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Singularity Cancellation in Fermion Loops Through Ward Identities

C. Kopper and J. Magnen

Abstract. Recently Neumayr and Metzner [1] have shown that the connected N-point density-correlation functions of the two-dimensional and the one-dimensional Fermi gas at one-loop order generically (i.e. for nonexceptional energy-momentum configurations) vanish/are regular in the small momentum/small energy-momentum limits. Their result is based on an explicit analysis in the sequel of the results of Feldman et al. [2]. In this note we use Ward identities to give a proof of the same fact - in a considerably shortened and simplified way - for any dimension of space.

The infrared properties of the connected N-point density-correlation function of the interacting Fermi gas at one-loop order, to be called N-loop for shortness, are important for the understanding of interacting Fermi systems, in particular in the low energy regime. The N-loops appear as Feynman (sub)diagrams or as kernels in effective actions. In two dimensions e.g., their properties are relevant for the analysis of the electron gas in relation with questions such as the breakdown of Fermi liquid theory and high temperature superconductivity. We refer to the literature in this respect, see [3,4] and references given there. Whereas the contribution of a single loop-diagram to the N-point function for $N \geq 3$ generally diverges in the small energy-momentum limit, these singularities have been known to cancel each other in various situations [3,4,5] in the symmetrized contribution, i.e. when summing over all possible orderings of the external momenta, a phenomenon called loop-cancellation. The two-loop has been known explicitly in one, two and three dimensions for quite some time [1], the calculation in two dimensions goes back to Stern [6]. We introduce the following notations adapted to those of [1] : $\Pi_N(q_1, \ldots, q_N)$ denotes the Fermionic N-loop for $N \geq 3$, see (2) below, as a function of the (outgoing) external energy-momentum variables $q_1, q_2 \ldots, q_{N-1}$ and $q_N = -(q_1 + \ldots + q_{N-1})$. Here the $(d+1)$-vector q stands for $(q_0, q_1, \ldots, q_d) = (q_0, \vec{q})$. We also introduce the variables

$$p_i = q_1 + q_2 + \ldots + q_{i-1}\,, \quad p_1 = 0\,, \quad 1 \leq i \leq N\,. \tag{1}$$

By definition we then have

$$\Pi_N(q_1, \ldots, q_N) = \int \frac{dk_0}{2\pi} \frac{d^d k}{(2\pi)^d}\, I_N(k; q_1, \ldots, q_N)$$
$$\text{with} \quad I_N(k; q_1, \ldots, q_N) = \prod_{j=1}^{N} G_0(k - p_j) \tag{2}$$

$$\text{and} \quad G_0(k) = \frac{1}{ik_0 - (\varepsilon_{\vec{k}} - \mu)}\,, \quad \varepsilon_{\vec{k}} = \frac{\vec{k}^2}{2m}\,, \ \mu \text{ being the Fermi energy.}$$

To have absolutely convergent integrals for $N \geq 3$, we restrict the subsequent considerations to the physically interesting cases $d \leq 3$. At the end of the paper we indicate how the same results can be obtained for $d \geq 4$. We also assume that the variables q_j have been chosen such that the integrand is not singular (see below (8)). In the following we will choose units such that $\mu = 1,\ 2m = 1$. By convention the vertex of q_1 will be viewed as the first vertex.
Symmetrization with respect to the external momenta $(q_1, \ldots, q_N)$ diminishes the degree of singularity of the Fermion loops. To prove this fact we have to introduce some notation on permutations. We denote by σ any permutation of the sequence $(2, \ldots, N)$. By $\Pi_N^\sigma(q_1, \ldots, q_N)$ we then denote $\Pi_N(q_1, q_{\sigma^{-1}(2)}, \ldots, q_{\sigma^{-1}(N)})$. For the completely symmetrized N-loop we write[1]

$$\Pi_N^S(q_1, \ldots, q_N) \;=\; \sum_\sigma \Pi_N^\sigma(q_1, \ldots, q_N)\;. \tag{3}$$

We will also have to consider subsets of permutations : For $n \leq N-2$ and $2 \leq j_1 < j_2 < \ldots < j_n \leq N$ we denote by $\sigma^{(i_1, \ldots, i_n)}_{(j_1, \ldots, j_n)}$ the permutation mapping $j_\nu \to i_\nu = \sigma(j_\nu) \in \{2, \ldots, N\}$, which preserves the order of the remaining sequence $\Big((2, \ldots, N) - (j_1, \ldots, j_n)\Big)$, i.e. $\sigma(\nu) < \sigma(\mu)$ for $\nu < \mu$, if $\nu, \mu \notin \{j_1, \ldots, j_n\}$. When the target positions $(i_1, \ldots, i_n)$ are summed over (see e.g. (4) below), we will write shortly $\sigma(j_1, \ldots, j_n)$, or also $\sigma_N(j_1, \ldots, j_n)$, σ_N^n, if we want to indicate the number N. Note that $n = N-2$ is already the most general case, since fixing the positions of $N-2$ variables (apart from q_1) fixes automatically that of the last. For the permutation $\sigma^{(i)}_{(j)}$, which maps j onto the i-th position in the sequence $(2, \ldots, N)$ (preserving the order of the other variables), we use the shorthands σ_j^i or σ_j. We then also introduce the N-loop, symmetrized with respect to the previously introduced subsets of permutations, i.e.[2]

$$\Pi_N^{S_n(j_1, \ldots, j_n)}(q_1, \ldots, q_N) = \sum_{\sigma_N(j_1, \ldots, j_n)} \Pi_N^{\sigma_N(j_1, \ldots, j_n)}(q_1, \ldots, q_N)\,,$$
$$\text{in particular} \quad \Pi_N^S \;=\; \Pi_N^{S_{N-2}}\;. \tag{4}$$

The notations corresponding to (3 - 4) will be applied in the same sense also to I_N.

The recent result [1] of Neumayr and Metzner, based on the exact expression for the N-loop from [2], which however is nontrivial to analyze, shows that for $N > 2$ and $d = 1,\, 2$ one has generically :

$$\Pi_N^S(\lambda q_1, \ldots, \lambda q_N) = O(1) \quad \text{for} \quad \lambda \to 0\,, \tag{5}$$

[1] We do not divide by the number of permutations, here $(N-1)!$, to shorten some of the subsequent formulae.

[2] Again we do not multiply by $\frac{(N-n-1)!}{(N-1)!}$.

$$\Pi_N^S(q_{10}\,,\,\lambda\vec{q}_1\,,\ldots,\,q_{N0}\,,\,\lambda\vec{q}_N) = O(\lambda^{2N-2}) \;\text{ for }\; \lambda\to 0\,, \tag{6}$$

$$\Pi_N^S(q_1,\ldots,q_N) = O(|\vec{q}_j|) \;\text{ for }\; \vec{q}_j\to 0\,. \tag{7}$$

We are not completely sure about the authors' definition of 'generically'. In any case their restrictions on the energy-momentum variables include the following one: The energy momentum set $\{q_1,\ldots,q_N\}$ is *nonexceptional*, if for all $J\subset_{\neq}\{1,\ldots,N\}$ we have

$$|\sum_{i\in J} q_{i0}| \geq \eta > 0\,. \tag{8}$$

Our bounds given in the subsequent proposition are based on this condition.[3]. Though we cannot exclude (and did not really try) that linear relations among the momentum variables $\{\vec{q}_1,\ldots,\vec{q}_N\}$ could even improve those bounds, it seems quite clear that they are saturated apart from subsets of momentum configurations of measure zero, cf. also the numerical results mentioned in [1]. Furthermore they deteriorate with the parameter η^{-1} (cf. the remarks in the end of the paper).

Proposition. *For nonexceptional energy-momentum configurations* $\{q_1,\ldots,q_N\}$ *(as defined through (8) with* η *fixed) and for* $N\geq 3$ *and* $n\leq N-2$ *the following bounds hold :*

A) *In the small* λ *limit* $q_{i0}\to\lambda\, q_{i0}\,,\ \vec{q}_i\to\lambda\,\vec{q}_i\,,\ \lambda\to 0$ (9)

A1) $|\Pi_N(\lambda q_1,\ldots,\lambda q_N)| \leq O(\lambda^{-(N-2)})\,,$ (10)

A2) $|\Pi_N^{S_n}(\lambda q_1,\ldots,\lambda q_N)| \leq O(\lambda^{-(N-2-n)}), |\Pi_N^S(\lambda q_1,\ldots,\lambda q_N)| \leq O(1).$ (11)

B) *In the dynamical limit* $\vec{q}_i\to\lambda\vec{q}_i\,,\ \lambda\to 0$ (12)

B1) $|\Pi_N(q_{10}\,,\,\lambda\vec{q}_1\,,\ldots,\,q_{N0}\,,\,\lambda\vec{q}_N)| \leq O(\lambda^2)\,,$ (13)

B2) $|\Pi_N^{S_n}(q_{10},\lambda\vec{q}_1\,,\ldots)| \leq O(\lambda^{2n+2}),\ |\Pi_N^S(q_{10},\lambda\vec{q}_1,\ldots)| \leq O(\lambda^{2N-2}).$ (14)

The functions $\lambda^{N-2}\,\Pi_N(\lambda q_1,\ldots,\lambda q_N)\,,\ \Pi_N^S(\lambda q_1,\ldots,\lambda q_N)\,,\ \Pi_N(q_{10}\,,\,\lambda\vec{q}_1\,,\ldots,$ $q_{N0},\,\lambda\vec{q}_N)$ *and* $\Pi_N^S(q_{10}\,,\,\lambda\vec{q}_1\,,\ldots,\,q_{N0}\,,\,\lambda\vec{q}_N)$ *are analytic functions of* λ *in a neighbourhood of* $\lambda=0$ *(depending on the momentum configuration, in particular on* η *).*

Proof. To prove A1) we perform the k_0-integration using residue calculus, so that (2) takes the form (cf. [2,1])

$$\Pi_N(q_1,\ldots,q_N) \;=\; \sum_{i=1}^{N}\int_{|\vec{k}-\vec{p}_i|<1}\frac{d^dk}{(2\pi)^d}\;\Big(\prod_{j=1,j\neq i}^{N} f_{ij}(\vec{k})\Big)^{-1}\,,\ \text{where} \tag{15}$$

$$f_{ij}(\vec{k}) = \varepsilon(\vec{k}-\vec{p}_i)-\varepsilon(\vec{k}-\vec{p}_j)+i(p_{i0}-p_{j0}) \;=\; 2\vec{k}\cdot(\vec{p}_j-\vec{p}_i)+(\vec{p}_i^{\,2}-\vec{p}_j^{\,2})+i(p_{i0}-p_{j0})\,.$$

[3] As for the spatial components $\vec{q}_i$ we only suppose that they lie in some fixed compact region $\{|\vec{q}_j|\leq K\}$.

This implies that

$$\Pi_N(\lambda q_1,\ldots,\lambda q_N) \;=\; \lambda^{-(N-1)} \sum_{i=1}^{N} \int_{|\vec{k}-\lambda\vec{p}_i|<1} \frac{d^dk}{(2\pi)^d} \;\Big(\prod_{j=1,j\neq i}^{N} f_{ij}^{\lambda}(\vec{k})\Big)^{-1} \;,\ \text{where}$$

$$f_{ij}^{\lambda}(\vec{k}) \;=\; 2\vec{k}\cdot(\vec{p}_j-\vec{p}_i)+\lambda(\vec{p}_i^{\,2}-\vec{p}_j^{\,2})+i(p_{i0}-p_{j0}) \;. \tag{16}$$

By Lemma 1 below we find

$$\sum_{i=1}^{N} \int_{|\vec{k}|<1} \frac{d^dk}{(2\pi)^d} \;\Big(\prod_{j=1,j\neq i}^{N} f_{ij}^{\lambda}(\vec{k})\Big)^{-1} \;=\; 0 \tag{17}$$

so that we obtain

$$\Pi_N(\lambda q_1,\ldots,\lambda q_N) \;=\; \lambda^{-(N-1)} \sum_{i=1}^{N} \Big\{\int_{|\vec{k}-\lambda\vec{p}_i|<1} - \int_{|\vec{k}|<1}\Big\} \frac{d^dk}{(2\pi)^d} \;\Big(\prod_{j=1,j\neq i}^{N} f_{ij}^{\lambda}(\vec{k})\Big)^{-1} \;. \tag{18}$$

Thus each entry in the sum in (16) has to be integrated only over a domain of measure $O(\lambda)$. Since the integrands are bounded in modulus by $O(1)$ due to the nonexceptionality of the momenta, this leads to the statement (10).

To prove B1), (13) we use again (15)

$$\Pi_N(q_{10},\lambda\vec{q}_1,\ldots,\, q_{N0},\lambda\vec{q}_N) = \sum_{i=1}^{N} \Big\{\int_{|\vec{k}-\lambda\vec{p}_i|<1} - \int_{|\vec{k}|<1}\Big\} \frac{d^dk}{(2\pi)^d} \Big(\prod_{j=1,j\neq i}^{N} f_{ij}(\lambda;\vec{k})\Big)^{-1}$$

$$\text{with}\quad f_{ij}(\lambda;\vec{k}) \;=\; 2\lambda\,\vec{k}\cdot(\vec{p}_j-\vec{p}_i)+\lambda^2\,(\vec{p}_i^{\,2}-\vec{p}_j^{\,2})+i(p_{i0}-p_{j0}) \;. \tag{19}$$

On performing the change of variables $\tilde{\vec{k}} = \vec{k}-\lambda\vec{p}_i$ in the integral over $|\vec{k}-\lambda\vec{p}_i|<1$ one realizes that the difference between the two integrals is of order λ^2. Or one may convince oneself that both integrals are even functions of λ. In any case this proves B1).

The previous considerations also imply that $\lambda^{N-2}\ \Pi_N(\lambda q_1,\ldots,\lambda q_N)$ and $\Pi_N(q_{10}\,,\,\lambda\vec{q}_1\,,\ldots)$ are analytic around $\lambda=0$: The imaginary parts of the denominators in (15, 16) stay bounded away from zero, and a convergent Taylor expansion for $\lambda^{N-2}\times(18)$ is easily obtained on performing the change of variables $\tilde{\vec{k}} = \vec{k}-\lambda\vec{p}_i$ in the first integrals.

For the proof of the proposition we will need also a slight generalization of the bounds on (15), which we have just obtained. We have to regard integrals of the type

$$\Pi_N(\varphi;\,q_1,\ldots,q_N) \;:=\int\!\frac{dk_0}{2\pi}\,\frac{d^dk}{(2\pi)^d}\,I_N(k;\,q_1,\ldots,q_N)\;\varphi(\vec{k};\,q_1,\ldots,q_N)\;, \tag{20}$$

where we demand that the functions $\varphi(\vec{k};\, q_1,\ldots,q_N)$ be continuous and uniformly bounded in the domain specified by (8) : $|\varphi(\vec{k};\, q_1,\ldots,q_N)| \le C$ for some suitable $C>0$, and furthermore

$$|\varphi(\vec{k};\, q_1,\ldots,q_N) \,-\, \varphi(\vec{k}+\vec{q};\, q_1,\ldots,q_N)| \le \sup_J |\vec{q}\cdot\vec{q}_J|\, C \quad \text{uniformly in } \vec{q}\in\mathbb{R}^d\,. \tag{21}$$

Here we set

$$\vec{q}_J = \sum_{j\in J}\vec{q}_j \quad \text{for } J\subset\{1,\ldots,N\}\,. \tag{22}$$

The scaling and dynamical limits A1) and B1) can also be studied for $\Pi_N(\varphi;\, q_1,\ldots,q_N)$ on introducing

$$\varphi_s(\vec{k};\, q_1,\ldots,q_N) = \varphi(\vec{k};\, \lambda q_1,\ldots,\lambda q_N)\,,\ \varphi_d(\vec{k};\, q_1,\ldots,q_N) = \varphi(\vec{k};\, q_{10},\lambda\vec{q}_1,\ldots,\, q_{N0},\lambda\vec{q}_N)\,. \tag{23}$$

To perform the k_0-integration as before, it is important to note that φ does not depend on k_0. The bounds

$$|\Pi_N(\varphi_s;\, \lambda q_1,\ldots,\lambda q_N)| \,\le O(\lambda^{-(N-2)})\,, |\Pi_N(\varphi_d;\, q_{10},\lambda\vec{q}_1,\ldots,\, q_{N0},\lambda\vec{q}_N)| \,\le O(\lambda^2) \tag{24}$$

are then proven as previously. In particular to prove (13) we perform the same change of variables as after (19) and use (after telescoping the integrand)

$$|\varphi_d(\tilde{\tilde{\vec{k}}}+\lambda\vec{p}_i;\, q_1,\ldots,q_N) \,-\, \varphi_d(\tilde{\tilde{\vec{k}}};\, q_1,\ldots,q_N)| \,\le\, O(\lambda^2)$$

for $|\vec{p}_i| \,\le\, K$, to obtain the factor of λ^2 required in B1). We finally note that (24) also holds for $N\ge 2$, by same method of proof, if the function φ assures the integrability of the integrand. On multiplying any admissible φ by the function Δ from (34), this is assured, and $\Delta\,\varphi$ has the properties required for φ above. Therefore we also find for $N\ge 2$

$$|\Pi_N((\Delta\,\varphi)_s;\, \lambda q_1,\ldots,\lambda q_N)| \le O(\lambda^{-(N-2)})\,,$$
$$|\Pi_N((\Delta\,\varphi)_d;\, q_{10},\lambda\vec{q}_1,\ldots,\, q_{N0},\lambda\vec{q}_N)| \le O(\lambda^2)\,, \tag{25}$$

and the same bound for $\Pi_N(\frac{1}{A}\,\varphi)$, using $\frac{1}{A}$ from (32) (in case $\vec{q}_j$ does not vanish). It remains to prove

Lemma 1. For any $n\ge 2$ and pairwise distinct complex numbers a_i, $i\in\{1,\ldots,n\}$, we set $a_{i,j} = a_i - a_j$. Then we have

$$\sum_{i=1}^{n}\prod_{j\ne i,j=1}^{n}\frac{1}{a_i-a_j} \,=\, 0\,. \tag{26}$$

Proof. By isolating the term $i=1$ in the sum

$$\sum_{i=1}^{n}\prod_{j\neq i,j=1}^{n}\frac{1}{a_i-a_j}=\prod_{j=2}^{n}\frac{1}{a_1-a_j}-\sum_{i=2}^{n}\Big(\prod_{j\neq i,j=2}^{n}\frac{1}{a_i-a_j}\Big)\frac{1}{a_1-a_i} \tag{27}$$

we obtain a presentation of (26) in terms of a difference of two rational functions of the complex variable a_1. They both have simple poles at $a_2,\ldots,a_n$ with identical residues. So the left hand side is an entire function of a_1 which vanishes for $|a_1|\to\infty$ and thus equals zero. (This implies that the second term on the right hand side is the partial fraction expansion of the first.) □

We now want to show how to obtain the statements A2) and B2) from A1) and B1) using the *Ward identity*,[4] in form of the simple propagator identity

$$(iq_0-2\vec q\cdot\vec k+\vec q^{\,2})\,G_0(k-q)\,G_0(k)\;=\;G_0(k-q)\;-\;G_0(k)\;. \tag{28}$$

When applying this identity to the product of the two subsequent propagators in I_N (2), which differ by the momentum q_j, and then summing over all possible positions of the momentum q_j in the loop, the sum telescopes, and we are left with the very first and very last contributions, the last being obtained from the first on shifting the variable k by q_j. Thus we obtain for $j\in\{2,\ldots,N\}$

$$\sum_{\sigma_j}\{iq_{j,0}-2\vec q_j\cdot(\vec k\,-\,\vec p^{\,\sigma_j})+\vec q_j^{\,2}\}\,I_N^{\sigma_j}(k;\,q_1,q_2,\ldots,q_N)\;= \tag{29}$$

$$I_{N-1}(k;\,q_1+q_j,q_2,\ldots,\not{q}_j,\ldots,q_N)\;-\;I_{N-1}(k+q_j;\,q_1+q_j,q_2,\ldots,\not{q}_j,\ldots,q_N)\;.$$

Note that the term on the r.h.s. vanishes on integration over k. The momentum p^{σ_j}, short for $p^{\sigma_j^i}$, is defined to be the momentum arriving at the vertex of q_j for the permutation σ_j^i, i.e.

$$\begin{aligned} p^{\sigma_j^i} &= q_1+\,q_{\sigma^{-1}(2)}\,+\ldots+\,q_{\sigma^{-1}(i-1)}\\ &= \begin{cases} q_1\,+\ldots+\,q_{j-1}\,+\,q_{j+1}\,+\ldots+\,q_i\,, & \text{for}\quad i>j\\ q_1\,+\ldots+\,q_{i-1}\,, & \text{for}\quad i<j\;. \end{cases} \end{aligned} \tag{30}$$

We can rewrite (29) as

$$A(\vec k,q_j)\,I_N^{S_1(j)}(k;\,q_1,\ldots,q_N)\;=\;-\sum_{\sigma_j}2\vec q_j\cdot\vec p^{\,\sigma_j}\,I_N^{\sigma_j}(k;\,q_1,\ldots,q_N)\,+ \tag{31}$$

$$I_{N-1}(k;\,q_1+q_j,\ldots,\not{q}_j,\ldots,q_N)\;-\;I_{N-1}(k+q_j;\,q_1+q_j,\ldots,\not{q}_j,\ldots,q_N)$$

[4] When regarding more general situations, a more general form of this identity can be derived from the functional integral defining the interacting fermion theory, in a way analogous to the famous Ward identity of QED. This identity between N- and $N-1$-point functions is related to fermion number conservation. In the present case we avoid introducing functional integrals and restrict to the simple propagator identity (28).

$$\text{with the definition} \qquad A(\vec{k}, q_j) \;=\; iq_{j0} - 2\vec{q}_j \cdot \vec{k} \,+\, \vec{q}_j^{\;2} \;. \tag{32}$$

On dividing by $A(\vec{k}, q_j)$, which is bounded away from 0 due to (8), we obtain (in shortened notation)

$$\begin{aligned} & I_N^{S_1(j)}(k;\, q_1, \ldots) = \\ & -\sum_{\sigma_j} \frac{2}{A(\vec{k}, q_j)}\, \vec{q}_j \cdot \vec{p}^{\,\sigma_j}\, I_N^{\sigma_j}(k;\, q_1, \ldots) \,+\, \Delta(\vec{k}, q_j)\, I_{N-1}(k+q_j;\, q_1+q_j, \ldots) \\ & \quad + \frac{1}{A(\vec{k}, q_j)}\, I_{N-1}(k;\, q_1+q_j, \ldots) - \frac{1}{A(\vec{k}+\vec{q}_j, q_j)}\, I_{N-1}(k+q_j;\, q_1+q_j, \ldots) \;. \end{aligned} \tag{33}$$

Here we used the definition

$$\Delta(\vec{k}, q_j) \;=\; \frac{1}{A(\vec{k}+\vec{q}_j, q_j)} \,-\, \frac{1}{A(\vec{k}, q_j)} \;=\; \frac{2\,\vec{q}_j^{\;2}}{A(\vec{k}+\vec{q}_j, q_j)\, A(\vec{k}, q_j)} \;. \tag{34}$$

Regarding (33) we realize that the last two terms give a vanishing contribution on integration over $\vec{k}$, by a shift of k. And the prefactors of the first two terms scale as λ^2 in the dynamical limit (see below (45, 46)), whereas there appears a gain of a factor of λ in the scaling limit when taking into account the change $N \to N-1$ in the second term and using an inductive argument based on A2) (11). To make work this inductive argument, we have to generalize (33) to symmetrization w.r.t. more than one variable. The Ward identity for $I_N^{\sigma_N^n}$ with $n \geq 2$ is obtained from (28) in the same way as (29) :

$$\begin{aligned} & \sum_{\sigma_N(j_1,\ldots,j_n)} \{iq_{j_\nu,0} - 2\vec{q}_{j_\nu} \cdot (\vec{k} - \vec{p}_{j_\nu}^{\,\sigma_N^n}) + \vec{q}_{j_\nu}^{\,2}\}\, I_N^{\sigma_N^n}(k;\, q_1, \ldots, q_N) \;= \\ & \sum_{\sigma_{N-1}(j_1,\ldots,\not{j}_\nu,\ldots,j_n)} \Big(I_{N-1}^{\sigma_{N-1}^{n-1}}(k;\, q_1+q_{j_\nu}, \ldots, q_N) - I_{N-1}^{\sigma_{N-1}^{n-1}}(k+q_{j_\nu};\, q_1+q_{j_\nu}, \ldots, q_N) \Big). \end{aligned} \tag{35}$$

Here the momentum $\vec{p}_{j_\nu}^{\,\sigma_N^n}$ is the one arriving at the vertex of q_{j_ν}, $j_\nu \in \{j_1, \ldots, j_n\}$, for the permutation σ_N^n. For each permutation σ_N^n appearing on the l.h.s. we sum on the r.h.s. over a permutation $[\sigma_{N-1}^{n-1}(\sigma_N^n, j_\nu)](j_1, \ldots, \not{j}_\nu, \ldots, j_n)$ which is defined as

$$[\sigma_{N-1}^{n-1}(\sigma_N^n, j_\nu)](j) \;=\; \begin{cases} \sigma_N^n(j)\,, & \text{if} \quad \sigma_N^n(j) < \sigma_N^n(j_\nu) \\ \sigma_N^n(j) - 1, & \text{if} \quad \sigma_N^n(j) > \sigma_N^n(j_\nu) \end{cases} \tag{36}$$

(so that $\sigma_{N-1}^{n-1}(\sigma_N^n, j_\nu)$ is indeed a map onto $\{2, \ldots, N-1\}$). To proceed to an identity in terms of $I_N^{S_n}$ we have to analyze and eliminate (as far as possible) the dependence of the term $\sim \vec{q}_{j_\nu} \cdot \vec{p}_{j_\nu}^{\,\sigma_N^n}$ on the permutations σ_N^n. We use the following

Lemma 2.

a) $$\sum_{\nu=1}^{n} \vec{q}_{j_\nu} \cdot \vec{p}_{j_\nu}^{\,\sigma(j_1,\dots,j_n)} = \sum_{\substack{i,j,i<j \\ i,j\in\{j_1,\dots,j_n\}}} \vec{q}_i \cdot \vec{q}_j \; + \sum_{\nu=1}^{n} \vec{q}_{j_\nu} \cdot \vec{\hat{p}}_{j_\nu}^{\,\sigma(j_1,\dots,j_n)} \,. \tag{37}$$

Here $\hat{p}$ is obtained from p by setting to zero the momenta $q_{j_1},\dots,q_{j_n}$, i.e.

$$\hat{p}_{j_\nu}^{\,\sigma(j_1,\dots,j_n)}(q_1,\dots,q_N) = p_{j_\nu}^{\,\sigma(j_1,\dots,j_n)}(\hat{q}_1,\dots,\hat{q}_N)\,, \tag{38}$$

where $\hat{q}_i = q_i$, if $i \notin \{j_1,\dots,j_n\}$ and $\hat{q}_i = 0$, if $i \in \{j_1,\dots,j_n\}$.

b) $$\hat{p}_{j_\nu}^{\,\sigma(j_1,\dots,j_n)}(q_1,\dots,q_N) = \hat{p}^{\,\hat{\sigma}_{j_\nu}} := \sum_{\substack{k\in(\{2,\dots,N\}-\{j_1,\dots,\not{j}_\nu,\dots,j_n\}), \\ k<\hat{\sigma}_{j_\nu}^{-1}(j_\nu)}} q_{\hat{\sigma}_{j_\nu}^{-1}(k)} \,. \tag{39}$$

Here $\hat{\sigma}_{j_\nu}$ is a permutation of the type $\sigma^1_{N-(n-1)}$, and it is defined as the permutation of the sequence $\Big((2,\dots,N)-(j_1,\dots,\not{j}_\nu,\dots,j_n)\Big)$ which transfers j_ν to the same position relative to $\Big((2,\dots,N)-(j_1,\dots,j_n)\Big)$ as $\sigma(j_1,\dots,j_n)$ does.

c) $$\sum_{\sigma(j_1,\dots,j_n)} I_N^{\sigma(j_1,\dots,j_n)} = \sum_{\hat{\sigma}_{j_\nu}} \big(I_N^{S_{n-1}(j_1,\dots,\not{j}_\nu,\dots,j_n)} \big)^{\hat{\sigma}_{j_\nu}} \,. \tag{40}$$

Proof. a) To extract all terms $\sim \vec{q}_i \cdot \vec{q}_j$, $i,j \in \{j_1,\dots,j_n\}$ from $\sum_{\nu=1}^n \vec{q}_{j_\nu} \cdot \vec{p}_{j_\nu}^{\,\sigma(j_1,\dots,j_n)}$, we go through the sum over j_ν according to the order in which q_{j_ν} appears in the N-loop for the permutation $\sigma(j_1,\dots,j_n)$, starting from the last momentum. We realize that we pick up exactly once each pair $\vec{q}_i \cdot \vec{q}_j$, $i,j \in \{j_1,\dots,j_n\}$. Once these terms have been extracted the remainder obviously takes the form from (37).

b) Since $\hat{p}_{j_\nu}^{\,\sigma(j_1,\dots,j_n)}(q_1,\dots,q_N)$ equals the sum of the momenta arriving at the vertex of q_{j_ν} in the permutation $\sigma(j_1,\dots,j_n)$, with all q_j, $j \in \{j_1,\dots,j_n\}$ set to zero, it is equal to the sum over those momenta, which lie in the complementary set and arrive at the vertex of q_{j_ν}. So it equals $\hat{p}^{\,\hat{\sigma}_{j_\nu}}$.

c) On the r.h.s. of (40), $I_N^{S_{n-1}}$, which has been symmetrized w.r.t. $(j_1,\dots,\not{j}_\nu,\dots,j_n)$, depends only on the sequence $\Big((2,\dots,N)-(j_1,\dots,\not{j}_\nu,\dots,j_n)\Big)$, which is acted upon by $\hat{\sigma}_{j_\nu}$. The statement (40) then follows from the observation that summing over all possible orderings of $(j_1,\dots,j_n)$ within $(2,\dots,N)$, keeping the order of the remaining variables fixed, can be achieved by summing, for fixed ordering of $\Big((2,\dots,N)-(j_1,\dots,\not{j}_\nu,\dots,j_n)\Big)$, over all possible orderings of $(j_1,\dots,\not{j}_\nu,\dots,j_n)$ within $(2,\dots,N)$, and then over the position of j_ν relative to $\Big((2,\dots,N)-(j_1,\dots,j_n)\Big)$. $\square$

Using Lemma 2 we come back to the analysis of (35). On summing over ν we obtain:

$$\Big[\sum_{\nu=1}^{n}\{iq_{j_\nu,0}-2\vec{q}_{j_\nu}\cdot\vec{k}+\vec{q}_{j_\nu}^{\,2}\}+\sum_{\substack{i,j,i<j\\ i,j\in\{j_1,\ldots,j_n\}}}\vec{q}_i\cdot\vec{q}_j\Big]I_N^{S_n(j_1,\ldots,j_n)}(k;\,q_1,\ldots,q_N)\;=\tag{41}$$

$$-2\sum_{\nu=1}^{n}\sum_{\hat\sigma_{j_\nu}}\vec{q}_{j_\nu}\cdot\vec{\hat p}_{j_\nu}^{\,\hat\sigma_{j_\nu}}\,\big(I_N^{S_{n-1}(j_1,\ldots,\not{j_\nu},\ldots,j_n)}\big)^{\hat\sigma_{j_\nu}}(k;\,q_1,\ldots,q_N)$$

$$+\sum_{\nu=1}^{n}\sum_{\sigma_{N-1}^{n-1}}\Big(I_{N-1}^{\sigma_{N-1}^{n-1}}(k;\,q_1+q_{j_\nu},\ldots,\not{q}_{j_\nu},\ldots,q_N)$$

$$-I_{N-1}^{\sigma_{N-1}^{n-1}}(k+q_{j_\nu};\,q_1+q_{j_\nu},\ldots,\not{q}_{j_\nu},\ldots,q_N)\Big)\,.$$

For the prefactor on the l.h.s. of (41) we write

$$A(\vec{k};\,q_{j_1},\ldots,q_{j_n}):=\sum_{\nu=1}^{n}\{iq_{j_\nu,0}-2\vec{q}_{j_\nu}\cdot\vec{k}+\vec{q}_{j_\nu}^{\,2}\}+\sum_{\substack{i,j,i<j\\ i,j\in\{j_1,\ldots,j_n\}}}\vec{q}_i\cdot\vec{q}_j\,,\tag{42}$$

and we also introduce

$$\Delta(\vec{k},\vec{q}_{j_\nu};\,q_{j_1},\ldots,q_{j_n}):=\frac{1}{A(\vec{k}+\vec{q}_{j_\nu};\,q_{j_1},\ldots)}-\frac{1}{A(\vec{k};\,q_{j_1},\ldots)}$$

$$=\frac{2\sum_{\mu=1}^{n}\vec{q}_{j_\mu}\cdot\vec{q}_{j_\nu}}{A(\vec{k};\,\ldots)\,A(\vec{k}+\vec{q}_{j_\nu};\ldots)}\,.\tag{43}$$

We divide by $A(\vec{k};\,q_{j_1},\ldots,q_{j_n})$ (similarly as in (33) above) and obtain

$$I_N^{S_n}=\frac{-2}{A(\vec{k};\,q_{j_1},\ldots)}\sum_{\nu=1}^{n}\sum_{\hat\sigma_{j_\nu}}\vec{q}_{j_\nu}\cdot\vec{\hat p}_{j_\nu}^{\,\hat\sigma_{j_\nu}}\,\big(I_N^{S_{n-1}}\big)^{\hat\sigma_{j_\nu}}$$

$$+\sum_{\nu=1}^{n}\Delta(\vec{k},\vec{q}_{j_\nu};\ldots)\,I_{N-1}^{S_{n-1}}(k+q_{j_\nu};q_1+q_{j_\nu},\ldots)$$

$$+\sum_{\nu=1}^{n}\Big(\frac{1}{A(\vec{k};\ldots)}I_{N-1}^{S_{n-1}}(k;q_1+q_{j_\nu},\ldots)-\frac{1}{A(\vec{k}+\vec{q}_{j_\nu};\ldots)}I_{N-1}^{S_{n-1}}(k+q_{j_\nu};q_1+q_{j_\nu},\ldots)\Big)\,.\tag{44}$$

In the terms on the r.h.s. there appear the terms $I_N^{S_{n-1}}$ and $I_{N-1}^{S_{n-1}}$. Regarding their prefactors, $\vec{q}_j\cdot\vec{p}^{\,\hat\sigma_{j_\nu}}$ scales as λ^2 in the small λ and dynamical limits, and, by (8),

$$|A(\vec{k};\,q_{j_1},\ldots,q_{j_n})|\;>\;\eta\,,|A(\vec{k};\,\lambda q_{j_1},\ldots,\lambda q_{j_n})|\;>\;\lambda\,\eta\,,|A(\vec{k};\,q_{j_1,0},\,\lambda\vec{q}_{j_1},\ldots)|\;>\;\eta\,,\tag{45}$$

$$|\Delta(\vec{k},\lambda\vec{q}_{j_\nu};\lambda q_{j_1},\ldots,\lambda q_{j_\nu})| \le \frac{K_1}{\eta^2}, \Delta(\vec{k},\lambda\vec{q}_{j_\nu};q_{j_1,0},\lambda\vec{q}_{j_1},\ldots,q_{j_\nu,0},\lambda\vec{q}_{j_\nu})| \le \lambda^2\,\frac{K_2}{\eta^2}\ , \tag{46}$$

(where K_1, K_2 depend on the (compact) sets of momenta considered).

Our inductive proof of A2) B2) is based on (44) together with (24, 25). We use an inductive scheme proceeding upwards in $N \ge 3$, and for fixed N upwards in n for $0 \le n \le N-2$. The induction hypotheses are

$$|\Pi_N^{S_n}(\varphi_s;\,\lambda q_1,\ldots,\lambda q_N)| \le O(\lambda^{-(N-2-n)})\,,\quad |\Pi_N^{S_n}(\varphi_d;\,q_{10},\lambda\vec{q}_1,\ldots,\,q_{N0},\lambda\vec{q}_N)| \le O(\lambda^{2+2n})\ . \tag{47}$$

For any N and $n = 0$ (the unsymmetrized case) the claim follows from (24). For $n > 0$ we regard the scaling resp. dynamical limit for (44), multiplied by φ and integrated over k. We can apply the induction hypothesis to the r.h.s. of (44) noting again that the functions $\Delta\,\varphi$ and $A\,\varphi$ have the properties required for φ. We also use (25) if $N = 3$. For each entry in the sum in the last line of (44) we perform in the second term the change of variables $\tilde{k} = k + \lambda q_{j_\nu}$ resp. $\tilde{k} = k + (q_{j_\nu,0},\lambda\vec{q}_{j_\nu})$ and then use (21). With the aid of (45, 46) and the induction hypothesis one then shows

$$|\int \frac{dk_0}{2\pi}\,\frac{d^dk}{(2\pi)^d}\,I_N^{S_n}(k;\,\lambda q_1,\ldots,\lambda q_N)\,\varphi(\vec{k};\,\lambda q_1,\ldots,\lambda q_N)\,| \le O(\lambda^{-(N-2-n)})\ , \tag{48}$$

$$|\int \frac{dk_0}{2\pi}\,\frac{d^dk}{(2\pi)^d}\,I_N^{S_n}(k;\,q_{1,0},\,\lambda\vec{q}_1,\ldots,q_{N,0},\,\lambda\vec{q}_N)\,\varphi(\vec{k};\,q_{1,0},\,\lambda\vec{q}_1,\ldots,q_{N,0},\,\lambda\vec{q}_N)\,| \le O(\lambda^{2+2n})\ . \tag{49}$$

On specializing to $\varphi \equiv 1$, this ends the proof of the proposition. □

We join a few comments on various extensions of the results obtained.
a) For *dimensions* $d \ge 4$ the N-loop integrals are absolutely convergent for $2N > d+1$ and can be obtained as limits $\Lambda_0 \to \infty$ of their regularized versions, which are defined on introducing a regulating function $\rho(\frac{\vec{k}^2}{\Lambda_0^2})$ in the propagators

$$G_0(k) \to G_0(\Lambda_0,k) \,=\, \frac{1}{ik_0\,-\,(\rho^{-1}(\frac{\vec{k}^2}{\Lambda_0^2})\,\vec{k}^2\,-\,1)}\ .$$

We suppose ρ to be smooth, monotonic, positive, of fast decrease and such that $\rho(x) \equiv 1$ for $x \le 1$. The regulator then appears in the A-factors when using the Ward identity, e.g. (32) changes into

$$iq_{j,0}\,+\,\rho^{-1}(\frac{(\vec{k}-\vec{q}_j)^2}{\Lambda_0^2})\,(\vec{k}-\vec{q}_j)^2\,-\,\rho^{-1}(\frac{\vec{k}^2}{\Lambda_0^2})\,\vec{k}^2\ .$$

But since these factors are still independent of k_0, the regulator disappears without leaving any trace after performing the k_0-integration, if $\Lambda_0 \ge \sum|\vec{q}_j| + 1$. So

we still obtain the same results for $d \geq 4$, if $2N > d+1$, and we obtain them without this last restriction in case we define the integrals as $\Lambda_0 \to \infty$ - limits of their regulated versions from the beginning.
b) Neumayr and Metzner [1] also prove $|\Pi_N^S(q_1, \dots, q_N)| \leq O(|\vec{q}_j|)$ for $\vec{q}_j \to 0$, keeping the other variables fixed. In our framework this result is obtained immediately from (33), and we realize that it holds already on symmetrization with respect to $\vec{q}_j$, full symmetrization is not required. This result can be generalized to several vanishing external momenta $\vec{q}_{j_1}, \dots, \vec{q}_{j_n}$, in the same way as we did for the proof of A2) and B2) in the proposition. Using (44) we obtain on induction

$$|\Pi_N^{S_n(j_1,\dots,j_n)}(q_1, \dots, q_N)| \leq O(\prod_{\nu=1}^{n} |\vec{q}_{j_\nu}|) \tag{50}$$

and of course the same bound on Π_N^S.
c) From the proof one can straightforwardly read off a bound w.r.t. the dependence on the parameter η from (8). This bound is in terms of $\eta^{-(N+n)}$, stemming from the contributions with a maximal number of factors of Δ. It is of course rather crude, since it does not take into account the effects of the nonvanishing spatial variables and can be improved, depending on the hypotheses made on those.

In conclusion we have recovered previous results on the infrared behaviour of the connected N-point density-correlation functions, in short N-loops, by simple, but rigorous arguments based on the *Ward identity.* We obtain bounds for the fully symmetrized N-loop, in showing, how successive symmetrization improves the infrared behaviour.[5] The bounds hold in any spatial dimension (taking into account the remarks from a) above). Since the Ward identities are explicit and easy to handle, they permit generalizations such as (50).

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C. Kopper and J. Magnen
Centre de Physique Théorique
CNRS UPR 14
Ecole Polytechnique
F–91128 Palaiseau Cedex
France
e-mail: kopper@cpht.polytechnique.fr
e-mail: magnen@cpht.polytechnique.fr

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