Probab. Theory Relat. Fields 117, 533–550 (2000)
Digital Object Identifier (DOI) 10.1007/s004400000062

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Martin–Kuramochi boundary and reflecting symmetric diffusion

Received: 23 June 1997 / Revised version: 28 September 1991/
Published online: 14 June 2000 – © Springer-Verlag 2000

Abstract. In this paper, we will give sufficient conditions for the existence of the reflecting diffusion process on a locally compact space. In constructing reflecting diffusion process, we consider the corresponding Martin–Kuramochi boundary as the reflecting barrier and introduce the notion of strong $(\mathscr{E}, u)$-Caccioppoli set. Our method covers reflecting diffusion processes with diffusion coefficient degenerating on the boundary.

1. Introduction

In this paper, we will present a sufficient condition for the existence of the reflecting diffusion process on a general locally compact space. In constructing reflecting diffusion process, we will consider the corresponding Martin–Kuramochi boundary as the reflecting barrier.

In Sect. 2, as the first step to our goal, we will attach the Martin–Kuramochi boundary to a locally compact Hausdorff space, starting with a Dirichlet space $(\mathscr{E}, \mathscr{F})$. The method was originally established by M. Fukushima [F-1] for a bounded Euclidean domain with the classical Dirichlet integral.

The definition of strong $(\mathscr{E}, u)$-Caccioppoli set which is opted for the situation with no coordinate functions will be introduced as an enhancement of strong Caccioppoli set in [C-F-W]. In Sect. 3, we will reorganize the results in their paper which gives sufficient and necessary condition for the reflecting Brownian motion to be a quasi-martingale. To be more precise, the notion of strong $(\mathscr{E}, u)$-Caccioppoli set which is dependent on function u in $\mathscr{F}$ provides a characterization for $u(X_t^*)$ to be an $\{\mathscr{M}_t\}$-quasi-martingale, where $\{X_t^*\}$ stands for the reflecting diffusion process on the state space compactified in Sect. 2 and $\{\mathscr{M}_t\}$-quasi-martingale is defined sightly differently from the quasi-martingale based upon the filtration $\{\mathscr{M}_t\}$ generated by the reflecting diffusion process.

In Sect. 4, we will see the Green's formula in [K] provides a criterion for domain as to whether it is a strong $(\mathscr{E}, u)$-Caccioppoli set. We will also have a generalization of the well-known Minkowski content condition. In fact, in the Euclidean space, this method covers reflecting diffusion processes with diffusion coefficient degenerating on the reflecting barrier.

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Mathematics Subject Classification (1991): 60J60, 60J55, 31C25, 31C35

For notations and full detail on Dirichlet space theory, the reader is referred to the book [F-O-T]. Finally, the author extends his heartfelt thanks to Professor M. Fukushima, Professor H. Kurata, Professor M. Takeda and the referee of the paper for their helpful comments.

2. Martin–Kuramochi boundary

We denote a connected locally compact separable metric space by X and denote the associate metric by d. Suppose that X has a Radon measure m satisfying $\mathrm{supp}[m] = X$ and $m(X) = 1$. We start with a Dirichlet space $(\mathscr{E}, \mathscr{F})$ on $L^2(X; m)$ satisfying the following conditions:

the completion $\mathscr{F}_0$ of $\mathscr{F} \cap C_0(X)$ with respect to the inner product $\mathscr{E}_1(u, v) = \mathscr{E}(u, v) + (u, v)_{L^2(X;m)}$ is a regular Dirichlet form $(\mathscr{E}, \mathscr{F}_0)$ on $L^2(X; m)$, (1)

$(\mathscr{E}, \mathscr{F}_0)$ has the strong local property. (2)

To construct the Martin–Kuramochi boundary, we focus only on the case that the absorbing diffusion process $\{X_t^0, \mathscr{M}_t^0, P_x^0\}$ corresponding to $(\mathscr{E}, \mathscr{F}_0)$ has continuous representation kernel for α-resolvent, that is, there exists a continuous function $G_\alpha(x, y)$ on $(X \times X) \setminus \Delta$ $(\Delta = \{(x, x) \mid x \in X\})$ such that

$$f \in L^2(X; m) \Rightarrow G_\alpha f(x) = \int_X G_\alpha(x, y) f(y) dm(y) \in \mathscr{F}_0,$$
$$\text{and } \mathscr{E}_\alpha(G_\alpha f, v) = (f, v) \quad \text{for all } v \in \mathscr{F}_0,$$

$$f \in \mathscr{C}_0(X) \Rightarrow G_\alpha f(x) = E_x[\int_0^\zeta e^{-\alpha t} f(X_t^0) dt],$$
$$\text{where } \zeta \text{ is the life time of the diffusion process.}$$

Here and elsewhere, we denote $\mathscr{E}(u, v) + \alpha(u, v)_{L^2(X;m)}$ by $\mathscr{E}_\alpha(u, v)$ and denote $(u, v)_{L^2(X;m)}$ simply by (u, v). In addition, we suppose that $G_\alpha f$ is continuous on X for each $f \in L^\infty_{comp}(X) = \{f \in L^\infty(X; m) \mid \mathrm{supp}[f] \text{ is compact in } X\}$ and $G_\alpha 1$ is continuous on X.

The following representation of the bilinear form $\mathscr{E}$ by signed measures of finite total variation will be crucially required:

- for any $u, v \in \mathscr{F}$, the associate co-energy measure $\mu_{<u,v>}$ satisfies

$$\mathscr{E}(u, v) = \mu_{<u,v>}(X).$$

To obtain the co-energy measure for the functions of the larger family $\overline{\mathscr{F}}$, we set $\mathscr{F}_{0,loc} = \{u \in L^2_{loc}(X; m) \mid$ for each domain $D \Subset X$ there exists $u_D \in \mathscr{F}_0$ such that $u|_D = u_D$ on $D\}$ and introduce Chen's L^2–reflected Dirichlet space $\overline{\mathscr{F}} = \{u \in L^2(X; m) \mid$ there exists a sequence of functions $\{u_n\} \subset \mathscr{F}_{0,loc}$ such that $\lim_{n\to\infty} u_n = u$ m-a.e. on X and $\{u_n\}$ is an $\mathscr{E}$-Cauchy sequence$\}$ so that $\overline{\mathscr{E}}(u, v)$ is well defined for u, v in $\overline{\mathscr{F}}$ ([C-1] and [K-T]). We put a basic assumption on $\mathscr{E}(u, v)$ and $(\overline{\mathscr{E}}, \overline{\mathscr{F}})$:

$$\mathscr{F} \subset \overline{\mathscr{F}}, \text{ for any } u, v \in \mathscr{F} \text{ the Dirichlet inner product } \mathscr{E}(u, v) \text{ coincides with } \overline{\mathscr{E}}(u, v) \text{ and } 1 \in \mathscr{F}. \tag{3}$$

Then, it is easy to see from assumption (2) and Lemma 5.6.1 [F-O-T] that, for any $u, v \in \overline{\mathscr{F}}$, $\mu_{<u,v>}$ is well-defined as the weak limit of $\mu_{<u_n,v_n>}$, where $\{u_n\}, \{v_n\} \subset \mathscr{F}_{0,loc}$ are $\overline{\mathscr{E}}$-Cauchy sequences satisfying $\lim_{n\to\infty} u_n = u$ and $\lim_{n\to\infty} v_n = v$ m-a.e. on X. In sect. 3, an extended version of the Green's formula in [K] based on this co-energy measure will be utilized.

To show the pointwise convergence of α-harmonic functions from $\mathscr{E}_\alpha$-convergence, we set the family of α-harmonic functions by

$$\mathscr{H}_\alpha(D) = \{u \in \mathscr{F} \mid \mathscr{E}(u, \varphi) + \alpha(u, \varphi)_{L^2(D;m)} = 0 \text{ for } \forall \varphi \in \mathscr{F} \cap \mathscr{C}_0(D)\},$$

and assume that

$$u \in \mathscr{H}_\alpha(D) \quad \text{and} \quad D_1 \Subset D \Rightarrow \sup_{x,y \in D_1} \frac{|u(x) - u(y)|}{d(x, y)^\beta} \le C\|u\|_{L^2(D;m)} \tag{4}$$

for some positive constants β and C depending on domains D_1 and D. We suppose that

$$m \text{ never charges on any point in } X \text{ and } \inf_{B_x(\delta)\subset D} m(B_x(\delta)) > 0 \text{ holds for } \forall\, \delta > 0, \forall\, D \Subset X, \tag{5}$$

where $B_x(\delta)$ stands for the ball with center x and radius δ.

Lemma 1.

(i) *If $\{u_n\}$ converges to u in $\mathscr{H}_\alpha(X)$ with respect to $\mathscr{E}_\alpha$, then $\{u_n\}$ converges to u uniformly on every compact set in X.*

(ii) *For any $x \in X$, there exists a unique element $R_{\alpha,x} \in \mathscr{H}_\alpha(X)$ satisfying*

$$\mathscr{E}_\alpha(R_{\alpha,x}, v) = v(x) \quad for all\ v \in \mathscr{H}_\alpha(X).$$

(iii) *$R_{\alpha,x}(y)$ is continuous on $X \times X$ as the function of (x, y).*

Proof. (i) is a consequence of assumptions (4) and (5).

(ii) (i) implies the continuity of the functional $\Phi_x(v) = v(x)$ for $v \in \mathscr{H}_\alpha(X)$ with respect to $\mathscr{E}_\alpha$. (ii) is clear from this continuity on Φ_x.

(iii) We first note that the map $R_{\alpha,\cdot} : X \to \mathscr{H}_\alpha(X)$ is continuous. This is because the estimate (4) shows

$$\sup_{\substack{\mathscr{E}_\alpha(w,w)\le 1\\ w\in\mathscr{H}_\alpha(X)}} |\mathscr{E}_\alpha(R_{\alpha,x} - R_{\alpha,x'}, w)| \le \sup_{\substack{\mathscr{E}_\alpha(w,w)\le 1\\ w\in\mathscr{H}_\alpha(X)}} |w(x) - w(x')| \le C d(x, x')^\beta,$$

whenever x and x' are both in some domain $D \Subset X$. In the inequality

$$\begin{aligned}\lim_{\substack{x'\to x\\ y'\to y}} |R_{\alpha,x'}(y')-R_{\alpha,x}(y)| &\leq \lim_{\substack{x'\to x\\ y'\to y}} (|R_{\alpha,x'}(y')-R_{\alpha,x}(y')| + |R_{\alpha,x}(y') - R_{\alpha,x}(y)|)\\ &\leq \lim_{x'\to x} \sup_{y'} |R_{\alpha,x'}(y') - R_{\alpha,x}(y')|\\ &\quad + \lim_{y'\to y} d(y', y)^{\beta} \|R_{\alpha,x}\|_{L^2(X;m)},\end{aligned}$$

it follows from (i) that the first term in the right-hand side vanishes. Since the second term vanishes as well, the assertion has been proved. □

For any open set $G \Subset X$, $\mathscr{H}_\alpha(G)$ is the orthogonal complement of $\{u \in \mathscr{F} \mid u = 0$ quasi-everywhere on $G^c\}$ in $\mathscr{F}$ with respect to the inner product $\mathscr{E}_\alpha$. Therefore, by using the orthogonal projection P_α^G on $\mathscr{H}_\alpha(G)$, the family $\mathscr{SH}_\alpha(X)$ of all continuous α-subharmonic functions is defined by $\mathscr{SH}_\alpha(X) = \{u \in \mathscr{F} \cap \mathscr{C}(X) \mid P_\alpha^G u \geq u$ quasi-everywhere on X, for every open set $G \Subset X\}$. Since the orthogonal projection $P_\alpha^G u$ takes the same value as u outside G for any function $u \in \mathscr{F}$, $P_\alpha^G u$ is non-negative for any non-negative u, otherwise by setting the non-empty open set $G_u = \{x \in G \mid P_\alpha^G u(x) < 0\}$ we are faced with a contradiction $0 > P_\alpha^G u = P_\alpha^{G_u} P_\alpha^G u = 0$ on G_u.

Lemma 2.

(i) $u \in \mathscr{SH}_\alpha(X) \Rightarrow \mathscr{E}_\alpha(R_{\alpha,x}, u) \geq u(x)$.
(ii) $R_{\alpha,x}$ *is a non-negative function on X and symmetric in the sense that* $R_{\alpha,x}(y) = R_{\alpha,y}(x)$ *for* $\forall\, x, \forall\, y \in X$.

Proof. (i) For any $u \in \mathscr{F}$, any sequence $\{D_j\}_{j=0}^{\infty}$ of domains satisfying $D_0 \Subset D_1 \Subset \cdots$ and $\cup_{j=0}^{\infty} D_j = X$ yields a Cauchy sequence $\{P_\alpha^{D_j} u\}_{j=0}^{\infty}$ with respect to the inner product $\mathscr{E}_\alpha$, since we have $j > \ell \Rightarrow \mathscr{E}_\alpha(P_\alpha^{D_j} u, P_\alpha^{D_j} u - P_\alpha^{D_\ell} u) = 0$ and $\|P_\alpha^{D_j} u\|_{\mathscr{E}_\alpha} \leq \|P_\alpha^{D_\ell} u\|_{\mathscr{E}_\alpha}$. The identity $\mathscr{E}_\alpha(u - P_\alpha^{D_j} u, h) = 0$ for $\forall\, h \in \mathscr{H}_\alpha(X)$ and $\forall\, j$ implies $\lim_{j\to\infty} P_\alpha^{D_j} u$ is the orthogonal projection $P_\alpha u$ of u on $\mathscr{H}_\alpha(X)$. From the definition of the subharmonicity, we deduce the following inequality:

$$\mathscr{E}_\alpha(R_{\alpha,x}, u) = \mathscr{E}_\alpha(R_{\alpha,x}, P_\alpha u) = P_\alpha u(x) \geq u(x)$$

for all $u \in \mathscr{SH}_\alpha(X)$.

(ii) For any open set $G \Subset X$, the orthogonal projection P_α^G sends $R_{\alpha,x} \vee 0$ to a non-negative function which dominates $R_{\alpha,x}$ and consequently $R_{\alpha,x} \vee 0$ is in $\mathscr{SH}_\alpha(X)$. On the other hand, thanks to the inequality in (i), it is easy to see that $R_{\alpha,x}$ is a unique minimizer of the functional $\Psi_x(v) = \mathscr{E}_\alpha(v, v) - 2v(x)$ for $v \in \mathscr{SH}_\alpha(X)$. This combined with the inequality $R_{\alpha,x}(x) = \mathscr{E}_\alpha(R_{\alpha,x}, R_{\alpha,x}) \geq 0$ implies that $R_{\alpha,x} \vee 0$ also minimizes the functional $\Psi_x(v)$. Therefore $R_{\alpha,x}$ is a non-negative function. The identity $R_{\alpha,x}(y) = \mathscr{E}_\alpha(R_{\alpha,y}, R_{\alpha,x}) = R_{\alpha,y}(x)$ $(\forall x, \forall y \in X)$ shows $R_{\alpha,x}(y)$ is symmetric. □

For any disjoint sequence of topological Borel subsets $E_1, \cdots, E_n$ of X satisfying $\cup_{k=1}^{n} E_k \Subset X$, let us use a notation $\|\{E_k\}\|$ to describe $\max_k \mathrm{diam}(\mathrm{E_k})$. Moreover, when a sequence $x_1, \cdots, x_n$ satisfies $x_k \in E_k$ $(k = 1, \cdots, n)$, we will denote $\sum_{k=1}^{n} R_{\alpha,x_k}(\cdot)u(x_k)m(E_k)$ by $R^{(E_1,\cdots,E_n)}_{\alpha,x_1,\cdots,x_n}u$.

Lemma 3. *For any $f \in L^2(X;m)$, there exists a unique element $R_\alpha f \in \mathscr{H}_\alpha(X)$ satisfying*

$$\mathscr{E}_\alpha(R_\alpha f, v) = (f, v) \quad for\ every\ v \in \mathscr{H}_\alpha(X), \tag{6}$$

which has a continuous representation $\int_X R_{\alpha,y}(\cdot)f(y)dm(y)$.

Proof. The continuity of the functional $\Phi^{[f]}(v) = (f, v)$ with respect to $\mathscr{E}_\alpha$ implies the existence of $R_\alpha f \in \mathscr{H}_\alpha(X)$ enjoying (6). To validate the continuous representation for $R_\alpha f$, let us start to prove $\lim_{\substack{\cup E_k \uparrow X \\ \|\{E_k\}\| \downarrow 0}} \|R^{(E_1,\cdots,E_n)}_{\alpha,x_1,\cdots,x_n}f - R_\alpha f\|_{\mathscr{E}_\alpha} = 0$, particularly for $f \in \mathscr{C}_0(X)$. For every $v \in \mathscr{H}_\alpha(X)$ and every $f \in \mathscr{C}_0(X)$, the relation

$$\mathscr{E}_\alpha(R^{(E_1,\cdots,E_n)}_{\alpha,x_1,\cdots,x_n}f(\cdot), v) = \sum_{k=1}^{n} f(x_k)v(x_k)m(E_k)$$

can be derived from Lemma 1 (ii). Hence, we obtain

$$\mathscr{E}_\alpha(R^{(E_1,\cdots,E_n)}_{\alpha,x_1,\cdots,x_n}f, R^{(F_1,\cdots,F_m)}_{\alpha,y_1,\cdots,y_m}f) = \sum_{k=1}^{n}\sum_{\ell=1}^{m} R_\alpha(x_k, y_\ell)f(x_k)f(y_\ell)m(E_k)m(F_\ell).$$

Since the Riemann sum in the right-hand side approximates $\int_{X\times X} R_{\alpha,y}(x)f(y)f(x)dm(y)dm(x)$, the convergence

$$\lim_{\substack{(\cup E_k)\cap(\cup F_\ell)\uparrow X \\ ||\{E_k\}||\downarrow 0, ||\{F_\ell\}||\downarrow 0}} ||R^{(E_1,\cdots,E_n)}_{\alpha,x_1,\cdots,x_n}f - R^{(F_1,\cdots,F_m)}_{\alpha,y_1,\cdots,y_m}f||_{\mathscr{E}_\alpha} = 0$$

can be justified by using this expression. As a result, the limit $\lim_{\substack{\cup E_k \uparrow X \\ ||\{E_k\}||\downarrow 0}} R^{(E_1,\cdots,E_n)}_{\alpha,x_1,\cdots,x_n}f$ exists in $\mathscr{F}$, which clearly coincides with $R_\alpha f$ in $L^2(X;m)$. By applying simple function approximation instead of the Riemann sum approximation, a similar argument works to verify that $\int_X R_{\alpha,y}(\cdot)f(y)dm(y)$ is a continuous representation of $R_\alpha f$ for any $f \in L^\infty_{comp}(X)$. Since $L^\infty_{comp}(X)$ is dense in $L^2(X;m)$, the continuous representation is extended to any function f in $L^2(X;m)$. □

Lemma 4. *For each $\alpha > 0$, we set $K_\alpha(x,y) = G_\alpha(x,y) + R_{\alpha,y}(x)$, then $\{K_\alpha(x,y)\}$ has the following properties:*

(i) *$K_\alpha(x,y)$ is non-negative continuous function on $(X \times X) \setminus \Delta$,*
(ii) *$K_\alpha(x,y) = K_\alpha(y,x)$ for any distinct two points $x, y \in X$,*
(iii) *$\alpha K_\alpha 1 = 1$,*
(iv) *$f \in L^2(X;m)$ and $v \in \mathscr{F} \Rightarrow \mathscr{E}_\alpha(K_\alpha f, v) = (f, v)$.*

Proof. (i) We first note that $G_\alpha u$ is a unique minimizer of the functional $\Psi^{[u]}(v) = \mathscr{E}_\alpha(v, v) - 2(v, u)$ for $v \in \mathscr{F}_0$. Now, we notice that $G_\alpha u \vee 0$ also minimizes the functional $\Psi^{[u]}(v)$, whenever u is non-negative. Therefore $G_\alpha(x, \cdot)$ is non-negative a.e. on X. Since we assumed the continuity of $G_\alpha(x, y)$ on $(X \times X) \setminus \Delta$, (i) is clear from Lemma 1 (iii) and Lemma 2 (ii).

(ii) (5) enables for us to take a sequence $\{f_n\} \subset \mathscr{C}_0(X)$ such that $f_n \to \delta_{\{x\}}$ with respect to the weak topology of $C_0(X)$. Letting $f \to \delta_{\{x\}}$ and $g \to \delta_{\{y\}}$ in the identity $(G_\alpha f, g) = \mathscr{E}_\alpha(G_\alpha f, G_\alpha g) = (f, G_\alpha g)$, we know that $G_\alpha(x, y) = G_\alpha(y, x)$ for any $x \neq y$. Together this with $R_\alpha(x, y) = R_\alpha(y, x)$ (for all $x, y \in X$), we obtain (ii).

(iii) By an easy calculation, we get

$$\mathscr{E}_\alpha(1 - \alpha G_\alpha 1, w) = \alpha\{(1, w) - \mathscr{E}_\alpha(G_\alpha 1, w)\}.$$

The right-hand side vanishes, whenever $w \in \mathscr{F} \cap \mathscr{C}_0$. This shows the $\mathscr{E}_\alpha$-harmonicity of $1 - \alpha G_\alpha 1$. By putting $w \in \mathscr{H}_\alpha(X)$ in the equality, in view of Lemma 3, we obtain that $\alpha R_\alpha 1 = 1 - \alpha G_\alpha 1$.

(iv) It suffices to show that $\mathscr{E}_\alpha(K_\alpha f, u) = (f, u)$ for any $f \in L^2(X; m)$ and any $u \in \mathscr{F}$ described as $u = w + v$, $w \in \mathscr{F} \cap \mathscr{C}_0(X)$ and $v \in \mathscr{H}_\alpha(X)$. It follows from the definition of the representation kernel of α-resolvent and Lemma 3 that

$$\begin{aligned}\mathscr{E}_\alpha(K_\alpha f, u) &= \mathscr{E}_\alpha(G_\alpha f, w) + \mathscr{E}_\alpha(R_\alpha f, v)\\ &= \mathscr{E}_\alpha(G_\alpha f, w) + \mathscr{E}_\alpha(R_\alpha f, v)\\ &= (f, w) + (u, v)\\ &= (f, u),\end{aligned}$$

from whence we can deduce (iv). □

Lemma 5. *If K^1 and K^2 are compact sets satisfying $K^1 \cap \overline{X \setminus K^2} = \emptyset$, then* $\sup_{x \in K^1, y \in X \setminus K^2} K_\alpha(x, y) < \infty$.

Proof. We take a domain D enjoying $K^1 \subset D$ and $\overline{D} \cap \overline{X \setminus K^2} = \emptyset$. We set $\mathscr{E}_{\alpha, X \setminus \overline{D}}(u, v) = \int_{X \setminus \overline{D}} \mu_{<u,v>} + \alpha(u, v)_{L_2(X \setminus \overline{D}; m)}$ and set $U_\alpha = K_\alpha f|_{X \setminus \overline{D}}$ for non-negative function $f \in \mathscr{C}_0(X)$ satisfying supp[f] $\subset K^1$ and $\int_{K^1} f(x) m(dx) = 1$. Then it turns out that

$$\mathscr{E}_{\alpha, X \setminus \overline{D}}(U_\alpha, U_\alpha - u) = \mathscr{E}_\alpha(K_\alpha f, K_\alpha f - u) = (f, K_\alpha f - u) = 0$$

for all $u \in \mathscr{F}_{K^2}^{[K_\alpha f]} = \{v \in \mathscr{F}_b \mid v = K_\alpha f \text{ on } K^2\}$. It follows from this identity that

$$\mathscr{E}_{\alpha, X \setminus \overline{D}}(U_\alpha, U_\alpha) = \mathscr{E}_{\alpha, X \setminus \overline{D}}(u, u) - \mathscr{E}_{\alpha, X \setminus \overline{D}}(U_\alpha - u, U_\alpha - u),$$

for all $u \in \mathscr{F}_{K^2}^{[K_\alpha f]}$. Since $\min\{K_\alpha f(x), \sup_{y \in K^2 \setminus \overline{D}} K_\alpha f(y)\}$ coincides with $K_\alpha f(x)$ on $K^2 \setminus \overline{D}$, by taking a function $\phi \in \mathscr{F}$ such that

$$\phi(x) = \begin{cases} 1\,, & \text{if } x \in \overline{D} \\ 0\,, & \text{if } x \in X \setminus K^2, \end{cases}$$

we can define a function $U_{\alpha,D} = \phi K_\alpha f + (1-\phi)\min\{K_\alpha f, \sup_{y\in K^2\setminus\overline{D}} K_\alpha f(y)\} \in \mathscr{F}_{K^2}^{[K_\alpha f]}$.

On the other hand, an equality in [O] shows that $\mu_{<U_\alpha \wedge \sup_{y\in K^2\setminus\overline{D}} K_\alpha f(y)>} = 1_{\{U_\alpha \le \sup_{y\in K^2\setminus\overline{D}} K_\alpha f(y)\}}\mu_{<U_\alpha>}$, which implies $U_\alpha = U_\alpha \wedge \sup_{y\in K^2\setminus\overline{D}} K_\alpha f(y) = U_{\alpha,D}$ on $X\setminus\overline{D}$. Consequently, we obtain $K_\alpha f(y) \le \sup_{y\in K^2\setminus\overline{D}} K_\alpha f(y) \le \sup_{x\in K_1, y\in K^2\setminus\overline{D}} K_\alpha(x,y) < \infty$, whenever y is in $X\setminus\overline{D}$. For any $x \in K^1$, by letting $u \to \delta_{\{x\}}$ with respect to the weak topology of $C_0(K^1)$, we can deduce $\sup_{x\in K^1, y\in X\setminus K^2} K_\alpha(x,y) \le \sup_{x\in K^1, y\in K^2\setminus\overline{D}} K_\alpha(x,y)$. The assertion has been proved. □

Lemma 6.

(i) *For each* $f \in \mathscr{C}_b(X)$, $K_\alpha f$ *is bounded continuous on* X *and* $K_\alpha f - K_\beta f + (\alpha-\beta)K_\alpha K_\beta f = 0$, *for any* $\alpha, \beta > 0$.

(ii) $f \in \mathscr{C}_b(X) \Rightarrow \lim_{\alpha\to\infty} \alpha K_\alpha f(x) = f(x)$, *for all* $x \in X$.

Proof. (i) Because of Lemma 3 and the assumption that $G_\alpha f$ is continuous on X for $f \in L^\infty_{comp}(X)$, we know that $K_\alpha f$ is continuous on X for $f \in L^\infty_{comp}(X)$. From Lemma 4 and Lemma 5, we obtain the boundedness and the continuity of $K_\alpha f$ for all $f \in \mathscr{C}_b(X)$. The remainder part can be verified by the following identity

$$\begin{aligned}
\mathscr{E}_\alpha(K_\alpha f - K_\beta f + (\alpha-\beta)K_\alpha K_\beta f, u) &= (f,u) - \mathscr{E}_\alpha(K_\beta f, u) + (\alpha-\beta)(K_\beta f, u) \\
&= (f,u) - \mathscr{E}_\beta(K_\beta f, u) \\
&= 0
\end{aligned}$$

for $\forall f \in \mathscr{C}_b(X)$ and $\forall u \in \mathscr{F}$.

(ii) We write for $\{X_t^0, \mathscr{M}_t^0, P_x^0\}$ the diffusion corresponding to the regular Dirichlet space $(\mathscr{E}, \mathscr{F}_0)$, then we have

$$\begin{aligned}
f \in \mathscr{C}_b(X) \Rightarrow \lim_{\alpha\to\infty} \alpha G_\alpha f(x) &= \lim_{\alpha\to\infty} \alpha E_x[\int_0^\zeta e^{-\alpha t} f(X_t^0)dt] \\
&= \lim_{t\to 0} E_x[f(X_t^0)] \\
&= f(x) \qquad (\forall\, x \in X),
\end{aligned}$$

where ζ is the life time of the diffusion process. On the other hand, $R_{\alpha,x}(\cdot)$ is a non-negative function on X. This implies that

$$\begin{aligned}
\lim_{\alpha\to\infty} |\alpha R_\alpha f(x)| &\le \|f\|_\infty \lim_{\alpha\to\infty} \alpha R_\alpha 1(x) \\
&= \|f\|_\infty \lim_{\alpha\to\infty} (1 - \alpha G_\alpha 1(x)) \\
&= 0 \qquad (\forall x \in X), \text{ for all } f \in \mathscr{C}_b(X).
\end{aligned}$$

The assertion is proved. □

By Lemma 5, every sequence $\{x_n\}$ with no accumulation point in X provides us with the family of 1-resolvent kernels $\{K_1(x_n, \cdot)\}$ which is uniformly bounded on every compact set. Here it turns out that $\{K_1(x_n,\cdot)\}$ is equi-continuous. In fact, the equation $\mathscr{E}_1(K_1u, v) = (u, v)$ shows that $K_\alpha u$ is 1-harmonic on $X \setminus \mathrm{supp}[u]$. By letting $u \to \delta_{\{x\}}$ in the weak topology of $C_0(X)$, it follows that $K_1(x,\cdot)$ is 1-harmonic on $X \setminus \{x\}$ and that the assumption (4) implies the equi-continuity of $\{K_1(x_n,\cdot)\}$.

A sequence $\{x_n\}$ with no accumulation point in X is called fundamental, if $\lim_{n\to\infty} K_1(x_n, z)$ exists for each $z \in X$. We can introduce an equivalence relation between two fundamental sequences: $\{x_n\}$,$\{x'_n\}$ are said to be equivalent if $\lim_{n\to\infty} K_1(x_n, z) = \lim_{n\to\infty} K_1(x'_n, z)$. Then the collection of equivalent classes $\partial_{M-K}X$ of fundamental sequences is called Martin-Kuramochi boundary of X and the 1-resolvent kernels $K_1(x,\cdot)$ are extended by $K_1(x, z) = \lim_{n\to\infty} K_1(x_n, z)$, where $z \in X$ and x stands for a point in $\partial_{M-K}X$ represented by a fundamental sequence $\{x_n\}$. For $x, y \in \overline{X} = X \cup \partial_{M-K}X$, we define a two point function

$$\rho(x, y) = \int_{\overline{X}} \frac{|K_1(x, z) - K_1(y, z)|}{1 + |K_1(x, z) - K_1(y, z)|} dm(z),$$

which is then clearly a metric on $\overline{X}$. Martin's proof works to deduce that

$$(\overline{X}, \rho) \text{ is a compactification of } X, \tag{7}$$

$$K_1(x, y) \text{ is } \rho\text{-continuous in } x \text{ on } \overline{X} \setminus \{y\}, \tag{8}$$

$$\text{the family of functions } \{K_1(x, y)\}_{y\in X} \text{ separates points of } \overline{X}. \tag{9}$$

Since we proved that $K_1 f$ is continuous on X for $f \in L^\infty_{comp}(X)$, we see

$$f \in L^\infty_{comp}(X) \Rightarrow K_1 f \in C(\overline{X}) \tag{10}$$

from the same argument in the proof of Lemma 3.2 in [C-F-W].

Remark. As for the relation between ideal reflecting boundaries and the relative boundary of Euclidean domains, the reader is referred to the papers [B-H], [C-1] and [C-2].

3. Characterization of quasi-martingale for reflecting diffusion

In this section, we start with the state space X and the Dirichlet space $(\mathscr{E}, \mathscr{F})$ considered in Sect. 2. It follows from (9) and Stone-Weierstrass theorem that $\{p(K_1g_1(x), \cdots, K_1g_\ell(x)) \mid p \text{ is polynomial and } g_1, \cdots, g_\ell \in \mathscr{C}_0(X)\}$ is dense in $\mathscr{C}(\overline{X})$. Consequently, $(\mathscr{F}, \mathscr{E})$ can be regarded as a regular Dirichlet space on $L^2(\overline{X}, m)$ by extending the measure m so that it satisfies $m(\partial_{M-K}X) = 0$. Let $X^* = (X^*_t, \mathscr{M}_t, P_x)$ denote the corresponding conservative diffusion.

Definition 1. For $u \in \mathscr{F}$, if there exists a positive constant $C(u)$ such that

$$|\mathscr{E}(u, g)| \le C(u) \sup_{x\in X} |g(x)| \qquad \text{for all } g \in \mathscr{F} \cap \mathscr{C}_b(X),$$

then the state space X is called a strong $(\mathscr{E}, u)$-Caccioppoli set.

Definition 2. A continuous 1-dimensional process Z is called an $\{\mathscr{F}_t\}$-quasi-martingale, if it admits a decomposition of the form

$$Z_t = Z_0 + M_t + N_t \qquad (\forall t \geq 0),$$

where M is a continuous $\{\mathscr{F}_t\}$-martingale satisfying

$$\sup\{E[|M_{T\wedge\tau}|] \mid \tau \text{ is a stopping time}] < \infty \qquad \text{for each } T \geq 0,$$

and N is an $\{\mathscr{F}_t\}$-adapted continuous process whose total variation on any compact time interval is integrable.

Proposition 1. *Suppose that $u(X_t^*)$ is an $\{\mathscr{M}_t\}$-quasi-martingale for $u \in \mathscr{F}$ and with respect to the probability measure P_m, then X is a strong $(\mathscr{E}, u)$-Caccioppoli set.*

Before the proof, we should note that Lemma 2.1 in [C-F-W] holds as the statement on $u, g \in \mathscr{F}$ in our situation:

Lemma 7. *For given $u \in \mathscr{F}$, we assume that $N_t^{[u]}$ in Fukushima's decomposition $u(X_t^*) - u(X_0^*) = M_t^{[u]} + N_t^{[u]}$ has an associated finite smooth signed measure $\mu^{[u]}$. Then $N_t^{[u]}$ is of bounded variation and its variation on each compact time interval is P_m−integrable, if and only if the finite smooth signed measure $\mu^{[u]}$ satisfies*

$$\mathscr{E}(u, g) = -\int_{\overline{X}} g(x)\mu^{[u]}(dx) \qquad \forall g \in \mathscr{F} \cap L_\infty(X).$$

Thanks to this lemma, we can mimic the discussion in the proof of Theorem 2.1 in [C-F-W] for giving the proof of Proposition 1.

Proof of Proposition 1. In Fukushima's decomposition $u(X_t^*) - u(X_0^*) = M_t^{[u]} + N_t^{[u]}$, the assumption in the proposition implies that the total variation of $N^{[u]}$ on each compact interval is P_m-integrable. Let $V_t^{[u]}$ denote the total variation of $N^{[u]}$ on $[0, t]$. Then $V^{[u]}$ is a positive continuous additive functional and since p_t is m-symmetric, there exists a constant $C(u)$ such that

$$E_m[V_t^{[u]}] = C(u)t \quad \text{ for each } t \geq 0. \tag{11}$$

We consider the excessive function $\rho(x) = P_x(V_t^{[u]} = \infty)$ for X^* restricted to $\overline{X} \setminus U$, where U stands for an exceptional set for Fukushima's decomposition. (11) implies that ρ vanishes m-a.e. and consequently $\rho(x) = 0$ for quasi-everywhere x. Hence we may assume

$$P_x(V_t^{[u]} < \infty) = 1, \qquad t \geq 0 \quad \text{for every} \quad x \notin U.$$

By (11), the function $\phi(x) = E_x[\int_0^\infty e^{-t} V_t^{[u]}\, dt]$ which coincides with $E_x[\int_0^\infty e^{-t}\, d V_t^{[u]}]$ outside U satisfies

$$\int_{\overline{X}} \phi(x)m(dx) = E_m[\int_0^\infty e^{-t} V_t^{[u]}\, dt] = C(u).$$

We know ϕ is finite m-a.e. on X. Since ϕ is excessive for X^* restricted to $\overline{X} \setminus U$, ϕ is finite quasi-everywhere on $\overline{X}$. By (11), there exists a finite smooth measure associated with the total variation process $V^{[u]}$, therefore $N^{[u]}$ also associates a finite smooth signed measure $\mu^{[u]}$ satisfying $|\mu^{[u]}|(\overline{X}) = E_m[V_1^{[u]}] = C(u) < \infty$.

It turns out from Lemma 7 that

$$\mathscr{E}(u, g) = \int_{\overline{X}} g(x)\mu^{[u]}(dx) \leq |\mu^{[u]}|(\overline{X}) \sup_{x \in X} |g(x)| = C(u) \sup_{x \in X} |g(x)|$$

for any $g \in \mathscr{F} \cap \mathscr{C}(\overline{X})$. Since $\mathscr{F} \cap \mathscr{C}(\overline{X})$ is dense in $\mathscr{F}$, one can derive the inequality in Definition 1. □

Since the Dirichlet space $(\mathscr{F}, \mathscr{E})$ is regular, we have the following assertion as in [C-F-W]:

Lemma 8. *If X is a strong $(\mathscr{E}, u)$-Caccioppoli set for $u \in \mathscr{F}$, then there exists measure $\mu^{[u]}$ satisfying $|\mu^{[u]}(\overline{X})| < \infty$ and*

$$\mathscr{E}(u, g) = -\int_{\overline{X}} g(x)\mu^{[u]}(dx), \qquad \forall g \in \mathscr{F} \cap \mathscr{C}(\overline{X}).$$

Proposition 2. *Suppose X is a strong $(\mathscr{E}, u)$-Caccioppoli set for $u \in \mathscr{F}$, then $u(X_t^*)$ is an $\{\mathscr{M}_t\}$-quasi-martingale with respect to P_m.*

This proposition can be proved in the same way as in Proof of Theorem 3.2 in [C-F-W]. Thanks to the property (10), the following assertion holds in our situation as in [C-F-W]:

Lemma 9. *If there exists a finite signed measure ν on $\overline{X}$ satisfying*

$$\mathscr{E}(u, g) = \int_{\overline{X}} g(x)\nu(dx), \qquad \forall g \in \mathscr{F} \cap \mathscr{C}(\overline{X}),$$

then ν is a smooth measure on $\overline{X}$.

4. Sufficient condition for strong $(\mathscr{E}, u)$-Caccioppoli set

In this section, we will deal with the case that X has an exhaustion function and give criteria as to whether X is a strong $(\mathscr{E}, u)$-Caccioppoli set. X is said to be exhausted by a continuous function ρ satisfying $\sup \rho = 0$, if $\{x \in X \mid \rho < \alpha\} \Subset X$ for any $\alpha < 0$ and $X = \cup_{\alpha<0}\{x \in X \mid \rho(x) < \alpha\}$.

Theorem 1. *Suppose that the state space X is exhausted by a continuous function ρ of $\mathscr{F}_0$ satisfying $\sup \rho = 0$. Then X is a strong $(\mathscr{E}, u)$-Caccioppoli set for $u \in \mathscr{F}$, if and only if there exists a finite signed measure $\mu_1^{[u]}$ on X such that*

$$\mathscr{E}(u, g) = -\int_X g(x)\mu_1^{[u]}(dx), \qquad \forall g \in \mathscr{F} \cap \mathscr{C}_0(X)$$

and there exists a finite signed measure $\mu_2^{[u]}$ on $\partial_{M-K}X$ such that

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{\{-\varepsilon<\rho<0\}} g \, d\mu_{<u,\rho>} = -\int_{\partial_{M-K}X} g(x)\mu_2^{[u]}(dx), \quad \forall g \in \mathscr{F} \cap \mathscr{C}_b(X).$$

Remark. In terms of Fukushima's decomposition $u(X_t^*) - u(X_0^*) = M^{[u]} + N^{[u]}$, $\mu_1^{[u]}$ is given as the smooth measure corresponding to $I_X \cdot N^{[u]}$.

Proof. Since we assumed $\mathscr{F} \subset \overline{\mathscr{F}}$, from Theorem 1.4.2, Lemma 3.2.5 and Lemma 5.6.1 in[F-O-T], we can deduce

$$\mu_{<u,g\rho_{-\varepsilon,0}>} = \rho_{-\varepsilon,0}\mu_{<u,g>} + g\mu_{<u,\rho_{-\varepsilon,0}>}$$

for any $u \in \mathscr{F}$, $g \in \mathscr{F} \cap \mathscr{C}_b(X)$ and $\rho_{-\varepsilon,0} = \varepsilon - (\rho + \varepsilon) \vee 0$ with $\varepsilon > 0$. Therefore, we see that the procedure in the proof of Main Lemma in [K] works to derive the Green's formula in [K] for these functions:

$$\int_{-\varepsilon}^0 \int_{\{\rho<s\}} d\mu_{<u,g>} ds = \int_X d\mu_{<u,g\rho_{-\varepsilon,0}>} + \int_{\{-\varepsilon<\rho<0\}} g d\mu_{<u,\rho>}.$$

By dividing both sides with ε and letting $\varepsilon \to 0$, the left-hand side yields

$$\lim_{\varepsilon\to 0} \frac{\int_{-\varepsilon}^0 \int_{\{\rho<s\}} d\mu_{<u,g>} ds}{\varepsilon} = \mathscr{E}(u,g).$$

The assumptions in the theorem ensure that

$$\begin{aligned} &\lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\Big(\int_X d\mu_{<u,g\rho_{-\varepsilon,0}>} + \int_{\{-\varepsilon<\rho<0\}} g d\mu_{<u,\rho>}\Big) \\ &= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\Big(\mathscr{E}(g\rho_{-\varepsilon,0}, u) + \int_{\{-\varepsilon<\rho<0\}} g d\mu_{<u,\rho>}\Big) \\ &= -\int_X g(x)\mu_1^{[u]}(dx) - \int_{\partial_{M-K}X} g(x)\mu_2^{[u]}(dx). \end{aligned}$$

Therefore, we have

$$|\mathscr{E}(u,g)| \le \sup_{x\in X} |g(x)|(|\mu_1^{[u]}| + |\mu_2^{[u]}|),$$

for any $g \in \mathscr{F} \cap \mathscr{C}_b(X)$. Hence, we can conclude that X is a strong $(\mathscr{E}, u)$-Caccioppoli set. Conversely, if X is a strong $(\mathscr{E}, u)$-Caccioppoli set, then Proposition 2 and Lemma 7 assure that there exists a finite smooth measure $\mu^{[u]}$ on $\overline{X}$ satisfying $|\mu^{[u]}(\overline{X})| < \infty$ and

$$\mathscr{E}(u,g) = -\int_{\overline{X}} g(x)\mu^{[u]}(dx), \qquad \forall g \in \mathscr{F} \cap \mathscr{C}_b(X).$$

This shows that $\mu_1^{[u]}$ in the theorem is obtained as the signed measure $\mu^{[u]}|_X$. Again by the Green formula in [K], we have

$$\begin{aligned} -\int_{\partial_{M-K}X} g(x)\mu^{[u]}(dx) &= \mathscr{E}(u,g) + \int_X g(x)\mu^{[u]}(dx) \\ &= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\Big(\int_{-\varepsilon}^0 \int_{\{\rho<s\}} d\mu_{<u,g>} ds - \int_X d\mu_{<u,g\rho_{-\varepsilon,0}>}\Big) \\ &= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_{\{-\varepsilon<\rho<0\}} g d\mu_{<u,\rho>}, \qquad \forall g \in \mathscr{F} \cap \mathscr{C}_b(X). \end{aligned}$$

Accordingly, $\mu_2^{[u]}$ is equal to $\mu^{[u]}|_{\partial_{M-K}X}$. □

Let $\mathscr{D}_{L^2(X;m)}(L)$ denote the domain of the generator L in $L^2(X;m)$ of the absorbing diffusion process corresponding to $(\mathscr{E},\mathscr{F}_0)$. We pick a function u out from the extended domain $\mathscr{D}(L)$ of the generator L which is defined by

$$\mathscr{D}(L)=\{u\in\mathscr{F}\mid \text{ for any compact set } K \text{ in } X \text{ there exists } \varphi_K\in\mathscr{F}\cap\mathscr{C}_0(X) \text{ such that } \varphi_K=1 \text{ on } K \text{ and } \varphi_K u\in\mathscr{D}_{L^2(X;m)}(L)\}.$$

Then, the strong local property shows that

$$\mathscr{E}(g,u)=\mathscr{E}(g,\varphi_K u)=-(g,L(\varphi_K u)),\quad g\in\mathscr{C}_0(D),$$

whenever D is a relatively compact open set satisfying $\overline{D}\subset K$. Again by the strong local property, we easily see that, for $u\in\mathscr{D}(L)$, Lu can be defined as a unique function Lu of $L^2_{loc}(X;m)$ such that, for any relatively compact open set D,

$$Lu=L(\varphi_K u)\quad \text{is valid on } D$$

independently of the choice of compact set K containing D. If Lu is in $L^1(X;m)$, then we obtain

$$\mathscr{E}(g,u)=-(g,Lu),$$

for any $g\in\mathscr{F}\cap\mathscr{C}(\overline{X})$ vanishing on $\partial_{M-K}X$.

Theorem 2. *If the state space X is exhausted by a continuous function ρ of $\mathscr{F}_0$ enjoying* $\sup\rho=0$ *and*

$$\liminf_{\varepsilon\to 0}\frac{\mu_{<\rho,\rho>}(-\varepsilon<\rho<0)}{\varepsilon}<\infty,$$

then X is a strong $(\mathscr{E},u)$-Caccioppoli set for any $u\in\mathscr{D}(L)$ satisfying $Lu\in L^1(X;m)$ and

$$\liminf_{\varepsilon\to 0}\frac{\mu_{<u,u>}(-\varepsilon<\rho<0)}{\varepsilon}<\infty.$$

Proof. Lemma 5.6.1 in [F-O-T] implies

$$\begin{aligned}|\mu_{<u,\rho>}|(-\varepsilon<\rho<0)&\le\sqrt{\mu_{<\rho,\rho>}(-\varepsilon<\rho<0)\mu_{<u,u>}(-\varepsilon<\rho<0)}\\&\le\frac{\mu_{<\rho,\rho>}(-\varepsilon<\rho<0)+\mu_{<u,u>}(-\varepsilon<\rho<0)}{2}.\end{aligned}$$

By dividing both sides with $\varepsilon>0$ and letting $\varepsilon\to 0$, we have

$$\liminf_{\varepsilon\to 0}\frac{|\mu_{<\rho,u>}|(-\varepsilon<\rho<0)}{\varepsilon}<\infty.$$

By the extended version of the Green's formula in [K] shown in the proof of Theorem 1, we obtain

$$\int_{-\varepsilon}^{0}\int_{\{\rho<s\}}d\mu_{<u,g>}ds-\int_X d\mu_{<u,g\rho_{-\varepsilon,0}>}=\int_{\{-\varepsilon<\rho<0\}}g\,d\mu_{<u,\rho>}$$

for any $g \in \mathscr{F} \cap \mathscr{C}_b(X)$, where $\rho_{-\varepsilon,0} = \varepsilon - (\rho + \varepsilon) \vee 0$. By dividing both sides with ε and letting $\varepsilon \to 0$, it turns out from

$$\lim_{\varepsilon \to 0} \frac{\int_{-\varepsilon}^{0} \int_{\{\rho < s\}} d\mu_{<u,g>} ds}{\varepsilon} = \mathscr{E}(u, g)$$

and

$$\begin{aligned}\lim_{\varepsilon \to 0} \frac{\int_X d\mu_{<u,g\rho_{-\varepsilon,0}>}}{\varepsilon} &= \lim_{\varepsilon \to 0} - \frac{(g\rho_{-\varepsilon,0}, Lu)}{\varepsilon} \\ &= -(g, Lu)\end{aligned}$$

that

$$\begin{aligned}|\mathscr{E}(u, g)| &\le |(g, Lu)| + \sup_{x \in X} |g(x)| \liminf_{\varepsilon \to 0} \frac{|\mu_{<\rho,u>}|(-\varepsilon < \rho < 0)}{\varepsilon} \\ &\le \sup_{x \in X} |g(x)| (\|Lu\|_{L^1(X;m)} + \liminf_{\varepsilon \to 0} \frac{|\mu_{<\rho,u>}|(-\varepsilon < \rho < 0)}{\varepsilon}),\end{aligned}$$

for any $g \in \mathscr{F} \cap \mathscr{C}_b(X)$. This shows that X is a strong $(\mathscr{E}, u)$-Caccioppoli set. □

5. Application to reflecting diffusion process on a domain in $\mathbf{R}^n$

The result of Sections 2, 3 and 4 will now be applied to obtain a large class of reflecting diffusions. In this section, we study the reflecting diffusion on a bounded domain D in $\mathbf{R}^n$ whose diffusion coefficient is determined by a symmetric matrix-valued function $A = (a_{i,j})$ on $\mathbf{R}^n$. In what follows, we assume that each of the functions $a_{i,j}$ is twice continuously differentiable on D and that there are measurable functions ρ_1 and ρ_2 on $\mathbf{R}^n$ such that

$$\rho_1(x)|\xi|^2 \le (A(x)\xi, \xi) \le \rho_2(x)|\xi|^2, \qquad \forall x, \forall \xi \in \mathbf{R}^n, \tag{12}$$

$$\rho_2 \text{ is continuous on } D \text{ and } \inf_{x \in D_1} \rho_1(x) > 0 \text{ for any domain} D_1 \Subset D. \tag{13}$$

By assuming $\int_D \rho_2(x)V(dx) = 1$ for Lebesgue measure V, we consider the measure $m(dx) = \rho_2(x)V(dx)$ on $\mathbf{R}^n$ which is normalized on D as before. Then, we can introduce a symmetric bilinear form $\mathscr{E}(u, v) = \int_D (A\nabla u, \nabla v) dV$ for $u, v \in \mathscr{C}_b^\infty(D)$ which is closable in $L^2(D; m)$, where $\mathscr{C}_b^\infty(D)$ is the family of all smooth functions on D whose all partial derivatives are bounded. We denote the smallest extension of the bilinear form by $(\mathscr{E}, \mathscr{F})$. From the definition, it is included in Chen's reflected Dirichlet space $(\overline{\mathscr{E}}, \overline{\mathscr{F}})$. All notations introduced in the earlier sections will now serve the analogous roles, except substituting D for the state space X and L_A for the generator L. For any $g \in L^\infty(D; V)$, we know from Theorem 8.24 in [G-T] that each function $u \in \mathscr{F}$ satisfying

$$\begin{aligned}&\mathscr{E}(u, \varphi) + \alpha(u, \varphi)_{L^2(D_1;m)} = (g, \varphi)_{L^2(D_1;m)}, \\ &\qquad\qquad \text{for all } \varphi \in \mathscr{C}_0^\infty(D) \text{ with supp}[u] \subset \mathrm{D}_1,\end{aligned}$$

for subdomain $D_1 \Subset D$ has a version enjoying

$$\sup_{x,y\in D_0} \frac{|u(x)-u(y)|}{|x-y|^{\beta}} \leq C(\|u\|_{L^2(D_1;V)} + \|g\|_{L^{\infty}(D_1;V)}) \tag{14}$$

for some positive constants β and C depending on domains D_1 and $D_0 \Subset D_1$. Recall that the function ρ_1 is away from zero on every compact subset in D. Accordingly, the assumption (4) in sect. 2 is satisfied. Other assumptions from (1) to (5) in sect. 2 are evidently fulfilled. In order to obtain continuous representation kernel for α-resolvent with properties in sect. 2, we further put some assumptions on ρ_1 and ρ_2:

$$\int_{B_x(2r)} \rho_2(x)V(dx) \leq C_1 \int_{B_x(r)} \rho_2(x)V(dx) < \infty, \quad r>0,\ x\in \mathbf{R}^n, \tag{15}$$

$$\sup_{x\in\mathbf{R}^n, r>0} \Big(\frac{1}{V(B_x(r))}\int_{B_x(r)} \rho_1(x)V(dx)\Big)\Big(\frac{1}{V(B_x(r))}\int_{B_x(r)} \frac{1}{\rho_1(x)}V(dx)\Big) < \infty, \tag{16}$$

$$(\frac{\rho_1}{\rho_2})^k \rho_2 \in L^1_{loc}(\mathbf{R}^n, V) \quad \text{for some } k > \frac{2q}{q-2},\ \text{where } q>4, \tag{17}$$

$$\frac{s}{r}\left(\frac{\int_{B_x(s)} \rho_2(x)V(dx)}{\int_{B_x(r)} \rho_2(x)V(dx)}\right)^{\frac{1}{q}} \leq C_2 \left(\frac{\int_{B_x(s)} \rho_1(x)V(dx)}{\int_{B_x(r)} \rho_1(x)V(dx)}\right)^{\frac{1}{2}}, \quad \forall\, r > \forall\, s > 0, \forall\, x\in\mathbf{R}^n. \tag{18}$$

By arguing exactly same as in [C-W], we see the α-resolvent of the corresponding absorbing diffusion has the representation kernel $G_\alpha(x,y)$ which is obtained as the limit $\lim_{\varepsilon\to 0} G_\alpha^{(\varepsilon)}(x,y)$ in $L^2_{loc}(D\setminus\{x\}; m_1)$, where $m_1(dx) = \rho_1(x)V(dx)$ and $G_\alpha^{(\varepsilon)}(x,y)$ is a unique element in $\mathscr{F}_0$ satisfying

$$\mathscr{E}(G_\alpha^{(\varepsilon)}, \varphi) + \alpha(G_\alpha^{(\varepsilon)}, \varphi)_{L^2(D_1;m)} = \frac{1}{\int_{B_x(\varepsilon)} \rho_2(y)V(dy)} \int_{B_x(\varepsilon)} (\varphi\rho_2)(y)V(dy)$$

for all $\varphi \in \mathscr{C}_0^\infty(D)$.

Proposition 3.

(i) *$G_\alpha(x,y)$ is continuous on $(D\times D)\setminus\Delta$ and satisfies the following estimate where the Green function $G_{B_y(2R)}$ with $R = \mathrm{diam(D)}$ is involved:*

$$\sup_{\substack{x\in D\\ r/2<|x-y|<r}} G_\alpha(x,y) \leq \operatorname{ess\,sup}_{\substack{x\in D\\ r/2<|x-y|<r}} G_{B_y(2R)}(x,y)$$
$$\leq C_2 \int_r^{2R} \left(\frac{\int_{B_x(t)} \rho_2(x)V(dx)}{\int_{B_x(t)} \rho_1(x)V(dx)}\right)^{\frac{q}{p(q-2)}} \frac{t}{\int_{B_x(t)} \rho_1(x)V(dx)} dt.$$

(ii) *For each* $f \in L^2(D; m)$, *there is an element* $u_f \in \mathscr{F}_0$ *satisfying*

$$\mathscr{E}(u_f, \varphi) + \alpha(u_f, \varphi)_{L^2(D;m)} = (f, \varphi)_{L^2(D;m)} \quad \textit{for all } \varphi \in \mathscr{C}_0^\infty(D),$$

which is equal to $G_\alpha f$ *a.e. on* D. *In particular, if* $f \in L^\infty(D; m)$ *then* u_f *is continuous and* $u_f = G_\alpha f$ *on* D.

(iii) G_α *is the representation kernel for the absorbing diffusion process* $\{X_t^0, \mathscr{M}_t^0, P_x^0\}$, *i.e.,*

$$f \in \mathscr{C}_b(D) \Rightarrow G_\alpha f(x) = E_x[\int_0^\zeta e^{-\alpha t} f(X_t^0) dt], \qquad \forall x \in D.$$

Proof. (i) By the observation in [C-W], our hypotheses (15)–(18) imply that

$$\begin{aligned}\operatorname*{ess\,sup}_{\substack{x \in D \\ r/2<|x-y|<r}} G_\alpha(x, y) &\le \operatorname*{ess\,sup}_{\substack{x \in D \\ r/2<|x-y|<r}} G_{B_y(2R)}(x, y) \\ &\le C_2 \int_r^{2R} \left(\frac{\int_{B_x(t)} \rho_2(x) V(dx)}{\int_{B_x(t)} \rho_1(x) V(dx)} \right)^{\frac{q}{p(q-2)}} \\ &\quad \times \frac{t}{\int_{B_x(t)} \rho_1(x) V(dx)} dt.\end{aligned}$$

The α-harmonicity of $G_\alpha(x, y)$ in $\{x \in D \mid r/2 < |x - y| < r\}$ and property (14) imply G_α has a continuous version. As a result, the version enjoys the estimate in (i).

(ii) It is easy to check that there is u_f satisfying the equation in (ii). Consequently, we see that

$$\begin{aligned}&\mathscr{E}(G_\alpha^{(\varepsilon)}(x, \cdot), u_f) + \alpha(G_\alpha^{(\varepsilon)}(x, \cdot), u_f)_{L^2(D;m)} \\ &= \frac{1}{\int_{B_x(\varepsilon)} \rho_2(y) V(dy)} \int_{B_x(\varepsilon)} (u_f \rho_2)(y) V(dy).\end{aligned}$$

Note that the left-hand side is equal to $(G_\alpha^{(\varepsilon)}(x, \cdot), f)_{L^2(D;m)}$ and pass to the limit as $\varepsilon \to 0$. Since the exponents in (17) and (18) are determined by the factor $q > 4$, the weak convergence $G_\alpha^{(\varepsilon)}(x, \cdot) \to G_\alpha(x, \cdot) \quad (\varepsilon \to 0)$ in $L^2(D; m)$ is established as in [C-W]. Thus, we observe that $u_f(x) = \int_D G_\alpha(x, y) f(y) m(dy)$ a.e. in D. In particular, if $f \in L^\infty(D; m)$, then we know from (14) that u_f is continuous in D. The continuity of u_f and ρ_2 ensures that $\frac{1}{\int_{B_x(\varepsilon)} \rho_2(y) V(dy)} \int_{B_x(\varepsilon)} (u_f \rho_2)(y) V(dy) \to u_f(x)$ as $\varepsilon \to 0$ for all $x \in D$. Accordingly, $u_f = G_\alpha f$ holds everywhere on D.

(iii) The basic theory of stochastic differential equation (see e.g., [I-W]) shows the existence of the absorbing diffusion process $\{X_t^0, \mathscr{M}_t^0, P_x^0\}$. The hypothesis (12) and (13) ensure that each domain $D' \Subset D$ admits Green's function $G_\alpha^{D'}$ satisfying

$$G_\alpha^{D'} f(x) = E_x[\int_0^{\tau_{D'}} e^{-\alpha t} f(X_t^0) dt] \quad (\forall x \in D')$$

for any non-negative function $f \in \mathscr{C}_b(D)$, where $\tau_{D'}$ is the exit time from D'. The monotone convergence theorem allows us to pass to the limit as $D' \uparrow D$, to obtain (iii). □

We are now ready for our main result.

Theorem 3. *If a bounded domain D satisfies*

$$\liminf_{\varepsilon \to 0} \frac{\int_{\{x \in D | \mathrm{dist}(x, \partial D) < \varepsilon\}} trace A(x) V(dx)}{\varepsilon} < \infty, \tag{19}$$

then X is a strong $(\mathscr{E}, u)$-Caccioppoli set for any function u in the extended domain $\mathscr{D}(L_A)$ of the L^2-generator L_A of the corresponding absorbing diffusion enjoying

$$\liminf_{\varepsilon \to 0} \frac{\mu_{<u,u>}(\{x \in D | \mathrm{dist}(x, \partial D) < \varepsilon\})}{\varepsilon} < \infty.$$

Proof. From [St] we know that every bounded domain D admits a smooth function d_D and positive constants c_1 , c_2 and C_3 satisfying

$$c_1 \mathrm{dist}(x, \partial D) \le d_D(x) \le c_2 \mathrm{dist}(x, \partial D) \quad \text{for all } x \in D \quad \text{and } \sup_{x \in D} \|\nabla u\| \le C_3$$

(see also Lemma 2.4 in [C-2]). By combining this with (19), it turns out that $d_D \in \mathscr{F}_0$ and that

$$\liminf_{\varepsilon \to 0} \frac{\mu_{<d_D, d_D>}(0 < d_D < \varepsilon)}{\varepsilon} < \infty.$$

Since the bounded domain D is exhausted by the smooth Lemma function $-d_D$, the assertion immediately follows from Theorem 2. □

Theorem 4. *If a domain D satisfies (19), moreover if $a_{i,j} \in L^2(D; V)$ and $\frac{\partial a_{i,j}}{\partial x_j} \in L^1(D; V)$ $(i, j = 1, \cdots, n)$, then D is a strong $(\mathscr{E}, x^i)$-Caccioppoli set for every component x^i of the canonical Euclidean coordinate $(x^1, \cdots, x^n)$, and for quasi-everywhere $x \in \overline{D}$ we have*

$$X_t^{*,i} - X_0^{*,i} = \sum_{j=1}^n \int_0^t \gamma_{i,j}(X_s^*) dB_s^j + \sum_{j=1}^n \int_0^t \frac{\partial a_{i,j}}{\partial x_j}(X_s^*) ds + L_{A,t}^i,$$

$$t \ge 0, \; i = 1, \cdots, n, \; P_x - \; a.s.$$

Here $(X_t^{*,1}, \cdots, X_t^{*,n}) = (x^1(X_t^*), \cdots, x^n(X_t^*))$, $B = (B^1, \cdots, B^n)$ is a Brownian motion martingale additive functional of X^*, $\gamma_{i,j}(x)$ is the symmetric positive definite $n \times n$ matrix whose square is $a_{i,j}(x)$, and $L_{A,t}^i$ is the positive continuous additive functional of $\overline{D}$ with associated smooth measure $1_{\partial_{M-K} D} \mu^{[x^i]}$.

Proof. If $a_{i,j} \in L^2(D; V)$ and $\frac{\partial a_{i,j}}{\partial x_j} \in L^1(D; V)$ $(i, j = 1, \dots, n)$, then for bounded function g in $\mathscr{F} \cap \mathscr{C}(\overline{X})$ we have

$$\begin{aligned} \mathscr{E}(x^i, g) &= \frac{1}{2}\sum_{j=1}^{n}\int_D a_{i,j}\frac{\partial g}{\partial x^i}dV \\ &= \frac{1}{2}\sum_{j=1}^{n}\int_D \frac{\partial(a_{i,j}g)}{\partial x^i}dV - \frac{1}{2}\sum_{j=1}^{n}\int_D g\frac{\partial a_{i,j}}{\partial x^i}dV. \end{aligned} \tag{20}$$

The first term in the right-hand side vanishes whenever g is in $\mathscr{C}_0(D)$. Therefore $x^i \in \mathscr{D}(L_A)$ and $L_A x^i \in L^1(D; V)$ $\quad (i = 1, \cdots, n)$. From (19), we know that

$$\liminf_{\varepsilon \to 0} \frac{\mu_{<x^i, x^i>}(\{x \in D | \mathrm{dist}(x, \partial D) < \varepsilon\})}{\varepsilon} < \infty, \quad i = 1, \cdots n.$$

Theorem 3 assures that D is a strong $(\mathscr{E}, x^i)$-Caccioppoli set for every component x^i, i.e., we get $\mathscr{E}(x^i, g) = \int_{\overline{D}} g d\mu^{[x^i]}$, $(i = 1, \cdots, n)$ for all $g \in \mathscr{F} \cap \mathscr{C}(\overline{X})$. It turns out that from (20) that $\frac{1}{2}\sum_{j=1}^{n}\int_D \frac{\partial(a_{i,j}g)}{\partial x^i}dV$ coincides with $\int_{\partial_{M-K}D} g \, d\mu^{[x^i]}$ and so we get $-\frac{1}{2}\sum_{j=1}^{n}\int_D g\frac{\partial a_{i,j}}{\partial x^i}dV = \int_D g d\mu^{[x^i]}$. By applying the same argument in the proof of Theorem 4.2 in [C-F-W], we can prove the formula in our theorem. □

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