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Non Exponential Law of Entrance Times in Asymptotically Rare Events for Intermittent Maps with Infinite Invariant Measure

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Abstract. We study piecewise affine maps of the interval with an indifferent fixed point causing the absolutely continuous invariant measure to be infinite. Considering the laws of the first entrance times of a point — picked at random according to Lebesgue measure — into a sequence of events shrinking to the strongly repelling fixed point, we prove that (when suitably normalized) they converge in distribution to the independent product of an exponential law to some power and a one-sided stable law.

Résumé Nous étudions une classe d'applications affines par morceaux de l'intervalle avec un point fixe indifférent dont la mesure invariante absolument continue est infinie. Nous considérons les lois des premiers temps d'entrée d'un point — choisi au hasard suivant la mesure de Lebesgue — dans une suite d'événements se concentrant autour du point fixe fortement répulsif. Nous prouvons que, correctement renormalisés, ces temps convergent en distribution vers le produit indépendant d'une loi exponentielle élevée à une certaine puissance et d'une loi stable unilatérale.

1 Introduction

There has been a recent interest in statistics of entrance - or return - times into rare events for chaotic dynamical systems. Given a sequence of sets in the phase space of some ergodic system with measures decaying to zero, one can ask about the asymptotic behaviour of the sequence of entrance times in these sets.

In the case of hyperbolic systems preserving a probability measure, entrance times typically converge to an exponential distribution when normalized by their expectations. The lack of memory property of the limit distribution is often interpreted as "unpredictability" of the occurence of rare events. Results of this type have been proved for different classes of systems and sequences of shrinking sets, see for example the survey in [5]. One basic family of examples is that of uniformly expanding maps of the interval. Interval maps with indifferent fixed points, frequently referred to as intermittent maps, perhaps give the simplest models beyond uniform hyperbolicity. For those situations where there still exists a finite absolutely continuous invariant measure, precise results again giving exponential limit laws have been given in [17].

The case of maps with an indifferent fixed point whose SRB measure is a Dirac mass at the fixed point - and where the only absolutely continuous invariant measure is infinite - is somewhat different. We refer to [1] for general ergodic properties of infinite measure preserving systems, and to [19] for specific information on interval maps with indifferent fixed points and further references. [8] considered a particular piecewise affine, i.e. Markov chain model, and proved convergence to an exponential distribution for entrance times close to the indifferent fixed point, which however are not rare in the sense of the invariant measure respectively the dynamics. The purpose of the present note is to similarly present a simple family of piecewise affine examples for which the entrance times to a particular sequence of sets, namely those shrinking to the strongly repelling fixed point, in general converge to a non exponential law which depends on the fine local behaviour at the fixed point. We also discuss what we expect to be the behaviour of these entrance times for more general sequences of cylinders.

The only other results known to us where a limit law different from the exponential distribution turns up are for systems of (very) low complexity, such as rotations (see [4]) and substitutions (see [10]). In these cases, the limit distributions are distributions of discrete random variables and the analysis has a different flavour.

2 Statement of the result

Let (I, λ) be the interval $I = [0, 1]$ endowed with Lebesgue measure λ. Let $(c_j)_{j\geq 0}$ be a sequence strictly decreasing to 0 with $c_0 = 1$ satisfying $c_{j+1}/c_j \to 1$. These points yield a partition (mod λ) of I into the intervals $I_j := (c_{j+1}, c_j)$, $j \geq 0$. We consider the map T on I which is affine and increasing on each I_j and maps I_0 onto I (with slope $s := (1 - c_1)^{-1}$) and I_j onto I_{j-1} for all $j \geq 1$, cf. Fig.1.

Since $T'(x) \to 1$ as $x \to 0$, transformations of this type frequently serve as simplified models for smooth 'intermittent' maps with an indifferent fixed point. The piecewise affine version T in fact is just a renewal Markov chain in a sense we shall make precise below. T is conservative ergodic and has a unique (up to a constant factor) absolutely continuous invariant measure μ (whose density is constant on each I_j) which is infinite if and only if $\sum_j c_j = +\infty$. Throughout we shall assume that this is the case (i.e. that the chain is null recurent) and we choose μ such that $\mu(I_0) = \lambda(I_0)$.

Example 1 *Specific examples which are frequently studied in the literature are given by* $c_j := const \cdot j^{-\alpha}$, $\alpha \in (0, 1]$, *which corresponds to* $Tx = x + ax^{1+\frac{1}{\alpha}} + o(x^{1+\frac{1}{\alpha}})$ *in the smooth setting.*

We are interested in the asymptotic distributional behaviour of the (first) entrance times to a sequence of asymptotically rare events. More precisely we consider the sequence (d_j) of the preimages of c_1 under the rightmost branch of T, i.e. $d_j := 1 - s^{-j}$ and the sequence of intervals $B_m := (d_{m+1}, 1)$, $m \geq 0$, with $\lambda(B_m) = \mu(B_m) = s^{-m}$. The variables τ_m, $m \geq 0$ we are interested in are the

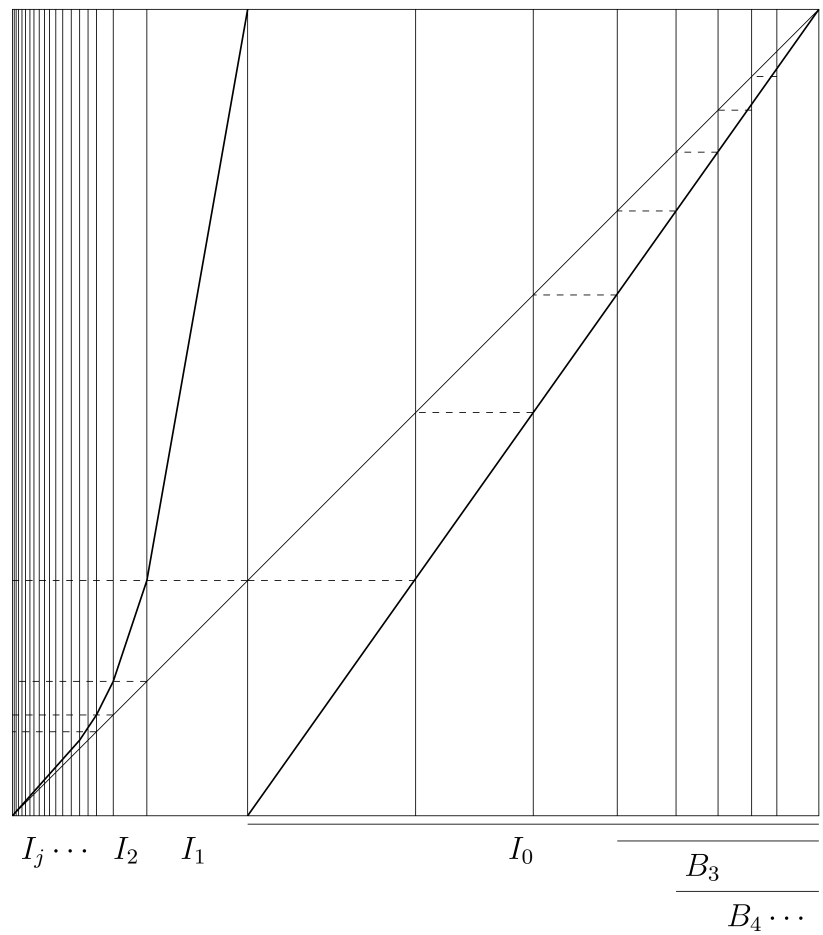

Figure 1: The map T.

numbers of steps needed to enter B_m, that is

$$\tau_m(x) := \min\{i \geq 1, T^i(x) \in B_m\}.$$

These *entrance times* obviously go to infinity almost surely and have infinite expectation with respect to λ. Still it is possible to understand their asymptotic behaviour.

To state the result, we let $\mathcal{E}$ denote the exponential law of parameter 1, and also use the same symbol for a generic random variable with this distribution, independent of all other variables that may appear. Similarly, $\mathcal{G}_\alpha$ denotes the (essentially unique) *one-sided stable law of index* $\alpha \in (0,1)$, i.e. the distribution on $\mathbf{R}_+ = (0,\infty)$ with Laplace transform $\widehat{G}_\alpha(t) = e^{-t^\alpha}$, see [12], pp.448, as well as the generic random variable with this distribution. For example, $\mathcal{G}_{\frac{1}{2}}$ (which naturally arises in return time problems for the simple coin-tossing random walk, cf. [11], p.90) is the law of $\frac{1}{\mathcal{N}^2}$, where $\mathcal{N}$ has a standard normal distribution. We shall say more about how these laws arise after the statement of the theorem, and it will become clear that it is natural to write $\mathcal{G}_1$ for the law with unit mass at 1.

The theorem below applies to the maps of Example 1, but we prefer to state the result in full generality since this causes no additional difficulties in the proof, and might turn the reader's attention to a classical probabilistic theory which is not particularly well known in the dynamics community.

When talking about asymptotic properties we shall identify a sequence (c_j) with its piecewise constant extension $c(x) := c_{[x]}$, $x \in \mathbf{R}_+$. Recall that a function $c : \mathbf{R}_+ \to \mathbf{R}_+$ is called *regularly varying* (at infinity) with *index* $\alpha \in \mathbf{R}$ if it is of the form $c(x) = x^\alpha l(x)$ where l is *slowly varying* in that it satisfies $\lim_{x\to+\infty} \frac{l(\sigma x)}{l(x)} = 1$ for all $\sigma > 0$ (e.g. if l is constant or $l(x) = \log x$). A function b *is asymptotically inverse* to c if $b(c(x)) \sim c(b(x)) \sim x$ as $x \to \infty$. Such functions exist and are unique up to asymptotic equivalence if $\alpha > 0$, see [2], pp.28.

Theorem 1 (Distributional convergence of the entrance times) *If the sequence (c_j) is regularly varying of index $-\alpha$ for some $\alpha \in (0,1)$, or if $(\sum_{j=0}^n c_j)_{n\geq 1}$ is slowly varying and $\alpha := 1$, then*

$$\frac{1}{b(s^m)} \cdot \tau_m \overset{d}{\Longrightarrow} \mathcal{E}^{\frac{1}{\alpha}} \cdot \mathcal{G}_\alpha$$

as $m \to \infty$, where the τ_m are considered as random variables distributed according to Lebesgue measure λ on I, and b is a function asymptotically inverse to $x \mapsto (c_1\Gamma(1-\alpha)c(x))^{-1}$ in the first case, and asymptotically inverse to $x \mapsto x/(c_1 \int_0^x c(y)dy)$ in the second. (Hence b is regularly varying with index $\frac{1}{\alpha}$ and satisfies $x = o(b(x))$ as $x \to \infty$).

Example 2 *In the case $\alpha = 1$, which lies at the threshold between the finite and the infinite measure regime, we still have an exponential distribution in the limit, although the normalizing sequence can no longer be given by the expectations of the τ_m which are already infinite. For the particular $\alpha = 1$ map from the family of example 1, we have (with κ a suitable constant)*

$$\kappa \cdot m^{-1} \cdot s^{-m} \cdot \tau_m \overset{d}{\Longrightarrow} \mathcal{E} \ .$$

Example 3 *In the $\alpha \in (0,1)$ cases of example 1, we have $b(s^m) = \kappa \cdot s^{\frac{m}{\alpha}}$. If, in particular, $\alpha = \frac{1}{2}$, we obtain*

$$\kappa \cdot s^{-2m} \cdot \tau_m \overset{d}{\Longrightarrow} \left(\frac{\mathcal{E}}{\mathcal{N}}\right)^2 .$$

Remark 1. The distribution $\mathcal{H}_\alpha := \mathcal{E}^{\frac{1}{\alpha}} \cdot \mathcal{G}_\alpha$ of the independent product of the $\frac{1}{\alpha}$-power of an exponential law of parameter 1 and the one-sided stable law of index α can more explicitely be described by its Laplace transform which is easily seen to be

$$\widehat{H}_\alpha(t) = \frac{1}{1+t^\alpha} \; .$$

Remark 2. A minor modification of our argument also gives the asymptotic distributional behaviour of the *first return times* $\varphi_m(x) := \min\{i \geq 1, T^i(x) \in B_m\}$, $x \in B_m$, regarded as random variables on the respective sets B_m with normalized Lebesgue measure $\lambda_m := \lambda(B_m)^{-1} \cdot \lambda$. We have

$$\frac{1}{b(s^m)} \cdot \varphi_m \stackrel{d}{\Longrightarrow} s^{-1}\delta_0 + (1-s^{-1})\,\mathcal{E}^{\frac{1}{\alpha}} \cdot \mathcal{G}_\alpha,$$

where δ_0 denotes unit point mass in zero. This is because $\{\varphi_m = 1\} = B_{m+1} \subseteq B_m$ always has λ_m−measure s^{-1} while under the condition that it should be larger than 1, φ_m behaves as τ_m above.

To get an intuitive understanding of the result we take a closer look at a Markov chain equivalent to T. It is a simple renewal chain (X_n) with states I_j, the renewal state being I_0, see Fig.2.

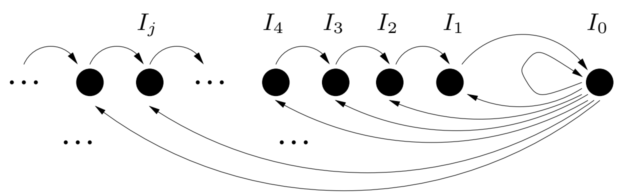

Figure 2: The Markov chain model.

The transition probabilities are given by $\mathbf{P}(X_{n+1} = I_j | X_n = I_0) = \lambda(I_j)/\lambda(I \setminus I_0)$. The precise relation to the interval map is as follows: if $Y_0 \in I$ is randomly chosen according to some probability density $\sum_j \pi_j 1_{I_j}$ constant on each I_j, and $Y_n := T^n(Y_0)$, $n \geq 1$, the resulting random sequence (X_n) with $X_n := I_j$ if $Y_n \in I_j$ is the renewal chain with initial distribution (π_j).

Any sample path of the renewal chain consists of a sequence of excursions to the left part. If we let L_k denote the time between the $k-1$st and kth visit in I_0, then (L_k) clearly is an iid sequence, and, when starting in I_0, the number of steps until we return to I_0 for the nth time is $\sum_{k=1}^n L_k$. This is where the stable laws enter:

By classical results, arithmetical averages of nonnegative iid variables L_k without expectation converge to some nondegenerate limit distribution iff the sequence of tail weights $t_j := \mathbf{P}(L_k \geq j)$ is regularly varying of index $-\alpha$ for some

$\alpha \in (0,1)$, in which case we have

$$\frac{1}{b(n)} \sum_{k=1}^{n} L_k \overset{d}{\Longrightarrow} \mathcal{G}_\alpha, \tag{1}$$

where b is asymptotically inverse to $x \mapsto (\Gamma(1-\alpha)t(x))^{-1}$, cf. [12], pp.448 or [2], pp.343. The same conclusion holds with $\alpha := 1$ provided $(\sum_{j=0}^{n} t_j)_{n\geq 1}$ is slowly varying and b is asymptotically inverse to $x \mapsto x/\int_0^x t(y)dy$, cf. [2],pp.372 or [12], pp.234. Observe that in the case $\alpha = 1$, which is closest to the situation of finite expectation (where the strong law of large numbers would give a.s. convergence of the averages $(\mathbf{E}(L_1)\cdot n)^{-1}\sum_{k=1}^{n} L_k \to 1$), (1) with $\mathcal{G}_1 = 1$ still gives a weak law of large numbers, while for $\alpha < 1$ stronger fluctuations cause the limit to become continuously distributed. In our particular situation we have $t_j = c_j$ showing that the conditions on (c_j) are most natural from a probabilist's point of view.

In fact (1) is essential for understanding how the limit law in the theorem arises. We give a rough heuristical sketch of the argument: Recall (cf. [12], pp.169) that α-stability of the law by definition means that the sum of n independent random variables $G_1, \cdots, G_n$ sharing this distribution has the same law as $n^{\frac{1}{\alpha}} G_1$. The target event B_m is to stay at I_0 for at least m steps. This can happen only at the end of an excursion when we are back at I_0, where we have a certain probability p_m (with $p_m \to 0$) for B_m to occur. If it does not, we are given another chance at our next return to I_0. The number θ_m of trials (and hence excursions) we need therefore will roughly have a geometric distribution and should thus converge to an exponential law as $m \to \infty$. On the other hand, the total number of steps done during that time will be given by the random sum $L_1 + \cdots + L_{\theta_m}$. Assume for the moment that the L_k were distributed according to $\mathcal{G}_\alpha$ (which they are not, but they share the same tail behaviour) and that they were independent of θ_m (in fact we shall see below that in a sense the major part of them is). Then, by the defining property of an α-stable law, this sum is distributed like ${\theta_m}^{\frac{1}{\alpha}} \cdot L_1$, so that

$$\tau_m \simeq L_1 + \cdots + L_{\theta_m} \simeq {\theta_m}^{\frac{1}{\alpha}} \cdot \mathcal{G}_\alpha,$$

where θ_m (when normalized by its expectation) is close to an exponential distribution.

3 Proof of the theorem

The adequate framework for proving a probabilistic result about a dynamical system metrically isomorphic to a Markov chain should be that of the latter. Instead of working with the simple renewal chain mentioned before, we shall find it more convenient to use a slightly refined Markov chain model in which the target events B_m appear explicitely. We let $J_j := (d_j, d_{j+1})$, $j \geq 0$, and consider the Markov chain $(X_i)_{i\geq 0}$ whose states are the I_j, $j \geq 1$ and J_j, $j \geq 1$, with the obvious transition probabilities $\mathbf{P}(X_{n+1} \in I_j | X_n = J_1) = \lambda(I_j)/\lambda(J_0) = (c_j - c_{j+1})/c_1$, and $\mathbf{P}(X_{n+1} \in J_j | X_n = I_1) = \lambda(J_j)/\lambda(I_0) = s^{-j}(s-1)$, cf. Fig.3.

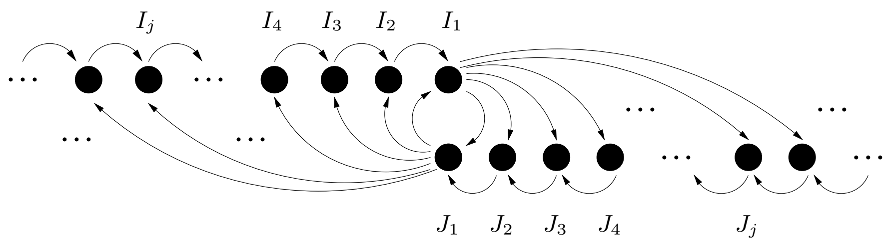

Figure 3: The refined Markov chain.

The relation between the chain and the map T is analogous to what we said before, the target event B_m is $\bigcup_{j>m} J_j$.

For convenience we shall first consider the chain starting with an initial distribution that in the interval map setting corresponds to normalized Lebesgue measure on I_0, that is, $\mathbf{P}(X_0 = J_j) = \lambda(J_j)/\lambda(I_0) = (1-s^{-1})s^{-j+1}$, $j \geq 1$. Again we consider

$$\tau_m := \min\{i \geq 1, X_i \in B_m\}.$$

To get an easy understanding of paths that enter B_m for the first time at a certain step we shall focus on the states J_1 and I_1 to separate excursions to the left and to the right. We let Θ_m denote the number of passages through J_1 (and hence through I_1) before time τ_m:

$$\Theta_m := \sum_{i=0}^{\tau_m - 1} 1_{J_1}(X_i).$$

Whether or not we hit B_m between two passages through J_1 depends on the edge we choose from I_1. Now, $p_m := \mathbf{P}(X_{i+1} \in B_m | X_i = I_1) = s^{-m} \to 0$ as $m \to +\infty$, and $\mathbf{P}(\Theta_m = 0) = \mathbf{P}(X_0 \in B_{m+1}) = p_{m+1}$, while $\mathbf{P}(\Theta_m = r) = (1-p_{m+1})p_m(1-p_m)^{r-1}$ for $r \geq 1$. Consequently, the Θ_m normalized by their expectations $\mathbf{E}[\Theta_m] = \frac{(1-p_{m+1})(1-p_m)}{p_m} \sim s^m$, converge to an exponential law of parameter 1:

$$\frac{1}{\mathbf{E}[\Theta_m]} \cdot \Theta_m \stackrel{d}{\Longrightarrow} \mathcal{E}. \tag{2}$$

Turning back to τ_m we are going to decompose it into the successive excursion times spent on either side. To formalize this, we set $S_0 := 0$, and for $k \geq 1$ let

$$T_k := \min\{i \geq S_{k-1} : X_i = J_1\}, \quad \text{and} \quad S_k := \min\{i \geq T_k : X_i = I_1\}.$$

The lengths of the kth excursion to the left and to the right are then respectively given by

$$L_k := S_k - T_k,\ k \geq 1 \quad \text{and} \quad R_k := T_{k+1} - S_k,\ k \geq 0.$$

(These L_k correspond morally - though not precisely - to those from the sketch above.) We can then represent the entrance time τ_m as

$$\tau_m = \sum_{k=1}^{\Theta_m} L_k + \sum_{k=0}^{\Theta_m - 1} R_k + 1. \tag{3}$$

This decomposition is useful because the sequences (L_k) and (R_k) are iid, and - most important for our purposes - the sequence (L_k) is independent of each Θ_m: the number of excursions to the left is independent of their lengths. Moreover we shall see later that the contribution of the R_k vanishes asymptotically, and we therefore concentrate on the first of the sums in (3).

As the the tail weights $t_j = \mathbf{P}(L_k > j)$ are now given by $\frac{c_j}{c_1}$, our assumptions on (c_j) ensure that (1) holds with b as in the theorem. Therefore the correct order of magnitude of $\sum_{k=1}^{\Theta_m} L_k$ is that of the random sequence $(b(\Theta_m))$ which in view of (2) we might hope to be given by $(b(\mathbf{E}[\Theta_m]))$. We therefore write

$$\frac{1}{b(s^m)} \cdot \sum_{k=1}^{\Theta_m} L_k = \frac{b(\mathbf{E}[\Theta_m])}{b(s^m)} \cdot \frac{b(\Theta_m)}{b(\mathbf{E}[\Theta_m])} \cdot \frac{1}{b(\Theta_m)} \sum_{k=1}^{\Theta_m} L_k. \tag{4}$$

The scalar factor in front converges to 1 because of the regular variation of b. The second factor exhibits good limiting behaviour, too: we have

$$\frac{b(\Theta_m)}{b(\mathbf{E}[\Theta_m])} \overset{d}{\Longrightarrow} \mathcal{E}^{\frac{1}{\alpha}}, \tag{5}$$

which is immediate from the following lemma.

Lemma 1 *Assume that E and E_m, $m \geq 0$, are random variables taking values in $\mathbf{R}_+ = (0, \infty)$, such that $\frac{1}{\gamma_m} E_m \overset{d}{\Longrightarrow} E$, for suitable normalizing constants $\gamma_m \to \infty$. If $b : \mathbf{R}_+ \to \mathbf{R}_+$ is regularly varying at infinity with index $\beta \neq 0$, then*

$$\frac{b(E_m)}{b(\gamma_m)} \overset{d}{\Longrightarrow} E^\beta.$$

Proof. Writing

$$\frac{b(E_m)}{b(\gamma_m)} = \left(\frac{E_m}{\gamma_m}\right)^\beta \cdot \frac{l\left(\frac{E_m}{\gamma_m}\gamma_m\right)}{l(\gamma_m)},$$

l being the slowly varying part of b, this is an easy application of the uniform convergence theorem for slowly varying functions which ensures that $\frac{l(\sigma x)}{l(x)} \to 1$, as $x \to +\infty$, uniformly in $\sigma \in [\Sigma^{-1}, \Sigma]$, for any $\Sigma > 1$. See [2], p.6. $\square$

Let us return to (4). Since we know that $\Theta_m \to \infty$ in probability and each is independent of (L_k) it is easy to see that the rightmost term will converge in law to a stable distribution $\mathcal{G}_\alpha$. However, as both random terms contain the Θ_m, they are not independent and we have to be careful about the distribution of their product. The reason why we will still have convergence to the independent product $\mathcal{E}^{\frac{1}{\alpha}} \cdot \mathcal{G}_\alpha$ is that the only thing that matters for the last term is that Θ_m is large. The precise distribution of Θ_m has hardly any effect on the distribution of the sum. This is made precise in the following lemma, the easy proof of which we omit.

Lemma 2 *Assume that Q_n, Q, H_m, H, and T_m are random variables such that*

1. *Q_n take values in $\mathbf{R}_+$ and $Q_n \overset{d}{\Longrightarrow} Q$,*
2. *T_m take values in $\mathbf{N}$ and $T_m \to \infty$ in probability,*
3. *$H_m \overset{d}{\Longrightarrow} H$,*
4. *Each of T_m, H_m, and H is independent of the sequence (Q_n) and of Q.*

Then

$$H_m \cdot Q_{T_m} \overset{d}{\Longrightarrow} H \cdot Q.$$

Of course, the important point here is that H_m and T_m need not be independent. Taking $H_m := \frac{b(\Theta_m)}{b(\mathbf{E}[\Theta_m])}$, $T_m := \Theta_m$ and $Q_n := \frac{1}{b(n)} \sum_{k=1}^{n} L_k$ we obtain

$$\frac{1}{b(s^m)} \sum_{k=1}^{\Theta_m} L_k \overset{d}{\Longrightarrow} \mathcal{E}^{\frac{1}{\alpha}} \cdot \mathcal{G}_\alpha. \tag{6}$$

To get the asymptotics of τ_m we still have to take care of the R_k, cf. (3). Recall that (R_k) is an iid sequence and that the R_k have finite expectation. Therefore $n^{-1} \sum_{k=0}^{n-1} R_k \to \mathbf{E}[R_1] \in \mathbf{R}_+$ almost surely. Since also $\Theta_m \to \infty$ a.s., we have $\Theta_m^{-1} \sum_{k=0}^{\Theta_m - 1} R_k \to \mathbf{E}[R_1]$ a.s. as $m \to \infty$. In view of $x/b(x) \to 0$ (which is clear from (1) as $\mathbf{E}[L_k] = \infty$) and (2), this implies

$$\frac{1}{b(s^m)} \sum_{k=1}^{\Theta_m - 1} R_k \to 0 \qquad \text{in probability.} \tag{7}$$

We therefore end up with

$$\frac{1}{b(s^m)} \cdot \tau_m \overset{\mathrm{d}}{\Longrightarrow} \mathcal{E}^{\frac{1}{\alpha}} \cdot \mathcal{G}_\alpha, \tag{8}$$

which shows that the distribution of the first entrance time in the small events B_m have the asserted limiting behaviour if we start our chain on the righthand half with the measure specified in the beginning.

To finally obtain the result for the case of the initial distribution which corresponds to Lebesgue measure for the interval map is almost trivial now: It is enough to notice that the shifted chain $(\bar{X}_i)_{i\geq 0}$ defined by $\bar{X}_i := X_{i+1}$ has this initial distribution, thus giving a realization of the process we are interested in, and to observe that for $\bar{\tau}_m := \min\{i \geq 0, \bar{X}_i \in B_m\}$ we have $\bar{\tau}_m - (\tau_m - 1) \to 0$ almost surely, so that (8) holds just as well with τ_m replaced by $\bar{\tau}_m$. □

4 A more general pattern

The following heuristic considerations suggest that the same limit laws should arise for a larger class of asymptotically rare events defined by prescribing the durations $k_1, k_2, \ldots \in \mathbf{N}$ of m consecutive excursions from I_0 and letting $m \to \infty$. (That is, we consider the nested sequence of cylinders around some point $x \in (0,1)$.) The situation is more intricate than before, since the excursions required to continue a successful attempt may change from step to step, and if we fail, we still need not necessarily start from scratch, as the last few excursions may well fit a shorter initial segment of (k_i).

We start from the Markov chain $(X_n)_{n\geq 0}$ with states I_j, $j \geq 0$, cf. Fig.2, and $\mathbf{P}(X_0 = I_0) = 1$. L_i, $i \geq 1$, will denote the duration of the ith excursion from I_0, and we let $S_n := \sum_{k=0}^{n-1} 1_{I_0}(X_k)$. To keep track of how many consecutive excursions of the prescribed lenghts we have done up to step n, we set $D_0 := 0$ and define $D_n := \max(\{0\} \cup \{r \geq 1 : L_{S_n - r+1} = k_1, \ldots, L_{S_n} = k_r\})$, $n \geq 1$. Observe then that $Z_n := (X_n, D_n)$, $n \geq 0$, again is a Markov chain. At step n we complete a series of m excursions of lengths $k_1, \ldots, k_m$ iff $Z_n = (I_0, m)$. The waiting time for this event is given by $\tau_m := \inf\{n \geq 1 : Z_n = (I_0, m)\}$. We decompose paths according to the visits of (Z_n) to $(I_0, 0)$. Let L_k^*, $k \geq 1$, denote the time between the $k-1$st and kth visit, and $\Theta_m := \sum_{k=0}^{\tau_m} 1_{(I_0,0)}(Z_k)$. Then τ_m is essentially given by $\sum_{k=1}^{\Theta_m} L_k^*$.

Θ_m is the waiting time until the first success (meaning that - with probability $p_m \to 0$ - we reach (I_0, m) before returning to $(I_0, 0)$) in a sequence of Bernoulli trials performed at each visit to $(I_0, 0)$. Hence $p_m\ \Theta_m \overset{d}{\Longrightarrow} \mathcal{E}$ as $m \to \infty$. Notice now that $\sum_{k=1}^{\Theta_m} L_k^*$ has the same distribution as $\sum_{k=1}^{\Theta_m} E_k^{(m)}$, where $(E_k^{(m)})_{k\geq 1}$ is an iid sequence independent of Θ_m, $E_k^{(m)}$ having the first return distribution $F^{(m)}$ of (Z_n) to $(I_0, 0)$ under the condition that we do not pass through (I_0, m). If the $F^{(m)}$ are *uniformly* in the domain of attraction of $\mathcal{G}_\alpha$ in the sense that both the $\mathcal{L}_\infty$-convergence of the distribution functions of $b^{(m)}(n)^{-1} \sum_{k=1}^{n} E_k^{(m)}$ to $\mathcal{G}_\alpha$, and the regular variation of the $b^{(m)}$ are uniform in m, then easy generalizations of the Lemmas above show that

$$\frac{1}{b^{(m)}(p_m^{-1})} \sum_{k=1}^{\Theta_m} E_k^{(m)} \overset{d}{\Longrightarrow} \mathcal{E}^{\frac{1}{\alpha}} \cdot \mathcal{G}_\alpha, \text{ as } m \to \infty.$$

We are however not going to rigorously discuss this question here.

Finally we notice that this pattern does not include the interesting case of cylinders shrinking to the indifferent fixed point $x = 0$. As remarked earlier, they do not constitute events which are asymptotically rare w.r.t. the invariant measure. A rough analysis suggests that the entrance times should behave rather differently. In effects, these entrance times can be written $\tau_m = \sum_{i=1}^{\Theta_m - 1} L_i$ where (L_i) is the sequence of iid random variables describing the durations of the excursions from I_0 and Θ_m is the first index i for which L_i is larger than m. For each m, we can consider an iid sequence $(E_i^{(m)})_{i \geq 1}$, independent of Θ_m, having the distribution of L_i given $\{L_i < m\}$. The random variable τ_m has the distribution of $\sum_{i=1}^{\Theta_m - 1} E_i^{(m)}$. Our point is that, at least in the simplest cases, one can use the theorem in Section IX.7 of [12] to identify the limit distribution of the triangular array $b(m)^{-1} \sum_{i=1}^{\mathbf{E}[\Theta_m]-1} E_i^{(m)}$ for suitable normalizing sequences b. It has finite expectation but is not trivial. So we believe that another class of limit laws may arise in this situation.

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