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Laplace approximations for sums of independent random vectors

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Abstract. Let $X_i, i \in \mathbf{N}$, be *i.i.d.* B-valued random variables, where B is a real separable Banach space. Let $\mathbf{\Phi}$ be a mapping $B \to \mathbf{R}$. Under a central limit theorem assumption, an asymptotic evaluation of $Z_n = E\left(\exp\left(n\mathbf{\Phi}(\sum_{i=1}^n X_i/n)\right)\right)$, up to a factor $(1 + o(1))$, has been gotten in Bolthausen [1]. In this paper, we show that the same asymptotic evaluation can be gotten without the central limit theorem assumption.

1. Introduction

Let B be a real separable Banach space with norm $\|\cdot\|$, μ be a probability measure on B. We assume that the smallest closed affined space that contains supp μ is B. Moreover we assume

(A1)

$$\int_B \exp(t\|x\|)\mu(dx) < \infty \ , \qquad \text{for all } t \in \mathbf{R} \ .$$

Let $\mathbf{\Phi} : B \to \mathbf{R}$ be a three times continuously Fréchet differentiable function satisfying the following:

(A2) There exist constants $C_1, C_2 > 0$, such that

$$\mathbf{\Phi}(x) \le C_1 + C_2\|x\|, \qquad \text{for any } x \in B \ .$$

Let X_n and S_n, $n \in \mathbf{N}$, be the random variables defined by $X_n(\underline{x}) = x_n$ and $S_n(\underline{x}) = \sum_{k=1}^n x_k$ for any $\underline{x} = (x_1, x_2, x_3, \cdots) \in B^{\mathbf{N}}$.

By Donsker-Varadhan [3], we have that

$$\lim_{n\to\infty} \frac{1}{n} \log E^{\mu^{\otimes\infty}}\left[\exp\left(n\mathbf{\Phi}\left(\frac{S_n}{n}\right)\right)\right] = \sup_{x\in B}\{\mathbf{\Phi}(x) - h(x)\} \ ,$$

where h is the entropy function of μ:

$$h(x) = \sup_{\phi\in B^*} \{\phi(x) - \log M(\phi)\}, \qquad x \in B \ ,$$

B^* is the dual Banach space of B and $M(\phi) = \int_B e^{\phi(x)}\mu(dx)$ for any $\phi \in B^*$.

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It has been shown by Bolthausen [1] that there is at least one $x^* \in B$ with $\mathbf{\Phi}(x^*) - h(x^*) = \sup_{x \in B}\{\mathbf{\Phi}(x) - h(x)\}$. Also, we assume the following as in Bolthausen [1]:

(A3) There is a unique $x^* \in B$ with $\mathbf{\Phi}(x^*) - h(x^*) = \sup_{x \in B}\{\mathbf{\Phi}(x) - h(x)\}$.

We will use x^* exclusively for this point.

Let ν be the probability measure on B given by

$$\nu(dx) = \frac{\exp(D\mathbf{\Phi}(x^*)(x))\mu(dx)}{M(D\mathbf{\Phi}(x^*))} \ .$$

As it has been shown by Bolthausen [1], the following proposition holds.

Proposition 1.1. *Under the assumptions* (A1), (A2), (A3),

$$x^* = \int_B x\nu(dx), \tag{1.1}$$

$$h(x^*) = D\mathbf{\Phi}(x^*)(x^*) - \log M(D\mathbf{\Phi}(x^*)) \ . \tag{1.2}$$

Let ν_0 be the 0-centered ν, *i.e.* $\nu_0 = \nu\theta_{x^*}^{-1}$, where $\theta_a : B \to B$ is defined by $\theta_a(x) = x - a$, $x \in B$.

Let $\Gamma(\varphi, \psi) = \int_B \varphi(x)\psi(x)\nu_0(dx)$ be the covariance of φ and ψ for any $\varphi, \psi \in B^*$. Then Γ becomes an inner product on B^*. Let $H \equiv (\overline{B^*}^{\Gamma})^*$, where $\overline{B^*}^{\Gamma}$ means the completion of B^* with respect to Γ. Then we can show that H can be regarded as a dense subset of B. (See Proposition 2.1.)

In this paper, we assume the following, which is a little stronger than (A1):

(A1′) There exists a constant $C_3 > 0$, such that

$$\int_B \exp\left(C_3\|x\|^2\right)\mu(dx) < \infty.$$

It has been shown by Bolthausen [1] that the following holds:

$$\Gamma(\phi, \phi) \geq D^2\mathbf{\Phi}(x^*)(\iota(\phi), \iota(\phi)), \qquad \text{for any } \phi \in B^*,$$

where $\iota(\phi) \equiv \int_B \phi(x)x\nu_0(dx)$, $\phi \in B^*$. From this, we see that all of the eigenvalues of the operator $D^2\mathbf{\Phi}(x^*)|_{H \times H}$ are smaller than or equal to 1. Furthermore we assume the following

(A4) All of the eigenvalues of $D^2\mathbf{\Phi}(x^*)|_{H \times H}$ are smaller than 1.

(A5) There exist constants $C_4 > 0$ and $\delta > 0$, and a continuous bilinear function $K : B \times B \to \mathbf{R}$, such that

$$|D^3\mathbf{\Phi}(x)(y, y, y)| \leq C_4\|y\|K(y, y)$$

for any $y \in B$ and any $x \in B$ with $\|x - x^*\| < \delta$.

The following is our main result:

Theorem 1.2. *Under the assumptions* (A1′), (A2) ∼ (A5) *above, we have*

$$\lim_{n\to\infty} \exp(-n(\mathbf{\Phi}(x^*) - h(x^*)))E^{\mu^{\otimes\infty}}\left[\exp\left(n\mathbf{\Phi}\left(\frac{S_n}{n}\right)\right)\right]$$
$$= \exp\left(\frac{1}{2}\int_B D^2\mathbf{\Phi}(x^*)(x,x)\nu_0(dx)\right)\cdot det_2\left(I_H - D^2\mathbf{\Phi}(x^*)\right)^{-\frac{1}{2}}.$$

Remark. The fact that $D^2\mathbf{\Phi}(x^*)|_{H\times H}$ is a Hilbert-Schmidt function, which ensures that the factor $\det_2\left(I_H - D^2\mathbf{\Phi}(x^*)\right)$ above is well-defined, can be gotten from Proposition 2.2 later.

Bolthausen [1] studied the same problem under the different assumption. He showed the following

Theorem 1.3. *Assume the following*

(B) ν_0 *satisfies central limit theorem*, i.e., ν_n *defined by* $\nu_n(A) = \nu_0^{*n}(\sqrt{n}A)$ *converges weakly to a Gaussian measure* γ *on* B.

Furthermore, assume (A1) ∼ (A4), *then*

$$\lim_{n\to\infty} \exp(-n(\mathbf{\Phi}(x^*) - h(x^*)))E^{\mu^{\otimes\infty}}\left[\exp\left(n\mathbf{\Phi}\left(\frac{S_n}{n}\right)\right)\right]$$
$$= \int_B \exp\left(\frac{1}{2}D^2\mathbf{\Phi}(x^*)[y,y]\right)\gamma(dy) .$$

If we assume that ν_0 satisfies central limit theorem as in Bolthausen [1], (H, B, γ) becomes an abstract Wiener Space, and so from Kuo [4] (Page 83, Theorem 4.6 (Goodman)), we can get that $D^2\mathbf{\Phi}(x^*)|_{H\times H}$ is a nuclear function. In this situation, the intergration $\int_B \exp(\frac{1}{2}D^2\mathbf{\Phi}(x^*)[y,y])\gamma(dy)$ appeared in Bolthausen's theorem is nothing but $\exp(\frac{1}{2}\int_B D^2\mathbf{\Phi}(x^*)(x,x)\nu_0(dx))\cdot \det_2(I - D^2\mathbf{\Phi}(x^*))^{-\frac{1}{2}}$, which is just the limit appeared in our theorem. And when the operator $D^2\mathbf{\Phi}(x^*)|_{H\times H}$ is not nuclear, but just a Hilbert-Schmidt function, Bolthausen's one is not defined, while our one is still well-defined. The point here is that the condition that a function is Hilbert-Schmidt can be easily checked be integration, while the condition nuclear is not. Moreover, if B is a Hilbert space, then (A5) is also satisfied.

As mentioned above, the central limit theorem assumption is actually a very strong assumption as we are dealing with infinite dimension space. Our theorem claims that without the assumption that ν_0 satisfies central limit theorem, the result still holds under the assumptions (A1′) and (A5).

Remark. In most of our proofs, (A1′) can be substituted by (A1), but in the proof of Lemma 3.5, we use (A1′) essentially to derive (3.18). We do not know whether one can weaken the assumption (A1′).

2. Preparations

Proposition 2.1. *H can be regarded as a dense subset of B.*

Proof. The fact that H can be regarded as a subset of B can be seen from the definition of H, the continuity of $\iota : (B^*, \|\cdot\|_{H^*}) \to (B, \|\cdot\|_B)$, and the completeness of B.

The denseness can be seen from the extension theory and the assumption that the closed affined space that contains supp ν_0 is B, by using contradiction. □

Proposition 2.2. *Under the assumption* (A1′) (*or just* (A1)), *for any continuous bilinear function* $A : B \times B \to \mathbf{R}$, $A\Big|_{H\times H}$ *is a Hilbert-Schmidt function.*

Proof. Since A is continuous, there exists a constant $C_0 > 0$, such that

$$|A(y_1, y_2)| \le C_0 \|y_1\| \cdot \|y_2\|, \qquad \text{for any } y_1, y_2 \in B \ .$$

Let $\{e_n\}_{n=1}^{\infty}$ be a complete orthonormal base of H^* with $\{e_n\}_{n=1}^{\infty} \subset B^*$. Then $\{\iota(e_n)\}_{n=1}^{\infty}$ is the corresponding base of H. Let $f_{n,m} : B \times B \to \mathbf{R}$ be defined as

$$f_{n,m}(y_1, y_2) :=< e_n, y_1 > \cdot < e_m, y_2 >, \qquad y_1, y_2 \in B \ ,$$

then $(f_{n,m}, f_{n',m'})_{L^2(d\nu_0^{\otimes 2})} = \delta_{nn'} \cdot \delta_{mm'}$ for any $n, m, n', m' \in \mathbf{N}$. Therefore,

$$\begin{aligned} \|A\|_{H.S.}^2 &= \sum_{n,m=1}^{\infty} A(\iota(e_n), \iota(e_m))^2 \\ &= \sum_{n,m=1}^{\infty} \left(\int_B \int_B A(y_1, y_2) f_{n,m}(y_1, y_2) \nu_0(dy_1)\nu_0(dy_2) \right)^2 \\ &\le \int_B \int_B |A(y_1, y_2)|^2 \nu_0(dy_1)\nu_0(dy_2) \le C_0^2 \left(\int_B \|y\|^2 \nu_0(dy) \right)^2 \ , \end{aligned}$$

which is finite by assumption (A1′) (or just (A1)). □

3. Basic lemmas

For any $R > 2$, let $\tilde{\nu}_R$ be the probability measure on $\mathbf{R}$ given by

$$\tilde{\nu}_R(\{R\}) = \frac{3}{4R^2 - 1}, \quad \tilde{\nu}_R\left(\left\{\frac{1}{2}\right\}\right) = \frac{R-2}{2R-1}, \quad \tilde{\nu}_R\left(\left\{-\frac{1}{2}\right\}\right) = \frac{R+2}{2R+1} \ .$$

By a simple calculation, we have

$$E^{\tilde{\nu}_R}[Y] = 0, \qquad E^{\tilde{\nu}_R}[Y^2] = 1 \ .$$

For any $a > 0$, let ρ_a be the probability measures on $\mathbf{R}$ given by

$$\rho_a(dR) = C_a \exp\left(-\frac{aR^2}{2}\right) dR, \quad R > 2 \ ,$$

where C_a is the normalizing constant, *i.e.* $C_a = \left(\int_2^\infty e^{-\frac{aR^2}{2}} dR\right)^{-1}$. Let γ_a be the probability measures on $\mathbf{R}$ given by

$$\gamma_a(dy) = \int \tilde{\nu}_R(dy)\rho_a(dR) \ ,$$

and let Y_i be *i.i.d.* random variables *s.t.* $P(Y_i \in dy) = \gamma_a(dy)$.

Lemma 3.1. *For any $a > 0$, there exists a constant D_a, depends only on a, such that*

$$P\left(\left|\frac{1}{\sqrt{n}}\sum_{i=1}^n Y_i\right| \geq z\right) \leq 2\exp\left(-\frac{1}{4D_a}z^2\right) \tag{3.1}$$

for any $z \geq 0$ and any $n \geq 1$.

Proof. Let $f(\xi) \equiv \int_{\mathbf{R}} e^{\xi y}\gamma_a(dy)$. Then it can be shown that

$$D_a \equiv \sup_{\xi\neq 0} \frac{1}{|\xi|^2}\log f(\xi) < \infty \ .$$

Therefore,

$$\begin{aligned} P\left(\left|\frac{1}{\sqrt{n}}\sum_{i=1}^n Y_i\right| \geq z\right) &\leq e^{-\xi\cdot\sqrt{n}z} E\left[e^{\xi\sum_{i=1}^n Y_i}\right] + e^{-\xi\cdot\sqrt{n}z} E\left[e^{-\xi\sum_{i=1}^n Y_i}\right] \\ &\leq 2e^{-\xi\cdot\sqrt{n}z}\cdot\exp\left(nD_a|\xi|^2\right) \end{aligned}$$

for any $\xi \neq 0$. Letting $\xi = \frac{z}{2D_a\sqrt{n}}$, we get (3.1). □

Lemma 3.2. *Under the assumption (A1′) in section 1, for any $c > 0$, there exists a $a_0 > 0$ small enough, such that for any $n \geq 3$ and any $a \in (0, a_0]$,*

$$c^n\left(\int_B \|x\|^{2n}\nu_0(dx)\right)^{1/2} \leq \int_{\mathbf{R}} y^n\gamma_a(dy) \ . \tag{3.2}$$

Proof. From assumption (A1′) and the definition of ν_0, there exists a constant $C_3' > 0$, such that $C_5 \equiv \int_B e^{C_3'\|x\|^2}\nu_0(dx) < \infty$. So for any $t > 0$,

$$f(x) \equiv \nu_0(\|X\| \geq t) \leq C_5 e^{-C_3' t^2} \ .$$

Therefore, for any $n \geq 3$,

$$\int_B \|x\|^n\nu_0(dx) \leq n\int_{(0,\infty)} y^{n-1}\cdot e^{-\frac{y^2}{2}}dy\cdot C_5\cdot\left(\frac{1}{\sqrt{2C_3'}}\right)^n \ . \tag{3.3}$$

On the other hand, from the definition of $\tilde{\nu}_R$, we can get by a calculation that for any $R > 2$,

$$\int_{\mathbf{R}} y^n \tilde{\nu}_R(dy) \geq \frac{3}{4} R^{n-2}$$

for any $n \geq 3$. So let $\rho_{0,a}$, $a > 0$ be the probability measures given by

$$\rho_{0,a}(dR) = \frac{2\sqrt{a}}{\sqrt{2\pi}} e^{-\frac{aR^2}{2}} dR, \qquad R > 0 \ .$$

then we have that for any $a < a_0$ and any $n \geq 3$,

$$\begin{aligned} \int_{\mathbf{R}} y^n \gamma_a(dy) &\geq \frac{3}{4} \int_{(0,\infty)} R^{n-2} \rho_a(dR) \geq \frac{3}{4} \int_{(0,\infty)} R^{n-2} \rho_{0,a_0}(dR) \\ &= \frac{3}{4} \frac{2}{\sqrt{2\pi}} \int_{(0,\infty)} y^{n-2} e^{-\frac{y^2}{2}} dy \cdot \left(\frac{1}{\sqrt{a_0}} \right)^{n-2} \quad . \end{aligned} \tag{3.4}$$

From (3.3) and (3.4), to prove the lemma, we only need to show that

$$\begin{aligned} & c^{2n} \cdot 2n \int_{(0,\infty)} y^{2n-1} e^{-\frac{y^2}{2}} dy \cdot C_5 \cdot \left(\frac{1}{2C_3'} \right)^n \\ & \leq \left(\frac{3}{2\sqrt{2\pi}} \right)^2 \cdot \left(\int_{(0,\infty)} y^{n-2} \cdot e^{-\frac{y^2}{2}} dy \right)^2 \cdot \left(\frac{1}{a_0} \right)^{n-2} \end{aligned} \tag{3.5}$$

holds for any $n \geq 3$ if $a_0 > 0$ is small enough. But this is easy to be seen by a simple calculation and Stirling's formula. □

Lemma 3.3. *Assume the assumption* (A1′) *in section* 1. *Let* $\mathbf{\Psi}$ *be a symmetric, bilinear function that satisfies the following conditions:*

1. *There exists a constant* $C_0 > 0$, *such that* $|\mathbf{\Psi}(x, y)| \leq C_0 \|x\| \cdot \|y\|$ *for any* $x, y \in B$,
2. $\int_B \mathbf{\Psi}(x, y)^2 \nu_0(dx)\nu_0(dy) = 1$.

Then, there exists an $a_0 > 0$, *depending only on* C_0 *and* $\int_B \|x\|^2 \nu_0(dx)$, *such that*

$$E^{\nu_0^{\otimes\infty}} \left[\prod_{k=1}^{m} \mathbf{\Psi}(X_{i_k}, X_{j_k}) \right] \leq E^{\gamma_a^{\otimes\infty}} \left[\prod_{k=1}^{m} Y_{i_k} Y_{j_k} \right] \tag{3.6}$$

holds for any $m \in \mathbf{N}$, *any* $i_1, \ldots, i_m, j_1, \ldots, j_m \in \mathbf{N}$ *with* $1 \leq i_k < j_k \leq n, k = 1, \cdots, m$, *and any* $a \in (0, a_0]$, *where* $\{X_i\}_{i=1}^{\infty}$ *is the sequence of random variables defined in section* 1, *and* $\{Y_i\}_{i=1}^{\infty}$ *is defined by* $Y_n(\underline{y}) = y_n$, $\underline{y} = (y_1, y_2, \ldots) \in \mathbf{R}^{\mathbf{N}}$.

Note. As ν_0 has mean 0, we get from the bilinearity of $\mathbf{\Psi}$ that $\int_B \mathbf{\Psi}(x, y)\nu_0(dy) = 0$ for any $x \in B$.

Proof. To simple the notation, in the proof of this lemma, we will write just E, which means the expectation with respect to $\nu_0^{\otimes\infty}$ when deal with $\{X_i\}_{i=1}^{\infty}$, and $\gamma_a^{\otimes\infty}$ when deal with $\{Y_i\}_{i=1}^{\infty}$, when there is no risk of being confused.

Let us consider the graph that consists all the i_k, j_k's as its nodes and all the $i_k j_k$'s as its lines. We may assume that the graph is connected, since if not, from the independent of the X_i's and Y_i's, we can consider each connected component, respectively.

Let

$$\alpha_\ell = \sharp\{k : i_k = \ell \text{ or } j_k = \ell\}, \qquad 1 \le \ell \le n \ .$$

If there exists a ℓ such that $\alpha_\ell = 1$, then (3.6) obviously holds as $0 = 0$. So, we may assume that $\alpha_\ell = 0$ or $\alpha_\ell \ge 2$ for all ℓ. Let

$$L = \{\ell; \alpha_\ell \ge 2\}, \qquad L_0 = \{\ell; \alpha_\ell \ge 3\} \ .$$

If $L = L\backslash L_0$, then all of the i_k's appear exactly twice, so from Schwartz's inequality and the independence of the X_i's and the assumptions, it could be seen that

$$\begin{aligned} E^{\nu_0^{\otimes\infty}}\left[\prod_{k=1}^m \mathbf{\Psi}(X_{i_k}, X_{j_k})\right] &\le \prod_{k=1}^m E\left[\mathbf{\Psi}(X_{i_k}, X_{j_k})^2\right]^{1/2} = 1 \qquad (3.7)\\ &= E^{\gamma_a^{\otimes\infty}}\left[\prod_{k=1}^m Y_{i_k} Y_{j_k}\right] \ . \end{aligned}$$

To see the inequality in the first line, we only need to notice that when r is an odd number,

$$\begin{aligned} &E\left[\mathbf{\Psi}(x, X_1)\mathbf{\Psi}(X_1, X_2)\cdots\mathbf{\Psi}(X_r, y)\right]\\ &\quad = E[(\mathbf{\Psi}(x, X_1)\mathbf{\Psi}(X_2, X_3)\cdots\mathbf{\Psi}(X_{r-1}, X_r))\\ &\qquad\qquad \cdot(\mathbf{\Psi}(X_1, X_2)\mathbf{\Psi}(X_3, X_4)\cdots\mathbf{\Psi}(X_r, y))]\\ &\quad \le E\left[(\mathbf{\Psi}(x, X_1)\mathbf{\Psi}(X_2, X_3)\cdots\mathbf{\Psi}(X_{r-1}, X_r))^2\right]^{1/2}\\ &\qquad\quad \cdot E\left[(\mathbf{\Psi}(X_1, X_2)\mathbf{\Psi}(X_3, X_4)\cdots\mathbf{\Psi}(X_r, y))^2\right]^{1/2}\\ &\quad = E[\mathbf{\Psi}(x, X_1)^2]^{1/2} E[\mathbf{\Psi}(X_1, X_2)^2]^{1/2}\cdots E[\mathbf{\Psi}(X_r, y)^2]^{1/2}\\ &\quad = E[\mathbf{\Psi}(x, X_1)^2]^{1/2}\cdot E[\mathbf{\Psi}(X_r, y)^2]^{1/2} \end{aligned}$$

for any x, y. The case when r is even is the same.

For the case when $L \ne L \setminus L_0$, by using (3.7), we have that

$$\begin{aligned} &E^{\nu_0^{\otimes\infty}}\left[\prod_{k=1}^m \mathbf{\Psi}(X_{i_k}, X_{j_k})\right]\\ &\quad = E\left[E\left[\prod_{k=1}^m \mathbf{\Psi}(X_{i_k}, X_{j_k})|\sigma\{X_x, x \in L_0\}\right]\right] \end{aligned}$$

$$\leq E\left[\prod_{k:i_k,j_k\in L_0}|\mathbf{\Psi}\left(X_{i_k},X_{j_k}\right)|\left(\prod_{k\in A}E\left[\mathbf{\Psi}(X_{i_k},X_{j_k})^2|\sigma\{X_x,x\in L_0\}\right]\right)^{1/2}\right], \tag{3.8}$$

where A in the third production is defined as

$$\begin{aligned}A=\{k:\ &(i_k\in L_0\quad\&\quad j_k\in L\setminus L_0),\\ &\text{or}\ \ (j_k\in L_0\quad\&\quad i_k\in L\setminus L_0)\}\ .\end{aligned}$$

So, let $g(x)=E[\mathbf{\Psi}(x,X_1)^2]^{1/2}$ and

$$\beta_\ell=\sharp\{k:(i_k=\ell\ \&\ j_k\in L\setminus L_0),\ \text{or}\ (i_k\in L\setminus L_0\ \&\ j_k=\ell)\}\ ,$$

then we can get from (3.8) that

$$\begin{aligned}E\left[\prod_{k=1}^m\mathbf{\Psi}(X_{i_k},X_{j_k})\right]\leq{}&E\left[\prod_{k:i_k,j_k\in L_0}|\mathbf{\Psi}(X_{i_k},X_{j_k})|^2\right]^{1/2}\\ &\cdot E\left[\prod_{k\in A}E[\mathbf{\Psi}(X_{i_k},X_{j_k})^2|\sigma\{X_x,x\in L_0\}]\right]^{1/2}\\ ={}&E\left[\prod_{k:i_k,j_k\in L_0}|\mathbf{\Psi}(X_{i_k},X_{j_k})|^2\right]^{1/2}\cdot E\left[\prod_{\ell\in L_0}g(X_\ell)^{2\beta_\ell}\right]^{1/2}.\end{aligned} \tag{3.9}$$

Since $|\mathbf{\Psi}(x,y)|\leq C_0\|x\|\cdot\|y\|$ for any $x,y\in B$ by the assumption,

$$\begin{aligned}&E\left[\prod_{k:i_k,j_k\in L_0}|\mathbf{\Psi}(X_{i_k},X_{j_k})|^2\right]^{1/2}\\ &\leq E\left[\prod_{k:i_k,j_k\in L_0}C_0^2\|X_{i_k}\|^2\|X_{j_k}\|^2\right]^{1/2}\\ &={C_0}^{\frac{1}{2}\sum_{\ell\in L_0}(\alpha_\ell-\beta_\ell)}\prod_{\ell\in L_0}E\left[\|X_\ell\|^{2(\alpha_\ell-\beta_\ell)}\right]^{1/2}\ .\end{aligned} \tag{3.10}$$

Also, from the definition of g, we have

$$\begin{aligned}g(x)&=E\left[|\mathbf{\Psi}(x,X_1)|^2\right]^{1/2}\leq E\left[{C_0}^2\|x\|^2\|X_1\|^2\right]^{1/2}\\ &=C_0\|x\|E^{\nu_0}\left[\|X_1\|^2\right]^{1/2}=C_6\|x\|,\end{aligned}$$

where $C_6 \equiv C_0 E^{\nu_0}[\|X_1\|^2]^{1/2}$. So,

$$E\left[\prod_{\ell\in L_0} g(X_\ell)^{2\beta_\ell}\right]^{1/2} = \prod_{\ell\in L_0} E\left[g(X_\ell)^{2\beta_\ell}\right]^{1/2}$$
$$\leq \prod_{\ell\in L_0} E\left[(C_6\|X_\ell\|)^{2\beta_\ell}\right]^{1/2} = \prod_{\ell\in L_0} C_6{}^{\beta_\ell} E\left[\|X_\ell\|^{2\beta_\ell}\right]^{1/2}$$
$$= C_6{}^{\sum_{\ell\in L_0}\beta_\ell} \prod_{\ell\in L_0} E\left[\|X_\ell\|^{2\beta_\ell}\right]^{1/2}. \tag{3.11}$$

Let $C_7 \equiv \max\{C_0, C_6, 1\}$, then from (3.9), (3.10), (3.11), we see that

$$E\left[\prod_{k=1}^{m} \mathbf{\Psi}(X_{i_k}, X_{j_k})\right]$$
$$\leq C_0{}^{\frac{1}{2}\sum_{\ell\in L_0}(\alpha_\ell-\beta_\ell)} \prod_{\ell\in L_0} E\left[\|X_\ell\|^{2(\alpha_\ell-\beta_\ell)}\right]^{1/2} \cdot C_6{}^{\sum_{\ell\in L_0}\beta_\ell} \prod_{\ell\in L_0} E\left[\|X_\ell\|^{2\beta_\ell}\right]^{1/2}$$
$$\leq C_7^{\frac{1}{2}\sum_{\ell\in L_0}(\alpha_\ell+\beta_\ell)} \prod_{\ell\in L_0} \left(E\left[\|X_\ell\|^{2(\alpha_\ell-\beta_\ell)}\right] E\left[\|X_\ell\|^{2\beta_\ell}\right]\right)^{1/2}$$
$$\leq C_7^{\sum_{\ell\in L_0}\alpha_\ell} \prod_{\ell\in L_0} E^{\nu_0^{\otimes\infty}}\left[\|X_\ell\|^{2\alpha_\ell}\right]^{1/2}.$$

On the other hand,

$$E^{\gamma_a^{\otimes\infty}}\left[\prod_{k=1}^{m}(Y_{i_k}Y_{j_k})\right] = \prod_{\ell\in L_0} E^{\gamma_a^{\otimes\infty}}\left[Y_\ell{}^{\alpha_\ell}\right].$$

So we only need to take a proper a_0, such that for any $a \leq a_0$, the following holds:

$$C_7^{\alpha_\ell} E^{\nu_0^{\otimes\infty}}\left[\|X_\ell\|^{2\alpha_\ell}\right]^{1/2} \leq E^{\gamma_a^{\otimes\infty}}\left[Y_\ell^{\alpha_\ell}\right], \qquad \text{for any } \ell \in L_0,$$

but this could be gotten from Lemma 3.2. □

The following lemma has been proved in Kusuoka-Tamura [5] (Lemma 2.1 in [5]). We write it here as it will be used later.

Lemma 3.4. *Let Z_i, $i \in \mathbf{N}$ be i.i.d. $\mathbf{R}^d$-valued random variables, with mean 0 and finite variance. Assume that there exist constants A_1, A_2, A_3, such that*

$$E[Z_1 \cdot {}^t Z_1] \leq A_1 \cdot I_d,$$

$$E[\exp(A_2|Z_1|)] \leq A_3.$$

Then for any $b < \frac{1}{2A_1}$, there exist constants $\delta > 0$ and $A_4 > 0$, such that

$$E\left[\exp\left(b\cdot\frac{1}{n}\left|\sum_{i=1}^{n} Z_i\right|^2\right), \left|\frac{1}{n}\sum_{i=1}^{n} Z_i\right| < \delta\right] \leq A_4, \qquad \textit{for any } n \in \mathbf{N}\ ,$$

where δ depends only on A_1, A_2, A_3 and b, and A_4 depends only on d, A_1, A_2, A_3 and b.

Lemma 3.5. *Assume the same assumptions and use the same notations as in Lemma 3.3. Then for any $b < \frac{1}{2}$, there exists a $\varepsilon > 0$, depends only on a_0 and b, where a_0 is the one chosen in Lemma 3.3, such that*

$$\sup_{n\in\mathbf{N}} E^{\nu_0^{\otimes\infty}}\left[\exp\left(b\cdot\frac{1}{n}\sum_{1\le i\ne j\le n}\boldsymbol{\Psi}(X_i,X_j)\right), \left|\frac{1}{n^2}\sum_{1\le i\ne j\le n}\boldsymbol{\Psi}(X_i,X_j)\right|<\varepsilon\right]<\infty\ . \tag{3.12}$$

Proof. First, since $n! \sim \sqrt{2\pi n}(\frac{n}{e})^n$, there exists a constant $C_8 > 0$, such that $n! \ge C_8^{-1}n^n e^{-2n}$. So, for $m = [n\varepsilon e^2]$,

$$\begin{aligned}\sum_{k=m+1}^{\infty}\frac{(n\varepsilon)^{2k}}{(2k)!} &\le C_8\sum_{k=m+1}^{\infty}\left(\frac{n\varepsilon e^2}{2k}\right)^{2k} \le C_8\sum_{k=0}^{\infty}\left(\frac{n\varepsilon e^2}{2m+2}\right)^{k}\\ &\le C_8\frac{1}{1-\frac{n\varepsilon e^2}{2m+2}} \le 2C_8\ .\end{aligned} \tag{3.13}$$

Also, in general, for any random variable Z,

$$\begin{aligned}&E[\exp(nZ), |Z|\le\varepsilon]\\ &\le 2E\left[\sum_{k=0}^{m}\frac{(nZ)^{2k}}{(2k)!}, |Z|\le\varepsilon\right]+2E\left[\sum_{k=m+1}^{\infty}\frac{(nZ)^{2k}}{(2k)!}, |Z|\le\varepsilon\right]\ ,\end{aligned} \tag{3.14}$$

and we can get from Lemma 3.3 that

$$E^{\nu_0^{\otimes\infty}}\left[\left(\sum_{1\le i\ne j\le n}\boldsymbol{\Psi}(X_i,X_j)\right)^m\right] \le E^{\gamma_a^{\otimes\infty}}\left[\left(\sum_{i=1}^{n}Y_i\right)^{2m}\right] \tag{3.15}$$

for any $m \in \mathbf{N}$ and any $a \le a_0$, where a_0 is the one chosen in Lemma 3.3.

So, let $P_m(\xi) = \sum_{k=0}^{m}\frac{|\xi|^{2k}}{(2k)!}$, $m \in \mathbf{N}$, and we can get from (3.13), (3.14), (3.15) that for $m = [bn\varepsilon e^2]$,

$$\begin{aligned}&E\left[\exp\left(b\cdot\frac{1}{n}\sum_{1\le i\ne j\le n}\boldsymbol{\Psi}(X_i,X_j)\right), \left|\frac{1}{n^2}\sum_{1\le i\ne j\le n}\boldsymbol{\Psi}(X_i,X_j)\right|\le\varepsilon\right]\\ &\le 4C_8+2E\left[\sum_{k=0}^{m}\frac{(b\frac{1}{n}\sum_{i\ne j}\boldsymbol{\Psi}(X_i,X_j))^{2k}}{(2k)!}\right]\end{aligned}$$

$$\leq 4C_8 + 2E\left[P_m\left(b\frac{1}{n}\left(\sum_{i=1}^{n} Y_i\right)^2\right), \left|\frac{1}{n}\sum_{i=1}^{n} Y_i\right| < \delta\right]$$
$$+2E\left[P_m\left(b\frac{1}{n}\left(\sum_{i=1}^{n} Y_i\right)^2\right), \left|\frac{1}{n}\sum_{i=1}^{n} Y_i\right| \geq \delta\right], \quad \text{for any } \delta > 0 \ . \tag{3.16}$$

For the second term in the last expression, from the definition of γ_a and the calculation in Lemma 3.1, we see that all of the conditions in Lemma 3.4 is satisfied. So, from Lemma 3.4, for any $b < \frac{1}{2}$, there exists a $\delta > 0$, such that

$$E\left[P_m\left(b\frac{1}{n}\left(\sum_{i=1}^{n} Y_i\right)^2\right), \left|\frac{1}{n}\sum_{i=1}^{n} Y_i\right| < \delta\right]$$
$$\leq E\left[\exp\left(b\frac{1}{n}\left(\sum_{i=1}^{n} Y_i\right)^2\right), \left|\frac{1}{n}\sum_{i=1}^{n} Y_i\right| < \delta\right] \leq^{\exists} C_9 \ . \tag{3.17}$$

Note that δ does not depend on ε here.

For the last term, since

$$P_m(\xi) \leq c^{-2m}\exp(c|\xi|)$$

for any $c \in (0, 1)$, we can get that

$$E\left[P_m\left(b\frac{1}{n}\left(\sum_{i=1}^{n} Y_i\right)^2\right), \left|\frac{1}{n}\sum_{i=1}^{n} Y_i\right| \geq \delta\right]$$
$$\leq c^{-2m}E\left[\exp\left(cb\frac{1}{n}\left(\sum_{i=1}^{n} Y_i\right)^2\right), \left|\frac{1}{n}\sum_{i=1}^{n} Y_i\right| \geq \delta\right]$$
$$\leq c^{-2m}E\left[\exp\left(2cb\frac{1}{n}\left(\sum_{i=1}^{n} Y_i\right)^2\right)\right]^{\frac{1}{2}} P\left(\left|\frac{1}{n}\sum_{i=1}^{n} Y_i\right| \geq \delta\right)^{\frac{1}{2}} \ . \tag{3.18}$$

But here, from the definition of Y_i, we can get from Lemma 3.1 that if $A \equiv \frac{1}{4D_a} - 2cb > 0$, which can be done for any fixed a and b by taking c small enough, then

$$E\left[\exp\left(2cb\frac{1}{n}\left(\sum_{i=1}^{n} Y_i\right)^2\right)\right] \leq \frac{4cb}{A} + 1 < \infty. \tag{3.19}$$

Also, by Cramér's Theorem (*c.f.* [6] page 29, Theorem 1.3.13), we see that

$$\gamma_a^{\otimes\infty}\left(\left|\frac{1}{n}\sum_{i=1}^{n}Y_i\right| \geq \delta\right) \leq \exp\left(-nI_{\gamma_a}(\delta)\right) + \exp\left(-nI_{\gamma_a}(-\delta)\right)$$
$$\leq 2e^{-n\alpha(\delta)} \ , \tag{3.20}$$

where

$$I_{\gamma_a}(\delta) = \sup\left\{\xi\delta - \log\int e^{\xi x}\gamma_a(dx), \xi \geq 0\right\} > 0,$$
$$I_{\gamma_a}(-\delta) = \sup\left\{-\xi\delta - \log\int e^{\xi x}\gamma_a(dx), \xi \leq 0\right\} > 0,$$
$$\alpha(\delta) \equiv I_{\gamma_a}(\delta) \wedge I_{\gamma_a}(-\delta) \ .$$

We have taken m to be $m = [bn\varepsilon e^2]$, so if we take $\varepsilon > 0$ small enough, such that

$$\frac{\alpha(\delta)}{2} + 2b\varepsilon e^2 \log c > 0 \ ,$$

then from (3.18), (3.19), (3.20), we have

$$E\left[P_m\left(b\frac{1}{n}\left(\sum_{i=1}^{n}Y_i\right)^2\right), \left|\frac{1}{n}\sum_{i=1}^{n}Y_i\right| \geq \delta\right]$$
$$\leq \left(\frac{4cb}{A}+1\right)^{\frac{1}{2}} e^{-2m\log c}\left(2e^{-n\alpha(\delta)}\right)^{\frac{1}{2}}$$
$$\leq \left(2\left(\frac{4cb}{A}+1\right)\right)^{\frac{1}{2}} e^{2\log c}\exp\left(-n\left(\frac{\alpha(\delta)}{2}+2b\varepsilon e^2\log c\right)\right)$$
$$<^{\exists} C_{10}, \qquad \text{for any } n \in \mathbf{N} \ , \tag{3.21}$$

the c here is the one chosen before.

(3.16), (3.17) and (3.21) completes the proof of the lemma. □

Lemma 3.6. *Assume the same conditions as in Lemma 3.5. Then for any* $b < \frac{1}{2}$, *there exist constants* $\varepsilon_1 > 0$ *and* $\varepsilon_2 > 0$, *such that the following holds:*

$$\sup_{n\in\mathbf{N}} E^{\nu_0^{\otimes\infty}}\left[\exp\left(b\cdot n\mathbf{\Psi}\left(\frac{S_n}{n},\frac{S_n}{n}\right)\right),\right.$$
$$\left.\left\{\left|\frac{1}{n^2}\sum_{i=1}^{n}\mathbf{\Psi}(X_i,X_i)\right| < \varepsilon_1\right\} \cap \left\{\|\frac{1}{n}\sum_{i=1}^{n}X_i\| < \varepsilon_2\right\}\right] < \infty \ .$$

Proof. Let $N_0 \equiv \frac{qbC_0}{C_3'}$. For $n = 1, \ldots, N_0$, the item is obviously bounded. So we only need to do with $n > N_0$. Since $b < \frac{1}{2}$, there exists a $p > 1$ small enough such that $p \cdot b < \frac{1}{2}$. Let q be the dual number of p, that is, $\frac{1}{p} + \frac{1}{q} = 1$, By Hölder's inequality and Lemma 3.5, we only need to show that

$$\sup_{n>N_0} E^{\nu_0^{\otimes\infty}}\left[\exp(qb\cdot\frac{1}{n}\sum_{i=1}^{n}\mathbf{\Psi}(X_i,X_i))\right]<\infty\ .$$

But by Hölder's inequality, for any $n > N_0 = \frac{qbC_0}{C_3'}$,

$$E^{\nu_0^{\otimes\infty}}\left[\exp\left(qb\cdot\frac{1}{n}\sum_{i=1}^{n}\mathbf{\Psi}(X_i,X_i)\right)\right]\le E^{\nu_0}\left[\exp(C_3'\|X\|^2)\right]^{\frac{qbC_0}{C_3'}}<\infty\ .$$

This completes the proof of the lemma. □

Lemma 3.7. *Assume the assumption* (A1′) *in section* 1. *Assume that* $\mathbf{\Psi}$ *is a symmetric, bilinear function that satisfies the following conditions:*

1. *There exists a constant* $C_0 > 0$*, such that*

$$|\mathbf{\Psi}(x,y)|\le C_0\|x\|\cdot\|y\|,\qquad \textit{for any } x,y\in B\ ,$$

2. $\int_B \mathbf{\Psi}(x,y)^2\nu_0(dx)\nu_0(dy)\equiv b<\frac{1}{2}$.

Then there exists a $\varepsilon > 0$*, such that*

$$\sup_{n\in\mathbf{N}} E^{\nu_0^{\otimes\infty}}\left[\exp(\frac{1}{n}\sum_{i,j=1}^{n}\mathbf{\Psi}(X_i,X_j)),\ \|\frac{1}{n}\sum_{i=1}^{n}X_i\|<\varepsilon\right]<\infty\ . \tag{3.22}$$

Proof. Since $\mathbf{\Psi}(x,y)\le C_0\|x\|\cdot\|y\|$ for any $x, y \in B$, we have

$$\begin{aligned}\nu_0^{\otimes\infty}\left(|\frac{1}{n^2}\sum_{i=1}^{n}\mathbf{\Psi}(X_i,X_i)|\ge\varepsilon_1\right)&\le\nu_0^{\otimes\infty}\left(\sum_{i=1}^{n}\|X_i\|^2\ge\frac{\varepsilon_1}{C_0}\cdot n^2\right)\\&\le e^{-\frac{\varepsilon_1}{C_0}\cdot n^2\cdot C_3'}\cdot(E^{\nu_0}[e^{C_3'\|X_1\|^2}])^n\ .\end{aligned}$$

Therefore,

$$\begin{aligned}&E^{\nu_0^{\otimes\infty}}\left[\exp(n\mathbf{\Psi}(\frac{S_n}{n},\frac{S_n}{n})),\left\{|\frac{1}{n^2}\sum_{i=1}^{n}\mathbf{\Psi}(X_i,X_i)|>\varepsilon_1\right\}\cap\left\{\|\frac{1}{n}\sum_{i=1}^{n}X_i\|<\varepsilon_2\right\}\right]\\&\le E^{\nu_0^{\otimes\infty}}\left[\exp(2n\mathbf{\Psi}(\frac{S_n}{n},\frac{S_n}{n})),\ \|\frac{1}{n}\sum_{i=1}^{n}X_i\|<\varepsilon_2\right]^{1/2}\\&\quad\cdot\nu_0^{\otimes\infty}\left(|\frac{1}{n^2}\sum_{i=1}^{n}\mathbf{\Psi}(X_i,X_i)|>\varepsilon_1\right)^{1/2}\\&\le\exp(nC_0\varepsilon_2^2)\cdot\left(\exp(-\frac{\varepsilon_1}{C_0}\cdot n^2\cdot C_3')\cdot(E^{\nu_0}[e^{C_3'\|X_1\|^2}])^n\right)^{\frac{1}{2}}\ ,\end{aligned}$$

which is obviously bounded for $n \in \mathbf{N}$.

This accompanied with Lemma 3.6 gives our assertion. □

4. Proof of the main theorem

In this section, we will give the proof of the main theorem.

As in Bolthausen [1], by a easy calculation and Proposition 1.1, we can get that

$$\exp\left(-n(\mathbf{\Phi}(x^*) - h(x^*))\right) E^{\mu^{\otimes\infty}}\left[\exp(n\mathbf{\Phi}(\frac{S_n}{n}))\right]$$
$$= E^{\nu_0^{\otimes\infty}}\left[\exp\left(\frac{n}{2}D^2\mathbf{\Phi}(x^*)(\frac{1}{n}\sum_{i=1}^n X_i, \frac{1}{n}\sum_{i=1}^n X_i) + nR(x^*, \frac{1}{n}\sum_{i=1}^n X_i)\right)\right] \ ,$$

where $R(x^*, \frac{1}{n}\sum_{i=1}^n X_i)$ is the 3rd remainder of the Taylor's formula.

Therefore, to proof Theorem 1.2, we only need to show that the following two lemmas hold:

Lemma 4.1 . *There exists a constant $\varepsilon > 0$, such that*

$$\lim_{n\to\infty} E^{\nu_0^{\otimes\infty}}\left[\exp\left(\frac{n}{2}D^2\mathbf{\Phi}(x^*)(\frac{1}{n}\sum_{i=1}^n X_i, \frac{1}{n}\sum_{i=1}^n X_i) + nR(x^*, \frac{1}{n}\sum_{i=1}^n X_i)\right), \|\frac{1}{n}\sum_{i=1}^n X_i\| < \varepsilon\right]$$
$$= \exp\left(\frac{1}{2}\int_B D^2\mathbf{\Phi}(x^*)(x,x)\nu_0(dx)\right)\cdot det_2(I - D^2\mathbf{\Phi}(x^*))^{-\frac{1}{2}} \equiv A \ . \quad (4.1)$$

Lemma 4.2 . *For any $\varepsilon > 0$,*

$$\limsup_{n\to\infty}\frac{1}{n}\log E^{\nu_0^{\otimes\infty}}\left[\exp\left(\frac{n}{2}D^2\mathbf{\Phi}(x^*)(\frac{1}{n}\sum_{i=1}^n X_i, \frac{1}{n}\sum_{i=1}^n X_i) + nR(x^*, \frac{1}{n}\sum_{i=1}^n X_i)\right), \|\frac{1}{n}\sum_{i=1}^n X_i\| \geq \varepsilon\right] < 0 \ . \quad (4.2)$$

Lemma 4.2 can be gotten from the following proposition, which has been shown by Donsker-Varadhan [3]:

Proposition 4.3. 1. *$h(x)$ is a lower semi-continuous function, and $\{x : h(x) \leq r\}$ is compact in B for $\forall r \in [0,\infty)$,*

2. *For any closed set $K \subset B$,*

$$\limsup_{n\to\infty}\frac{1}{n}\log\mu^{\otimes\infty}\left(\{\underline{x}; \frac{1}{n}\sum_{i=1}^n x_i \in K\}\right) \leq -\inf\{h(x); x \in K\} \ ,$$

3. *For any open set $G \subset B$,*

$$\liminf_{n\to\infty}\frac{1}{n}\log\mu^{\otimes\infty}\left(\{\underline{x}; \frac{1}{n}\sum_{i=1}^n x_i \in G\}\right) \geq -\inf\{h(x); x \in G\} \ .$$

To prove Lemma 4.1 , we will give the following proposition and lemma first:

Proposition 4.4. *Let* $\mathbf{\Psi} : B \times B \to \mathbf{R}$ *be a function such that* $\mathbf{\Psi}|_{H\times H}$ *is a Hilbert-Schmidt function with eigenvalues* $a_\ell, \ell = 1, 2, \cdots$ *and eigenvectors* $e_\ell, \ell = 1, 2, \cdots$*, i.e.*

$$\mathbf{\Psi}(x, y) = \sum_{k=1}^{\infty} a_k(e_k, x)(e_k, y), \qquad \text{for all } x, y \in H \ .$$

Then e_k *can be extended to the whole* B *for any* k *that satisfies* $a_k \neq 0$*, and* $\sum_{k=1}^{N} a_k(e_k,x)(e_k,y)$ *converges to* $\mathbf{\Psi}(x,y)$ *in* $L^2(d\nu_0^{\otimes 2}, B \times B)$ *as* $N \to \infty$*.*

Proof. $\{e_\ell\}_{\ell\in\mathbf{N}}$ is a complete orthogal normalized base of H^*. Let $f_\ell, \ell \in \mathbf{N}$ be the dual base of H. Since $\mathbf{\Psi}(f_\ell, x) = a_\ell(e_\ell, x)$ for any $x \in H$ for each ℓ, and the left hand side is continuous with respect to $x \in B$, we can extend e_ℓ to the whole B in this way if $a_\ell \neq 0$. The others are easy. □

Lemma 4.5. *Under the assumptions* (A1′), (A2) ∼ (A5) *in section* 1*, there exist constants* $p > 1$ *and* $\varepsilon > 0$*, such that*

$$\sup_{n\in\mathbf{N}} E^{\nu_0^{\otimes\infty}}\left[\exp\left(p\cdot\frac{n}{2}D^2\mathbf{\Phi}(x^*)(\frac{1}{n}\sum_{i=1}^{n}X_i, \frac{1}{n}\sum_{i=1}^{n}X_i)\right), \|\frac{1}{n}\sum_{i=1}^{n}X_i\| < \varepsilon\right] < \infty \ .$$

Proof. Let $a_\ell \in \mathbf{R}$ and $e_\ell \in H^*$, $\ell \in \mathbf{N}$ be the eigenvalues and the corresponding eigenvectors of $D^2\mathbf{\Phi}(x^*)\Big|_{H\times H}$, then

$$D^2\mathbf{\Phi}(x^*)(x, y) = \sum_{\ell=1}^{\infty} a_\ell(e_\ell, x)(e_\ell, y), \qquad \text{for any } x, y \in H \ .$$

$e_\ell, \ell = 1, 2, \cdots$ becomes a orthonormal base of H^*. Let $f_\ell, \ell = 1, 2, \cdots$ be the dual base of H, then as done in Proposition 4.4, for any ℓ with $a_\ell \neq 0$, we can assume that $e_\ell \in B^*$.

For any $N \in \mathbf{N}$, let

$$\mathbf{\Psi}_1^{(N)}(x, y) = \sum_{k=1}^{N} a_k(e_k, x)(e_k, y),$$
$$\mathbf{\Psi}_2^{(N)}(x, y) = D^2\mathbf{\Phi}(x^*)(x, y) - \mathbf{\Psi}_1(N)(x, y), \qquad x, y \in B \ .$$

Since $D^2\mathbf{\Phi}(x^*)$ is a Hilbert-Schmidt function from Proposition 2.2, we can see that $\mathbf{\Psi}_2^{(N)}$ is also a Hilbert-Schmidt function. Also, from Proposition 4.4, for any $\delta > 0$, there exists a $N_0 \in \mathbf{N}$ large enough, such that $\int_{B\times B}\mathbf{\Psi}_2^{(N_0)}(x, y)^2\nu_0(dx)\nu_0(dy) < \delta$. For the sake of simply, from now on, we will write $\mathbf{\Psi}_i$ for $\mathbf{\Psi}_i^{(N_0)}$, $i = 1, 2$. From the definition of $\mathbf{\Psi}_1$ and $\mathbf{\Psi}_2$, we see that they are bilinear and symmetric.

From Hölder's inequation, for any $r, s > 1 : \frac{1}{r} + \frac{1}{s} = 1$, we have

$$E^{\nu_0^{\otimes\infty}}\left[\exp\left(p\left\{\frac{n}{2}D^2\mathbf{\Phi}(x^*)(\frac{1}{n}\sum_{i=1}^{n}X_i, \frac{1}{n}\sum_{i=1}^{n}X_i) + nR(x^*, \frac{1}{n}\sum_{i=1}^{n}X_i)\right\}\right),\right.$$
$$\left.\|\frac{1}{n}\sum_{i=1}^{n}X_i\| < \varepsilon\right]$$

$$\leq E^{\nu_0^{\otimes\infty}}\left[\exp\left(p \cdot r \frac{n}{2} \mathbf{\Psi}_1(\frac{S_n}{n}, \frac{S_n}{n})\right), \|\frac{S_n}{n}\| < \varepsilon\right]^{\frac{1}{r}} \tag{4.3}$$

$$\cdot E^{\nu_0^{\otimes\infty}}\left[\exp\left(p \cdot s \cdot \frac{n}{2} \mathbf{\Psi}_2(\frac{S_n}{n}, \frac{S_n}{n})\right), \|\frac{S_n}{n}\| < \varepsilon\right]^{\frac{1}{s}} \quad . \tag{4.4}$$

For (4.3), since $\mathbf{\Psi}_1$ is a finite type, we can consider X_i's as finite dimensional valued random variables. Also, since $a_k < 1, k \in \mathbf{N}$ from the assumption (A4), and $a_n \to 0$ as $n \to \infty$ from the fact that $\sum_{n=1}^{\infty} a_n^2 < \infty$, there exists a constant $a < 1$, such that $a_n < a$ for any $n \in \mathbf{N}$. Take $p > 1$ such that $a \cdot p < 1$, and fix it. Then take $r > 1$ small enough, and we can get from Lemma 3.4 that this term is bounded for $n \in \mathbf{N}$, for $\varepsilon > 0$ small enough. Note that the $p > 1$ and $r > 1$ here depend only on $a_k, k \in \mathbf{N}$, and are independent to N.

For (4.4), as mentioned above, $\mathbf{\Psi}_2$ satisfies all of the conditions in Lemma 3.7 except (3). But for any fixed s, we can take δ small enough such that (3) is being satisfied. So, from Lemma 3.7, (4.4) is bounded for $n \in \mathbf{N}$, for N_0 large enough such that $\delta > 0$ is small enough.

This completes the proof of the lemma. □

Now, we will give the proof of Lemma 4.1 , using the proposition and lemma above.

Proof of Lemma 4.1 . Here, from Lemma 4.2 , we have that

$$\lim_{n\to\infty}\left|E\left[\exp\left(\frac{n}{2}D^2\mathbf{\Phi}(x^*)(\frac{S_n}{n}, \frac{S_n}{n}) + nR(x^*, \frac{S_n}{n})\right), \|\frac{S_n}{n}\| < \varepsilon\right] - A\right|$$

$$= \lim_{\varepsilon\to 0}\lim_{n\to\infty}\left|E\left[\exp\left(\frac{n}{2}D^2\mathbf{\Phi}(x^*)(\frac{S_n}{n}, \frac{S_n}{n}) + nR(x^*, \frac{S_n}{n})\right), \|\frac{S_n}{n}\| < \varepsilon\right] - A\right|$$

$$\leq \lim_{\varepsilon\to 0}\lim_{n\to\infty}|E\left[\exp\left(\frac{n}{2}D^2\mathbf{\Phi}(x^*)(\frac{S_n}{n}, \frac{S_n}{n}) + nR(x^*, \frac{S_n}{n})\right), \|\frac{S_n}{n}\| < \varepsilon\right]$$

$$- E\left[\exp\left(\frac{n}{2}D^2\mathbf{\Phi}(x^*)(\frac{S_n}{n}, \frac{S_n}{n})\right), \|\frac{S_n}{n}\| < \varepsilon\right]| \tag{4.5}$$

$$+ \lim_{\varepsilon\to 0}\lim_{n\to\infty} E\left[\exp\left(\frac{n}{2}D^2\mathbf{\Phi}(x^*)(\frac{S_n}{n}, \frac{S_n}{n})\right), \|\frac{S_n}{n}\| < \epsilon\right] - A| \ , \tag{4.6}$$

so the lemma will be shown if we can show that (4.5) equals 0, and that there exists a constant $\varepsilon_0 > 0$, such that for any $\varepsilon < \varepsilon_0$,

$$E^{\nu_0^{\otimes\infty}}\left[\exp\left(\frac{n}{2}D^2\mathbf{\Phi}(x^*)(\frac{1}{n}\sum_{i=1}^{n}X_i, \frac{1}{n}\sum_{i=1}^{n}X_i)\right), \|\frac{1}{n}\sum_{i=1}^{n}X_i\| < \varepsilon\right] \to A,\ n \to \infty. \tag{4.7}$$

Let us show (4.7) first. Here, as in Kusuoka-Tamura [5], we can take a seperable Hilbert space H_1 such that H is a dense linear subspace of H_1, and the inclusion map from H to H_1 is a Hilbert-Schmidt operator. Then, let W be an H_1-valued random variable such that

$$E\left[\exp(\sqrt{-1}(W,u))\right] = \exp\left(-\frac{1}{2}\|u\|^2_{H^*}\right), \quad \text{for all } u \in H_1^* \subset H^* \ .$$

Since

$$E^{\nu_0^{\otimes\infty}}\left[nu(\frac{S_n}{n})^2\right] = \|u\|^2_{H^*} \ ,$$

$\frac{1}{\sqrt{n}} S_n$ can be regarded as H_1-valued random variables with respect to $\nu_0^{\otimes\infty}$. Therefore, from the central limit theorem for independently identically distributed Hilbert space valued random variables, we see that the law of $\frac{1}{\sqrt{n}} \sum_{i=1}^n X_i$ under $\nu_0^{\otimes\infty}$ converges to W in distribution as $n \to \infty$.

So,

$$\begin{aligned}
&\frac{1}{n} \sum_{1\le i\ne j\le n} \sum_{k=1}^N a_k(e_k, X_i)(e_k, X_j) \\
&= \sum_{k=1}^N a_k(e_k, \frac{1}{\sqrt{n}} S_n)(e_k, \frac{1}{\sqrt{n}} S_n) - \frac{1}{n}\sum_{i=1}^n \sum_{k=1}^N a_k(e_k, X_i)^2 \\
&\to \sum_{k=1}^N a_k(e_k, W)^2 - \sum_{k=1}^N a_k = \sum_{k=1}^N a_k\left((e_k, W)^2 - 1\right), \qquad \text{for any } N \in \mathbf{N} \ ,
\end{aligned}$$

where the "$\to$" above means the convergence in distribution. Therefore, since

$$E\left[\{\frac{1}{n} \sum_{1\le i\ne j\le n} \left(D^2\mathbf{\Phi}(x^*)(X_i, X_j) - \sum_{k=1}^N a_k(e_k X_i)(e_k, X_j)\right)\}^2\right] \to 0, N \to \infty \ ,$$

which is uniformly in n, we see that $\frac{1}{n}\sum_{1\le i\ne j\le n} D^2\mathbf{\Phi}(x^*)(X_i, X_j)$ under $\nu_0^{\otimes\infty}$ converges to : $D^2\mathbf{\Phi}(x^*)(W, W)$: in distribution as $n \to \infty$, where : $D^2\mathbf{\Phi}(x^*)(x, x)$: is defined as the $L^2(d\tilde{\mu})$-limit of $\sum_{\ell=1}^N a_\ell((e_\ell, x)^2 - 1)$ as $N \to \infty$. $\tilde{\mu}$ is the distribution of W. Also, $\frac{1}{n}\sum_{n=1}^{n} D^2\mathbf{\Phi}(x^*)(X_i, X_i)$ under $\nu_0^{\otimes\infty}$ converges to $\int_B D^2\mathbf{\Phi}(x^*)(x, x)\nu_o(dx)$ almost surely.

Therefore, (4.7) can be gotten from Lemma 4.5.

Now, let us show that (4.5) equals 0. Write it as $\lim_{\varepsilon\to 0}\lim_{n\to\infty}\phi(n, \varepsilon)$. Let $p > 1$ be the one chosen in Lemma 4.5, and let q be determined by $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\phi(n,\varepsilon) \le E\left[\exp\left(p\cdot\frac{n}{2}D^2\mathbf{\Phi}(x^*)(\frac{S_n}{n}, \frac{S_n}{n})\right), \|\frac{S_n}{n}\| < \varepsilon\right]^{1/p} \tag{4.8}$$

$$\cdot E\left[\left|\exp(nR(x^*, \frac{S_n}{n})) - 1\right|^q, \|\frac{S_n}{n}\| < \varepsilon\right]^{1/q} \ . \tag{4.9}$$

The boundness of (4.8) for $n \in \mathbf{N}$ has been estabilished. As for (4.9), by Lemma 3.7,

$$\sup_{n\in\mathbf{N}} E\left[e^{p\cdot qC_4\varepsilon nK(\frac{S_n}{n},\frac{S_n}{n})}, \|\frac{S_n}{n}\| < \varepsilon\right] < \infty$$

if $\varepsilon > 0$ is small enough, so from the fact that $|e^x - 1|^q \leq (e^{|x|} - 1)^q \leq e^{q|x|} - 1$, we have

$$(4.9)^q \leq E\left[e^{qnR(x^*,\frac{S_n}{n})}, \|\frac{S_n}{n}\| < \varepsilon\right] - \nu_0(\|\frac{S_n}{n}\| < \varepsilon)$$

$$\leq E\left[e^{qC_4\varepsilon K(\frac{S_n}{\sqrt{n}},\frac{S_n}{\sqrt{n}})}, \|\frac{S_n}{n}\| < \varepsilon\right] - \nu_0(\|\frac{S_n}{n}\| < \varepsilon)$$

$$\to E\left[\exp(qC_4\varepsilon : K(W,W) :)\right] \cdot e^{C_4\varepsilon \int_B K(y,y)\nu_0(dy)} - 1, \quad \text{as } n \to \infty ,$$

which converges to 0 as $\varepsilon \to 0$.

This completes the proof of the lemma. □

5. Remark

Let $U \equiv \det_2(I - D^2\mathbf{\Phi}(x^*))^{-\frac{1}{2}} < \infty$, and let $P_n, n \in \mathbf{N}$, be the probability measures given by

$$dP_n/d\mu^{\otimes\infty}(\underline{x}) = \exp\left(n\mathbf{\Phi}(\frac{S_n}{n})\right)/E^{\mu^{\otimes\infty}}\left[\exp(n\mathbf{\Phi}(\frac{S_n}{n}))\right], \quad \underline{x} = (x_1, x_2, \cdots) .$$

Since we did not assume the existence of the Gaussian measure on B as in Bolthausen [1], we can not write in B the limit of the disribution of $\sqrt{n}(\frac{S_n}{n} - x^*)$ under P_n, but we can still get the following:

Theorem 5.1. *Assume the same conditions as in Theorem 1.2, then for any $n \in \mathbf{N}$, and any $u_k \in B^*$, $k = 1, 2, \cdots, n$, the distribution of $\{_{B^*}(u_k, \sqrt{n}(\frac{S_n}{n} - x^*))_B, u_k \in B^*, k = 1, 2, \cdots, n\}$ under P_n converge weakly to the Normal distribution $N(0, (\sum_{k=1}^{\infty} u_i^k u_j^k \frac{1}{1-a_k})_{i,j})$, where $a_\ell, e_\ell, \ell \in \mathbf{N}$ are the ones defined in the proof of lemma 4.5, and $u_i = \sum_k u_i^k e_k, \quad i = 1, 2, \cdots$*

The proof is similar with the one above, and will be omitted.

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