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A-Priori Decay for Eigenfunctions of Perturbed Periodic Schrödinger Operators

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We dedicate this work to Werner Amrein for his 60-th anniversary

Abstract. In this paper we use a general procedure [11] allowing to study the asymptotic behavior of eigenfunctions (even for eigenvalues that are embedded in the continuous spectrum) and prove exponential decay of eigenfunctions for a large class of perturbed periodic Schrödinger Hamiltonians.

1 Introduction

In this paper we consider the problem of obtaining upper bounds for the rate of decay at infinity for eigenfunctions of perturbed periodic Schrödinger operators. More precisely, let us fix a Hamiltonian of the form $H_I := H + V_I$ where $H := -\Delta + V$ is a periodic Schrödinger operator in dimension n and V_I is a perturbation decaying at infinity (faster then $|x|^{-1}$). We shall suppose that the spectrum of H has an isolated part at the bottom that can be described by N analytic eigenvalues with analytic associated eigenprojectors (for example if the first band is isolated), more precisely we shall impose our Hypothesis 1.1 below. Under these conditions we show that any eigenvalue of the perturbed Hamiltonian H_I that is a regular value (more precisely see Definition 1.2), has eigenfunctions that decay exponentially at infinity, with an exponent linear in $|x|$ (see Theorem 1.4). Let us remark that our result covers also the case of embedded eigenvalues as long as they are regular.

Let us point out that the existence of embedded eigenvalues for perturbations of periodic Schrödinger operators has been subject to intensive work. In [10] it is shown that for any continuous V and any number E belonging to the spectrum of H, there exists a function V_I which is $O(< x >^{-1})$ at infinity such that E is an eigenvalue of $H + V_I$. In more than one dimension the situation is less clear. Anyway, if $n = 2$ or 3, for some classes of periodic V's, eigenvalues embedded into the spectrum of H are forbidden if one imposes the very restrictive condition

$$|V_I(x)| \leq Cexp(-|x|^{4/3+\epsilon})$$

for a strictly positive ϵ (see [9]).

We obtain our result (Theorem 1.4) by first proving a weighted estimation of Hardy type (with exponential weights) for the unperturbed periodic Hamiltonian

H (Theorem 1.3). In fact, inspired by [1], [2], [3], [4], [5], [8], we elaborate a general scheme (see also [11]) for obtaining Hardy type inequalities for a Hamiltonian starting from a conjugate operator of a special form imposed by the form of the weight function. In [11] we have used this general method for Hamiltonians given by convolution with analytic functions. In our case we shall isolate the bounded energy region of the first N bands for which we shall apply a generalization of our previous method and the rest of the spectrum for which we shall use a variant of the general method of Agmon [1].

We shall denote by ∇ and Δ the usual gradient and Laplace operators on $C_0^\infty(\mathbb{R}^n)$ and by $\mathcal{H}^2(\mathbb{R}^n)$ the Sobolev space of second order. Let $p = 2$ for n=1,2,3, $p > n/2$ for $n \geq 4$ and let $V \in L^p_{loc}(\mathbb{R}^n;\mathbb{R})$ be $\mathbb{Z}^n$-periodic on $\mathbb{R}^n$. By some obvious modifications one can also consider a general type of lattice. We consider the Hamiltonian :

$$H = -\Delta + V \tag{1.1}$$

to be the usual self-adjoint operator in $L^2(\mathbb{R}^n)$ (having domain $\mathcal{H}^2(\mathbb{R}^n)$, see [12]). The well-known Floquet representation allows one to decompose H as a direct integral corresponding to the representation : $L^2(\mathbb{R}^n) \cong L^2(\mathbb{T}^n; L^2(\Omega))$, where :

$$\mathbb{T}^n := \mathbb{R}^n/\mathbb{Z}^n \cong (S^1)^n; \qquad \Omega := [0,1)^n \tag{1.2}$$

are the n-dimensional torus and the fundamental domain associated to $\mathbb{Z}^n$. In the following we identify functions defined on $\mathbb{T}^n$ with periodic functions on $\mathbb{R}^n$.

The Hamiltonian H is decomposable with respect to the above representation and each "fibre Hamiltonian" $H(\tau)$ (for $\tau \in \mathbb{T}^n$) has compact resolvent and thus a discrete spectrum $\{\lambda_a(\tau)\}_{a\in\mathbb{N}}$, defining the so-called "band functions". Due to the fact that our procedure relies on the regularity of the functions : $\mathbb{T}^n \ni \tau \longmapsto \lambda_a(\tau)$ and being well known that for $n > 1$ some difficult problems appear in this context, we are obliged to impose some implicit conditions that we now formulate.

We shall constantly denote :

$$\begin{array}{c} \mathbb{C}^n_\delta := \{z \in \mathbb{C}^n \mid |\mathcal{I}m z_j| < \delta, \quad \forall\, j \in \{1,...,n\}\}, \quad \delta > 0 \\ \mathcal{P}(L^2(\Omega)) := \left\{P \in \mathcal{B}(L^2(\Omega)) \mid P^2 = P = P^*\right\}. \end{array} \tag{1.3}$$

Hypothesis 1.1. *By denoting $\sigma(H)$ the spectrum of the operator H, we assume :*

a) $\sigma(H) = \sigma_0 \cup \sigma_\infty$, where : $(\inf \sigma_\infty) - (\sup \sigma_0) = d_0 > 0$;

b) there is some $N \in \mathbb{N}^*$ and for each $a \in \{1,...,N\}$ two functions :

$$\lambda_a : \mathbb{T}^n \to \mathbb{R} \qquad , \qquad \pi_a : \mathbb{T}^n \to \mathcal{P}(L^2(\Omega)) \tag{1.4}$$

that are analytic (with respect to the uniform topology on $\mathcal{P}(L^2(\Omega))$ in the second case) and admit holomorphic extensions to some strip $\mathbb{C}^n_\delta$ for some $\delta > 0$, such that the Hamiltonian H reduced to the spectral subspace associated to σ_0 is unitarily equivalent, in the Floquet representation, to multiplication with the following *operator-valued function of* $\tau \in \mathbb{T}^n$*:*

$$\sum_{a=1}^{N} \lambda_a(\tau)\pi_a(\tau). \tag{1.5}$$

Let us remark that our Hypothesis covers the usual case in which the spectrum of H has an isolated band at its bottom, but also the situation of several bands, even overlapping, as long as one can assure the analyticity of the eigenvalues and of the eigenprojections.

Definition 1.2. Let us denote by $\mathcal{E}_0(H)$, the set of points $t < \inf \sigma_\infty$ such that $\exists \varepsilon > 0$, $\exists \alpha_0 > 0$ for which $|(\nabla\lambda_a)(\tau)| \geq \alpha_0$, $\forall \tau \in \lambda_a^{-1}((t-\varepsilon, t+\varepsilon))$ and $\forall a \in \{1, ..., N\}$. We call this set, the *regular set of* H *below* σ_∞.

Let us remark that $\mathcal{E}_0(H)$ is the complement in $(-\infty, \inf \sigma_\infty)$ of the set of critical values of the functions $\{\lambda_1, ..., \lambda_N\}$. With these notations we can state now the main results of our work, that will be proved in Section 3.

Theorem 1.3. *Let* H *be a periodic Schrödinger Hamiltonian satisfying the Hypothesis 1.1 and let* $E \in \mathcal{E}_0(H)$*. Then there exists a constant* $\kappa_0 \in (0, 2\pi\delta)$ *such that for any* $\kappa \in (0, \kappa_0)$ *there exists a positive constant* C *(depending on* E *and* κ*) for which :*

$$\left\| e^{\kappa<Q>} f \right\|_{\mathcal{D}(H)} \leq C \left\| \sqrt{<Q>} e^{\kappa<Q>} (H-E) f \right\|, \qquad \forall f \in \mathcal{D}(H). \tag{1.6}$$

We have denoted by $\|.\|_{\mathcal{D}(H)}$ *the graph norm with respect to* H*.*

Theorem 1.4. *Let* H *be a periodic Schrödinger operator (1.1) for which Hypothesis 1.1 stands true. Let* V_I *be a potential of class* $L^p_{loc}(\mathbb{R}^n)$ *(with* p *as defined before (1.1)), such that* $\lim_{|x|\to\infty} < x > |V_I(x)| = 0$*. Then for any eigenvalue* E *of the Hamiltonian* $H_I := H + V_I$ *that belongs to* $\mathcal{E}_0(H)$ *there exists* $\kappa \in (0, \delta)$ *such that for any corresponding eigenvector* g *:*

$$e^{\kappa<Q>} g \in L^2(\mathbb{R}^n). \tag{1.7}$$

An Appendix is dedicated to some technical lemmas needed in the proof of Theorem 1.3.

2 Some Developments in the Floquet Representation

Let H be a periodic Schrödinger Hamiltonian as in the preceding section. We shall briefly recall some facts concerning the Floquet representation in order to fix our notations and to put into evidence some objects and properties that we shall need in the sequel.

For $x \in \mathbb{R}^n$ let $x = [x] + \underline{x}$ with $[x] \in \mathbb{Z}^n$, $\underline{x} \in \Omega$. Then, if we denote $\mathcal{K} = L^2(\Omega)$, we can define the unitary isomorphism :

$$\begin{gathered} L^2(\mathbb{R}^n) \ni f \mapsto U_0 f \in L^2(\mathbb{T}^n;\mathcal{K}) \\ (U_0 f)(\tau,\xi) := (2\pi)^{n/2} \sum_{\alpha\in\mathbb{Z}^n} e^{-i2\pi\alpha\cdot\tau} f(\alpha+\xi). \end{gathered} \tag{2.8}$$

For further use let us also give the explicit form of its inverse :

$$\forall \overset{\circ}{f} \in L^2(\mathbb{T}^n;\mathcal{K}), \qquad (U_0^{-1} \overset{\circ}{f})(x) = (2\pi)^{-n/2} \int_{\mathbb{T}^n} e^{i2\pi[x]\cdot\tau} \overset{\circ}{f}(\tau,\underline{x}) d\tau. \tag{2.9}$$

We constantly distinguish between the two unitarily equivalent representations $\mathcal{H} = L^2(\mathbb{R}^n)$ and $\overset{\circ}{\mathcal{H}} = L^2(\mathbb{T}^n;\mathcal{K})$ and we use notations of the form $\overset{\circ}{H} := U_0 H U_0^{-1}$. For the position operators :

$$\begin{gathered} \mathcal{D}(Q_j) := \left\{ f \in \mathcal{H} \mid \int_{\mathbb{R}^n} |x_j f(x)|^2 dx < \infty \right\} \\ (Q_j f)(x) := x_j f(x), \qquad Q := (Q_j)_{j=1,\dots,n} \end{gathered} \tag{2.10}$$

we have the explicit form in the representation $\overset{\circ}{\mathcal{H}}$:

$$\left(\overset{\circ}{Q}\overset{\circ}{f}\right)(\tau,\xi) := \left(U_0 Q U_0^{-1} \overset{\circ}{f}\right)(\tau,\xi) = \left(\left(\frac{i}{2\pi}\nabla_\tau + M_\xi\right)\overset{\circ}{f}\right)(\tau,\xi) \tag{2.11}$$

for any $\overset{\circ}{f} \in C^\infty(\mathbb{T}^n;\mathcal{K})$, where ∇_τ is the gradient operator with respect to the variable $\tau \in \mathbb{T}^n$ and M_ξ is the operator of multiplication with the variable in $\mathcal{K}$.

For any n commuting variables $\{X_1,\dots,X_n\}$ let $< X > := \left\{ \sum_{j=1}^n X_j^2 \right\}^{1/2}$. Then $< Q >$ defines a self-adjoint operator on the domain $\mathcal{D}(Q) := \bigcap_{j=1}^n \mathcal{D}(Q_j)$ that is a domain of essential self-adjointness for each Q_j.

It is useful to observe that for $j = 1,\dots,n$, one can define the operators :

$$([Q_j] f)(x) := [x_j] f(x), \qquad \mathcal{D}([Q_j]) := \mathcal{D}(Q_j) \tag{2.12}$$

and they satisfy the relation :

$$[Q_j] = -\frac{1}{2\pi} U_0^{-1} (-i\nabla_\tau) U_0. \tag{2.13}$$

Associated to these operators we have a third representation that we shall frequently use $\widetilde{\mathcal{H}} := l^2(\mathbb{Z}^n;\mathcal{K})$, obtained by the inverse discrete Fourier transform :

$$\begin{gathered} \mathcal{F}_0 : l^2(\mathbb{Z}^n;\mathcal{K}) \to L^2(\mathbb{T}^n;\mathcal{K}) \\ (\mathcal{F}_0 \tilde{u})(\tau,\xi) := (2\pi)^{n/2} \sum_{\alpha\in\mathbb{Z}^n} e^{-i2\pi\alpha\cdot\tau} \tilde{u}(\alpha,\xi). \end{gathered} \tag{2.14}$$

We shall also use the following unitary operator :

$$U := \mathcal{F}_0^{-1} U_0 : L^2(\mathbb{R}^n) \to l^2(\mathbb{Z}^n; \mathcal{K}), \qquad (Uf)(\alpha,\xi) = f(\alpha+\xi)$$
$$\left(U^{-1}\tilde{f}\right)(x) = \tilde{f}\left([x], \underline{x}\right). \tag{2.15}$$

For any functions $F : \mathbb{Z}^n \to \mathcal{B}(\mathcal{K})$ and $\lambda : \mathbb{T}^n \to \mathcal{B}(\mathcal{K})$ we can define the multiplication operators $\tilde{M}_F$ on $\widetilde{\mathcal{H}}$ and $\overset{\circ}{M}_\lambda$ on $\overset{\circ}{\mathcal{H}}$, with evident domains, given by :

$$\left(\tilde{M}_F\tilde{f}\right)(\alpha,\xi) := \left(F(\alpha)\tilde{f}\right)(\alpha,\xi)$$
$$\left(\overset{\circ}{M}_\lambda \overset{\circ}{f}\right)(\tau,\xi) := \left(\lambda(\tau)\ \overset{\circ}{f}\right)(\tau,\xi). \tag{2.16}$$

For $\lambda : \mathbb{T}^n \to \mathcal{B}(\mathcal{K})$ we can define its Fourier transform :

$$\hat{\lambda}(\alpha) := (2\pi)^{-n/2} \int_{\mathbb{T}^n} e^{i2\pi\alpha\cdot\tau}\lambda(\tau)d\tau \tag{2.17}$$

(with integrals defined in weak sense in $\mathcal{B}(\mathcal{K})$) and we define the convolution operator on $\widetilde{\mathcal{H}}$:

$$\left(\lambda_*\tilde{f}\right)(\alpha,\xi) := \left(\mathcal{F}_0^{-1}\ \overset{\circ}{M}_\lambda\ \mathcal{F}_0\tilde{f}\right)(\alpha,\xi) = \sum_{\beta\in\mathbb{Z}^n}\left(\hat{\lambda}(\alpha-\beta)\tilde{f}(\beta)\right)(\xi). \tag{2.18}$$

Thus for any bounded function λ we have $\|\lambda_*\|_{\mathcal{B}(\tilde{\mathcal{H}})} = \|\lambda\|_{L^\infty(\mathbb{T}^n;\mathcal{B}(\mathcal{K}))}$.

In order to simplify some formulae let us define the discrete translations in $\widetilde{\mathcal{H}}$. For $j = 1, ..., n$ let $\epsilon_j \in \mathbb{Z}^n$ be given by $(\epsilon_j)_k := \delta_{jk}$ and :

$$\left(V_j\tilde{f}\right)(\alpha,\xi) := \tilde{f}(\alpha-\epsilon_j,\xi). \tag{2.19}$$

Due to the fact that $\{V_1, ..., V_n\}$ commute, for any $\alpha \in \mathbb{Z}^n$ one can define :

$$V(\alpha) \equiv V^\alpha = \prod_{j=1}^{n} V_j^{\alpha_j} \tag{2.20}$$

so that :

$$\lambda_* = \sum_{\beta\in\mathbb{Z}^n} \hat{\lambda}(\beta)V(\beta). \tag{2.21}$$

In the sequel we shall frequently need to estimate the norm of the operator λ_* between spaces with weights (growing exponentially at infinity). Even the definition of the conjugate operator that we shall propose asks for the control of such objects. Formally one has :

$$\left(\tilde{M}_F\lambda_*\tilde{M}_{F^{-1}}\tilde{f}\right)(\alpha,\xi) = \sum_{\beta\in\mathbb{Z}^n}\left(F(\alpha)\hat{\lambda}(\alpha-\beta)F(\beta)^{-1}\tilde{f}\right)(\beta,\xi). \tag{2.22}$$

Lemma 2.5. *Let $\rho : \mathbb{T}^n \to \mathcal{B}(\mathcal{K})$ be an analytic function having a holomorphic extension to a strip $\mathbb{C}^n_\delta$ for some strictly positive constant δ. Then for $\kappa \in [0, 2\pi\delta)$ we have :*

$$\|\rho\|^2_{2,-\kappa} := \sum_{\beta\in\mathbb{Z}^n} \left(e^{\kappa|\beta|} \|\hat{\rho}(\beta)\|_{\mathcal{B}(\mathcal{K})}\right)^2 < \infty. \tag{2.23}$$

Proof. Let us remark that for $\beta \in \mathbb{Z}^n$ and $\nu \in \mathbb{N}^n$:

$$\beta^\nu \hat{\rho}(\beta) = (2\pi)^{-n/2} (2\pi i)^{-|\nu|} \int_{\mathbb{T}^n} e^{i2\pi\beta\cdot\tau} (\partial^\nu \rho)(\tau) d\tau,$$

$$\|\beta^\nu \hat{\rho}(\beta)\|_{\mathcal{B}(\mathcal{K})} \leq (2\pi)^{n-(|\nu|+n/2)} \left(\sup_{\tau\in\mathbb{T}^n} \|(\partial^\nu\rho)(\tau)\|_{\mathcal{B}(\mathcal{K})}\right) \leq M_\rho (2\pi)^{n/2-|\nu|} \frac{\nu!}{\delta^{|\nu|}}$$

due to the analyticity assumption on ρ and the Cauchy inequalities. On the other hand one has for any $\theta \in \mathbb{R}_+$ and $l \in \mathbb{N}$:

$$(\theta|\beta|)^l \leq \theta^l \sum_{|\nu|=l} |\beta^\nu| \frac{l!}{\nu!}$$

so that :

$$(\theta|\beta|)^l \|\hat{\rho}(\beta)\|_{\mathcal{B}(\mathcal{K})} \leq C_{n,\varepsilon} \frac{M_\rho (2\pi)^{n/2}}{(n-1)!} l! \left(\frac{(1+\varepsilon)\theta}{2\pi\delta}\right)^l$$

for any $\varepsilon > 0$. By summing up we get that for any $\theta > \kappa$:

$$\begin{aligned} e^{\theta|\beta|} \|\hat{\rho}(\beta)\|_{\mathcal{B}(\mathcal{K})} &\leq C_{n,\varepsilon} \tfrac{M_\rho(2\pi)^{n/2}}{(n-1)!} \sum_{l\in\mathbb{N}} \left(\tfrac{(1+\varepsilon)\theta}{2\pi\delta}\right)^l \\ \sum_{\beta\in\mathbb{Z}^n} \left(e^{\kappa|\beta|} \|\hat{\rho}(\beta)\|_{\mathcal{B}(\mathcal{K})}\right)^2 &\leq C \left\{\sum_{\beta\in\mathbb{Z}^n} e^{-2(\theta-\kappa)|\beta|}\right\} \left\{\sum_{l\in\mathbb{N}} \left(\tfrac{(1+\varepsilon)\theta}{2\pi\delta}\right)^l\right\}^2 \end{aligned} \tag{2.24}$$

and this is finite for $(1+\varepsilon)\theta < 2\pi\delta$. □

Definition 2.6. Let $\rho : \mathbb{T}^n \to \mathcal{B}(\mathcal{K})$ admit a holomorphic extension to the strip $\mathbb{C}^n_\delta$ (with respect to the uniform topology) for some $\delta > 0$. Assume given a function $m : \mathbb{Z}^n \to \mathbb{R}$ satisfying : $m(\alpha) \geq 1$, $m(\alpha+\beta) \leq C_1 m(\alpha)m(\beta)$. For any function $G : \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{C}$ such that for some $\kappa \in [0, 2\pi\delta)$:

$$\sup_{\alpha,\beta\in\mathbb{Z}^n} e^{-\kappa|\alpha|} m(\beta) |G(\alpha,\beta)| \equiv \|G\|_{\infty,\kappa,m} < \infty \tag{2.25}$$

we define in $\tilde{\mathcal{H}}$ the following operators :

$$\begin{aligned} \left((\rho\diamondsuit G)\tilde{f}\right)(\alpha,\xi) &:= \sum_{\beta\in\mathbb{Z}^n} G(\beta,\alpha) \left(\widehat{\rho}(\beta)\tilde{f}(\alpha-\beta)\right)(\xi) \\ \left((\rho\diamondsuit G)^\dagger \tilde{f}\right)(\alpha,\xi) &:= \sum_{\beta\in\mathbb{Z}^n} G(\beta,\alpha-\beta) \left(\widehat{\rho}(\beta)\tilde{f}(\alpha-\beta)\right)(\xi). \end{aligned} \tag{2.26}$$

If $m(\beta) = 1$ for every β we denote $\|G\|_{\infty,\kappa,1} = \|G\|_{\infty,\kappa}$.

Proposition 2.7. *For ρ, m and G as in Definition 2.6 let $\tilde{\mathcal{H}}_m$ denote the domain of the operator of multiplication with the function m provided with the graph-norm. Then for any $\kappa' \in (\kappa, 2\pi\delta)$ (for κ the exponent associated to the function G), we have the estimation :*

$$\|\rho\diamondsuit G\|_{\mathcal{B}(\tilde{\mathcal{H}};\tilde{\mathcal{H}}_m)} \leq C\,\|G\|_{\infty,\kappa,m}\,\|\rho\|_{2,-\kappa'}\,. \tag{2.27}$$

Proof.

$$\left\|(\rho\diamondsuit G)\,\tilde{f}\right\|^2_{\tilde{\mathcal{H}}_m} := \sum_{\alpha\in\mathbb{Z}^n} m(\alpha)^2 \left\|\sum_{\beta\in\mathbb{Z}^n} G(\beta,\alpha)\left(\widehat{\rho}(\beta)\tilde{f}\right)(\alpha-\beta,.)\right\|^2_{\mathcal{K}} \leq$$

$$\leq \|G\|^2_{\infty,\kappa,m}\sum_{\alpha\in\mathbb{Z}^n}\left(\sum_{\beta\in\mathbb{Z}^n}\frac{e^{\kappa'|\beta|}}{<\beta>^{n/2+\varepsilon}}\,\|\hat{\rho}(\beta)\|_{\mathcal{B}(\mathcal{K})}\left\|\tilde{f}(\alpha-\beta,.)\right\|_{\mathcal{K}}\right)^2 \leq$$

$$\leq C^2\,\|G\|^2_{\infty,\kappa,m}\,\|\rho\|^2_{2,-\kappa'}\left\|\tilde{f}\right\|_{\tilde{\mathcal{H}}}.$$

□

In computing commutators we use a slight generalization of the above result.

Definition 2.8. Let $\lambda : \mathbb{T}^n \to \mathcal{B}(\mathcal{K})$ and $\rho : \mathbb{T}^n \to \mathcal{B}(\mathcal{K})$ admit holomorphic extensions to $\mathbb{C}^n_\delta$ (with respect to the uniform topology) for some $\delta > 0$. Assume given a function $m : \mathbb{Z}^n \to \mathbb{R}$ satisfying : $m(\alpha) \geq 1,\ m(\alpha+\beta) \leq Cm(\alpha)m(\beta)$. For any function $\Gamma : \mathbb{Z}^n \times \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{C}$ such that for some $\kappa \in [0, 2\pi\delta)$:

$$\sup_{\alpha,\beta,\gamma\in\mathbb{Z}^n} e^{-\kappa(|\alpha|+|\beta|)} m(\gamma)\,|\Gamma(\alpha,\beta,\gamma)| \equiv \|\Gamma\|_{\infty,\kappa,m} < \infty \tag{2.28}$$

we define in $\tilde{\mathcal{H}}$ the following operator :

$$\left(((\lambda\star\rho)\diamondsuit\Gamma)\,\tilde{f}\right)(\alpha,\xi) := \sum_{\beta,\gamma\in\mathbb{Z}^n}\left(\hat{\lambda}(\beta)\hat{\rho}(\gamma)\tilde{M}_{\Gamma(\gamma,\beta,.)}V(\beta+\gamma)\tilde{f}\right)(\alpha,\xi). \tag{2.29}$$

Proposition 2.9. *For λ, ρ, m and Γ as in Definition 2.8 let $\tilde{\mathcal{H}}_m$ denote the domain of the operator of multiplication with the function m provided with the graph-norm. Then for any $\kappa' \in (\kappa, 2\pi\delta)$ (with κ the exponent associated to the function Γ), we have the estimation :*

$$\|(\lambda\star\rho)\diamondsuit\Gamma\|_{\mathcal{B}(\tilde{\mathcal{H}};\tilde{\mathcal{H}}_m)} \leq C\,\|\Gamma\|_{\infty,\kappa,m}\,\|\lambda\|_{2,-\kappa'}\,\|\rho\|_{2,-\kappa'}\,. \tag{2.30}$$

The proof is similar to the previous one. Let us give now the application of this result in computing commutators. In the sequel we use the restriction to $\mathbb{Z}^n$

of functions defined on $\mathbb{R}^n$ and we need some bounds on their variation on $\mathbb{Z}^n$. It is convenient to express this variation by using the Leibnitz formula applied to the initial function defined on $\mathbb{R}^n$.

Corollary 2.10. *Let $\lambda : \mathbb{T}^n \to \mathcal{B}(\mathcal{K})$ and $\rho : \mathbb{T}^n \to \mathcal{B}(\mathcal{K})$ admit holomorphic extensions to the strip $\mathbb{C}^n_\delta$ (with respect to the uniform topology) for some $\delta > 0$. Let $m : \mathbb{Z}^n \to \mathbb{R}_+$ and $G : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ be given such that the restriction of G to $\mathbb{Z}^n \times \mathbb{Z}^n$ satisfies the assumptions of Definition 2.6 and also the following estimation :*

$$\sup_{\alpha,\beta\in\mathbb{Z}^n} e^{-\kappa|\alpha|} m'(\beta) \left|(\nabla G)(\alpha,\beta)\right| \equiv \|\nabla G\|_{\infty,\kappa,m'} < \infty \tag{2.31}$$

for a function m' satisfying the same conditions as the function m. Then :

$$[\lambda_*, \rho\Diamond G] = (\lambda \star \rho)\,\Diamond\Gamma \tag{2.32}$$

with :

$$\Gamma(\alpha,\beta,\gamma) := G(\alpha,\beta-\gamma) - G(\alpha,\beta) = -\int_0^1 ds\left\{\gamma\cdot\left(\nabla G_{(2)}\right)(\alpha,\beta-s\gamma)\right\}$$

(here $\nabla_{(2)}$ represents the gradient with respect to the second variable) so that we can apply Proposition 2.9.

Proof.

$$\left([\lambda_*, \rho\Diamond G]\,\tilde f\right)(\alpha,\xi) = \sum_{\beta\in\mathbb{Z}^n}\widehat{\lambda}(\beta)\left(\left[V(\beta), \sum_{\gamma\in\mathbb{Z}^n}\widehat{\rho}(\gamma)G(\gamma,\alpha)V(\gamma)\right]\tilde f\right)(\alpha,\xi) =$$
$$= \sum_{\beta,\gamma\in\mathbb{Z}^n}\{G(\gamma,\alpha-\beta) - G(\gamma,\alpha)\}\left(\widehat{\lambda}(\beta)\widehat{\rho}(\gamma)V(\beta+\gamma)\tilde f\right)(\alpha,\xi)\,.$$

□

As it is well known [6], [7], [12],[13], the operator $\overset{\circ}{H}$ is analytically decomposable, i.e. $\overset{\circ}{H}$ may be viewed as a multiplication operator with a function $\overset{\circ}{H}(\tau)$ defined on $\mathbb{T}^n$ with values self-adjoint operators on $\mathcal{K}$, with compact resolvent that depends analytically on $\tau \in \mathbb{T}^n$. We shall suppose that $\sigma(H) = \sigma_0 \cup \sigma_\infty$ with $(\inf \sigma_\infty) - (\sup \sigma_0) = d_0 > 0$ and consider the spectral projection P_0 of H corresponding to σ_0. We denote : $K := P_0HP_0,\ H_\infty := H - K,\ P_\infty := \mathbf{1} - P_0$. By our Hypothesis 1.1 there exists a number $N \in \mathbb{N}^*$ such that the operator $\overset{\circ}{K} = U_0KU_0^{-1}$ has the following expression :

$$\overset{\circ}{K} = \sum_{a=1}^{N} \overset{\circ}{M}_{\lambda_a}\overset{\circ}{M}_{\pi_a} \equiv \sum_{a=1}^{N}\overset{\circ}{K}_a \equiv \overset{\circ}{M}_k \tag{2.33}$$

where :

$$k(\tau) := \sum_{a=1}^{N}\lambda_a(\tau)\pi_a(\tau). \tag{2.34}$$

We shall sometimes use the notations: $P_a := U_0^{-1} \overset{\circ}{M}_{\pi_a} U_0,\ \Lambda_a := U_0^{-1} \overset{\circ}{M}_{\lambda_a} U_0$.

Let us observe that $\overset{\circ}{P}_0 := U_0 P_0 U_0^{-1}$ is an operator of multiplication with the analytic function :

$$p_0(\tau) := -\frac{1}{2\pi i} \oint_\Gamma \left(\overset{\circ}{H} (\tau) - \zeta \right)^{-1} d\zeta \tag{2.35}$$

for any contour Γ separating σ_0 from the rest of the spectrum. We remark that $p_0(\tau)$ and $k(\tau)$ are analytic functions of τ even without the condition (b) of our Hypothesis 1.1. Moreover we have :

$$\begin{array}{c} p_0(\tau) = \sum_{a=1}^{N} \pi_a(\tau), \qquad \sigma_0 = \bigcup_{a=1}^{N} \lambda_a(\mathbb{T}^n), \\ \pi_a(\tau)\pi_b(\tau) = 0 \qquad for \quad a \neq b. \end{array} \tag{2.36}$$

As in our previous paper [11], in order to define the conjugate operator we shall need the derivatives of the function $k(\tau)$ (in the uniform topology). We shall use the following notations :

$$\begin{array}{l} l_a : \mathbb{T}^n \ni \tau \to l_a(\tau) := (\nabla\lambda_a)(\tau) \in \mathbb{R}^n \\ \overset{\circ}{L} := \sum_{a=1}^{N} \overset{\circ}{M}_{l_a} \overset{\circ}{M}_{\pi_a} \equiv \sum_{a=1}^{N} \overset{\circ}{L}_a \ . \end{array} \tag{2.37}$$

An important difficulty in extending our previous results [11] from the case of a scalar analytic function $\lambda : \mathbb{T}^n \to \mathbb{R}$ to an analytic operator valued function $k : \mathbb{T}^n \to \mathcal{B}(\mathcal{K})$ of the form (2.34), comes from terms like : $\pi_a (\nabla\pi_b) \pi_c$, appearing when computing commutators. Nevertheless, a simple calculus shows that :

$$\pi_a (\nabla\pi_b) \pi_a = 0, \qquad \forall (a,b) \in \{1,...,N\}^2 . \tag{2.38}$$

Thus in our developments a very important role will be played by the following linear projection :

$$\mathcal{B}(\mathcal{H}) \ni S \mapsto \mathbb{P}_K(S) := \sum_{a=1}^{N} P_a S P_a \in \mathcal{B}(\mathcal{H}). \tag{2.39}$$

Proposition 2.11. *Let $\mathbb{P}_K$ be the projection defined above (2.39). Then :*

1. $\mathbb{P}_K(KS) = \mathbb{P}_K(SK) = K\mathbb{P}_K(S)$,
2. $\mathbb{P}_K^2 = \mathbb{P}_K$,
3. $\mathbb{P}_K(S^*) = \mathbb{P}_K(S)^*$,
4. $\mathbb{P}_K(S\mathbb{P}_K(T)) = \mathbb{P}_K(S)\mathbb{P}_K(T)$,
5. $\mathbb{P}_K([K,T]) = [K, \mathbb{P}_K(T)]$.

We concentrate now on the study of the weight functions that we shall use. In order to control the exponential growth of the weight we are interested in, we shall need to use a cut-off procedure and work with a class of bounded weights for which we shall prove estimations that are uniform with respect to the cut-off.

Definition 2.12. Given some constant $\kappa > 0$ we define Φ_κ as the class of functions $\tilde{\varphi} : [1, \infty) \to \mathbb{R}_+$ that are of class C^∞ and satisfy the properties :

$$|\tilde{\varphi}(t)| \leq \kappa t; \qquad 0 < (\partial\tilde{\varphi})(t) \leq \kappa; \qquad |(\partial^p \tilde{\varphi})(t)| \leq \frac{\kappa}{t}, \quad \forall p \geq 2.$$

Notation 2.13.

$\varphi(x) := \tilde{\varphi}(< x >)$; $\qquad w(x) := e^{\varphi(x)}$; $\qquad X(x) := (\nabla\varphi)(x) \equiv x\eta(x)$;
$W := w(Q)$; $\qquad W_0 := w([Q])$.

Proposition 2.14. *We have the estimations :*

$$|X(x)| \leq \kappa; \qquad |\eta(x)| \leq \frac{\kappa}{< x >}; \qquad |(\nabla\eta)(x)| \leq \frac{\kappa C}{< x >^2}.$$

In the following we shall need to compare the weights W and W_0.

Lemma 2.15. *There exists a strictly positive constant C such that we have :*

$$C^{-1} w(x) \leq w([x]) \leq C w(x), \qquad \forall x \in \mathbb{R}^n.$$

Proof.

$$|\varphi(x) - \varphi([x])| = \left| \underline{x} \cdot \int_0^1 (\nabla\varphi)([x] + s\underline{x})\, ds \right| \leq \kappa,$$
$$e^{-\kappa} w([x]) \leq w(x) \leq e^{\kappa} w([x]).$$

□

Lemma 2.16. *There is a constant C such that $\forall a \in \{1, ..., N\}$:*

$$\left\| [P_a, W_0]\, W_0^{-1} \right\|_{\mathcal{B}(\mathcal{H})} \leq C\kappa.$$

Proof. We study the element $[P_a, W_0]\, W_0^{-1} f$ in the representation $\tilde{\mathcal{H}}$. Denoting :

$$\theta_{\alpha,\beta}(s) := \int_0^s \beta \cdot X(\alpha - s\beta)\, ds,$$

we have :

$$|\theta_{\alpha,\beta}(s)| \leq s\kappa\, |\beta|,$$
$$\left([(\pi_a)_*, W_0]\, W_0^{-1} \tilde{f} \right)(\alpha, \xi) = -\sum_{\beta \in \mathbb{Z}^n} (\theta_{\alpha,\beta}(1))\, e^{\theta_{\alpha,\beta}(s)} \left(\widehat{(\pi_a)}(\beta)\, V(\beta)\, \tilde{f} \right)(\alpha, \xi),$$
$$\left\| [(\pi_a)_*, w([Q])]\, w([Q])^{-1} \tilde{f} \right\|^2_{\tilde{\mathcal{H}}} \leq \kappa^2 C^2 \left\| \pi_a \right\|_{2,-\kappa'} \left\| \tilde{f} \right\|_{\tilde{\mathcal{H}}}.$$

□

We come now to the problem of defining a conjugate operator for K.

Proposition 2.17. *P_a (for a=1,...,N), K and L leave $\mathcal{D}(< Q >)$ invariant.*

Proof. We have :

$$U\mathcal{D}(< Q >) = U\mathcal{D}(< [Q] >) = \mathcal{D}(< \nabla_\tau >)$$
$$\left(\nabla_\tau \left(\overset{\circ}{P}_a \overset{\circ}{f}\right)\right)(\tau,\xi) = (\nabla_\tau \pi_a)(\tau)\, \overset{\circ}{f}(\tau,\xi) + \pi_a(\tau) \left(\nabla_\tau\, \overset{\circ}{f}\right)(\tau,\xi)$$

and all the functions $\lambda_a(\tau)$, $l_a(\tau)$ and $\pi_a(\tau)$ are analytic on $\mathbb{T}^n$. □

Definition 2.18. On $\mathcal{D}(< Q >)$ we define the following symmetric operator :

$$A_0 := \frac{1}{2}\left\{[Q]\cdot L + L\cdot[Q]\right\}.$$

Once we have fixed $E \in \mathcal{E}_0(H)$ (as in the statement of Theorem 1.3) let us choose a bounded open interval I such that : $E \in I \subset \bar{I} \subset \mathcal{E}_0(H)$. We would like to use the operator $\mathbb{P}_K(A_0)$ as a conjugate operator for K on I.

Proposition 2.19. *With the above notations we have :*

$$i\left[K, \mathbb{P}_K(A_0)\right] = \tfrac{1}{2\pi} L^2 \in \mathcal{B}(\mathcal{H})$$
$$E_K(I)\, i\left[K, \mathbb{P}_K(A_0)\right] E_K(I) = \tfrac{1}{2\pi} E_K(I)\, L^2 E_K(I) \geq \omega_I^2 E_K(I)$$

where $E_K(I)$ is the spectral projection of K corresponding to the interval I and

$$\omega_I := \frac{1}{2\pi} \min_a \left(\inf_{\tau\in\lambda_a^{-1}(I)} \left|(\nabla_\tau \lambda_a)(\tau)\right| \right) > 0. \tag{2.40}$$

Proof. Using the properties of the projection $\mathbb{P}_K$ we observe that :

$$i\left[K, \mathbb{P}_K(A_0)\right] = \tfrac{i}{2} \sum_{a=1}^{N} \left\{P_a\left[K_a, [Q]\right]\cdot L P_a + P_a L\cdot \left[K_a, [Q]\right] P_a\right\} +$$
$$+ \tfrac{i}{2} \sum_{a=1}^{N} \left\{P_a [Q]\cdot [K_a, L]\, P_a + P_a [K_a, L]\cdot [Q]\, P_a\right\};$$
$$L P_a = \sum_{b=1}^{N} \left(U_0^{-1}\, \overset{\circ}{M}_{l_b}\, U_0 P_b\right) P_a = P_a \left(U_0^{-1}\, \overset{\circ}{M}_{l_a}\, U_0\right) P_a = P_a L;$$
$$P_a\left[K_a, [Q]\right] P_a = \tfrac{i}{2\pi} U^{-1} \left\{ \left(\overset{\circ}{M}_{\pi_a} \left[\overset{\circ}{M}_{\lambda_a}, \nabla_\tau\right] \overset{\circ}{M}_{\pi_a}\right) + \right.$$
$$\left. + \overset{\circ}{M}_{\lambda_a} \left(\overset{\circ}{M}_{\pi_a} \left[\overset{\circ}{M}_{\pi_a}, \nabla_\tau\right] \overset{\circ}{M}_{\pi_a}\right) \right\} U = -\tfrac{i}{2\pi} L_a;$$
$$[K_a, L] = [K_a, L_a] = 0$$

(in the last line both operators being multiplication with scalar functions in the subspace corresponding to $\pi_a(\tau)$). □

In order to derive a Hardy type inequality with exponential weights one has to define a conjugate operator that is very intimately related to the commutator of the Hamiltonian with the weight function. Thus we need a more complicated conjugate operator for K on the interval I; the definition we propose is motivated by the results of the Appendix. Let $X : \mathbb{R}^n \to \mathbb{R}^n$ be a vector field of class $C^\infty(\mathbb{R}^n)$ satisfying :

$$|X(x)| \leq \kappa; \qquad |(\partial_x^\nu X)(x)| \leq \frac{\kappa}{< x >},\ |\nu| \geq 1.$$

We shall denote by the same letter X its restriction to $\mathbb{Z}^n$. Later we shall take X to be the field defined in Definition 2.13.

Notation 2.20.

$$Z_\pm(\alpha, \beta) := \int_0^1 e^{\pm s\alpha\cdot X(\beta)} ds = \pm\frac{e^{\pm\alpha\cdot X(\beta)} - 1}{\alpha \cdot X(\beta)}.$$

Let us observe that :

$$|Z_\pm(\alpha, \beta)| \leq e^{\kappa|\alpha|}, \qquad \forall(\alpha, \beta) \in \mathbb{Z}^n \times \mathbb{Z}^n \tag{2.41}$$

so that it satisfies the assumptions on the function G (with $m(\beta) \equiv 1$) made in Definition 2.6. For any $a = 1, ..., N$ we define now :

$$L_X^+ := \sum_{a=1}^N P_a\, (l_a \diamondsuit Z_+)^\dagger\, P_a, \qquad\qquad L_X^- := \sum_{a=1}^N P_a\, (l_a \diamondsuit Z_-)\, P_a. \tag{2.42}$$

Definition 2.21. On $\mathcal{D}(< Q >)$ we define the following symmetric operator :

$$A_X := \frac{1}{2}\left\{[Q] \cdot L_X^+ + L_X^- \cdot [Q]\right\}.$$

By Proposition 2.17, $\mathbb{P}_K(A_X)$ is well defined and symmetric on $\mathcal{D}(< Q >)$.

Proposition 2.22. *On $\mathcal{D}(< Q >)$ we have the following equality :*

$$[K, \mathbb{P}_K(A_X)] = [K, \mathbb{P}_K(A_0)] + \mathcal{R}_X$$

where for some constant C (independent of κ) : $\|\mathcal{R}_X\|_{\mathcal{B}(\mathcal{H})} \leq C\kappa$.

Remark 2.23. For a given interval I as above, if κ is small enough, the operator $\mathbb{P}_K(A_X)$ is still conjugate to K on I.

Proof. Let us observe that :

$$|Z_\pm(\alpha, \beta) - 1| \leq |\alpha \cdot X(\beta)| \left|\int_0^1 \int_0^1 e^{\pm st\alpha\cdot X(\beta)} ds dt\right| \leq \kappa\, |\alpha|\, e^{\kappa|\alpha|} \leq \kappa e^{\kappa'|\alpha|}$$

for any $\kappa' \in (\kappa, 2\pi\delta)$. Moreover :

$$i\left[K, \mathbb{P}_K(A_X)\right] = \frac{i}{2} \sum_{a=1}^{N} \left\{ P_a \left[K_a, [Q]\right] \cdot L_X^{+} P_a + P_a L_X^{-} \cdot \left[K_a, [Q]\right] P_a \right\} + \\ + \frac{i}{2} \sum_{a=1}^{N} \left\{ P_a [Q] \cdot \left[K_a, L_X^{+}\right] P_a + P_a \left[K_a, L_X^{-}\right] \cdot [Q] P_a \right\},$$

$$P_a [Q] \cdot \left[K_a, L_X^{+}\right] P_a = P_a [Q] \cdot P_a \left[U^{-1} \overset{\circ}{M}_{\lambda_a} U, (l_a \diamondsuit Z_+)^{\dagger}\right] P_a,$$

To compute this commutator we make use of the Corollary 2.10. We define :

$$\Gamma_+(\gamma, \alpha, \beta) := Z_+(\gamma, \alpha - \beta - \gamma) - Z_+(\gamma, \alpha - \gamma) \tag{2.43}$$

and observe that it satisfies the estimation :

$$|\Gamma_+(\gamma, \alpha, \beta)| \leq \int_0^1 ds \left| e^{s\gamma \cdot X(\alpha - \beta - \gamma)} - e^{s\gamma \cdot X(\alpha - \gamma)} \right| \leq \\ \leq \int_0^1 ds \int_0^1 dts \left| \gamma\beta \left(\nabla X\right)(\alpha - t\beta - \gamma)\right| e^{s\gamma \cdot X(\alpha - t\beta - \gamma)} \leq \kappa < \alpha >^{-1} e^{\kappa'(|\beta| + |\gamma|)}.$$

Thus a direct use of the Corollary 2.10 gives us the expected result. □

3 The Exponential Weighted Estimation

In this Section we prove Theorem 1.3 and Theorem 1.4 of the Introduction. Our strategy is to follow the procedure elaborated in [11] . Thus we shall make a cut-off on the weight in order to make it bounded and also a cut-off on the support of the test function. Our main technical result is an estimation for compactly supported test functions, with bounded weights associated to the class Φ_κ, but with constants depending only on κ (the upper bound on the derivative of the phase function from Φ_κ). In dealing with this situation we shall separate a neighborhood of σ_∞, for which we shall apply the well-known Agmon method [1] and the neighborhood of σ_0 for which we shall extend our method [11] from a case of scalar analytic functions to that of a function $k : \mathbb{T}^n \to \mathcal{B}(\mathcal{K})$ of the type (2.34).

From now on we shall use Definition 2.12 and Notation 2.13 assuming that :

$$\tilde{\varphi} \in \Phi_\kappa \cap L^\infty\left([1, \infty)\right). \tag{3.44}$$

Our first step is to prove the following estimation.

Proposition 3.24. *For $\kappa \in [0, 2\pi\delta)$ and any $E \in \mathcal{E}_0(H)$ there exists a constant C such that for any $f \in \mathcal{H}^2_{comp}(\mathbb{R}^n)$ one gets :*

$$\|Wf\|_{\mathcal{D}(H)} \leq C \left\| \psi(< Q >)^{-1} W(H - E) f \right\|$$

(the function ψ is defined by $\psi(x) := \sqrt{\kappa < x >^{-2} + 2\eta(x)}$).

The proof of this estimation is based on the following two Propositions dealing separately with $P_\infty \mathcal{H}$ and $P_0 \mathcal{H}$.

Proposition 3.25. *For $E < \inf \sigma_\infty$ there exist two positive constants C_κ and C (the second one being independent of κ) such that for any $f \in \mathcal{H}^2_{comp}(\mathbb{R}^n)$ the following estimation holds :*

$$\|P_\infty W f\|^2_{\mathcal{D}(H)} - \kappa C \|P_0 W f\|^2 \leq C_\kappa \|W(H-E)f\|^2 .$$

Proof. Evidently, the fact that $f \in \mathcal{H}^2_{comp}(\mathbb{R}^n)$ implies that $Wf \in \mathcal{H}^2_{comp}(\mathbb{R}^n)$. Let $d := dist(E, \sigma_\infty)$. Let us observe that $H_\infty = P_\infty H = H P_\infty$ so that by hypothesis our value of E is beneath the spectrum of H_∞ and we can follow [1].

$$2 < Wf, (H_\infty - E P_\infty) W f > \geq 2d \|P_\infty W f\|^2 \tag{3.45}$$

$$d \|P_\infty W f\|^2 \leq \mathcal{R}e < P_\infty W f, W(H-E)f > + \mathcal{R}e < P_\infty W f, [H, W] f >$$

For the first term on the right-hand side we use the Schwartz inequality and for any $\theta > 0$ we write :

$$2\mathcal{R}e < P_\infty W f, W(H-E)f > \leq \theta \|P_\infty W f\|^2 + \theta^{-1} \|P_\infty W(H-E)f\|^2 .$$

For the second term we observe that on $\mathcal{D}(H)$:

$$[H, W] W^{-1} = (-i\nabla\varphi) \cdot D + D \cdot (-i\nabla\varphi) + (\nabla\varphi)^2 ,$$

thus :

$$\mathcal{R}e < P_\infty W f, \left([H, W] W^{-1}\right) P_\infty W f > = \|(\nabla\varphi) P_\infty W f\|^2 .$$

Using once again the Schwartz inequality we obtain that for $\theta_0 > 0$:

$$\begin{gathered} 2\mathcal{R}e < P_\infty W f, \left([H, W] W^{-1}\right) P_0 W f > \leq \\ \leq \theta_0 \|P_\infty W f\|^2 + \theta_0^{-1} \left\|\left([H, W] W^{-1}\right) P_0 W f\right\|^2 . \end{gathered} \tag{3.46}$$

In order to estimate the second term above let us observe that for $\mathcal{I}mz \neq 0$:

$$\|D P_0 g\|^2 = \left\|D(H+z)^{-1}(H+z) P_0 g\right\|^2 \leq C^2 \|P_0 g\|^2 ,$$

due to the fact that $D(H+z)^{-1}$ is a bounded operator and P_0 projects on a bounded spectral region of H. Moreover by Hypothesis 1.1 we have $|\nabla\varphi| \leq \kappa$ so that choosing $\theta_0 = 2C^2\kappa$ we get :

$$\theta_0^{-1} \left\|\left([H, W] W^{-1}\right) P_0 W f\right\|^2 \leq \kappa \left(1 + \frac{\kappa^2}{2C^2}\right) \|P_0 W f\|^2 .$$

Choosing finally $\theta < \kappa^2$ we get :

$$2 (d - \kappa C) \|P_\infty W f\|^2 - 2\kappa \|P_0 W f\|^2 \leq d^{-1} \|P_\infty W(H-E)f\|^2 .$$

Let us obtain now the graph norm of H on the left hand side :

$$\begin{aligned}\|g\|^2_{\mathcal{D}(H)} = \|g\|^2 + \|Hg\|^2\,, \\ \|P_\infty Wf\|^2_{\mathcal{D}(H)} \le \left(1+2E^2\right)\|P_\infty Wf\|^2 + 2\,\|(H-E)\,Wf\|^2 \le \\ \le \left(1+2E^2\right)\|P_\infty Wf\|^2 + 2\,\|W\,(H-E)\,f\|^2 + \\ +2\left\|[H,W]\,W^{-1}\,(H+z)^{-1}\right\|^2 \|Wf\|^2_{\mathcal{D}(H)}\,, \\ \left\|[H,W]\,W^{-1}\,(H+z)^{-1}\right\| \le \kappa^2 C^2, \\ \|Wf\|^2_{\mathcal{D}(H)} = \|P_\infty Wf\|^2_{\mathcal{D}(H)} + \|P_0 Wf\|^2_{\mathcal{D}(H)}\,.\end{aligned} \tag{3.47}$$

Putting all these together we get the result. □

For the neighborhood of σ_0 we shall obtain an estimation for the operator K with "weight operator" $\mathbb{P}_K\,(W_0)$.

Proposition 3.26. *Let $E \in I \subset \bar{I} \subset \mathcal{E}_0(H)$, η be defined by Notation 2.13 and*

$$\psi(x) := \sqrt{\kappa < x >^{-2} + 2\eta(x)}.$$

Then there exists a constant C_0 such that for any $f \in \mathcal{H}^2_{comp}(\mathbb{R}^n)$ one has :

$$\|\mathbb{P}_K\,(W_0)\,f\|^2 \le C_0\left\|\psi^{-1}([Q])\mathbb{P}_K\,(W_0)\,(K-E)f\right\|^2.$$

Proof. Let us first remark that : $\mathbb{P}_K\,(W_0)\,(K-E) = \mathbb{P}_K\,(W_0)\,(H-E)$. As in our previous paper [11] we shall consider the following expression :

$$\begin{aligned}2\mathcal{I}m < \mathbb{P}_K\,(A_X)\,\mathbb{P}_K\,(W_0)\,f, (H-E)\mathbb{P}_K\,(W_0)\,f >= \\ = -i < \mathbb{P}_K\,(W_0)\,f, [\mathbb{P}_K\,(A_X)\,, H]\,\mathbb{P}_K\,(W_0)\,f > .\end{aligned} \tag{3.48}$$

But (see Proposition 2.22) :

$$\begin{aligned}[\mathbb{P}_K\,(A_X)\,, H] = [\mathbb{P}_K\,(A_X)\,, K] = [\mathbb{P}_K(A_0), K] - \mathcal{R}_X \\ \|\mathcal{R}_X\|_{\mathcal{B}(\mathcal{H})} \le C\kappa.\end{aligned} \tag{3.49}$$

Using now Proposition 2.19 we can write :

$$\begin{aligned}[\mathbb{P}_K(A_0), K] = E_H\,(I)\,i\,[\mathbb{P}_K(A_0), K]\,E_H\,(I) + \\ +(P_0 - E_H\,(I))\,[\mathbb{P}_K(A_0), K]\,E_H\,(I) + P_0\,[\mathbb{P}_K(A_0), K]\,(P_0 - E_H\,(I))\end{aligned}$$

and $[\mathbb{P}_K(A_0), K] = \frac{i}{2\pi} L^2 \in \mathcal{B}(\mathcal{H})$. We have the inequality : $\|E_H\,(I)\,g\| \le \|P_0 g\|$, so that by using the Schwartz inequality we obtain :

$$\begin{aligned}|< \mathbb{P}_K\,(W_0)\,f, (P_0 - E_H\,(I))\,[\mathbb{P}_K(A_0), K]\,E_H\,(I)\,\mathbb{P}_K\,(W_0)\,f > + \\ + < \mathbb{P}_K\,(W_0)\,f, P_0\,[\mathbb{P}_K(A_0), K]\,(P_0 - E_H\,(I))\mathbb{P}_K\,(W_0)\,f >| \le \\ \le \tfrac{1}{2\pi}\,\|L\|^2\left\{\theta\,\|(P_0 - E_H\,(I))\mathbb{P}_K\,(W_0)\,f\|^2 + \theta^{-1}\,\|\mathbb{P}_K\,(W_0)\,f\|^2\right\}.\end{aligned}$$

Let us observe that :

$$(P_0 - E_H(I)) = (P_0 - E_H(I))(K-E)^{-1}(K-E),$$
$$\|(P_0 - E_H(I))\mathbb{P}_K(W_0) f\| \leq C_E \{\|\mathbb{P}_K(W_0)(K-E) f\| + \|[K, \mathbb{P}_K(W_0)] f\|\}.$$

For the last term on the right hand side we use Proposition 4.31 from the Appendix and the Remark following it. This gives us the following estimation :

$$\|(P_0 - E_H(I))\mathbb{P}_K(W_0) f\| \leq C_E \{\|\mathbb{P}_K(W_0)(K-E) f\| + \kappa C \|\mathbb{P}_K(W_0) f\|\}.$$

If we choose $\theta > \kappa^{-1}$, we obtain that the left hand side is bounded by :

$$\|L\|^2 \left\{ C'_E \|\mathbb{P}_K(W_0)(K-E) f\|^2 + (\kappa C_E)^2 \|\mathbb{P}_K(W_0) f\|^2 \right\}.$$

Using the Mourre estimation (Proposition 2.19 and Proposition 2.22), we obtain :

$$\begin{aligned} 2\mathcal{I}m < \mathbb{P}_K(A_X)\mathbb{P}_K(W_0) f, (H-E)\mathbb{P}_K(W_0) f > &\geq \omega_I \|\mathbb{P}_K(W_0) f\|^2 - \\ - C''_E \|L\|^2 &\left\{ \|\mathbb{P}_K(W_0)(K-E) f\|^2 + \kappa^2 \|\mathbb{P}_K(W_0) f\|^2 \right\} \end{aligned} \tag{3.50}$$

(for the first term of the second line we used the same procedure as above). For the first term in (3.50), we observe that $H\mathbb{P}_K(W_0) = K\mathbb{P}_K(W_0)$ and commute K with $\mathbb{P}_K(W_0)$. The Schwartz inequality gives :

$$\begin{aligned} &2\mathcal{I}m < \mathbb{P}_K(A_X)\mathbb{P}_K(W_0) f, \mathbb{P}_K(W_0)(K-E) f > \leq \\ &\leq \|\psi([Q])\mathbb{P}_K(A_X)\mathbb{P}_K(W_0) f\|^2 + \left\|\psi([Q])^{-1}\mathbb{P}_K(W_0)(K-E) f\right\|^2. \end{aligned} \tag{3.51}$$

For the term with the commutator we use the Conclusion 4.35 of the Appendix :

$$\begin{aligned} &2\mathcal{I}m < \mathbb{P}_K(A_X)\mathbb{P}_K(W_0) f, (H-E)\mathbb{P}_K(W_0) f > - \\ &- \left\| \sqrt{\psi([Q])^2 - 2\eta([Q])}\, \mathbb{P}_K(A_X)\mathbb{P}_K(W_0) f \right\|^2 \leq \\ &\leq \left\|\psi([Q])^{-1}\mathbb{P}_K(W_0)(K-E) f\right\|^2 + \kappa C \|\mathbb{P}_K(W_0) f\|^2. \end{aligned} \tag{3.52}$$

If we chose now $\psi(x)$ as in the statement of the theorem, we obtain the inequality

$$\begin{aligned} c \left\| \sqrt{\psi([Q])^2 - 2\eta([Q])}\, \mathbb{P}_K(A_X)\mathbb{P}_K(W_0) f \right\|^2 = \\ = \kappa \left\| < [Q] >^{-1} \mathbb{P}_K(A_X)\mathbb{P}_K(W_0) f \right\|^2 \leq \kappa C \|\mathbb{P}_K(W_0) f\|^2. \end{aligned}$$

From this and (3.50) we get the expected result for κ small enough. □

Proof of Proposition 3.24

For $f \in \mathcal{H}^2_{comp}(\mathbb{R}^n)$ we get from the previous two propositions :

$$\begin{aligned} \|\mathbb{P}_K(W_0) f\|^2 &\leq C_0 \left\|\psi^{-1}([Q])\mathbb{P}_K(W_0)(H-E) f\right\|^2, \\ \|P_\infty W f\|^2_{\mathcal{D}(H)} - \kappa C \|P_0 W f\|^2 &\leq C_\kappa \|W(H-E) f\|^2. \end{aligned} \tag{3.53}$$

We shall begin with the first inequality and obtain an estimation for $P_0 W f$.

$$\mathbb{P}_K (W_0) f = P_0 W_0 f + \sum_{a=1}^{N} P_a [W_0, P_a] f.$$

Using Lemma 2.16 for the terms of the sum on the right hand side we obtain :

$$\|P_0 W_0 f\| - \kappa N C \|W_0 f\| \leq \|\mathbb{P}_K (W_0) f\| . \tag{3.54}$$

By Lemma 2.15 from Section 2 we have :

$$\|P_0 W f\| \leq \|W P_0 f\| + \|[P_0, W] f\| \leq C_1 \|W_0 P_0 f\| + \|[P_0, W] f\| \leq$$

$$\leq C_1 \|P_0 W_0 f\| + C_1 \|[P_0, W_0] f\| + \|[P_0, W] f\| .$$

Let us compute now the commutator : $[P_0, W]$.

$$([P_a, W] f)(x) = \sum_{\alpha \in \mathbb{Z}^n} U^{-1} \widehat{\pi_a} ([x] - \alpha) \left\{ \widetilde{W} (\alpha + \underline{x}) - \widetilde{W} (x) \right\} \widetilde{f} (\alpha + \underline{x}),$$
$$\varphi (x + s(\alpha - [x])) - \varphi (\alpha + \underline{x}) = (\alpha - [x]) \cdot \int_0^{s-1} dt X (\alpha + \underline{x} + t(\alpha - [x])) .$$

Putting all these together we get the estimations :

$$\begin{gathered} \|[P_0, W] f\| \leq \kappa C \|W f\| , \\ \|[P_0, W_0] f\| \leq \kappa C \|W_0 f\| \leq \kappa C' \|W f\| \end{gathered} \tag{3.55}$$

for some constants C, C' independent of κ. Our first estimation in (3.53) implies :

$$\|P_0 W f\| \leq C'_\kappa \left\| \psi ([Q])^{-1} \mathbb{P}_K (W_0) (H - E) f \right\| . \tag{3.56}$$

Now we have to repeat the arguments above in order to treat the right hand side and eliminate the projection $\mathbb{P}_K$. We shall use the following notations :

$$\widetilde{W_0} := \psi ([Q])^{-1} W_0; \qquad \widetilde{W} := \psi (Q)^{-1} W.$$

Then we have :

$$\begin{gathered} \left\| \psi ([Q])^{-1} \mathbb{P}_K (W_0) (H - E) f \right\| \leq \\ \leq C' \sum_{a=1}^{N} \left\{ \left\| P_a \widetilde{W_0} (H - E) f \right\| + \left\| P_a \left[\widetilde{W_0}, P_a \right] (H - E) f \right\| + \right. \\ \left. + \left\| \left[\psi ([Q])^{-1}, P_a \right] W_0 P_a (H - E) f \right\| \right\}, \\ \left[\psi ([Q])^{-1}, P_a \right] \psi ([Q]) = \sum_{\alpha \in \mathbb{Z}^n} U^{-1} \widehat{\pi_a} (\alpha) \left[\psi ([Q])^{-1}, V(\alpha) \right] \psi ([Q]) = \\ = - \sum_{\alpha \in \mathbb{Z}^n} U^{-1} \widehat{\pi_a} (\alpha) \left\{ \int ds \alpha \cdot \left(\psi^{-2} \nabla \psi \right) ([Q] - s\alpha) \right\} \psi ([Q] - \alpha) V(\alpha) . \end{gathered} \tag{3.57}$$

Let us recall the definition of the function ψ and observe that :

$$\left|\left(\psi^{-2}\nabla\psi\right)(\beta - s\alpha)\,\psi\,(\beta-\alpha)\right| \leq \kappa C \frac{<\alpha>}{<\beta>},$$

so that by using Proposition 2.7 we get the estimation :

$$\left\|\left[\psi\left([Q]\right)^{-1}, P_a\right]\psi\left([Q]\right)\right\| \leq \kappa C. \tag{3.58}$$

By similar arguments we obtain the bound $\left\|\left[\widetilde{W_0}, P_a\right]\widetilde{W_0}^{-1}\right\| \leq \kappa C$. Putting all these estimations together we obtain the following inequality :

$$\|P_0 W f\| \leq C_\kappa^{''}\left\|\widetilde{W}(H-E)f\right\| \tag{3.59}$$

and combining with the inequality (3.53) we finally obtain :

$$\|Wf\|_{\mathcal{D}(H)} \leq C\left\|\widetilde{W}(H-E)f\right\|. \tag{3.60}$$

□

In view of our Theorem 1.4 we shall now obtain a similar "local estimation" for the perturbed Hamiltonian $H_I = H + V_I$, where V_I satisfies the conditions of Theorem 1.4. We have for f supported outside the ball of radious R :

$$\|\tilde{W}V_I f\| \leq \| <Q> \chi_R V_I (H+i)^{-1}\|\|Wf\|_{\mathcal{D}(H)} \leq \theta C\|\tilde{W}(H-E)f\|$$

for any chosen $\theta > 0$, once we take R large enough. Thus :

$$\|\tilde{W}(H_I - E)f\| \geq (1-\theta C)\|\tilde{W}(H-E)f\| \geq \frac{1-\theta C}{C}\|Wf\|_{\mathcal{D}(H)}. \tag{3.61}$$

We present now the cut-off procedure that allows us to obtain our main result (Theorem 1.3) from Proposition 3.24. We fix $\kappa > 0$ and the phase function $\tilde{\varphi}_0(t) = \kappa t$ for $t \in [1,\infty)$. Let f belong to :

$$\mathcal{M} := \left\{f \in \mathcal{D}(H) \mid \sqrt{<Q>}e^{\varphi_0(<Q>)}(H-E)f \in L^2(\mathbb{R}^n)\right\}. \tag{3.62}$$

We shall approximate the function f with functions with compact support, but in order to control the limit we shall need to work first with bounded phase functions $\tilde{\varphi} \in \Phi_\kappa$ that converge to $\widetilde{\varphi}_0$. Let us fix $\chi \in C_0^\infty(\mathbb{R})$ such that :

$$0 \leq \chi(t) \leq 1, \quad \chi(t) = 0 \quad for\ |t| \geq 1, \quad \chi(t) = 1 \quad for\ |t| \leq 1/2. \tag{3.63}$$

For $f \in \mathcal{M}$, $x \in \mathbb{R}^n$ and $\theta \in (0,1]$ we set :

$$\chi_\theta(x) := \chi(\theta <x>); \qquad f_\theta := \chi_\theta f. \tag{3.64}$$

Let :

$$j(t) := \begin{cases} \left(\int_{\mathbb{R}} e^{-\frac{1}{1-t^2}} dt\right)^{-1} e^{-\frac{1}{1-t^2}}, \ for \ |t| < 1 \\ 0, \ for \ |t| \geq 1 \end{cases} . \tag{3.65}$$

For $N \in \mathbb{N}$ let :

$$\tilde{\eta}_N(t) := \begin{cases} \kappa \ , \ for \ t \leq 2N \\ 0 \ , \ for \ t > 2N \end{cases} , \tag{3.66}$$

$$j_N(t) := \frac{1}{N} j(t/N), \quad \eta_N := j_N * \tilde{\eta}_N \ , \quad \varphi_N(t) := \int_0^t \eta_N(s) ds, \ \forall t \geq 0. \tag{3.67}$$

Lemma 3.27. *The following relations are true :*

1. $j \in C_0^\infty(\mathbb{R}^n), \quad 0 \leq j(t), \quad \int_{\mathbb{R}} j(t)dt = 1, \quad j(-x) = j(x),$
2. $\int_{\mathbb{R}} j_N(t)dt = 1, \qquad j_N(t) = 0$ *for* $|t| \geq N$,
3. $\eta_N \in C^\infty(\mathbb{R}), \qquad \eta_N(t) \leq \kappa, \qquad |t(\partial\eta_N)(t)| \leq C_1\kappa,$
   $\left|(\partial^k \eta_N)(t)\right| \leq C_k \kappa \ \ \forall t \in \mathbb{R}$ *, for* $k \in \mathbb{N}$ *and with* C_k *independent of* κ*,*
4. $\varphi_N(t) \leq \varphi_0(t), \quad \lim_{N\to\infty} \varphi_N(t) = \varphi_0(t), \quad \forall t \in \mathbb{R}.$

Proof. We shall prove only those estimations that are not completely obvious. First we observe that $0 \leq \eta_N(t) \leq \kappa$ and that for $t \leq N$ we get $\eta_N(t) = \kappa$ and for $t \geq 3N$ we get $\eta_N(t) = 0$. For the first derivative of $\eta_N(t)$ we see that :

$$t(\partial\eta_N)(t) = t\kappa \int_{t-2N}^{N} (\partial j_N)(\tau)d\tau = -\kappa \frac{t}{N} j(t/N - 2); \tag{3.68}$$

but $j(\tau - 2) \neq 0$ implies that $1 < \tau < 3$ so that $|t(\partial\eta_N)(t)| \leq 3c\kappa$. For the higher derivatives we observe that :

$$(\partial^k \eta_N)(t) = -\kappa(\partial^{k-1} j_N)(t - 2N) = -\kappa \frac{1}{N^k} (\partial^{k-1} j)(t/N - 2), \tag{3.69}$$

so that $\left|(\partial^k \eta_N)(t)\right| \leq C_k \kappa$ for any $k > 1$, with C_k independent of κ. □

Corollary 3.28. *For any* $N \in \mathbb{N}$ *the phase function* φ_N *defined by (3.67) belongs to the class* $\Phi_{\kappa'}$ *for some* $\kappa' > \kappa$*.*

We fix now the value of κ small enough (as in the statement of Proposition 3.24), $f \in \mathcal{M}$, $\theta \in (0, 1]$ and $N \in \mathbb{N}$ large enough so that we can apply Proposition 3.24 with the phase function φ_N for the function f_θ (with compact support). Thus :

$$\|e^{\varphi_N} f_\theta\|^2_{\mathcal{D}(H)} \leq C_\kappa \left\|\psi_N(Q)^{-1} e^{\varphi_N}(H - E) f_\theta\right\|^2, \tag{3.70}$$

where ψ_N is given by the same formula as in Proposition 3.26 but with φ replaced by φ_N. We remove the cut-off in f by letting $\theta \to 0$ and we use Fatou Lemma on the left hand side of the inequality (3.70) and the Dominated Convergence Theorem on the right hand side (the boundedness of e^{φ_N} is crucial at this step). This leads us to an estimation for any $f \in \mathcal{M}$ with phase function φ_N . A similar procedure allows us to control the limit $N \to \infty$ and to finish the proof of Theorem 1.3. Let us consider the limit of the right hand side of (3.70) when $\theta \to 0$.

$$\psi_N^{-1} e^{\varphi_N}(H-E) f_\theta = \chi_\theta \psi_N^{-1} e^{\varphi_N}(H-E) f + \psi_N^{-1} e^{\varphi_N} \left[H, \chi_\theta\right] f. \tag{3.71}$$

When $\theta \to 0$ the first term converges in L^2-norm to $\psi_N(Q)^{-1} e^{\varphi_N}(H-E)f$. Concerning the second term, we observe that for any $N \in \mathbb{N}$ we can find a finite constant C_N (diverging with N) such that : $\left\| e^{\varphi_N} \psi_N^{-1}(Q) < Q >^{-1} \right\| \leq C_N$. For a fixed N we study the family $\{< Q > [H, \chi_\theta(Q)]\, f\}_{\theta > 0}$ of L^2 -functions. We denote :

$$\zeta_\theta(x) := -2i\theta x \chi'(\theta < x >), \qquad \tilde{\zeta}_\theta = -i\nabla \left(\frac{1}{2 < x >} \zeta_\theta \right) \tag{3.72}$$

and observe that we can write :

$$< Q > [H, \chi_\theta(Q)]\, f = \zeta_\theta(Q) D f + \tilde{\zeta}_\theta(Q) f \tag{3.73}$$

We shall now estimate the norm $\| < Q > [H, \chi_\theta(Q)]\, f \|$. If we take into account that $\chi'(t)$ has support in the set $\{1/2 \leq t \leq 1\}$ and if we denote h_θ the characteristic function of the set $\left\{ \tau \in \mathbb{R}_+ \mid \frac{1}{2\theta} \leq \tau \leq \frac{1}{\theta} \right\}$ (that evidently converges pointwise to 0 for $\theta \to 0$) we finally get that :

$$|\zeta_\theta(x)| \leq C h_\theta(< x >); \qquad \left| \tilde{\zeta}_\theta(x) \right| \leq C\theta. \tag{3.74}$$

We use the fact that for $f \in \mathcal{D}(H)$ the vector Df belongs to $L^2(\mathbb{R}^n)$ in order to show that the second term in (3.71) converges to zero for $\theta \to 0$. We have thus proved that :

$$\lim_{\theta \to 0} \| < Q > [H, \chi_\theta(Q)]\, f \| = 0. \tag{3.75}$$

In conclusion, for a fixed $N \in \mathbb{N}$, the cut-off in f on the right hand side of (3.70) can be removed. For the left hand side we observe that for any $y \in \mathbb{R}$:

$$\lim_{\theta \to 0} e^{\varphi_N(y)} f_\theta(y) = e^{\varphi_N(y)} f(y). \tag{3.76}$$

Let us point out that in the left hand side of (3.70) we have to control the behavior of the graph norm $\|e^{\varphi_N} \chi_\theta f\|_{\mathcal{D}(H)}$ when $\theta \to 0$. For that we commute H with χ_θ and use once again the calculus done above (where now the factor $< z >$ in the definition of ζ_θ is absent so that the convergence to zero with θ follows immediately).

We still have to study the behavior of the inequality (3.70) with f_θ replaced by f, when $N \to \infty$. For this we prove the following lemma.

Lemma 3.29. *There exists a constant C such that for any $N \in \mathbb{N}$ we have :*

$$\frac{e^{\varphi_N(<x>)}}{\psi_N(x)} \leq C\sqrt{<x>}e^{\kappa<x>}.$$

Proof. For $N \in \mathbb{N}$ we define the function :

$$g_N(t) := \frac{e^{\tilde{\varphi}_N(t)}}{\sqrt{\kappa t^{-2} + 2t^{-1}\tilde{\varphi}'_N(t)}} = \frac{te^{\tilde{\varphi}_N(t)}}{\sqrt{\kappa + 2t\tilde{\varphi}'_N(t)}}. \tag{3.77}$$

We have :

$$\tilde{\varphi}'_N(t) = \eta_N(t) = \frac{\kappa}{N}\int_{-\infty}^{2N} j((t-s)/N)ds = \kappa \int_{\frac{t}{N}-2}^{1} j(\tau)d\tau. \tag{3.78}$$

Since $\tilde{\varphi}'_N$ is decreasing and $\tilde{\varphi}'_N(2N) = \kappa/2$, one has : $\tilde{\varphi}'_N(t) \geq \kappa/2$ for $t \leq 2N$ and $\tilde{\varphi}'_N(t) \leq \kappa/2$ for $t \geq 2N$. Hence, for $t \leq 2N$ we have $\{\kappa + 2t\tilde{\varphi}'_N(t)\}^{1/2} \geq (\kappa t)^{1/2}$, which implies $g_N(t) \leq (t/\kappa)^{1/2}e^{\tilde{\varphi}_0(t)}$. For $t \geq 2N$ we get $\varphi_N(t) \leq \kappa t/2$, which gives $g_N(t) \leq \omega\sqrt{t}e^{\tilde{\varphi}_0(t)}$, with $\omega := \sup_{t\geq 1} \left(\sqrt{t}e^{-\kappa t/2}\right)$. □

Using this result we see that the right hand side of (3.70) (with f_θ replaced by f) is uniformly bounded by :

$$\left\|\psi_N(Q)^{-1}e^{\tilde{\varphi}_N}(H-E)f\right\|^2 \leq C\left\|\sqrt{<Q>}e^{\kappa<Q>}(\lambda(D)-E)f\right\|^2, \forall N \in \mathbb{N} \tag{3.79}$$

with C independent of N, the right hand side being finite due to the hypothesis $f \in \mathcal{M}$. But evidently :

$$\psi_N(x)^{-1}e^{\tilde{\varphi}_N(x)} \underset{N\to\infty}{\to} \sqrt{<x>}e^{\kappa<x>}, \tag{3.80}$$

so that we can use the Dominated Convergence Theorem. For the first term on the left hand side one can immediately use the Fatou Lemma in a way similar to the argument we gave for the $\theta \to 0$ limit. Thus we obtain the expected inequality :

$$\left\|e^{\kappa<Q>}f\right\|_{\mathcal{D}(H)} \leq C_\kappa \left\|\sqrt{<Q>}e^{\kappa<Q>}(H-E)f\right\| \tag{3.81}$$

and this finishes the proof of Theorem 1.3.

Proof of Theorem 1.4

First let us consider the set $\mathcal{M}_I$ defined as in (3.62) but with H_I replacing H. Let us fix some $f \in \mathcal{M}_I$ with support far enough from the origin (so that after a cut-off to a compact support we can apply the estimation in (3.61)). Then we can repeat the above cut-off procedure. Due to the fact that V_I commutes with all the cut-off functions, it follows that all the above procedure of removing cut-offs extends to the perturbed case without any modification. We thus obtain (see (3.61)) :

$$\left\|e^{\kappa<Q>}f\right\|_{\mathcal{D}(H)} \leq C_\kappa \left\|\sqrt{<Q>}e^{\kappa<Q>}(H_I - E)f\right\|.$$

Suppose H_I has an eigenvalue E belonging to $\mathcal{E}_0(H)$ with eigenfunction g. Denoting by χ the smoothed characteristic function of a ball of sufficiently large radius R in $\mathbb{R}^n$, by $\chi^\perp = 1 - \chi$ and by $f = \chi^\perp g$ we see that :

$$\begin{gathered}\left\|e^{\kappa<Q>}g\right\| \leq C\left\|\sqrt{<Q>}e^{\kappa<Q>}(H_I - E)f\right\| + \left\|e^{\kappa<Q>}\chi g\right\|, \\ (H_I - E)f = (H_I - E)g - (H_I - E)\chi g = -(H_I - E)\chi g\end{gathered}$$

so that :

$$\left\|e^{\kappa<Q>}g\right\| \leq C\left\|\sqrt{<Q>}e^{\kappa<Q>}(H_I - E)\chi g\right\| + \left\|e^{\kappa<Q>}\chi g\right\| < \infty,$$

due to the fact that H_I is a differential operator. □

4 Appendix

In this appendix we shall study the commutator $[K, \mathbb{P}_K(W_0)]$ and show that it can be written in a special form that allows one to compare it with the conjugate operator $\mathbb{P}_K(A_X)$. We have :

$$[K, \mathbb{P}_K(W_0)] = \sum_{a=1}^{N} P_a \left[\Lambda_a P_a, W_0\right] P_a = \sum_{a=1}^{N} P_a \left[\Lambda_a, W_0\right] P_a. \tag{4.82}$$

Let us observe that :

$$P_a \left[\Lambda_a, W_0\right] P_a = P_a \left[\Lambda_a, W_0\right] W_0^{-1} P_a W_0 P_a + P_a \left[P_a, \left[\Lambda_a, W_0\right] W_0^{-1}\right] W_0 P_a. \tag{4.83}$$

Lemma 4.30. *The operator* $\left[\Lambda_a, W_0\right] W_0^{-1}$ *defines a bounded operator in* $\mathcal{H}$ *and*

$$\left\|\left[\Lambda_a, W_0\right] W_0^{-1}\right\|_{\mathcal{B}(\mathcal{H})} \leq \kappa C.$$

Proof. We have :

$$\left[\Lambda_a, W_0\right] W_0^{-1} = U^{-1}\left(\lambda_a \diamondsuit F\right) U,$$

where we denoted :

$$F(\alpha, \beta) := -\int_0^1 ds \alpha \cdot \int_0^1 dt X\left(\beta - t\alpha\right) \exp\left\{s\alpha \cdot \int_0^1 dt X\left(\beta - t\alpha\right)\right\}. \tag{4.84}$$

We observe that we have the estimation : $|F(\alpha, \beta)| \leq \kappa < \alpha > e^{\kappa|\alpha|} \leq \kappa e^{\kappa'|\alpha|}$ for any $\kappa' \in (\kappa, 2\pi\delta)$. Using now Proposition 2.7 we get the expected result. □

An important difficulty in our technical developments comes from the fact that we have to consider the product of the operator $[\Lambda_a, W_0]\, W_0^{-1}$ with some unbounded operator and the above lemma does not give sufficient information in order to control this product. More precisely, our method of obtaining Hardy type inequalities from a Mourre estimation relies heavily on the study of the following object :

$$2\mathcal{I}m < \mathbb{P}_K(A_X)\mathbb{P}_K(W_0)f, [K, \mathbb{P}_K(W_0)]\, f > . \tag{4.85}$$

The next proposition gives a technical result concerning the structure of the commutator of K with $\mathbb{P}_K(W_0)$, that will allow us to treat the expression (4.85).

Proposition 4.31. *The following relation holds :*

$$\begin{aligned}[K, \mathbb{P}_K(W_0)] = -\tfrac{i}{2\pi}\eta\,([Q])\,\mathbb{P}_K(A_X)\mathbb{P}_K(W_0) + T\mathbb{P}_K(W_0) + \\ + R_0\mathbb{P}_K(W_0) + \sum_{a=1}^{N} R_a W_0 P_a\end{aligned}$$

where $T = T^* \in \mathcal{B}(\mathcal{H})$, $R_a \in \mathcal{B}(\mathcal{H};\mathcal{H}_m)$ *for* $m(x) :=< x >$, *a=0,…,N and :*

$$\|T\|_{\mathcal{B}(\mathcal{H})} + \max_{a=0,1,\dots,N} \|R_a\|_{\mathcal{B}(\mathcal{H};\mathcal{H}_m)} \leq \kappa C,$$

for some constant C *independent of* κ.

Proof. We consider once again (4.82) and (4.83) and we observe that :

$$\left[P_a, [\Lambda_a, W_0]\, W_0^{-1}\right] = U^{-1}\left[(\pi_a)_*, (\lambda_a \diamondsuit F)\right] U =: U^{-1}\left((\pi_a \star \lambda_a) \diamondsuit \Psi\right) U$$

where :

$$\Psi(\alpha, \beta, \gamma,) := -\int_0^1 ds\beta \cdot \left(\nabla_{(2)}F\right)(\alpha, \gamma - s\beta). \tag{4.86}$$

We denote :

$$Y(\alpha,\beta) := \alpha \cdot \int_0^1 dtX\left(\beta - t\alpha\right), \quad Y_1(\alpha,\beta) := \alpha \cdot \int_0^1 dt\left(\partial X\right)\left(\beta - t\alpha\right), \tag{4.87}$$

so that :

$$\begin{aligned}\left(\nabla_{(2)}F\right)(\alpha,\beta) = -\textstyle\int_0^1 dsY_1(\alpha,\beta)\exp\left\{sY(\alpha,\beta)\right\} - \\ -\textstyle\int_0^1 dsY(\alpha,\beta)\left(sY_1(\alpha,\beta)\right)\exp\left\{sY(\alpha,\beta)\right\},\end{aligned}$$

hence we have the estimation (with $\kappa' > \kappa$) :

$$\left|\left(\nabla_{(2)}F\right)(\alpha,\beta)\right| \leq \kappa\left\{\int_0^1 dt\frac{|\alpha|}{< \beta - t\alpha >}\right\} e^{\kappa|\alpha|} \leq \frac{\kappa}{< 2\beta >} e^{\kappa'|\alpha|}. \tag{4.88}$$

In order to treat the first term in (4.83) we have to make a more detailed analysis of the factor $[\Lambda_a, W_0]\, W_0^{-1}$ and separate it into its hermitian and antihermitian parts :

$$\frac{1}{2}\left\{2\Lambda_a - W_0\Lambda_a W_0^{-1} - W_0^{-1}\Lambda_a W_0\right\} + \frac{1}{2}\left\{W_0^{-1}\Lambda_a W_0 - W_0\Lambda_a W_0^{-1}\right\},$$

$$2\Lambda_a - W_0\Lambda_a W_0^{-1} - W_0^{-1}\Lambda_a W_0 = U^{-1}\left(\lambda_a \diamondsuit G_+\right) U,$$

where :

$$G_+(\alpha,\beta) := \left(1 - e^{(\varphi(\beta)-\varphi(\beta-\alpha))}\right) + \left(1 - e^{(\varphi(\beta-\alpha)-\varphi(\beta))}\right). \tag{4.89}$$

Some algebra, using the Leibnitz formula, shows that G_+ satisfies :

$$|G_+(\alpha,\beta)| \le \kappa < \alpha >^2 e^{\kappa|\alpha|} \le \kappa e^{\kappa'|\alpha|}$$

for any $\kappa' \in (\kappa, 2\pi\delta)$. Then :

$$\begin{aligned} & W_0^{-1}\Lambda_a W_0 - W_0\Lambda_a W_0^{-1} \\ & = U^{-1} \sum_{\alpha\in\mathbb{Z}^n} \widehat{\lambda_a}\,(\alpha) \left\{ V\,(\alpha)\, e^{(\varphi([Q])-\varphi([Q]+\alpha))} - e^{(\varphi([Q])-\varphi([Q]-\alpha))} V\,(\alpha) \right\} U. \end{aligned} \tag{4.90}$$

Let us observe that :

$$e^{(\varphi(\beta)-\varphi(\beta\pm\alpha))} = \left(e^{(\varphi(\beta)-\varphi(\beta\pm\alpha))} - e^{\mp\alpha\cdot X(\beta)}\right) + \left(e^{\mp\alpha\cdot X(\beta)} - 1\right) + 1,$$

$$\begin{aligned} \varphi\,(\beta) \quad &- \quad \varphi\,(\beta\pm\alpha) \ \pm \ \alpha\cdot X\,(\beta) \\ &= \quad - \sum_{j,k=1}^{n} \int_0^1 tdt \int_0^1 du \left\{\alpha_j\alpha_k\partial_j X_k\,(\beta \pm ut\alpha)\right\} \equiv Y_\pm(\beta,\alpha). \end{aligned} \tag{4.91}$$

Let us introduce the notations :

$$\begin{aligned} G_1(\alpha,\beta) \quad &:= \quad -\frac{1}{2}\sum_{j,k=1}^{n}\int_0^1 ds\alpha_j\alpha_k\left(\partial_k X_j\right)(\beta - s\alpha) e^{-s\alpha\cdot X(\beta-\alpha)}, \\ G_2(\alpha,\beta) \quad &:= \quad -\frac{1}{2}\int_0^1 ds\alpha\cdot\left(\nabla\eta\right)(\beta - s\alpha), \\ G_-(\beta,\alpha) \quad &:= \quad \frac{1}{2}\int_0^1 ds \left\{ Y_+(\beta,\alpha)e^{-\alpha\cdot X(\beta)} \exp\left\{sY_+(\beta,\alpha)\right\} \right. \\ &- \quad \left. Y_-(\beta,\alpha)e^{\alpha\cdot X(\beta)} \exp\left\{-sY_-(\beta,\alpha)\right\} \right\}. \end{aligned} \tag{4.92}$$

We have the estimations :

$$\begin{aligned} |G_1(\alpha,\beta)| \quad &\le \quad \kappa C\frac{< \alpha >^3}{< \beta >} e^{\kappa|\alpha|} \ \le \ \frac{\kappa C}{< \beta >} e^{\kappa'|\alpha|}, \\ |G_2(\alpha,\beta)| \quad &\le \quad \kappa C\frac{< \alpha >^3}{< \beta >^2} \ \le \ \frac{\kappa C}{< \beta >^2} e^{\kappa''|\alpha|}, \\ |G_-(\alpha,\beta)| \quad &\le \quad \kappa C\frac{< \alpha >^3}{< \beta >} e^{2\kappa|\alpha|} \ \le \ \frac{\kappa C}{< \beta >} e^{2\kappa'|\alpha|}. \end{aligned} \tag{4.93}$$

for any strictly positive constants κ'' and $\kappa' > \kappa$. Then we can write :

$$\frac{1}{2}P_a\left\{W_0^{-1}\Lambda_a W_0 - W_0\Lambda_a W_0^{-1}\right\}P_a = -\frac{i}{2\pi}\eta\left([Q]\right)P_a(A_X)P_a$$
$$+P_aU^{-1}\left(\lambda_a\diamondsuit(G_1+G_-)\right)UP_a + U^{-1}\left(\pi_a\diamondsuit G_2\right)UP_a(A_X)P_a.$$

In conclusion we obtain :

$$[K,\mathbb{P}_K(W_0)] = \eta\left([Q]\right)\sum_{a=1}^{N}P_aA_XP_a\mathbb{P}_K(W_0)+$$
$$+\sum_{a=1}^{N}\left\{P_aU^{-1}\left(\lambda_a\diamondsuit(G_1+G_-)\right)UP_a + U^{-1}\left(\pi_a\diamondsuit G_2\right)UP_aA_XP_a\right\}\mathbb{P}_K(W_0)+$$
$$+\tfrac{1}{2}\sum_{a=1}^{N}P_aU^{-1}\left(\lambda_a\diamondsuit G_+\right)U\mathbb{P}_K(W_0) + \sum_{a=1}^{N}P_aU^{-1}\left(\left(\pi_a\star\lambda_a\right)\diamondsuit\Psi\right)UW_0P_a.$$

We introduce now the notations :

$$T := \tfrac{1}{2}\sum_{a=1}^{N}P_aU^{-1}\left(\lambda_a\diamondsuit G_+\right)UP_a,$$
$$R_0 := \sum_{a=1}^{N}\left\{P_aU^{-1}\left(\lambda_a\diamondsuit(G_1+G_-)\right)UP_a + U^{-1}\left(\pi_a\diamondsuit G_2\right)UP_aA_XP_a\right\}, \tag{4.94}$$
$$R_a := P_aU^{-1}\left(\left(\pi_a\star\lambda_a\right)\diamondsuit\Psi\right)U.$$

Taking into account Proposition 2.7, Proposition 2.9 and the estimations proved above for G_+, G_1, G_2, G_- and Ψ we get the stated result. □

Remark 4.32. Let us finally remark that for a=1,...,N :

$$\begin{aligned}
W_0P_af &= P_aW_0P_af + \left([W_0,P_a]W_0^{-1}\right)W_0P_af,\\
[W_0,P_a]W_0^{-1} &= -U^{-1}\left(\pi_a\diamondsuit F\right)U,\\
\left\|[W_0,P_a]W_0^{-1}\right\|_{\mathcal{B}(\mathcal{H})} &\le \kappa\left\|\pi_a\right\|_{2,\kappa'},\\
\left\|W_0P_af\right\|_{\mathcal{H}} &\le \left(1-\kappa\left\|\pi_a\right\|_{2,\kappa'}\right)^{-1}\left\|P_aW_0P_af\right\|_{\mathcal{H}}.
\end{aligned}$$

Summing upon $a \in \{1,...,N\}$ we re-obtain the term $\mathbb{P}_K(W_0)f$.

Remark 4.33. We have the following relations :

$$\begin{aligned}
&2 \quad \mathcal{I}m < \mathbb{P}_K(A_X)\mathbb{P}_K(W_0)f, [K,\mathbb{P}_K(W_0)]\,f >\\
+&\quad 2 < \mathbb{P}_K(A_X)\mathbb{P}_K(W_0)f, \eta\left([Q]\right)\mathbb{P}_K(A_X)\mathbb{P}_K(W_0)f >\\
\le&\quad 2\mathcal{I}m < \mathbb{P}_K(A_X)\mathbb{P}_K(W_0)f, T\mathbb{P}_K(W_0)f > + \kappa C\left\|\mathbb{P}_K(W_0)f\right\|^2\\
=&\quad (-i) < \mathbb{P}_K(W_0)f, [\mathbb{P}_K(A_X),T]\,\mathbb{P}_K(W_0)f > + \kappa C\left\|\mathbb{P}_K(W_0)f\right\|^2.
\end{aligned}$$

Lemma 4.34. *We have* $[\mathbb{P}_K(A_X),T] \in \mathcal{B}(\mathcal{H})$ *and* $\|[\mathbb{P}_K(A_X),T]\| \le \kappa C$.

Proof. Let us remind that :

$$T := \sum_{a=1}^{N} P_a U^{-1} \left(\lambda_a \diamondsuit G_+\right) U P_a \tag{4.95}$$

so that the commutator takes the form :

$$[\mathbb{P}_K(A_X), T] = -\frac{1}{2}\sum_{a=1}^{N} P_a U^{-1} \left[\left\{[Q]\left(l_a \diamondsuit Z_+\right)^\dagger + \left(l_a \diamondsuit Z_-\right)[Q]\right\}, \lambda_a \diamondsuit G_+\right] U P_a.$$

$$\begin{aligned}
&\left[\left\{[Q]\left(l_a \diamondsuit Z_+\right)^\dagger + \left(l_a \diamondsuit Z_-\right)[Q]\right\}, \lambda_a \diamondsuit G_+\right] = \\
&\qquad = \left[[Q], \lambda_a \diamondsuit G_+\right]\left(l_a \diamondsuit Z_+\right)^\dagger + \left(l_a \diamondsuit Z_-\right)\left[\lambda_a \diamondsuit G_+, [Q]\right] + \\
&\qquad + [Q]\left[\left(l_a \diamondsuit Z_+\right)^\dagger, \lambda_a \diamondsuit G_+\right] + \left[l_a \diamondsuit Z_-, \lambda_a \diamondsuit G_+\right][Q], \\
&\left[[Q], \lambda_a \diamondsuit G_+\right] = \tfrac{i}{2\pi}\left(l_a \diamondsuit G_+\right), \\
&\left[l_a \diamondsuit Z_-, \lambda_a \diamondsuit G_+\right] = \sum_{\beta,\gamma\in\mathbb{Z}^n}\left[\widehat{l_a}(\gamma)\tilde{M}_{Z_-(\gamma,[Q])}V(\gamma), \widehat{\lambda_a}(\beta)\tilde{M}_{G_+(\beta,[Q])}V(\beta)\right] = \\
&\qquad = \sum_{\beta,\gamma\in\mathbb{Z}^n}\widehat{l_a}(\gamma)\widehat{\lambda_a}(\beta)\left\{\textstyle\int_0^1 ds\beta\cdot\nabla_{(2)}Z_-\left(\gamma, [Q]-s\beta\right) - \right. \\
&\qquad\qquad \left. - \textstyle\int_0^1 ds\gamma\cdot\nabla_{(2)}G_+\left(\beta, [Q]-s\gamma\right)\right\} V(\beta+\gamma), \\
&\left|\left(\nabla_{(2)}G_+\right)(\alpha,\beta)\right| \le \tfrac{\kappa C}{<\beta>} < \alpha >^2 e^{\kappa|\alpha|} \le \tfrac{\kappa C}{<\beta>} e^{\kappa'|\alpha|}
\end{aligned}$$

(by some obvious calculations). Proposition 2.7 gives the expected estimation. □

Conclusion 4.35. *Putting together the above results we get the following relation :*

$$\begin{aligned}
&2\mathcal{I}m < \mathbb{P}_K(A_X)\mathbb{P}_K(W_0)f, [K, \mathbb{P}_K(W_0)]\, f > + \\
&+ \left\|\sqrt{2\eta\left([Q]\right)}\mathbb{P}_K(A_X)\mathbb{P}_K(W_0)f\right\|^2 \le \kappa C \left\|\mathbb{P}_K(W_0)f\right\|^2.
\end{aligned}$$

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