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Vikram K. Srimurthy*

On the equivalence of measures on loop space

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Abstract. Let K be a simply-connected compact Lie Group equipped with an Ad_K -invariant inner product on the Lie Algebra $\mathfrak{K}$, of K. Given this data, there is a well known left invariant "H^1-Riemannian structure" on $L(K)$ (the infinite dimensional group of continuous based loops in K), as well as a heat kernel $\nu_T(k_0, \cdot)$ associated with the Laplace-Beltrami operator on $L(K)$. Here $T > 0$, $k_0 \in L(K)$, and $\nu_T(k_0, \cdot)$ is a certain probability measure on $L(K)$. In this paper we show that $\nu_1(e, \cdot)$ is equivalent to Pinned Wiener Measure on K on $\mathfrak{G}_{s_0} \equiv \sigma \langle x_t : t \in [0, s_0] \rangle$ (the σ-algebra generated by truncated loops up to "time" s_0).

1. Introduction

In this paper we consider the equivalence of two measures on the loop space of a compact Lie group. This so-called "loop group" is the space of continuous paths in the Lie group based at the identity equipped with a certain well-known left-invariant "H^1-Riemannian structure". The study of Loop groups is motivated primarily by physics and the theory of group Representations. They have been studied extensively in both the mathematics and the physics literature. See for example [25], [17], [23], [3], [13], [14], [1], [20], [16], [11] and the references therein.

Heat Kernel and pinned Wiener measure are two natural measures that have been advocated as the "right" measure on Loop groups. Pinned Wiener measure on a Loop group is the law of a group-valued Brownian motion that has been conditioned on loops. This measure has been extensively studied in [15], [22], [2], [21]. Heat Kernel measure has been studied in [12], [10] as another natural measure on Loop Space. In [12], Driver and Lohrenz showed that there exists a certain process that deserves to be called "Brownian motion" on the path space of a Loop group. The Heat Kernel measures on the Loop Space are the time t, $t > 0$ distributions of this Brownian motion. Thus it is a natural question to consider the equivalence of these two measures.

A further motivation comes from logarithmic Sobolev inequalities and the papers of Getzler [14], Gross [15], Driver [10], Hsu, Aida, and Elworthy. The classical Sobolev inequalities are a fundamental tool in analyzing finite-dimensional manifolds. For infinite-dimensional manifolds logarithmic Sobolev inequalities, because of their dimension-independent character, are seen to be the proper analogues of

V. K. Srimurthy: 99 Brookline Street, Cambridge, MA 02139, USA. e-mail: vsrimurt@mediaone.net

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classical Sobolev inequalities. Logarithmic Sobolev inequalities have been studied extensively over infinite-dimensional linear spaces as well as finite-dimensional manifolds (see [7], [8] for surveys and [18]). If a logarithmic Sobolev inequality does hold for pinned Wiener measure, μ_0, then the Dirichlet form $\mu_0 \langle \nabla f, \nabla f \rangle$ associated with pinned Wiener measure will have a spectral gap (the so-called "Mass Gap inequality").

In [14], Getzler showed that the Bakry and Emery criteria (see [4] and [5]) for proving a logarithmic Sobolev inequality does not hold in general for loop groups when the "underlying measure" is pinned Wiener measure. In [15], using pinned Wiener measure, Gross showed that a logarithmic Sobolev inequality on Loop space does hold, but with an added potential term (a so-called "defective" logarithmic Sobolev inequality). Using Heat Kernel measure instead, Driver and Lohrenz proved in [12] that a logarithmic Sobolev inequality does hold on Loop groups, without Gross' potential. If Heat Kernel and pinned Wiener measures were equivalent with Radon-Nikodym derivatives bounded above and below then the Holley-Stroock Lemma (see [18])would tell us that pinned Wiener measure admits a classical (i.e. "non-defective") logarithmic Sobolev inequality. Even if the equivalence were not so nice, it might still be possible to use the Driver-Lohrenz result of [12] to eliminate the Gross' potential term and thereby prove a logarithmic Sobolev inequality for pinned Wiener measure.

In Section 5 we show that pinned Wiener measure is equivalent to Heat Kernel measure on $\mathfrak{F}_s$, the σ-algebra of functions depending on the loop up to time $s < 1$. We view the Loop-Space-valued Brownian motion, developed by Driver and Lohrenz in [12], as a group-valued two-parameter process. Viewing one of the parameters fixed, the resulting process has the same distribution as Heat Kernel measure. In Section 4, using extensively the two-parameter calculus developed by Norris in [24], we show that in the other parameter this process is a Brownian semimartingale on the path space of the Lie group. The fact that we can pull back this process to a Lie algebra valued Brownian Semimartingale together with Girsanov's Theorem, and the fact that Wiener measure and pinned Wiener measure are equivalent on $\mathfrak{F}_s$; gives us our result that on $\mathfrak{F}_s$ Heat Kernel measure and pinned Wiener measure are equivalent. In our proof, the analysis is done in a bigger space (the Wiener space of the compact Lie group) which is why we require s to be strictly less than one.

Heat Kernel measure is a time t distribution of a process on the path space of a Loop group which is started from the identity loop (i.e. the constant loop). This describes a homotopy between the endpoint of this process and the identity loop. As a consequence, Heat Kernel measure concentrates all its mass on null-homotopic loops. On the other hand pinned Wiener measure is quasi-invariant under translations by finite-energy loops. Thus Pinned Wiener measure must assign non-zero mass to all homotopy classes. Therefore if the Lie group is not simply connected, pinned Wiener measure is not equivalent to Heat Kernel measure. Thus our result showing absolute continuity on $\mathfrak{F}_s$ for $s < 1$ is in a sense the best result that can be obtained in the non-simply-connected case.

2. Statement of results

2.1. Loop group geometry

Let K be a connected compact Lie group, $\mathfrak{K} \equiv T_e K$ be the Lie algebra of K, and $\langle \cdot, \cdot \rangle_{\mathfrak{K}}$ be an Ad_K-invariant inner product on $\mathfrak{K}$. For $\xi \in \mathfrak{K}$, let $|\xi|_{\mathfrak{K}} \equiv \sqrt{\langle \xi, \xi \rangle_{\mathfrak{K}}}$. Let ℓ_g and ρ_g be left and right translations on K respectively. (i.e. ℓ_g and ρ_g are maps taking K to K so that $\ell_g(x) = gx$ while $\rho_g(x) = xg$). Let

$$L(K) \equiv \{\sigma \in C([0,1] \to K) \,|\sigma(0) = \sigma(1) = e\}$$

denote the based loop group on K consisting of continuous paths $\sigma : [0,1] \to K$ such that $\sigma(0) = \sigma(1) = e$, where $e \in K$, is the identity element.

Definition 2.1. (Tangent Space of $L(K)$). We will need the following definitions:-

- Given a function $h : [0,1] \to \mathfrak{K}$ such that $h(0) = 0$, define $(h,h)_H = \infty$ if h is not absolutely continuous and set $(h,h)_H = \int_0^1 \left|h'(s)\right|^2 ds$ otherwise.
- Define

$$H \equiv H(\mathfrak{K}) \equiv \{h : [0,1] \to \mathfrak{K} | h(0) = 0 \quad \text{and} \quad (h,h) < \infty\}.$$

Then $H(\mathfrak{K})$ is a Hilbert space under $(\cdot,\cdot)_H$.
- Define

$$H_0 \equiv H_0(\mathfrak{K}) \equiv \{h \in H(\mathfrak{K}) \,| h(1) = 0\}.$$

Then $(H_0(\mathfrak{K}), (\cdot,\cdot)_H)$ is also a Hilbert space.

In order to define the tangent space $TL(K)$ of $L(K)$ let θ denote the Maurer-Cartan form. That is $\theta\langle\xi\rangle \equiv \left(\ell_{k^{-1}}\right)_* \xi$ for all $\xi \in T_k K$, and $k \in K$. Let $\theta\langle X\rangle(s) \equiv \theta\langle X(s)\rangle$ and $p : TK \to K$ be the canonical projection. We now define

$$TL(K) \equiv \{X : [0,1] \to TK | \theta\langle X\rangle \in H_0 \quad \text{and} \quad p \circ X \in L(K)\}.$$

By abuse of notation, use the same p to denote the canonical projection from $TL(K) \to L(K)$. As usual, define the tangent space at $k \in L(K)$ by $T_k L(K) \equiv p^{-1}\{k\}$. Using left translations, we extend the inner product $(\cdot,\cdot)_{H_0}$ on H_0 to a Riemannian metric on $TL(K)$. Explicitly set

$$(X,X)_{L(K)} \equiv (\theta\langle X\rangle, \theta\langle X\rangle)_{H_0(\mathfrak{K})} \text{ where } X \in TL(K).$$

In this way, $L(K)$ is to be thought of as an infinite-dimensional Riemannian manifold. Viewing the Lie algebra $(\mathfrak{K}, 0)$ as a commutative Lie group with Lie algebra $\mathfrak{K}$, we obtain definitions for

$$L(\mathfrak{K}) \equiv \{\sigma \in C([0,1] \to \mathfrak{K}) \,|\sigma(0) = \sigma(1) = 0\}$$

as the "Lie group" with Lie algebra $H_0(\mathfrak{K})$ thought of as a commutative Lie algebra.

Definition 2.2. (The Laplacian $\triangle_{L(K)}$ and $\triangle_{L(\Re)}$). Take an orthonormal basis of $H_0(\Re)$. Then define an operator $\triangle_{L(K)}$ on functions f on $L(K)$ by setting

$$\triangle_{L(K)} f \equiv \sum \partial_h^2 f, \text{ where } (\partial_h f)(\gamma) \equiv \partial_\varepsilon f(\gamma \exp \varepsilon h)|_{\varepsilon=0}.$$

Define the Laplacian $\triangle_{L(\Re)}$ on functions f on $L(\Re)$ in the same way above by setting

$$\triangle_{L(\Re)} f \equiv \sum \partial_h^2 f, \text{ where } (\partial_h f)(\gamma) \equiv \partial_\varepsilon f(\gamma + \varepsilon h)|_{\varepsilon=0}.$$

Remark 2.3. (Motivation for defintion 2.2). In analogy with finite-dimensional Riemannian geometry, given a function f on $L(K)$, we should expect the Laplacian $\triangle_{L(K)} f$ to be $div(\nabla f)$ where the divergence $div \equiv tr\nabla$, with ∇ being the Levi-Civita covariant derivative. So given an orthonormal basis $\{h\}$ of $H_0(\Re)$, and letting ∂_h, in $TL(K)$, be the left-invariant vector field associated to h, we should expect

$$\text{“}\triangle f = div(\nabla f) = \sum_h \partial_h \cdot \nabla_{\partial_h}(\nabla f) = \sum_h \partial_h^2 f - \left(\nabla_{\partial_h} \partial_h\right) \cdot \nabla f.\text{”}$$

However, the sum $\sum_h \partial_h^2 f - \left(\nabla_{\partial_h} \partial_h\right) \cdot \nabla f$ is not well-defined independent of orthonormal basis. As Driver and Lohrenz showed in [12] $\sum_h \partial_h^2 f - \left(\nabla_{\partial_h} \partial_h\right) \cdot \nabla f$ is defined independent of good orthonormal bases. Here an orthonormal basis $\{h\}$ is good if the Lie bracket $\left[h(s), h'(s)\right] = 0$ s-a.s. In that case $\sum_h \partial_h^2 - \left(\nabla_{\partial_h} \partial_h\right)$ reduces to $\sum \partial_h^2$ (which is independent of any orthonormal basis).

2.2. Measures on the loop group

2.2.1. Pinned Wiener measure

Let the Wiener space $W_e(K)$ denote the space of all continuous paths in K starting at the identity. Explicitly

$$W_e(K) \equiv \{\sigma \in C([0,1] \to K) \,|\, \sigma(0) = e\}.$$

Definition 2.4. (Heat Kernel measure on K). Let $t > 0$. The Heat Kernels P_t^K on K are the unique functions so that for any smooth f on K, the function u on $[0,\infty) \times K$ defined by setting $u(t,x) \equiv \int_K f(y) P_t^K\left(x^{-1}y\right) dy$ is a solution to the Heat equation with initial condition f. Explicitly

$$\partial_t u = \frac{1}{2}\triangle_K u$$
$$u(t,x) \to f(x) \text{ as } t \to 0.$$

It is well known that $x \to P_t^K$ are smooth function on K and that $P_t^K(x) = P_t^K\left(x^{-1}\right)$. Here “dy” denotes Haar measure on K.

Definition 2.5. (Wiener Measure on $W_e(K)$). Wiener Measure, μ_t, on $W_e(K)$ with parameter t, is the unique measure so that for any bounded cylinder function f of the form $f(x) = F\left(x_{s_1}, \cdots, x_{s_n}\right)$ we have

$$\mu_t[f] \equiv \int_{K^n} F(x_1, \cdots, x_n) \prod_{i=1}^{n} P^K_{t(s_i - s_{i-1})}\left(x_{i-1}^{-1} x_i\right) dx_i,$$

where $x_0 = e$ and $s_0 = 0$. [The measure μ_1 will also be denoted by μ in the sequel.]

Definition 2.6. (Brownian motion on K). We will state two equivalent definitions. A process $s \to \beta(s)$ is a Brownian motion on K starting at e with parameter t iff:

1. β is a $W_e(K)$-valued random variable distributed according to Wiener measure μ_t
2. the process $s \to \beta(s)$ is a diffusion starting at e with generator $\frac{t}{2}\triangle_K$. This means that the process $s \to \beta(s)$ is a martingale so that $\beta(0) = e$ a.s. and

$$d_s(\phi \circ \beta) = \left(\phi' \circ \beta(s)\right) d_s\beta + \frac{t}{2}(\triangle_K \phi) \circ \beta(s)\, ds$$

for any smooth ϕ on K. Here $\triangle_K$ is the Laplacian on K with respect to the metric $\langle \cdot, \cdot \rangle_K$ on K while $d_s\beta$ denotes the Ito differential of β in the s variable.

The first definition is easier in simpler cases like $\mathbb{R}^d$ or compact Lie groups. The second definition is easier to extend to the infinite-dimensional cases and manifolds. See Definition 2.11.

Definition 2.7. (Pinned Wiener Measure) Pinned Wiener Measure, $\mu_{0,t}$, on $L(K)$ with parameter t is the unique measure on $L(K)$ so that for any bounded cylinder functions f of the form $f(x) = F\left(x_{s_1}, \cdots, x_{s_n}\right)$ where $F \in C^\infty(K)$, then

$$\mu_{0,t}[f] \equiv \int_{K^n} F(x_1, \cdots, x_n) \frac{P^K_{t(1-s_n)}\left(x_n^{-1}\right)}{P^K_t(e)} \prod_{i=1}^{n} P^K_{t(s_i - s_{i-1})}\left(x_{i-1}^{-1} x_i\right) dx_i, \tag{2.1}$$

where $x_0 = e$ and $s_0 = 0$.[We will use the notation μ_0 to denote $\mu_{0,1}$.]

Definition 2.8. (Brownian bridge on K). $s \to \chi(s)$ is a Brownian bridge on K with parameter t if χ is an $L(K)$-valued random variable distributed according to pinned Wiener measure $\mu_{0,t}$.

2.2.2. Heat kernel measure

Definition 2.9. (Brownian Bridge Sheet on $\mathfrak{K}$). A Gaussian process $\{\chi(t)\}_{t\in[0,1]}$ is a Brownian bridge Sheet on $\mathfrak{K}$ if for (t, s) in $[0, 1]^2$, $\chi(t, s)$ is a $\mathfrak{K}$-valued mean-zero Gaussian process with covariance given by

$$E\langle A, \chi(t,s)\rangle_{\mathfrak{K}} \langle B, \chi(\tau,\sigma)\rangle_{\mathfrak{K}} = \langle A, B\rangle_{\mathfrak{K}} (t \wedge \tau)\, G_0(s,\sigma),$$

where $\chi(t,s) \equiv \chi(t)(s) \in \mathfrak{K}$; $A, B \in \mathfrak{K}$; $t, \tau, s, \sigma \in [0,1]$; and $G_0(s,\sigma) \equiv s \wedge \sigma - s\sigma$.

Remark 2.10. It turns out that if χ is a Brownian bridge sheet on $\mathfrak{K}$ then χ_{ts} has a version which is continuous in both its parameters, $t \to \chi_{ts}$ is a Brownian motion on $\mathfrak{K}$ with parameter $G_0(s, s)$ and $s \to \chi_{ts}$ is a Brownian bridge on $\mathfrak{K}$ with parameter t. We will always choose such a jointly-continuous version of χ.

Definition 2.11. (Brownian motion on $L(K)$). A process $t \to \Sigma(t, \cdot)$ is an $L(K)$-valued Brownian motion if and only if for any smooth cylinder function $f : L(K) \to \mathbb{R}$, there is a real-valued martingale M_t so that

$$d_t [f(\Sigma(t, \cdot))] = d_t M + \frac{1}{2}\left(\triangle_{L(K)} f\right)(\Sigma(t, \cdot))\, dt.$$

See Theorem 2.14 for the existence of this Brownian motion. So $t \to \Sigma(t, \cdot)$ is a diffusion on $L(K)$ with generator $\frac{1}{2}\triangle_{L(K)}$. [Define a Brownian motion on $L(\mathfrak{K})$ by thinking of $\mathfrak{K}$ as a Lie group and applying the above definition]

We will need the the following Theorem:

Theorem 2.12. (Malliavin). *Let* $\left(\Omega_0, \mathfrak{F}^0, \{\mathfrak{F}^0_{ts}\}_{(t,s)\in[0,1]^2}, P_0\right)$ *be a filtered complete probability space where*

$$\mathfrak{F}^0_{ts} \equiv \sigma\langle \chi_{\tau u} : \tau \in [0, t], u \in [0, s]\rangle,$$

$\mathfrak{F}^0 \equiv \vee_{(t,s)\in[0,1]^2} \mathfrak{F}^0_{ts}$, *and* χ *is a* $\mathfrak{K}$*-valued Brownian bridge sheet in the sense of Definition* 2.9. *Let* ∂_t *denote Stratonowicz differentiation in the* t *variable. Then given* $k_0 \in L(K)$ *there is a jointly continuous solution* $\Sigma(t, s)$ *to the stochastic differential equation*

$$\partial_t \Sigma(t, s) = \sum_{A\in ONB(\mathfrak{K})} \left(\ell_{\Sigma(t,s)*} A\right) \partial_t \chi^A(t, s) \tag{2.2}$$
$$\text{with } \Sigma(0, s) = k_0(s), \forall s \in [0, 1],$$

where the A *run through an orthonormal basis of* $\mathfrak{K}$ *and* $\chi^A(t, s) \equiv \langle \chi(t, s), A\rangle_{\mathfrak{K}}$. *Henceforth we write Eq.* (2.2) *more concisely as*

$$\partial_t \Sigma(t, s) = \left(\ell_{\Sigma(t,s)}\right)_* \partial_t \chi(t, s) \text{ with } \Sigma(0, s) = k_0(s), \forall s \in [0, 1]. \tag{2.3}$$

[*see Malliavin* [23]; *see also Theorem* 3.8 *of* [10] *and Baxendale* [6]]

Remark 2.13. (Explicit Matrix Representation of Eq. [2.3]). Let $\mathscr{M}_m(\mathbb{R})$ be all $m \times m$ matrices on $\mathbb{R}$ and $GL_m(\mathbb{R})$ be all invertible matrices in $\mathscr{M}_m(\mathbb{R})$. We will work with an explicit matrix representation of our Lie group K. K will be thought of as a subgroup of $GL_m(\mathbb{R}) \subset \mathscr{M}_m(\mathbb{R})$ for some m. Such a representation exists as a consequence of the Peter-Weyl Theorem. Hence Eq. (2.3) can be rewritten as

$$\partial_t \Sigma(t, s) = \Sigma(t, s)\, \partial_t \chi(t, s) \tag{2.4}$$
$$\text{with } \Sigma(0, \cdot) = k_0, \forall s \in [0, 1],$$

where we have used matrix multiplication to define $\Sigma(t, s)\, \partial_t \chi(t, s)$. Explicitly if we let B_{ij} denote the i, j entry of the matrix B we have

$$\partial_t (\Sigma(t, s))_{ij} = \sum_k (\Sigma(t, s))_{ik}\, \partial_t (\chi(t, s))_{kj}.$$

Theorem 2.14. (Brownian motion on $L(K)$). *Let $\Sigma(t,s)$ be the process from Theorem 2.12 and Remark 2.13. Theorem 2.12 tells us that $s \to \Sigma(t,s)$ is a Loop a.s. Let Σ_t denote this loop $s \to \Sigma(t,s)$. Then $t \to \Sigma_t$ is a Brownian motion on $L(K)$ in the sense of Definition 2.11.*

Proof. See Theorem 3.10 of Driver [10]. □

Now that we know that Brownian motion on $L(K)$ exists, we can define Heat Kernel measure on $L(K)$.

Definition 2.15. (Heat Kernel measure on $L(K)$). Let $k_0 \in L(K)$ be a loop and let $t > 0$. Let $\Sigma(t,\cdot)$ be an $L(K)$-valued Brownian motion so that $\Sigma(0,\cdot) = k_0$ in $L(K)$ a.s. Then, as in the finite-dimensional manifold case, Heat Kernel measure $\nu_t(k_0, dk)$ is defined to be the law of $\Sigma(t,\cdot)$. Explicitly

$$\int_{L(K)} f(k)\,\nu_t(k_0, dk) = Ef \circ \Sigma(t,\cdot)\,.$$

Remark 2.16. (Heat Kernel measure is a Heat Kernel). Driver and Lohrenz showed for any $t > 0$, for all bounded cylinder functions f on $L(K)$; the function u on $(0,\infty) \times L(K)$ defined by

$$u(t,k_0) \equiv \int_{L(K)} f(k)\,\nu_t(k_0, dk)\,,$$

is the unique solution to the heat equation

$$\partial u(t,\cdot)/\partial t = \frac{1}{2}\triangle_{L(K)} u(t,\cdot) \text{ with } \lim_{t\downarrow 0} u(t,k) = f(k_0)\,.$$

Here $\triangle_{L(K)}$ denotes the operator from Definition 2.2. See Theorem 1.1 of [12]. See also Definitions 3.10 and 4.17 in [12]. In [12], results on Heat kernel measures are obtained for groups of compact type, and not merely compact Lie groups.

2.3. The stochastic framework

We shall use the results of Section 2.2.2 to obtain our probability space.

Definition 2.17. (Ambient probability space). $\left(\Omega, \mathfrak{F}, \{\mathfrak{F}_{ts}\}_{(t,s)\in[0,1]^2}, P\right)$ is going to be our biparametrically-filtered probability space where

- $\Omega \equiv C([0,1] \to L(K))$ equipped with $\mathfrak{F}$, the completion of the Borel σ-algebra under P.
- Let Σ be the process from Theorem 2.12 so that $\Sigma_0 = e$, where e denotes the identity loop.
- P is defined to be Wiener Measure on $C([0,1] \to L(K))$. Explicitly, $P \equiv Law\ \Sigma$.
- $g_t : C([0,1] \to L(K)) \to L(K)$ by $x \to x(t)$ for any $x \in C([0,1] \to L(K))$
- By Theorem 2.14 we see that $dLaw\ g_t = d\nu_t(e,\cdot)$.

- $g_{ts}(x) = [x(t)](s)$ in K.
- $\mathfrak{F}_{00}$ is a σ-algebra containing all the null sets of $\mathfrak{F}$.
- $\mathfrak{F}_{ts} \equiv \sigma \langle g_{\tau\sigma} : \tau \in [0,t] \text{ and } \sigma \in [0,s] \rangle \vee \mathfrak{F}_{00}$.

Theorem 2.18. (Semimartingale properties of $g_{\cdot s}$). *The process g of Definition 2.17 has the following properties:-*

1. *The process $t \to g_{ts}$ is a semimartingale.*
2. *Let $X_{ts} \equiv \int_0^t g_{\tau s}^{-1} \partial_\tau g_{\tau s}$. Then $t \to X_{t\cdot}$ is a Brownian bridge sheet on $\mathfrak{K}$ with respect to the measure P. Furthermore, X can be taken to be continuous in both its parameters.*

Proof. $t \to \Sigma_{ts}$ a Brownian motion on $K \Rightarrow t \to g_{ts}$ a Brownian motion on K. In particular, g is a semimartingale and $X_{ts} \equiv \int_0^t g_{\tau s}^{-1} \partial_\tau g_{\tau s}$ is well-defined. By Proposition 8.3 of [9] we know that $X_{ts} \circ \Sigma = \int_0^t \Sigma_{\tau s}^{-1} \partial_\tau \Sigma_{\tau s} = \chi_{ts}$. Thus X is a Brownian bridge sheet with respect to the measure P. □

Remark 2.19. We shall never again refer to χ, Σ or the underlying abstract probability space. Also we will always use the version of X that is continuous in both parameters t and s.

We are now in a position to state the main result of this paper.

Theorem 2.20. (Semimartingale properties of $g_{t\cdot}$). *Let g be an $L(K)$-valued Brownian motion as in Defintion 2.17. Then:-*

1. *$s \to g_{ts}$ is a K-valued $\mathfrak{F}_{ts}$-semimartingale.*
2. $$\int_0^s \partial_\sigma g_{t\sigma} g_{t\sigma}^{-1} = W_{ts} - \int_0^s \frac{d\sigma}{1-\sigma} \int_0^t Ad_{g_{\tau\sigma}} d_\tau X_{\tau\sigma},$$
   where $s \to W_{ts}$ is a Brownian motion on $\mathfrak{K}$ with parameter t.

Proof. Theorem 2.20 is a special case of Theorem 4.1 proved in Section 4. □

Theorem 2.21. *Let $z < 1$ and let $\mathfrak{G}_z \equiv \sigma \langle x_s : s \in [0,z] \rangle$ where $x_s : L(K) \to K$ is the evaluation map at time s. Then pinned Wiener measure, μ_0, is absolutely continuous with respect to Heat Kernel measure, $\nu_1(e,\cdot)$, on the σ-algebra $\mathfrak{G}_z$.*

Proof. This Theorem is proved as Theorem 5.1 in Section 5. □

3. Motivation for theorem 2.20

Definition 3.1. (Brownian Sheet on $\mathfrak{K}$). A Gaussian process $\{\beta(t)\}_{t\in[0,1]}$ is a $\mathfrak{K}$-valued Brownian sheet if for (t,s) in $[0,1]^2$, $\beta(t,s)$ is a $\mathfrak{K}$-valued mean-zero Gaussian process with covariance given by

$$E\langle A, \beta(t,s)\rangle_{\mathfrak{K}} \langle B, \beta(\tau,\sigma)\rangle_{\mathfrak{K}} = \langle A, B\rangle_{\mathfrak{K}} (t \wedge \tau) G(s,\sigma),$$

where $\beta(t,s) \equiv \beta(t)(s) \in \mathfrak{K}$; $A, B \in \mathfrak{K}$; $t,\tau,s,\sigma \in [0,1]$; and $G(s,\sigma) \equiv \min(s,\sigma)$.

Remark 3.2. (Theorem 2.20 is reasonable). g satisfies

$$\partial_t g_{ts} = g_{ts} \partial_t X_{ts} \text{ with } g_{0s} = e, \tag{3.1}$$

where $X_{\cdot s}$ is the Brownian bridge sheet from Theorem 2.18. By Theorem 3.7, there is a Brownian sheet b on $\mathfrak{K}$ so that

$$X_{ts} = b_{ts} - \int_0^s b_{t\sigma} \frac{(1-s)}{(1-\sigma)^2} d\sigma. \tag{3.2}$$

If we replace X by b in Eq. (3.1), then Lemma 3.3 shows that $s \to g_{ts}$ would be a K-valued Brownian motion with variance t and hence $\int_0^{\cdot} \partial_s g_{ts} g_{ts}^{-1}$ would be a $\mathfrak{K}$-valued Brownian motion with variance t. In reality, because $X_{t\cdot}$ contains an extra finite-variation term, it turns out that the law of $\int_0^{\cdot} \partial_s g_{ts} g_{ts}^{-1}$ is equivalent (but not equal) to the law of a Brownian motion on $\mathfrak{K}$.

Lemma 3.3. (Semimartingale properties of $h_{t\cdot}$). *Let b be a $\mathfrak{K}$-valued Brownian Sheet* (*see Definition* 3.1). *Let h_{ts} be the solution to*

$$\partial_t h_{ts} = h_{ts} \partial_t b_{ts} \text{ with } h_{0s} = e. \tag{3.3}$$

Then the process $s \mapsto h_{ts}$ is a K-valued Brownian motion with parameter t. Furthermore one can choose a version of h which is jointly continuous in both parameters s and t. In future, h will be taken to be this jointly continuous solution. Note:- Eq. [3.3] *is to be interpreted like Eq.* [2.2].

Proof. Let $s_i = i/n$. Then $\{0 = s_0 < s_1 < \cdots < s_n = 1\}$ is a partition of $[0, T]$. For convenience, let $\triangle_i b\,(t) \equiv b_{ts_i} - b_{ts_{i-1}}$. We compute

$$\begin{aligned}
&\partial_t \left(h_{ts_i} h_{ts_{i-1}}^{-1}\right) \\
&= h_{ts_i} \partial_t b_{ts_i} h_{ts_{i-1}}^{-1} - h_{ts_i} \partial_t b_{ts_{i-1}} h_{ts_{i-1}}^{-1} \\
&= h_{ts_i} \partial_t \triangle_i b\,(t)\, h_{ts_{i-1}}^{-1} \\
&= \left(h_{ts_i} h_{ts_{i-1}}^{-1}\right) Ad_{h_{ts_{i-1}}} \partial_t \triangle_i b\,(t) \\
&= \left(h_{ts_i} h_{ts_{i-1}}^{-1}\right) Ad_{h_{ts_{i-1}}} d_t \triangle_i b\,(t) + \frac{1}{2} d_t \left(h_{ts_i} h_{ts_{i-1}}^{-1}\right) Ad_{h_{ts_{i-1}}} d_t \triangle_i b\,(t) \\
&\quad + \frac{1}{2} Ad_{h_{ts_{i-1}}} \left[d_t b_{ts_{i-1}}, d_t \triangle_i b\,(t)\right] \\
&= \left(h_{ts_i} h_{ts_{i-1}}^{-1}\right) Ad_{h_{ts_{i-1}}} d_t \triangle_i b\,(t) + \frac{1}{2} d_t \left(h_{ts_i} h_{ts_{i-1}}^{-1}\right) Ad_{h_{ts_{i-1}}} d_t \triangle_i b\,(t) \\
&= \left(h_{ts_i} h_{ts_{i-1}}^{-1}\right) \partial_t \int_0^t Ad_{h_{\tau s_{i-1}}} \triangle_i b\,(d\tau)\,,
\end{aligned}$$

where we have used that fact that $b_{ts_{i-1}} \in \mathfrak{F}_{1s_{i-1}}$ and that $\triangle_i b\,(\cdot)$ is independent of $\mathfrak{F}_{1s_{i-1}}$. Thus

$$\partial_t \left(h_{ts_i} h_{ts_{i-1}}^{-1}\right) = \left(h_{ts_i} h_{ts_{i-1}}^{-1}\right) \partial_t \int_0^t Ad_{h_{\tau s_{i-1}}} d_\tau \triangle_i b\,(\tau) \text{ with } h_{0s_i} h_{0s_{i-1}}^{-1} = e.$$

It suffices to show that $\left\{\int_0^{\cdot} Ad_{h_{ts_{i-1}}} d_t \triangle_i b(t)\right\}_{i\in\{1,\cdots,n\}}$ is a $\mathfrak{R}^n$-valued Brownian motion with parameter $1/n$, since this will imply that $t \to \left\{h_{ts_i} h_{ts_{i-1}}^{-1}\right\}_{i\in\{1,\cdots,n\}}$ is a K^n-valued Brownian motion with the same parameter. But this is true by Levy's criterion and the following computation of quadratic variations.

Let J_t denote the joint quadratic variation

$$\int_0^t Ad_{h_{\tau s_{i-1}}} d_\tau \triangle_i b(\tau)\, Ad_{h_{\tau s_{j-1}}} d_\tau \triangle_j b(\tau)\,.$$

Then

$$\begin{aligned} dJ_t &= Ad_{h_{ts_{i-1}}}\left(d_t \triangle_i b(t)\right) Ad_{h_{ts_{j-1}}}\left(d_t \triangle_j b(t)\right) \\ &= \sum_{A,B}\left(Ad_{h_{ts_{i-1}}} A \otimes Ad_{h_{ts_{j-1}}} B\right) d_t \triangle_i b^A(t)\, d_t \triangle_j b^B(t) \\ &= \delta_{ij} \triangle_i s dt \sum_A \left(Ad_{h_{ts_{i-1}}} A\right)^{\otimes 2} \\ &= \frac{\delta_{ij}}{n} \sum_A \left(Ad_{h_{ts_{i-1}}} A\right)^{\otimes 2} dt \\ &= \frac{\delta_{ij}}{n} \sum_A A^{\otimes 2} dt. \end{aligned}$$

We still have to show that h_{ts} has a jointly continuous version. That is by Kolmogorov's continuity criterion we must show that

$$P\left[d\left(h_{ts}, h_{\tau\sigma}\right)^p\right] \le C\left[(t-\tau)^2 + (s-\sigma)^2\right]^{\frac{m+\beta}{2}},$$

where $d(x,y)$ denotes the distance between points x and y in K. The proof is essentially the same as that done in Theorem 3.8 of Driver [10] with the modification that $G(s,\sigma)$ is used in place of $G_0(s,\sigma)$. in particular, see Eq. [3.12] of [10]. □

3.1. Semimartingale properties of X_{ts}

Let X_{ts} be as in Theorem 2.18. Then X is a Brownian bridge sheet on $\mathfrak{R}$. Brownian Sheets are easier to work with than Brownian bridge Sheets (they are martingales in both their parameters for instance). The goal of this section is to write $X_{t\cdot}$ as a linear functional of $b_{t\cdot}$, a Brownian sheet.

To motivate this decomposition we recall the decomposition of a Brownian bridge $\widetilde{X}$ (below in Remark 3.4) in terms of a Brownian motion and a finite-variation part. The Brownian bridge $\widetilde{X}_\cdot$ is supposed to play the role of $X_{t\cdot}$ but with one fewer parameter.

Remark 3.4. (Doob's h transform). Let $\widetilde{X}$ be a Brownian bridge from 0 to 0 on $\mathbb{R}$. Then there is a Brownian motion $\widetilde{b}$ which can be written as a linear function of $\widetilde{X}$

$$\text{(i.e. } \widetilde{b}_s \equiv \widetilde{X}_s - \int_0^s \left(\nabla \ln P_{1-\sigma}^{\mathbb{R}}\right)(\widetilde{X}_\sigma) d\sigma = \widetilde{X}_s + \int_0^s \frac{\widetilde{X}_\sigma}{1-\sigma} d\sigma).$$

Definition 3.5. Define continuous $\Re$-valued linear maps on paths,

$$T_s, S_s : C([0,1] \to \Re) \to \Re,$$

by setting

$$T_s(y) = y(s) - \int_0^s y(\sigma)\frac{(1-s)}{(1-\sigma)^2}d\sigma \text{ if } s \in [0,1).$$
$$S_s(x) \equiv x(s) + \int_0^s \frac{x(\sigma)}{(1-\sigma)}d\sigma \text{ if } s \in [0,1).$$

Remark 3.6. Notice that in Remark 3.4 we wrote the underlying Brownian motion $\widetilde{b}.$ as $S\left(\widetilde{X}.\right)(\cdot)$. Similarly we shall prove the process $b_{t\cdot} \equiv S(X_{t\cdot})$ is a Brownian Sheet and that $X_{t\cdot}$ can be written as $T(b_{t\cdot})$.

Theorem 3.7. (Decomposition of the Brownian bridge sheet). *Let X be the Brownian bridge sheet from Theorem 2.18. Define b by setting*

$$b_{ts} \equiv S_s(X_{t\cdot}) = X_{ts} + \int_0^s \frac{X_{t\sigma}d\sigma}{1-\sigma} \text{ for any } t, s \in [0,1].$$

Then b is a Brownian sheet on $\Re$ and X_{ts} can be recovered from b as:

$$X_{ts} = T_s(b_{t\cdot}) = b_{ts} - \int_0^s b_{t\sigma}\frac{(1-s)}{(1-\sigma)^2}d\sigma. \tag{3.4}$$

We shall defer the proof of Theorem 3.7 until after Lemma 3.8 below.

Lemma 3.8. (Properties of the transformations S and T). *There exist unitary maps $T : H(\Re) \to H_0(\Re)$ and $S : H_0(\Re) \to H(\Re)$ so that $T(y)(s) = T_s(y)$ and $S(x)(s) = S_s(x)$ for any $s \in [0,1)$. Furthermore $S = T^{-1}$.*

Proof. Define a subset of $H(\Re)$ by setting

$$U \equiv \left\{y : y' \in C_c^\infty((0,1) \to \Re)\right\}.$$

$C_c^\infty((0,1) \to \Re)$ is dense in $L^2([0,1] \to \Re)$ and therefore, by the isometry between L^2 and $H(\Re)$,U is dense in $H(\Re)$.

Define $x(s) = T_s(y)$ for $y \in U$. Then

$$x(s) = \int_0^s \frac{(1-s)}{1-\sigma}y'(\sigma)d\sigma,$$

and

$$x'(s) = y'(s) - \int_0^s \frac{y'(\sigma)}{1-\sigma}d\sigma. \tag{3.5}$$

Since y' is zero near 1, $s \to y'(s)/(1-s)$ is bounded on $[0,1]$ and x' is constant near $s = 1$. The boundedness of $y'(s)/(1-s)$ implies that $x(s) \to 0$ as $s \to 1$. So we can define a map $T : U \to H_0(\Re)$.

We claim that T is a norm-preserving and so can be extended to a map from $H(\mathfrak{R})$ onto a closed subspace of $H_0(\mathfrak{R})$. Integrating by parts,

$$\begin{aligned}
&\int_0^1 ds\left[\int_0^s \frac{y'(\sigma)}{1-\sigma}d\sigma\right]^2 - 2\int_0^1 y'(s)\cdot\int_0^s \frac{y'(\sigma)}{1-\sigma}d\sigma ds\\
&\quad=\left[\int_0^1 \frac{y'(\sigma)}{1-\sigma}d\sigma\right]^2 - 2\int_0^1 \frac{y'(s)}{1-s}\cdot\int_0^s \frac{y'(\sigma)}{1-\sigma}d\sigma ds\\
&\quad=\left[\int_0^1 \frac{y'(\sigma)}{1-\sigma}d\sigma\right]^2 - 2\int_0^1 \left[\int_0^s \frac{y'(\sigma)}{1-\sigma}d\sigma\right]\cdot d_s\left[\int_0^s \frac{y'(\sigma)}{1-\sigma}d\sigma\right]\\
&\quad=0,
\end{aligned}$$

and so expanding with Eq. [3.5] we see that

$$\int_0^1 \left|x'(s)\right|^2 ds = \int_0^1 \left|y'(s)\right|^2 ds.$$

If $x \perp \operatorname{Im} T$, then for any $y \in H(\mathfrak{R})$

$$\begin{aligned}
0 &= \int_0^1 x'(s)\cdot\left[y'(s) - \int_0^s \frac{y'(\sigma)}{1-\sigma}d\sigma\right]ds\\
&\Longleftrightarrow \int_0^1 x'(s)\cdot y'(s)\,ds = \int_0^1 (x(1)-x(\sigma))\cdot\frac{y'(\sigma)}{1-\sigma}d\sigma\\
&\Longleftrightarrow \int_0^1 \left[x'(s)+\frac{x(s)}{1-s}\right]\cdot y'(s)\,ds = 0.
\end{aligned}$$

Since y' can be any arbitrary element of $L^2([0,1]\to\mathfrak{R})$, we must have

$$x'(s) = \frac{-x(s)}{1-s} \Longleftrightarrow x(s) = x(\varepsilon)\frac{1-s}{1-\varepsilon}.$$

Letting $\varepsilon \to 0$ and using $x(0)=0$, we see that x is identically 0. Thus T is surjective and so provides an isometry between $H(\mathfrak{R})$ and $H_0(\mathfrak{R})$.

Now we find the inverse of T. Let x in $H(\mathfrak{R})$, $y = T(x)$ in $H_0(\mathfrak{R})$ and $z_s = S_s(y)$ so that $z_s = S_s \circ T(x)$. Since $x(0) = z_0 = 0$ and

$$\begin{aligned}
\frac{d}{ds}z_s &= y'(s) + \frac{y(s)}{1-s}\\
&= x'(s) - \int_0^s \frac{x'(\sigma)\,d\sigma}{1-\sigma} + \frac{y(s)}{1-s}\\
&= x'(s) - \frac{T(x)(s)}{1-s} + \frac{y(s)}{1-s}\\
&= x'(s),
\end{aligned}$$

we conclude that $S_s \circ T(x) = x(s)$ for any x in $H(\mathfrak{R})$.

Define $S : H_0(\mathfrak{R}) \to H(\mathfrak{R})$ so that $S(y)(s) = S_s(y)$ for any $s < 1$ and $S(y)(1) = \lim_{s\to 1} S(y)(s)$. Then by definition, S is the inverse of T. □

Proof of Theorem 3.7. First we show that

$$E\,\langle b_{ts}, A\rangle_{\Re}\,\langle b_{\tau\sigma}, B\rangle_{\Re} = (t\wedge\tau)\,G\,(s,\sigma)\,\langle B, A\rangle_{\Re}$$

Recall $b^A_{ts} \equiv \langle b_{ts}, A\rangle_{\Re}$ and $X^A_{ts} \equiv \langle X_{ts}, A\rangle_{\Re}$. Let

$$l_s\,(x) \equiv \int_0^1 d_u\alpha_s\,(u)\,x\,(u)\,,$$

where

$$d_u\alpha_s\,(u) = \left[\delta\,(u-s) + 1_{[0,s]}\frac{1}{1-u}\right]du$$

is a positive measure on [0, 1]. Here δ denotes the Dirac delta measure. Then

$$l_s\,(x) = x\,(s) + \int_0^1 x\,(u)\,\frac{du}{1-u} = S_s\,(x)\,.$$

Define $b_{ts} \equiv S_s\,(X_{t\cdot})$ as in Definition 3.5. So

$$Eb^A_{ts}b^B_{\tau\sigma} = E\int d_u\alpha_s\,(u)\,d_v\alpha_\sigma\,(\nu)\,X^A_{tu}X^B_{\tau\nu}. \tag{3.6}$$

By Tonelli's Theorem and Hölder's inequality, we have

$$\begin{aligned} E\ &\int d_u\alpha_s\,(u)\,d_v\alpha_\sigma\,(\nu)\left|X^A_{tu}X^B_{\tau\nu}\right| \\ &\le \int d_u\alpha_s\,(u)\,d_v\alpha_\sigma\,(\nu)\sqrt{E\left(X^A_{tu}\right)^2 E\left(X^B_{\tau\nu}\right)^2} \\ &= \int d_u\alpha_s\,(u)\,d_v\alpha_\sigma\,(\nu)\sqrt{t\tau G_0\,(u,u)\,G_0\,(\nu,\nu)} < \infty. \end{aligned}$$

Thus applying Fubini to Eq. [3.6] we see that

$$\begin{aligned} Eb^A_{ts}b^B_{\tau\sigma} &= \int d_u\alpha_s\,(u)\,d_v\alpha_\sigma\,(\nu)\,EX^A_{tu}X^B_{\tau\nu} \\ &= (t\wedge\tau)\,\langle A, B\rangle_{\Re}\int d_u\alpha_s\,(u)\int d_v\alpha_\sigma\,(\nu)\,G_0\,(u,\nu)\,. \end{aligned} \tag{3.7}$$

Since $G_0\,(u,\nu)$ is the reproducing kernel for $H_0\,(\Re)$, we have for any orthonormal basis h of $H_0\,(\Re)$,

$$G_0\,(u,\nu) = \sum h\,(u)\,h\,(\nu)\,.$$

Returning to Eq. [3.7] we get

$$\begin{aligned} E\,\langle b_{ts}, A\rangle_{\Re}\,\langle b_{\tau\sigma}, B\rangle_{\Re} &= (t\wedge\tau)\,\langle A, B\rangle_{\Re}\sum\int h\,(u)\,d_u\alpha_s\,(u)\int h\,(\nu)\,d_v\alpha_\sigma\,(\nu) \\ &= (t\wedge\tau)\,\langle A, B\rangle_{\Re}\sum S_s\,(h)\,S_\sigma\,(h) \\ &= (t\wedge\tau)\,\langle A, B\rangle_{\Re}\sum S\,(h)\,(s)\,S\,(h)\,(\sigma)\,. \end{aligned}$$

By Lemma 3.8, $S(h)$ runs through an orthonormal basis of $H(\mathfrak{K})$. This, together with the fact that $G(s,\sigma)$ is the reproducing kernel for $H(\mathfrak{K})$ yields,

$$\sum S(h)(s)\,S(h)(\sigma) = G(s,\sigma) = s \wedge \sigma.$$

Thus

$$E\,\langle b_{ts}, A\rangle_{\mathfrak{K}}\,\langle b_{\tau\sigma}, B\rangle_{\mathfrak{K}} = (t \wedge \tau)\,\langle A, B\rangle_{\mathfrak{K}}\,(s \wedge \sigma)\,.$$

Thus b is a $\mathfrak{K}$-valued Brownian sheet.

It remains to show that $T_s(b_{t\cdot}) = X_{ts}$. Let $x \in L(\mathfrak{K})$. Then for any $\varepsilon > 0$, can choose a δ so that $\sup_{[0,\delta]\cup[1-\delta,1]} |x| < \varepsilon$. There is an $\widetilde{x}$ on $C^\infty((\delta, 1-\delta) \to \mathfrak{K})$ so that $\left\|\widetilde{x} - x\downarrow_{(\delta,1-\delta)}\right\|_\infty < \varepsilon$. Define $\overline{x}$ by setting

$$\overline{x}(s) = \frac{s}{\delta}\widetilde{x}(\delta)\,1_{[0,\delta]} + \widetilde{x}(s)\,1_{(\delta,1-\delta)} + \frac{1-s}{\delta}\widetilde{x}(1-\delta)\,1_{[1-\delta,1]}.$$

Then

$$\|\overline{x} - x\|_\infty < \left(2 \sup_{[0,\delta]\cup[1-\delta,1]} |x| + \varepsilon\right) \vee \varepsilon < 3\varepsilon.$$

Furthermore, $\overline{x} \in H_0(\mathfrak{K})$.

So now take $x \in H_0(\mathfrak{K})$ so that $\|x - X_{t\cdot}\|_\infty < \varepsilon$. Then $T_s(b_{t\cdot}) = T_s(b_{t\cdot} - S(x)) + x(s)$.

$$|T_s(b_{t\cdot} - S(x))| \le \sup_{[0,s]} |b_{t\cdot} - S(x)| \left(1 + \int_0^s \frac{(1-s)}{(1-\sigma)^2} d\sigma\right).$$

$$\begin{aligned}\sup_{[0,s]} |b_{t\cdot} - S(x)| &\le \sup_{u\in[0,s]} \left| X_{tu} - x(u) + \int_0^u \frac{X_{t\sigma} - x(\sigma)}{(1-\sigma)} d\sigma \right| \\ &\le \sup_{u\in[0,s]} |X_{tu} - x(u)| \left(1 + \int_0^s \frac{d\sigma}{(1-\sigma)}\right).\end{aligned}$$

Thus as $\varepsilon \to 0$ we have $T_s(b_{t\cdot}) = X_{ts}$ and we are done. □

4. Semi martingale properties of g_T.

Let X be the Brownian bridge sheet from subsection 3.1. In this section we shall show that X is a semimartingale in the sense of Norris (see [24]) and then proceed to find $\int_0^\cdot \partial_s g_{Ts} g_{Ts}^{-1}$ using his powerful two-parameter calculus methods. Throughout this section we shall use the term “semimartingale” to mean “semimartingale in the sense of Norris”. We shall also stick to Norris’ notation as far as possible in this section.

Theorem 4.1. (Semimartingale properties of $g_{T\cdot}$). *Let g be our $L(K)$-valued Brownian motion. Then:-*

1. *g is a semimartingale.*

2. *There is a* $\mathfrak{K}$*-valued Brownian sheet* $\widetilde{b}$ *with the same law as* b *so that*

$$\int_0^s \partial_\sigma g_{T\sigma} g_{T\sigma}^{-1} = \widetilde{b}_{Ts} - \int_0^s \frac{d\sigma}{1-\sigma} \int_0^T Ad_{g_{t\sigma}} d_t X_{t\sigma}.$$

We defer the proof of this Theorem to the end of the section.

Theorem 4.2. (X is a semimartingale). *On the domain* $\mathfrak{D} \equiv \{(t,s) \in [0,\infty) \times [0,1)\}$, *$X$ is a uniform semimartingale satisfying the equation*

$$\partial_t \partial_s X = \partial_t \partial_s b - \frac{1}{1-s} \partial_t X \partial s \text{ with } X_{0s} = X_{t0} = 0. \tag{4.1}$$

Proof. First we show X is a uniform (s,t)-semimartingale in the sense of Norris. By Theorem 3.7 we have

$$X_{ts} = b_{ts} - \int_0^s b_{t\sigma} \frac{(1-s)}{(1-\sigma)^2} d\sigma.$$

Since b is clearly a semimartingale, it will suffice to show that the expression

$$C_{ts} = \int_0^s b_{t\sigma} \frac{(1-s)}{(1-\sigma)^2} d\sigma$$

is an (s,t)-semimartingale. Differentiating, we see that

$$C_{ts} = \int_0^s d\sigma \left[\frac{b_{t\sigma}}{(1-\sigma)} - \int_0^\sigma b_{tu} \frac{1}{(1-u)^2} du \right]. \tag{4.2}$$

By Ito

$$b_{t\sigma} \frac{1}{(1-\sigma)} = \int_0^\sigma b_{tu} \frac{1}{(1-u)^2} du + \int_0^\sigma d_u b_{tu} \frac{1}{(1-u)},$$

and so

$$\begin{aligned} C_{ts} &= \int_0^s d\sigma \int_0^\sigma \frac{1}{(1-u)} d_u b_{tu} \\ &= \int_{-1}^t \int_0^s \left[\int_0^t \int_{-1}^s \frac{1}{(1-u')} 1_{[0,u]}(u') \, 1_{[-1,0]}(r') \, d_r d_{u'} b_{ru'} \right] dr' du. \end{aligned}$$

We still need to check Eq. (2.15) of Norris; that for any $p \in [1,\infty)$, the expression

$$\int_{-1}^t \int_0^s \left[\int_0^t \int_{-1}^s \left\| \frac{1}{(1-u')} 1_{[0,u]}(u') \, 1_{[-1,0]}(r') \right\|_{L^p}^2 dr du' \right]^{1/2} dr' du$$

is finite. A direct computation establishes that this is so for any $(t,s) \in \mathfrak{D}$. Thus X is a uniform (s,t)-semimartingale. By Norris (see Pg 282 of [24]) we know that a uniform (s,t)-semimartingale which is constant on both the t and s axes is both a uniform t-semimartingale as well as a uniform s-semimartingale. Since $X_{0s} = X_{t0} = 0$, we conclude that X is a uniform semimartingale.

Now that we have shown X is a semimartingale, we can verify by direct computation that X satisfies Eq. [4.1].

$$\begin{aligned}\partial_t \partial_s X - \partial_t \partial_s b + \frac{1}{1-s} \partial_t X \partial s \\ = \partial_t X \partial s - \partial_t \partial_s C_{ts}.\end{aligned}$$

From Eq. [4.2] we see that

$$\begin{aligned}&\partial_t \partial_s X - \partial_t \partial_s b + \frac{1}{1-s} \partial_t X \partial s \\ &= \frac{\partial_t X \partial s}{1-s} - \left[\partial_t b_{ts} - \int_0^s \partial_t b_{t\sigma} \frac{1-s}{(1-\sigma)^2} d\sigma \right] \frac{\partial s}{1-s} \\ &= 0.\end{aligned}$$

Hence we are done. □

Lemma 4.3. (g is a semimartingale). *Let g be our $L(K)$-valued Brownian motion from Definition* 2.17*. Then on $\mathfrak{D}$, g is a semimartingale satisfying*

$$D_s \partial_t \psi = \psi \partial_t \partial_s b - \frac{1}{1-s} \partial_t \psi \partial s \text{ with } g_{0s} = g_{t0} = e.$$

Proof. Define ψ to be the solution of

$$D_s \partial_t \psi = \psi \partial_t \partial_s b - \frac{1}{1-s} \partial_t \psi \partial s, \tag{4.3}$$

where D denotes covariant differentiation with respect to the left connection on K. Apply Theorem 3.2.6 of [24] to the $\mathbb{R} \times K$-valued process $\widetilde{\psi}_{ts} \equiv (s, \psi_{ts})$. To facilitate this define coordinate projections $\pi_1 : (s, x) \to s$ and $\pi_2 : (s, x) \to x$. Then we have

$$\widetilde{D}_s \partial_t \widetilde{\psi} = \left(0, \pi_2\left(\widetilde{\psi}\right) \partial_t \partial_s b\right) - \frac{1}{1-s} \partial_t \widetilde{\psi} \partial s,$$

where $\widetilde{D}$ denotes covariant differentiation with respect to the left connection on $\mathbb{R} \times K$. In Norris notation, we would have

$$\widetilde{D}_s \partial_t \widetilde{\psi} = \alpha \langle \partial_t \partial_s b \rangle + \beta_{01} \left\langle \partial_t \widetilde{\psi} \partial s \right\rangle,$$

with

$$\alpha \langle \omega \rangle = (0, \pi_2(x) \omega) \ \forall \omega \in \mathfrak{K},$$

and

$$\beta_{01} \langle \omega \rangle = \left(\frac{-1}{1 - \pi_1(x)} \right) \omega \ \forall \omega \in T_x(\mathbb{R} \times K).$$

Thus $\widetilde{\psi}$ is a semimartingale, and hence so is ψ.

Define

$$\begin{aligned}\chi_{\cdot s} &= \int_0^{\cdot} \psi_{ts}^{-1}\partial_t\psi_{ts} \\ &= \int_0^{\cdot} \psi_{ts}^{-1}d_t\psi_{ts} - \frac{1}{2}\int_0^{\cdot}\left(\psi_{ts}^{-1}d_t\psi_{ts}\right)\left(\psi_{ts}^{-1}d_t\psi_{ts}\right) \\ &= I - J.\end{aligned}$$

The first term I is a semimartingale by Theorem 2.3.1 of [24], while the second term J is a semimartingale by Theorem 2.3.2 of [24]. Thus χ is a semimartingale. We will now show that χ satisfies Eq. [4.1]. Computing directly, we see that

$$\begin{aligned}\partial_t\partial_s\chi + \frac{1}{1-s}\partial_t\chi\partial s &= \partial_s\left(\psi^{-1}\partial_t\psi\right) + \frac{1}{1-s}\psi^{-1}\partial_t\psi\partial s \\ &= \psi^{-1}D_s\partial_t\psi + \frac{1}{1-s}\psi^{-1}\partial_t\psi\partial s.\end{aligned}$$

Applying Eq. [4.3], we see that

$$\partial_t\partial_s\chi + \frac{1}{1-s}\partial_t\chi\partial s = \partial_t\partial_s b.$$

Thus $\chi = X$ and $\psi = g$. Thus g is a semimartingale satisfying the equation

$$D_s\partial_t g = g\partial_t\partial_s b - \frac{1}{1-s}\partial_t g\partial s. \qquad \square$$

We are now able to return to the proof of Theorem 4.1.

Proof of Theorem 4.1. We have already shown that g is a semimartingale. Let D and $\widehat{D}$ denote covariant differentiation with respect to the left and right connections on K respectively. Thus

$$\begin{aligned}\widehat{D}_t\partial_s g &= \left(\partial_t\left(\left(\partial_s g\right)g^{-1}\right)\right)g \\ &= \partial_t\partial_s g - \left(\partial_s g\right)g^{-1}\left(\partial_t g\right) \\ &= g\partial_s\left(g^{-1}\partial_t g\right) \\ &= D_s\partial_t g \\ &= g\partial_t\partial_s b - \frac{1}{1-s}\partial_t g\partial s.\end{aligned}$$

Now define a process $\widetilde{b}$ by setting

$$\partial_t\partial_s\widetilde{b} = Ad_g\left\langle\partial_t\partial_s b\right\rangle \text{ with } \widetilde{b}_{0s} = \widetilde{b}_{t0} = 0.$$

Ad_g is previsible and preserves the inner product on $\mathfrak{K}$ and so by Theorem 2.4.1 of [24], we see that $\widetilde{b}$ is a Brownian sheet on $\mathfrak{K}$ (i.e. $\widetilde{b}$ and b have the same distribution). Therefore,

$$\left(\partial_t\left(\left(\partial_s g\right)g^{-1}\right)\right)g = \widehat{D}_t\partial_s g = \left(\partial_t\partial_s\widetilde{b}\right)g - \partial_t g\frac{ds}{1-s},$$

which implies that

$$\partial_t \left((\partial_s g)\, g^{-1} \right) = \left(\partial_t \partial_s \widetilde{b} \right) - Ad_g \langle \partial_t X \rangle \frac{ds}{1-s}.$$

This means that

$$\int_0 (\partial_s g)\, g^{-1} = \widetilde{b} - \int_0 \int_0 Ad_g \langle \partial_t X \rangle \frac{ds}{1-s}.$$

Now using the fact that $d_t g = g d_t X + \frac{1}{2} g d_t X d_t X$, we see that

$$\begin{aligned} Ad_g \partial_t X &= Ad_g d_t X + \frac{1}{2} (d_t g d_t X)\, g^{-1} - \frac{1}{2} g\, (d_t X)\, g^{-1} (d_t g)\, g_{ts}^{-1} \\ &= Ad_g d_t X. \end{aligned}$$

Thus we have shown that

$$\int_0 (\partial_s g)\, g^{-1} = \widetilde{b} - \int_0 \frac{ds}{1-s} \int_0 Ad_g \langle d_t X \rangle . \qquad \square$$

5. $HKM \downarrow_{\mathfrak{G}_z} \sim PWM \downarrow_{\mathfrak{G}_z}$

Let Ω, P, and g_t be as in Definition 2.17 and let $\mathfrak{G}_z$ be as in Theorem 2.21. Then $t \to g_t$ is an $L(K)$-valued Brownian motion and thus $Law\ g_t$ equals Heat Kernel measure $\nu_t(e, \cdot)$. From Section 4 we know that $\int_0^s \partial_\sigma g_{t\sigma} g_{t\sigma}^{-1}$ is a Brownian semimartingale. In this Section we will show that $\int_0^s \partial_\sigma g_{t\sigma} g_{t\sigma}^{-1}$ has a law equivalent to that of a Brownian motion. We will then know that Pinned Wiener measure is equivalent to Heat Kernel measure on $\mathfrak{G}_z$ for any $z < 1$, by the equivalence of Wiener measure and Pinned Wiener measure.

Theorem 5.1. *$\nu_1(e, \cdot)$[Heat Kernel measure on $L(K)$] is equivalent to μ_0 [Pinned Wiener Measure on $L(K)$] as measures on $\left(L(K), \mathfrak{G}_z\right)$ where $\mathfrak{G}_z \equiv \sigma \langle x_t : t \in [0, z] \rangle$, for any $z < 1$.*

We supply the proof of this result after the statement of Lemma 5.4.

Definition 5.2. Let B_{ts} be defined to solve the Fisk-Stratonowicz equation $\partial_s B_{ts} = \partial_s b_{ts} B_{ts}$ with $B_{t0} = e$ where b is the Brownian sheet from Theorem 3.7.

Theorem 5.3. *Let $t \mapsto g_{t\cdot}$ be our $L(K)$-valued Brownian motion from Definition* 2.17 *and let $s \to B_{ts}$ be the K-valued Brownian motion of Definition* 5.2*. Then $g_{T\cdot}$ and $B_{T\cdot}$ have equivalent laws as measures on $C([0, s] \to K)$ for any $s < \frac{\sqrt{1+4T}-1}{2T}$.*

We prove this result in Section 5.1.

Lemma 5.4. *If $\kappa_1 \sim \kappa_2$ then $\kappa_1 \otimes \nu \sim \kappa_2 \otimes \nu$, where κ_1, κ, ν are probability measures.*

We prove this Lemma after the proof of Theorem 5.1 below.

Lemma 5.5. *Let $0 < t_1 < \cdots < t_k$. Then $B^{-1}_{t_{i-1}\cdot} B_{t_i\cdot}$ are independent $\mathfrak{K}$-valued Brownian motions with parameters $t_i - t_{i-1}$.*

The proof of this Lemma is supplied after the proof of Lemma 5.4

Proof of Theorem 5.1. Let μ_0 be Pinned Wiener measure on $L(K)$ and let μ be Wiener measure on $C([0,1] \to K)$ as in Definitions 2.5 and 2.7. Then $\mu = Law[B_{1\cdot}]$ since $B_{1\cdot}$ is a standard K-valued Brownian motion by Definition 5.2. A key fact that we shall exploit in this proof is μ_0 is equivalent to μ on $\mathfrak{G}_z$ for any $z < 1$.

Fix $z < 1$. Now

$$\lim_{T \to 0} \frac{\sqrt{1+4T} - 1}{2T} = 1$$

so there exists an $N \in \mathbb{N}$ large so that

$$z < \frac{\sqrt{1+4/N} - 1}{2/N}.$$

Let $T \equiv 1/N$. We know that $t \to g_{t\cdot}$ is an $L(K)$-valued Brownian motion and so has independent increments. Suppose we can show for $t_2 > t_1$ that $B^{-1}_{t_1 s} B_{t_2 s}$ is independent of $\mathfrak{F}_{t_1 1}$. Then letting $A' = \{(k_1, \cdots, k_n) : k_1 \cdots k_N \in A\}$ we have

$$\begin{aligned}
\nu_1(e, A) &= P\{g_{1\cdot} \in A\} \\
&= P\left\{g_{(1/N)\cdot}\left(g^{-1}_{(1/N)\cdot} g_{(2/N)\cdot}\right) \cdots \left(g^{-1}_{(N-1/N)\cdot} g_{1\cdot}\right) \in A\right\} \\
&= \left(\otimes_{i=1}^{N} Law_{g_{T\cdot}}\right)(A'),
\end{aligned}$$

while

$$\begin{aligned}
\mu(A) &= P\{B_{1\cdot} \in A\} \\
&= P\left\{B_{(1/N)\cdot}\left(B^{-1}_{(1/N)\cdot} B_{(2/N)\cdot}\right) \cdots \left(B^{-1}_{(N-1/N)\cdot} B_{1\cdot}\right) \in A\right\} \\
&= \left(\otimes_{i=1}^{N} Law_{B_{T\cdot}}\right)(A').
\end{aligned}$$

Now by Theorem 5.3 $g_{T\cdot}$ has a law equivalent to that of $B_{T\cdot}$, on the restricted σ-algebra $\mathfrak{G}_z$. Invoking Lemma 5.4 repeatedly, we see that $\otimes_{i=1}^{N} Law_{g_{T\cdot}} \sim \otimes_{i=1}^{N} Law_{B_{T\cdot}}$, on the restricted σ-algebra $\mathfrak{G}_z$. Thus if A is $\mathfrak{G}_z$-measurable, $A' \in \mathfrak{G}_z^{\otimes N}$ and

$$\begin{aligned}
\nu_1 \ (e, A) = \left(\otimes_{i=1}^{N} Law_{g_{T\cdot}}\right)(A') . = 0 \\
\iff \mu(A) = \left(\otimes_{i=1}^{N} Law_{B_{T\cdot}}\right)(A') = 0.
\end{aligned}$$

Hence we are done if we show that $B^{-1}_{t_1 s} B_{t_2 s}$ is independent of $\mathfrak{F}_{t_1 1}$.

By Lemma 5.5, if $\tau \le t_1 < t_2$ then $B_{\tau\cdot}$ and $B_{t_1\cdot}^{-1}B_{t_2\cdot}$ are independent Brownian motions and so $B_{t_1\cdot}^{-1}B_{t_2\cdot}$ is independent of $\sigma\left\langle B_{ts} : t \le t_1 \text{ and } s \le 1\right\rangle$. However since $b_{ts} = \int_0^s B_{t\delta u}B_{tu}^{-1}$, we see that

$$\begin{aligned}\mathfrak{F}_{t_1 1} &= \sigma\left\langle b_{ts} : t \le t_1 \text{ and } s \le 1\right\rangle \\ &\subset \sigma\left\langle B_{ts} : t \le t_1 \text{ and } s \le 1\right\rangle.\end{aligned}$$

Therefore $B_{t_1\cdot}^{-1}B_{t_2\cdot}$ is independent of $\mathfrak{F}_{t_1 1}$ and we are finished. □

Proof of Theorem 5.4. It will suffice to show that if $\kappa_1 \ll \kappa_2$ then $\kappa_1 \otimes \nu \ll \kappa_2 \otimes \nu$. For rectangles, it is clear that $(\kappa_1 \otimes \nu)(1_A(x)1_B(y)) = (\kappa_2 \otimes \nu)(1_A(x)f(x)1_B(y))$. This extends to linear combinations of rectangles by linearity and all bounded measurable functions by dominated convergence. Thus $d(\kappa_1 \otimes \nu)/d(\kappa_2 \otimes \nu)(x,y) = d\kappa_1/d(\kappa_2)(x)$. Thus $\kappa_1 \otimes \nu \ll \kappa_2 \otimes \nu$. □

Proof of Theorem 5.5. From Definition 5.2 we see that

$$\partial_s\left(B_{t_{i-1}s}^{-1}B_{t_is}\right) = \left[Ad_{B_{t_{i-1}s}^{-1}}\partial_s\left(b_{t_is} - b_{t_{i-1}s}\right)\right]B_{t_{i-1}s}^{-1}B_{t_is}. \tag{5.1}$$

Let $\widetilde{b}_i \equiv \int_0^{\cdot} Ad_{B_{t_{i-1}s}^{-1}}\partial_s\left(b_{t_is} - b_{t_{i-1}s}\right)$. Then

$$\begin{aligned}\widetilde{b}_i &= \int_0^{\cdot} Ad_{B_{t_{i-1}s}^{-1}} d_s\left(b_{t_is} - b_{t_{i-1}s}\right) + \frac{1}{2}\int_0^{\cdot} Ad_{B_{t_{i-1}s}^{-1}}\left[d_s\left(b_{t_is} - b_{t_{i-1}s}\right), d_s b_{t_{i-1}s}\right] \\ &= \int_0^{\cdot} Ad_{B_{t_{i-1}s}^{-1}} d_s\left(b_{t_is} - b_{t_{i-1}s}\right).\end{aligned}$$

Thus since the Adjoint action is norm preserving, we have that $\widetilde{b}_i$ is a Brownian motion on $\mathfrak{K}$. Computing quadratic variations, we see that

$$\begin{aligned}&d_s\widetilde{b}_i(s) \otimes d_s\widetilde{b}_j(s) \\ &\quad = Ad_{B_{t_{i-1}s}^{-1}} d_s\left(b_{t_is} - b_{t_{i-1}s}\right) \otimes Ad_{B_{t_{j-1}s}^{-1}} d_s\left(b_{t_js} - b_{t_{j-1}s}\right) \\ &\quad = \sum_A\left(Ad_{B_{t_{i-1}s}^{-1}}A \otimes Ad_{B_{t_{j-1}s}^{-1}}A\right) d_s\left(b_{t_js}^A - b_{t_{j-1}s}^A\right) d_s\left(b_{t_is}^A - b_{t_{i-1}s}^A\right) \\ &\quad = \delta_{ij}\left(\sum_A A^{\otimes 2}\right)(t_i - t_{i-1})\,ds.\end{aligned}$$

□

5.1. *Proof of Theorem 5.3*

Theorem 5.6. *Let Y_{Ts} be defined to be $\int_0^s \partial_\sigma g_{T\sigma}g_{T\sigma}^{-1}$ as in Theorem* 4.1. *Then Law $Y_{T\cdot} \sim$ Law $b_{T\cdot}$ as measures on $C([0,s] \to \mathfrak{K})$ for any $s < \frac{\sqrt{1+4T}-1}{2T}$.*

The proof of Theorem 5.6 is given in Section 5.1.1.

Proof of Theorem 5.3 Fix s. Pick T so that $s < \frac{\sqrt{1+4T}-1}{2T}$.

Let x_σ be the evaluation map at σ on $C([0,s] \to \Re)$. Define the probability spaces Ω^b (resp. Ω^Y) as the set $C([0,s] \to \Re)$ as equipped with $Law\left(b_{T\cdot} \downarrow_{[0,s]}\right)$ (resp. $Law\left(Y_{T\cdot} \downarrow_{[0,s]}\right)$) and filtration generated by x. Let η^b and η^Y be the solutions to the stochastic differential equation

$$\partial\eta = \eta\partial x \text{ with } \eta_0 = 1 \tag{*}$$

in the probability spaces Ω^b and Ω^y respectively. Then Proposition 8.3 of [9] implies that $\eta^b\left(b_{T\cdot} \downarrow_{[0,s]}\right) = B_{T\cdot} \downarrow_{[0,s]}$ and $\eta^Y\left(Y_{T\cdot} \downarrow_{[0,s]}\right) = g_{T\cdot} \downarrow_{[0,s]}$. Theorem 5.6 implies that $b_{T\cdot} \downarrow_{[0,s]}$ and $Y_{T\cdot} \downarrow_{[0,s]}$ have equivalent laws and hence $\eta \equiv \eta^b = \eta^Y$ a.s. Thus if

$$\begin{aligned}
&E1_A\left(g_{T\cdot} \downarrow_{[0,s]}\right) = 0\\
\iff\ &E1_A \circ \eta\left(Y_{T\cdot} \downarrow_{[0,s]}\right) = 0\\
\iff\ &E1_A \circ \eta\left(b_{T\cdot} \downarrow_{[0,s]}\right) = 0\\
\iff\ &E1_A\left(B_{T\cdot} \downarrow_{[0,s]}\right) = 0.
\end{aligned}$$

Hence by the Radon-Nikodym Theorem $Law\ g_{T\cdot}$ is equivalent to $Law\ B_{T\cdot}$ as measures on $C([0,s] \to K)$. □

5.1.1. Proof of Theorem 5.6

Remark 5.7. Let $\pi_s : C([0,1] \to L) \to C([0,s] \to L)\,; \pi_s(x.)(r) = x(r)$ for any $r \leq s$. We make no distinction between a measure ν_1 on $(C([0,s] \to L), \sigma\langle x_r : r \leq s\rangle)$ and a measure ν_2 on $(C([0,1] \to L), \sigma\langle x_r : r \leq s\rangle)$ so long as $\nu_1(F \circ \pi_s) = \nu_2(F)$ for any $F : C([0,s] \to L) \to \mathbb{R}$. where L stands for either K or $\Re$.

Remark 5.8. (Theorem 5.6 is not obvious). Since for $s < 1$, $Law\ X_{T.} \sim Law\ b_{T.}$ (as measures on $C([0,s] \to \Re)$), one might suspect $Law\ X \sim Law\ b$ (as measures on $C([0,1] \times [0,s] \to \Re)$) which should then indicate that

$$\begin{aligned}
Law\,(Y_T.) &= Law \int_{R_{T\cdot}} Ad_{g_{t\sigma}} d_t d_\sigma X_{t\sigma}\\
&\sim Law \int_{R_{T\cdot}} Ad_{g_{t\sigma}} d_t d_\sigma b_{t\sigma} = Law\,(b_T.)\,.
\end{aligned}$$

Unfortunately in the t-variable, $X_{.s}$ and $b_{.s}$ are Brownian motions with parameters $s - s^2$ and s respectively. Thus $Law X \perp Law b$ since

$$P_X\left(\sum_i |\Delta_i\omega(s)|^2 \to s - s^2\right) = 1,$$

while

$$P_b\left(\sum_i |\Delta_i\omega(s)|^2 \to s\right) = 1.$$

Hence these two measures live on different sets.

Theorem 5.9. (Girsanov, see [19]). *Let* $(\Omega, \mathfrak{F}, \{\mathfrak{F}_\cdot\}, P)$ *be a filtered probability space. Let* $\beta_\cdot$ *be a* d*-dimensional Brownian motion and let* $Z_\cdot$ *be an* $\mathbb{R}^d$*-valued adapted process so that* $E \exp \frac{1}{2} \int_0^S |Z_s|^2 \, ds$ *is finite and* $\int_0^S \left(Z_s^i\right)^2 ds < \infty$ *almost surely for any* $i \in \{1, \cdots, d\}$*. Define*

$$\widetilde{Z}_\cdot \equiv \exp\left[\int_0^\cdot Z_s \cdot d_s\beta_s - \frac{1}{2}\int_0^\cdot |Z_s|^2 \, ds\right].$$

Define a new measure $\widetilde{P}_S$ *on* $\mathfrak{F}_S$ *by setting* $\widetilde{P}(A) = E 1_A \widetilde{Z}_S$*. Then* $\widetilde{P}_S$ *is a probability equivalent to* P *and the process* $\{Y_t, \mathfrak{F}_t; 0 \le s \le S\}$ *is a* d*-dimensional Brownian motion on* $\left(\Omega, \mathfrak{F}_S, \widetilde{P}\right)$ *where* $Y_\cdot \equiv \beta_\cdot + \int_0^\cdot Z_s ds$*.*

We will use the following two Lemmas which are proven in Section 5.1.2.

Lemma 5.10. *Let* X *be the* $\mathfrak{K}$*-valued Brownian bridge sheet of Theorem 2.18. Then the expression* $\widetilde{X}_{t\sigma} \equiv \int_0^t Ad_{g_{u\sigma}} d_u X_{u\sigma}$ *has the same law as* $X_{t\sigma}$*.*

Lemma 5.11. *Let* X *be the* $\mathfrak{K}$*-valued Brownian bridge sheet of Theorem 2.18. Then*

$$P \exp\left[\frac{1}{2}\int_0^s d\sigma \left|\int_0^T Ad_{g_{t\sigma}} \frac{d_t X_{t\sigma}}{1-\sigma}\right|_{\mathfrak{K}}^2\right] < \infty, \text{ if } s < \frac{\sqrt{1+4T}-1}{2T}.$$

Proof of Theorem 5.6. Define

$$Z_T(\sigma) \equiv \frac{-1}{(1-\sigma)} \int_0^T Ad_{g_{t\sigma}} d_t X_{t\sigma}.$$

By definition of $Y_{T\cdot}$ in Theorem 4.1

$$Y_{T\cdot} \equiv \widetilde{b}_{T\cdot} + \int_0^\cdot d\sigma Z_T(\sigma).$$

By Lemma 5.11,

$$E \exp \int_0^S |Z_T(\sigma)|_{\mathfrak{K}}^2 \, d\sigma < \infty \text{ whenever } S < \frac{\sqrt{1+4T}-1}{2T}.$$

Thus the measure

$$d\widetilde{P}_S \equiv \exp\left[\int_0^S Z_T(s) \cdot d_s\widetilde{b}_{Ts} - \frac{1}{2}\int_0^S |Z_T(s)|^2 \, ds\right] dP$$

is a probability on $\mathfrak{F}_{TS}$ and the process $\{Y_{Ts}, \mathfrak{F}_{Ts}; 0 \le s \le S\}$ is a $\widetilde{P}_S$-Brownian motion on $\mathfrak{K}$. Thus for any set $\mathscr{A} \subset (C[0,S] \to \mathfrak{K})$

$$E 1_{\mathscr{A}} \circ \widetilde{b}_{T\cdot} = 0 \iff \widetilde{E} 1_{\mathscr{A}} \circ Y_{T\cdot} = 0 \iff E 1_{\mathscr{A}} \circ Y_{T\cdot} = 0,$$

since the measures $\widetilde{P}_S$ and P are equivalent on $\mathfrak{F}_{TS}$. [Note:- it is essential that $\mathscr{A}$ only depend on the path to time S or else $1_{\mathscr{A}} \circ Y_{T\cdot}$ will cease to be $\mathfrak{F}_{TS}$-measurable.]

□

5.1.2. Proofs of Lemmas 5.10 and 5.11

Proof of Theorem 5.10. $\widetilde{X}_{t\sigma}$ is a $\mathfrak{F}_{t\sigma}$ martingale. To show $X_{\cdot s}$ and $\widetilde{X}_{\cdot s}$ have the same law it will suffice to show $\widetilde{X}_{\cdot\sigma}$ is a $\mathfrak{K}$-valued Brownian motion with parameter $\sigma - \sigma^2$. To this end, let $\{A\}$ run through an orthonormal basis of $\mathfrak{K}$. Then

$$\begin{aligned} d_t\widetilde{X}_{t\sigma} \otimes d_t\widetilde{X}_{t\sigma} &= Ad_{g_{t\sigma}} d_t X_{t\sigma} \otimes Ad_{g_{t\sigma}} d_t X_{t\sigma} \\ &= \left(\sigma - \sigma^2\right) dt \sum_{A,B} \delta_{AB} Ad_{g_{t\sigma}} A \otimes Ad_{g_{t\sigma}} B \\ &= \left(\sigma - \sigma^2\right) dt \sum_{A} \left(Ad_{g_{t\sigma}} A\right)^{\otimes 2} \\ &= \left(\sigma - \sigma^2\right) dt \sum_{A} A^{\otimes 2}. \end{aligned}$$

Thus we are done.

Proof of Theorem 5.11. By using Jensen's inequality on the probability $1_{[0,s]}\frac{d\sigma}{s}$ we have

$$\begin{aligned} &\exp\left[\frac{1}{2}\int_0^s d\sigma \left|\int_0^T Ad_{g_{t\sigma}} \frac{d_t X_{t\sigma}}{1-\sigma}\right|_{\mathfrak{K}}^2\right] \\ &= \exp\left[\int_0^s \frac{d\sigma}{s}\frac{s}{2}\left|\int_0^T Ad_{g_{t\sigma}} \frac{d_t X_{t\sigma}}{1-\sigma}\right|_{\mathfrak{K}}^2\right] \\ &\le \int_0^s \frac{d\sigma}{s} \exp\left[\frac{s}{2}\left|\int_0^T Ad_{g_{t\sigma}} \frac{d_t X_{t\sigma}}{1-\sigma}\right|_{\mathfrak{K}}^2\right]. \end{aligned}$$

Thus

$$\begin{aligned} J &\equiv P\exp\left[\frac{1}{2}\int_0^s d\sigma \left|\int_0^T Ad_{g_{t\sigma}} \frac{d_t X_{t\sigma}}{1-\sigma}\right|_{\mathfrak{K}}^2\right] \\ &\le \int_0^s \frac{d\sigma}{s} P\exp\left[\frac{s}{2(1-\sigma)^2}\left|\int_0^T Ad_{g_{t\sigma}} d_t X_{t\sigma}\right|_{\mathfrak{K}}^2\right]. \end{aligned}$$

Now by Lemma 5.10, $\widetilde{X}_{\cdot} = \int_0^{\cdot} Ad_{g_{t\sigma}} d_t X_{t\sigma}$ is a Brownian motion on $\mathfrak{K}$ with parameter $G_0(\sigma,\sigma)$. Thus in the expectation, we can replace $\left|\int_0^T Ad_{g_{t\sigma}} d_t X_{t\sigma}\right|_{\mathfrak{K}}^2$ by $TG_0(\sigma,\sigma)\,|N|_{\mathfrak{K}}^2$, where N is a standard normal $\mathfrak{K}$-valued random variable. So we get

$$J \le \int_0^s \frac{d\sigma}{s} P\exp\frac{sT\sigma}{2(1-\sigma)}\,|N|_{\mathfrak{K}}^2\,.$$

So we see that

$$\begin{aligned} J < \infty &\iff \frac{sT\sigma}{(1-\sigma)} < \infty \text{ for } \sigma \in [0, s] \\ &\iff \frac{\sigma}{1-\sigma} < \frac{1}{sT} \text{ for } \sigma \in [0, s] \\ &\iff \frac{s}{1-s} < \frac{1}{sT} \\ &\iff Ts^2 + s - 1 < 0 \\ &\iff s \in \left[0, \frac{\sqrt{1+4T}-1}{2T}\right). \end{aligned}$$

□

References

1. Aida, S., Elworthy, D.: Differential calculus on path and loop spaces. I. Logarithmic Sobolev inequalities on path spaces. C. R. Acad. Sci. Paris Sér. I Math. **321(1)**, 97–102 (1995)
2. Airault, H., Malliavin, P.: Integration on loop groups. II. Heat equation for the Wiener measure. J. Funct. Anal. **104(1)**, 71–109 (1992)
3. Albeverio, S., Høegh-Krohn, R.: The energy representation of Sobolev-Lie groups. Compositio Math. **36(1)**, 37–51 (1978)
4. Bakry, D., Emery, M.: Diffusions hypercontractives. In Séminaire de probabilités, XIX, 1983/84, pages 177–206, Springer, Berlin-New York, 1985
5. Bakry, D., Emery, M.: Hypercontractivité de semi-groupes de diffusion. C. R. Acad. Sci. Paris Sér. I Math. **299(15)**, 775–778 (1984)
6. Baxendale, P.: Wiener processes on manifolds of maps. Proc. Roy. Soc. Edinburgh Sect. A **87(1-2)**, 127–152 (1980/81)
7. Davies, E.B., Gross, L., Simon, B.: Hypercontractivity: a bibliographic review. In Ideas and methods in quantum and statistical physics (Oslo, 1988), pages 370–389, Cambridge Univ. Press, Cambridge, 1992
8. Deuschel, J.-D., Stroock, D.W.: Large deviations. Academic Press, Inc., Boston, MA, 1989
9. Bruce K. Driver. A Cameron-Martin type quasi-invariance theorem for Brownian motion on a compact Riemannian manifold. J. Funct. Anal. **110(2)**, 272–376 (1992)
10. Driver, B.K.: Integration by parts and quasi-invariance for heat kernel measures on loop groups. J. Funct. Anal. **149(2)**, 470–547 (1997)
11. Driver, B.K., Hall, B.C.: Yang–Mills theory and the Segal–Bargmann transform. Comm. Math. Phys. **201(2)**, 249–290 (1999)
12. Driver, B.K., Lohrenz, T.: Logarithmic Sobolev inequalities for pinned loop groups. J. Funct. Anal., **140(2)**, 381–448 (1996)
13. Frenkel, I.B.: Orbital theory for affine Lie algebras. Invent. Math. **77(2)** 301–352 (1984)
14. Getzler, E.: Dirichlet forms on loop space. Bull. Sci. Math. (2) **113(2)**, 151–174 (1989)
15. Gross, L.: Logarithmic Sobolev inequalities on loop groups. J. Funct. Anal. **102(2)**, 268–313 (1991)
16. Gross, L.: Logarithmic Sobolev inequalities and contractivity properties of semigroups. In Dirichlet forms (Varenna, 1992), pp. 54–88, Springer, Berlin, 1993
17. Gross, L.: Uniqueness of ground states for Schrödinger operators over loop groups. J. Funct. Anal. **112(2)**, 373–441 (1993)

18. Holley, R., Stroock, D.: Logarithmic Sobolev inequalities and stochastic Ising models. J. Statist. Phys., **46(5-6)**, 1159–1194 (1987)
19. Karatzas, I., Shreve, S.E.: Brownian motion and stochastic calculus. Springer-Verlag, New York, second edition, 1991
20. Landsman, N.P., Wren, K.K.: Constrained quantization and θ-angles. Nuclear Phys. B **502(3)**, 537–560 (1997)
21. Malliavin, M.-P., Malliavin, P.: Integration on loop group. III. Asymptotic Peter-Weyl orthogonality. J. Funct. Anal. **108(1)**, 13–46 (1992)
22. Malliavin, M.-P., Malliavin, P.: Integration on loop groups. III. Quasi invariant measures. J. Funct. Anal. **93(1)**, 207–237 (1990)
23. Malliavin, P.: Hypoellipticity in infinite dimensions. In Diffusion processes and related problems in analysis, Vol. I (Evanston, IL, 1989), pages 17–31, Birkhäuser Boston, Boston, MA, 1990
24. Norris, J.R.: Twisted sheets. J. Funct. Anal. **132(2)**, 273–334 (1995)
25. Pressley, A., Segal, G.: Loop groups. The Clarendon Press, Oxford University Press, New York, 1986. Oxford Science Publications.