \sigma \otimes \delta_0 \otimes \mathscr{W}$, it is easy to check that $\bar{S}(\lambda^*, 0, \int c_T d\mu_T^*)$ is exactly $\mathscr{C}^*$ of (5.5) corresponding to (IC3).

Proof. Observe that $\mathscr{C}^N = \bar{S}(\lambda_N^*, \frac{1}{\sqrt{N}} B_u^1, \widetilde{\mathscr{C}}^N)$ for

$$\widetilde{\mathscr{C}}^N := (0, 0, \frac{1}{\beta N} \sum_{i=2}^{N} B_u^i B_v^i (\lambda_i)^k e^{w\lambda_i}) \ .$$

Note that the couple $\left(\int c_T d\widetilde{\mu}_T^N, \widetilde{\mu}_T^N \right)$ for

$$\widetilde{\mu}_T^N := \frac{1}{N-1} \sum_{i=2}^{N} \delta_{\lambda_i, 0, B^i_{[0,T]}} \ ,$$

is exponentially equivalent in $(\mathscr{Y}_T, d)$ to $(\widetilde{\mathscr{C}}^N, \hat{\mu}_T^N)$ (c.f. [15, Definition 4.2.10]). By Schilder's theorem $\frac{1}{\sqrt{N}} B_u^1$, independent of $(\lambda_N^*, \int c_T d\widetilde{\mu}_T^N, \widetilde{\mu}_T^N)$, satisfies the LDP with GRF $\mathscr{J}(\cdot)$ in $H_{0,T}^1$. Thus, by the contraction principle, it suffices to show that $(\lambda_N^*, \int c_T d\widetilde{\mu}_T^N, \widetilde{\mu}_T^N)$ satisfies the LDP in $\mathbb{R} \times \mathscr{Y}_T$ with the GRF $\kappa_*(r) + I_T^1(\cdot)$ (c.f. [15, Theorem 4.2.13]).

In the $\boldsymbol{\lambda}$-quenched case, assuming **(H0)–(H1)** and **(H2)**, the latter LDP is a consequence of Theorem 5.4 since almost surely, $\lambda_N^* \to \lambda^*$ and $\widetilde{\mu}^N := (\widetilde{\mu}_T^N)_1 \to \sigma$ in $\mathscr{P}(\mathbb{R})$. Assuming instead **(H0a)–(H1a)** and **(H2)**, it follows from **(H1a)** that $|\lambda_N^*| \leq \lambda^*$, whereas by **(H0a)** and **(H2)**, for any $\delta > 0$,

$$\limsup_{N\to\infty} \frac{1}{N} \log \mathbb{P}(\lambda_N^* \leq \lambda^* - \delta) = -\infty$$

(for example, see the proof of (6.3)). Consequently, $(\lambda_N^*, \int c_T d\widetilde{\mu}_T^N, \widetilde{\mu}_T^N)$ is exponentially equivalent in $\mathbb{R} \times \mathscr{Y}_T$ to $(\lambda^*, \int c_T d\widetilde{\mu}_T^N, \widetilde{\mu}_T^N)$. Moreover, $\widetilde{\mu}^N$ is exponentially equivalent in $\mathscr{P}(\mathbb{R})$ to $\hat{\sigma}^N$, hence also satisfying **(H0a)**. In particular, Theorem 5.4 applies to $(\int c_T d\widetilde{\mu}_T^N, \widetilde{\mu}_T^N)$, yielding the annealed LDP for $(\lambda_N^*, \int c_T d\widetilde{\mu}_T^N, \widetilde{\mu}_T^N)$ with the GRF $\kappa_*(r) + I_T^1(\cdot)$. □

5.5. LDP for the free measure with Gibbs initial conditions

In this section, we consider (IC4) where the law of $\boldsymbol{v}_0$ is the diagonalized Gibbs measure $\mu_{\boldsymbol{\lambda}}^N$ of (2.1). We then establish the following result

Theorem 5.9. *Assuming* **(H0)–(H1)** *(or* **(H0a)–(H1a)***) and* **(H2)**, *the couple* $(\mathscr{C}^N, \hat{\mu}_T^N)$ *satisfies a quenched (respectively, annealed) LDP in* $\mathscr{Y}_T$ *with respect to the law* σ^N *of the modes, under the law induces on* $(\boldsymbol{v}_0, \boldsymbol{B})$ *by (IC4) and* $\mathscr{W}^{\otimes N}$*. The GRF of this LDP is given by*

$$I_T^4(\mathscr{C}, \mu) := J(\mathscr{C}, \mu) - \inf_{(\widetilde{\mathscr{C}}, \widetilde{\mu}) \in \mathscr{Y}_T} J(\widetilde{\mathscr{C}}, \widetilde{\mu}) \ ,$$

where

$$J(\mathscr{C},\mu) := \; I(\mu|\sigma\otimes\gamma\otimes\mathscr{W}) + F_\gamma\left(\mathscr{C} - \int c_T d\mu\right) + \beta f(\mathscr{C}_{1,0}(0)) \\ -\mathscr{C}_{1,0}(0)/2 + \beta\mathscr{C}_{1,1}(0)\ ,$$

in case $\mu_1 = \sigma$ *and* $I(\mu|\sigma\otimes\gamma\otimes\mathscr{W}) < \infty$ *and* $J(\mathscr{C},\mu) := \infty$ *otherwise. Here*

$$F_\gamma(x) = \sup_{\alpha\in\mathscr{D}_o} \langle \alpha, x\rangle\ , \qquad x\in\mathscr{X}_T$$

is a non-negative convex function, where

$$\mathscr{D}_o := \left\{\alpha\in\mathscr{M}_a : \int e^{\langle\alpha, c_T(\lambda, v_0, B.)\rangle} d\gamma(v_0) d\mathscr{W}(B.) < \infty \quad \forall|\lambda|\le\lambda^*\right\}\ ,$$

for the vector space $\mathscr{M}_a$ *of all* $\mathbb{R}^9$*-weighted finite sums of atomic (dirac) measures on* $[0,T]^3$ *(and* c_T *is given in (5.2)). When* f *is strictly convex, the unique minimizer of* $J(\cdot)$ *in* $\mathscr{Y}_T$ *is* $(\mathscr{C}^*,\mu_T^*)$ *of (5.5).*

Proof. Noting that $\mu^N_{\boldsymbol{\lambda}}$ is a special case of the spherical law (2.14), our starting point is Corollary 5.5. When applied for $\hat g(x) = \beta f(x) - x/2$ and $h(x) = \beta x$, it results with $(\mathscr{C}^N, \hat\mu^N_T)$ satisfying the stated LDP with the GRF $I^4_T(\cdot)$. The identification of the function $F_\gamma(x)$ is done by examining Step 5 of the proof of Theorem 5.4 and its use of Lemma 6.5.

Turning to find the minimizers of $I^4_T(\cdot)$ when f is strictly convex, recall that the unique minimizer of the GRF of the LDP for $((\mathscr{C}^N)_{1,0}(0), (\mathscr{C}^N)_{1,1}(0), (\hat\mu^N_0)_{1,2})$ is shown in Theorem 2.2 to be $(u_\beta, v_\beta, \pi^*)$. Thus, the minimizers $(\mathscr{C}^*,\mu^*_T)$ of $I^4_T(\cdot)$ are the minimizers of $J(\cdot)$ subject to the constraints $(\mathscr{C}^*)_{1,0}(0) = u_\beta$, $(\mathscr{C}^*)_{1,1}(0) = v_\beta$ and $(\mu^*_T)_{1,2} = \pi^*$. Since $\{(c_T)_{1,k}(\cdot), k = 0,1,2\}$ is independent of $B.$, writing $\mathscr{C}^* = \int c_T d\mu^*_T + x^*$, it follows that x^* minimizes $F_\gamma(x)$ subject to the constraints $x_{1,0}(0) = u_\beta - \int (c_T)_{1,0}(0) d\pi^*$ and $x_{1,1}(0) = v_\beta - \int (c_T)_{1,1}(0) d\pi^*$, whereas μ^*_T minimizes $I(\mu|\sigma\otimes\gamma\otimes\mathscr{W})$ subject to the given marginal $(\mu)_{1,2} = \pi^*$. Thus, necessarily, $\mu^*_T = \pi^*\otimes\mathscr{W}$.

As for the optimization problem in x, recall that in the course of proving Theorem 2.2 we have also characterized v_β. In particular, this characterization implies that the constraints on x are such that

$$x_{1,1}(0) = \lambda^* x_{1,0}(0) = \lambda^* c_{EA} \tag{5.20}$$

for $c_{EA} := u_\beta - \int v^2 d\nu^*_0(v) \ge 0$. To complete the proof, it suffices to show that the unique minimizer of $F_\gamma(\cdot)$ subject to (5.20) is given by

$$x^*_{1,k}(w) = c_{EA}(\lambda^*)^k e^{\lambda^* w}\ , \qquad x^*_{2,k}(\cdot)\equiv 0, \qquad x^*_{3,k}(\cdot)\equiv 0, \quad k = 0,1,2\,. \tag{5.21}$$

To this end, we first show that for every $x\in\mathscr{X}_T$,

$$F_\gamma((x_1,x_2,x_3)) \ge F_\gamma((x_1,0,x_3)) \ge F_\gamma((x_1,0,0)) \tag{5.22}$$

with strict inequality whenever $x_3 \neq 0$. Indeed, observe that $c_T = ((c_T)_1, (c_T)_2, (c_T)_3)$ equals in $\sigma \otimes \gamma \otimes \mathscr{W}$-law to $((c_T)_1, -(c_T)_2, (c_T)_3)$ due to the symmetry of the law $\mathscr{W}$ of $B_\cdot$. Consequently, if $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathscr{D}_o$, then so does $(\alpha_1, -\alpha_2, \alpha_3)$. By convexity it then follows that $(\alpha_1, 0, \alpha_3) \in \mathscr{D}_o$, resulting with the left inequality of (5.22). By the conditional independence of $(c_T)_1$ and $(c_T)_3$ given λ under the $\sigma \otimes \gamma \otimes \mathscr{W}$-law, it follows that $(\alpha_1, 0, \alpha_3) \in \mathscr{D}_o$ if and only if $(\alpha_1, 0, 0) \in \mathscr{D}_o$ and $(0, 0, \alpha_3) \in \mathscr{D}_o$. Hence,

$$F_\gamma((x_1, 0, x_3)) = F_\gamma((x_1, 0, 0)) + F_\gamma((0, 0, x_3)) \geq F_\gamma((x_1, 0, 0))$$

as we claimed in (5.22), with a strict inequality whenever $x_3 \neq 0$ (since $F_\gamma(x) > 0$ for all $x \neq 0$).

Suppose next that there is a minimizer $(\mathscr{C}^*, \mu_T^*)$ of $I_T^4(\cdot)$ such that $x_{2,k}^*(u, w) \neq 0$ for some $k \in \{0, 1, 2\}$, $u \in [0, T]$ and $w \in [0, T]$. Let $\hat{I}_T^1(\cdot)$ denote the GRF for the LDP of $((\mathscr{C}^N)_{1,0}(0), (\mathscr{C}^N)_{1,1}(0), (\mathscr{C}^N)_{2,k}(u, w), \hat{\mu}_T^N)$ in $\mathbb{R}^3 \times \mathscr{Q}_T$ when starting with (IC1) initial conditions, with $\hat{I}_T^4(\cdot)$ denoting the GRF for the LDP of the same object under (IC4) initial conditions. Going over Step 5 of the proof of Theorem 5.4, it is not hard to show that $\hat{I}_T^1(\cdot)$ is finite only when $\mu_1 = \sigma$ and $I(\mu|\sigma \otimes \nu_0 \otimes \mathscr{W}) < \infty$, in which case it is given by

$$\hat{I}_T^1(\mathscr{C}, \mu) = I(\mu|\sigma \otimes \nu_0 \otimes \mathscr{W}) + \hat{F}_\gamma(\mathscr{C} - \int \hat{c} d\mu) ,$$

where $\hat{c} := ((c_T)_{1,0}(0), (c_T)_{1,1}(0), (c_T)_{2,k}(u, w))$ and for any $\boldsymbol{y} \in \mathbb{R}^3$,

$$\begin{aligned} \hat{F}_\gamma(\boldsymbol{y}) &= \sup\{\langle \boldsymbol{\alpha}, \boldsymbol{y}\rangle : \int \exp((\alpha_1 + \lambda\alpha_2)v^2 + \alpha_3\beta^{-1/2}\sqrt{u}Bv\lambda^k e^{w\lambda}) d\gamma(v) d\gamma(B) \\ &\qquad < \infty \ \ \forall |\lambda| \leq \lambda^*\} \\ &= \sup\{\alpha_1 y_1 + \alpha_2 y_2 + \alpha_3 y_3 \ : \ \alpha_1 + \lambda\alpha_2 + 0.5\alpha_3^2\beta^{-1}u\lambda^{2k}e^{2w\lambda} < 0.5 \\ &\qquad \forall |\lambda| \leq \lambda^*\} . \end{aligned} \tag{5.23}$$

It follows as before that the minimizers of $\hat{I}_T^4(\cdot)$ are given by $(\int \hat{c} d\mu_T^* + \boldsymbol{y}^\star, \mu_T^*)$, where $\boldsymbol{y}^\star = (y_1^*, y_2^*, y_3^*) \in \mathbb{R}^3$ for

$$y_2^* = \lambda^* y_1^* = \lambda^* c_{EA} \geq 0 , \tag{5.24}$$

and y_3^* is a minimizer of $\hat{F}_\gamma(y_1^*, y_2^*, y_3)$. It follows immediately from (5.23) that for any $y_3 \neq 0$

$$\begin{aligned} \hat{F}_\gamma(y_1^*, y_2^*, y_3) &= \frac{y_1^*}{2} + \sup_{\alpha_2, \alpha_3} \inf_{|\lambda| \leq \lambda^*} \{\alpha_2(\lambda^* - \lambda) y_1^* + \alpha_3 y_3 \\ &\quad - 0.5\alpha_3^2\beta^{-1}u\lambda^{2k}e^{2w\lambda} y_1^*\} \\ &\geq \frac{y_1^*}{2} + \sup_{\alpha_3} \{|\alpha_3||y_3| - 0.5\alpha_3^2\beta^{-1}u(\lambda^*)^{2k}e^{2w\lambda^*} y_1^*\} > \frac{y_1^*}{2} \\ &= \hat{F}_\gamma(y_1^*, y_2^*, 0) . \end{aligned}$$

By the contraction principle, necessarily $y_3^* = x_{2,k}^*(u, w) \neq 0$ should also be a minimizer of $\hat{F}_\gamma(y_1^*, y_2^*, y_3)$, a contradiction.

Suppose then that there is a minimizer $(\mathscr{C}^*, \mu_T^*)$ of $I_T^4(\cdot)$ such that $x_{1,k}^*(w) \neq y_3^* := c_{EA}(\lambda^*)^k e^{\lambda^* w}$ for some $k \in \{0, 1, 2\}$ and $w \in [0, T]$. Applying the same strategy for the GRF $\widetilde{I}_T^4(\cdot)$ of the LDP of $((\mathscr{C}^N)_{1,0}(0), (\mathscr{C}^N)_{1,1}(0), (\mathscr{C}^N)_{1,k}(w), \hat{\mu}_T^N)$ we have that necessarily $x_{1,k}^*(w)$ is a minimizer of $\widetilde{F}_\gamma(y_1^*, y_2^*, y_3)$ for

$$\widetilde{F}_\gamma(y_1, y_2, y_3) = \sup\{\alpha_1 y_1 + \alpha_2 y_2 + \alpha_3 y_3 \ : \ \alpha_1 + \lambda\alpha_2 + \alpha_3 \lambda^k e^{w\lambda} < 0.5 \\ \forall |\lambda| \leq \lambda^* \},$$

and y_1^*, y_2^* of (5.24). In case of $\lambda^* c_{EA} > 0$, taking $\alpha_2 \to \infty$ yields, for all $y_3 \neq y_3^*$,

$$\begin{aligned}\widetilde{F}_\gamma(y_1^*, y_2^*, y_3) &= \frac{y_1^*}{2} + \sup_{\alpha_2, \alpha_3} \inf_{|\lambda| \leq \lambda^*} \{\alpha_2(\lambda^* - \lambda) y_1^* + \alpha_3(y_3 - \lambda^k e^{w\lambda} y_1^*)\} \\ &\geq \frac{y_1^*}{2} + \sup_{\alpha_3} \{\alpha_3(y_3 - y_3^*)\} > \frac{y_1^*}{2} = \widetilde{F}_\gamma(y_1^*, y_2^*, y_3^*),\end{aligned}$$

in contradiction with $x_{1,k}^*(w) \neq y_3^*$ minimizing $\widetilde{F}_\gamma(y_1^*, y_2^*, \cdot)$. In case $\lambda^* = 0$, it follows from Theorem 2.2 that $c_{EA} = 0$ and then clearly $\widetilde{F}_\gamma(0, 0, \cdot)$ has a unique global minimum at $y_3^* = 0$. We have thus completed the proof of (5.21) and with it, that of the theorem. □

6. LDPs for the GOE and related quantities

6.1. Large deviations of eigenvalues of the GOE

Let us recall that, if J_{ij} is a symmetric matrix of centered Gaussian random variables such that

$$\mathbb{E}[J_{ij}^2] = \frac{1}{N} \qquad \mathbb{E}[J_{ii}^2] = \frac{2}{N},$$

then the law σ_N of the N real valued eigenvalues of $\mathbf{J}$ is given by

$$\sigma^N(d\lambda_1, \cdots, d\lambda_N) = \frac{1}{Z^N} \prod_{1 \leq i < j \leq N} |\lambda_i - \lambda_j| \exp\left\{-\frac{1}{4} N \sum_{i=1}^N \lambda_i^2\right\} \prod_{i=1}^N d\lambda_i,$$

where Z^N is the normalizing constant. It has been proven in [4] that

Theorem 6.1. *The spectral measures $\hat{\sigma}^N = \frac{1}{N}\sum_{i=1}^N \delta_{\lambda_i}$ of the GOE* $\mathbf{J}$ *satisfy the LDP with speed N^2 and with the GRF*

$$I(\mu) = \frac{1}{4}\int x^2 d\mu(x) + \frac{1}{2}\int \log|x-y|^{-1} d\mu(x) d\mu(y) - 3/8,$$

whose unique minimizer is the semicircular probability measure

$$\sigma = \frac{1}{2\pi} 1\!\!1_{|x| \leq 2} \sqrt{4 - x^2}\, dx. \tag{6.1}$$

We next state and prove the LDP for the maximum eigenvalue of the GOE $\mathbf{J}$.

Theorem 6.2. *The maximal eigenvalues* $\lambda_N^* = \max_{i=1}^N \lambda_i$ *of the GOE* $\mathbf{J}$ *satisfy the LDP in* $\mathbb{R}$ *with speed* N *and the GRF*

$$I^*(x) = \begin{cases} \int_2^x \sqrt{(z/2)^2 - 1}dz, & x \geq 2\,, \\ \infty, & \textit{otherwise}\,. \end{cases} \tag{6.2}$$

The next estimate is key to the proof of Theorem 6.2.

Lemma 6.3. *For every* M *large enough and all* N*,*

$$\sigma^N \left(\max_{i=1}^N |\lambda_i| \geq M \right) \leq e^{-NM^2/9}\,.$$

Proof. Observe that for any $|x| \geq M \geq 8$ and $\lambda_i \in \mathbb{R}$,

$$|x - \lambda_i| e^{-\lambda_i^2/4} \leq (|x| + |\lambda_i|) e^{-\lambda_i^2/4} \leq 2|x| \leq e^{x^2/8}\,.$$

Therefore, integrating with respect to λ_1 yields, for $M \geq 8$,

$$\begin{aligned}
\sigma^N(|\lambda_1| \geq M) &\leq e^{-\frac{1}{8}NM^2} \frac{Z^{N-1}}{Z^N} \int_{|x|\geq M} e^{-x^2/8} dx \\
&\quad \times \int \prod_{i=2}^N (|x - \lambda_i| e^{-\lambda_i^2/4} e^{-x^2/8}) d\sigma^{N-1}(\lambda_j, j \geq 2) \\
&\leq e^{-\frac{1}{8}NM^2} \frac{Z^{N-1}}{Z^N} \int e^{-x^2/8} dx
\end{aligned}$$

Further, following Selberg (c.f. [22, Theorem 4.1.1]), the explicit formula for Z^N shows that $Z^{N-1}/Z^N \leq e^{C'N}$ for some finite C' and all N (see also proof of [4, Property 3.1]). Taking $C = \max(C', \int e^{-x^2/8}dx)$, it follows that for any $M \geq 8$

$$\sigma^N(\max_{i=1}^N |\lambda_i| \geq M) \leq N\sigma^N(|\lambda_1| \geq M) \leq e^{-\frac{1}{8}NM^2 + 2CN}\,,$$

and the lemma follows since $C < \infty$ is independent of M. □

Proof of Theorem 6.2. Obviously, $I^*(x)$ is a GRF. Moreover, with $I^*(x)$ continuous and strictly increasing on $[2, \infty)$ it clearly suffices to show that for any $x < 2$,

$$\limsup_{N\to\infty} \frac{1}{N} \log \sigma^N \left(\lambda_N^* \leq x \right) = -\infty \tag{6.3}$$

whereas for any $x > 2$

$$\lim_{N\to\infty} \frac{1}{N} \log \sigma^N \left(\lambda_N^* \geq x \right) = -I^*(x)\,. \tag{6.4}$$

Starting with (6.3), fix $x < 2$ and $f \in C_b(\mathbb{R})$ such that $f(y) = 0$ for all $y \leq x$ whereas $\int f d\sigma > 0$. Note that $\{\lambda_N^* \leq x\} \subseteq \{\int f d\hat{\sigma}^N = 0\}$, so (6.3) follows by applying the upper bound of the LDP of Theorem 6.1 for the closed set $F = \{\mu : \int f d\mu = 0\}$, such that $\sigma \notin F$. Turning to the upper bound in (6.4), fix $M \geq x > 2$, noting that

$$\sigma^N \left(\lambda_N^* \geq x\right) = \sigma^N (\max_{i=1}^N |\lambda_i| > M) + \sigma^N \left(\lambda_N^* \geq x, \max_{i=1}^N |\lambda_i| \leq M\right) \tag{6.5}$$

By Lemma 6.3, the first term is exponentially negligible for all M large enough. To deal with the second term, let $\sigma_N^{N-1}(\lambda \in \cdot) = \sigma^{N-1}((1 - N^{-1})^{1/2}\lambda \in \cdot)$, $\hat{\mu}^{N-1} = (N-1)^{-1} \sum_{i=2}^N \delta_{\lambda_i}$ and

$$C_N := \frac{Z^{N-1}}{Z^N}(1 - N^{-1})^{N(N-1)/4}.$$

Further, let $B(\sigma, \delta)$ denote an open ball in $\mathscr{P}(\mathbb{R})$ of radius $\delta > 0$ and center σ, with $B_M(\sigma, \delta)$ its intersection with $\mathscr{P}([-M, M])$. Observe that for any $z \in [-M, M]$ and $\mu \in \mathscr{P}([-M, M])$,

$$\Phi(z, \mu) := \int \log |z - y| d\mu(y) - \frac{1}{4} z^2 \leq \log(2M)\ .$$

Thus, for the second term in (6.5),

$$\begin{aligned}
&\sigma^N \left(\lambda_N^* \geq x, \max_{i=1}^N |\lambda_i| \leq M\right) \\
&\leq N C_N \int_x^M d\lambda_1 \int_{[-M,M]^{N-1}} e^{(N-1)\Phi(\lambda_1, \hat{\mu}^{N-1})} d\sigma_N^{N-1}(\lambda_j, j \geq 2) \\
&\leq N C_N \Bigg(\int_x^M e^{(N-1) \sup_{\mu \in B_M(\sigma,\delta)} \Phi(z,\mu)} dz \\
&\qquad + (2M)^N \sigma_N^{N-1}(\hat{\mu}^{N-1} \notin B(\sigma, \delta)) \Bigg)
\end{aligned} \tag{6.6}$$

For any h of Lipschitz norm at most 1 and $N \geq 2$,

$$|(N-1)^{-1} \sum_{i=2}^N (h((1 - N^{-1})^{1/2}\lambda_i) - h(\lambda_i))| \leq 3N^{-1} \max_{i=2}^N |\lambda_i|\ .$$

Thus, by Lemma 6.3, the spectral measures $\hat{\mu}^{N-1}$ under σ^{N-1} are exponentially equivalent in $\mathscr{P}(\mathbb{R})$ to the spectral measures $\hat{\mu}^{N-1}$ under σ_N^{N-1}, so Theorem 6.1 applies also for the latter (c.f. [15, Theorem 4.2.13]). In particular, the second term in (6.6) is exponentially negligible as $N \to \infty$ for any $\delta > 0$ and $M < \infty$. Therefore,

$$\begin{aligned}
&\limsup_{N \to \infty} \frac{1}{N} \log \sigma^N \left(\lambda_N^* \geq x, \max_{i=1}^N |\lambda_i| \leq M\right) \\
&\leq \limsup_{N \to \infty} \frac{1}{N} \log C_N + \lim_{\delta \downarrow 0} \sup_{\substack{z \in [x,M] \\ \mu \in B_M(\sigma,\delta)}} \Phi(z, \mu)
\end{aligned} \tag{6.7}$$

Note that $\Phi(z,\mu) = \inf_{\eta>0} \Phi_\eta(z,\mu)$ with $\Phi_\eta(z,\mu) := \int \log(|z-y| \vee \eta) d\mu(y) - \frac{1}{4}z^2$ continuous on $[-M, M] \times \mathscr{P}([-M, M])$. Thus, $(z,\mu) \mapsto \Phi(z,\mu)$ is upper semi-continuous, so

$$\lim_{\delta\downarrow 0} \sup_{\substack{z\in[x,M] \\ \mu\in B_M(\sigma,\delta)}} \Phi(z,\mu) = \sup_{z\in[x,M]} \Phi(z,\sigma) \tag{6.8}$$

With σ supported on $[-2, 2]$, clearly $D(z) := \frac{d}{dz}\Phi(z,\sigma)$ exists for $z \geq 2$. Moreover, $D(z) = -\sqrt{(z/2)^2 - 1} \leq 0$ as shown for example in [4, Proof of Lemma 2.7]. It is also shown in [4, Lemma 2.7] that $\Phi(2,\sigma) = -1/2$. Hence, for $x > 2$,

$$\sup_{z\geq x} \Phi(z,\sigma) = \Phi(x,\sigma) = -\frac{1}{2} - I^*(x)\,. \tag{6.9}$$

Again by means of Selberg's formula, it is not hard to verify that $N^{-1}\log C_N \to 1/2$ (c.f. the proof of [4, Property 3.1]). Combining this with (6.7)–(6.9) completes the proof of the upper bound for (6.4). To prove the complementary lower bound, fix $y > x > r > 2$ and $\delta > 0$, noting that for all N,

$$\begin{aligned}
\sigma^N\left(\lambda_N^* \geq x\right) &\geq \sigma^N\left(\lambda_1 \in [x,y], \max_{i=2}^{N} |\lambda_i| \leq r\right) \\
&= C_N \int_x^y e^{-\lambda_1^2/4} d\lambda_1 \int_{[-r,r]^{N-1}} e^{(N-1)\Phi(\lambda_1,\hat{\mu}^{N-1})} d\sigma_N^{N-1}(\lambda_j, j\geq 2) \\
&\geq kC_N \exp\left((N-1) \inf_{\substack{z\in[x,y] \\ \mu\in B_r(\sigma,\delta)}} \Phi(z,\mu)\right) \sigma_N^{N-1}\left(\hat{\mu}^{N-1} \in B_r(\sigma,\delta)\right)
\end{aligned}$$

with $k = k(x, y) > 0$. Recall that the LDP with speed N^2 and GRF $I(\cdot)$ applies for the measures $\hat{\mu}^{N-1}$ under σ_N^{N-1}. It follows by this LDP's upper bound that $\sigma_N^{N-1}(\hat{\mu}^{N-1} \notin B(\sigma,\delta)) \to 0$, whereas by the symmetry of $\sigma^N(\cdot)$ and the upper bound of (6.4),

$$\sigma_N^{N-1}\left(\hat{\mu}^{N-1} \notin \mathscr{P}([-r,r])\right) \leq 2\sigma^{N-1}\left(\lambda_N^* \geq r\right) \to 0$$

as $N \to \infty$. Consequently,

$$\liminf_{N\to\infty} \frac{1}{N} \log \sigma^N\left(\lambda_N^* \geq x\right) \geq \frac{1}{2} + \inf_{\substack{z\in[x,y] \\ \mu\in B_r(\sigma,\delta)}} \Phi(z,\mu)$$

Observe that $(z,\mu) \mapsto \Phi(z,\mu)$ is continuous on $[x, y] \times \mathscr{P}([-r, r])$, for $y > x > r > 2$. Hence, considering $\delta \downarrow 0$ followed by $y \downarrow x$ results with the required lower bound

$$\liminf_{N\to\infty} \frac{1}{N} \log \sigma^N\left(\lambda_N^* \geq x\right) \geq \frac{1}{2} + \Phi(x,\sigma)\,. \qquad \square$$

6.2. Identification of GRF for non-continuous contractions

Let $c : \Sigma \to \mathscr{X}$ be a continuous function from a Polish space Σ to a separable Banach space $(\mathscr{X}, \|\cdot\|)$. Let $C^{a.b}(\Sigma)$ denote the class of $\mathbb{R}$-valued, continuous, bounded above functions on Σ. Let $\mathscr{X}^*$ denote the topological dual of $\mathscr{X}$ and $\langle \alpha, x\rangle = \alpha(x)$ the duality map for $\alpha \in \mathscr{X}^*$ and $x \in \mathscr{X}$. Endow the set of finite (probability) measures $\mathscr{M}(\Sigma)$ ($\mathscr{P}(\Sigma)$, respectively), with the $C_b(\Sigma)$-topology. Let $\{\mathscr{L}_N\}$ be a sequence of $\mathscr{P}(\Sigma)$-valued random variables, and denote the law of $\mathscr{L}_N$ by $\mathbb{P}_N$. For every $W \in C(\Sigma)$ such that $\mathbb{P}_N$-a.s. $\int (W \vee 0) d\mathscr{L}_N < \infty$ and every $r > 0$, let

$$\Lambda_r(W) = \limsup_{N\to\infty} \frac{1}{N} \log \int e^{N(\int W d\mathscr{L}_N \wedge r)} d\mathbb{P}_N \ , \tag{6.10}$$

with $\bar{\Lambda}(W) := \sup_{r>0} \Lambda_r(W)$. In particular, $\bar{\Lambda}(W)$ is well defined for all $W \in C^{a.b}(\Sigma)$ and $\bar{\Lambda}(W) = \Lambda_\infty(W)$ for any such W.

For c that is bounded with respect to the norm of $\mathscr{X}$, the Bochner integral $\int c d\mu$ is well defined on $\mathscr{P}(\Sigma)$ with $\mu \mapsto \int c d\mu : \mathscr{P}(\Sigma) \to (\mathscr{X}, \|\cdot\|)$ a continuous function. Let $\mathscr{C}_N := \int c d\mathscr{L}_N$ and $F(0) = 0$, while $F(x) = \infty$ for $x \neq 0$. If $(\mathscr{C}_N, \mathscr{L}_N)$ satisfies the LDP in $\mathscr{X} \times \mathscr{P}(\Sigma)$ with a convex GRF $\mathbb{I}(\mathscr{C}, \mu)$, then by the contraction principle (c.f [15, Theorem 4.2.1]), $\{\mathscr{L}_N\}$ satisfies the LDP in $\mathscr{P}(\Sigma)$ with some convex GRF $I_0(\mu)$ such that

$$\mathbb{I}(\mathscr{C}, \mu) = \begin{cases} I_0(\mu) + F(\mathscr{C} - \int c d\mu) & \text{if } I_0(\mu) < \infty, \\ +\infty & \text{otherwise.} \end{cases} \tag{6.11}$$

The main result of this section is the following extension of the identity (6.11), for an appropriate choice of F, to *unbounded* continuous functions c such that $\Lambda_\infty(\eta\|c\|) < \infty$ for some $\eta > 0$. Note that then $\mathbb{E}(\int \|c\| d\mathscr{L}_N) < \infty$ and the Bochner integrals $\mathscr{C}_N = \int c d\mathscr{L}_N$ are (again) well defined $\mathscr{X}$-valued random variables.

Proposition 6.4. *Suppose* $\Lambda_\infty(\eta||c||) < \infty$ *for some* $\eta > 0$, *and that* $(\mathscr{C}_N, \mathscr{L}_N)$ *satisfies the LDP in* $\mathscr{X} \times \mathscr{P}(\Sigma)$ *with a convex GRF* $\mathbb{I}(\mathscr{C}, \mu)$. *Let* $\mathscr{D} := \{\alpha \in \mathscr{Y} : \bar{\Lambda}(\langle \alpha, c\rangle) < \infty\}$ *for a separating vector space* $\mathscr{Y} \subseteq \mathscr{X}^*$, *and*

$$F(x) := \sup_{\alpha \in \mathscr{D}_o} \langle \alpha, x\rangle \ , \qquad x \in \mathscr{X}$$

where $\mathscr{D}_o := \{\alpha \in \mathscr{Y} : \exists p > 1,\ p\alpha \in \mathscr{D}\}$. *Suppose also that for any* $W_n \in C_{\mathscr{D}}(\Sigma) := \{V + \langle \alpha, c\rangle : V \in C_b(\Sigma), \alpha \in \mathscr{D}_o\}$ *such that* $W_n \downarrow W_\infty \in C^{a.b}(\Sigma)$

$$\limsup_{n\to\infty} \bar{\Lambda}(W_n) \le \bar{\Lambda}(W_\infty). \tag{6.12}$$

Then, $\mathbb{I}(\mathscr{C}, \mu)$ *satisfies the identity (6.11) for* $I_0(\cdot)$ *the GRF for the LDP of* $\{\mathscr{L}_N\}$ *in* $\mathscr{P}(\Sigma)$. *In particular,* $\mathbb{I}(\mathscr{C}, \mu) = 0$ *iff* $\mathscr{C} = \int c d\mu$ *and* $I_0(\mu) = 0$.

Remarks.

- Clearly, $F(0) = 0$ and when $\mathscr{D} = \mathscr{Y}$ then $F(x) = \infty$ for every $x \neq 0$. This is indeed the case when $c : \Sigma \to (\mathscr{X}, \|\cdot\|)$ is bounded, or more generally, when $\bar{\Lambda}(\eta||c||) < \infty$ for *all* $\eta < \infty$.
- By Varadhan's lemma, the LDP with GRF for $(\mathscr{C}_N, \mathscr{L}_N)$ implies the existence of limits in (6.10) for $W = V + \langle \alpha, c\rangle$ and the convexity of $(\alpha, V) \mapsto \bar{\Lambda}(V + \langle \alpha, c\rangle)$. Moreover, then necessarily $\bar{\Lambda}(V) = \Lambda_\infty(V)$ is finite for every $V \in C_b(\Sigma)$, with

$$I_0(\mu) = \sup_{V\in C_b(\Sigma)} \left\{ \int V d\mu - \bar{\Lambda}(V) \right\} \tag{6.13}$$

(see [15, Lemma 4.1.5(a) and Theorem 4.5.10], noting that $\int V d\mathscr{L}_N$ is bounded in N).
- It is not clear from (6.11) that $\mathbb{I}(\cdot)$ is lower semi-continuous. In fact, $(\mathscr{C}, \mu) \mapsto F(\mathscr{C} - \int c d\mu)$ might be upper semi-continuous but not lower semi-continuous (for such an example, see Section 6.3). However, it is well known that

$$\mathbb{I}(\mathscr{C}, \mu) = \sup_{\alpha\in\mathscr{D}_o} \sup_{V\in C_b(\Sigma)} \left\{ \langle \alpha, \mathscr{C}\rangle + \int V d\mu - \bar{\Lambda}(V + \langle \alpha, c\rangle) \right\}. \tag{6.14}$$

Recall that $\mathbb{I}(\mathscr{C}, \mu)$ is also the convex GRF for the LDP in the vector space $\mathscr{X} \times \mathscr{M}(\Sigma)$ endowed with the coarser $\mathscr{Y} \times C_b(\Sigma)$-topology. Thus, to get (6.14) apply for example [16, Theorem 3.1] in the latter space, noting that $\bar{\Lambda}(V + \langle \alpha, c\rangle) < \infty$ iff $\alpha \in \mathscr{D}$ regardless of the value of $V \in C_b(\Sigma)$. In the original formula, the supremum over α is achieved on the whole set $\mathscr{D}$. However, $t\alpha \in \mathscr{D}_o$ for any $\alpha \in \mathscr{D}$ and $t \in [0, 1)$. The convexity of $t \mapsto h(t) := \bar{\Lambda}(tV + \langle t\alpha, c\rangle) : [0, 1] \to \mathbb{R}$ implies that $\lim_{t\uparrow 1} h(t) \le h(1)$. Hence, suffices to consider $\alpha \in \mathscr{D}_o$ in (6.14). Being a supremum of continuous functions, $\mathbb{I}(\cdot)$ is lower semi-continuous. Comparing with (6.11) this is due to some cancellation between the $I_0(\cdot)$ and $F(\cdot)$ terms.
- Suppose $(\mathscr{C}_N, \mathscr{L}_N)$ are exponentially tight in $\mathscr{X} \times \mathscr{P}(\Sigma)$, and that the limit in (6.10) exists for $r = \infty$ and $W = V + \langle \alpha, c\rangle$. If in addition $h(\boldsymbol{\theta}) := \Lambda_\infty(\sum_{i=1}^d \theta_i \langle \alpha_i, c\rangle + \theta_{i+d} V_i)$, is an essentially smooth, lower semicontinuous function that is finite in some neighborhood of 0, for any $\alpha_i \in \mathscr{X}^*$, $V_i \in C_b(\Sigma)$, $d \in \mathbb{N}$, then $(\mathscr{C}_N, \mathscr{L}_N)$ satisfies the LDP with the convex GRF $\mathbb{I}(\mathscr{C}, \mu)$ of (6.14), or alternatively, with $\Lambda_\infty(V + \langle \alpha, c\rangle)$ replacing there $\bar{\Lambda}(V + \langle \alpha, c\rangle)$ (this is a simple adaptation of the proof of [15, Corollary 4.6.14], where the restriction of the LDP to $\mathscr{X} \times \mathscr{P}(\Sigma)$ is by [15, Lemma 4.1.5(b)]).

Proof. By the contraction principle $\mathbb{I}(\mathscr{C}, \mu) \ge I_0(\mu)$ for any $\mathscr{C} \in \mathscr{X}$, $\mu \in \mathscr{P}(\Sigma)$. Thus, hereafter fix without loss of generality $\mu \in \mathscr{P}(\Sigma)$ such that $I_0(\mu) < \infty$. By (6.13), then,

$$\eta \int (\|c\| \wedge M) d\mu \le I_0(\mu) + \bar{\Lambda}(\eta||c||) < \infty$$

for every $M < \infty$, so by monotone convergence theorem we see that $\int \|c\| d\mu < \infty$. Therefore, $\langle \alpha, c\rangle \in L^1(\mu)$ for any $\alpha \in \mathscr{X}^*$, $\int c d\mu$ is well defined as a Bochner integral, and we have the Fubini property that

$$\int \langle \alpha, c\rangle d\mu = \langle \alpha, \int c d\mu\rangle.$$

Consequently, (6.14) becomes

$$\mathbb{I}(\mathscr{C}, \mu) = \sup_{\alpha \in \mathscr{D}_o} \left\{ \langle \alpha, \mathscr{C} - \int c d\mu\rangle + I_{\langle \alpha, c\rangle}(\mu) \right\},$$

where

$$I_g(\mu) := \sup_{V \in C_b(\Sigma)} \{ \int (V + g) d\mu - \bar{\Lambda}(V + g) \}, \tag{6.15}$$

for $g \in L^1(\mu) \cap C_{\mathscr{D}}(\Sigma)$. We establish the identity (6.11) as soon as we show that $I_g(\mu) = I_0(\mu)$ for any such g, that is, when $V + g \in C_{\mathscr{D}}(\Sigma)$ for all $V \in C_b(\Sigma)$. To this end, let

$$\phi_{n,m}(x) = x \mathbb{1}_{x \in (-m,n)} + n \mathbb{1}_{x \geq n} - m \mathbb{1}_{x \leq -m}$$

for $x \in \mathbb{R}$, with $\phi_m := \phi_{\infty,m}$. Then, $\phi_{n,m}(g) \in C_b(\Sigma)$ so for any $n, m \in \mathbb{N}$ and $V \in C_b(\Sigma)$,

$$I_g(\mu) \geq \int (V + g - \phi_{n,m}(g)) d\mu - \bar{\Lambda}(V + g - \phi_{n,m}(g)).$$

Since $g \in L^1(\mu)$, by dominated convergence

$$\lim_{m \to \infty} \lim_{n \to \infty} \int (g - \phi_{n,m}(g)) d\mu = 0 .$$

Applying (6.12) to $W_n := V + g - \phi_{n,m}(g) \downarrow V + g - \phi_m(g) \in C^{a.b}(\Sigma)$, it follows that for every $V \in C_b(\Sigma)$,

$$\limsup_{n \to \infty} \bar{\Lambda}(V + g - \phi_{n,m}(g)) \leq \bar{\Lambda}(V + g - \phi_m(g)) \leq \bar{\Lambda}(V)$$

(recall that $\bar{\Lambda}(W) \leq \bar{\Lambda}(W')$ whenever $W \leq W' \in C^{a.b}(\Sigma)$). Hence, $I_g(\mu) \geq I_0(\mu)$. Similarly, $\phi_{n,m}(V + g) \in C_b(\Sigma)$, so for any $n, m \in \mathbb{N}$ and $V \in C_b(\Sigma)$,

$$I_0(\mu) \geq \int \phi_{n,m}(V + g) d\mu - \bar{\Lambda}(\phi_{n,m}(V + g)) .$$

With $\int \phi_{n,m}(V + g) d\mu \to \int (V + g) d\mu$ by dominated convergence, and applying (6.12) for $W_m = \phi_{n,m}(V + g) \downarrow \phi_{n,\infty}(V + g) \in C^{a.b}(\Sigma)$, we get that

$$\limsup_{m \to \infty} \bar{\Lambda}(\phi_{n,m}(V + g)) \leq \bar{\Lambda}(\phi_{n,\infty}(V + g)) \leq \bar{\Lambda}(V + g) ,$$

and deduce that $I_0(\mu) \geq I_g(\mu)$. This completes the proof of the identity (6.11).

Since $\Lambda_\infty(\eta||c||) < \infty$, the convex set $\mathscr{D}$ contains the intersection of $\mathscr{Y}$ with an open ball centered at the origin. If $x \neq 0$, then $\langle \alpha, x \rangle > 0$ for some $\alpha \in \mathscr{Y}$. Taking $\epsilon > 0$ small enough such that $\epsilon\alpha \in \mathscr{D}_o$ thus results with $F(x) > 0$. Since $F(0) = 0$, by (6.11) we see that $\mathbb{I}(\mathscr{C}, \mu) = 0$ iff $\mathscr{C} = \int c d\mu$ and $I_0(\mu) = 0$. □

The following lemma describes the typical application of Proposition 6.4 in this paper.

Lemma 6.5. *Suppose that* $\Sigma = \mathbb{R} \times \Sigma'$ *and* $\mathscr{L}_N = N^{-1} \sum_{i=1}^N \delta_{\lambda_i, x_i}$ *for* x_i *i.i.d.* Σ'*-valued which are independent of* $\{\lambda_i\}$ *and* $\hat{\sigma}^N := N^{-1} \sum_{i=1}^N \delta_{\lambda_i}$ *satisfies the LDP in* $\mathscr{P}(K)$ *for some* K *compact, with a GRF that is zero at* σ *and* $+\infty$ *otherwise. Let* $\psi_W(\lambda) := \log \mathbb{E}[e^{W(\lambda,\cdot)}]$ *for* $W \in C(\Sigma)$.
(a). If $\psi_{pW}(\cdot)$ *is bounded above on* K *for some* $p > 1$, *then*

$$\Lambda_\infty(W) = \bar{\Lambda}(W) = \widetilde{\Lambda}(W) := \int \psi_W(\lambda) d\sigma(\lambda) \tag{6.16}$$

(b). If $\psi_{\langle \alpha, c \rangle}(\cdot)$ *is bounded above on* K *whenever* $\widetilde{\Lambda}(p\langle \alpha, c \rangle) < \infty$ *for some* $p > 1$, *then* $\mathscr{D}_o = \{\alpha \in \mathscr{Y} : \inf_{p>1} \widetilde{\Lambda}(p\langle \alpha, c \rangle) < \infty\}$, *the identity (6.16) applies to any* $W \in C_{\mathscr{D}}(\Sigma) \cup C^{a.b}(\Sigma)$ *and condition (6.12) holds.*

Remarks.
(a) Lemma 6.5 applies also for non-random $\boldsymbol{\lambda}$ if $\hat{\sigma}^N \to \sigma$ for some $\sigma \in \mathscr{P}(K)$, for all $\delta > 0$ eventually $\{\lambda_i\} \subset K^\delta$ (the closed δ-blowup of K), and the boundedness above of $\psi_{\langle \alpha, c \rangle}(\cdot)$ is established over K^δ for some $\delta = \delta(\alpha, p) > 0$. Indeed, carrying the whole proof in K^δ, the only modification needed is in (6.18) where now $\Lambda_\infty(W) = \limsup_{N\to\infty} \int \psi_W d\hat{\sigma}^N = \widetilde{\Lambda}(W)$.
(b) In the context of Proposition 6.4, part (b) of Lemma 6.5 results with $\widetilde{\Lambda}(\cdot)$ replacing $\bar{\Lambda}(\cdot)$ in (6.14) and with the convex GRF for the LDP of $\mathscr{C}_N$ being

$$\mathbb{I}(\mathscr{C}) = \sup_{\alpha \in \mathscr{D}_o} \left\{ \langle \alpha, \mathscr{C} \rangle - \widetilde{\Lambda}(\langle \alpha, c \rangle) \right\}. \tag{6.17}$$

(c) If Lemma 6.5 applies for $\mathscr{Y} = \mathscr{X}^*$, then the convex set $\mathscr{D}_o$ contains a centered open ball for the operator norm on $\mathscr{X}^*$. Consequently $\mathscr{D}_o$ is in this case the interior of $\{\alpha \in \mathscr{X}^* : \widetilde{\Lambda}(\langle \alpha, c \rangle) < \infty\}$ for the latter topology.

Proof. (a) Let $p > 1$ and $C < \infty$ be such that $\sup_{\theta \in K} \psi_{pW}(\theta) \leq C$. Then, for any $\theta_n \to \lambda \in K$, by Hölder's inequality,

$$\mathbb{E}[e^{W(\theta_n,\cdot)} \mathbb{1}_{W(\theta_n,\cdot) \geq W(\lambda,\cdot)+1}] \leq e^{C/p} \mathbb{P}(W(\theta_n, \cdot) \geq W(\lambda, \cdot) + 1)^{1/q}$$

By continuity of W, we thus see that $\mathbb{E}[e^{W(\theta_n,\cdot)} \mathbb{1}_{W(\theta_n,\cdot) \geq W(\lambda,\cdot)+1}] \to 0$ as $n \to \infty$. Our assumption also implies that $\psi_W(\cdot) \leq \psi_{pW}(\cdot)/p$ is finite throughout K, so by continuity of W and dominated convergence,

$$\mathbb{E}[e^{W(\theta_n,\cdot)} \mathbb{1}_{W(\theta_n,\cdot) \leq W(\lambda,\cdot)+1}] \to_{n\to\infty} \mathbb{E}[e^{W(\lambda,\cdot)}] \,.$$

With $\theta_n \to \lambda \in K$ an arbitrary sequence, by continuity of $y \mapsto \log y$ on $\mathbb{R}^+$ we see that $\psi_W \in C(K)$. Hence, $\psi_W \in C_b(K)$ by compactness of K. As $\psi_{p(W\wedge r)}(\theta) \leq$

$pr < \infty$, the preceding argument applies to $W \wedge r$. Thus, $\psi_{W\wedge r} \in C_b(K)$ for any $r < \infty$. By monotone convergence, $\psi_{W\wedge r} \uparrow \psi_W$ for every $\lambda \in K$. Moreover, $\psi_{W\wedge r} \in C_b(K)$ are uniformly bounded below on the compact K, hence $\widetilde{\Lambda}(W\wedge r) \uparrow \widetilde{\Lambda}(W)$. By definition

$$\Lambda_\infty(W) = \limsup_{N\to\infty} \frac{1}{N} \log \mathbb{E}\big[e^{N\int_K \psi_W(\lambda)d\hat{\sigma}^N(\lambda)}\big] \geq \bar{\Lambda}(W) \geq \lim_{r\to\infty} \bar{\Lambda}(W\wedge r)$$
$$= \lim_{r\to\infty} \Lambda_\infty(W\wedge r)\,. \tag{6.18}$$

Applying Varadhan's lemma for the continuous, bounded mapping $\mu \mapsto \int \psi_W d\mu$ on $\mathscr{P}(K)$ it follows by the trivial LDP for $\hat{\sigma}^N$ that $\Lambda_\infty(W) = \widetilde{\Lambda}(W)$. Likewise, $\psi_{W\wedge r} \in C_b(K)$ results with $\Lambda_\infty(W\wedge r) = \widetilde{\Lambda}(W\wedge r)$ for all $r < \infty$ and (6.16) follows out of (6.18).

(b) The proof of part (a) shows that $\bar{\Lambda}(W) \geq \widetilde{\Lambda}(W)$ for any $W \in C(\Sigma)$, with equality for $W \in C^{a.b}(\Sigma)$ and by our assumption also for $W = V + \langle\alpha, c\rangle$ with α such that $\widetilde{\Lambda}(p^2\langle\alpha, c\rangle) < \infty$ for some $p > 1$. If $\alpha \in \mathscr{D}_o$ then $p^2\alpha \in \mathscr{D}$ for some $p > 1$, so we see that $\bar{\Lambda}(W) = \widetilde{\Lambda}(W)$ for all $W \in C_{\mathscr{D}}(\Sigma) \cup C^{a.b}(\Sigma)$. Fix $W_n \in C_{\mathscr{D}}(\Sigma)$ such that $W_n \downarrow W_\infty \in C^{a.b}(\Sigma)$. Then, $\widetilde{\Lambda}(W_1) < \infty$, implying that $E(e^{W_1(\lambda,\cdot)}) < \infty$ for σ-a.e. λ. This immediately gives $\psi_{W_n} \downarrow \psi_{W_\infty}$ by dominated convergence. As $\int(\psi_{W_1}\vee 0)d\sigma < \infty$, by dominated convergence $\int(\psi_{W_n}\vee 0)d\sigma \downarrow \int(\psi_{W_\infty}\vee 0)d\sigma$. Since $\int(\psi_{W_n}\wedge 0)d\sigma \downarrow \int(\psi_{W_\infty}\wedge 0)d\sigma$ by monotone convergence, we now deduce that (6.12) holds for any such $W_n \downarrow W_\infty$. We have already seen that $\inf_{p>1}\widetilde{\Lambda}(p\langle\alpha, c\rangle) < \infty$ whenever $\alpha \in \mathscr{D}_o$. Moreover, if $\widetilde{\Lambda}(p^3\langle\alpha, c\rangle) < \infty$ for some $p > 1$ then by our assumption and (6.16) it follows that $p\alpha \in \mathscr{D}$, that is $\alpha \in \mathscr{D}_o$. □

6.3. LDP for the uniform law on the sphere

We wish to stress a simple corollary of our work, namely, the LDP for the empirical measure under the uniform law $s_N^{\sqrt{N}}$ on the sphere S^{N-1} of radius $\sqrt{N}$. This is a direct consequence of the strategy of Sections 5.3 and 6.2.

Theorem 6.6. *The law of the empirical measure under $s_N^{\sqrt{N}}$ satisfies a LDP of speed N and the GRF $H_1(\cdot)$, where*

$$H_u(\mu) = \begin{cases} I(\mu|\gamma) + \frac{1}{2}(1 - \int x^2 d\mu(x) + \log u) & \text{if } \int x^2 d\mu(x) \leq u \\ +\infty & \text{otherwise.}\end{cases} \tag{6.19}$$

Proof. Fix $u > 0$ and let $\boldsymbol{g}$ be of law $\gamma^{\otimes N}$. By Cramèr's theorem, $(u^{-1}N^{-1}|\boldsymbol{g}|^2, N^{-1}\sum_{i=1}^N \delta_{g_i})$ satisfies the LDP in $\mathbb{R} \times \mathscr{P}(\mathbb{R})$ with some convex GRF. In this setting Lemma 6.5 trivially applies, and it is easy to check that the formula (6.11) for this GRF is then

$$\mathbb{I}_u(r, \mu) = \begin{cases} I(\mu|\gamma) + \frac{1}{2}(ru - \int x^2 d\mu) & \text{if } \int x^2 d\mu \leq ru, \\ +\infty & \text{otherwise.}\end{cases}$$

In Section 5.3 we provided a representation of the empirical measure under $s_N^{\sqrt{N}}$ as the contraction $S : \mathbb{R}^+ \times \mathscr{P}(\mathbb{R}) \to \mathscr{P}(\mathbb{R})$ of $(N^{-1}|\boldsymbol{g}|^2, N^{-1}\sum_{i=1}^N \delta_{g_i})$

for $S(r, \mu(\cdot)) = \mu(\sqrt{r}\,\cdot)$. It is not hard to check that $H_u(\mu) = \inf_{(r,\widetilde{\mu}) \in S^{-1}(\mu)} \mathbb{I}_u(r, \widetilde{\mu})$. □

Remark. This proof shows also that the LDP for the empirical measure of $\boldsymbol{v}_0 = \boldsymbol{G}\boldsymbol{u}_0$ of (IC2) has the GRF

$$H(\mu) = \begin{cases} \kappa(0) & \text{if } \mu = \delta_0 \\ \inf\limits_{u>0} \{\kappa(u) + H_u(\mu)\} & \text{otherwise.} \end{cases}$$

6.4. *Strong solutions of the SDS (1.1)*

The next lemma provides the existence and uniqueness of strong solutions of the SDS (1.1), hence also of the diagonalized SDS (2.2).

Lemma 6.7. *If f' is globally Lipschitz and bounded from below, then for any integer N, $\beta \in (0, \infty]$, $T < \infty$, any symmetric $N \times N$ matrix $\mathbf{J}$, and any given initial condition $(u_0^i)_{1 \le i \le N} \in \mathbb{R}^N$, which is independent of $\{B_t^i\}$, the SDS (1.1) admits a unique strong solution on $\mathscr{C}([0, T], \mathbb{R}^N)$.*

Proof. Since f' is globally Lipschitz, for any $M < \infty$ it is easy to check that $b_i(\mathbf{u}) = (\mathbf{J}\mathbf{u})_i - f'(N^{-1}|\mathbf{u}|^2 \wedge M)u^i$ results with a globally Lipschitz drift $b(\mathbf{u}) = (b_1(\mathbf{u}), \ldots, b_N(\mathbf{u}))$. The existence and uniqueness of a square-integrable strong solution $\mathbf{u}^{(M)}$ for the SDS

$$du_t^i = \sum_{j=1}^{N} J_{ij} u_t^j dt - f'\left(N^{-1}|\mathbf{u}_t|^2 \wedge M\right) u_t^i dt + \beta^{-1/2} dB_t^i \tag{6.20}$$

is thus standard (for example, see [19, Theorems 5.2.5, 5.2.9]). With $\mathbf{u}^{(M)}$ defined on the same probability space and filtration, consider the stopping times $\tau_M = \inf\{t : |\mathbf{u}_t^{(M)}| \ge \sqrt{NM}\}$. Note that $\mathbf{u}^{(M)}$ is the unique strong solution of (1.1) for $t \in [0, \tau_M]$, with τ_M a non-decreasing sequence. By the Borel-Cantelli lemma, it suffices for the existence of a unique strong solution $\mathbf{u} = \lim_{M \to \infty} \mathbf{u}^{(M)}$ of the SDS (1.1) in $[0, T]$, to show that

$$\sum_{M=1}^{\infty} \mathbb{P}\left(\tau_M \le T\right) < \infty. \tag{6.21}$$

To this end, fix M and let $\mathbf{x}_t = \mathbf{u}_{t \wedge \tau_M}^{(M)}$ with $Z_s = 2\beta^{-1/2} \int_0^{s \wedge \tau_M} \sum_{i=1}^N x_t^i dB_t^i$. Applying Ito's formula for $g_t = |\mathbf{x}_t|^2$ insures that

$$\begin{aligned} g_s &\le g_0 + 2\int_0^{s\wedge\tau_M} \langle \mathbf{J}\mathbf{x}_t, \mathbf{x}_t \rangle dt - 2\int_0^{s\wedge\tau_M} f'(N^{-1}g_t) g_t dt + Z_s + \beta^{-1} N s \\ &\le g_0 + \beta^{-1} N s + 2(\lambda_N^* + c) \int_0^s g_t dt + Z_s \end{aligned} \tag{6.22}$$

where we have used the lower bound $f' \geq -c$ for some $c \in \mathbb{R}$ and denoted by $\lambda_N^* \geq 0$ the spectral radius of $\mathbf{J}$. As the quadratic variation of the martingale Z_s is $(4/\beta)\int_0^{s\wedge\tau_M} g_t dt \leq 4\beta^{-1}s\sqrt{NM}$, applying Doob's inequality for the martingale $L_s = \exp(Z_s - (2/\beta)\int_0^{s\wedge\tau_M} g_t dt)$, yields for any $A > 0$,

$$\mathbb{P}\left(\sup_{s\leq T}\left\{Z_s - (2/\beta)\int_0^s g_t dt\right\} \geq A\right) \leq \mathbb{P}\left(\sup_{s\leq T} L_s \geq e^A\right) \leq e^{-A}$$

Therefore, (6.22) shows that with probability greater than $1 - e^{-A}$, for any $t \leq T$,

$$g_t \leq g_0 + \beta^{-1}NT + A + 2(\beta^{-1} + \lambda_N^* + c)\int_0^t g_s ds \ ,$$

and by Gronwall's lemma then also

$$\sup_{t\leq T} |\mathbf{u}_{t\wedge\tau_M}^{(M)}|^2 \leq (|\mathbf{u}_0|^2 + \beta^{-1}NT + A)e^{2(\beta^{-1}+\lambda_N^*+c)T}. \tag{6.23}$$

For large enough M one may set $A = A(M) > 0$ such that the right-side of (6.23) is $NM/2$, resulting with

$$\mathbb{P}(\tau_M \leq T) \leq e^{-A(M)}.$$

Since $A(M) \geq \eta M$ for some $\eta > 0$ and all large M, this is enough to give (6.21). □

7. Proofs of Theorems 3.1, 3.4 and Lemma 3.3

Proof of Theorem 3.1. By (2.5)–(2.8) we have that for any $\beta \in (0, \infty)$,

$$F_\beta = \sup_{u,v}\inf_{\rho,\alpha}\{(\beta - \alpha)v + (\frac{1}{2} - \rho)u - \frac{c\beta}{2}u^2 + L(\rho, \alpha)\} + \frac{1}{2}\log(2\pi). \tag{7.1}$$

The value of F_β is the one obtained by exchanging the supremum and the infimum in the above formula. Even though the Min-Max theorem does not apply directly, one may prove its conclusion by cutting wisely the sets over which the extrema are taken. We provide here a different argument, which though less transparent, is much shorter and more elegant. This argument is based on the formula (4.9) that represents the free energy F_β in terms of an extremum involving the extra parameter $\mu \in \mathcal{P}(\mathbb{R})$.

Setting $\alpha = \beta$ and $\rho = \frac{1}{2} - \beta s$ for fixed $s \geq \lambda^*$ leads to

$$F_\beta \leq \sup_u(s\beta u - \frac{c\beta}{2}u^2) - \frac{1}{2}\int \log(s - \lambda)d\sigma(\lambda) + \frac{1}{2}\log(\pi\beta^{-1}).$$

Hence, with optimal $u = s/c$ we see that $F_\beta \leq G_\beta$ where

$$G_\beta := \frac{1}{2}\inf_{s\geq\lambda^*}\{\frac{\beta s^2}{c} - \int \log(s - \lambda)d\sigma(\lambda)\} + \frac{1}{2}\log(\pi\beta^{-1}). \tag{7.2}$$

Since $p(s,\beta)$ is strictly increasing in $s \ge \lambda^*$ and in $\beta \ge 0$ with $p(s,0) < 0$ for all $s \in [\lambda^*, \infty)$, we find the following two cases depending upon the existence of a (unique) $s = s_\beta > \lambda^*$ so that

$$\frac{2\beta}{c} s = \mathbf{L}(s) \,. \tag{7.3}$$

• For $\beta \in (0, \beta_c)$ there exists $s_\beta > \lambda^*$ satisfying (7.3). Such s_β is unique by strict monotonicity of $s \mapsto p(s,\beta)$. Note that the infimum in (7.2) is attained where the infimum of $\int_{\lambda^*}^s p(t,\beta)dt$ is attained, that is at $s = s_\beta$.
• For $\beta \ge \beta_c$ there is no solution $s > \lambda^*$ of (7.3). Then, $\int_{\lambda^*}^s p(t,\beta)dt > 0$ for all $s > \lambda^*$ and the infimum in (7.2) is attained at $s_\beta := \lambda^*$.

In view of the above, it suffices to show that $F_\beta \ge G_\beta$ in order to establish (3.4). To this end, set $u = s_\beta / c$, $v = u s_\beta - 1/(2\beta)$ and $\mu = \mu^{\rho,\alpha}$ of (4.11) for $\alpha = \beta$ and $\rho = \frac{1}{2} - \beta s_\beta$. Obviously, $(s_\beta - \lambda^*)\mathbf{L}(s_\beta) \le 1$, and with $\sigma(\cdot)$ symmetric, it follows that

$$\frac{2\beta s_\beta^2}{c} \ge s_\beta \mathbf{L}(s_\beta) \ge 1 \,, \tag{7.4}$$

implying that $\lambda^* u \ge v \ge 0$. Moreover, (7.4) and the choice of s_β result with

$$u = \frac{s_\beta}{c} \ge \frac{1}{2\beta}\mathbf{L}(s_\beta) + \frac{s_\beta}{\lambda^*} |u - \frac{1}{2\beta}\mathbf{L}(s_\beta)| \,.$$

The latter inequality amounts to the condition that $\mu \in \mathscr{A}(u,v)$ of (2.12). Consequently, by (4.9), (2.11), (4.12), (2.5) and (2.7) we see that

$$\begin{aligned} F_\beta &\ge h(u,v) - \mathscr{K}(u,v,\mu) = h(u,v) + \frac{1}{2}(1-u) + L(\rho,\alpha) \\ &= \beta v + \frac{1}{2} - \frac{\beta c u^2}{2} + \frac{1}{2}\log(2\pi) - \frac{1}{2}\int \log(1 - 2(\alpha\lambda + \rho)) d\sigma(\lambda) \\ &= \frac{\beta s_\beta^2}{2c} - \frac{1}{2}\int \log(s_\beta - \lambda) d\sigma(\lambda) + \frac{1}{2}\log(\pi\beta^{-1}), \end{aligned}$$

as needed to complete the proof of (3.4).

Recall that by Theorem 2.1,

$$\nu_0^* = \int \gamma_{1-2(\alpha_\beta\lambda + \rho_\beta)} d\sigma(\lambda) = \int \gamma_{2\beta(s_\beta - \lambda)} d\sigma(\lambda).$$

Thus, for each $\beta \in (0, \beta_c)$, ν_0^* is a mixture of Gaussian laws with uniformly bounded variances, so ν_0^* itself has a sub-Gaussian tail. For any $\beta \ge \beta_c$, the $2k$-th moment of ν_0^* is $c_{k,\beta}\int(\lambda^* - \lambda)^{-k} d\sigma(\lambda)$, for some finite, positive constant $c_{k,\beta}$. To conclude the proof, observe that $\int (\lambda^* - \lambda)^{-(k+1)} d\sigma(\lambda) = \infty$ whenever $|\mathbf{L}^{(k)}(s)| \to \infty$ for $s \downarrow \lambda^*$ (see (3.1)). □

The proof of Lemma 3.3 is reminiscent of the approach of Wong and Wong [30, 31]. Since $\sigma(\cdot)$ is supported on the compact set $[-\lambda^*, \lambda^*]$, the Stieljes transform $\mathbf{L}(\cdot)$ of (3.1) has an analytic continuation

$$\mathbf{L}(z) := \int \frac{1}{z-\lambda} d\sigma(\lambda) \tag{7.5}$$

to $z \in \mathbb{C} \setminus [-\lambda^*, \lambda^*]$, which for $\Re\{z\} > \lambda^*$ may also be expressed as

$$\mathbf{L}(z) = \int_0^\infty e^{-z\theta} \mathscr{L}(\theta) d\theta \ . \tag{7.6}$$

The first step in our analysis is to study the asymptotics of $\mathbf{L}(\lambda^* + z)$ in the sectors

$$S_\theta := \left\{ z \neq 0 \ : \ |\arg(z)| < \frac{\pi}{2} + \theta \right\} \ , \tag{7.7}$$

of $\mathbb{C}$, for $|z| \downarrow 0$, as summarized in the next lemma.

Lemma 7.1. *Fixing $\theta \in (0, \frac{\pi}{2})$, for all $k < q - 1$,*

$$\limsup_{z \in S_\theta, |z| \downarrow 0} |\mathbf{L}^{(k)}(\lambda^* + z) - \mathbf{L}^{(k)}(\lambda^*)| = 0, \tag{7.8}$$

where

$$(-1)^k \mathbf{L}^{(k)}(\lambda^*) = \int_0^\infty e^{-\lambda^* \theta} \theta^k \mathscr{L}(\theta) d\theta < \infty \tag{7.9}$$

while for $k = n = [q]$ and $b_2 = (-1)^n b_1 \Gamma(n + 1 - q) \neq 0$,

$$\limsup_{z \in S_\theta, |z| \downarrow 0} |z^{n+1-q} \mathbf{L}^{(n)}(\lambda^* + z) - b_2| = 0 \ , \tag{7.10}$$

and in case $q = n$ is integer also,

$$\limsup_{z \in S_\theta, |z| \downarrow 0} |z \mathbf{L}^{(n-1)}(\lambda^* + z)| = 0 \ . \tag{7.11}$$

Proof of Lemma 7.1. From (7.5) it follows that for all $z \in \mathbb{C} \setminus [-\lambda^*, \lambda^*]$, $k = 1, \dots, n = [q]$,

$$\mathbf{L}^{(k)}(z) = (-1)^k k! \int (z - \lambda)^{-(k+1)} d\sigma(\lambda) \ . \tag{7.12}$$

When $\Re\{z\} > \lambda^*$, it follows from (7.6) that

$$\mathbf{L}^{(k)}(z) = (-1)^k \int_0^\infty e^{-z\theta} \theta^k \mathscr{L}(\theta) d\theta$$

which by monotone convergence (for $z = s \in \mathbb{R}$ such that $s \downarrow \lambda^*$), and (3.8) yields (7.9). In case $q = n$ is an integer, (3.8) similarly implies that

$$\lim_{s \downarrow 0} |s \mathbf{L}^{(n-1)}(\lambda^* + s)| = 0 \ . \tag{7.13}$$

Fixing hereafter $\theta \in (0, \frac{\pi}{2})$, let $\kappa = \sqrt{(1-\sin\theta)/2} > 0$. Note that for all $z \in S_\theta$ and $\lambda \in [-\lambda^*, \lambda^*]$,

$$|z + \lambda^* - \lambda| \geq \kappa(|z| + \lambda^* - \lambda) \; . \tag{7.14}$$

Combining (7.12) (for $k = n-1$) and (7.14) it follows that $|\mathbf{L}^{(n-1)}(\lambda^* + z)| \leq \kappa^{-n}|\mathbf{L}^{(n-1)}(\lambda^* + |z|)|$ for all $z \in S_\theta$. Hence, (7.13) implies (7.11) in case q is an integer.

Fixing next $k < q - 1$, let

$$h_z(\lambda) := |(z + \lambda^* - \lambda)^{-(k+1)} - (\lambda^* - \lambda)^{-(k+1)}| \, ,$$

and note that by (7.14), $h_z(\lambda) \leq C_0(\lambda^* - \lambda)^{-(k+1)}$ for some $C_0 = C_0(k, \kappa) < \infty$, all $z \in S_\theta$ and all $\lambda \in [-\lambda^*, \lambda^*]$. Moreover, it is easy to check that $h_z(\lambda) \leq \delta(\lambda^* - \lambda)^{-(k+1)}$ whenever $\lambda^* - \lambda \geq M|z|$, for all $\delta > 0$ and some $M = M(\delta) < \infty$ independent of λ and z. Hence, by (7.12), for all $z \in S_\theta$ and $\delta > 0$,

$$\begin{aligned} |\mathbf{L}^{(k)}(\lambda^* + z) - \mathbf{L}^{(k)}(\lambda^*)| &\leq k! \int h_z(\lambda) d\sigma(\lambda) \\ &\leq \delta |\mathbf{L}^{(k)}(\lambda^*)| + C_0 k! \int_{(\lambda^* - \lambda) \leq M|z|} (\lambda^* - \lambda)^{-(k+1)} d\sigma(\lambda) \; . \end{aligned}$$

By (7.9) and (7.12) we know that $\int (\lambda^* - \lambda)^{-(k+1)} d\sigma(\lambda) < \infty$, whereas $\sigma([\lambda^* - M|z|, \lambda^*]) \to 0$ as $|z| \to 0$ (by (3.9)). It thus follows that

$$\limsup_{z \in S_\theta, |z| \downarrow 0} |\mathbf{L}^{(k)}(\lambda^* + z) - \mathbf{L}^{(k)}(\lambda^*)| \leq \delta |\mathbf{L}^{(k)}(\lambda^*)| \, ,$$

leading to (7.8) when taking $\delta \downarrow 0$.

Turning to prove (7.10), integration by parts results for $z \in \mathbb{C} \setminus \mathbb{R}^-$ with

$$\begin{aligned} &\int_{-\lambda^*}^{\lambda^*} (z + \lambda^* - \lambda)^{-(n+1)} d\sigma(\lambda) \\ &\quad = (z + 2\lambda^*)^{-(n+1)} + (n+1) \int_0^{2\lambda^*} (z + x)^{-(n+2)} \sigma([\lambda^* - x, \lambda^*]) dx \; , \end{aligned} \tag{7.15}$$

and by a change of variable, for all $\delta > 0$, then

$$z^{n+1-q} \int_0^\delta (z + x)^{-(n+2)} x^q dx = \int_{\Gamma_{0,w}} (1 + \xi)^{-(n+2)} \xi^q d\xi := -A_{w,0} \, ,$$

where $w := \delta z^{-1}$ and $\Gamma_{v,w} := \{(1-x)v + xw : x \in [0,1]\}$ is the line segment connecting $v \in \mathbb{C}$ to $w \in \mathbb{C}$. Take $r = |w| \in \mathbb{R}^+$, $r' \in (0, r)$ and $w' = (r'/r)w$. Noting that $h(\xi) := (1+\xi)^{-(n+2)}\xi^q$ is analytic inside the trapezoid connecting r', r, w and w', Cauchy's formula yields

$$A_{r',r} + A_{r,w} + A_{w,w'} + A_{w',r'} = 0 \, ,$$

where $A_{v,w} := \int_{\Gamma_{v,w}} h(\xi)d\xi$. Since $n+1 > q$, it follows by Euler's integral of the first kind, that

$$\lim_{r\uparrow\infty, r'\downarrow 0} A_{r',r} = \frac{\Gamma(q+1)\Gamma(n+1-q)}{\Gamma(n+2)} \in (0,\infty)\,.$$

With $\eta = \arg(w)$ it is easy to check that

$$|A_{r,w}| \le 2r^{1+q}(r\cos(\eta/2)-1)^{-(n+2)} \to_{r\to\infty} 0\,,$$

uniformly in $|\eta| < \pi/2+\theta$. Similarly, $|A_{w',r'}| \le 2(r')^{1+q}(1-r')^{-(n+2)} \to 0$ as $r' \to 0$. It thus follows that for any fixed $\delta > 0$,

$$\limsup_{|z|\to 0, z\in S_\theta} |z^{n+1-q}\int_0^\delta (z+x)^{-(n+2)}x^q dx - \frac{\Gamma(q+1)\Gamma(n+1-q)}{\Gamma(n+2)}| = 0\,. \tag{7.16}$$

By (3.9), for any $\delta' > 0$ there exists $\delta > 0$ small enough such that

$$|\int_0^\delta (z+x)^{-(n+2)}\{\sigma([\lambda^*-x,\lambda^*]) - \frac{b_1 x^q}{\Gamma(q+1)}\}dx| \le \delta' \int_0^\delta |z+x|^{-(n+2)}x^q dx\,, \tag{7.17}$$

where for all $z \in S_\theta$, by (7.14),

$$|z|^{n+1-q}\int_0^\delta |z+x|^{-(n+2)}x^q dx \le \kappa^{-(n+2)}\int_0^\infty (1+s)^{-(n+2)}s^q ds < \infty\,. \tag{7.18}$$

Since for any fixed $\delta > 0$,

$$\limsup_{|z|\to 0} |z|^{n+1-q}\int_\delta^{2\lambda^*} |z+x|^{-(n+2)}dx = 0 \tag{7.19}$$

it thus follows from (7.16)–(7.19) that

$$\limsup_{|z|\to 0, z\in S_\theta} |z^{n+1-q}\int_0^{2\lambda^*} (z+x)^{-(n+2)}\sigma([\lambda^*-x,\lambda^*])dx - \frac{b_1\Gamma(n+1-q)}{\Gamma(n+2)}| = 0\,.$$

By (7.12) and (7.15) we then easily establish (7.10). □

We state and prove next a direct inversion lemma for Laplace transforms.

Lemma 7.2. *Suppose that the (one-sided) Laplace transform*

$$\mathbf{f}(z) := \int_0^\infty e^{-zx} f(x)dx$$

of an absolutely integrable, continuous function $f(x)$*, defined for* $\Re\{z\} > 0$*, has an analytic continuation in* S_θ *of (7.7) for some* $\theta \in (0, \frac{\pi}{2})$ *and is such that* $|\mathbf{f}(z)| \to 0$ *when* $|z| \to \infty$ *for* $z \in S_\theta$*. If for some* $\xi \in \mathbb{C}$ *and* $r \in (0, \infty)$*,*

$$\limsup_{z \in S_\theta, |z| \downarrow 0} |z^r \mathbf{f}(z) - \xi| = 0 \,, \tag{7.20}$$

then,

$$\limsup_{x \uparrow \infty} |x^{1-r} f(x) - \frac{\xi}{\Gamma(r)}| = 0. \tag{7.21}$$

Proof of Lemma 7.2. This proof is an adaptation of [30, proof of Theorem 1]. With $f(x)$ continuous and absolutely integrable, we have from the Laplace inversion formula that for all $x \geq 0$,

$$f(x) = \frac{1}{2\pi i} \int_\Gamma e^{zx} \mathbf{f}(z) dz \,, \tag{7.22}$$

where $\Gamma = \{z : \Re\{z\} = s_0 > 0\}$. Since $\mathbf{f}(z)$ is in fact analytic in S_θ and $|\mathbf{f}(z)| \to 0$ when $|z| \to \infty$, Cauchy's theorem implies that we can replace Γ by the contour

$$\Gamma_{\rho,\eta} = D_{\rho,\eta} \cup C_{\rho,\eta} \tag{7.23}$$

for any $(\rho, \eta) \in (0, \infty) \times (0, \theta)$, where

$$\begin{aligned} D_{\rho,\eta} &= \left\{ \arg(z) = \frac{\pi}{2} + \eta, |z| \geq \rho \right\} \cup \left\{ \arg(z) = -\frac{\pi}{2} - \eta, |z| \geq \rho \right\} \\ C_{\rho,\eta} &= \left\{ |z| = \rho, \arg(z) \in \left(-\frac{\pi}{2} - \eta, \frac{\pi}{2} + \eta \right) \right\}. \end{aligned}$$

It follows by Hankel's integral representation of the gamma function and a change of variable, that for any loop Γ around the negative axis, and any $r > 0$, $x > 0$,

$$\frac{1}{2\pi i} \int_\Gamma z^{-r} e^{zx} dz = \frac{x^{r-1}}{\Gamma(r)} \,.$$

Let $\boldsymbol{\zeta}(z) := \mathbf{f}(z) - \xi z^{-r}$ for $\xi \in \mathbb{C}$ and $r \in (0, \infty)$ of (7.20). Then, by (7.22),

$$x^{1-r} f(x) - \frac{\xi}{\Gamma(r)} = \frac{x^{1-r}}{2\pi i} \int_{\Gamma_{\rho,\eta}} \boldsymbol{\zeta}(z) e^{zx} dz \,, \tag{7.24}$$

where $\Gamma_{\rho,\eta}$ is any of the contours of (7.23). Note that $\boldsymbol{\zeta}(z)$ is also analytic on S_θ with $|\boldsymbol{\zeta}(z)| \to 0$ as $|z| \to \infty$, $z \in S_\theta$. Fixing $\eta \in (0, \theta)$ it thus follows that $\sup_{z \in D_{\rho,\eta}} |\boldsymbol{\zeta}(z)|$ is finite for any $\rho > 0$. Moreover, by (7.20), $\sup_{z \in D_{\rho,\eta}} |\boldsymbol{\zeta}(z)| \leq \delta \rho^{-r}$ for all $\delta > 0$ and $\rho \leq \rho_1(\delta)$. Consequently, for all $\rho \leq \rho_1$

$$\begin{aligned} \left| \int_{D_{\rho,\eta}} \boldsymbol{\zeta}(z) e^{zx} dz \right| &\leq \int_{D_{\rho,\eta}} |\boldsymbol{\zeta}(z)| e^{x \Re\{z\}} d|z| \\ &\leq 2\delta \rho^{-r} \int_\rho^\infty e^{-xy \sin \eta} dy \leq 2\delta \rho^{-r} (x \sin \eta)^{-1} \,, \end{aligned} \tag{7.25}$$

and likewise,

$$\Big|\int_{C_{\rho,\eta}} \zeta(z)e^{zx}dz\Big| \le 2\pi\delta\rho^{1-r}e^{\rho x} . \tag{7.26}$$

Choosing $\rho = x^{-1}$ it follows from (7.24), (7.25) and (7.26) that

$$|x^{1-r}f(x) - \frac{\xi}{\Gamma(r)}| \le \delta C$$

for some $C = C(\eta) < \infty$, all $\delta > 0$ and any $x \ge x_0(\delta)$. Taking $\delta \downarrow 0$ we establish (7.21). □

Proof of Lemma 3.3. For all $\tau \ge 0$ let

$$g(\tau) = e^{-2s_\beta\tau}R(\tau) .$$

With $f'(x) = cx$ it follows from (3.5) and (3.6) that $R(t)$ satisfies the linear Volterra integrodifferential equation,

$$R'(t) = 2cK_d(t)R(t) = 2c\mathscr{L}(2t) + 2c\beta^{-1}\int_0^t R(\tau)\mathscr{L}(2(t-\tau))d\tau . \tag{7.27}$$

By Fubini's theorem and integration by parts, it follows that for all $s \ge 0$,

$$2(s_\beta + s)\int_0^T e^{-2s\tau}g(\tau)d\tau - 1 \le c\mathbf{L}(s_\beta + s)(1 + \beta^{-1}\int_0^T e^{-2s\tau}g(\tau)d\tau) . \tag{7.28}$$

Recall that $\mathbf{L}(s_\beta + s) < \infty$ and $p(s_\beta + s, \beta)$ of (3.3) is strictly positive for all $s > 0$. Therefore, by (7.28), the (one-sided) Laplace transform

$$\boldsymbol{g}(z) := \int_0^\infty e^{-2z\tau}g(\tau)d\tau \tag{7.29}$$

of $g(\tau) \ge 0$ converges absolutely whenever $s = \Re\{z\} > 0$. Hence, considering the (one-sided) Laplace transform of (7.27) for $\Re\{z\} > 0$, leads by (7.6), Fubini's theorem and integration by parts to

$$2(s_\beta + z)\boldsymbol{g}(z) - 1 = c\mathbf{L}(s_\beta + z)(1 + \beta^{-1}\boldsymbol{g}(z)) . \tag{7.30}$$

Note that for all $s \ge \lambda^*$ and $w \neq 0$,

$$\Re\{\mathbf{L}(s + iw)\} = \int_{-\lambda^*}^{\lambda^*} \frac{s-\lambda}{(s-\lambda)^2 + w^2}d\sigma(\lambda) < \mathbf{L}(s) .$$

Hence, $\Re\{p(s_\beta + z, \beta)\} > 0$ whenever $\Re\{z\} > 0$, in which case by (7.30),

$$\boldsymbol{g}(z) = \frac{\beta(c\mathbf{L}(s_\beta + z) + 1)}{cp(s_\beta + z, \beta)} . \tag{7.31}$$

Note that $|\mathbf{L}(z)| \to 0$ as $|z| \to \infty$, so that $p(s_\beta + z, \beta) = \frac{2\beta}{c}(s_\beta + z) - \mathbf{L}(s_\beta + z)$ does not vanish for large values of $|z|$. Furthermore, $p(s_\beta + z, \beta)$ is analytic in $\mathbb{C} \setminus \mathbb{R}^-$ and has no zeros in the right half plane $\Re\{z\} \geq 0$ except possibly in $z = 0$. Since an analytic function can have only a finite number of zeros in any compact subset of the complex plane, there must exist a $\theta \in (0, \pi/2)$ so that $p(s_\beta + z, \beta)$ does not vanish in the domain S_θ of (7.7). Fixing hereafter this value of θ, we have that $\boldsymbol{g}(z)$ of (7.31) is well defined for any $z \in S_\theta$ where it is an analytic function, such that $|\boldsymbol{g}(z)| \to 0$ as $|z| \to \infty, z \in S_\theta$. With $\mathbf{L}^{(k)}(s_\beta + z), k = 1, 2, \ldots, n = [q]$ also analytic on $\mathbb{C} \setminus \mathbb{R}^-$ and $|\mathbf{L}^{(k)}(z)| \to 0$ as $|z| \to \infty$, it follows by the same argument that

$$\boldsymbol{g}^{(n)}(z) = \int_0^\infty e^{-2z\tau}(-2\tau)^n g(\tau) d\tau \tag{7.32}$$

converges absolutely whenever $s = \Re\{z\} > 0$ and has an analytic continuation to S_θ, given by

$$\boldsymbol{g}^{(n)}(z) = \frac{\beta}{c} \frac{d^n}{dz^n} \left\{ \frac{c\mathbf{L}(s_\beta + z) + 1}{p(s_\beta + z, \beta)} \right\} ,$$

such that $|\boldsymbol{g}^{(n)}(z)| \to 0$ as $|z| \to \infty, z \in S_\theta$.

We shall next verify that $\boldsymbol{g}(z)$ satisfies (7.20) for some $\xi_{q,\beta} \neq 0$ and $r_{q,\beta} \in (0, \infty)$ when $\beta \in (0, \beta_c]$. Lemma 7.2 then applies to $\mathbf{f} = \boldsymbol{g}$, resulting by (7.29) with

$$x^{1-r_{q,\beta}} g(x) \sim_{x\uparrow\infty} C_{q,\beta} := \frac{2^{r_{q,\beta}} \xi_{q,\beta}}{\Gamma(r_{q,\beta})} \tag{7.33}$$

We shall also verify that $\boldsymbol{g}^{(n)}(z)$ satisfies (7.20) for some $\xi_{q,\beta} \neq 0$ and $r_{q,\beta} \in (0, \infty)$ when $\beta > \beta_c$. Lemma 7.2 then applies to $\mathbf{f} = \boldsymbol{g}^{(n)}$, resulting by (7.32) with

$$x^{1-r_{q,\beta}}(-2x)^n g(x) \sim_{x\uparrow\infty} C_{q,\beta} . \tag{7.34}$$

The statement of Lemma 3.3 is exactly (7.33)–(7.34) with $r_{q,\beta} = 1$ when $\beta < \beta_c$ or $\beta = \beta_c, q > 2$; $r_{q,\beta} = q - 1$ when $\beta = \beta_c, q \in (1, 2)$; and $r_{q,\beta} = n + 1 - q$ when $\beta > \beta_c$.

Turning to verify (7.20) for $\boldsymbol{g}, \boldsymbol{g}^{(n)}, \xi \neq 0$ and stated values of r, we have the following cases.

- If $\beta \in (0, \beta_c)$ then $s_\beta > \lambda^*$ is such that $p(s_\beta, \beta) = 0$, whereas $\mathbf{L}(s_\beta + z)$ is analytic in a neighborhood of $z = 0$. Thus, $\boldsymbol{g}(z)$ of (7.31) has a simple pole at $z = 0$, and (7.20) holds for $r = 1$ and $\xi = \beta(c\mathbf{L}(s_\beta) + 1)/(c\frac{\partial p}{\partial s}(s_\beta, \beta))$. It is easy to check that $\frac{\partial p}{\partial s}(s_\beta, \beta) = 2\beta/c - \mathbf{L}'(s_\beta) > 0$, hence $\xi \neq 0$.
- If $\beta = \beta_c$, then $s_\beta = \lambda^*$ with $\mathbf{L}(\lambda^* + z)$ analytic only for $z \in S_\theta$ and not in a whole neighborhood of $z = 0$. With $q > 1$ it suffices by (7.9) and (7.8) (for $k = 0$), to consider the scaled limit of $p(\lambda^* + z, \beta_c)$ for $z \in S_\theta$. To this end observe that

$$p(\lambda^* + z, \beta_c) = \frac{2\beta_c}{c} z + (\mathbf{L}(\lambda^*) - \mathbf{L}(\lambda^* + z)) = \frac{2\beta_c}{c} z - \int_{\Gamma_{0,z}} \mathbf{L}'(\lambda^* + \xi) d\xi ,$$

where the line segment $\Gamma_{0,z}$ connecting 0 and $z \in S_\theta$ is inside S_θ. If $q > 2$, then (7.8) applies for $k = 1$, thus implying that

$$\limsup_{|z|\downarrow 0, z\in S_\theta} |z^{-1} p(\lambda^* + z, \beta_c) - b_3| = 0 \tag{7.35}$$

for $b_3 := 2\beta_c/c - \mathbf{L}'(\lambda^*) \in (0, \infty)$ (recall by (7.9) that $\mathbf{L}'(\lambda^*) < 0$ is finite). So in this case $|z\boldsymbol{g}(z) - \xi| \to 0$ as $|z| \to 0$, uniformly in S_θ, where $\xi = \beta(c\mathbf{L}(\lambda^*) + 1)/(cb_3) > 0$. If $q \in (1, 2)$, then applying (7.10) for $\mathbf{L}'(\lambda^* + \xi)$, it follows that for some $\delta(|z|) \downarrow 0$ as $|z| \downarrow 0$,

$$\Big| \int_{\Gamma_{0,z}} \mathbf{L}'(\lambda^* + \xi) d\xi - \frac{b_2 z^{q-1}}{q-1} \Big| \le |z|^{q-1} \delta(|z|)$$

for all $z \in S_\theta$. Hence, in this case, for $b_3 = -b_2/(q-1) > 0$,

$$\limsup_{|z|\downarrow 0, z\in S_\theta} |z^{1-q} p(\lambda^* + z, \beta_c) - b_3| = 0\,, \tag{7.36}$$

and now $|z^{q-1}\boldsymbol{g}(z) - \xi| \to 0$ as $|z| \to 0$, uniformly in S_θ.

• If $\beta > \beta_c$ then again $s_\beta = \lambda^*$. Thanks to (7.9), (7.8) (and (7.11) in case q is an integer), it follows from (7.30) that

$$\limsup_{|z|\downarrow 0, z\in S_\theta} |z|^{n+1-q} \Big| \boldsymbol{g}^{(n)}(z) - \frac{\beta + \boldsymbol{g}(z)}{p(\lambda^* + z, \beta)} \mathbf{L}^{(n)}(z) \Big| = 0\,.$$

Recall that $p(\lambda^*, \beta) > 0$ and $\boldsymbol{g}(0)$ is finite in this case. Applying (7.8) (for $k = 0$) and (7.10) we thus have that

$$\limsup_{|z|\downarrow 0, z\in S_\theta} |z^{n+1-q} \boldsymbol{g}^{(n)}(z) - \xi| = 0\,,$$

where $\xi = b_2(\beta + \boldsymbol{g}(0))/(p(\lambda^*, \beta)) \neq 0$. □

Proof of Theorem 3.4. In analogy with (7.27), here we have that

$$R'(t) = 2ce^{2t\lambda^*} + 2c\beta^{-1} \int_0^t R(\tau) \mathscr{L}(2(t-\tau)) d\tau\,.$$

Proceeding as in Section 3.2.3 it suffices to establish that

$$R(x) \sim_{x\uparrow\infty} C_{q,\beta} x^\rho e^{2s_\beta x}\,, \tag{7.37}$$

where $\rho = \min(q-1, 1)$ for $\beta = \beta_c$, $q \neq 2$ and $\rho = 0$ otherwise. To this end, note that similarly to the derivation of (7.31), here

$$\boldsymbol{g}(z) = \frac{\frac{c\beta}{s_\beta + z - \lambda^*} + \beta}{cp(s_\beta + z, \beta)}\,,$$

whenever $\Re\{z\} > 0$. The function $\boldsymbol{g}(z)$ then admits the same singularities as the one of (7.31), except for an additional simple pole at $z = \lambda^* - s_\beta$. We thus observe the following three regimes.

• For $\beta \in (0, \beta_c)$, the simple pole at $\lambda^* - s_\beta < 0$ affects neither the asymptotics of $\boldsymbol{g}(z)$ near $z = 0$, nor that of $g(x)$ as $x \to \infty$.
• For $\beta = \beta_c$ we have $s_\beta = \lambda^*$. Using (7.35) when $q > 2$ and (7.36) for $q \in (1, 2)$, now the additional pole at $z = 0$ results with

$$\limsup_{|z|\downarrow 0, z \in S_\theta} |z^{(1+\rho)}\boldsymbol{g}(z) - \xi| = 0 \ , \tag{7.38}$$

for $\xi = \beta/b_3 > 0$, leading to the estimates (7.37) on the asymptotics of $g(x)$ as $x \to \infty$.
• For $\beta > \beta_c$ the simple pole at $z = 0$ results with (7.38) holding for $\rho = 0$ and $\xi = \beta/p(\lambda^*, \beta) > 0$. Consequently, applying Lemma 7.2 for $\mathbf{f} = \boldsymbol{g}$ leads here to $R(x) \sim c_{EA}^{-1} e^{2\lambda^* x}$ as $x \to \infty$. It is not hard to check that then $K(t, s)$ of (3.14) converges to $c_{EA} \in (0, \infty)$ whenever $t - s \to \infty$ and $s \to \infty$. □

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