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# On the connectivity properties of the complementary set in fractal percolation models

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**Abstract.** We study the connectivity properties of the complementary set in Poisson multiscale percolation model and in Mandelbrot's percolation model in arbitrary dimension. By using a result about majorizing dependent random fields by Bernoulli fields, we prove that if the selection parameter is less than certain critical value, then, by choosing the scaling parameter large enough, we can assure that there is no percolation in the complementary set.

## 1. Introduction and statement of the results

In this paper we study two models from the theory of fractal percolation: Poisson multiscale percolation model and Mandelbrot's percolation model. The latter, which is a generalization of the concept of Cantor set, was introduced by B. Mandelbrot [9] and subsequently studied by J.T. Chayes, L. Chayes, R. Durrett, G. Grimmett and others, see e.g. [1–5, 12, 14, 15]. In particular, Chayes et al. in [3] introduced the concept of "sheet percolation" in the random Cantor set, which is in fact equivalent to the absence of connected path in the complementary set. Their results were generalized to the case of arbitrary dimension by M.E. Orzechowski in [14]. Here we give an alternative (and shorter) proof of his result. In fact, the difference between our proof and that of [14] is that, thanks to the result about majorizing dependent fields, we were able to use a much simpler geometrical construction. We note also that the problem of percolation in the complementary set was considered by M.V. Menshikov, S.A. Molchanov and A.F. Sidorenko in Chapter 10 of [12]. Using combinatorial argument, they proved the absence of percolation for small parameter.

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The main result of this paper concerns the analogous Poissonian problem. The selected set here is constructed by composition of Poisson fields of different intensities of balls of different radii. We note that the method of [14] can hardly be applied to this model. Two-dimensional variant of this model was considered by R. Meester and R. Roy in Section 8.1 of [10]. We generalise their results to the case of arbitrary dimension and also we remove their restriction that the scaling parameter could go to infinity only along some fixed subsequence.

One of the key points of our approach is a result (Theorem 2.1) about majorizing dependent random fields by Bernoulli fields. We expect that this theorem may be useful for studying other percolation models. In fact, results about majorization by Bernoulli fields are important in percolation and were used by many authors, see e.g. [6]. A result closely related to ours was proved by T.M. Liggett, R.H. Schonmann and A.M. Stacey in [8]. We discuss this in more detail in Section 2.1.

### *1.1. Poisson multiscale model*

Let us construct Poisson multi-scale percolation model in $\mathbf{R}^d$, $d \geq 2$. Fix $R > 1$. Level-$i$ balls are balls in $\mathbf{R}^d$ with radius $R^{-i}$, $i = 0, 1, \ldots$ Centers of level-$i$ balls form Poisson field in $\mathbf{R}^d$ with rate $\lambda_i = cR^{id}$ independently of the others. Denote by $U^{(i)}$ the union of level-$i$ balls. The object of interest is the set $U = \cup_{i=0}^{\infty} U^{(i)}$. We say that in this model percolation occurs if almost surely there exists a continuous path $\gamma : \mathbf{R} \mapsto U$, such that $\gamma$ is not contained in any finite box. It is known that there exists $c_{cr} = c_{cr}(d)$, $0 < c_{cr} < \infty$ (cf. [10, 12, 13]) such that if $c < c_{cr}$, then $U^{(i)}$ is the union of nonintersecting finite components a.s. and if $c > c_{cr}$, then $U^{(i)}$ percolates for all $i = 0, 1, \ldots$ (from the rescaling argument it is evident that $c_{cr}$ does not depend on $i$, i.e. $c_{cr}$ is the critical intensity for the model of Poissonian balls of radius 1).

Trivially, if $c > c_{cr}$, then the set $U$ percolates for any $R$; it is not difficult to get that it is so for $c = c_{cr}$ too. Indeed, let us show that for $c = c_{cr}$, balls of levels 0 and 1 are enough to percolate. Trivially, the set $U^{(0)}$ majorizes a field of level-1 balls with intensity $c_{cr}$. Together with $U^{(1)}$, they form a field of level-1 balls with intensity $c_{cr} R^d (1 + R^{-d})$. This is in the supercritical phase due to the rescaling argument. So, our main result is the following

**Theorem 1.1.** *For any $c < c_{cr}$ there exists $R_0 = R_0(c)$ such that for all $R \geq R_0$ there is no percolation in the random set $U$.*

### *1.2. Mandelbrot's percolation model*

We construct the random set $D \subset \mathbf{R}^d$, $d \geq 2$, by the following iteration process. Fix some $N \in \mathbf{N}$ and $p \in (0, 1)$. Divide the space into closed (hyper)cubes with side 1, which are called level-0 cubes. Each of these cubes is selected with probability $p$ independently of the others; the union of all the selected level-0 cubes is denoted by $D_0$. Then, divide every nonselected level-0 cube into $N^d$ equal cubes with side $N^{-1}$, which are called level-1 cubes. Again, each of level-1 cubes is selected with probability $p$ independently of the others; together with the selected level-0 cubes they form the set $D_1$. Iterating this construction, we obtain the sequence of sets $D_0 \subset D_1 \subset D_2 \subset \cdots$, where $D_i = D_{i-1} \cup \{\text{selected level-}i \text{ cubes}\}$. The side of level-$i$ cube equals $N^{-i}$ for $i = 0, 1, \ldots$ Finally, we define a random set $D = D(d, N, p) := \bigcup_{i=0}^{\infty} D_i$.

We say that in this model percolation occurs if almost surely there exists a continuous path $\gamma : \mathbf{R} \mapsto D$, such that $\gamma$ is not contained in any finite box.

Note that $\overline{(\mathbf{R}^d \setminus D)} \cap [0, 1]^d$ is the standard Mandelbrot's random fractal. In fact, considering this model in the whole space instead of $[0, 1]^d$ do not bring new nontrivial results; we made it like this to underline the similarity of this model (and of method of studying it) with the Poisson multiscale model.

To formulate the result, we define a $d$-dimensional lattice $\mathbf{M}^d$ as the lattice with vertex set $\mathbf{Z}^d$ and the edge set given by the following adjacency relation. Two vertices $x = (x_1, \ldots, x_d)$ and $y = (y_1, \ldots, y_d)$ of $\mathbf{M}^d$ are connected iff $|x_i - y_i| \le 1, i = 1, \ldots, d$. Note that two cubes are connected if they have at least one common point, so the percolation by selected level-0 cubes is equivalent to the site percolation in $\mathbf{M}^d$. Let $p_{cr} = p_{cr}(d)$ be the critical probability for the site percolation in $\mathbf{M}^d$. The main result of this section is the following

**Theorem 1.2 (Orzechowski [14]).** *For any $p < p_{cr}$ there exist $N_0 = N_0(p)$ such that for all $N \ge N_0$ there is no percolation in the random set $D$.*

## 2. Proofs

### 2.1. Majorizing dependent fields

**Definition 2.1.** *We say that one random set $G_1$ is stochastically smaller than the other random set $G_2$ iff it is possible to couple them in such a manner that $G_1 \subset G_2$.*

We need the following result:

**Theorem 2.1.** *Let $\{\eta(x)\}_{x \in \mathbf{Z}^d}$, $\eta(x) \in \{0, 1\}$ be a translation invariant random field. Denote by $M_a = \{-a, \ldots, a\}^d \subset \mathbf{Z}^d$ the cube of side $2a + 1$ centered in $0$. Denote*

$$\delta = \sup \mathbf{P}\{\eta(0) = 1 \mid \eta(x_1) = 0, \ldots, \eta(x_l) = 0, \eta(y_1) = b_1, \ldots, \eta(y_m) = b_m\}, \tag{1}$$

*where the supremum is taken over all (including the empty) finite subsets $\{x_1, \ldots, x_l\} \subset M_a$, $\{y_1, \ldots, y_m\} \subset \mathbf{Z}^d \setminus M_a$, and $b_1, \ldots, b_m \in \{0, 1\}$. Then the random set $G = \{x : \eta(x) = 1\}$ is stochastically smaller than the Bernoulli random set with parameter $p = 1 - (1 - \delta^{(2a+1)^{-d}})^{(2a+1)^d}$.*

*Proof.* We prove this theorem in two steps.

*Step 1.* (Note that similar construction was used in Lemma 8 of [7].) Consider a field of Bernoulli (i.e. independent) random variables $\{\eta'(x)\}_{x \in \mathbf{Z}^d}$ with $\mathbf{P}\{\eta'(x) = 1\} = \delta$. Let

$$G' = \{x \in \mathbf{Z}^d : \text{ there exists } y \in M_a(x) \text{ such that } \eta'(y) = 1\},$$

where $M_a(x) = \{-a+x, \dots, a+x\}^d$. We construct a coupling such that $G \subset G'$. To this end, enumerate $\mathbf{Z}^d = \{x_1, x_2, \dots\}$ and let $\xi_{x_1}, \xi_{x_2}, \dots$ be i.i.d. random variables uniformly distributed in [0, 1]. Put

$$\eta'(x_i) = \mathbf{1}_{\{\xi_{x_i} \le \delta\}},$$

for $i = 1, 2, \dots$ and

$$\eta(x_1) = \mathbf{1}_{\{\xi_{x_1} \le \mathbf{P}\{\eta(x_1)=1\}\}}.$$

So $\eta(x_1) \le \eta'(x_1)$ as $\mathbf{P}\{\eta(x_1) = 1\} \le \delta$. If $\eta'(x_1) = 0$ then we pass to $x_2$. Using (1) we have

$$\eta(x_2) = \mathbf{1}_{\{\xi_{x_2} \le \mathbf{P}\{\eta(x_2)=1|\eta(x_1)=0\}\}} \le \mathbf{1}_{\{\xi_{x_2} \le \delta\}} = \eta'(x_2).$$

If $\eta'(x_1) = 1$ then all the sites from $M_a(x_1)$ already belong to $G'$, so we just exclude them from the "tail" $\{x_2, x_3, \dots\}$ and reenumerate this tail in natural way. We continue to construct the field $\{\eta(x)\}$ in this way. At each step $m$ we have

$$\begin{aligned} &\mathbf{P}\{\eta(x_m) = 1 \mid \eta(x_1), \dots, \eta(x_{m-1})\} \\ &= \mathbf{P}\{\eta(x_m) = 1 \mid \eta(x_{i_1}) = 0, \dots, \eta(x_{i_l}) = 0, \eta(x_{i_{l+1}}), \dots, \eta(x_{i_{m-1}})\} \le \delta, \end{aligned}$$

where $x_{i_1}, \dots, x_{i_l} \in M_a(x_m)$ and $x_{i_{l+1}}, \dots, x_{i_{m-1}} \notin M_a(x_m)$. So

$$G \cap \{x_1, \dots, x_m\} \subset G' \cap \{x_1, \dots, x_m\}.$$

Hence, by induction, $G \subset G'$.

*Step 2.* Now we will construct a coupling such that the Bernoulli random set $B(p)$ with parameter $p$ will contain $G'$. For $z \in M_a$ denote

$$G'_z = \{y \in M_a(x) : x = (2a+1)k + z, \;\; k \in \mathbf{Z}^d, \text{ and } \eta'(x) = 1\}.$$

There are $(2a+1)^d$ such sets $G'_z$ and $G' = \cup_{z \in M_a} G'_z$. Let $\zeta_x$, $x \in \mathbf{Z}^d$ be i.i.d. random variables uniformly distributed in [0, 1] and $\eta''(x) = \mathbf{1}_{\{\zeta(x) \le \delta^{(2a+1)^{-d}}\}}$. Define $B(\delta^{(2a+1)^{-d}}) = \{x \in \mathbf{Z}^d : \eta''(x) = 1\}$. Since

$$\mathbf{P}\{\zeta(y) \le \delta^{(2a+1)^{-d}} \quad \text{for all } y \in M_a(x)\} = \delta,$$

we can write

$$G'_z = \{y \in M_a(x) : x = (2a+1)k + z, \;\; k \in \mathbf{Z}^d, \text{ and } \eta'_z(x) = 1\}$$

where

$$\eta'_z(x) = \mathbf{1}_{\{\eta''(y)=1 \text{ for all } y \in M_a(x)\}}.$$

Thus, $G'_z \subset B(\delta^{(2a+1)^{-d}})$. This implies that $G$ is stochastically smaller than $B(p)$ for $p = 1 - (1 - \delta^{(2a+1)^{-d}})^{(2a+1)^d}$. □

In comparison with our Theorem 2.1, Theorem 0.0 (i) of [8] relaxes the condition on the supremum in (1), but gives the proof only for $\delta$ sufficiently small. Thus, Theorem 2.1 is of independent interest, although the result from [8] could have been used for our needs too. Note also that just before Theorem 0.0 in [8] it is stated that for one-dimensional 1-dependent positively correlated fields the majorization could *always* be obtained; our result together with an argument analogous to Lemma 2.2 below show that it is true for many-dimensional fields arising from typical block (rescaling) arguments in percolation (at least when the event "the block is good" is increasing, as it often is).

Another reason which motivated us to include this theorem, is the following. Applying Theorem 2.1 to the set $\tilde{G} = \{x : \eta(x) = 0\}$ and making suitable changes in (1), one gets that the field $\eta$ is also *minorized* by some Bernoulli field, so Theorem 2.1 in fact allows us to make two-sided bounds. To illustrate this, consider the following

**Example.** Let $F(\cdot, \dots, \cdot)$ be some *increasing* Boolean function of $(2a+1)^d$ arguments such that $F(\mathbf{0}) = 0$ and $F(\mathbf{1}) = 1$. Let $\eta_1$ be a Bernoulli field with parameter $q_1 < p_{cr}$ and $\eta_2$ be a dependent field constructed in the following way: If $\eta_3$ is a Bernoulli field with parameter $q_2$, let $\eta_2(x) = F(\eta_3(y), y \in M_a(x))$. Denote $G_i = \{x : \eta_i(x) = 1\}$, $i = 1, 2$. One may be interested in proving the following:

(i) for fixed $q_1 < p_{cr}$ and small $q_2$ the set $G_1 \cup G_2$ do not percolate;
(ii) for fixed $q_2$ and $q_1$ close enough to $p_{cr}$ the set $G_1 \cup G_2$ percolates.

Now, Theorem 0.0 (i) of [8] allows us to prove (i), but is applicable to (ii) only when $q_2$ is close to 1. On the other hand, Theorem 2.1 together with an argument analogous to Lemma 2.2 gives the proof of both (i) and (ii).

### *2.2. Proof of Theorem 1.2*

We prefer to put the proof of this theorem first, because it is more illustrative and the proof of Theorem 1.1 contains some artificial constructions.

Following [5, 14], it suffices to show that the set $D_n$ is in the subcritical phase uniformly in $n$ (i.e. that the probability of existence of the path from the origin to the boundary of big box is small uniformly in $n$). Indeed, note that the absence of percolation in the complementary set is *full-sheet percolation* of [14], so we may apply Lemma 1 of [14].

To proceed, we will need some definitions and lemmas. Level-$k$ cube $K$ is labeled in the following way:

$$\begin{aligned} K &= (k, i_1, \dots, i_d) \\ &= \{x = (x_1, \dots, x_d) \in \mathbf{R}^d : x_j \in [i_j N^{-k}, (i_j + 1)N^{-k}], j = 1, \dots, d\}, \end{aligned}$$

$i_l \in \mathbf{Z}$, $l = 1, \dots, d$, where $i_1, \dots, i_d$ are "coordinates" of the cube. Suppose also that we are given a family of i.i.d. random variables $Z(k, i_1, \dots, i_d)$, each taking the value 1 with probability $p$ and 0 otherwise. Introduce the random set $D^{(i)}$ as the union of all level-$i$ cubes $K$ for which $Z(K) = 1$. Clearly, $D_n = \cup_{i=0}^n D^{(i)}$ (random sets $D_n$ were defined in Section 1.2).

The distance between two cubes $K_1 = (k, i_1, \dots i_d)$ and $K_2 = (k, j_1, \dots j_d)$ is $\mathrm{dist}(K_1, K_2) := \max\{|i_l - j_l|, l = 1, \dots, d\}$. Note that it is defined for the cubes of the same level only.

By induction we define passable and good cubes.

**Definition 2.2.** *The level-$n$ cube $K$ is good iff $Z(K) = 1$. For $k < n$ the level-$k$ cube $K'$ is passable iff it intersects with a sequence of at least $N$ adjacent good level-$(k+1)$ cubes. The cube $K'$ is good iff it is passable or $Z(K') = 1$.*

**Lemma 2.1.** *If there exists an infinite path in $D_n$, then there exists an infinite sequence of adjacent good level-0 cubes.*

*Proof.* Consider an infinite path $\gamma$ in $D_n$. Without restricting of generality one can suppose that this path do not pass two or more times through the same selected cube. Consider a connected subpath $\gamma'$ of $\gamma$ which passes only through cubes from $D^{(n)}$. As $\gamma$ do not pass two or more times through the same selected cube, then two cases are possible: either the number of cubes through which $\gamma'$ passes is not less than $N$, or $\gamma'$ connects two selected cubes $K_1, K_2$ of levels $i_1, i_2 < n$ such that $K_1 \cap K_2 \neq \emptyset$. In the second case $\gamma$ can be modified in such a way that it will not contain $\gamma'$ anymore. In the first case all the level-$(n-1)$ cubes through which it passes are passable and therefore good. Thus we obtain from $\gamma$ the sequence $\gamma_1$ of adjacent good cubes of levels $0, \dots, n-1$ (note that those of level at most $n-2$ are in fact selected cubes). Iterating this construction, we obtain the infinite sequence $\gamma_n$ of adjacent good level-0 cubes. $\square$

Let us introduce some notation. For fixed $k$ identify the level-$k$ cubes with the vertices of $\mathbf{Z}^d$, i.e. cube $(k, i_1, \dots, i_d)$ is identified with the point $(i_1, \dots, i_d) \in \mathbf{Z}^d$. Note that the graph obtained by this identification and by the connectivity by one common point is $\mathbf{M}^d$. We associate two random fields $\eta_k^s = \{\eta_k^s(x)\}_{x\in\mathbf{Z}^d}$ and $\eta_k^g = \{\eta_k^g(x)\}_{x\in\mathbf{Z}^d}$ to the set of level-$k$ cubes, where

$$\eta_k^s(x) = \begin{cases} 1, & \text{if the cube } (k,x) \text{ is passable,} \\ 0, & \text{otherwise,} \end{cases}$$

$$\eta_k^g(x) = \begin{cases} 1, & \text{if the cube } (k,x) \text{ is good,} \\ 0, & \text{otherwise.} \end{cases}$$

Denote also by $\xi_{q,k} = \{\xi_{q,k}(x)\}_{x\in\mathbf{Z}^d}, \xi_{q,k}(x) \in \{0,1\}$ the Bernoulli random field of level-$k$ cubes with parameter $q \in [0,1]$. Let $\mathbf{P}_{\eta_k^g}$ and $\mathbf{P}_{q,k} := \mathbf{P}_{\xi_{q,k}}$ be the probability measures induced by the respective random fields.

**Lemma 2.2.** *Suppose that for some $0 \leq k < n$ the random field $\eta_{k+1}^g$ of good level-$(k+1)$ cubes is stochastically smaller than the Bernoulli field $\xi_{p',k+1}$. Then the random field $\eta_k^s$ of passable level-$k$ cubes is stochastically smaller than the Bernoulli field $\xi_{p_1,k+1}$ with $p_1 = 1 - (1 - \delta^{5^{-d}})^{5^d}$, where $\delta = \mathbf{P}_{p',k+1}\{\eta_k^s(0) = 1\}$, i.e. $\delta$ is the probability that a given cube is passable computed with the assumption that the field of good cubes of the next level is Bernoulli with parameter $p'$.*

*Proof of Lemma 2.2.* By the assumption of the lemma, we can majorize the field $\eta_{k+1}^g$ by Bernoulli field $\xi_{p',k+1}$. Denote by $\tilde{\eta}_k^s$ the field of passable level-$k$ cubes constructed on $\xi_{p',k+1}$ (i.e. suppose that we declare level-$(k+1)$ cubes good independently with probability $p'$ and construct the level-$k$ passable cubes starting from them). Clearly, $\eta_k^s$ is majorized by $\tilde{\eta}_k^s$, so it suffices to prove the lemma for $\tilde{\eta}_k^s$. Fix any $l \ge 0$, $m \ge 0$, $x_1, \dots, x_l \in M_2$, $y_1, \dots, y_m \notin M_2$, $b_1, \dots, b_m \in \{0, 1\}$ (see Theorem 2.1 for the definition of $M_2$). Let us introduce the events

$$\begin{aligned} A &= \{\tilde{\eta}_k^s(0) = 1\}, \\ B &= \{\tilde{\eta}_k^s(x_1) = 0, \dots, \tilde{\eta}_k^s(x_n) = 0\}, \\ C &= \{\tilde{\eta}_k^s(y_1) = b_1, \dots, \tilde{\eta}_k^s(y_m) = b_m\}. \end{aligned}$$

Note that $\mathbf{P}(A) = \delta$. We need to prove only that $\mathbf{P}(A \mid B \cap C) \le \mathbf{P}(A)$. Denote $\Delta(C) = \{y \in \mathbf{Z}^d : \mathrm{dist}(y, y_j) \le 1 \text{ for some } j = 1, \dots, m\}$. We can write $C = \cup_i C_i$ where $C_i$ are all possible configurations of good level-$(k+1)$ cubes contained in $\Delta(C)$ such that the event $C$ occurs. It is clear that $C_i \cap C_j = \emptyset$ for $i \ne j$, so, denoting $\mathbf{P}^{C_i}(\cdot) = \mathbf{P}(\cdot \mid C_i)$,

$$\begin{aligned} \mathbf{P}(A \cap B \cap C) &= \sum_i \mathbf{P}^{C_i}(A \cap B)\mathbf{P}(C_i) \\ &\le \sum_i \mathbf{P}^{C_i}(A)\mathbf{P}^{C_i}(B)\mathbf{P}(C_i) \\ &= \mathbf{P}(A) \sum_i \mathbf{P}^{C_i}(B)\mathbf{P}(C_i) \\ &= \mathbf{P}(A)\mathbf{P}(B \cap C). \end{aligned}$$

Here we used that $A$ is an increasing event, $B$ is decreasing, and the conditional measure $\mathbf{P}^{C_i}$ is FKG, so $\mathbf{P}^{C_i}(A \cap B) \le \mathbf{P}^{C_i}(A)\mathbf{P}^{C_i}(B)$. The event $A$ only depends on the level-$(k+1)$ cubes in $M_1$, so if $y_1, \dots, y_m$ are outside $M_2$, then $M_1 \cap \Delta(C) = \emptyset$ and $\mathbf{P}^{C_i}(A) = \mathbf{P}(A)$. Now it remains to apply Theorem 2.1 with $a = 2$ and $\delta$ defined above. Lemma 2.2 is proved. $\square$

The key point in the course of the proof of Theorem 1.2 is the following

**Lemma 2.3.** *If $p < p' < p_{cr}$, then there exists $N_0 \in \mathbf{N}$ such that for any $N \ge N_0$ and any $n > 1$ the random field of good level-$0$ cubes can be stochastically majorized by Bernoulli random field with parameter $p'$.*

*Proof.* We will prove this lemma by induction. Fix $\varepsilon < p' - p$.

Clearly, for level-$n$ cubes, $\eta_n^g = \xi_{p,n}$ is stochastically smaller than $\xi_{p',n}$ so the statement of the lemma holds.

Suppose that for the level-$(k+1)$ cubes, $\eta_{k+1}^g$ is stochastically smaller than the Bernoulli field $\xi_{p',k+1}$.

We need the following definition:

**Definition 2.3.** *For random field $\{\zeta(x)\}_{x \in \mathbf{Z}^d}$, $\zeta(x) \in \{0, 1\}$ define the cluster of the vertex $x \in \mathbf{Z}^d$*

$$\begin{aligned} W_\zeta(x) = \{y \in \mathbf{Z}^d &: \textit{there exist } m \textit{ and a sequence} \\ & y_0 = x, y_1, \dots, y_m = y \textit{ such that } \zeta(y_i) = 1 \\ & \textit{and } y_{i-1} \textit{ is a neighbor of } y_i \textit{ in } \mathbf{M}^d, i = 1, \dots, m\}. \end{aligned}$$

As it is proved in [11], $p' < p_{cr}$ implies that there exists $\beta(p') > 0$ such that for all $\nu$ and all $x \in \mathbf{Z}^d$

$$\mathbf{P}\{|W_{\xi_{p'}}(x)| \geq \nu\} \leq \exp(-\beta(p')\nu). \tag{2}$$

Fix some level-$k$ cube $K$ with coordinates $x \in \mathbf{Z}^d$. Using (2) and the induction assumption (i.e. that the field $\eta^g_{k+1}$ is majorized by Bernoulli field $\xi_{p',k+1}$), we have

$$\begin{aligned}\mathbf{P}\{K \text{ is passable}\} = \mathbf{P}\{&\text{there exist a path by good level-}(k+1)\text{ cubes}\\ &\text{of length not less than } N \text{ intersecting } K\}\\ &\leq N^d \mathbf{P}_{\eta^g_{k+1}}\{W_{\eta^g_{k+1}}(0) \geq N\}\\ &\leq N^d \mathbf{P}_{p',k+1}\{W_{\xi_{p',k+1}}(0) \geq N\}\\ &\leq N^d \exp(-\beta(p')N) =: \varepsilon_1(N).\end{aligned} \tag{3}$$

By Lemma 2.2, the random field of passable level-$(k+1)$ cubes is majorized by Bernoulli random field with parameter

$$\varepsilon_2(N) = 1 - (1 - (\varepsilon_1(N))^{5^{-d}})^{5^d}$$

(which can be made arbitrary small by choosing small $\varepsilon_1 = \varepsilon_1(N)$).

Choose $N$ such that $\varepsilon_2(N) < \varepsilon$, i.e. $p + \varepsilon_2(N) < p'$. As the random field of good level-$k$ cubes is the random field of passable level-$k$ cubes together with $D^{(k)}$, we get that $\eta^g_k$ is stochastically smaller than $\xi_{p',k}$.

Note that the choice of $N$ depends only on $p'$. So we can take it the same for all levels of cubes. Thus, Lemma 2.3 is proved. □

Now we can finish the proof of Theorem 1.2. By Lemma 2.3, one can choose $N$ such that the random field of good level-0 cubes is majorized by Bernoulli field with parameter $p'$, $p < p' < p_{cr}$. By Lemma 2.1, this implies that the set $D_n$ is in the subcritical phase uniformly in $n$. □

### 2.3. *Proof of Theorem 1.1*

Denote $U_n = \cup_{i=0}^n U^{(i)}$. Similarly to the previous model, we need to show that the set $U_n$ is in the subcritical phase uniformly in $n$. Indeed, take a finite path $\gamma : [0,1] \mapsto U$. Note that, due to the fact that the balls never "touch", we may suppose that they are open without changing the problem. So, $\gamma$ is covered by open sets, and, choosing a finite subcovering, one gets that $\gamma$ lies in $U_i$ for some finite $i$.

Choose $\alpha$, $0 < \alpha < 1/2\sqrt{d}$, and $c'$, $c < c' < c_{cr}$, such that

$$c'(1+\alpha)^d < c_{cr}. \tag{4}$$

Denote $\lambda'_i = c' R^{id}$. Consider also the balls of radius $R^{-i}(1+\alpha)$ in $\mathbf{R}^d$, whose centers form Poisson field of rate $\lambda'_i$; denote their union by $W^{(i)}$. Note that, due to (4), the set $W^{(i)}$ does not percolate for $i = 1, \ldots, n$.

For $i = 1, \ldots, n-1$ consider a partition of the space into the cubes with side $R^{-i}/\sqrt{d}$, which we call the level-$i$ cubes. Note that the side of level-$i$ cube is chosen in such a way that if the center of level-$i$ ball is inside the cube, then the latter is completely covered by the ball. Also, denote by $V^{(i)}$ the union of the balls with the centers at the centers of the level-$i$ balls and radius $R^{-i}(1+\alpha)$, so the set $V^{(i)}$ is in fact the "expanded" set $U^{(i)}$.

We define now *passable sets* $P_0, \ldots, P_{n-1}$, and *good sets* $G_0, \ldots, G_n$.

**Definition 2.4.** *First, define the good level-n set $G_n$ by $G_n := U^{(n)}$. For $i < n$, level-i cube is passable if it intersects with some connected component of diameter greater than $2\alpha R^{-i}$ of $G_{i+1}$. The passable level-i set $P_i$ is the union of all passable level-i cubes. The good level-i set $G_i$ is defined by $G_i := P_i \cup V^{(i)}$.*

**Lemma 2.4.** *Percolation in $U_n$ implies percolation in $G_0$.*

*Proof.* Consider a path $\gamma$ in $U_n$. Let $\gamma'$ be some subpath of level-$n$ balls (i.e. from $G_n$) connecting some two balls $S_1, S_2$ of levels $i_1, i_2 < n$. If the diameter of $\gamma'$ is less than $2\alpha R^{-n}$, than $\gamma'$ is contained in $V^{(i_1)} \cup V^{(i_2)}$. Otherwise, by Definition 2.4, $\gamma'$ is covered by passable level-$(n-1)$ cubes. Iterating this argument, we get the proof of Lemma 2.4. □

The main ingredient of the proof of Theorem 1.1 is the following:

**Proposition 2.1.** *If for fixed $c < c' < c_{cr}$ the scaling parameter $R$ is large enough, then $G_i$ can be stochastically majorized by $W^{(i)}$, $i = 0, \ldots, n$.*

*Proof.* We prove this proposition by induction. Clearly, $G_n = U^{(n)}$ can be majorized by $W^{(n)}$. Suppose that the hypothesis of the proposition holds for the level $k+1$; let us prove it for the level $k$.

Note that, since the Poisson field of balls $W^{(k+1)}$ is subcritical, the result of [13] imply that for any $\alpha > 0$, $i \in \mathbf{N}$ there exists $\beta(\alpha, c') > 0$ such that for any level-$k$ cube $K$ (compare with (3))

$$\mathbf{P}\{K \text{ is passable}\} \leq R^d e^{-\beta(\alpha,c')R}. \tag{5}$$

To proceed with the proof of Proposition 2.1, we need two additional lemmas.

**Lemma 2.5.** *The random field of passable level-k cubes can be stochastically majorized by Bernoulli random field of cubes with parameter $\varepsilon(R)$ which can be made arbitrary close to 0 by choosing $R$ large enough. Note that the choice of $R$ depends only on $d, c, c'$, but not on $n$.*

*Proof of Lemma 2.5.* Note that two cubes are passable or not independently if there are at least two cubes between them, because we are interested in the connected components of $G_{k+1}$ with diameter greater than $2\alpha R^{-k}$, and $\alpha < 1/2\sqrt{d}$. Analogously with Lemma 2.2, substituting the sum by the integral and using FKG for Poisson fields, one can prove that the random field of passable level-$k$ cubes satisfies the conditions of Theorem 2.1 with $a = 2$ and, by (5), $\delta = R^d e^{-\beta(\alpha,c')R}$. So it can be stochastically majorized by Bernoulli random field with parameter $\varepsilon(R) = 1 - (1 - (R^d e^{-\beta(\alpha,c')R})^{5^{-d}})^{5^d}$. □

**Lemma 2.6.** *Bernoulli random field of level-$k$ cubes with parameter $\varepsilon(R)$ can be stochastically majorized by balls of radius $R^{-k}(1+\alpha)$, centers of which form Poisson field in $\mathbf{R}^d$ with rate $\varepsilon' R^{kd}$, and $\varepsilon'$ can be made arbitrarily close to* 0 *by choosing $R$ large enough.*

*Proof of Lemma 2.6.* Consider the following coupling. The cube is selected if there is some point of Poisson field in it. Note that the cubes are selected or not independently and

$$\mathbf{P}\{\text{the cube is selected}\} = \varepsilon(R) = 1 - \exp(\varepsilon'((1+\alpha)/\sqrt{d})^d).$$

If there is a center of the ball in the cube, then the cube is completely covered by the ball. By Lemma 2.5, choosing $R$ large we can make $\varepsilon(R)$ arbitrary close to 0 and thus $\varepsilon'$ will be arbitrary close to 0 (and, in particular, $\varepsilon' < c' - c$). $\square$

We continue proving Proposition 2.1. By Lemmas 2.5 and 2.6, the good level-$k$ set $G_k$ is majorized by the union of $V^{(k)}$ with the field of balls of radius $R^{-k}(1+\alpha)$, centers of which form Poisson field in $\mathbf{R}^d$ with rate $\varepsilon' R^{kd}$. Since $c + \varepsilon' < c'$, Proposition 2.1 is proved. $\square$

Now we finish the proof of Theorem 1.1. By Lemma 2.4,

$$\{\text{no percolation in } G_0\} \Rightarrow \{\text{no percolation in } U_n\}.$$

By (4) and Proposition 2.1, the set $G_0$ (and therefore $U_n$) is in the subcritical phase uniformly in $n$. Thus, Theorem 1.1 is proved. $\square$

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