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On Static Stars in Newtonian Gravity and Lane-Emden Type Equations

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Abstract. The equations governing static stellar models in Newtonian gravity are equivalent to a Lane-Emden type equation. For such equations existence, uniqueness, and regularity of global solutions is shown for a large class of right-hand sides, including a subclass of non-Lipschitz continuous equations of state which is relevant if e.g. phase transitions occur. Furthermore, it is shown that for a star of finite radius the polytropic index of the equation of state is not necessarily bounded near the star's surface.

Résumé Les équations qui régissent les modèles stellaires statiques en gravité newtonienne sont équivalentes à une équation de type Lane-Emden. Pour de telles équations, l'existence, l'unicité et la régularité de solutions globales sont montrées pour une large classe de membres de droite, incluant une sous-classe d'équations d'état continues et non lipschitziennes qui s'appliquent lorsque par exemple une transition de phase a lieu. De plus, il est montré que pour une étoile de rayon fini, l'indice polytropique de l'équation d'état n'est pas nécessairement fini près de la surface.

1 Introduction

The field equations for equilibrium configurations of rotating stars, within general relativity or Newtonian gravity, can be written in suitable coordinates as a system of semilinear elliptic partial differential equations where the elliptic operators on the left-hand side are equivalent to Laplacians in flat space (see [2], [19]). Due to Poisson's integral formulas this system (with a free boundary!) is equivalent to a fixed point problem $u = \mathcal{T}u$ in suitable function spaces. The principal idea, namely to start with "reasonable" functions u_0 and iteratively apply the mapping $\mathcal{T}$ in order to get approximate solutions of the fixed point problem, led to one of the most efficient numerical solution techniques [2] for rotating stellar models and for more general configurations. Therefore, it is a natural task (which is also important for the reliability of such numerical solutions) to prove rigorously that these approximate solutions converge to the solution of the original problem.

In this article the most simple case toward this goal is investigated, namely static (i.e. non-rotating) stars within the framework of Newton's theory of gravitation. In this case the above mentioned fixed point problem is equivalent to a singular ordinary differential equation of second order of so-called Lane-Emden type (see e.g. [8] and [7]). It turns out that, apart from "essentially" polytropic equations of state (see [8], [7], [11], [18] and [16]), there are still some interesting open questions for this classical problem. For example, I have found in the literature no general existence or uniqueness results for global solutions if the equation of state is not Lipschitz continous, as is the case if e.g. phase transitions occur (and for realistic models such situations cannot be ruled out). Furthermore, the question for which equations of state the stellar radius is finite is not yet completely solved.

Below, it is shown that existence, uniqueness, and regularity of global solutions of Lane-Emden type equations can be established (along the line of the principal idea of the iterative scheme, and with some a priori error estimates) for a large class of right-hand sides, including equations of state with phase transitions. Moreover, it is shown that essentially polytropic behaviour with index strictly less than five of the equation of state near the star's surface is not necessary for the star to be of finite size.

The article is organized as follows: In Sect. 2 the physical problem is brought into an appropriate mathematical form. In Sect. 3 the existence, uniqueness, and regularity results for global solutions of Lane-Emden type equations are presented. In Sect. 4 the relation between the equation of state and the finiteness of the stellar radius is investigated. Finally, in Sect. 5 the obtained results are summarized and some generalizations are pointed out.

2 Mathematical Formulation of the Problem

A static star of ideal fluid has three physical degrees of freedom in Newtonian gravity: the gravitational potential U, the mass density ϵ, and the pressure p. These three quantities are scalar fields on 3-dimensional flat space $\mathbb{R}^3$. The three basic equations governing such an equilibrium configuration are (i) Poisson's equation, (ii) Euler's equation, and (iii) an equation of state (EOS). Using the notations Δ for the Laplacian and ∇ for the gradient in $\mathbb{R}^3$, Poisson's and Euler's equation read[1]:

$$\Delta U = 4\pi\,\epsilon\ , \tag{1}$$

$$\nabla p = -\epsilon\,\nabla U\ , \tag{2}$$

respectively. For most astrophysically interesting objects sufficiently close to equilibrium it is permissible to presuppose an EOS of the form (see e.g. [7], [20])

$$\epsilon = \epsilon(p)\ . \tag{3}$$

Basically, in this article it is assumed that the real-valued function $p \mapsto \epsilon(p)$, defined on the interval $[0, p_{\max}] \subset \mathbb{R}_0^+$, obeys the following properties:

1. $\epsilon(p) > 0$ for all $p > 0$,

[1] Throughout this article "geometrized units" are used where the gravitational constant is set equal to one.

2. $\epsilon(0) = 0$,

3. $\epsilon(p)$ is increasing,

4. $\epsilon(p)$ is piecewise continuous, and

5. the integral
$$F(p) := \int_0^p \frac{1}{\epsilon(\tilde{p})}\, d\tilde{p} \tag{4}$$
exists for all $p \in [0, p_{\max}]$.

Remark 1. Property 1. ensures that the mass density is positive. Property 2. is used for convenience. In the following $\lim_{p\downarrow 0} \epsilon(p) > 0$ is permissible (corresponding to stiff matter at the star's surface)! Property 3. implies that the matter described by the EOS is "microscopically stable" (see e.g. [10]). The regularity property 4. will be slightly strengthened below (see Definition 2). Since $\epsilon(0) = 0$, property 5. is essentially a condition on the behaviour of $\epsilon(p)$ as $p \to 0$.

Furthermore, it is presupposed that the pressure p, as a function on $\mathbb{R}^3$, is at least continuous everywhere. Let $\mathcal{I} := \{x \in \mathbb{R}^3 \,|\, p(x) > 0\}$ be the interior of the star[2], $\mathcal{S} := \partial\mathcal{I}$ (the boundary of $\mathcal{I}$, i.e.) the star's surface, and $\mathcal{E} := \mathbb{R}^3 \setminus (\mathcal{I} \cup \mathcal{S})$ the exterior of the star. Then Euler's Eq. (2) can be integrated in $\mathcal{I}$: To this end, let $\Gamma \subset \mathcal{I}$ be any C^1-path from a point x_S in $\mathcal{S}$ to a point x in $\mathcal{I}$. According to the assumptions, $p > 0$ and $\epsilon = \epsilon(p) > 0$ in $\mathcal{I}$, and $p = 0$ on $\mathcal{S}$. Therefore

$$F(p(x)) = \int_\Gamma \langle \epsilon^{-1} \nabla p, ds\rangle = -\int_\Gamma \langle \nabla U, ds\rangle = U(x_S) - U(x) \tag{5}$$

if U is at least C^1 on $\mathcal{I} \cup \mathcal{S}$. This equation has immediate consequences:

1. U is constant on every connected component of $\mathcal{S}$ (since F, p, and U are continuous, and $F(0) = 0$). Carleman [6] (for incompressible matter) and Lichtenstein [14] (for the general case) proved that $\mathcal{S}$ consists of only one component. Thus, let $U_S := U(x_S)$ be the gravitational potential on the star's surface $\mathcal{S}$.

2. $U(x) < U_S$ for all $x \in \mathcal{I}$, since $F(p) > 0$ for $p > 0$.

3. Since $F'(p)$ exists almost everywhere (a.e.), and $F'(p) = \frac{1}{\epsilon(p)} > 0$ a.e. (due to the assumptions), the function $[0, p_{\max}] \ni p \mapsto F(p) \in \mathbb{R}_0^+$ is invertible. Thus, for all $x \in \mathcal{I} \cup \mathcal{S}$
$$p(x) = F^{-1}(U_S - U(x))\ , \tag{6}$$
i.e. in the interior of the star and on the star's surface the pressure can be expressed in terms of the gravitational potential.

[2]Note that $\mathcal{I} = p^{-1}(\mathbb{R}^+)$ is an open subset of $\mathbb{R}^3$, for p is C^0.

For the following it is convenient to use

$$u(x) := U_S - U(x) \tag{7}$$

as the basic potential[3]. Assuming at present that $U(x) > U_S$ for all $x \in \mathcal{E}$ (this will be shown below, see Corollary 2) relation (6) can be extended to all $x \in \mathcal{I} \cup \mathcal{S} \cup \mathcal{E} = \mathbb{R}^3$:

$$p(x) = F^{-1}(u(x)_+) \tag{8}$$

where the abbreviation $u_+ := \sup\{u, 0\}$ is used for the restriction of u to its positive part. Therefore, introducing the function

$$[0, F(p_{\max}) =: u_{\max}] \ni u \mapsto \boxed{\mu(u) := \epsilon(F^{-1}(u))} \in [0, \epsilon(p_{\max}) =: \epsilon_{\max}]\ , \tag{9}$$

the three basic Eqs. (1)–(3) can be condensed into the single equation

$$\Delta u = -4\pi\, \mu(u_+) \tag{10}$$

on $\mathbb{R}^3$. Again, Carleman [6] and Lichtenstein [14] proved that every bounded (i.e. physically relevant) solution of this equation is necessarily *spherically symmetric*[4]. Remembering that in Eq. (10) one integration constant is free (corresponding to U_S) it is convenient to fix this constant by demanding that u takes a given value $0 < u_c \leq u_{\max}$ at the center of symmetry. According to Eq. (8), this is equivalent to fixing the pressure at the star's center:

$$p_c \stackrel{!}{=} F^{-1}(u_c) \Leftrightarrow \boxed{u_c \stackrel{!}{=} F(p_c)}\,. \tag{11}$$

It is shown below (see Corollary 2) that $p_c = p_{\max}$.

Convention: In the following, the center of symmetry is always chosen as the origin of the particular coordinate system.

Due to its scaling property Eq. (10) can be transformed into a "standard form". To this end let $a > 0$ and $x = a\,\xi$. Then in the ξ-coordinates Eq. (10) reads:

$$\Delta_\xi \left(\frac{u(\xi)}{u_c}\right) = -4\pi a^2\, \frac{\mu(u_c)}{u_c} \cdot \frac{\mu(u_c \left(\frac{u(\xi)}{u_c}\right)_+)}{\mu(u_c)}\ . \tag{12}$$

Therefore, let $a := (\frac{u_c}{4\pi\,\mu(u_c)})^{1/2}$, $\tilde{u} := u/u_c$, and $\tilde{\mu}(\tilde{u}) := \mu(u_c\tilde{u})/\mu(u_c)$. Then the problem takes the following form:

Given: An increasing function $\tilde{u} \mapsto \tilde{\mu}(\tilde{u})$ with $\tilde{\mu}(0) = 0$ and $\tilde{\mu}(1) = 1$.

[3]Note that only such a difference of U has a physical invariant meaning, for if U is a solution of Eqs. (1) and (2), so is $U + c$ ($c \in \mathbb{R}$). Usually, c is fixed by demanding that $\lim_{|x|\to\infty} U(x) = 0$.

[4]If $u \leq 0$, this is a consequence of Liouville's theorem.

Wanted: A bounded function $\tilde{u}$ on $\mathbb{R}^3$ satisfying $\tilde{u}(0) = \tilde{u}_c = 1$ and

$$\Delta\tilde{u} = -\tilde{\mu}(\tilde{u}_+) \; . \tag{13}$$

Remark 2. From the mathematical point of view it is not necessary in the following that the function $\tilde{\mu}(\tilde{u})$ corresponds to an EOS according to Eq. (9).

Convention: For simplicity, the symbols "˜" on u and μ are omitted in the following.

Remark 3. Considering that a solution for the above problem is necessarily spherically symmetric Eq.(13) reads in spherical coordinates:

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{du(r)}{dr}\right) = u''(r) + \frac{2}{r}u'(r) = -\mu(u(r)_+) \; . \tag{14}$$

For a polytropic EOS, i.e. $\mu(u) = u^\nu$ with $\nu > 0$ (see Lemma 5 below), this is the so-called *Lane-Emden equation*[5] (see [8], [7]).

With Poisson's integral, Eq. (13) is equivalent (at least in the distributional sense; see e.g. [15], Theorem 6.21) to

$$u(\xi) = 1 - \frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{\mu(u(y)_+)}{|y|}\,dy + \frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{\mu(u(y)_+)}{|\xi - y|}\,dy \; , \quad \forall\xi\in\mathbb{R}^3 \; . \tag{15}$$

Using the spherical symmetry of u and the abbreviation $u(r)$ instead of $u(r\,\xi/|\xi|)$ for any $\xi \neq 0$ in $\mathbb{R}^3$ and $r \geq 0$, a straightforward computation of the integrals in Eq. (15) yields the following (generally) *nonlinear integral equation of Volterra type*:

$$\boxed{u(r) = 1 - \int_0^r g(r,s)\,\mu(u(s)_+)\,ds =: (\mathcal{T}u)(r)} \; , \tag{16}$$

for all $r := |\xi| \geq 0$, where

$$\boxed{g(r,s) := s\left(1 - \frac{s}{r}\right)} \tag{17}$$

(for $r = 0$ see Lemma 1 below). Note that $0 \leq g(r,s) < s$ for all $0 < s \leq r$ and g is not symmetric.

In summary, it has been demonstrated that the *global* problem of a static star in Newtonian gravity is equivalent to the *fixed point problem* (16) for a 1-dimensional real-valued function u. The rest of this article is devoted to the investigation of this fixed point problem and its solutions. This investigation will be similar to the treatment of Picard and Lindelöf of the initial value problem for a non-singular ordinary differential equation of first order.

[5] Usually, this equation is considered only in the interior and it is understood that the interior solution is matched to an exterior solution where the interior solution vanishes, i.e. the "+"-subscript for u on the right hand side of the equation is omitted.

Remark 4. A similar integral equation approach to static Newtonian stellar models is taken in the work of Bucerius [3]–[5]. In contrast to this article, where the pressure at the center of symmetry, p_c, is fixed, Bucerius fixes the stellar radius. However, if the stellar radius is fixed the solutions are not unique in general as the linear case $\mu(u) = u$ shows: In this case the solutions are given in the interior $\mathcal{I} \cong [0, \pi)$ by $u(r) = a\,\frac{\sin(r)}{r}$ where $a > 0$ is arbitrary. Furthermore, only a few explicit EOS are treated to obtain approximate solutions in form of truncated series in terms of eigenfunctions of the corresponding linear problem.

3 Existence, Uniqueness, and Regularity Results

3.1 A Priori Properties

Definition 1. *For $v \in L^\infty(\mathbb{R}_0^+)$ let*

$$\boxed{(\mathcal{Q}v)(r) := \int_0^r g(r,s)v(s)\,ds}\,, \quad \forall r \geq 0. \tag{18}$$

Lemma 1 (Properties of $\mathcal{Q}$). *For all $v, w \in L^\infty(\mathbb{R}_0^+)$:*

1. *The mapping $v \mapsto \mathcal{Q}v$ is linear.*
2. *If $v \leq w$ then $\mathcal{Q}v \leq \mathcal{Q}w$.*
3. $(\mathcal{Q}v)(r) = r^2 \int_0^1 g(1,\sigma)v(r\sigma)\,d\sigma$.
4. $\lim_{r \downarrow 0}(\mathcal{Q}v)(r) = 0$.
5. *$r \mapsto (\mathcal{Q}v)(r)$ is (at least) Hölder continuous differentiable on every compact subset $K \subset \mathbb{R}_0^+$, i.e. $\mathcal{Q}v \in C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}_0^+)$ with $\alpha \in (0,1)$, and*

$$(\mathcal{Q}v)'(r) = \int_0^r \left(\frac{s}{r}\right)^2 v(s)\,ds = r\int_0^1 \sigma^2\, v(r\sigma)\,d\sigma\,, \quad \forall r \geq 0\,. \tag{19}$$

   Especially $(\mathcal{Q}v)'(0) = 0$.

6. *Let $\delta > 0$. If $v \in C^0([0,\delta])$ then the second derivative of $\mathcal{Q}v$ at $r = 0$ exists and $(\mathcal{Q}v)''(0) = v(0)/3$. Especially for all $r \in [0,\delta]$:*

$$(\mathcal{Q}v)(r) = \left(\frac{v(0)}{6} + q_v(r)\right) r^2\,,$$

   with $q_v \in C^0([0,\delta])$ and $q_v(0) = 0$.

Proof.

1. Distributive law and linearity of the integral.
2. Since $g(r,s) \geq 0$ for all $0 < s \leq r$, the assertion follows from the analogous property of the integral.
3. Substitute $\sigma = s/r$ in (18).
4. Let $\delta_0 > 0$, and $c_0 := \sup_{s\in[0,\delta_0]} |v(s)|$. By assumption $0 \leq c_0 < \infty$. Then for all $0 < r \leq \delta_0$
$$|(\mathcal{Q}v)(r)| = \left| \int_0^r s\left(1 - \frac{s}{r}\right) v(s)\, ds \right| \leq c_0 \int_0^r s\left(1-\frac{s}{r}\right) ds = c_0 \frac{r^2}{6} \ .$$
Thus $\lim_{r\downarrow 0} (\mathcal{Q}v)(r) = 0$.
5. Since by assumption $(\mathcal{Q}v)(r := |\xi|) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{v(|y|)}{|y|}\,dy - \frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{v(|y|)}{|\xi - y|}\,dy$ for all $\xi \in \mathbb{R}^3$ and $v \in L^\infty(\mathbb{R})$ the regularity statement follows directly from the regularity properties of the Poisson integral (see e.g. [15], Theorem 10.2). Moreover in the previous equation, and thus in Eq. (18), differentiation commutes with the integral sign. Hence for all $r > 0$: $(\mathcal{Q}v)'(r) = r(1 - r/r)v(r) + \int_0^r \frac{d}{dr} s(1-s/r)v(s)\,ds = \int_0^r (s/r)^2 v(s)\,ds = r\int_0^1 \sigma^2 v(r\sigma)\,d\sigma$ where in the last step the substitution $\sigma = s/r$ was used. An analogous argument as for $\lim_{r\downarrow 0} \mathcal{Q}v(r) = 0$ shows that $\lim_{r\downarrow 0}(\mathcal{Q}v)'(r) = 0$.
6. Let $r > 0$. Then
$$\frac{(\mathcal{Q}v)'(r) - (\mathcal{Q}v)'(0)}{r - 0} = \frac{r\int_0^1 \sigma^2 v(r\sigma)\,d\sigma - 0}{r} = \int_0^1 \sigma^2 v(r\sigma)\,d\sigma =: f(r) \ .$$
Note that $f(0) = v(0)/3$. Therefore, to prove the assertion it is sufficient to show that the function f is continuous on $[0,\delta]$. To this end let $\varepsilon > 0$. For v is continuous on the compact interval $[0,\delta]$ (by assumption) it is even uniformly continuous, i.e. $\exists \eta_\varepsilon > 0$ such that $|v(r_1) - v(r_2)| < \varepsilon$ for all $r_1, r_2 \in [0,\delta]$ with $|r_1 - r_2| < \eta_\varepsilon$. Since $r_1\sigma, r_2\sigma \in [0,\delta]$ and $|r_1\sigma - r_2\sigma| \leq |r_1 - r_2| < \eta_\varepsilon$ for all $\sigma \in [0,1]$, it follows that $|f(r_1) - f(r_2)| \leq \int_0^1 \sigma^2 |v(r_1\sigma) - v(r_2\sigma)| d\sigma < \varepsilon \int_0^1 \sigma^2 d\sigma = \varepsilon/3 < \varepsilon$, i.e. f is continuous on $[0,\delta]$. □

Corollary 1. *Every solution u of the fixed point problem (16) has the following properties:*

1. $\lim_{r\downarrow 0} u(r) = 1$.
2. $u \in C^{1,\alpha}_{\text{loc}}(\mathbb{R})$ *with* $\alpha \in (0,1)$ *and*
$$u'(r) = -\int_0^r \left(\frac{s}{r}\right)^2 \mu(u(s)_+)\, ds \ , \quad \forall r \geq 0 \ . \tag{20}$$
Especially $\boxed{u'(0) = 0}$ *and if* $\mu \not\equiv 0$ *a.e. then* $\boxed{u'(r) < 0 \text{ for all } r > 0}$.

3. *Let* $\mu_1 := \lim_{u\uparrow 1}\mu(u)$*. Then* $u(r) = 1 - \frac{\mu_1}{6}r^2 + o(r^2)$ *as* $r \downarrow 0$.

Proof. Since $u = \mathcal{T}u = 1 - \mathcal{Q}\mu(u_+)$, $0 \le \mu \le 1$, and the function $u \mapsto \mu(u)$ is increasing by assumption, the assertions are immediate consequences of Lemma 1. Note: If $\mu \not\equiv 0$ a.e. then $\mu(u_+) > 0$ at least on a neighbourhood of $r = 0$ (since u is continuous and $u(0) = 1$.)

Corollary 2. *Let* u *be a solution of the fixed point problem (16) and* $\mu \not\equiv 0$ *a.e.. Then:*

1. *The functions* u, $p = F^{-1}(u_+)$, *and* $\epsilon = \mu(u_+)$ *take their maximal values only at* $r = 0$*, i.e. at the center of symmetry.*
2. *There is at the most one* $0 < r_S \le \infty$ *with* $u(r_S) = 0$*. If such an* r_S *exists it is the stellar radius (otherwise let* $r_S := \infty$*). If* $r_S < \infty$ *then* $u'(r_S) < 0$.
3. *If* $r_S < \infty$*, the following holds for the gravitational potential* U *(corresponding to* u*):*

$$U(x) \left\{ \begin{array}{c} < \\ = \\ > \end{array} \right\} U_S\ , \quad \textit{for all}\ x \in \left\{ \begin{array}{c} \mathcal{I} \\ \mathcal{S} \\ \mathcal{E} \end{array} \right\} . \tag{21}$$

4. $|u|$ *is bounded.*

Proof. 1.–3. are immediate consequences of $u'(r) < 0$ for all $r > 0$. To 4.: $u \le 1$ by 1. If $r_S = \infty$ then $u \ge 0$. If $r_S < \infty$ then[6] $\mu(u(r)_+) \le 1_{[0,r_S]}(r)$ by assumption. Hence $u(r) \ge 1 - \int_0^{r_S} s(1 - s/r)\,ds \ge 1 - \int_0^{r_S} s\,ds = 1 - r_S^2/2 > -\infty$ for all $r > 0$.

3.2 Existence, Uniqueness, and Regularity for Lipschitz Continuous μ

Lemma 2. *For all* $v, w \in \mathbb{R}$: $|v_+ - w_+| \le |v - w|$.

Proof. Since $v_+ = \frac{1}{2}(v + |v|)$, it follows that $|v_+ - w_+| = \frac{1}{2}|(v + |v|) - (w + |w|)| = \frac{1}{2}|(v - w) + (|v| - |w|)| \le \frac{1}{2}(|v - w| + ||v| - |w||) \le \frac{1}{2}(|v - w| + |v - w|)| = |v - w|$.

Lemma 3. *Let* $0 < r_0 \le \infty$, $c, j \in \mathbb{R}_0^+$*, and* $\mathbb{R}_0^+ \ni r \mapsto \hat{\mu}(r) := \begin{cases} c\,r^j & : r < r_0 \\ 0 & : r \ge r_0 \end{cases}$.
Then

$$0 \le (\mathcal{Q}\hat{\mu})(r) = \begin{cases} \frac{c}{(j+2)(j+3)}\, r^{j+2} & : r \in [0, r_0) \\ \le \frac{c}{j+2}\, r_0^{j+2} & : r \in \mathbb{R}_0^+ \ . \end{cases} \tag{22}$$

Proof. Let $r_0 < \infty$. Evaluating the integral in Eq. (18) yields

$$(\mathcal{Q}\hat{\mu})(r) \quad = \quad \begin{cases} \frac{c}{(j+2)(j+3)}\, r^{j+2} & : 0 \le r \le r_0 \\ \frac{c}{(j+2)(j+3)}\, r_0^{j+2} + \frac{c}{j+3}\, r_0^{j+3}\left(\frac{1}{r_0} - \frac{1}{r}\right) & : r \ge r_0\ . \end{cases}$$

[6] Let $1_X(x) := \{1 \text{ if } x \in X,\ 0 \text{ if } x \notin X\}$ be the characteristic function of a set X.

Since $[r_0,\infty) \ni r \mapsto \frac{c}{(j+2)(j+3)}\, r_0^{j+2} + \frac{c}{j+3}\, r_0^{j+3}\left(\frac{1}{r_0} - \frac{1}{r}\right)$ is increasing and tends to $\frac{c}{j+2}\, r_0^{j+2}$ as $r \to \infty$ the assertion follows. Eq. (22) still holds if $r_0 = \infty$. □

Lemma 4 (Contraction). *Assume that:*

1. *The function μ is Lipschitz continuous on $[0,1]$: $\mu \in C^{0,1}([0,1])$, i.e. $\exists \ell \in \mathbb{R}_0^+$ such that $|\mu(v) - \mu(w)| \le \ell |v-w|$ for all $v, w \in [0,1]$.*
2. *$\exists \mathbf{X} \subset \mathbf{B}_R := \left(C^0([0,R)), \|.\|_R := \sup_{r\in[0,R)} |.|\right)$ with $R > 0$ and $\mathcal{T}(\mathbf{X}) \subset \mathbf{X}$.*

Then for all $v, w \in \mathbf{X}$:

$$\|\mathcal{T}^n(v) - \mathcal{T}^n(w)\|_R \le \frac{\left(\sqrt{\ell}\, R =: C(\ell, R)\right)^{2n}}{(2n+1)!} \|v - w\|_R \,. \tag{23}$$

If in addition $R = \infty$ and

$$r_* := \sup\left\{ r_v := \sup\{r \in \mathbb{R}_0^+ \mid v(r) \ge 0\} \;\middle|\; v \in \mathbf{X} \right\} < \infty$$

then for all $v, w \in \mathbf{X}$:

$$\|\mathcal{T}^n(v) - \mathcal{T}^n(w)\|_\infty \le \frac{C(\ell, r_*)^{2n}}{(2n)!} \|v - w\|_\infty \,. \tag{24}$$

Therefore, $\exists n_0 \in \mathbb{N}$ such that $\mathcal{T}^{n_0}$ is a contraction on $\mathbf{X}$.

Proof. If $R < \infty$, let $r_* := R$. To prove estimates (23) and (24) it is sufficient to establish the inequality

$$|(\mathcal{T}^n v)(r) - (\mathcal{T}^n w)(r)| \le \begin{cases} \frac{\ell^n}{(2n+1)!}\, \|v - w\|\, r^{2n} & : r \in [0, r_*) \\ \frac{\left(\sqrt{\ell}\, r_*\right)^{2n}}{(2n)!} \|v - w\| & : r \in [0, R)\,, \end{cases} \tag{25}$$

where the index R of $\|.\|_R$ is omitted for simplicity. By induction:

1. $n = 0$: Inequality (25) is true.
2. Step $n \to n+1$: Assume that (25) is valid for an n, then

$$\begin{aligned}
|\mathcal{T}^{n+1}(v) - \mathcal{T}^{n+1}(w)| &= |\mathcal{T}(\mathcal{T}^n v) - \mathcal{T}(\mathcal{T}^n w)| \\
\text{by (16), (18)} &= |\mathcal{Q}\mu((\mathcal{T}^n v)_+) - \mathcal{Q}\mu((\mathcal{T}^n w)_+)| \\
\text{by Lemma 1(1.)} &= \left|\mathcal{Q}\Big(\mu((\mathcal{T}^n v)_+) - \mu((\mathcal{T}^n w)_+)\Big)\right| \\
\text{by Lemma 1(2.)} &\le \mathcal{Q}\,|\mu((\mathcal{T}^n v)_+) - \mu((\mathcal{T}^n w)_+)| \\
\mu \text{ Lipschitz}^7 &\le \mathcal{Q}\left(\ell\,|(\mathcal{T}^n v)_+ - (\mathcal{T}^n w)_+|\right) \\
\text{by Lemma 2} &\le \mathcal{Q}\left(\ell\,|\mathcal{T}^n v - \mathcal{T}^n w|\right) \\
\text{by (25)} &\le \mathcal{Q}\hat{\mu}\,,
\end{aligned}$$

[7] Note that $(\mathcal{T}^n v)(r)_+, (\mathcal{T}^n w)(r)_+ \in [0,1]$, since $\mathcal{T}(\mathbf{X}) \subset \mathbf{X}$ by assumption.

where $\hat{\mu}(r) = \begin{cases} c\,r^j & : r < r_0 \\ 0 & : r \geq r_0 \end{cases}$, with $c = \ell\,\frac{\ell^n}{(2n+1)!}\,\|v-w\|$, $j = 2n$, and $r_0 = r_*$. Hence, with (22):

$$|(\mathcal{T}^{n+1}v)(r) - (\mathcal{T}^{n+1}w)(r)| \leq$$

$$\begin{cases} \dfrac{\ell\,\frac{\ell^n}{(2n+1)!}\,\|v-w\|}{(2n+2)(2n+3)}\,r^{2n+2} = \dfrac{\ell^{(n+1)}}{(2(n+1)+1)!}\,\|v-w\|\,r^{2(n+1)} & : r \in [0, r_*) \\ \dfrac{\ell\,\frac{\ell^n}{(2n+1)!}\,\|v-w\|}{2n+2}\,r_*^{2n+2} = \dfrac{\left(\sqrt{\ell}\,r_*\right)^{2(n+1)}}{(2(n+1))!}\,\|v-w\| & : r \in [0, R)\,, \end{cases}$$

i.e. inequality (25) is valid for $n+1$. □

Remark 5. In Lemma 4, μ is assumed to be continuous in particular. Since $\mu(0) = 0$ and $\mu(1) = 1$ by assumption, $\lim_{v\downarrow 0}\mu(v) = 0$ and $\lim_{v\uparrow 0}\mu(v) = 1$. Note that it is not necessary that μ is increasing.

Proposition 1 (Existence, uniqueness, regularity).
Assume $\mu \in C^{0,1}([0,1])$. Then the fixed point problem (16)*, $u = \mathcal{T}u$, has a unique solution $u \in C^{2,\alpha}(\mathbb{R}_0^+)$ with $\alpha < 1$. If in addition the trivially extended*[8] *function $\mu \in C^{k,\alpha}((-\infty,1])$ with $k \geq 1$ then $u \in C^{k+2,\alpha}(\mathbb{R}_0^+)$.*

Proof.

1. Let $R > 0$ and $v \in \mathbf{X}_R := \{w \in C^0([0,R)) \mid w \leq 1\}$. Then $v(r)_+ \in [0,1]$ for all $r \in [0,R)$. Thus $\mu(v_+)$ is defined, and $\mu(v_+) \geq 0$. Hence $\mathcal{Q}\mu(v_+) \geq 0 \Rightarrow \mathcal{T}v = 1 - \mathcal{Q}\mu(v_+) \leq 1$, i.e. $\mathcal{T}v \in \mathbf{X}_R$. Therefore by Lemma 4, there is an n_0 such that $\mathcal{T}^{n_0}$ is a contraction on $\mathbf{X}_R$ which is a subset of the Banach space $\mathbf{B}_R := (C^0([0,R)), \|.\|_R := \sup_{r\in[0,R)}|.|)$. By virtue of the Banach fixed point theorem (see e.g. [13], Theorem 5.1-2 & Lemma 5.4-3) $v = \mathcal{T}v$ has a (unique) fixed point $u_R \in \mathbf{X}_R$ and $\lim_{n\to\infty}\mathcal{T}^n u_0 = u_R$ for every $u_0 \in \mathbf{X}_R$.

2. Let $0 < R_1 \leq R_2$ and u_{R_1}, u_{R_2} be the corresponding solutions, which exist by 1. Then $u_{R_1} \equiv u_{R_2}$ on $[0,R_1)\cap[0,R_2) = [0,R_1)$: By assumption, $\mathcal{T}^n u_{R_1} \equiv u_{R_1}$ and $u_{R_2} \equiv \mathcal{T}^n u_{R_2}$ for all $n \in \mathbb{N}$. Thus, due to the estimate (23) it follows that $\|u_{R_1} - u_{R_2}\|_{R_1} = \|\mathcal{T}^n u_{R_1} - \mathcal{T}^n u_{R_2}\|_{R_1} \leq C_n\|u_{R_1} - u_{R_2}\|_{R_1}$ with $\lim_{n\to\infty} C_n = 0$. Hence $\|u_{R_1} - u_{R_2}\|_{R_1} = 0 \Leftrightarrow u_{R_1} \equiv u_{R_2}$ on $[0,R_1)$.

3. Since $R_1 > 0$ is arbitrary in 2., there is a unique fixed point $u \in C^0(\mathbb{R}_0^+)$.

4. By Corollary 1, $u \in C^1(\mathbb{R}_0^+)$ and $u \leq 1$. Since $\mu \in C^{0,1}([0,1])$ and $\mu(0) = 0$ it follows that $\mu(u_+) \in C^{0,1}(\mathbb{R}_0^+) \subset C^{0,\alpha}(\mathbb{R}_0^+)$ with $\alpha < 1$. Therefore, due to the regularity properties of the Poisson integral $u \in C^{2,\alpha}(\mathbb{R}_0^+)$ (cf. Corollary 1, and see e.g. [15], Theorem 10.3). If $k \geq 1$, the assertion follows by induction using the above argument again. □

[8] I.e. $\mu(v) = 0$ for all $v < 0$.

Remark 6.

1. u is bounded by Corollary 2, i.e. $\|u\|_\infty < \infty$.
2. Due to the regularity properties, differentiation commutes with the integral sign for the solution u of the fixed point problem (16). Therefore, u is a classical solution of Eq. (14).
3. If $\mu \equiv 0$ a.e. then $u \equiv 1$ is the unique solution of the fixed point problem (16).
4. In case that $r_* < \infty$ (cf. Lemma 4), only the analogue of steps 1. and 4. is needed in the above proof. Furthermore, then the stellar radius r_S is finite. For more details on the question whether the stellar radius is finite or not see Sect. 4 below.

Corollary 3 (A priori error estimates). *Assumptions as in Proposition 1. Then for every* $1 \geq u_0 \in C^0(\mathbb{R}_0^+)$ *the sequence* $u_i := \mathcal{T} u_{i-1}$, $\forall i \in \mathbb{N}$, *converges uniformly to the unique fixed point* u *on every intervall* $[0, R > 0)$ *and*

$$\boxed{\|u - u_n\|_R \leq \cosh C(\ell, R) \cdot \frac{C(\ell, R)^{2n}}{(2n+1)!} \cdot \|u_1 - u_0\|_R} \tag{26}$$

with $C(\ell, R) = \sqrt{\ell} R$ *(where* ℓ *is the Lipschitz constant of* μ*). If* $r_* < \infty$ *then*

$$\|u - u_n\|_\infty \leq \cosh C(\ell, r_*) \cdot \frac{C(\ell, r_*)^{2n}}{(2n)!} \cdot \|u_1 - u_0\|_\infty \,. \tag{27}$$

Proof. Using the triangle inequality it follows that for all $n, m \in \mathbb{N}$:

$$\begin{aligned} \|\mathcal{T}^{n+m} u_0 - \mathcal{T}^n u_0\|_R &\leq \sum_{k=0}^{m-1} \|\mathcal{T}^{n+k+1} u_0 - \mathcal{T}^{n+k} u_0\|_R \\ \text{by (23)} &\leq \sum_{k=0}^{m-1} \frac{C(\ell, R)^{2(n+k)}}{(2(n+k)+1)!} \|u_1 - u_0\|_R \\ &\leq \frac{C(\ell, R)^{2n}}{(2n+1)!} \|u_1 - u_0\|_R \cdot \sum_{k=0}^{\infty} \frac{C(\ell, R)^{2k}}{(2k)!} \,. \end{aligned}$$

Since $\lim_{m\to\infty} \mathcal{T}^{n+m} u_0 = u$ by Banach's fixed point principle and $\sum_{k=0}^\infty \frac{C(\ell,R)^{2k}}{(2k)!} = \cosh C(\ell, R)$ the first a priori estimate follows. If $r_* < \infty$, the argument is analogous.

Remark 7. Note that the estimates (26) and (27) imply that the convergence is even faster than exponential.

3.3 "Characteristic" Examples of EOS

A physically important class of EOS are the polytropes. In the literature, nearly exclusively this class has been used for static stellar models in Newtonian gravity (see e.g. [8], [7], [11]).

Lemma 5 (Polytropic EOS). *Given a polytropic equation of state:*

$$p(\epsilon) = k\,\epsilon^{\gamma} \iff \epsilon(p) = \left(\frac{p}{k}\right)^{\frac{1}{\gamma}} , \tag{28}$$

with $k > 0$ and $\gamma > 1$. Then

$$\mu(u) = \kappa\, u^{\nu} \text{ , i.e. } \boxed{\tilde{\mu}(\tilde{u}) = \tilde{u}^{\nu}} , \tag{29}$$

where $\nu := \frac{1}{\gamma-1} > 0$ ($\Leftrightarrow \gamma = 1 + \frac{1}{\nu}$) is the index[9] of the polytropic EOS, and

$$\kappa = (\nu\gamma\, k)^{-\nu} \iff k = \frac{1}{\nu\gamma\,\kappa^{(\gamma-1)}} . \tag{30}$$

Proof. By definition (4):

$$F(p) = \int_0^p \epsilon(\tilde{p})^{-1} d\tilde{p} = \tfrac{k^{1/\gamma}}{(1-\frac{1}{\gamma})}\, p^{1-\frac{1}{\gamma}} = \nu\gamma\, k^{1/\gamma}\, p^{1/(\nu\gamma)} .$$

The integral exists iff $1 - \frac{1}{\gamma} > 0 \Leftrightarrow \gamma > 1$, and the inverse of the function F is given by

$$u = F(p) = \nu\gamma\, k^{1/\gamma}\, p^{1/(\nu\gamma)} \iff p = \frac{u^{\nu\gamma}}{(\nu\gamma)^{\nu\gamma}\, k^{\nu}} = F^{-1}(u) .$$

Since $\mu(u) := (\epsilon \circ F^{-1})(u)$ (by definition (9)) it follows that

$$\mu(u) = \left(\frac{F^{-1}(u)}{k}\right)^{\frac{1}{\gamma}} = \frac{u^{\nu}}{(\nu\gamma)^{\nu}\, k^{\frac{\nu+1}{\gamma}}} = \frac{1}{(\nu\gamma\, k)^{\nu}}\, u^{\nu} ,$$

and $\tilde{\mu}(\tilde{u}) = \kappa u^{\nu}/\kappa u_c^{\nu} = (u/u_c)^{\nu} = \tilde{u}^{\nu}$. □

Therefore, Proposition 1 applies to a polytropic EOS only if the index $\nu \geq 1$ since for $0 \leq \nu < 1$ the corresponding function $[0,1] \ni u \mapsto \mu(u) = u^{\nu}$ is *no more* Lipschitz continuous at $u = 0$, i.e. at the star's surface[10]. To include this case $\nu < 1$, and furthermore, to permit EOS with phase transitions, Proposition 1 must be generalized. This generalization will be developed in Sect. 3.4. It is helpful to have an idealized EOS modeling matter with N phases:

[9] In the literature, the notation N or n is frequently used for the index of a polytropic EOS, instead of ν.

[10] It is known, that this surface exists if $\nu \in [0, 5)$; see e.g. [7], [11].

Lemma 6 (Step function EOS). *Given an EOS being a step function:*

$$\epsilon(p) = \{\epsilon_i \text{ , if } p \in (p_{i-1}, p_i), \text{ and } i = 1, \dots, N\} , \tag{31}$$

with $N \geq 1$, $0 < \epsilon_1 \leq \dots \leq \epsilon_N$, *and* $0 =: p_0 < p_1 < \dots < p_N =: p_{\max}$. *Then*

$$\mu(u) = \{\epsilon_i \text{ , if } u \in (u_{i-1}, u_i), \text{ and } i = 1, \dots, N\} , \tag{32}$$

with $u_0 = 0$, and $u_i = \sum_{j=1}^{i} \frac{p_j - p_{j-1}}{\epsilon_j}$ for $i = 1, \dots, N$. Note that the values of ϵ at p_i, and μ at u_i, are irrelevant for the fixed point problem (16).

Proof. Straightforward, following the proof of the preceding Lemma.

3.4 More General μ: Existence, Uniqueness, and Regularity

Definition 2 (Admissible μ). *In the following* $\mu : [0,1] \to [0,1]$ *is called admissible iff*

1. $\mu(0) = 0$ *and* $\mu(1) = 1$.
2. μ *is an increasing function.*
3. *Extend* μ *by* $\mu(v) := 0$ *for all* $v < 0$. *Then, for every* $v \leq 1$ *there are constants* $\delta_v(\mu) > 0$, $\ell_v(\mu) \geq 0$, $\alpha_v(\mu) \in (0,1]$ *such that for all* w, z *in the open interval* $(v - \delta_v(\mu), v)$ *the inequality*

$$\boxed{|\mu(w) - \mu(z)| \leq \ell_v(\mu) |(v-w)^{\alpha_v(\mu)} - (v-z)^{\alpha_v(\mu)}|} \tag{33}$$

   holds.
4. $D := \{v \leq 1 \mid \mu$ *is not continuous in* $v\} \subset [0,1]$ *is a set of measure zero.*

Since μ *is increasing the left- and right-hand limit exist for all* $v \leq 1$*:*

$$0 \leq \lim_{w \uparrow v} \mu(w) =: \mu^-(v) \leq \mu(v) \leq \mu^+(v) := \lim_{w \downarrow v} \mu(w) \leq 1 \; .$$

Especially

$$\lim_{v \downarrow 0} \mu(v) =: \mu_0 \geq 0 \quad \textit{and} \quad \lim_{v \uparrow 1} \mu(v) =: \mu_1 \leq 1 \; .$$

Remark 8.

1. With the trivial extension of μ for negative values the index "+" (for the positive part of a function) can be omitted in the definition of $\mathcal{T}$ (cf. (16)). Furthermore, for all $v \leq 0$ condition (33) is trivial. (This extension is only introduced to simplify some of the following proofs.)
2. The functions μ corresponding to a polytropic (especially for $\nu \in [0,1)$) or a step function[11] EOS are admissible.
3. If μ corresponds to an EOS then $\mu(v) > 0$ for all $v \in (0,1]$.

[11]Note that if μ is a step function, then $\ell_v(\mu) = 0$, for all $v \leq 1$.

Lemma 7. *Assume that $\mu \not\equiv 0$ a.e. is admissible. Then:*

1. $w \le v \le 1 \Rightarrow \mathcal{T}v \le \mathcal{T}w \le 1$.
2. *Let $u_0 :\equiv 1$ and $u_i := \mathcal{T}u_{i-1}$ for all $i \in \mathbb{N}$. Then:*
   (a) *For all $i \in \mathbb{N}$: $u_i \in C^{1,\alpha<1}_{\text{loc}}(\mathbb{R}^+_0)$ and $u_i'(r) < 0$ for all $r > 0$.*
   (b) *For all $i \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$:*

$$u_{2i+1} \le u_{2i+3} \le u_{2i+2} \le u_{2i} \le 1 \; . \tag{34}$$

   Especially, the subsequence $\{u_{2i}\}_{i\in\mathbb{N}_0}$ is decreasing and the subsequence $\{u_{2i+1}\}_{i\in\mathbb{N}_0}$ is increasing. Therefore, both subsequences converge (at least pointwise on $\mathbb{R}^+_0$):

$$\lim_{i\to\infty} u_{2i+1} =: \underline{u} \;\le\; \bar{u} := \lim_{i\to\infty} u_{2i} \le 1 \; .$$

   (c) *$\underline{u} = \mathcal{T}\bar{u}$ and $\bar{u} = \mathcal{T}\underline{u}$, i.e. $\underline{u}$ and $\bar{u}$ are fixed points of $\mathcal{T}^2$.*
   (d) *If v is a fixed point of $\mathcal{T}^2$ then $v \in C^{1,\alpha<1}_{\text{loc}}(\mathbb{R}^+_0)$ and $v(r)' < 0$ for all $r > 0$; i.e. $\underline{u}, \bar{u} \in C^{1,\alpha<1}_{\text{loc}}(\mathbb{R}^+_0)$ and $\underline{u}'(r), \bar{u}(r)' < 0$ for all $r > 0$.*

Proof.

1. If $w(r) \le v(r) \le 1$ for all $r \in \mathbb{R}^+_0$ then $0 \le \mu(w(r)) \le \mu(v(r)) \le 1$ for all $r \in \mathbb{R}^+_0$ since $0 \le \mu \le 1$ is an increasing function. Thus by Lemma 1(2.), $1 \ge \mathcal{T}w = 1 - \mathcal{Q}\mu(w) \ge 1 - \mathcal{Q}\mu(v) = \mathcal{T}v$.
2. (a) Since $0 \le \mu \le 1$ and $\mu \not\equiv 0$ a.e., the assertion is an immediate consequence of Lemma 1(5.).
   (b) Since $0 \le u_0 \equiv 1$, it follows by 1. that $u_1 = \mathcal{T}u_0 \le \mathcal{T}0 = 1 = u_0$. Then again by 1., $u_0 = 1 \ge u_2 = \mathcal{T}u_1 \ge \mathcal{T}u_0 = u_1$, $u_1 = \mathcal{T}u_0 \le \mathcal{T}u_2 = u_3$, and $u_3 = \mathcal{T}u_2 \le \mathcal{T}u_1 = u_2$. Therefore $u_1 \le u_3 \le u_2 \le u_0$. By induction the assertion follows.
   (c) For all $r > 0$ the sequence $\{v_i(.) := g(r,.)\mu(u_{2i}(.))\}_{i\in\mathbb{N}_0}$ of functions on $(0,r)$ is bounded: $0 \le v_i \le r$. Let $D := \{v \le 1 \mid \mu \text{ is not continuous in } v\} \subset [0,1]$. By assumption D is a set of measure zero. Since $u_{2i}' < 0$ on $(0,r)$ for all $i \in \mathbb{N}$, the set $\tilde{D} := \bigcup_{i\in\mathbb{N}} u_{2i}^{-1}(D) \subset \mathbb{R}^+_0$ has measure zero and v_i is continuous on $\mathbb{R}^+_0 \setminus \tilde{D}$ for all $i \in \mathbb{N}$. Therefore, $\lim_{i\to\infty} v_i(.) = \lim_{i\to\infty} g(r,.)\mu(u_{2i}(.)) = g(r,.)\mu(\bar{u}(.))$ a.e. on $(0,r)$. By Lebesgue's dominated convergence theorem (see e.g. [15], Theorem 1.8) limit and integration sign commute. Thus for all $r > 0$:

$$\begin{aligned}\underline{u}(r) &= \lim_{i\to\infty} u_{2i+1}(r) = \lim_{i\to\infty} (\mathcal{T}u_{2i})(r) \\ &= \lim_{i\to\infty} (1 - \int_0^r g(r,s)\mu(u_{2i}(s))\, ds)\end{aligned}$$

$$= \quad 1 - \int_0^r g(r,s)\mu(\bar{u}(s))\, ds$$
$$= \quad (\mathcal{T}\bar{u})(r)\ ,$$

and $\bar{u} = \mathcal{T}\underline{u}$ by analogy. Therefore, $\underline{u} = \mathcal{T}\bar{u} = \mathcal{T}(\mathcal{T}\underline{u}) = \mathcal{T}^2\underline{u}$ and $\bar{u} = \mathcal{T}\underline{u} = \mathcal{T}^2\bar{u}$.

(d) Again an immediate consequence of Lemma 1(5.). □

Remark 9. For all compact intervals $[0, R > 0]$ the sequences of functions $\{u_{2i}\}_{i\in\mathbb{N}}$, $\{u_{2i+1}\}_{i\in\mathbb{N}}$ are subsets of $C^1([0,R])$. These sequences and its derivatives are uniformly bounded since $1-\frac{1}{6}R^2 \le u_1 \le u_i \le 1$ and $|u_i'(r)| = r\int_0^1 \sigma^2\mu(u_{i-1}(r\sigma)_+)\, d\sigma \le r\int_0^1 \sigma^2\, d\sigma = \frac{r}{3} \le \frac{R}{3}$ for all $i \in \mathbb{N}$. Hence these sequences are equicontinuous (see e.g. [12], Theorem 5.19). Thus by the Arzelà-Ascoli theorem (see e.g. [12], Theorem 5.20) and by the monotonicity of both sequences it follows that $u_{2i} \to \bar{u}$ and $u_{2i+1} \to \underline{u}$ *uniformly* on $[0,R]$.

Lemma 8. *Let $0 < R \le \infty$. If $\mathcal{T}^2$ has a unique fixed point on $[0,R)$, so does $\mathcal{T}$ and both fixed points are equal on $[0,R)$.*

Proof. Let $u = \mathcal{T}^2 u$. Then $\mathcal{T}^2(\mathcal{T}u) = \mathcal{T}(\mathcal{T}^2(u)) = \mathcal{T}(u)$. Hence $\mathcal{T}u$ is a fixed point of $\mathcal{T}^2$. Since the fixed point of $\mathcal{T}^2$ is unique $\mathcal{T}u = u$, i.e. u is a fixed point of $\mathcal{T}$. Let $\tilde{u}$ be another fixed point of $\mathcal{T}$. Then $\tilde{u}$ is also a fixed point of $\mathcal{T}^2$. However, this fixed point is unique. Thus $\tilde{u} = u$. □

Remark 10. Note that the converse of Lemma 8 is *not* true in general.

Lemma 9. *Assume that $\mu \not\equiv 0$ a.e. is admissible. Let*

$$X := \mathcal{T}^2(\{\tilde{v} : \mathbb{R}_0^+ \to \mathbb{R} \mid \tilde{v} \le 1\}) \subset C^1(\mathbb{R}_0^+)\ ,$$

and for all $r_0 > 0$, $w \in X$

$$X_{r_0}(w) := \{v \in X \mid v(r) = w(r)\ ,\ \forall r \in [0, r_0]\}\ .$$ [12]

Then there exist injective mappings

$$\begin{array}{rrl} q: & X \to C^0(\mathbb{R}_0^+), & v \mapsto q(v) \\ l_{r_0}^w: & X_{r_0}(w) \to C^0(\mathbb{R}_0^+), & v \mapsto l_{r_0}^w(v) \end{array}$$

such that:

1. $\forall r \in \mathbb{R}_0^+$*:*

$$v(r) \quad = \quad 1 - \Big(\frac{\mu_1}{6} + q(v)(r)\Big) r^2\ , \quad \forall v \in X \tag{35}$$
$$v(r_0 + r) \quad = \quad w(r_0) - \Big(|w'(r_0)| + l_{r_0}^w(v)(r)\Big) r\ , \quad \forall v \in X_{r_0}(w)\ . \tag{36}$$

[12] Note that $X_{r_0}(w)$ is the equivalence class containing w with respect to the equivalence relation $\sim_{r_0}$ on X: $v \sim_{r_0} w :\Leftrightarrow v = w$ on $[0, r_0]$.

2. There are constants $\eta > 0$, $H, L^w_{r_0} \in \mathbb{R}^+$, $\beta \in (0,2]$ such that $\forall r \in [0,\eta]$:

$$\text{(a)} \qquad |q(v)(r)| \leq H\, r^{\beta}\,, \quad \forall v \in X \tag{37}$$

$$\text{(b)} \qquad |l^w_{r_0}(v)(r)| \leq L^w_{r_0}\, r\,, \quad \forall v \in X_{r_0}(w)\,. \tag{38}$$

3. There are constants $\delta_0, \delta^z_{r_0} > 0$, $C_0, C^z_{r_0} \in \mathbb{R}^+_0$, $\alpha_0, \alpha^z_{r_0} \in (0,2]$ such that

(a) $\forall r \in [0,\delta_0]$ and $\forall v, w \in X$:

$$|q(\mathcal{T}v)(r) - q(\mathcal{T}w)(r)| \leq C_0 \int_0^r \frac{1}{s^{1-\alpha_0}} |q(v)(s) - q(w)(s)|\, ds\,. \tag{39}$$

(b) $\forall r \in [0,\delta^z_{r_0}]$, $\forall z \in X$, and $\forall v, w \in X_{r_0}(z)$:

$$|l^{\mathcal{T}z}_{r_0}(\mathcal{T}v)(r) - l^{\mathcal{T}z}_{r_0}(\mathcal{T}w)(r)| \leq C^z_{r_0} \int_0^r \frac{1}{s^{1-\alpha^z_{r_0}}} |l^z_{r_0}(v)(s) - l^z_{r_0}(w)(s)|\, ds\,. \tag{40}$$

Proof.

1. If $v \in X$ then (by Lemma 1) $v \in C^1(\mathbb{R}^+_0)$, $v'(0) = 0$, $v'(r) < 0$ for all $r > 0$, and $v''(0) = \lim_{w\uparrow 1} \mu(w)/3 = \mu_1/3 > 0$. Therefore the mappings q and l exist. These mappings are injective since Eq. (35) resp. (36) can be uniquely solved for q resp. l.

2. (a) Let $v \in X$. By definition there is a $\hat{v} \leq 1$ with $v = \mathcal{T}^2\hat{v}$. Let $\tilde{v} := \mathcal{T}\hat{v}$, then $v = \mathcal{T}\tilde{v}$ and $\tilde{v}$ is bounded by 1 from above, and by $r \mapsto 1 - \frac{\mu_1}{6} r^2$ from below. Let $\eta := \sqrt{\frac{6\delta_1(\mu)}{\mu_1}} > 0$, i.e. $\forall r \in (0,\eta)$: $\tilde{v}(r) \in (1 - \delta_1(\mu), 1)$. Then, it follows that for all $r \in (0,\eta)$:

$$\begin{aligned}
|q(v)(r)| &= r^{-2} \left| v(r) - \left(1 - \frac{\mu_1}{6}\, r^2\right) \right| \\
&= r^{-2} \left| \Big(1 - (\mathcal{Q}\mu(\tilde{v}))(r)\Big) - \Big(1 - (\mathcal{Q}\mu_1)(r)\Big) \right| \\
&\quad \text{(by definition of } \mathcal{T}) \\
&\leq r^{-2} \lim_{\varepsilon\downarrow 0} \left(\mathcal{Q}\, |\mu(\tilde{v}) - \mu(1-\varepsilon)|\right)(r) \\
&\quad \text{(using Lemma 1)} \\
&\leq r^{-2} \lim_{\varepsilon\downarrow 0} \ell_1(\mu) \left(\mathcal{Q} \left| (1-\tilde{v})^{\alpha_1(\mu)} - (1-(1-\varepsilon))^{\alpha_1(\mu)} \right| \right)(r) \\
&\quad \text{(using inequality (33))} \\
&= r^{-2} \ell_1(\mu) \left(\mathcal{Q} \left| (1-\tilde{v})^{\alpha_1(\mu)} \right| \right)(r)
\end{aligned}$$

$$\begin{aligned}
&\leq \; r^{-2}\ell_1(\mu)\int_0^r g(r,s)\left(\frac{\mu_1}{6}\,s^2\right)^{\alpha_1(\mu)} ds \\
&\quad (\text{since } 0 \leq (1-\tilde{v}(s)) \leq \tfrac{\mu_1}{6}\,s^2) \\
&= \; \frac{\ell_1(\mu)\mu_1^{\alpha_1(\mu)}}{6^{\alpha_1(\mu)}(2+2\alpha_1(\mu))(3+2\alpha_1(\mu))}\,r^{2\alpha_1(\mu)} \\
&\leq \; \frac{\ell_1(\mu)\mu_1^{\alpha_1(\mu)}}{6^{1+\alpha_1(\mu)}}\,r^{2\alpha_1(\mu)}\;.
\end{aligned}$$

Therefore, with $H := \frac{\ell_1(\mu)\mu_1^{\alpha_1(\mu)}}{6^{1+\alpha_1(\mu)}} \in \mathbb{R}^+$, and $\beta := 2\alpha_1(\mu) \in (0,2]$ inequality (37) follows. Note that H and β are independent of v.

(b) Let $r_0 > 0$, $w \in X$, and $v \in X_{r_0}(w)$. Then by definition, $v(r) = w(r)$ for all $r \in [0, r_0]$, $v'(r_0) = w'(r_0) < 0$, and $\exists \tilde{w} \leq 1$ such that $w = \mathcal{T}\tilde{w}$. Thus, the function $\mathbb{R}_0^+ \ni r \mapsto v(r+r_0)$ is bounded by $w(r_0)$ from above. Since $\mu(\tilde{w}) \leq \mu_1 \leq 1$ a.e., it is bounded from below by the strictly monotone function

$$\begin{aligned}
r \mapsto h^w_{r_0}(r) \;&:=\; 1-\Big(\mathcal{Q}\left(\mu(\tilde{w})\cdot 1_{[0,r_0)} + \mu_1\cdot 1_{[r_0,\infty)}\right)\Big)(r_0+r) \\
&=\; w(r_0) + w'(r_0)r_0 + \frac{\mu_1}{2}r_0^2 \\
&\quad -w'(r_0)\frac{r_0^2}{r_0+r} - \frac{\mu_1}{3}\frac{r_0^3}{r_0+r} - \frac{\mu_1}{6}(r_0+r)^2\;. \qquad (41)
\end{aligned}$$

Therefore

$$\begin{aligned}
|l^w_{r_0}(v)(r)| \;&=\; \left| r^{-1}\Big(w(r_0) - v(r_0+r)\Big) - |w'(r_0)|\right| \\
&\quad (\text{by } (36)) \\
&\leq\; \left| r^{-1}\Big(w(r_0) - h^w_{r_0}(r)\Big) + w'(r_0)\right| \\
&\quad (\text{since } h^w_{r_0}(r) \leq v(r_0+r) \leq w(r_0), \text{ and } w'(r_0) < 0) \\
&=\; \left|\frac{\mu_1(3r_0+r)+6w'(r_0)}{6(r_0+r)}\right| r =: L^w_{r_0}(r)\,r\;.
\end{aligned}$$

Hence, with $0 \leq L^w_{r_0} := \sup_{r\in[0,\eta]} L^w_{r_0}(r) < \infty$ for $\eta > 0$, inequality (38) follows. Note that $L^w_{r_0}$ is independent of v.

3. (a) Let $v, w \in X$. Then $\mathcal{T}v, \mathcal{T}w \in X$ by definition. Furthermore, the functions v, w, $\mathcal{T}v, \mathcal{T}w$ are bounded by 1 from above and from below by $r \mapsto 1 - \frac{\mu_1}{6}r^2$. Let $\tilde{\delta}_0 := \sqrt{\frac{6\delta_1(\mu)}{\mu_1}} > 0$. Hence $\forall r \in (0, \tilde{\delta}_0)$: $v(r)$, $w(r)$, $(\mathcal{T}v)(r)$, and $(\mathcal{T}w)(r) \in (1-\delta_1(\mu), 1)$.

Therefore, due to the properties of μ it follows $\forall v, w \in X$ and $r \in (0, \tilde{\delta}_0)$:

$$\begin{aligned}
&|q(\mathcal{T}v)(r) - q(\mathcal{T}w)(r)|\, r^2 = \\
&= \quad |(\mathcal{T}v)(r) - (\mathcal{T}w)(r)| \\
&\qquad \text{(by (35))} \\
&= \quad \left| \int_0^r g(r,s)\Big(\mu(v(s)) - \mu(w(s))\Big) ds \right| \\
&\leq \quad \ell_1(\mu) \int_0^r s \left| (1 - v(s))^{\alpha_1(\mu)} - (1 - w(s))^{\alpha_1(\mu)} \right| ds \\
&\qquad (g(r,s) \leq s, \text{ and property (33))} \\
&= \quad \ell_1(\mu) \int_0^r s^{1+2\alpha_1(\mu)} \Big| (\frac{\mu_1}{6} + q(v)(s))^{\alpha_1(\mu)} - \\
&\qquad\qquad - (\frac{\mu_1}{6} + q(w)(s))^{\alpha_1(\mu)} \Big| ds \\
&\qquad \text{(by (35))}.
\end{aligned}$$

Due to estimate (37) and $\frac{\mu_1}{6} > 0$, for every $0 < \lambda < \frac{\mu_1}{6}$ there is a $\delta_0 = \delta_0(\lambda) \in (0, \tilde{\delta}_0]$ (independent of v, w!) such that $\frac{\mu_1}{6} + q(v)(s)$ and $\frac{\mu_1}{6} + q(w)(s)$ are in $(\frac{\mu_1}{6} - \lambda, \frac{\mu_1}{6} + \lambda)$ for all $s \in [0, \delta_0)$. Since $|(c+x)^\alpha - (c+y)^\alpha| \leq \frac{\alpha}{(c-d)^{1-\alpha}} |x-y|$, $\forall \alpha < 1$, $c > 0$, and $|x|, |y| \leq d < c$, it follows that for all $r \in (0, \delta_0)$:

$$\begin{aligned}
&|q(\mathcal{T}v)(r) - q(\mathcal{T}w)(r)|\, r^2 \leq \\
&\leq \quad \frac{\ell_1(\mu)\alpha_1(\mu)}{(\frac{\mu_1}{6} - \lambda)^{1-\alpha_1(\mu)}} \int_0^r s^{1+2\alpha_1(\mu)} \, |q(v)(s) - q(w)(s)| \, ds \ .
\end{aligned}$$

Thus, with $C_0 := \frac{\ell_1(\mu)\alpha_1(\mu)}{(\frac{\mu_1}{6} - \lambda)^{1-\alpha_1(\mu)}} < \infty$, $\alpha_0 := 2\alpha_1(\mu) \in (0, 2]$, and $r^{-2} \leq s^{-2}$ estimate (39) follows.

(b) Let $r_0 > 0$, $z \in X$, and $v, w \in X_{r_0}(z)$. Then $\mathcal{T}v, \mathcal{T}w \in X_{r_0}(\mathcal{T}z)$. Furthermore as in 2.(b), the functions $\mathbb{R}_0^+ \ni r \mapsto v(r + r_0), w(r + r_0)$ are bounded by $z(r_0)$ from above and from below by the strictly monotone function $r \mapsto h^z_{r_0}(r)$ (see Eq. (41)). Let $\tilde{\delta}^z_{r_0} := (h^z_{r_0})^{-1}(z(r_0) - \delta_{z(r_0)}(\mu)) > 0$, then $v(r_0 + r), w(r_0 + r) \in (z(r_0) - \delta_{z(r_0)}(\mu), z(r_0))$ for all $r \in (0, \tilde{\delta}^z_{r_0})$. Therefore, due to the properties of μ it follows for all $v, w \in X_{r_0}(z)$ and $r \in (0, \tilde{\delta}^z_{r_0})$:

$$\begin{aligned}
&|l^{\mathcal{T}z}_{r_0}(\mathcal{T}v)(r) - l^{\mathcal{T}z}_{r_0}(\mathcal{T}w)(r)|\, r = \\
&= \quad |(\mathcal{T}v)(r_0 + r) - (\mathcal{T}w)(r_0 + r)| \\
&\qquad \text{(by (36))} \\
&= \quad \left| \int_{r_0}^{r_0 + r} g(r_0 + r, s)\Big(\mu(v(s)) - \mu(w(s))\Big) ds \right|
\end{aligned}$$

$$\begin{aligned}
& \quad \text{(since } v = w \text{ on } [0, r_0]) \\
\leq & \quad (r_0 + \tilde{\delta}^z_{r_0}) \int_0^r |\mu(v(r_0+s)) - \mu(w(r_0+s))| \, ds \\
& \quad (s \to r_0 + s, \text{ and } g(r_0 + r, r_0 + s) \leq (r_0 + s) \leq (r_0 + \tilde{\delta}^z_{r_0})) \\
\leq & \quad (r_0 + \tilde{\delta}^z_{r_0}) \ell_{z(r_0)}(\mu) \int_0^r \Big| (z(r_0) - v(r_0+s))^{\alpha_{z(r_0)}(\mu)} - \\
& \qquad\qquad - (z(r_0) - w(r_0+s))^{\alpha_{z(r_0)}(\mu)} \Big| \, ds \\
& \quad \text{(by (33))} \\
= & \quad (r_0 + \tilde{\delta}^z_{r_0}) \ell_{z(r_0)}(\mu) \int_0^r s^{\alpha_{z(r_0)}(\mu)} \Big| (|z'(r_0)| + l^z_{r_0}(v)(s))^{\alpha_{z(r_0)}(\mu)} - \\
& \qquad\qquad - (|z'(r_0)| + l^z_{r_0}(w)(s))^{\alpha_{z(r_0)}(\mu)} \Big| \, ds \\
& \quad \text{(by (36))}.
\end{aligned}$$

In analogy to (a): Due to estimate (38) and $|z'(r_0)| > 0$, for every $\lambda \in (0, |z'(r_0)|)$ there is a $\delta^z_{r_0} = \delta^z_{r_0}(\lambda) \in (0, \tilde{\delta}^z_{r_0}]$ (independent of v, w!) such that $|z'(r_0)| + l^z_{r_0}(v)(s)$ and $|z'(r_0)| + l^z_{r_0}(w)(s)$ are in $(|z'(r_0)| - \lambda, |z'(r_0)| + \lambda)$ for all $s \in [0, \delta^z_{r_0})$. Since $|(c+x)^\alpha - (c+y)^\alpha| \leq \frac{\alpha}{(c-d)^{1-\alpha}} |x - y|$, $\forall \alpha < 1$, $c > 0$, and $|x|, |y| \leq d < c$, it follows that for all $r \in (0, \delta^z_{r_0})$:

$$\begin{aligned}
& |l^{\mathcal{T}z}_{r_0}(\mathcal{T}v)(r) - l^{\mathcal{T}z}_{r_0}(\mathcal{T}w)(r)| \, r \leq \\
& \quad \leq \frac{(r_0 + \tilde{\delta}^z_{r_0}) \ell_{z(r_0)}(\mu) \alpha_{z(r_0)}(\mu)}{(|z'(r_0)| - \lambda)^{1 - \alpha_{z(r_0)}(\mu)}} \int_0^r s^{\alpha_{z(r_0)}(\mu)} \left| l^z_{r_0}(v)(s) - l^z_{r_0}(w)(s) \right| ds \, .
\end{aligned}$$

Thus, with $C^z_{r_0} := \frac{(r_0 + \tilde{\delta}^z_{r_0}) \ell_{z(r_0)}(\mu) \alpha_{z(r_0)}(\mu)}{(|z'(r_0)| - \lambda)^{1 - \alpha_{z(r_0)}(\mu)}} < \infty$, $\alpha^z_{r_0} := \alpha_{z(r_0)}(\mu) \in (0, 1] \subset (0, 2]$, and $r^{-1} \leq s^{-1}$ estimate (40) follows. □

Lemma 10 (Contraction). *Let $R > 0$, $\mathbf{X} \subset C^0([0, R])$, and $\mathcal{R} : \mathbf{X} \to \mathbf{X}$. If there are constants $K \in \mathbb{R}^+_0$, $\alpha > 0$ such that for all $x, y \in \mathbf{X}$ and $r \in [0, R]$*

$$|(\mathcal{R}x)(r) - (\mathcal{R}y)(r)| \leq K \int_0^r \frac{1}{s^{1-\alpha}} |x(s) - y(s)| \, ds \, ,$$

then for all $n \in \mathbb{N}_0$ and $r \in [0, R]$:

$$|(\mathcal{R}^n x)(r) - (\mathcal{R}^n y)(r)| \leq \frac{\left(\frac{K}{\alpha}\right)^n}{n!} r^{n\alpha} \|x - y\|_R \leq \frac{\left(\frac{K R^\alpha}{\alpha}\right)^n}{n!} \|x - y\|_R \, . \tag{42}$$

Proof. By induction :

1. $n = 0$: Since x, y are continuous functions on the compact interval $[0, R]$ the supremum norm $\|x - y\|_R := \sup_{r \in [0,R]} |x(r) - y(r)|$ is finite and inequality (42) is true.

2. Step $n \to n+1$:

$$\begin{aligned}
|(\mathcal{R}^{n+1}x)(r) - (\mathcal{R}^{n+1}y)(r)| = \\
&= |(\mathcal{R}(\mathcal{R}^n x))(r) - (\mathcal{R}(\mathcal{R}^n y))(r)| \\
&\leq K \int_0^r \frac{1}{s^{1-\alpha}} |(\mathcal{R}^n x)(s) - (\mathcal{R}^n y)(s)|\, ds \\
&\leq K \frac{\left(\frac{K}{\alpha}\right)^n}{n!} \|x-y\|_R \int_0^r \frac{s^{n\alpha}}{s^{1-\alpha}}\, ds \\
&= \frac{\left(\frac{K}{\alpha}\right)^{n+1}}{(n+1)!} r^{(n+1)\alpha} \|x-y\|_R\,.
\end{aligned}$$

Proposition 2 (Existence, uniqueness, regularity).
For every admissible μ the fixed point problem (16), $u = \mathcal{T}u$, has a unique solution in $C^{1,\alpha<1}_{\text{loc}}(\mathbb{R}_0^+)$. If $\mu \not\equiv 0$ a.e. the solution u is a strictly decreasing function. Moreover, if for $0 \leq r_1 < r_2 \leq \infty$ the function $\mu \in C^{k,\alpha<1}(u((r_1,r_2)))$ with $k \geq 0$ then $u \in C^{k+2,\alpha}((r_1,r_2))$.

Proof. If $\mu \equiv 0$ a.e., then due to the definition of $\mathcal{T}$ (see (16)) $u \equiv 1$ is the unique fixed point. Therefore, let $\mu \not\equiv 0$ a.e. in the following.

1. Existence and uniqueness: Due to Lemma 7 and Lemma 8 it is sufficient to show that $\mathcal{T}^2$ has at the most one fixed point on $\mathbb{R}_0^+$ since then $\underline{u} = \bar{u} =: u$ is the unique solution of $u = \mathcal{T}u$. To this end, let $v, w \in \mathcal{F}_2 := \{z \in C^0(\mathbb{R}_0^+) \mid z = \mathcal{T}^2 z \text{ on } \mathbb{R}_0^+\}$ and $A := \{r \in \mathbb{R}_0^+ \mid v(s) = (\mathcal{T}v)(s) = w(s) = (\mathcal{T}w)(s)\,,\ \forall s \in [0,r]\}$. $A \neq \emptyset$ because $0 \in A$ by Lemma 1. Furthermore, A is closed since v, w and $\mathcal{T}v, \mathcal{T}w$ are continuous. Moreover A is open in $\mathbb{R}_0^+$: Let $r_0 \in A$. By definition of A, $\mathcal{T}^n v, \mathcal{T}^n w \in X_{r_0}(v) = X_{r_0}(w)$ for all $n \in \mathbb{N}_0$. Then due to Lemma 9 :

   (a) If $r_0 = 0$: Let $\mathbf{X} := \bigcup_{n \in \mathbb{N}_0} \{q(\mathcal{T}^n v), q(\mathcal{T}^n w)\} \subset q(X) \subset C^0(\mathbb{R}_0^+)$ and $\mathcal{R} := q \circ \mathcal{T} \circ q^{-1}$. Then $\mathcal{R}(\mathbf{X}) \subset \mathbf{X}$ and for all $x, y \in \mathbf{X}$:

   $$|(\mathcal{R}x)(r) - (\mathcal{R}y)(r)| \leq K \int_0^r \frac{1}{s^{1-\alpha}} |x(s) - y(s)|\, ds$$

   $\forall r \in [0,R]$ by (39), with $R := \delta_0 > 0$, $K := C_0 \in \mathbb{R}_0^+$, $\alpha := \alpha_0 > 0$.

   (b) If $r_0 > 0$: Then $l^v_{r_0} = l^{\mathcal{T}v}_{r_0} = l^w_{r_0} = l^{\mathcal{T}w}_{r_0}$ since $\mathcal{T}v, \mathcal{T}w \in X_{r_0}(\mathcal{T}v) = X_{r_0}(v)$. Let $\mathbf{X} := \bigcup_{n \in \mathbb{N}_0} \{l^v_{r_0}(\mathcal{T}^n v), l^v_{r_0}(\mathcal{T}^n w)\} \subset l^v_{r_0}(X_{r_0}(v)) \subset C^0(\mathbb{R}_0^+)$ and $\mathcal{R} := l^v_{r_0} \circ \mathcal{T} \circ (l^v_{r_0})^{-1}$. Then $\mathcal{R}(\mathbf{X}) \subset \mathbf{X}$ and for all $x, y \in \mathbf{X}$:

   $$|(\mathcal{R}x)(r) - (\mathcal{R}y)(r)| \leq K \int_0^r \frac{1}{s^{1-\alpha}} |x(s) - y(s)|\, ds$$

   $\forall r \in [0,R]$ by (40), with $R := \delta^v_{r_0} > 0$, $K := C^v_{r_0} \in \mathbb{R}_0^+$, $\alpha := \alpha^v_{r_0} > 0$.

Therefore, in both cases Lemma 10 applies: Let $x := q(v)$, $y := q(w)$ if $r_0 = 0$ and $x := l^v_{r_0}(v)$, $y := l^v_{r_0}(w)$ if $r_0 > 0$. Then by estimate (42):

$$\|x - y\|_R = \|\mathcal{R}^{2n}x - \mathcal{R}^{2n}y\|_R \leq c_{2n}\|x - y\|_R$$

where $v = \mathcal{T}^2 v$, $w = \mathcal{T}^2 w \Rightarrow x = \mathcal{R}^2 x$, $y = \mathcal{R}^2 y$ was used. Since $c_{2n} \to 0$ as $n \to \infty$ it follows that $\|x - y\|_R = 0$, i.e. $v = w$ on $[0, r_0] \cup [r_0, r_0 + R] = [0, r_0 + R]$. Then by Lemma 8, $\mathcal{T}v = v = w = \mathcal{T}w$ on $[0, r_0 + R]$. Thus, the set A is open because $R > 0$. In summary, A is open, closed, and not empty. Hence, $A = \mathbb{R}^+_0$, i.e. if $v, w \in \mathcal{F}_2$ then $v \equiv w$ on $\mathbb{R}^+_0$.

2. The conditional higher regularity properties follow as in Propositon 1.

Remark 11.

1. The proof shows that
   (a) the sequence $u_i = \mathcal{T}u_{i-1}$ with $u_0 \equiv 1$ converges to the unique fixed point u and due to (34) the following a posteriori estimates hold for all $i \in \mathbb{N}$:
   $$u_{2i+1} \leq u \leq u_{2i}\ . \tag{43}$$
   (b) there is (at least) a neighbourhood $[0, \delta_0 > 0]$ of $r_0 = 0$ such that the uniform convergence of u_i on $[0, \delta_0]$ is even faster than exponential.
2. If there is an $r_S < \infty$ with $u(r_S) = 0$, i.e. the stellar radius r_S is finite, then $u \in C^\infty((r_S, \infty))$. More precisely, then Eq. (16) implies $u(r) = a + b/r$ for $r \geq r_S$ with $a < 0$ and $b = M := \int_0^{r_S} \mu(u(s))s^2\, ds > 0$.
3. If μ is "merely" continuous, then it is straightforward to prove existence of solutions of Eq. (16) in analogy to the Peano existence theorem for non-singular ordinary differential equations: The set $\mathbf{X} := \{v \in C^0([0, R]) \mid v \leq 1\}$ is closed and convex[13]. Furthermore, the mapping $\mathbf{X} \ni u \mapsto \mathcal{T}u \in \mathbf{X}$ is continuous[14], $\mathcal{T}(\mathbf{X}) \subset \mathbf{X}$, and $\mathcal{T}(\mathbf{X}) \subset C^{1,\alpha<1}([0, R]) \subset C^1([0, R])$ is precompact by the Arzelà-Ascoli theorem[15]. Hence, Schauder's fixed point theorem (see e.g. [9], Corollary 11.2) guarantees that $\mathcal{T}$ has a fixed point in $\mathbf{X}$. However, such a fixed point is not necessarily unique.

Corollary 4. *If μ is a step function with N "phases"[16] then the sequence $u_0 :\equiv 1$, $u_{i\geq 1} := \mathcal{T}u_{i-1}$ coincides(!) with the unique fixed point $u = \mathcal{T}u$ for all $i \geq 2N + 2$.*

[13] Since for all $v, w \in \mathbf{X}$, and $\lambda \in [0, 1]$: $\lambda v + (1 - \lambda)w \leq \lambda + (1 - \lambda) = 1$.

[14] Note that μ is uniformly continuous for it is continuous on the compact interval $[0, 1]$ by assumption.

[15] Because $1 - \frac{R^2}{6} \leq v \leq 1$ and $|v'| \leq \frac{R}{3}$ for all $v \in \mathcal{T}(\mathbf{X})$, it follows that $\mathcal{T}(\mathbf{X})$ is a set of uniformly bounded and equicontinuous functions (see e.g. [12], Theorem 5.19).

[16] Here, it is not necessary that μ corresponds to a step function EOS, i.e. $\lim_{v\downarrow 0} \mu(v) = 0$ is allowed.

Proof. Due to Proposition 2, $u = \mathcal{T}u$ exists and is unique. If $\mu \equiv 0$ a.e. then $u_i \equiv 1 = \mathcal{T}1$ for all $i \in \mathbb{N}$. Therefore let $\mu(v) \not\equiv 0$ a.e. in the following, i.e. $\mu(v) = \{\epsilon_k$, if $v \in (v_{k-1}, v_k)$, $k = 1, \ldots, N\}$ with $-\infty =: v_{-1} < 0 =: v_0 < v_1 < \ldots < v_N := 1$ and $0 =: \epsilon_0 \le \epsilon_1 < \ldots < \epsilon_N \le 1$. By Lemma 7, the functions $u_{i\ge 1}$ are strictly decreasing. Thus, for every $k = 0, \ldots, N$ and $i \in \mathbb{N}$ there is a unique $0 < r_i^k \le \infty$ such that either $u_i(r_i^k) = v_k$ or $u_i > v_k$ on $[0, \infty =: r_i^k)$. Especially $r_i^N = 1$ for all $i \ge 1$. Since $u_{2i+1} \le u_{2i+3} \le u_{2i+2} \le u_{2i}$, it follows that $r_{2i+1}^k \le r_{2i+3}^k \le r_{2i+2}^k \le r_{2i}^k$.

Assumption : $\exists k_0 \in \{1, \ldots, N\}$ and $j_0 = 2i_0 + 1 \ge 1$ such that $r_{j_0}^{k_0} < \infty$ and $u_j = u_{j_0}$ on $[0, r_{j_0}^{k_0}]$ for all $j \ge j_0$ ($\Rightarrow r_j^{k_0} = r_{j_0}^{k_0}, \forall j \ge j_0$). Then $u_{j\ge j_0} \in (v_{k_0-1}, v_{k_0})$ ($\Rightarrow \mu(u_j) = \epsilon_{k_0}$) on $(r_{j_0}^{k_0}, r_j^{k_0-1})$. Since $r_j^{k_0-1} \le r_{j+2}^{k_0-1} \le r_{j+3}^{k_0-1} \le r_{j+1}^{k_0-1}$ for all $j \ge j_0 + 2n$ (with $n \in \mathbb{N}_0$), it follows that $\mu(u_{j\ge j_0+1}) = \epsilon_{k_0}$ on $(r_{j_0}^{k_0-1}, r_{j_0+2}^{k_0-1})$. Hence, $\mu(u_{j\ge j_0+1}) = \mu(u_{j_0+1})$ on $[0, r_{j_0+2}^{k_0-1})$. This implies (since $u_{i+1} = 1 - \mathcal{Q}\mu(u_i)$) that for all $j \ge j_0+2$: $u_j = u_{j_0+2}$ on $[0, r_{j_0+2}^{k_0-1})$. If $r_{j_0+2}^{k_0-1} = \infty$, then u_{j_0+2} is already the (unique) fixed point u. If $r_{j_0+2}^{k_0-1} < \infty$, then (by continuity) $u_{j\ge j_0+2} = u_{j_0+2}$ on $[0, r_{j_0+2}^{k_0-1}]$. In summary, either u_{j_0+2} is the fixed point u or the assumption holds for $k_0 - 1 \in \{0, \ldots, N-1\}$ and $j_0 + 2 \ge 3$. If the assumption is true for $k_0 = 0$, then $u_{j\ge j_0+1} = u_{j_0+1}$ on $\mathbb{R}_0^+$ (i.e. the fixed point is reached) because $\mu(u_j) = 0$ on $(r_{j_0}^0, \infty)$ for all $j \ge j_0$. Since $u_{i\ge 1}(0) = 1$, the assumption is true for $k_0 = N$ and $j_0 = 1$. Therefore (note that $k_0 \in \{0, \ldots, N\}$) it follows that at the most after $1 + N \cdot 2 + 1 = 2N + 2$ iterations the fixed point of $\mathcal{T}$ is reached.

Remark 12.

1. It can be shown by similar arguments that even for any $u_0 \le 1$ the sequence $u_i = \mathcal{T}u_{i-1}$ coincides with the fixed point after *finite* steps of iterations. This "amazing" convergence property is closely related to the fact that in (33) the constants $\ell_v(\mu) = 0$ for all $v \le 1$.

2. If μ is a step function, the fixed point u can be constructed explicitly (at least in principe): If $u(r) \in (v_{k-1}, v_k)$ then u must have the form

$$u(r) = a_k + \frac{b_k}{r} - \frac{\epsilon_k}{6}\, r^2$$

(because $u''(r) + \frac{2}{r}\, u'(r) = -\epsilon_k$, cf. Eq. (14)). Since $u \in C^1(\mathbb{R}_0^+)$, the constants $a_k, b_k \in \mathbb{R}$ are uniquely determined by the condition that u and u' are continuous on $\mathbb{R}_0^+$. Starting with $u(0) = 1$ and $u'(0) = 0$, i.e. $u(r) = 1 - \frac{\epsilon_N}{6}\, r^2$ as long as $u(r) \ge v_{N-1}$, these conditions lead at the most to cubic equations.

4 Relations between μ and Finiteness of r_S

For physical applications, one of the most important question is whether there is a finite r_S with $u(r_S) = 0$ for a given admissible μ or not[17], i.e. in the context of static stars in Newtonian gravity whether for a given EOS the stellar radius r_S is finite or not. This question will be investigated in this section. The following facts are known about this question in the literature: For a polytropic EOS the stellar radius is finite if the index $\nu \in [0, 5)$ and it is infinite if $\nu \geq 5$ (see [7] and [11]). If μ behaves "essentially" polytropic near $\mu = 0$, then the stellar radius is finite if the polytropic index $\nu \in [0, 3]$. And if $\nu \in (3, 5)$ the radius can be finite or infinite (see [18], [16], and [17], p. 20). Rendall and Schmidt [18] raised the question: Is it *necessary* for a finite stellar radius that the function μ behaves essentially polytropic with index strictly less than five near $\mu = 0$ (i.e. at the star's surface)? Below (see Remark 15), it is shown that the answer for Newtonian gravity is: *No.*

Lemma 11 (A priori criteria). *Assume that μ is admissible. By Proposition 2 the corresponding fixed point problem (16) has a unique solution u. Then the following holds:*

1. *If $\lim_{v\downarrow 0} \mu(v) =: \mu_0 > 0$ then r_S is finite.*
2. *Let $u_0 :\equiv 1$ and $u_{i\geq 1} := \mathcal{T} u_{i-1}$. If $\exists\, j = 2i$ with $i \geq 1$ such that $\exists\, r_j < \infty$ with $u_j(r_j) = 0$ then $r_S \leq r_j$, i.e. r_S is finite.*

Proof.

1. Assume, in contrary to the assertion that r_S is not finite, i.e. $u > 0$ on $\mathbb{R}_0^+$. Therefore, $\mu(u) \geq \mu_0 > 0$ on $\mathbb{R}_0^+$ by assumption. Hence $u(r) = (\mathcal{T} u)(r) = 1 - (\mathcal{Q}\mu(u))(r) \leq 1 - (\mathcal{Q}(\mu_0 \cdot 1_{[0,\infty)}))(r) = 1 - \frac{\mu_0}{6} r^2 =: u_1(r)$, for all $r \in \mathbb{R}_0^+$, which is a contradiction since $u_1(r) < 0$ for $r > \sqrt{6/\mu_0} < \infty$.
2. Since $\lim_{i\to\infty} u_i = u$ by the proof of Proposition 2 and $u \leq u_{2i\geq 0}$ due to the estimates (34) in Lemma 7 the assertion follows (because u_{2i} is decreasing). $\square$

Lemma 12 (General sub- and supersolution). *For every admissible μ and corresponding fixed point u the following holds:*

1. *Let $\check{u} : \mathbb{R}_0^+ \to \mathbb{R}$, $r \mapsto \check{u}(r) := 1 - \frac{\mu_1}{6} r^2$. Then $\check{u} \leq u$, i.e. $\check{u}$ is a subsolution.*
2. *Let $\mu \not\equiv 0$ a.e., $\hat{D} := \{v \in (0, 1] \mid \mu(v) > 0\} \ni 1$, and*

$$\begin{aligned} \hat{D} \ni v &\mapsto \hat{r}(v) := \sqrt{\tfrac{6(1-v)}{\mu(v)}} \\ \mathbb{R}_0^+ \ni r &\mapsto \hat{u}_v(r) := \begin{cases} 1 - \frac{\mu(v)}{6} r^2 \geq v & : r \leq \hat{r}(v) < \infty \\ v & : r \geq \hat{r}(v) \end{cases} \\ \mathbb{R}_0^+ \ni r &\mapsto \hat{u}(r) := \inf_{v\in\hat{D}} \hat{u}_v(r) \geq 0\,. \end{aligned}$$

Then $u \leq u_+ \leq \hat{u}$, i.e. $\hat{u}$ is a supersolution.

[17] Note that r_S is unique if it exists (if not, let $r_S := \infty$), since u is strictly monotone.

Proof.

1. $\mu(u_+) \le \mu_1$ a.e. because $u(0) = 1$, u is decreasing (by Corollary 1), and $\mu : [0,1] \to [0,1]$ is increasing with $\mu \le \mu_1$ a.e. (by assumption). Hence by Lemma 1, $u(r) = (\mathcal{T}u)(r) = 1 - (\mathcal{Q}\mu(u_+))(r) \ge 1 - (\mathcal{Q}(\mu_1 \cdot 1_{[0,\infty)}))(r) = 1 - \frac{\mu_1}{6}\, r^2 =: \check{u}(r)$ for all $r \in \mathbb{R}_0^+$.

2. $\hat{D} \subset u(\mathbb{R}_0^+)_+$. For assume this is false, i.e. $\exists v_0 \in \hat{D}$ with $v_0 \notin u(\mathbb{R}_0^+)_+$. Then by definition, $\hat{r}(v_0) < \infty$ and $u > v_0$ on $\mathbb{R}_0^+$ (because u is decreasing). Therefore (since μ is increasing) $0 \le v_0 < u(r) = (\mathcal{T}u)(r) \le 1 - (\mathcal{Q}(\mu(v_0) \cdot 1_{[0,\infty)}))(r) = 1 - \frac{\mu(v_0)}{6}\, r^2 \to -\infty$ as $r \to \infty$, which is a contradiction. It remains to show that $\hat{u}_v \ge u_+$ for all $v \in \hat{D}$. To this end let $v_0 \in \hat{D}$. Hence by the preceding $v_0 \in u(\mathbb{R}_0^+)_+$, i.e. $\exists r_0 \in \mathbb{R}_0^+$ with $u(r_0) = v_0 > 0$. Since u is strictly decreasing, r_0 is unique and $u(r) \gtrless v_0$ for all $r \lessgtr r_0$. Thus $\mu(u_+) \ge \mu(v_0)$ on $[0, r_0]$. Then for all $r \in [0, r_0]$: $0 < v_0 \le u(r)_+ = u(r) = (\mathcal{T}u)(r) \le 1 - (\mathcal{Q}(\mu(v_0) \cdot 1_{[0,r_0]}))(r) = 1 - \frac{\mu(v_0)}{6}\, r^2 = \hat{u}_{v_0}(r)$ ($\Rightarrow r_0 \le \hat{r}(v_0)$). Since $\hat{u}_{v_0} \ge v_0 > 0$ on $\mathbb{R}_0^+$ and $v_0 \ge u(r)_+ \ge u(r)$ for all $r \ge r_0$, it follows that $\hat{u}_{v_0} \ge u_+ \ge u$ on $[0, r_0] \cup [r_0, \infty) = \mathbb{R}_0^+$. □

Corollary 5. *If $\mu(v) > 0$ for all $v > 0$ and $\lim_{v \downarrow 0} \mu(v) \ge 0$ for an admissible function μ, then*

$$\lim_{r\to\infty} u(r)_+ = 0$$

for the corresponding fixed point u of $\mathcal{T}$.

Proof. Let $\varepsilon > 0$ and $v_\varepsilon := \min\{\varepsilon, 1\} \in (0,1]$. By assumption $v_\varepsilon \in \hat{D}$. Then by Lemma 12(2.), $0 < \hat{r}(v_\varepsilon) < \infty$ and $u(r)_+ \le \hat{u}(r) \le \hat{u}_{v_\varepsilon}(r) = v_\varepsilon \le \varepsilon$ for all $r \ge \hat{r}(v_\varepsilon)$, i.e. $\lim_{r\to\infty} u_+(r) = 0$. □

Remark 13. For instance, the Corollary applies to all EOS, especially to polytropic with index $\nu > 5$.

Lemma 13 (Necessary and sufficient condition). *Let u be the unique fixed point of $\mathcal{T}$ for an admissible μ. Equivalent are:*

1. *r_S is finite.*
2. *$\exists\, r_0 \in (0,\infty)$ such that* $$\frac{u(r_0)}{r_0} < |u'(r_0)| = -u'(r_0)$$
3. *$\exists\, r_0 \in (0,\infty)$ such that* $$\boxed{1 < \int_0^{r_0} s\,\mu(u(s)_+)\,ds}\,.$$

Proof.

(1. ⇒ 2.): Assume r_S is finite. Then $r_0 := r_S > 0$ since $u(0) = 1$ and $u \in C^0$. Hence $u(r_0) = 0$ and $u'(r_0) < 0$ by Corollary 1. Thus inequality 2. holds.

(2. $\Rightarrow$ 1.): Assume inequality 2. holds, i.e. $\exists r_0 \in (0,\infty)$ such that $u(r_0) + r_0 u'(r_0) < 0$. Let $0 \le v := u_+ \cdot 1_{[0,r_0]}$. Then $v \le u_+$ and $\mu(v) \le \mu(u_+)$ (since μ is increasing by assumption). Therefore (by Lemma 1) $\mathcal{T}v = 1 - \mathcal{Q}\mu(v) \ge 1 - \mathcal{Q}\mu(u_+) = \mathcal{T}u = u$ and $\forall r \ge r_0$ (using Eqs. (16) and (20)):

$$\begin{aligned}(\mathcal{T}v)(r) &= 1 - \int_0^{r_0} s\left(1 - \frac{s}{r}\right)\mu(u(s)_+)\,ds \\ &= \left(1 - \textstyle\int_0^{r_0} s\,\mu(u(s)_+)\,ds\right) + \frac{\int_0^{r_0} s^2 \mu(u(s)_+)\,ds}{r} =: a + \frac{b}{r} \\ (\mathcal{T}v)'(r) &= -\int_0^{r_0}\left(\frac{s}{r}\right)^2 \mu(u(s)_+)\,ds = -\frac{b}{r^2}\,.\end{aligned}$$

Since $\mathcal{T}v = \mathcal{T}u = u$ on $[0, r_0]$ and $\mathcal{T}v, u \in C^1(\mathbb{R}_0^+)$, it follows that $u(r_0) = (\mathcal{T}v)(r_0) = a + \frac{b}{r_0}$ and $u'(r_0) = (\mathcal{T}v)'(r_0) = -\frac{b}{r_0^2}$. Thus

$$0 > u(r_0) + r_0 u'(r_0) = a\,.$$

Therefore $\lim_{r\to\infty} u \le \lim_{r\to\infty} \mathcal{T}v = \lim_{r\to\infty} a + b/r = a < 0$. Because $u(0) = 1$ and u is decreasing there is an $r_S \in (0,\infty)$ with $u(r_S) = 0$, i.e. 1. holds.

(2. $\Leftrightarrow$ 3.): $0 > u(r_0) + r_0 u'(r_0) = a = 1 - \int_0^{r_0} s\,\mu(u(s)_+)\,ds$. □

Remark 14.

1. Note that $\lim_{r_0 \downarrow 0} \frac{u(r_0)}{r_0} = \infty$ and $\lim_{r_0\downarrow 0} u'(r_0) = 0$.
2. If $u(r_0) > 0$, then $r_0 < r_S$ and the knowledge of the fixed point $u(r)$ for $r > r_0$ is not needed in order to guarantee the finiteness of the stellar radius r_S!
3. For the polytropic EOS with index $\nu = 5$ (i.e. $\mu(u) = u^5$): $u(r) = (1 + \frac{1}{3}r^2)^{-1/2}$ ($\Rightarrow r_S = \infty$) and $\int_0^\infty s\,\mu(u(s)_+)\,ds = \int_0^\infty s\,(1 + \frac{1}{3}s^2)^{-5/2}\,ds = 1$.

Corollary 6. *Let u be the unique fixed point of $\mathcal{T}$ for an admissible μ. Then*

$$\lim_{r\to\infty} u'(r) = 0\,.$$

Proof. If $r_S < \infty$, then Eq. (16) implies that $u(r) = a + b/r$ for all $r > r_S$ with $a, b \in \mathbb{R}$. Hence the assertion follows. If $r_S = \infty$ then $u \ge 0$. Thus $u_+ = u$ and $\int_0^r s\,\mu(u(s))\,ds \le 1$ for all $r \in \mathbb{R}_0^+$ by Lemma 13(3.). Therefore $|u'(r)| = \int_0^r (s/r)^2\,\mu(u(s))\,ds \le r^{-1}\int_0^r s\,\mu(u(s))\,ds \le 1/r$. Hence the assertion follows. □

Corollary 7 (Sufficient conditions). *Assume μ is admissible and u is the corresponding fixed point of $\mathcal{T}$.*

1. *If there is* $\tilde{u} \le u$ *and* $r_0 < \infty$ *with* $1 < \int_0^{r_0} s\,\mu(\tilde{u}(s)_+)\,ds \Rightarrow r_S$ *is finite.*

2. *If* $\boxed{\frac{\mu_1}{3} < \int_0^1 \mu(v)\,dv} \Rightarrow r_S$ *is finite.*

3. *If there is* $\tilde{u} \ge u$ *with* $\int_0^{\infty} s\,\mu(\tilde{u}(s)_+)\,ds \le 1 \Rightarrow r_S$ *is infinite.*

Proof.

1. Since μ is increasing by assumption, $\mu(\tilde{u}_+) \le \mu(u_+)$ and therefore $1 < \int_0^{r_0} s\,\mu(\tilde{u}(s)_+)\,ds \le \int_0^{r_0} s\,\mu(u(s)_+)\,ds$. Hence by Lemma 13(3.) the assertion follows.

2. If $0 < \mu_1 \le 1$, then by Lemma 12(1.) $\check{u}(s) = 1 - \frac{\mu_1}{6}\,s^2 \le u(s)$ and $\check{u} \ge 0$ on $[0, r_0 := \sqrt{6/\mu_1} < \infty]$. Thus, using the substitution $v = \check{u}(s)$

$$\int_0^{r_0} s\,\mu(\check{u}(s)_+)\,ds = \frac{3}{\mu_1}\int_0^1 \mu(v)\,dv > 1\ .$$

   Hence by 1. the assertion follows. If $\mu_1 = 0$ then $\mu \equiv 0$ a.e. by definition, i.e. the assumption does not hold.

3. $\mu(\tilde{u}_+) \ge \mu(u_+)$ since μ is increasing by assumption. Therefore, it follows that $1 \ge \int_0^{\infty} s\,\mu(\tilde{u}(s)_+)\,ds \ge \int_0^{\infty} s\,\mu(u(s)_+)\,ds$, i.e. $1 \ge \int_0^{r_0} s\,\mu(u(s)_+)\,ds$ for all $r_0 \in (0,\infty)$. Hence by Lemma 13(3.) the assertion follows. □

Remark 15.

1. Note that statement 2. in Corollary 7 is an *a priori criterion*. Furthermore, this criterion answers (within Newtonian gravity) the mentioned question raised by Rendall and Schmidt: Since there are admissible μ having polytropic behaviour with index $\nu \ge 5$ near $\mu = 0$ and satisfying this criterion 2. (e.g. $\mu(u) = u^{\nu \ge 5} \cdot 1_{[0,v<\frac{2}{3})} + 1_{[v,1]}$), the answer to their question is: The essentially polytropic behaviour with index strictly less than five near $\mu = 0$ is *not necessary* for a finite stellar radius. Joseph and Lundgren remark (see [11], p. 243, Footnote $\star$) that Lebovitz made a similar observation.

2. Criterion 2. in Corollary 7 reads for polytropic EOS with index $\nu \in \mathbb{R}_0^+$: $\int_0^1 v^{\nu}\,dv = \frac{1}{\nu+1} > \frac{1}{3}$, which is valid for $\nu \in [0,2)$. Since it is known that for all polytropic EOS with $\nu < 5$ the stellar radius r_S is finite (see [7] and [11]), this shows that the criterion 2. in Corollary 7 is *not necessary*. Another sufficient a priori criterion for finite radius, which is sharp in the polytropic case, was given by Simon [21], Eq. (14) : $F(p) = \int_0^p \frac{d\tilde{p}}{\epsilon(\tilde{p})} \le 6\frac{p}{\epsilon(p)}$ for all $p \in (0, p_c)$. Since for polytropic EOS this ("pointwise") condition is valid only if the index $\nu \le 5$, the argument in 1. shows that this conditon is not necessary either.

3. An illuminating example for the general case is the following: For the polytropic EOS $\mu(v) = v^\nu$ with index $\nu \geq 5$ the stellar radius is infinite. However, for all $\mu_\delta(v) = \delta^\nu \cdot 1_{[\delta,1]}(v)$ (note that $\mu_\delta \leq \mu$) with $0 < \delta < \frac{2}{3}$ the stellar radius is finite!

4. The supersolution $\hat{u}$ in Lemma 12 is too weak in order to give (together with condition 3. in Corollary 7) a reasonable (sufficient) criterion for an (realistic) EOS so that the radius is infinite. However, if μ is admissible and of the form $\mu = 0$ on $[0, v_0]$ with $0 < v_0 < 1$ and $0 < \mu_0 := \mu^+(v_0) \leq \mu \in C^1((v_0, 1))$ on $(v_0, 1]$, then $\int_0^\infty s\,\mu(\hat{u}(s)_+)\,ds = \int_{v_0}^1 \hat{r}(v)\,\mu(v)|\,\hat{r}'(v)|\,dv = 3\int_{v_0}^1 (1 + (1 - v)\frac{\mu'(v)}{\mu(v)})\,dv = 3(1-v_0)(1+|\ln\mu_0|) - 3\int_{v_0}^1 |\ln\mu(v)|\,dv$, where the substitution $s = \hat{r}(v) = \sqrt{\frac{6(1-v)}{\mu(v)}}$ and $\hat{u}(\hat{r}(v)) = v$ was used. For example, if $\mu = \mu_0 \cdot 1_{[v_0,1]}$ then $\int_0^\infty s\,\mu(\hat{u}(s)_+)\,ds = 3(1 - v_0)$. Hence, if $v_0 \geq \frac{2}{3}$ then $3(1 - v_0) \leq 1$ and r_S is infinite by condition 3. in Corollary 7, which is the optimal result in this special case.

5. Since the subsolutions u_{2i+1} and the supersolutions u_{2i} (cf. (43)) converge monotonically to the unique solution u as $i \to \infty$, these sub- and supersolutions can be used (at least in principle) in condition 1. and 3. of Corollary 7 in order to provide sequences of (sufficient) conditions of increasing sharpness for a given admissible μ. Since criterion 3. in Lemma 13 is sufficient *and* necessary these sequences of conditions are optimal in the "limit" $i \to \infty$ (e.g. for the polytropic EOS with "critical" index $\nu = 5$ it was already mentioned that $\int_0^\infty s\,u(s)^5 ds = 1$).

Lemma 14 (Gronwall type). *Let* $d \in C^0([0, R))$ *with*

$$d(r) \leq a\,r^2 + b\int_0^r g(r,s)d(s)\,ds\,, \quad \forall r \in [0, R)$$

and $0 < R \leq \infty$, $a \in \mathbb{R}$, $b \geq 0$. *Then for all* $r \in [0, R)$*:*

$$d(r) \leq \frac{6a}{b}\left(\frac{\sinh(\sqrt{b}\,r)}{\sqrt{b}\,r} - 1\right) = a\,r^2\left(1 + \tfrac{1}{20}b\,r^2 + O(b^2r^4)\right)\,. \tag{44}$$

Proof. For every $\varepsilon > 0$ let $f_\varepsilon(r) := (\varepsilon + \frac{6a}{b})\frac{\sinh(\sqrt{b}\,r)}{\sqrt{b}\,r} - \frac{6a}{b}$. Then, an elementary integration shows that for all $r \in \mathbb{R}_0^+$:

$$f_\varepsilon(r) = \varepsilon + a\,r^2 + b\textstyle\int_0^r g(r,s)f_\varepsilon(s)\,ds\,. \tag{45}$$

Hence, in order to prove estimate (44) it is sufficient to show that $d(r) < f_\varepsilon(r)$ for all $r \in [0, R)$: Since $d(0) \leq 0 < \varepsilon = \lim_{r\downarrow 0} f_\varepsilon(r)$ this inequality is true for $r = 0$. Assume $r_0 := \inf\{r \in [0, R) \mid d(r) = f_\varepsilon(r)\} > 0$. Since $d, f_\varepsilon \in C^0$, it follows that

$d(r_0) = f_\varepsilon(r_0)$ and $d \le f_\varepsilon$ on $[0, r_0]$ (otherwise, the intermediate value theorem yields a contradiction). Therefore,

$$\begin{aligned} d(r_0) &\le a\,r_0^2 + b\int_0^{r_0} g(r_0,s)\,d(s)\,ds \\ &\quad \text{(by assumption)} \\ &< \varepsilon + a\,r_0^2 + b\int_0^{r_0} g(r_0,s)\,f_\varepsilon(s)\,ds \\ &\quad \text{(since } \varepsilon > 0,\ b \ge 0, \text{ and } \forall s \in [0,r_0]\text{: } g(r_0,s) \ge 0,\ d(s) \le f_\varepsilon(s)) \\ &= f_\varepsilon(r_0) \\ &\quad \text{(by (45))}\ , \end{aligned}$$

which is a contradiction. □

Remark 16. Since $g(r,s) \le s$, Gronwall's Lemma (see e.g. [1], p. 99) can be used to obtain a similar estimate. However, this estimate is weaker than (44).

Corollary 8 ("Approximation"). *Assume μ_1, μ_2 are admissible functions and u_1, u_2 are the corresponding unique fixed points due to Proposition 2. Let* $\delta := \|\mu_1 - \mu_2\|_1 := \sup_{v\in[0,1]} |\mu_1(v) - \mu_2(v)| \in [0,1]$. *If in addition μ_1 is Lipschitz continuous on* $[0,1]$, *with Lipschitz constant $\ell > 0$, then for all $r \in \mathbb{R}_0^+$:*

$$\boxed{|u_1(r) - u_2(r)| \le \frac{\delta}{\ell}\left(\frac{\sinh(\sqrt{\ell}\,r)}{\sqrt{\ell}\,r} - 1\right)} = \delta\,r^2\left(\tfrac{1}{6} + \tfrac{1}{120}\ell\,r^2 + O(\ell^2 r^4)\right)\ . \tag{46}$$

Proof. Let $d(r) := |u_1(r) - u_2(r)|$ (note that $d \in C^0(\mathbb{R}_0^+)$). Then

$$\begin{aligned} d(r) &= \left|\int_0^r g(r,s)\Big(\mu_2(u_2(s)_+) - \mu_1(u_1(s)_+)\Big)\right| \\ &\quad \text{(by Eq. (16)} \\ &\le \int_0^r g(r,s)|\mu_2(u_2(s)_+) - \mu_1(u_2(s)_+)|\,ds \\ &\quad + \int_0^r g(r,s)|\mu_1(u_2(s)_+) - \mu_1(u_1(s)_+)|\,ds \\ &\le \delta\int_0^r g(r,s)\,ds + \ell\int_0^r g(r,s)d(s)\,ds \\ &= \frac{\delta}{6}\,r^2 + \ell\int_0^r g(r,s)d(s)\,ds\ . \end{aligned}$$

By Lemma 14 the assertion follows. □

Remark 17. If, for example, μ_2 is a step function with finite "stellar radius" (note that this can be explicitly decided, at least in principle), then estimate (46) implies that all Lipschitz continuous μ_1 with $\|\mu_1 - \mu_2\|_1 \le \delta \ll 1$ have also finite stellar radius.

4.1 Relation between EOS and Surface Potential

In the following, the omitted symbols "˜" on μ and u are restored for accuracy (see Eq. (12) and the following convention). Let $\tilde{\mathbf{A}} := \{\tilde{\mu}$ admissible $\mid \exists\, p_c > 0$ and $\exists$ EOS ϵ such that $\mu = \epsilon \circ F^{-1}$ and $\tilde{r}_S(\tilde{\mu}) < \infty\}$. If $\tilde{\mu} \in \tilde{\mathbf{A}}$, it follows that the support of the positive part of the corresponding solution $\tilde{u} = \tilde{u}(\tilde{\mu})$ and the mass density $\tilde{\epsilon} = \tilde{\mu}(\tilde{u}_+)$, viewed as spherically symmetric functions on $\mathbb{R}^3$, is a ball with radius $\tilde{r}_S < \infty$. Then, due to Poisson's equation (1) the corresponding Newtonian gravitational potential $U(x) \propto -\int_{\mathbb{R}^3} \frac{\tilde{\epsilon}(y)}{|x-y|}\, dy$ vanishes at infinity. Therefore, by Eq. (7) it follows that $\lim_{r\to\infty} \tilde{u}(r) = \tilde{U}_S := U(r_S\, \xi/|\xi|)/u_c$ (for every $0 \neq \xi \in \mathbb{R}^3$). Since the solution $\tilde{u}$ is unique, the mapping

$$\tilde{\phi} : \tilde{\mathbf{A}} \longrightarrow (-\infty, 0) = \mathbb{R}^- \,, \tag{47}$$

$$\tilde{\mu} \longmapsto \tilde{\phi}(\tilde{\mu}) \quad := \quad \lim_{r\to\infty} \tilde{u}(\tilde{\mu}, r) = 1 - \int_0^\infty s\, \tilde{\mu}(\tilde{u}(\tilde{\mu}, s)_+)\, ds$$

(by Eq. (16) ,

which assigns to every admissible $\tilde{\mu} \in \tilde{\mathbf{A}}$ the corresponding normalized Newtonian surface potential $\tilde{U}_S$, is well-defined.

Remark 18. The following diagram holds :

$$(p_c, \epsilon) \longmapsto (u_c, \mu) \longmapsto \tilde{\mu} \overset{\tilde{\phi}}{\longmapsto} \tilde{U}_S \,.$$

Note that all symbols "$\mapsto$" represent well-defined mappings. It is known that $\tilde{\phi}$ restricted to polytropic EOS with index $\nu \in (0, 5)$ is injective.

Lemma 15. *The mapping $\tilde{\phi} : \tilde{\mathbf{A}} \to \mathbb{R}^-$ is not injective, i.e. in general the value of the normalized surface potential of a solution does not uniquely determine the standard form of the EOS.*

Proof. Let $\tilde{\mu}_\epsilon := \epsilon \cdot 1_{[0,1]}$ with $\epsilon \in (0, 1]$. Then, it is straightforward to show that $\tilde{\phi}(\tilde{\mu}_\epsilon) = -2$ for all $\epsilon \in (0, 1]$. □

Remark 19. Another more interesting example, which shows that $\tilde{\phi}$ is not injective in general, is the following: Let $\tilde{\mu}_1(\tilde{u}) = \tilde{u}$, and $\tilde{\mu}_2 = \frac{9-\sqrt{33}}{18} \cdot 1_{[0,\frac{1}{2})} + 1_{[\frac{1}{2},1]}$ (where $\frac{9-\sqrt{33}}{18} \approx 0.18$). Again, it is straightforward to show that $\tilde{\phi}(\tilde{\mu}_1) = \tilde{\phi}(\tilde{\mu}_2) = -1$. (Note that $\tilde{u}(\tilde{\mu}_1, r) = \frac{\sin(r)}{r}$ for $r \in (0, \pi]$.)

5 Conclusions

It was shown that existence, uniqueness, and regularity of global solutions for Lane-Emden type equations can be established using a simple iterative scheme for quite general right-hand sides, including equations of state with phase transitions. The iteration converges uniformly for Lipschitz continuous right-hand sides at a rate even faster than exponential (at least on every compact set). For a large subclass of the non-Lipschitz continuous right-hand sides, the same convergence behaviour could be established only near the center of symmetry. Whether this rate of convergence still holds outside a neighbourhood of the center of symmetry or not remains an open question (apart for step function, where the solution is reached after finite steps of iteration!). Furthermore, two equivalent criteria were given which are necessary and sufficient so that the stellar radius is finite. These criteria lead to a sufficient (however not necessary) a priori condition on the equation of state which shows that essentially polytropic behaviour with index strictly less than five of the equation of state near the star's surface is not necessary in order to have a star of finite size. Still, the question for a "practicable" sufficient and necessary a priori criterion is open. Moreover, it was shown that the relation between the equation of state and the surface potential is not injective in general.

Since the field equations for equilibrium states of rotating stars in Newton's as well as in Einstein's theory of gravitation have essentially the same structure as the field equations in the static Newtonian case (as was pointed out in the introduction), the numerical results obtained with the method [2] give hope that some of the main ideas in this article can be generalized to obtain existence results for realistic models of rotating stars within general relativity.

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References

[1] H. Amann, Gewöhnliche Differentialgleichungen. Berlin, New York: de Gruyter 1983

[2] S. Bonazzola, E. Gourgoulhon, M. Salgado, J.A. Marck, Axisymmetric rotating relativistic bodies: a new numerical approach for "exact" solutions. *Astron. Astrophys.* **278** (1993) 421–443.

[3] H. Bucerius, Integralgleichungstheorie des Sternaufbaus. I. Die polytrope Gaskugel. *Astron. Nachr.* **265** (1938) 145–158.

[4] H. Bucerius, Integralgleichungstheorie des Sternaufbaus. II. Die numerische Lösungsmethode. *Astron. Nachr.* **266** (1938) 49–62.

[5] H. Bucerius, Integralgleichungstheorie des Sternaufbaus. III. Zusammengesetzte polytrope Gaskugeln. *Astron. Nachr.* **267** (1939) 253–272.

[6] T. Carleman, Über eine isoperimetrische Aufgabe und ihre physikalischen Anwendungen. *Math. Zeitschr.* **3** (1918) 1–7.

[7] S. Chandrasekhar, An Introduction to the Study of Stellar Structure. New York: Dover Publications, Inc. 1967.

[8] R. Emden, Gaskugeln. Leipzig: Teubner-Verlag 1907.

[9] D. Gilbarg, N.S. Trudinger, Elliptic Partial Differential Equations of Second Order. Berlin, Heidelberg, New York: Springer-Verlag 1998.

[10] B.K. Harrison, K. Thorne, M. Wakano, J.A. Wheeler, Gravitation Theory and Gravitational Collapse. Chicago: University of Chicago Press 1965 Wiley & Sons, Inc. 1983.

[11] D.D Joseph, T.S. Lundgren, Quasilinear Dirichlet Problems Driven by Positive Sources. *Arch. Rational Mech. Anal.* **49** (1973) 241–269.

[12] J. Jost, Postmodern Analysis. Berlin, Heidelberg, New York: Springer-Verlag 1998.

[13] E. Kreyszig, Introductory functional analysis with applications. New York: John Wiley & Sons, Inc. 1978.

[14] L. Lichtenstein, Über eine Eigenschaft der Gleichgewichtsfiguren rotierender Flüssigkeiten, deren Teilchen einander nach dem Newtonschen Gesetze anziehen. *Math. Zeitschr.* **28** (1928) 635–640.

[15] E.H. Lieb, M. Loss, Analysis. *Graduate Studies in Mathematics*, Vol. 14. American Mathematical Society 1997.

[16] T. Makino, On the existence of positive solutions at infinity for ordinary differential equations of Emden type. *Funkcialaj Ekvacioj.* **27** (1984) 319–329.

[17] G. Rein, A.D. Rendall, Compact support of spherically symmetric equilibria in non-relativistic and relativistic galactic dynamics. *Math. Proc. Camb. Phil. Soc.* **128** (2000) 363–380.

[18] A.D. Rendall, B.G. Schmidt, Existence and properties of spherically symmetric static fluid bodies with a given equation of state. *Classical Quantum Grav.* **8** (1991) 985–1000.

[19] U.M. Schaudt, On the Dirichlet Problem for the Stationary and Axisymmetric Einstein Equations. *Commun. Math. Phys.* **190** (1998) 509–540.

[20] S.L. Shapiro, S.A. Teukolsky, Black Holes, White Dwarfs, and Neutron Stars. The Physics of Compact Objects. New York: John Wiley & Sons, Inc. 1983.

[21] W. Simon, On journeys to the Moon by balloon. *Class. Quantum Grav.* **10** (1993) 177–181.

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