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Asymptotic properties of the electromagnetic field in the external Schwarzschild spacetime

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Abstract. We study the asymptotic behaviour of the solutions to the vacuum Maxwell equations in the external Schwarzschild spacetime. The results are based on the extensive use of geometric considerations and the introduction of generalized energy estimates. We obtain the asymptotic behaviour along the null outgoing directions and we prove also some partial results concerning the behaviour along the timelike curves. Our techniques can be also used to control the asymptotic behaviour of the various derivatives of the Maxwell field and to obtain the asymptotic behaviour of the Weyl tensor fields, solutions of the "spin 2" equations.

1 Introduction

In this paper we study the asymptotic behaviour of the solutions of the vacuum Maxwell equations in the external Schwarzschild spacetime. The results are based on the extensive use of geometric considerations and the introduction of generalized energy estimates.

These ideas and techniques have been introduced by D.Christodolou and S.Klainerman, [Ch-Kl1], in the case of the Minkowski spacetime for the Maxwell equations and for the linear spin 2 equations of the Weyl tensor field, the Bianchi equations [1]. Their generalization has been subsequently used by the same authors to prove the much more complicated problem of the non linear stability of the Minkowski space, see [Ch-Kl2], [Kl-Ni] and [Ch-Kl-Ni].

To obtain the asymptotic behaviour of the solutions using geometric considerations is much more complicated in the Schwarzschild spacetime as, differently from the Minkowski spacetime, the conformal group is not anymore a group of isometries. The only Killing vectors are those associated to the rotation group and the one associated to time translations. Moreover there are not conformal Killing vectors. This is the reason why we have a complete control of the asymptotic behaviour only outside a cone of directions. In other words we are able to control the asymptotic behaviour along the null outgoing geodesics, while we have only partial results for the behaviour along a generic timelike curve.

[1] In the flat case one could also try to obtain the same results just looking at the fundamental solution of the equations, while this turns out much more complicated in the Schwarzschild background spacetime. In fact the "strong" Huygens principle is not true in the Schwarzschild spacetime, see [McL], [Fr]. This implies that the value of the solution at a generic point does not depend only on the values at the boundary of the domain of influence intersected with the hypersurface $\Sigma_{t=0}$ where the initial data are given.

Apart from the geometric considerations, our results are based on the construction of a family of generalized energy-type norms and on the control of their boundedness. From it, using Sobolev estimates, we control the L^∞ norms of the electromagnetic field.

Recalling the expression of the metric tensor, in t, r, θ, ϕ coordinates,

$$ds^2 = -(1 - \frac{2m}{r})dt^2 + (1 - \frac{2m}{r})^{-1}dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2),$$

it turns out that the null ingoing geodesics tend asymptotically to the boundary $\partial\mathcal{M} = \{p \in \overline{\mathcal{M}} | r(p) = 2m\}$ of the external Schwarzschild spacetime, denoted hereafter by $\mathcal{M}$. Viceversa the outgoing null geodesics starting near to the boundary remain for a "long time" at a small distance from it, the longer the nearer to the boundary the geodesic starts. This is the reason why, as we discuss in detail later on, the asymptotic behaviour of the Maxwell solutions, although with the same power decay as the one in the Minkowski case, it is not "uniform" with respect to the distance of the null curves from $\partial\mathcal{M}$, at the time $t = 0$ [2].

Using similar techniques we can also control the asymptotic behaviour of the various derivatives of the Maxwell solutions. This is possible, of course, if we assume sufficiently regular initial data. These results will be carefully discussed in a subsequent paper.

The central result of this paper is Theorem 3.6 :

Let $\mathcal{M}_{\delta_0}$ be the region of the spacetime outside the "cone" made by the null outgoing geodesics which at $\Sigma_{t=0}$ have $r = 2m + \delta_0$. We denote $\alpha, \underline{\alpha}, \rho, \sigma$ the null decomposition of the Maxwell tensor field with respect to a moving frame adapted to the null outgoing and ingoing "cones" of the Schwarzschild spacetime and assume the inital data sufficiently regular; then there exists a positive function C_2 depending on the initial data norms, the mass m and the distance δ_0 such that [3] [4]

$$\sup_{\mathcal{M}_{\delta_0}} |r^{\frac{5}{2}}\alpha| \le C_2(m, \delta_0)$$

$$\sup_{\mathcal{M}_{\delta_0}} |r\tau_-^{\frac{3}{2}}\underline{\alpha}| \le C_2(m, \delta_0)$$

$$\sup_{\mathcal{M}_{\delta_0}} |r^2\tau_-^{\frac{1}{2}}(|\rho - \bar{\rho}|, |\sigma - \bar{\sigma}|)| \le C_2(m, \delta_0) \ .$$

[2] Our results are obtained in a coordinate independent way. Nevertheless the behaviour near the boundary of the spacetime has a partial dependance on the choice of the moving frame, see the discussion in subsection 2.2.

[3] τ_- is the equivalent, in the Schwarzschild spacetime, of the function $\left(1 + (t - r)^2\right)^{\frac{1}{2}}$ in the Minkowski case.

[4] $\bar{\rho}$ is the average of ρ over S, the two dimensional surface intersection of the outgoing and ingoing cones. As we discuss later on, r can be defined in a coordinate independent way as $r \equiv \left(\frac{|S|}{4\pi}\right)^{\frac{1}{2}}$ where $|S|$ is the area of S.

Moreover, if the initial data satisfy also the following conditions

$$\sup_{\Sigma_{t=0}} |r^2\overline{\rho}| \leq \frac{q_0}{4\pi}$$
$$\sup_{\Sigma_{t=0}} |r^2\overline{\sigma}| \leq \frac{h_0}{4\pi}$$

then there exists a function C_3 depending on the initial data, m and δ_0 such that

$$\sup_{\mathcal{M}_{\delta_0}} |r^2(|\rho|, |\sigma|)| \leq C_3(m, \delta_0) \ .$$

Finally the functions $C_2(m, \delta_0)$ and $C_3(m, \delta_0)$ diverge as $\delta_0 \to 0$.

Although in the seventies and in the eighties a considerable effort has been done studying the behaviour of scalar, spin 1 and spin 2 wave equations in curved spacetimes and in particular in the Schwarzschild spacetime, see, for instance [Bar-Pre], [Por-St], [St-Sch], [St], [Pr1], [Pr2], nevertheless general results of this type are, in fact, absent. Moreover, although the asymptotic behaviour we find can be considered the expected one as the asymptotic decay is the same as the one in the Minkowski case, nevertheless there are various aspects which is worthwhile to point out.

a) This result seems in disagreement with the expectations associated to the Penrose compactification method [Pe1], [Pe2].

b) Differently from the flat case, the asymptotic estimates in the null directions are not uniform. We have a partial control of the non uniformity.

c) The initial conditions on $\overline{\rho}$: $\sup_{\Sigma_{t=0}} |r^2\overline{\rho}| \leq \frac{q_0}{4\pi}$ can be interpreted as describing the electric charge inside the internal region of the extended Schwarzschild spacetime. Therefore this approach can be thought as a first (linear) step toward the study of the Einstein equations coupled with the Maxwell equations in the presence of a charged "black hole" ([Haw-El]). It can be seen also as a counterpart of the Reissner-Norsdstrom model.

d) A technical, but crucial aspect, discussed in detail along the paper, is the use of integral norms performed along the null outgoing and ingoing cones, instead of the more familiar ones, done over the constant time hypersurfaces.

e) The same techniques can be applied to the spin 2 equations obtaining similar results. In this case they can be seen as a preliminary step toward the proof of the much more complicated problem of the global nonlinear stability of the spacetime outside the domain of dependance of a compact region at $t = 0$, see [Kl-Ni] and [Ch-Kl-Ni].

f) An extension of the work of D. Christodolou and S. Klainerman [Ch-Kl1] to general spin field equations, always with the Minkowski spacetime as background spacetime, has been developed by Wei-Ton Shu [Shu]. In his final remarks he also suggests an extension to the Schwarzschild spacetime proposing some of the modified pseudo Killing vector fields we use here. Nevertheless he does not seem to realize the serious difficulty arising from the fact that they are not anymore Killing and moreover, in the case of K_0, not even asymptotically Killing.

2 General properties of the Schwarzschild spacetime and some analytic tools

In the spherical coordinates the Schwarzschild metric has the form:

$$ds^2 = -(1 - \tfrac{2m}{r})dt^2 + (1 - \tfrac{2m}{r})^{-1}dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2) \tag{2.1}$$

where m is the gravitational mass (units with $c = G = 1$). This metric is singular for $r = 0$ and for $r = 2m$, therefore one has to cut out of the manifold, defined by the coordinates (t, r, θ, ϕ), the regions $r = 0$ and $r = 2m$. The $r > 2m$ region, denoted by $\mathcal{M}$, is called the external Schwarzschild spacetime [5].

In this section we describe the causal structure of the Schwarzschild spacetime with its exact or approximate symmetries and some global Sobolev estimates which are needed to obtain the asymptotic behaviour.

2.1 The foliations of the Schwarzschild spacetime

We denote $C(u)$ the null outgoing hypersurfaces, we will call, hereafter, null outgoing cones. They are described by the equations $u(p) = const$ where u is a solution of the eikonal equation $g^{\mu\nu}\partial_\mu w \partial_\nu w = 0$, satisfying initial conditions on the spacelike hypersurface $\Sigma_0 = \{p \in \mathcal{M} | t(p) = 0\}$ such that the null geodesics generating it are outgoing ones. Their tangent vector field is $L^\mu = -g^{\mu\sigma}\partial_\sigma u$. Analogously $\underline{C}(\underline{u})$ are the null ingoing hypersurfaces, or ingoing cones, described by the equations $\underline{u}(p) = const$ and $\underline{u}$ is a solution of the eikonal equation satisfying initial conditions such that the null geodesics generating it are ingoing ones. Their tangent vector field is $\underline{L}^\mu = -g^{\mu\sigma}\partial_\sigma \underline{u}$. It is immediate to realize that in the Schwarzschild spacetime the functions $u(p)$ and $\underline{u}(p)$ are

$$\underline{u} = t + r_* \ , \ u = t - r_*$$

where

$$r_* \equiv r + 2m \log(\frac{r}{2m} - 1) \tag{2.2}$$

Defining $\Phi^2 = \left(1 - \frac{2m}{r}\right)$, the null geodesic vector fields are

$$\begin{aligned} L &= \Phi^{-2}\frac{\partial}{\partial t} + \frac{\partial}{\partial r} = \Phi^{-2}(\frac{\partial}{\partial t} + \frac{\partial}{\partial r_*}) = 2\Phi^{-2}\frac{\partial}{\partial \underline{u}} \\ \underline{L} &= \Phi^{-2}\frac{\partial}{\partial t} - \frac{\partial}{\partial r} = \Phi^{-2}(\frac{\partial}{\partial t} - \frac{\partial}{\partial r_*}) = 2\Phi^{-2}\frac{\partial}{\partial u} \end{aligned} \tag{2.3}$$

and $g(L, \underline{L}) = -2\Phi^{-2}$. From L and $\underline{L}$ we define the null vector fields

$$\begin{aligned} e_4 &= \Phi^{-1}(\frac{\partial}{\partial t} + \frac{\partial}{\partial r_*}) = 2\Phi^{-1}\frac{\partial}{\partial \underline{u}} \\ e_3 &= \Phi^{-1}(\frac{\partial}{\partial t} - \frac{\partial}{\partial r_*}) = 2\Phi^{-1}\frac{\partial}{\partial u} \end{aligned} \tag{2.4}$$

[5] For more details see [Haw-El] and [M-Th-W].

forming a null pair satisfying

$$g(e_3, e_3) = g(e_4, e_4) = 0\ ,\ g(e_3, e_4) = -2\ .$$

The null pair $\{e_3, e_4\}$ and the corresponding null frame is called "coordinate stationary observer frame", see for instance [Pr1], [Pr2]. In fact it is obtained starting from the orthonormal vector fields $\Phi^{-1}\frac{\partial}{\partial t}$ and $\Phi\frac{\partial}{\partial r}$ associated to the t, r coordinates of a stationary observer. Later on we will discuss the null frames associated to moving observers.

The null cones $C(u)$ and $\underline{C}(\underline{u})$ foliate the Schwarzschild spacetime. From them it is possible to define the two dimensional spacelike surfaces

$$S(u, \underline{u}) = C(u) \cap \underline{C}(\underline{u}) \tag{2.5}$$

which generate a two dimensional foliation. Another foliation we will use is the one made by the constant time hypersurfaces

$$\Sigma_t = \{p \in \mathcal{M} | t(p) = t\}$$

and, obviously, $S(u, \underline{u}) = S(u, t)$ where $S(u, t) = \Sigma_t \cap C(u)$. Adding to e_3, e_4 an orthonormal frame $\{e_a\}_{a=1,2}$ relative to the tangent space of the $S(u, \underline{u})$ surfaces we obtain a null frame "adapted" to this foliation. For instance

$$\begin{aligned} e_4 &= \Phi^{-1}(\frac{\partial}{\partial t} + \frac{\partial}{\partial r_*}) = \Phi L \\ e_3 &= \Phi^{-1}(\frac{\partial}{\partial t} - \frac{\partial}{\partial r_*}) = \Phi \underline{L} \\ e_\theta &= \frac{1}{r}\frac{\partial}{\partial \theta}\ ,\ e_\phi = \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi} \end{aligned} \tag{2.6}$$

The choice of the adapted null frame is not unique. In particular we can consider the following "scaling transformation" for the null pair e_3, e_4:

$${e_4}' = a e_4,\ \ {e_3}' = a^{-1} e_3 \tag{2.7}$$

where a is a scalar function on $\mathcal{M}$. The vectors of the null pair

$$\begin{aligned} e_4 &= \Phi^{-1}(\frac{\partial}{\partial t} + \frac{\partial}{\partial r_*}) = \Phi^{-1}\frac{\partial}{\partial t} + \Phi\frac{\partial}{\partial r} \\ e_3 &= \Phi^{-1}(\frac{\partial}{\partial t} - \frac{\partial}{\partial r_*}) = \Phi^{-1}\frac{\partial}{\partial t} - \Phi\frac{\partial}{\partial r} \end{aligned} \tag{2.8}$$

are a combination of the Schwarzschild coordinate basis normalized vectors. They are interpreted as the null pair associated to a stationary observer. If, instead, we consider a "freely falling observer", that is an observer moving along a radial geodesic, for instance from the spatial infinity toward the "origin" of the Schwarzschild spacetime, we can associate to it a different null pair e''_4, e''_3 [6]

$$e''_4 = \left(\frac{1-\beta}{1+\beta}\right)^{\frac{1}{2}} e_4 \ , \ e''_3 = \left(\frac{1+\beta}{1-\beta}\right)^{\frac{1}{2}} e_3 \tag{2.9}$$

where $\beta = -\left(\frac{2m}{r}\right)^{\frac{1}{2}}$ is his speed at the points p with coordinate $r(p) = r$. This null pair is connected to the previous null pair by the scaling transformation of coefficient $a = \left(\frac{1-\beta}{1+\beta}\right)^{\frac{1}{2}}$. This allows to reinterpret the asymptotic decay as seen by different observer.

2.2 The symmetries of the Schwarzschild spacetime

The Schwarzschild spacetime is static which means that the diffeomorphisms associated to the time translations are isometries, and is spherically symmetric, that is the group $SO(3)$ is a group of isometries, the two spheres $S(u, \underline{u})$ being their transitivity surfaces.

A family of one parameter diffeomorphisms, Φ_t, generated by a vector field X is a one parameter group of isometries if ${\Phi_t}^* g = g$ which implies $L_X g = 0$, where $L_X g$ is the Lie derivative of the metric with respect to X. In this case X is called a Killing vector field and $L_X g = 0$ is equivalent to the equation [7] $D_{(\alpha} X_{\beta)} = 0$. If the diffeomorphisms Φ_t are such that ${\Phi_t}^* g = \Omega_t^2 g$, with Ω_t a real, regular scalar

[6] The pair e''_4, e''_3 is obtained in the following way: first one proves that an observer starting at the "spatial" infinity with zero speed moves radially toward the origin with radial velocity $\beta = -\left(\frac{2m}{r}\right)^{\frac{1}{2}}$. Next we recall that the components of the vector fields of the tangent space at the generic point p of $\mathcal{M}$ can be interpreted as the normal coordinates associated to the point p of the manifold and, as they describe the local inertial frames of General Relativity, different normal coordinates are connected through Lorentz transformations. The Lorentz transformation associated to the previous β transforms $e_T = \Phi^{-1}\frac{\partial}{\partial t}$, $e_R = \Phi\frac{\partial}{\partial r}$ into the new vectors

$$\begin{aligned} e'_T &= \Phi^{-1} e_T + \Phi^{-1}\beta e_R \\ e'_R &= -\Phi^{-1}\beta e_T + \Phi^{-1} e_R \end{aligned}$$

recalling that $\Phi^{-1} = \left(1-\beta^2\right)^{\frac{1}{2}}$. Differently from the previous pair these vectors are not orthonormal, we form an orthonormal pair in the standard way obtaining

$$e''_T = \left(\frac{1-\beta^2}{1+\beta^2}\right) e'_T - \frac{2\beta}{1+\beta^2} e'_R \ , \ e''_R = e'_R \ .$$

The null pair $e''_4 = \frac{1}{\sqrt{2}}\left(e''_T + e''_R\right), e''_3 = \frac{1}{\sqrt{2}}\left(e''_T - e''_R\right)$ satisfies the following relation with the stationary observer's one

$$e''_4 = \left(\frac{1-\beta}{1+\beta}\right)^{\frac{1}{2}} e_4 \ , \ e''_3 = \left(\frac{1+\beta}{1-\beta}\right)^{\frac{1}{2}} e_3 \ .$$

[7] D_α is the covariant derivative $D_\alpha \equiv D_{\frac{\partial}{\partial x^\alpha}}$ associated to the metric g and $(\alpha\ \beta)$ is the symmetrisation symbol.

function on $\mathcal{M}$ different from zero, then $\{\Phi_t\}$ describes a conformal isometry and, in this case,

$$L_X g_{\alpha\beta} \equiv^{(X)} \pi_{\alpha\beta} = D_\alpha X_\beta + D_\beta X_\alpha = \lambda_X g_{\alpha\beta}$$

where $\lambda_X(p) = \frac{d}{dt}\Omega_{X,t}(p)\big|_{t=0}$ is a regular function on the manifold $\mathcal{M}$ and X is a conformal Killing vector field. The tensor ${}^{(X)}\pi$ is called the deformation tensor of X.

As shown in [Ch-Kl1], see also [Kl1] and [Kl2], and discussed in the next subsection, the presence of isometries or conformal isometries is crucial to obtain generalized energy-type estimates. These are used to derive, via Sobolev type theorems, the asymptotic decays of the solutions of the linear equations. Therefore one has to ask which vector fields can be used in the Schwarzschild spacetime, as, in this case, there are not conformal Killing vector fields.

We introduce the following vector fields which, although not conformal Killing vectors in $\mathcal{M}$ are such that their deformation tensors have nice properties. We call them "pseudo-Killing vectors" [8] [9]:

$$\begin{aligned}
&T_0 = \frac{\partial}{\partial t}, \quad S = t\frac{\partial}{\partial t} + r_*\frac{\partial}{\partial r_*}\\
&\Omega_{(i,j)} = x^i\frac{\partial}{\partial x^j} - x^j\frac{\partial}{\partial x^i}\\
&K_0 = 2tS + (r_*^2 - t^2)T_0
\end{aligned} \tag{2.10}$$

With the exception of T_0, generator of the time translations, and of the $\Omega_{(i,j)}$'s, generating the spatial rotations, the vector fields defined in eqs. 2.11 are not Killing

[8] In the eqs. 2.11, the coordinates x^i are the usual Cartesian ones $x^1 = r\sin\theta\cos\phi$, $x^2 = r\sin\theta\sin\phi$, $x^3 = r\cos\theta$.

[9] Adding to the previous ones the vector fields

$$\begin{aligned}
&T_r = \frac{\partial}{\partial r_*}\ ,\ \Omega_{(0,r)} = -t\left(\frac{\partial}{\partial r_*} + r_*\frac{\partial}{\partial t}\right)\\
&K_r = 2r_*S + (r_*^2 - t^2)T_r
\end{aligned}$$

the following commutation relations hold

$$\begin{aligned}
&[T_0, T_r] = 0\ ,\quad [T_0, \Omega_{(i,j)}] = 0\ ,\quad [T_r, \Omega_{(i,j)}] = 0\\
&[T_0, \Omega_{(0,r)}] = \eta_{00}T_r\ ,\quad [\Omega_{(i,j)}, S] = 0\\
&[T_r, \Omega_{(0,r)}] = -\eta_{rr}T_0\ ,\quad [S, \Omega_{(0,r)}] = 0\\
&[T_0, S] = T_0\ ,\quad [K_0, \Omega_{(0,r)}] = \eta_{00}K_r\ ,\quad [K_0, S] = -K_0\\
&[T_r, S] = T_r\ ,\quad [K_r, \Omega_{(0,r)}] = -\eta_{rr}K_0\ ,\quad [K_r, S] = K_r\\
&[K_0, T_r] = 2\Omega_{(0,r)}\ ,\quad [K_0, T_0] = 2\eta_{00}S\\
&[K_r, T_0] = -2\Omega_{(0,r)}\ ,\quad [K_r, T_r] = 2\eta_{rr}S\ ,\quad [\Omega_{(0,r)}, \Omega_{(i,j)}] = 0\\
&[\Omega_{(i,j)}, K_0] = 0\ ,\quad [\Omega_{(i,j)}, K_r] = 0\ ,\ [K_0, K_r] = -4r_*(r_*^2 - t^2)T_0\\
&[\Omega_{(i,j)}\ , \Omega_{(l,k)}] = \eta_{il}\Omega_{(k,j)} - \eta_{ik}\Omega_{(l,j)} + \eta_{jl}\Omega_{(i,k)} - \eta_{jk}\Omega_{(i,l)}\ .
\end{aligned}$$

where $\eta_{\mu\nu}$ is the metric of the flat space. The whole algebra will be used in a following paper to obtain the asymptotic behaviour of the higher derivatives.

nor conformal Killing and their deformation tensors satisfy the following lemma [10]:

Lemma 2.1. *Let* ${}^{(X)}\hat{\pi}$ *be the traceless part of the deformation tensor* ${}^{(X)}\pi = L_X g$. *Chosing as* X *the vectors* S, K_0, *the following expressions hold*

$$ {}^{(X)}\hat{\pi}_{\alpha\beta} = \mu_{(X)} sign(\alpha) g_{\alpha\beta} \tag{2.11} $$

where

$$ sign(\alpha) = \begin{cases} +1, & if \;\; \alpha : 0, r \\ -1, & if \;\; \alpha : \theta, \phi \end{cases} $$

and

$$ \begin{aligned} \mu_{(S)} &= 1 + r_*(\Phi\partial_r\Phi - \frac{\Phi^2}{r}) \\ \mu_{(K_0)} &= 2t\mu_{(S)} \end{aligned} \tag{2.12} $$

Moreover, for $\frac{m}{r}$ *small, the previous quantities are, approximately,*

$$ \mu_{(S)} = O(\frac{m}{r}\log\frac{r}{2m}) \ , \ \mu_{(K_0)} = O(m\frac{t}{r}\log\frac{r}{2m}) \tag{2.13} $$

showing that S *and* K_0, *in the limit* $m \to 0$, *are conformal Killing vector fields.*

It will be useful to express $T_0,\ S,\ K_0$ and $\bar{K} = K_0 + T_0$ in terms of the null pair $\{e_4, e_3\}$:

$$ \begin{aligned} T_0 &= \frac{\Phi}{2}(e_4 + e_3) \ , \ S = \frac{\Phi}{2}(\underline{u}e_4 + ue_3) \\ K_0 &= \frac{\Phi}{2}(\underline{u}^2 e_4 + u^2 e_3) \ , \ \bar{K} = \frac{\Phi}{2}(\tau_+^2 e_4 + \tau_-^2 e_3) \end{aligned} \tag{2.14} $$

where we define $\tau_\pm^2 \equiv l_0^2 + (r_* \pm t)^2$ and l_0, chosen $= 1$ unless explicitely stated, has the dimension of a length. We will often use also the following vector fields:

$$ \tilde{T}_0 = \frac{1}{2}(e_4 + e_3) = \Phi^{-1}T_0 \ , \ \tilde{N} = \frac{1}{2}(e_4 - e_3) = \Phi\frac{\partial}{\partial r} \equiv \Phi N \tag{2.15} $$

2.3 The connection coefficients of the external Schwarzschild spacetime

The way the submanifolds $S(u, \underline{u})$, see eq. 2.5, are embedded in $\mathcal{M}$ is determined by their null second fundamental forms χ, $\underline{\chi}$, two-covariant tensors on $S(u, \underline{u})$,

$$ \chi(X, Y) = g(\mathbf{D}_X e_4, Y), \ \ \underline{\chi}(X, Y) = g(\mathbf{D}_X e_3, Y) \tag{2.16} $$

[10]The μ_X functions for the fields, $K_r, \Omega_{(0,r)}$, introduced in footnote 9, are

$$ \mu_{(K_r)} = 2r_* + ({r_*}^2 + t^2)(\Phi\partial_r\Phi - \frac{\Phi^2}{r}) \ , \ \mu_{(\Omega_{(0,r)})} = t(\Phi\partial_r\Phi - \frac{\Phi^2}{r}) $$

and do not tend to zero for $\frac{m}{r} \to 0$.

where $X, Y \in TS$ and $\mathbf{D}$ denotes the connection on $(\mathcal{M}, g)$. In the general case the 1-form $\zeta(X) = \frac{1}{2}g(\mathbf{D}_X e_4, e_3)$, called the torsion of S, is also needed to describe the embedding. In the Schwarzschild spacetime, due to its rotational symmetry, ζ and the traceless parts of χ and $\underline{\chi}$ are identically zero so that

$$\chi_{ab} = \frac{1}{2}\delta_{ab} tr\chi = \delta_{ab}\frac{\Phi}{r}\ ,\ \underline{\chi}_{ab} = \frac{1}{2}\delta_{ab} tr\underline{\chi} = -\delta_{ab}\frac{\Phi}{r} \tag{2.17}$$

The second fundamental forms $\chi, \underline{\chi}$ and the torsion ζ are a subset of the whole family of connection coefficients which describe the geometric structure of the whole spacetime. The other ones different from zero are

$$\omega \equiv -\frac{1}{2}\mathbf{D}_4 \log\Phi\ ,\ \underline{\omega} \equiv -\frac{1}{2}\mathbf{D}_3 \log\Phi \tag{2.18}$$

These coefficients satisfy the null structure equations of the manifold.

2.4 Global Sobolev estimates

We introduce on $\mathcal{M}$ the following Euclidean pointwise norm for the generic $\binom{m}{n}$ tensor field U:

$$|U|^2 \quad \equiv \quad U^{\rho_1\ldots\rho_m}_{\mu_1\ldots\mu_n} U^{\sigma_1\ldots\sigma_m}_{\nu_1\ldots\nu_n}\, \bar{g}^{\mu_1\nu_1}\, \ldots\, \bar{g}^{\mu_n\nu_n}\, \bar{g}_{\rho_1\sigma_1}\, \ldots\, \bar{g}_{\rho_m\sigma_m} \tag{2.19}$$

where $\bar{g}_{\mu\nu} \equiv g_{\mu\nu} + 2(\tilde{T}_0)_\mu(\tilde{T}_0)_\nu$. We denote $\not{\mathbf{D}}_4, \not{\mathbf{D}}_3$ the projections over the tangent space TS of $\mathbf{D}_4 \equiv \mathbf{D}_{e_4}$ and $\mathbf{D}_3 \equiv \mathbf{D}_{e_3}$ and $\not\nabla$ the Levi-Civita connection relative to the induced metric on $S(u, \underline{u})$.
The proofs of the following lemmas and propositions are in the Appendix.

Lemma 2.2. *Let G be a C^∞ tensor field tangent to $S(u, \underline{u})$, then the following Sobolev estimate holds*

$$\sup_{S(u,\underline{u})} |G| \leq c r^{-\frac{1}{2}} \left(\int_{S(u,\underline{u})} |G|^4 + |r\not\nabla G|^4 \right)^{\frac{1}{4}} \tag{2.20}$$

Here and in the sequel c denotes a constant independent from the relevant parameters.

Proposition 2.3. *Let U be a C^∞ tensor field tangent at each point to the corresponding $S(u, \underline{u})$, let us denote $C(u; [\underline{u}_0, \underline{u}])$ the portion of the null outgoing cone where $\underline{u}'$ varies in the interval $[\underline{u}_0, \underline{u}]$ and introduce the analogous definition for $\underline{C}(\underline{u}, [u_0, u])$. The following estimates hold :*

$$\begin{aligned} \sup_{S(u,\underline{u})}(r^{\frac{3}{2}}|U|) &\leq c \left(\int_{S(u,\underline{u}_0)} r^4|U|^4 + r^4|r\not\nabla U|^4 \right)^{\frac{1}{4}} \\ &+ \left(\int_{C(u;[\underline{u}_0,\underline{u}])} |U|^2 + r^2|\not\nabla U|^2 + r^2|\not{\mathbf{D}}_4 U|^2 + r^4|\not\nabla^2 U|^2 + r^4|\not\nabla\not{\mathbf{D}}_4 U|^2 \right)^{\frac{1}{2}} \end{aligned} \tag{2.21}$$

$$\sup_{S(u,\underline{u})}(r\tau_-^{\frac{1}{2}}|U|) \leq c\left(\int_{S(u,\underline{u}_0)} r^2\tau_-^2|U|^4 + r^2\tau_-^2|r\nabla\!\!\!/ U|^4\right)^{\frac{1}{4}}$$
$$+\left(\int_{C(u;[\underline{u}_0,\underline{u}])} |U|^2 + r^2|\nabla\!\!\!/ U|^2 + \tau_-^2|\mathbf{D}\!\!\!\!/_4 U|^2 + r^4|\nabla\!\!\!/^2 U|^2 + r^2\tau_-^2|\nabla\!\!\!/\mathbf{D}\!\!\!\!/_4 U|^2\right)^{\frac{1}{2}} \tag{2.22}$$

where the integrals over the null cones $C(u;[\underline{u}_0,\underline{u}])$ and $\underline{C}(\underline{u};[u_0,u])$ are defined in the following way [11]

$$\int_{C(u;[\underline{u}_0,\underline{u}])} F \equiv \int_{\underline{u}_0}^{\underline{u}} d\underline{u}' \int_{S(u,\underline{u}')} \Phi F$$
$$\int_{\underline{C}(\underline{u};[u_0,u])} F \equiv \int_{u_0}^{u} du' \int_{S(u',\underline{u})} \Phi F. \tag{2.23}$$

A similar result holds expressing the sup-norms in terms of integrals along the $\underline{C}(\underline{u},[u_0,u])$ null cones:

$$\sup_{S(u,\underline{u})}(r^{\frac{3}{2}}|U|) \leq c\left(\int_{S(u_0,\underline{u})} r^4|U|^4 + r^4|r\nabla\!\!\!/ U|^4\right)^{\frac{1}{4}}$$
$$+\left(\int_{\underline{C}(\underline{u},[u_0,u])} |U|^2 + r^2|\nabla\!\!\!/ U|^2 + r^2|\mathbf{D}\!\!\!\!/_3 U|^2 + r^4|\nabla\!\!\!/^2 U|^2 + r^4|\nabla\!\!\!/\mathbf{D}\!\!\!\!/_3 U|^2\right)^{\frac{1}{2}} \tag{2.24}$$

$$\sup_{S(u,\underline{u})}(r\tau_-^{\frac{1}{2}}|U|) \leq c\left(\int_{S(u_0,\underline{u})} r^2\tau_-^2|U|^4 + r^2\tau_-^2|r\nabla\!\!\!/ U|^4\right)^{\frac{1}{4}}$$
$$+\left(\int_{\underline{C}(\underline{u},[u_0,u])} |U|^2 + r^2|\nabla\!\!\!/ U|^2 + \tau_-^2|\mathbf{D}\!\!\!\!/_3 U|^2 + r^4|\nabla\!\!\!/^2 U|^2 + r^2\tau_-^2|\nabla\!\!\!/\mathbf{D}\!\!\!\!/_3 U|^2\right)^{\frac{1}{2}} \tag{2.25}$$

Together with this proposition we will use another proposition, similar to the previous one, where the sup-norm of the function U is estimated in terms of L^2 norms relative to a Σ_t hypersurface.

Proposition 2.4. *Let U be a C^∞ tensor field tangent at each point p to the corresponding $S(t,r)$*[12] *and satisfying*

$$\lim_{r\to\infty} r|U(r,\omega)| = 0$$

[11] As $C(u;[\underline{u}_0,\underline{u}])$ and $\underline{C}(\underline{u};[u_0,u])$ are null hypersurfaces, there is not a "canonical" definition of their volume so that we are free to choose an appropriate definition for

$$\int_{C(u;[\underline{u}_0,\underline{u}])} \quad \text{and} \quad \int_{\underline{C}(\underline{u};[u_0,u])}.$$

The definition we use differs by a factor Φ, from the one used in [Ch-Kl2], page 221.

[12] $S(t,r) = S(t(u,\underline{u}), r(u,\underline{u})) = S(u,\underline{u})$.

where with ω we indicate the angular coordinates, then the following estimates hold:
Nondegenerate version:

$$\sup_{S(t,r)}(r^{\frac{3}{2}}|U|)$$
$$\leq c\left(\int_{\Sigma_t}|U|^2+r^2|\nabla\!\!\!/ U|^2+r^2|\mathbf{D}\!\!\!\!/_{\tilde{N}}U|^2+r^4|\nabla\!\!\!/^2U|^2+r^4|\nabla\!\!\!/\mathbf{D}\!\!\!\!/_{\tilde{N}}U|^2\right)^{\frac{1}{2}} \tag{2.26}$$

Degenerate version:

$$\sup_{S(t,r)}(r\tau_-^{\frac{1}{2}}|U|)$$
$$\leq c\left(\int_{\Sigma_t}|U|^2+r^2|\nabla\!\!\!/ U|^2+\tau_-^2|\mathbf{D}\!\!\!\!/_{\tilde{N}}U|^2+r^4|\nabla\!\!\!/^2U|^2+r^2\tau_-^2|\nabla\!\!\!/\mathbf{D}\!\!\!\!/_{\tilde{N}}U|^2\right)^{\frac{1}{2}} \tag{2.27}$$

where the integral over the spacelike hypersurface Σ_t is [13]

$$\int_{\Sigma_t}H\equiv\int_{2m}^{\infty}dr'\int_{S(t,r')}\Phi^{-1}H$$

An immediate corollary of this proposition is

Corollary 2.5. *Let U be a C^∞ tensor field tangent at each point p to the corresponding $S(t,r)$ and satisfying*

$$\lim_{r\to\infty}r|U(r,\omega)|=0$$

where with ω we indicate the angular coordinates, then the following estimates hold:
Nondegenerate version:

$$\sup_{r\geq 2m}\int_{S(t,r)}(r^4|U|^4+r^4|r\nabla\!\!\!/ U|^4)$$
$$\leq c\left(\int_{\Sigma_t}|U|^2+r^2|\nabla\!\!\!/ U|^2+r^2|\mathbf{D}\!\!\!\!/_{\tilde{N}}U|^2+r^4|\nabla\!\!\!/^2U|^2+r^4|\nabla\!\!\!/\mathbf{D}\!\!\!\!/_{\tilde{N}}U|^2\right)^{\frac{1}{2}} \tag{2.28}$$

[13] In this case
$dV=\theta^{(t)}\wedge\theta^{(r)}\wedge\theta^{(\theta)}\wedge\theta^{(\phi)}(\tilde{T}_0,\cdot,\cdot,\cdot)=\tilde{T}_0^\nu\tilde{\epsilon}_{\nu\alpha\beta\gamma}dx^\alpha\wedge dx^\beta\wedge dx^\gamma=\Phi^{-1}|detg|^{\frac{1}{2}}\epsilon_{ijk}dx^i\wedge dx^j\wedge dx^k$
and

$$\int_{\Sigma_t}dV=\int_{r=2m}^{\infty}dr\int_{S(t,r)}\Phi^{-1}r^2 sin\theta d\theta d\phi$$

Degenerate version:

$$\sup_{r\geq 2m}\int_{S(t,r)}(r^2\tau_-^2|U|^4+r^2\tau_-^2|r\nabla\!\!\!/ U|^4)$$
$$\leq c\left(\int_{\Sigma_t}|U|^2+r^2|\nabla\!\!\!/ U|^2+\tau_-^2|D\!\!\!\!/_{\tilde N}U|^2+r^4|\nabla\!\!\!/^2U|^2+r^2\tau_-^2|\nabla\!\!\!/ D\!\!\!\!/_{\tilde N}U|^2\right)^{\frac12} \tag{2.29}$$

3 The asymptotic behaviour of the solutions of the Maxwell equations

3.1 The null decomposition of the electromagnetic field

The electromagnetic field is a two form F. We denote by ${}^*F$ its Hodge dual, whose components are ${}^*F_{\mu\nu}=\frac12\tilde\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$ where $\tilde\epsilon=dV$ and, for a generic choice of coordinates, $\tilde\epsilon_{\mu\nu\rho\sigma}=(g)^{\frac12}\delta^1_{[\mu}\delta^2_{\nu}\delta^3_{\rho}\delta^4_{\sigma]}$. g denotes the absolute value $|det\{g_{\mu\nu}\}|$. The vacuum Maxwell equations are [14]:

$$D_{[\lambda}F_{\mu\nu]}=0\quad,\quad D_{[\lambda}{}^*F_{\mu\nu]}=0 \tag{3.30}$$

The following lemma, whose elementary proof is in [Ch-Kl1], will be used over and over:

Lemma 3.1. *The Maxwell equations are invariant under isometries and conformal isometries. In particular, if X is the Killing or conformal Killing vector field associated to the isometry and F is a solution of the Maxwell equations, then also L_XF is a solution.*

Given a vector field X we define the one form $i_XF\equiv F(\cdot,X)$. The one form $i_X{}^*F$ is defined in the same way. i_XF ,$i_X{}^*F$ completely determine the two form F at any point where $\langle X,X\rangle\equiv g(X,X)$ is different from zero. If $X=\tilde T_0$ the one forms $E=i_{\tilde T_0}F, H=i_{\tilde T_0}{}^*F$ are called the electric and magnetic parts of F, respectively, and they are tangential to the hypersurfaces Σ_t.

The null decomposition of the electromagnetic tensor F in terms of one forms and scalar functions on S is defined in the following way [15]:

$$\alpha_a\equiv\alpha(F)(e_a)=F(e_a,e_4)$$

[14] The vacuum Maxwell equations can also be written as

$$D^\mu F_{\mu\nu}=0\quad,\quad D^{\mu\,*}F_{\mu\nu}=0\ .$$

[15] $\alpha,\underline{\alpha},\rho,\sigma$ as tensor fields on $\mathcal M$ can be written as

$$\underline{\alpha}_\mu=\Pi^\lambda_\mu F_{\lambda\nu}e_3^\nu\ ,\ \alpha_\mu=\Pi^\lambda_\mu F_{\lambda\nu}e_4^\nu\ ,\ \rho=\frac12F_{\mu\nu}e_3^\mu e_4^\nu\ ,\ \sigma=\frac12\tilde\epsilon^{\mu\nu}F_{\mu\nu}$$

where $\tilde\epsilon_{\mu\nu}\equiv\frac12\tilde\epsilon_{\mu\nu\rho\sigma}e_3^\rho e_4^\sigma$ is the area form of the two spheres $S(u,\underline{u})$ and Π^λ_μ is the projection tensor over TS: $\Pi^{\mu\nu}=g^{\mu\nu}+\frac12(e_3^\mu e_4^\nu+e_4^\mu e_3^\nu)$.

$$
\begin{aligned}
&\underline{\alpha}_a \equiv \underline{\alpha}(F)(e_a) = F(e_a, e_3) \\
&\rho \equiv \rho(F) = \frac{1}{2}F(e_3, e_4) \\
&\sigma \equiv \sigma(F) = F(e_\theta, e_\phi)
\end{aligned}
\tag{3.31}
$$

${}^*F$ can be similarily decomposed in terms of ${}^{\otimes}\alpha$, ${}^{\otimes}\underline{\alpha}$, ${}^{\otimes}\rho$, ${}^{\otimes}\sigma$ with

$$
{}^{\otimes}\underline{\alpha} = {}^*\underline{\alpha} \ , \ {}^{\otimes}\alpha = -{}^*\alpha \ , \ {}^{\otimes}\rho = \sigma \ , \ {}^{\otimes}\sigma = -\rho
\tag{3.32}
$$

and * indicates the Hodge dual on $S(u, \underline{u})$. Finally the null components can be expressed in terms of electric and magnetic parts of F:

$$
\begin{aligned}
&\alpha_a = F(e_a, e_4) = (E_a + \epsilon_{ab}H_b) \\
&\underline{\alpha}_a = F(e_a, e_3) = (E_a - \epsilon_{ab}H_b) \\
&\rho = F(\tilde{T}_0, \tilde{N}) = -E_\perp \\
&\sigma = {}^*F(\tilde{T}_0, \tilde{N}) = -H_\perp
\end{aligned}
\tag{3.33}
$$

3.2 The Maxwell equations in the null decomposition

With respect to the null decomposition, the Maxwell equations have the following form [16]:

$$
\begin{aligned}
&\not{\mathbf{D}}_4\underline{\alpha} + (\partial_r\Phi + \frac{\Phi}{r})\underline{\alpha} + \not\nabla\rho -{}^* \not\nabla\sigma = 0 \\
&\not{\mathbf{D}}_3\alpha - (\partial_r\Phi + \frac{\Phi}{r})\alpha - \not\nabla\rho -{}^* \not\nabla\sigma = 0 \\
&\mathbf{D}_4\sigma + 2\frac{\Phi}{r}\sigma + \mathrm{c\not{u}rl}\,\alpha = 0 \ , \ \mathbf{D}_4\rho + 2\frac{\Phi}{r}\rho - \mathrm{d\not{i}v}\,\alpha = 0 \\
&\mathbf{D}_3\sigma - 2\frac{\Phi}{r}\sigma + \mathrm{c\not{u}rl}\,\underline{\alpha} = 0 \ , \ \mathbf{D}_3\rho - 2\frac{\Phi}{r}\rho + \mathrm{d\not{i}v}\,\underline{\alpha} = 0
\end{aligned}
\tag{3.34}
$$

which, written in terms of the null components, become

$$
\begin{aligned}
&\partial_{e_4}\underline{\alpha}(e_a) + (\partial_r\Phi + \frac{\Phi}{r})\underline{\alpha}(e_a) + \partial_{e_a}\rho - \epsilon_{ab}\partial_{e_b}\sigma = 0 \\
&\partial_{e_3}\alpha(e_a) - (\partial_r\Phi + \frac{\Phi}{r})\alpha(e_a) - \partial_{e_a}\rho - \epsilon_{ab}\partial_{e_b}\sigma = 0 \\
&\partial_{e_4}\sigma + 2\frac{\Phi}{r}\sigma + \partial_{e_\theta}\alpha(e_\phi) - \partial_{e_\phi}\alpha(e_\theta) + \frac{\cot\theta}{r}\alpha(e_\phi) = 0 \\
&\partial_{e_4}\rho + 2\frac{\Phi}{r}\rho - \partial_{e_\theta}\alpha(e_\theta) - \partial_{e_\phi}\alpha(e_\phi) - \frac{\cot\theta}{r}\alpha(e_\theta) = 0
\end{aligned}
\tag{3.35}
$$

[16] We use the following definitions

$$
\mathrm{d\not{i}v}\, f = \not\nabla^a f_a \ , \ \mathrm{c\not{u}rl}\, f = \epsilon_{ab}\not\nabla^a f^b.
$$

$$
\begin{aligned}
&\partial_{e_3}\sigma - 2\frac{\Phi}{r}\sigma + \partial_{e_\theta}\underline{\alpha}(e_\phi) - \partial_{e_\phi}\underline{\alpha}(e_\theta) + \frac{\cot\theta}{r}\underline{\alpha}(e_\phi) = 0\\
&\partial_{e_3}\rho - 2\frac{\Phi}{r}\rho + \partial_{e_\theta}\underline{\alpha}(e_\theta) + \partial_{e_\phi}\underline{\alpha}(e_\phi) + \frac{\cot\theta}{r}\underline{\alpha}(e_\theta) = 0
\end{aligned}
$$

3.3 The energy-momentum tensor and the energy-type norms

The control of the asymptotic behaviour of the solutions of Maxwell equations in the Schwarzschild spacetime is based, as discussed in the introduction, on two results:
a) The existence of bounded "energy-type" norms.
b) The existence of exact or, at least, approximated symmetries of the Schwarzschild spacetime.

The starting ingredient to define these "energy-type" norms is the Maxwell energy-momentum tensor Q:

$$
Q(X,Y) = \langle i_X F, i_Y F\rangle + \langle i_X {}^*F, i_Y {}^*F\rangle \tag{3.36}
$$

for generic vector fields X, Y. Its components are

$$
Q_{\mu\nu} = F_{\mu\rho}F_\nu{}^\rho + {}^*F_{\mu\rho}{}^*F_\nu{}^\rho = 2F_{\mu\rho}F_\nu{}^\rho - \tfrac{1}{2}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} \quad .
$$

Lemma 3.2. *The energy-momentum tensor field Q has the following properties:*

I) Q is symmetric, traceless and for any non spacelike future directed vector fields X, Y satisfies $Q(X,Y) \geq 0$.

II) If F is a solution of the Maxwell equations then Q has vanishing divergence

$$
D^\mu Q_{\mu\nu} = 0.
$$

The proof of $I)$ and $II)$ is elementary, see [Ch-Kl1]. The following expressions hold:

$$
\begin{aligned}
&Q(e_3,e_3) = 2|\underline{\alpha}|^2\ ,\ Q(e_4,e_4) = 2|\alpha|^2\\
&Q(e_3,e_4) = 2(\rho^2+\sigma^2)
\end{aligned} \tag{3.37}
$$

As $T_0, \bar{K}$ are timelike future directed vector fields, the following quantities are non negative [17] [18]

$$
\begin{aligned}
&Q(T_0,e_4) = \Phi\{|\alpha|^2 + (\rho^2+\sigma^2)\}\\
&Q(\bar{K},e_4) = \Phi\{\tau_+^2|\alpha|^2 + \tau_-^2(\rho^2+\sigma^2)\}\\
&Q(T_0,e_3) = \Phi\{|\underline{\alpha}|^2 + (\rho^2+\sigma^2)\}
\end{aligned}
$$

[17] The last two are in fact strictly positive.

[18] The choice of $\tilde{T}_0$ in the second argument of $Q(\cdot,\cdot)$ is due to the fact that $\tilde{T}_0$ is the unit vector normal to Σ_t and plays the same role as e_4, e_3 in the previous expressions. Viceversa in the first argument, T_0 is a Killing vector field and $\bar{K}$ approximates the corresponding conformal Killing vector field of the Minkowski spacetime.

$$
\begin{aligned}
&Q(\bar{K},e_3)=\Phi\{\tau_-^2|\underline{\alpha}|^2+\tau_+^2(\rho^2+\sigma^2)\} \qquad (3.38)\\
&Q(T_0,\tilde{T}_0)=\Phi/2\{|\underline{\alpha}|^2+|\alpha|^2+2(\rho^2+\sigma^2)\}\\
&Q(\bar{K},\tilde{T}_0)=\Phi/2\{{\tau_-}^2|\underline{\alpha}|^2+{\tau_+}^2|\alpha|^2+(\tau_+^2+\tau_-^2)(\rho^2+\sigma^2)\}
\end{aligned}
$$

The integrals of $Q(T_0,\tilde{T}_0)$, $Q(\bar{K},\tilde{T}_0)$ over the hypersurfaces Σ_t, the integrals of $Q(T_0,e_4)$, $Q(\bar{K},e_4)$ along the null cones $C(u)$ and those of $Q(T_0,e_3)$, $Q(\bar{K},e_3)$ along the null cones $\underline{C}(\underline{u})$ are the energy-type norms we are going to use. Their relevance follows from the fact that the properties of the tensor Q and the asymptotically approximate symmetries of the Schwarzschild spacetime allow to prove their boundedness once they are bounded on the initial hypersurface. This result, crucial for proving the asymptotic behaviour, is thoroughly investigated in the next section. Here we assume that a family of integral norms is bounded and derive from this assumption the asymptotic behaviour for the solutions of the Maxwell equations. Let us consider the following integrals [19]:

$$\int_{\Sigma_t} Q(L^a_{\mathcal{O}}F)(T_0,\tilde{T}_0) \quad , \quad \int_{\Sigma_t} Q(L^b_{T_0}L^a_{\mathcal{O}}F)(T_0,\tilde{T}_0) \qquad (3.39)$$

$$
\begin{aligned}
&\int_{C(u;[\underline{u}_0,\underline{u}])} Q(L^a_{\mathcal{O}}F)(\bar{K},e_4) \quad , \quad \int_{\underline{C}(\underline{u};[u_0,u])} Q(L^a_{\mathcal{O}}F)(\bar{K},e_3)\\
&\int_{C(u;[\underline{u}_0,\underline{u}])} Q(L_S L^a_{\mathcal{O}}F)(\bar{K},e_4) \quad , \quad \int_{\underline{C}(\underline{u};[u_0,u])} Q(L_S L^a_{\mathcal{O}}F)(\bar{K},e_3) \qquad (3.40)\\
&\int_{C(u;[\underline{u}_0,\underline{u}])} Q(L^b_S L^a_{\mathcal{O}}F)(\bar{K},e_4) \quad , \quad \int_{\underline{C}(\underline{u};[u_0,u])} Q(L^b_S L^a_{\mathcal{O}}F)(\bar{K},e_3)
\end{aligned}
$$

where $a\geq 1$, $b\geq 0$, $Q(L^a_{\mathcal{O}}F)$ is defined in the following way

$$Q(L^b_S L^a_{\mathcal{O}}F)=\sum_{i_1j_1,..,i_aj_a} Q(L^b_S L_{\Omega_{i_1j_1}}...L_{\Omega_{i_aj_a}}F) \qquad (3.41)$$

where $\Omega_{(ij)}$ are the Killing vector fields associated to the rotation group and in the sum: $i_l<j_l$. From these integrals we define the following norms:

$$
\begin{aligned}
&\mathcal{I}^{\mathcal{O}}_k(u;[\underline{u}_0,\underline{u}])=\sum_{1\leq a\leq k+2}\int_{C(u;[\underline{u}_0,\underline{u}])} Q(L^a_{\mathcal{O}}F)(\bar{K},e_4)\\
&\mathcal{I}^{S}_k(u;[\underline{u}_0,\underline{u}])=\sum_{1\leq a\leq k+1}\int_{C(u;[\underline{u}_0,\underline{u}])} Q(L_S L^a_{\mathcal{O}}F)(\bar{K},e_4)\\
&\underline{\mathcal{I}}^{\mathcal{O}}_k(\underline{u};[u_0,u])=\sum_{1\leq a\leq k+2}\int_{\underline{C}(\underline{u};[u_0,u])} Q(L^a_{\mathcal{O}}F)(\bar{K},e_3) \qquad (3.42)\\
&\underline{\mathcal{I}}^{S}_k(\underline{u};[u_0,u])=\sum_{1\leq a\leq k+1}\int_{\underline{C}(\underline{u};[u_0,u])} Q(L_S L^a_{\mathcal{O}}F)(\bar{K},e_3)
\end{aligned}
$$

[19] The reason why we do not assume the integrals $\int_{C,\underline{C}} Q(F)(\bar{K},e_{4,3})$ and $\int_{\Sigma} Q(F)(\bar{K},\tilde{T}_0)$ bounded follows from the remark b) after Theorem 3.7.

We introduce also similar quantities relative to Σ_t:

$$\begin{aligned}\mathcal{I}_k^{\mathcal{O}}(\Sigma_t) &= \sum_{1\leq a\leq k+2}\int_{\Sigma_t} Q(L_{\mathcal{O}}^a F)(\bar{K},\tilde{T}_0)\\ \mathcal{I}_k^{S}(\Sigma_t) &= \sum_{1\leq a\leq k+1}\int_{\Sigma_t} Q(L_S L_{\mathcal{O}}^a F)(\bar{K},\tilde{T}_0)\end{aligned} \tag{3.43}$$

and [20]

$$\begin{aligned}\mathcal{E}_k^{\mathcal{O}}(\Sigma_t) &= \sum_{1\leq a\leq k+2}\int_{\Sigma_t} Q(L_{\mathcal{O}}^a F)(T_0,\tilde{T}_0)\\ \mathcal{E}_k^{T_0}(\Sigma_t) &= \sum_{1\leq a\leq k+1}\int_{\Sigma_t} Q(L_{T_0} L_{\mathcal{O}}^a F)(T_0,\tilde{T}_0)\end{aligned} \tag{3.44}$$

Fixed $(u,\underline{u})$ let $\bar{t}\in[0,\frac{1}{2}(u+\underline{u})]$; we denote $u_0(\bar{t})$ and $\underline{u}_0(\bar{t})$ the values [21] of the functions $u(p)$, $\underline{u}(p)$ at the intersections $\underline{C}(\underline{u})\cap\Sigma_{\bar{t}}$ and $C(u)\cap\Sigma_{\bar{t}}$ respectively. Finally $\Sigma_t(\geq r)$ is the portion of the hypersurface Σ_t made by points whose radial coordinates are greater or equal to r.

The following proposition allows to estimate the sup-norms of the null components of the Maxwell fields in terms of the energy-type norms, eqs. 3.42, 3.43, relative to the null cones and to the $\Sigma_{\bar{t}}$ hypersurface.

Proposition 3.3. *Let $t=\frac{1}{2}(u+\underline{u})$ sufficiently large. There exists $\bar{t}\in[0,\frac{1}{2}(u+\underline{u}))$ such that every regular [22] solution of the vacuum Maxwell equations in the external Schwarzschild spacetime satisfies the following inequalities*

$$\begin{aligned}
r^{\frac{5}{2}}|\alpha(u,\underline{u})| &\leq c\Phi(r(u,\underline{u}_0(\bar{t})))^{-2}\Big[\big(\mathcal{I}_0^{\mathcal{O}}(u;[\underline{u}_0(\bar{t}),\underline{u}])+\mathcal{I}_0^{S}(u;[\underline{u}_0(\bar{t}),\underline{u}])\big)^{\frac{1}{2}}\\
&\quad+\big(\mathcal{I}_0^{\mathcal{O}}(\Sigma_{\bar{t}}(\geq r(u,\underline{u}_0(\bar{t})))+\mathcal{I}_0^{S}(\Sigma_{\bar{t}}(\geq r(u,\underline{u}_0(\bar{t})))\big)^{\frac{1}{2}}\Big]\\
r\tau_-^{\frac{3}{2}}|\underline{\alpha}(u,\underline{u})| &\leq c\Phi(r(u,\underline{u}))^{-2}\Big[\big(\underline{\mathcal{I}}_0^{\mathcal{O}}(\underline{u};[u_0(\bar{t}),u])+\underline{\mathcal{I}}_0^{S}(\underline{u};[u_0(\bar{t}),u])\big)^{\frac{1}{2}}\\
&\quad+\big(\mathcal{I}_0^{\mathcal{O}}(\Sigma_{\bar{t}}(\geq r(u_0(\bar{t}),\underline{u}))+\mathcal{I}_0^{S}(\Sigma_{\bar{t}}(\geq r(u_0(\bar{t}),\underline{u}))\big)^{\frac{1}{2}}\Big]\\
r^2(|\rho(u,\underline{u})|,|\sigma(u,\underline{u})|) &\leq c\sup\nolimits_{\Sigma_{\bar{t}}}|r^2(\bar{\rho},\bar{\sigma})|\\
&\quad+c\tau_-^{-\frac{1}{2}}\Phi(r(u,\underline{u}))^{-\frac{1}{2}}\Big[\underline{\mathcal{I}}_0^{\mathcal{O}}(\underline{u};[u_0(\bar{t}),u])^{\frac{1}{2}}\\
&\quad+\big(\mathcal{I}_0^{\mathcal{O}}(\Sigma_{\bar{t}}(\geq r(u_0(\bar{t}),\underline{u}))+\mathcal{I}_0^{S}(\Sigma_{\bar{t}}(\geq r(u_0(\bar{t}),\underline{u}))\big)^{\frac{1}{2}}\Big]
\end{aligned} \tag{3.45}$$

where c is a generic constant independent from $u,\underline{u}$.

[20] The $\mathcal{E}$ norms are introduced as they are used to obtain a weaker, but more general asymptotic result as discussed later on.

[21] In fact $u_0(\bar{t})$ and $\underline{u}_0(\bar{t})$ depend also on $\underline{u}$ and u respectively and should be written $u_0(\bar{t},\underline{u})$ and $\underline{u}_0(u,\bar{t})$, but we omit this dependence to simplify the notation.

[22] The exact meaning of "regular" is discussed in Theorem 3.6.

Proof. We sketch the proof for the null component α and leave in the Appendix the detailed proof for each component. We apply first Proposition 2.3 with $U = r\alpha$. Using the explicit expressions of the $Q(L_{\mathcal{O}}^{1;2}F)(\bar{K}, e_4)$ and $Q(L_S L_{\mathcal{O}} F)(\bar{K}, e_4)$, see eqs. 3.41, it follows that the $C(u, [\underline{u}_0, \underline{u}])$ integrals of $r^2|\alpha|^2, r^2|r\nabla\!\!\!/\alpha|^2, \ldots$ are bounded in terms of the norm integrals $\mathcal{I}_0^{\mathcal{O}}(u; [\underline{u}_0, \underline{u}])$ and $\mathcal{I}_0^S(u; [\underline{u}_0, \underline{u}])$. It is here that the factor $\Phi(r(u, \underline{u}_0(\bar{t})))$ appears. Next we use Corollary 2.5 to express the $L^4(S(u, \underline{u}_0(\bar{t})))$ norms of $r\alpha$ and $r^2\nabla\!\!\!/\alpha$ in terms of integrals [23] along the hypersurface $\Sigma_{\bar{t}}(\geq r(u, \underline{u}_0(\bar{t})))$ and, proceeding as before, these integrals are expressed in terms of the norms 3.43 producing, also in this case, the factor $\Phi(r(u, \underline{u}_0(\bar{t})))$. Analogous arguments, described in the Appendix, apply for the other null components. We observe that in the case of $\underline{\alpha}$ we are forced to use the norms $\underline{\mathcal{I}}_0^{\mathcal{O}}(\underline{u}; [u_0, u])$ and $\underline{\mathcal{I}}_0^S(\underline{u}; [u_0, u])$ and, in this case, the factor $\Phi(r(u_0(\bar{t}), \underline{u}))$ appears.

Remark. This Proposition is a preliminary step to the control of the asymptotic behaviour of the null components of the electromagnetic field. Here the need for $\bar{t}$ being large comes from the control of $\mathbf{D}\!\!\!\!/_4\alpha$ in terms of $\mathbf{D}\!\!\!\!/_S\alpha$ and $\mathbf{D}\!\!\!\!/_3\alpha$, see eq. 5.129. The main result, still to prove, is to show that the energy-type norms in 3.45 are bounded in terms of (the norms of) the initial data. These estimates also require an appropriate choice of $\bar{t}$.

The next two propositions describe the central technical result of the paper and together with the previous one allow to prove the main theorem. They prove that the $\mathcal{I}$ norms appearing in Proposition 3.3, although not conserved, can be bounded in terms of the initial data. Let

$$r_*(\delta_0) = \left((2m + \delta_0) + 2m \log \frac{\delta_0}{2m}\right) \tag{3.46}$$

and denote $\mathcal{M}_{\delta_0}$ the region of the external Schwarzschild spacetime

$$\mathcal{M}_{\delta_0} = \{p \in \mathcal{M} | t(p) \geq 0,\ u(p) \leq -r_*(\delta_0)\} \tag{3.47}$$

We define $V(u, \underline{u})$ the part of the domain of dependance of $S(u, \underline{u})$ above the initial hypersurface

$$V(u, \underline{u}) = \{p \in J^-(S(u, \underline{u})) | t(p) \geq 0\} \tag{3.48}$$

whose boundary is formed by the union of the portions of the null cones $C(u)$ and $\underline{C}(\underline{u})$ lying in $V(u, \underline{u})$ and a finite region of Σ_0. Moreover we decompose $V(u, \underline{u})$ as

$$V(u, \underline{u}) = V_{\leq \bar{t}}(u, \underline{u}) \cup V_{\geq \bar{t}}(u, \underline{u}) \tag{3.49}$$

where $V_{\geq \bar{t}}(u, \underline{u})$ denotes the part of $V(u, \underline{u})$ above $\Sigma_{\bar{t}}$.

[23] We have, nevertheless, to prove that the assumption

$$\lim_{r \to \infty} r|U(r, \omega)| = 0$$

required in Proposition 2.4 is satisfied for the choice of the U tensor done here. These asymptotic spatial behaviour can be proved using this same Proposition 3.3 with a δ_0 sufficiently large to choose $\bar{t} = 0$ and the fact that, due to Proposition 3.5 the r.h.s. of eqs. 3.45 are bounded.

Proposition 3.4. *Let $\delta_0 > 0$ and assume $t = \frac{1}{2}(u + \underline{u})$ sufficiently large* [24]. *Given a positive constant C_0 sufficiently large, there exists a time* [25] $\bar{t}_0 = \bar{t}(m, \delta_0)$ *such that for any $(u, \underline{u}) \in \mathcal{M}_{\delta_0}$, satisfying the assumption on t, and $\bar{t} \in [\bar{t}_0, t)$ the following estimates hold*

$$\begin{aligned}
\mathcal{I}_0^{\mathcal{O}}(u; [\underline{u}_0(\bar{t}), \underline{u}]) &\leq C_0 \mathcal{I}_0^{\mathcal{O}}(\Sigma_{\bar{t}} \cap V(u, \underline{u})) \leq C_0 \mathcal{I}_0^{\mathcal{O}}(\Sigma_{\bar{t}} \cap \mathcal{M}_{\delta_0}) \\
\mathcal{I}_0^{S}(u; [\underline{u}_0(\bar{t}), \underline{u}]) &\leq C_0 \left(\mathcal{I}_0^{\mathcal{O}}(\Sigma_{\bar{t}} \cap V(u, \underline{u})) + \mathcal{I}_0^{S}(\Sigma_{\bar{t}} \cap V(u, \underline{u})) \right) \\
&\leq C_0 \left(\mathcal{I}_0^{\mathcal{O}}(\Sigma_{\bar{t}} \cap \mathcal{M}_{\delta_0}) + \mathcal{I}_0^{S}(\Sigma_{\bar{t}} \cap \mathcal{M}_{\delta_0}) \right) \qquad (3.50) \\
\underline{\mathcal{I}}_0^{\mathcal{O}}(\underline{u}; [u_0(\bar{t}), u]) &\leq C_0 \mathcal{I}_0^{\mathcal{O}}(\Sigma_{\bar{t}} \cap V(u, \underline{u})) \leq C_0 \mathcal{I}_0^{\mathcal{O}}(\Sigma_{\bar{t}} \cap \mathcal{M}_{\delta_0}) \\
\underline{\mathcal{I}}_0^{S}(\underline{u}; [u_0(\bar{t}), u]) &\leq C_0 \left(\mathcal{I}_0^{\mathcal{O}}(\Sigma_{\bar{t}} \cap V(u, \underline{u})) + \mathcal{I}_0^{S}(\Sigma_{\bar{t}} \cap V(u, \underline{u})) \right) \\
&\leq C_0 \left(\mathcal{I}_0^{\mathcal{O}}(\Sigma_{\bar{t}} \cap \mathcal{M}_{\delta_0}) + \mathcal{I}_0^{S}(\Sigma_{\bar{t}} \cap \mathcal{M}_{\delta_0}) \right)
\end{aligned}$$

where $\Sigma_{\bar{t}} \cap V(u, \underline{u})$ is the subset of $\Sigma_{\bar{t}}$ with $r(p) \in [r(u, \underline{u}_0(\bar{t})), r(u_0(\bar{t}), \underline{u})]$ and $\Sigma_{\bar{t}} \cap \mathcal{M}_{\delta_0}$ is the subset of $\Sigma_{\bar{t}}$ with $r(p) \geq r(u = -r_*(\delta_0), \underline{u}_0(\bar{t}))$. From the previous equation it follows immediately that [26]

$$\begin{aligned}
\sup_{(u,\underline{u}) \in \mathcal{M}_{\delta_0} \cap V_{\geq \bar{t}}(u,\underline{u})} \mathcal{I}_0^{\mathcal{O}}(u; [\underline{u}_0(\bar{t}), \underline{u}]) &\leq C_0 \mathcal{I}_0^{\mathcal{O}}(\Sigma_{\bar{t}} \cap \mathcal{M}_{\delta_0}) \\
\sup_{(u,\underline{u}) \in \mathcal{M}_{\delta_0} \cap V_{\geq \bar{t}}(u,\underline{u})} \mathcal{I}_0^{S}(u; [\underline{u}_0(\bar{t}), \underline{u}]) &\leq C_0 \left(\mathcal{I}_0^{\mathcal{O}}(\Sigma_{\bar{t}} \cap \mathcal{M}_{\delta_0}) + \mathcal{I}_0^{S}(\Sigma_{\bar{t}} \cap \mathcal{M}_{\delta_0}) \right) \\
\sup_{(u,\underline{u}) \in \mathcal{M}_{\delta_0} \cap V_{\geq \bar{t}}(u,\underline{u})} \underline{\mathcal{I}}_0^{\mathcal{O}}(\underline{u}; [u_0(\bar{t}), u]) &\leq C_0 \mathcal{I}_0^{\mathcal{O}}(\Sigma_{\bar{t}} \cap \mathcal{M}_{\delta_0}) \qquad (3.51) \\
\sup_{(u,\underline{u}) \in \mathcal{M}_{\delta_0} \cap V_{\geq \bar{t}}(u,\underline{u})} \underline{\mathcal{I}}_0^{S}(\underline{u}; [u_0(\bar{t}), u]) &\leq C_0 \left(\mathcal{I}_0^{\mathcal{O}}(\Sigma_{\bar{t}} \cap \mathcal{M}_{\delta_0}) + \mathcal{I}_0^{S}(\Sigma_{\bar{t}} \cap \mathcal{M}_{\delta_0}) \right)
\end{aligned}$$

Moreover the function $\bar{t}(m, \delta_0)$ diverges as $\delta_0 \to 0$.

Proposition 3.5. *For a generic $\bar{t} > 0$ the following inequality holds*

$$\left(\mathcal{I}_0^{\mathcal{O}}(\Sigma_{\bar{t}} \cap \mathcal{M}_{\delta_0}) + \mathcal{I}_0^{S}(\Sigma_{\bar{t}} \cap \mathcal{M}_{\delta_0}) \right) \leq C_1(m, \delta_0; \bar{t}) \left(\mathcal{I}_0^{\mathcal{O}}(\Sigma_0) + \mathcal{I}_0^{S}(\Sigma_0) \right) \qquad (3.52)$$

$C_1(m, \delta_0; \bar{t})$ is a positive function increasing in m and $\bar{t}$ and decreasing in δ_0, such that, for $m > 0$

$$\begin{aligned}
\lim_{\delta_0 \to 0} C_1(m, \delta_0; \bar{t}) &= \infty \\
\lim_{\bar{t} \to \infty} C_1(m, \delta_0; \bar{t}) &= \infty \qquad (3.53)
\end{aligned}$$

The proofs of Propositions 3.4, 3.5 are discussed in section 4.

[24] The meaning of "sufficiently large" will be clear in the course of the proof.

[25] The time $\bar{t}_0$ depends, obviously, also on C_0.

[26] It is worthwhile to observe that the estimates for $\sup_{(u,\underline{u}) \in \mathcal{M}_{\delta_0}} \underline{\mathcal{I}}_0^{\mathcal{O}}(\underline{u}; [u_0(\bar{t}), u])$ and $\sup_{(u,\underline{u}) \in \mathcal{M}_{\delta_0}} \underline{\mathcal{I}}_0^{S}(\underline{u}; [u_0(\bar{t}), u])$ are very rough.

3.4 The asymptotic behaviour

Using the results discussed in the previous propositions we can state the main theorems of this paper concerning the asymptotic behaviour of a class of solutions of the Maxwell equations.

We consider the Cauchy problem for the Maxwell equations 3.30 in the external Schwarzschild spacetime where the initial data are given on the Σ_0 hypersurface. We specify them in terms of the norms 3.43, 3.44 relative to the Σ_0 hypersurface, with $k = 0$:

$$\begin{aligned} \mathcal{I}_0^{\mathcal{O}}(\Sigma_0) &= \sum_{1\leq a\leq 2} \int_{\Sigma_0} Q(L_{\mathcal{O}}^a F)(\bar{K}, \tilde{T}_0) \\ \mathcal{I}_0^{S}(\Sigma_0) &= \sum_{1\leq a\leq 1} \int_{\Sigma_0} Q(L_S L_{\mathcal{O}}^a F)(\bar{K}, \tilde{T}_0) \end{aligned} \tag{3.54}$$

$$\begin{aligned} \mathcal{E}_0^{\mathcal{O}}(\Sigma_0) &= \sum_{1\leq a\leq 2} \int_{\Sigma_0} Q(L_{\mathcal{O}}^a F)(T_0, \tilde{T}_0) \\ \mathcal{E}_0^{T_0}(\Sigma_0) &= \sum_{1\leq a\leq 1} \int_{\Sigma_0} Q(L_{T_0} L_{\mathcal{O}}^a F)(T_0, \tilde{T}_0) \end{aligned} \tag{3.55}$$

Theorem 3.6. *Let the initial data be such that $\mathcal{I}_0^{\mathcal{O}}(\Sigma_0)$ and $\mathcal{I}_0^{S}(\Sigma_0)$ are bounded, let $\delta_0 > 0$ be fixed, then there exists a positive function C_2 depending on the initial data $\mathcal{I}$-norms, m and δ_0 such that*

$$\begin{aligned} &\sup_{\mathcal{M}_{\delta_0}} |r^{\frac{5}{2}}\alpha| \leq C_2(m, \delta_0) \\ &\sup_{\mathcal{M}_{\delta_0}} |r\tau_-^{\frac{3}{2}}\underline{\alpha}| \leq C_2(m, \delta_0) \\ &\sup_{\mathcal{M}_{\delta_0}} |r^2\tau_-^{\frac{1}{2}}(|\rho - \bar{\rho}|, |\sigma - \bar{\sigma}|)| \leq C_2(m, \delta_0)\ . \end{aligned} \tag{3.56}$$

Moreover, if the initial data satisfy also the following conditions

$$\sup_{\Sigma_0} |r^2\overline{\rho}| \leq \frac{q_0}{4\pi}\ ,\ \sup_{\Sigma_0} |r^2\overline{\sigma}| \leq \frac{h_0}{4\pi} \tag{3.57}$$

it follows that

$$\sup_{\Sigma_t} |r^2\overline{\rho}| \leq \frac{q_0}{4\pi}\ ,\ \sup_{\Sigma_t} |r^2\overline{\sigma}| \leq \frac{h_0}{4\pi} \tag{3.58}$$

and there exists a constant C_3, depending on m, δ_0 and the initial data, such that

$$\sup_{\mathcal{M}_{\delta_0}} |r^2(|\rho|, |\sigma|)| \leq C_3(m, \delta_0) \tag{3.59}$$

Finally the constants $C_2(m,\delta_0)$ and $C_3(m,\delta_0)$ are bounded by a positive function which, as $\delta_0 \to 0$, diverges at most as:

$$C_{2,3}(m,\delta_0) = O\left(\left(\frac{2m}{\delta_0}\right)^{|m\log\frac{\delta_0}{2m}|^3}\right) \tag{3.60}$$

We recall that, for any $\delta_0 > 0$, the boundary of the region $\mathcal{M}_{\delta_0}$ is the part of the null cone $C(u = -r_*(\delta_0))$ with $t > 0$. Therefore to control the asymptotic behaviour of the Maxwell solution in $\mathcal{M}_{\delta_0}$ amounts to controlling the asymptotic behaviour along the null outgoing geodesics, that is moving on the null outgoing cones $C(u)$ or, obviously, along the spacelike curves inside $\mathcal{M}_{\delta_0}$ [27].

Using the $\mathcal{E}$ norms, eq.3.44, it is easier to prove a weaker proposition analogous to Proposition 3.3 and obtain the following

Theorem 3.7. *Let the initial data of the Maxwell equations be such that $\mathcal{E}_0^{\mathcal{O}}(\Sigma_0)$, $\mathcal{E}_0^{T_0}(\Sigma_0)$ are bounded, then, fixed $\delta_0 > 0$, there exists a positive constant C_4 depending on m, δ_0 and the initial data $\mathcal{E}$-norms such that for $r \geq 2m + \delta_0 \equiv r_0(\delta_0)$*

$$\begin{aligned}
&\sup_{r\geq 2m+\delta_0} |r^{\frac{1}{2}}\alpha| \leq C_4(m,\delta_0)\\
&\sup_{r\geq 2m+\delta_0} |r^{\frac{1}{2}}\underline{\alpha}| \leq C_4(m,\delta_0)\\
&\sup_{r\geq 2m+\delta_0} |r^{\frac{3}{2}}(|\rho|,|\sigma|)| \leq C_5(m,\delta_0)
\end{aligned}$$

where $C_4(m,\delta_0) \leq C\Phi(r_0(\delta_0))^{-\frac{3}{2}}$ and $C_5(m,\delta_0) \leq C\Phi(r_0(\delta_0))^{-\frac{1}{2}}$.

Remarks.

a) The difference between the last theorem and the previous one is that, in this case, the curves along which we consider the asymptotic behaviour can go out from the region $\mathcal{M}_{\delta_0}$ as, for instance, the time like curves $r = const$. The result is, nevertheless, much weaker and, probably, not optimal.

b) The rational behind the assumption on $\overline{\rho}$ and $\overline{\sigma}$ on Σ_0 is the following one: if we assume that $\sup_{\Sigma_0}|r^2\overline{\rho}|$ is bounded and different from zero, from $F(\tilde{N},\tilde{T}_0) = -\rho$ and $divE = 0$ it follows that

$$q_0 = \int_{S(t=0,r=2m+\delta_0)} \rho \neq 0.$$

This can be interpreted as the global electric charge contained in the internal part of the Schwarzschild spacetime. The requirement that $\sup_{\mathcal{M}_{\delta_0}}|r^2\overline{\rho}|$ is bounded can, therefore, be interpreted as the request that the global electric charge contained

[27] In fact one can also build a timelike curve totally contained in $\mathcal{M}_{\delta_0}$ which asymptotically approximate a null geodesic. Along this curve the result of Theorem 3.6 holds.

in the extended spacetime be finite. An analogous argument can be done for the assumption on $\overline{\sigma}$, and physically $\int_{S(t=0,r=2m+\delta_0)}\sigma$ can be interpreted as the global magnetic charge contained in the extended spacetime.

c) Proposition 3.4 has, in fact, a more general version allowing to bound a larger family of norms containing $\mathcal{I}_k^{\mathcal{O}}$, $\mathcal{I}_k^S$, $\underline{\mathcal{I}}_k^{\mathcal{O}}$, $\underline{\mathcal{I}}_k^S$ for $k \geq 0$ in terms of the corresponding quantities relative to the initial hypersurface. Using this general version we will be able to control the asymptotic behaviour of the derivatives of the solutions of the Maxwell equations in the region $\mathcal{M}_{\delta_0}$, if we control the analogous quantities on the initial hypersurface, that is provided that the initial data are sufficiently regular. This will be discussed in a next paper where the full algebra of the pseudo Killing vector fields will be used.

d) The main difference between the results proved here and the analogous ones proved in the flat case, see [Ch-Kl1], is that the asymptotic behaviour here is not uniform. This is expressed by the fact that the constants $C_2(m,\delta_0)$ and $C_3(m,\delta_0)$ diverge as δ_0 tends to zero [28]. Moreover we have to remark that the results obtained here are in disagreement with those expected using the compactification arguments, see [Pe1], [Pe2], concerning the asymptotic behaviour of α.

e) It is important to observe that if we choose a different null pair, for instance the one associated to the "freely falling observer",

$$e_4{}'' = \Lambda e_4,\ \ e_3{}'' = \Lambda^{-1}e_3$$

where $\Lambda = \sqrt{\frac{1-\beta}{1+\beta}}$, the null components of the Maxwell tensor field transform in the following way:

$$\begin{aligned}&\alpha''_a = \Lambda\alpha_a\ ,\ \underline{\alpha}''_a = \Lambda^{-1}\underline{\alpha}_a\\ &\rho'' = \rho\ ,\ \sigma'' = \sigma\end{aligned} \tag{3.61}$$

This remark shows that we cannot eliminate the non uniformity, for $\delta_0 \to 0$, of the functions $C_2(m,\delta_0)$ and $C_3(m,\delta_0)$ just "changing the observer".

4 The control of the energy-type norms

In this section we prove that the energy-type norms introduced in the previous section are bounded in terms of analogous norms for the initial data. This is the content of Propositions 3.4 and 3.5 which are the main technical results of this work and allow to prove Theorems 3.6, 3.7 discussed in the previous section. We recall a Proposition, whose simple proof is in [Ch-Kl1]

Proposition 4.1. *Let $Q(G)$ be the energy-momentum tensor field of an antisymmetric two form G and let X be a vector field. Define the covariant vector field*

[28] One has also to recall that the null cones of the Schwarzschild spacetime differ from the corresponding ones of the Minkowski spacetime and diverge from them asymptotically.

P associated to X, $P_\alpha = Q(G)_{\alpha\beta}X^\beta$, then, as Q is symmetric and traceless, it follows that[29]

$$divP = \left[(\mathbf{Div}Q(G))_\beta X^\beta + \frac{1}{2}Q_{\alpha\beta}{}^{(X)}\hat{\pi}^{\alpha\beta}\right] \tag{4.62}$$

where the deformation tensor ${}^{(X)}\pi = L_X g$, measures how much the diffeomorphism generated by X differs from an isometry or a conformal isometry. ${}^{(X)}\hat{\pi}$ is its traceless part.

Corollary 4.2. *Let the tensor field G satisfy the vacuum Maxwell equations and X be a Killing or conformal Killing vector field, then $divP = 0$.*

Integrating $divP$ in the region $V_{\geq\bar{t}}(u,\underline{u})$ and using Stokes theorem we obtain the following Lemma:

Lemma 4.3. *Let $P_\alpha = Q(G)_{\alpha\beta}X^\beta$ be defined as in Proposition 4.1, then Stokes theorem implies*

$$\left\{\int_{\underline{C}(\underline{u})\cap V_{\geq\bar{t}}(u,\underline{u})} \Phi^{-1}Q(G)(X,e_3) + \int_{C(u)\cap V_{\geq\bar{t}}(u,\underline{u})} \Phi^{-1}Q(G)(X,e_4) - \int_{\Sigma_{\bar{t}}\cap V(u,\underline{u})} Q(G)(X,\tilde{T}_0)\right\} = -\int_{V_{\geq\bar{t}}(u,\underline{u})}\left[(\mathbf{Div}Q(G))_\beta X^\beta + \frac{1}{2}Q(G)^{\alpha\beta\,(X)}\hat{\pi}_{\alpha\beta}\right] \tag{4.63}$$

Choosing as G, $L_{\Omega_{ij}}F$ and $L^2_{\Omega_{ij}}F$, with F solution of the Maxwell equations, and observing that, due to the spherical symmetry of the spacetime, $L^n_{\Omega_{ij}}F$ is also a solution for any ij and $n \geq 0$, it follows that, posing $X = T_0$, the $\mathcal{E}$ norms defined in eq. 3.44 are conserved [30]. Viceversa, posing $X = \bar{K}$ we have, see eq. 3.42,

$$\mathcal{I}_0^{\mathcal{O}}(u;[\underline{u}_0(\bar{t}),\underline{u}]) + \underline{\mathcal{I}}_0^{\mathcal{O}}(\underline{u};[u_0(\bar{t}),u]) - \mathcal{I}_0^{\mathcal{O}}(\Sigma_{\bar{t}}\cap V(u,\underline{u})) \leq Err^{(\mathcal{O})}(V_{\geq\bar{t}}(u,\underline{u})) \tag{4.64}$$

where $$Err^{(\mathcal{O})}(V_{\geq\bar{t}}(u,\underline{u})) = \frac{1}{2}\sum_{1\leq a\leq 2}\int_{V_{\geq\bar{t}}(u,\underline{u})} |{}^{(\bar{K})}\hat{\pi}^{\alpha\beta}Q(\,L^a_{\mathcal{O}}F)_{\alpha\beta}| \tag{4.65}$$

The analogous inequality is more complicated for $G = L_S L_{\Omega_{ij}}F$ as this form is not a solution of the Maxwell equations. We obtain

$$\mathcal{I}_0^{S}(u;[\underline{u}_0(\bar{t}),\underline{u}]) + \underline{\mathcal{I}}_0^{S}(\underline{u};[u_0(\bar{t}),u]) - \mathcal{I}_0^{S}(\Sigma_{\bar{t}}\cap V(u,\underline{u})) \leq Err^{(S)}(V_{\geq\bar{t}}(u,\underline{u})) \tag{4.66}$$

[29] $divP \equiv D^\alpha P_\alpha$.

[30] With an obvious modification of the region where the Stokes theorem applies.

where

$$Err^{(S)}(V_{\geq\bar{t}}(u,\underline{u})) = \frac{1}{2}\int_{V_{\geq\bar{t}}(u,\underline{u})}|{}^{(\bar{K})}\hat{\pi}^{\alpha\beta}Q(L_S\,L_{\mathcal{O}}F)_{\alpha\beta}| + \int_{V_{\geq\bar{t}}(u,\underline{u})}|(\mathbf{Div}Q(L_S\,L_{\mathcal{O}}F))_\alpha\bar{K}^\alpha| \tag{4.67}$$

Defining

$$\begin{aligned}\mathcal{H}^{\mathcal{O}}(\geq\bar{t};u,\underline{u}) &\equiv \sup_{(u',\underline{u}')\in V_{\geq\bar{t}}(u,\underline{u})}\left(\mathcal{I}_0^{\mathcal{O}}(\underline{u}';[u_0(\bar{t}),u'])+\underline{\mathcal{I}}_0^{\mathcal{O}}(\underline{u}';[u_0(\bar{t}),u'])\right)\\ \mathcal{H}^{S}(\geq\bar{t};u,\underline{u}) &\equiv \sup_{(u',\underline{u}')\in V_{\geq\bar{t}}(u,\underline{u})}\left(\mathcal{I}_0^{S}(\underline{u}';[u_0(\bar{t}),u'])+\underline{\mathcal{I}}_0^{S}(\underline{u}';[u_0(\bar{t}),u'])\right)\end{aligned} \tag{4.68}$$

$$\begin{aligned}\mathcal{H}(\geq\bar{t};u,\underline{u}) &\equiv \mathcal{H}^{\mathcal{O}}(\geq\bar{t};u,\underline{u})+\mathcal{H}^{S}(\geq\bar{t};u,\underline{u})\\ \mathcal{H}(\Sigma_{\bar{t}}\cap V(u,\underline{u})) &\equiv \mathcal{I}_0^{\mathcal{O}}(\Sigma_{\bar{t}}\cap V(u,\underline{u}))+\mathcal{I}_0^{S}(\Sigma_{\bar{t}}\cap V(u,\underline{u}))\end{aligned} \tag{4.69}$$

the following inequality holds

$$\mathcal{H}(\geq\bar{t};u,\underline{u})-\mathcal{H}(\Sigma_{\bar{t}}\cap V(u,\underline{u}))\leq Err^{(\mathcal{O})}(V_{\geq\bar{t}}(u,\underline{u}))+Err^{(S)}(V_{\geq\bar{t}}(u,\underline{u})) \tag{4.70}$$

In conclusion to bound $\mathcal{H}(\geq\bar{t};u,\underline{u})$ *in terms of* $\mathcal{H}(\Sigma_{\bar{t}}\cap V(u,\underline{u}))$ *we have to control*

$$Err(V_{\geq\bar{t}}(u,\underline{u}))\equiv Err^{(\mathcal{O})}(V_{\geq\bar{t}}(u,\underline{u}))+Err^{(S)}(V_{\geq\bar{t}}(u,\underline{u})) \tag{4.71}$$

From the inequality 4.70 it follows that we can control $\mathcal{H}(\geq\bar{t};u,\underline{u})$ in terms of $\mathcal{H}(\Sigma_{\bar{t}}\cap V(u,\underline{u}))$ if we are able to control the error term: $Err(V_{\geq\bar{t}}(u,\underline{u}))$. In a similar way to estimate $\mathcal{H}(\Sigma_{\bar{t}}\cap V(u,\underline{u}))$ in terms of $\mathcal{H}(\Sigma_0\cap V(u,\underline{u}))$ we have to control the corresponding error term relative, now, to the region $V_{\leq\bar{t}}$. This is the core of the technical part. The control of these error terms allow to prove the following two Propositions which, at their turn, imply Propositions 3.4, 3.5.

Proposition 4.4. *Fixed* m, $\delta_0>0$ *let* $V(u,\underline{u})\subset\mathcal{M}_{\delta_0}$ *with* $t=\frac{1}{2}(u+\underline{u})$ *very large* [31]*, then a sufficiently large* $\bar{t}_0=\bar{t}(m,\delta_0)<t$ *exists, depending on* m *and* δ_0*, but independent from* t*, such that for any* $\bar{t}\geq\bar{t}_0$ *the following inequality holds*

$$\mathcal{H}(\geq\bar{t};u,\underline{u})\leq C_5(m,\delta_0)\mathcal{H}(\Sigma_{\bar{t}}\cap V(u,\underline{u})) \tag{4.72}$$

where $C_5(m,\delta_0)$ *is a positive bounded function. Moreover, if* δ_0 *is sufficiently large* [32]*, it is possible to choose* $\bar{t}(m,\delta_0)=0$*.*

[31] As m is the only intrinsic length unit, t sufficiently large means $t=Mm$ with $M>>1$

[32] How much large is understood looking at the proof of the Proposition. In particular $\underline{u}_0(\bar{t}=0)=r_*(\delta_0)$ must be such that eqs. 4.82, 4.99 are satisfied.

Next Proposition implies immediately Proposition 3.5:

Proposition 4.5. *For any given m, $\delta_0 > 0$ and a generic $\bar{t}$ the following inequality holds*

$$\mathcal{H}(\Sigma_{\bar{t}} \cap \mathcal{M}_{\delta_0}(u, \underline{u})) \leq C_1(m, \delta_0, \bar{t})\mathcal{H}(\Sigma_0) \tag{4.73}$$

where $C_1(m, \delta_0; \bar{t})$ is a positive function increasing in m and $\bar{t}$ and decreasing in δ_0, such that, for $m > 0$

$$C_1(m, \delta_0) = O\left(\left(\frac{2m}{\delta_0}\right)^{|m \log \frac{\delta_0}{2m}|^3}\right) \tag{4.74}$$

Remark. It is important to observe that we have chosen to estimate the energy-type norms using the "flux-norms" above $\Sigma_{\bar{t}}$ and the energy-norms on the Σ_t's hypersurfaces, below $\Sigma_{\bar{t}}$. The reason is that to prove Proposition 4.4 the use of the "flux-norms" is required and the analogous result in terms of the Σ_t norms is false, see the remark at the end of the proof. The advantage of considering the flux-norms only above $\Sigma_{\bar{t}}$ lies in the fact that all the quantities Φ have, in this case, a lower bound strictly greater than zero and independent from u. Below $\Sigma_{\bar{t}}$ there is no advantage in using the flux-norms instead of the energy-type norms.

4.1 Proof of Proposition 4.4

The proof of Proposition 4.4 is divided in various parts. We start estimating $\mathcal{H}^{\mathcal{O}}(\geq \bar{t}; u, \underline{u})$, see eq. 4.68, in terms of $\mathcal{I}_0^{\mathcal{O}}(\Sigma_{\bar{t}} \cap V(u, \underline{u}))$.

Proposition 4.6. *For any $m, \delta_0 > 0$, fixed $\epsilon_{0,1}$ small, it is possible to find $\bar{t} \geq \bar{t}_0$ such that for any $V(u, \underline{u}) \subset \mathcal{M}_{\delta_0}$*

$$\mathcal{H}^{\mathcal{O}}(\geq \bar{t}; u, \underline{u}) - \mathcal{H}^{\mathcal{O}}(\Sigma_{\bar{t}} \cap V(u, \underline{u})) \leq \epsilon_{0,1}\mathcal{H}^{\mathcal{O}}(\geq \bar{t}; u, \underline{u}) \tag{4.75}$$

Proof. We have to control, see eq. 4.64,

$$Err^{(\mathcal{O})}(V_{\geq \bar{t}}(u, \underline{u})) = \frac{1}{2} \sum_{1 \leq a \leq 2} \int_{V_{\geq \bar{t}}(u, \underline{u})} |{}^{(\bar{K})}\hat{\pi}^{\alpha\beta} Q(\mathcal{L}_{\mathcal{O}}^a F)_{\alpha\beta}| \ .$$

As $Q(\mathcal{L}_{\mathcal{O}}^a F)_{\alpha\beta}$ is traceless and recalling the expression of ${}^{(\bar{K})}\hat{\pi}$, eq. 2.11, we have

$${}^{(\bar{K})}\hat{\pi}^{\alpha\beta} Q(\mathcal{L}_{\mathcal{O}}^a F)_{\alpha\beta} = -4t\mu_{(S)}(r) Q(\mathcal{L}_{\mathcal{O}}^a F)(e_3, e_4) \tag{4.76}$$

where, see eq. 2.12,

$$\mu_{(S)}(r) = 1 + r_*(\Phi\partial_r\Phi - \frac{\Phi^2}{r}) = \frac{m}{r}\left[3 - 2\left(1 - 3\frac{m}{r}\right)\log(\frac{r}{2m} - 1)\right] \ .$$

From eq. 3.38, as $Q(e_4, e_3)$, $Q(e_3, e_3)$ and $Q(e_4, e_4)$ are non negative,

$$Q(\,L^a_{\mathcal{O}}F)(e_3, e_4) \leq \frac{2}{\Phi}\frac{1}{\tau_+^2}Q(\,L^a_{\mathcal{O}}F)(\bar{K}, e_3)$$

and, therefore,

$$\begin{aligned}\int_{V_{\geq\bar{t}}(u,\underline{u})} |{}^{(\bar{K})}\pi^{\alpha\beta}Q(\,L^a_{\mathcal{O}}F)_{\alpha\beta}| &\leq 8\int_{V_{\geq\bar{t}}(u,\underline{u})} t\mu_{(S)}(r)\frac{1}{\Phi}\frac{1}{\tau_+^2}Q(\,L^a_{\mathcal{O}}F)(\bar{K}, e_3)\\ \leq c\frac{m}{\Phi(r(u,\underline{u}_0(\bar{t})))}\int_{\underline{u}_0(\bar{t})}^{\underline{u}}\frac{d\underline{u}'}{\tau_+^2}&\int_{\underline{C}(\underline{u}';[u_0(\bar{t}),u])}\frac{t\left(1+|\log(\frac{r}{2m}-1)|\right)}{r}Q(\,L^a_{\mathcal{O}}F)(\bar{K}, e_3)\end{aligned} \tag{4.77}$$

where c is a generic constant. To estimate the factor $\frac{t}{r}$ in 4.77 we use the following lemma easy to prove.

Lemma 4.7. *Fixed $m, \delta_0 > 0$, defining*

$$\begin{aligned}&\bar{t}_0 = 2|r_*(\delta_0)| = 2|(2m+\delta_0) + 2m\log\frac{\delta_0}{2m}|\ ,\ \textit{for}\ \delta_0 < 2m\\ &\bar{t}_0 = 0\ ,\ \textit{for}\ \delta_0 \geq 2m\end{aligned} \tag{4.78}$$

then, for $t > \bar{t}_0$, on any $C(u) \subset \mathcal{M}_{\delta_0}$, the following inequality holds

$$c_2 \leq \frac{t}{r} \leq c_1 \tag{4.79}$$

where c_1, c_2 are generic constants, moreover there exist constants c_3, c_4, independent from m, δ_0, such that $c_3 r \leq r_* \leq c_4 r$.

Substituting inequality 4.79 in the r.h.s. of 4.77 we obtain [33],

$$\begin{aligned}&\int_{V_{\geq\bar{t}}(u,\underline{u})} |{}^{(\bar{K})}\hat{\pi}^{\alpha\beta}Q(\,L^a_{\mathcal{O}}F)_{\alpha\beta}|\\ &\leq c\frac{m}{\Phi(r(u,\underline{u}_0(\bar{t})))}\int_{\underline{u}_0(\bar{t})}^{\underline{u}} d\underline{u}'\frac{\left(1+|\log(\frac{r(u_0(\bar{t}),\underline{u}')}{2m}-1)|\right)}{\underline{u}'^2}\\ &\qquad\qquad\int_{\underline{C}(\underline{u}';[u_0(\bar{t}),u])}Q(\,L^a_{\mathcal{O}}F)(\bar{K}, e_3)\\ &\leq c\frac{m}{\Phi(r(u,\underline{u}_0(\bar{t})))}\left[\sup_{(u',\underline{u}')\in V_{\geq\bar{t}}(u,\underline{u})}\left(\int_{\underline{C}(\underline{u}';[u_0(\bar{t}),u'])}Q(\,L^a_{\mathcal{O}}F)(\bar{K}, e_3)\right)\right]\cdot\\ &\qquad\qquad\cdot\int_{\underline{u}_0(\bar{t})}^{\infty} d\underline{u}'\frac{\left(1+|\log(\frac{r(u_0(\bar{t}),\underline{u}')}{2m}-1)|\right)}{\underline{u}'^2}\end{aligned}$$

[33] Recall that the generic constant c can be different in different inequalities.

$$\leq c\frac{m}{\Phi(r(u,\underline{u}_0(\bar{t})))}\left(\sup_{(u',\underline{u}')\in V_{\geq\bar{t}}(u,\underline{u})}\underline{\mathcal{I}}_0^{\mathcal{O}}(\underline{u}';[u_0(\bar{t}),u'])\right)\cdot$$
$$\cdot\int_{\underline{u}_0(\bar{t})}^{\infty}d\underline{u}'\frac{\left(1+|\log(\frac{r(u_0(\bar{t}),\underline{u}')}{2m}-1)|\right)}{\underline{u}'^2} \tag{4.80}$$

and from it

$$Err^{(\mathcal{O})}(V_{\geq\bar{t}}(u,\underline{u}))\leq c\left(\frac{m\left(1+|\log(\frac{r(u_0(\bar{t}),\underline{u}_0(\bar{t}))}{2m}-1)|\right)}{\Phi(r(u,\underline{u}_0(\bar{t})))\underline{u}_0(\bar{t})}\right)\mathcal{H}^{\mathcal{O}}(\geq\bar{t};u,\underline{u}) \tag{4.81}$$

Therefore, fixed m,δ_0, given $\epsilon_{0,1}$ small, it is possible to find $\bar{t}$ and, consequently, $\underline{u}_0(\bar{t})$ such that

$$c\left(\frac{m\left(1+|\log(\frac{r}{2m}-1)|\right)}{\Phi(r(u,\underline{u}_0(\bar{t})))\underline{u}_0(\bar{t})}\tilde{c}_0(\bar{t},r_0)\right)\leq\epsilon_{0,1}$$

proving the proposition. The estimate of $\mathcal{H}^S(\geq\bar{t};u,\underline{u})$ is provided by the following

Proposition 4.8. *For any $m,\delta_0>0$, fixed $\epsilon_{0,2}$ small, it is possible to find $\bar{t}\geq\bar{t}_0$ such that for any $V(u,\underline{u})\subset\mathcal{M}_{\delta_0}$*

$$\begin{aligned}&\mathcal{H}^{\mathcal{S}}(\geq\bar{t};u,\underline{u})-\left(\mathcal{H}^{\mathcal{O}}(\Sigma_{\bar{t}}\cap V(u,\underline{u}))+\mathcal{H}^{\mathcal{S}}(\Sigma_{\bar{t}}\cap V(u,\underline{u}))\right)\\&\quad\leq\epsilon_{0,2}\left(\mathcal{H}^{\mathcal{O}}(\geq\bar{t};u,\underline{u})+\mathcal{H}^{\mathcal{S}}(\geq\bar{t};u,\underline{u})\right)\end{aligned} \tag{4.82}$$

Proof. We have to control the various integrals in

$$\begin{aligned}Err^{(S)}(V_{\geq\bar{t}}(u,\underline{u})) &= \frac{1}{2}\int_{V_{\geq\bar{t}}(u,\underline{u})}|^{(\bar{K})}\hat{\pi}^{\alpha\beta}Q(L_S\,L_{\mathcal{O}}F)_{\alpha\beta}|\\&+\int_{V_{\geq\bar{t}}(u,\underline{u})}|(\mathbf{Div}Q(L_S\,L_{\mathcal{O}}F))_\alpha\bar{K}^\alpha|\end{aligned} \tag{4.83}$$

The first term is controlled as in the previous case obtaining, for $\bar{t}$ sufficiently large,

$$\begin{aligned}&\frac{1}{2}\int_{V_{\geq\bar{t}}(u,\underline{u})}|^{(\bar{K})}\hat{\pi}^{\alpha\beta}Q(L_S\,L_{\mathcal{O}}F)_{\alpha\beta}|\\ \leq\;&\frac{\epsilon_{0,2}}{2}\sup_{\underline{u}'\in[\underline{u}_0,\underline{u}]}\left(\int_{\underline{C}(\underline{u}')\cap V_{\geq\bar{t}}(u,\underline{u})}Q(L_S\,L_{\mathcal{O}}F)(\bar{K},e_3)\right)\\ \leq\;&\frac{\epsilon_{0,2}}{2}\left(\sup_{(u',\underline{u}')\in V_{\geq\bar{t}}(u,\underline{u})}\underline{\mathcal{I}}_0^S(u',\underline{u}')\right)\leq\frac{\epsilon_{0,2}}{2}\mathcal{H}^S(\geq\bar{t};u,\underline{u})\end{aligned} \tag{4.84}$$

The control of the second integral in 4.83 is more delicate. We need the explicit expression of $\mathbf{Div}Q(L_S\, L_{\mathcal{O}}F)$ which is provided by the following lemma, whose proof is in the Appendix,

Lemma 4.9. *Let F be a solution of the Maxwell equations, then, denoting $\tilde{F} \equiv L_S L_{\Omega_{ij}}F$ and $\tilde{Q}_{\mu\nu} \equiv Q(L_S L_{\Omega_{ij}}F)_{\mu\nu}$, the following relation holds*

$$D^\mu \tilde{Q}_{\mu\nu} = 2(J^{(1)}_\rho + J^{(2)}_\rho + J^{(3)}_\rho)\tilde{F}_\nu{}^\rho \tag{4.85}$$

$$= \sum_{l=1}^{3}\left[-J^{(l)}(e_3)e_4^\sigma - J^{(l)}(e_4)e_3^\sigma + 2J^{(l)}(e_\theta)e_\theta^\sigma + 2J^{(l)}(e_\phi)e_\phi^\sigma\right](L_S L_{\Omega_{ij}}F)_{\nu\sigma}$$

where

$$\begin{aligned} J^{(1)}_\rho &\equiv J^{(1)}_\rho(L_{\Omega_{ij}}F) = {}^{(S)}\hat{\pi}^{\mu\sigma}D_\sigma L_{\Omega_{ij}}F_{\mu\rho} \\ J^{(2)}_\rho &\equiv J^{(2)}_\rho(L_{\Omega_{ij}}F) = {}^{(S)}\Gamma_\lambda\, L_{\Omega_{ij}}F^\lambda{}_\rho \\ J^{(3)}_\rho &\equiv J^{(3)}_\rho(L_{\Omega_{ij}}F) = {}^{(S)}\Gamma_{\sigma\rho\,\lambda}\, L_{\Omega_{ij}}F^{\sigma\lambda} \end{aligned} \tag{4.86}$$

$$\begin{aligned} {}^{(S)}\Gamma_{\sigma\rho\,\lambda} &= \frac{1}{2}\left[g_{\lambda\sigma}\triangle_{(sign(\lambda))}\delta^r_\mu + g_{\lambda\mu}\triangle_{(sign(\lambda))}\delta^r_\sigma - g_{\sigma\mu}\triangle_{(sign(\sigma))}\delta^r_\lambda\right] \\ {}^{(S)}\Gamma_\lambda &= g^{\sigma\rho\,(S)}\Gamma_{\sigma\rho\,\lambda} \end{aligned} \tag{4.87}$$

with

$$\begin{aligned} \triangle_{(sign(\lambda))} &= \triangle_+ = \partial_r({}^{(S)}\mu + \frac{1}{4}tr^{(S)}\pi) = O(m)\frac{1}{r^2}, \quad if\ \lambda \in \{0, r\} \\ \triangle_{(sign(\lambda))} &= \triangle_- = \partial_r(-{}^{(S)}\mu + \frac{1}{4}tr^{(S)}\pi) = O(m)\frac{\log r}{r^2}, \quad if\ \lambda \in \{\theta, \phi\} \end{aligned} \tag{4.88}$$

Moreover it is simple to obtain the following expression [34]

$$\begin{aligned} (\mathbf{Div}Q(L_S\, L_{\mathcal{O}}F))_\nu \bar{K}^\nu &= \sum_{l=1}^{3}\Big\{[-J^{(l)}(e_3)e_4^\sigma - J^{(l)}(e_4)e_3^\sigma \\ &+ 2J^{(l)}(e_\theta)e_\theta^\sigma + 2J^{(l)}(e_\phi)e_\phi^\sigma](L_S\, L_{\mathcal{O}}F)_{\nu\sigma}\Big\}(\Phi/2)(\tau_+^2 e_4^\nu + \tau_-^2 e_3^\nu) \\ &= \sum_{l=1}^{3}\Phi\Big\{\tau_+^2[J^{(l)}(e_4)\rho(L_S\, L_{\mathcal{O}}F) - I^{(l)}\cdot\alpha(L_S\, L_{\mathcal{O}}F)] \\ &- \tau_-^2[J^{(l)}(e_3)\rho(L_S\, L_{\mathcal{O}}F) - I^{(l)}\cdot\underline{\alpha}(L_S\, L_{\mathcal{O}}F)]\Big\} \end{aligned} \tag{4.89}$$

where $I^{(l)}\cdot\alpha(L_S\, L_{\mathcal{O}}F) \equiv \sum_a J^{(l)}(e_a)\cdot\alpha(L_S\, L_{\mathcal{O}}F)(e_a)$. A similar expression holds for $I^{(l)}\cdot\underline{\alpha}(L_S\, L_{\mathcal{O}}F)$.

[34] This expression is written in a slightly simbolical way. For instance, with $J^{(l)}(e_4)\rho(L_S\, L_{\mathcal{O}}F)$ we mean

$$J^{(l)}(e_4)\rho(L_S\, L_{\mathcal{O}}F) = \sum_{i<j}\left(J^{(l)}_\nu(L_{\Omega_{ij}}F)e_4^\nu\right)\rho(L_S L_{\Omega_{ij}}F).$$

Hereafter with $L_{\mathcal{O}}F$ we mean $L_{\Omega_{ij}}F$ for a generic i, j.

To estimate $\int_{V_{\geq \bar{t}(u,\underline{u})}} |(\mathbf{Div}Q(L_S\, L_{\mathcal{O}}F))_\nu \bar{K}^\nu|$ we decompose the currents $J^{(l)}$ in terms of the null components of the Maxwell tensor field. The explicit computation is in the Appendix and the result is

$$\begin{aligned}
J^{(1)}(e_4) &= \mu_{(S)}(r)[4\frac{\Phi}{r}\rho - 2\,\mathrm{d}\!\!/\mathrm{iv}\,\alpha] \\
J^{(1)}(e_3) &= \mu_{(S)}(r)[4\frac{\Phi}{r}\rho - 2\,\mathrm{d}\!\!/\mathrm{iv}\,\underline{\alpha}] \\
J^{(1)}(e_\theta) &= \mu_{(S)}(r)[2\partial_{e_\phi}\sigma + \frac{\Phi}{r}(\alpha(e_\theta) - \underline{\alpha}(e_\theta))] \\
J^{(1)}(e_\phi) &= \mu_{(S)}(r)[-2\partial_{e_\theta}\sigma + \frac{\Phi}{r}(\alpha(e_\phi) - \underline{\alpha}(e_\phi))]
\end{aligned} \tag{4.90}$$

where, here, $\alpha, \underline{\alpha}, \rho, \sigma$ denote $\alpha(L_{\mathcal{O}}F), \underline{\alpha}(L_{\mathcal{O}}F), \rho(L_{\mathcal{O}}F), \sigma(L_{\mathcal{O}}F)$. The estimates of the various components of these currents are given in the next Lemma. They use the fact that, due to the symmetries of the Schwarzschild spacetime, $\alpha(L_{\Omega_{ij}}F) = L_{\Omega_{ij}}\alpha(F)$ and

$$|\,L_{\mathcal{O}}\alpha|^2 = r^2|\nabla\!\!\!/\alpha|^2 + |\alpha|^2\ ,\ |\,L_{\mathcal{O}}\underline{\alpha}|^2 = r^2|\nabla\!\!\!/\underline{\alpha}|^2 + |\underline{\alpha}|^2\ ,\ |\,L_{\mathcal{O}}(\rho,\sigma)|^2 = r^2|\nabla\!\!\!/(\rho,\sigma)|^2.$$

Lemma 4.10. *Let F be a solution of the Maxwell equations, let us denote* [35] *with $L^a_{\mathcal{O}}F$ a tensor $L_{\Omega_{i_1j_1}}...L_{\Omega_{i_aj_a}}F$, for an arbitrary choice of the indices $i_1...i_a, j_1...j_a$, then, using the explicit expression of ${}^{(S)}\hat{\pi}^{\mu\nu}$ and the Maxwell equations, the various components of the current $J^{(1)}, J^{(2)}, J^{(3)}$ associated to $L^a_{\mathcal{O}}F$ have the following estimates*

$$\begin{aligned}
|J^{(1)}(e_3)| &\leq c\frac{\mu_{(S)}(r)}{r}(|r\nabla\!\!\!/\underline{\alpha}(L^a_{\mathcal{O}}F)| + |r\nabla\!\!\!/\rho(L^a_{\mathcal{O}}F)|) \\
|J^{(1)}(e_4)| &\leq c\frac{\mu_{(S)}(r)}{r}(|r\nabla\!\!\!/\alpha(L^a_{\mathcal{O}}F)| + |r\nabla\!\!\!/\rho(L^a_{\mathcal{O}}F)|) \\
|J^{(1)}(e_\theta)| &\leq c\frac{\mu_{(S)}(r)}{r}(|r\nabla\!\!\!/\sigma(L^a_{\mathcal{O}}F)| + |\alpha(L^a_{\mathcal{O}}F)| + |\underline{\alpha}(L^a_{\mathcal{O}}F)| \\
&+ |r\nabla\!\!\!/\alpha(L^a_{\mathcal{O}}F)| + |r\nabla\!\!\!/\underline{\alpha}(L^a_{\mathcal{O}}F)|) \\
|J^{(1)}(e_\phi)| &\leq c\frac{\mu_{(S)}(r)}{r}(|r\nabla\!\!\!/\sigma(L^a_{\mathcal{O}}F)| + |\alpha(L^a_{\mathcal{O}}F)| + |\underline{\alpha}(L^a_{\mathcal{O}}F)| \\
&+ |r\nabla\!\!\!/\alpha(L^a_{\mathcal{O}}F)| + |r\nabla\!\!\!/\underline{\alpha}(L^a_{\mathcal{O}}F)|)
\end{aligned} \tag{4.91}$$

and, for the generic component of $J^{(2,3)}$,

$$|J^{(2,3)}| \leq c\frac{\mu_{(S)}(r)}{r}(|\alpha(L^a_{\mathcal{O}}F)| + |\underline{\alpha}(L^a_{\mathcal{O}}F)| + |\rho(L^a_{\mathcal{O}}F)|) \tag{4.92}$$

[35] Remark that this definition is slightly different from the one in eq. 3.41.

Applying this Lemma to $\mathbf{Div}Q(L_S\, L_{\mathcal{O}}F)$ and recalling the relations between the Lie derivatives and the covariant derivatives, the following estimates hold

$$
\begin{aligned}
|J^{(1)}(e_3)| &\leq c\frac{\mu_{(S)}(r)}{r}\Big(|r\nabla\!\!\!/\underline{\alpha}(F)|+|r^2\nabla\!\!\!/^2\underline{\alpha}(F)|+|r\nabla\!\!\!/\rho(F)|\Big)\\
|J^{(1)}(e_4)| &\leq c\frac{\mu_{(S)}(r)}{r}(|r\nabla\!\!\!/\alpha(F)|+|r^2\nabla\!\!\!/^2\alpha(F)|+|r\nabla\!\!\!/\rho(F)|) \qquad (4.93)\\
|J^{(1)}(e_\theta)| &\leq c\frac{\mu_{(S)}(r)}{r}(|r^2\nabla\!\!\!/^2\sigma(F)|+|\alpha(F)|+|\underline{\alpha}(F)|+|r\nabla\!\!\!/\alpha(F)|+|r\nabla\!\!\!/\underline{\alpha}(F)|)\\
|J^{(1)}(e_\phi)| &\leq c\frac{\mu_{(S)}(r)}{r}(|r^2\nabla\!\!\!/^2\sigma(F)|+|\alpha(F)|+|\underline{\alpha}(F)|+|r\nabla\!\!\!/\alpha(F)|+|r\nabla\!\!\!/\underline{\alpha}(F)|)
\end{aligned}
$$

$$
|J^{(2,3)}|\leq c\frac{\mu_{(S)}(r)}{r}\left(|\alpha(F)|+|r\nabla\!\!\!/\alpha(F)|+|\underline{\alpha}(F)|+|r\nabla\!\!\!/\underline{\alpha}(F)|+|r\nabla\!\!\!/\rho(F)|\right) \qquad (4.94)
$$

Using these estimates we obtain the following bound

$$
\begin{aligned}
&\int_{V_{\geq\bar{t}}(u,\underline{u})}|(\mathbf{Div}Q(L_S\, L_{\mathcal{O}}F))_\alpha\bar{K}^\alpha|\leq \qquad (4.95)\\
&c\int_{V_{\geq\bar{t}}(u,\underline{u})}\frac{\mu_{(S)}(r)}{r}\Big\{\tau_+^2\Big[\Big(|r\nabla\!\!\!/\alpha(F)|+|r^2\nabla\!\!\!/^2\alpha(F)|+|r\nabla\!\!\!/\rho(F)|\Big)|\rho(L_S\, L_{\mathcal{O}}F)|\\
&+\Big(|r^2\nabla\!\!\!/^2\sigma(F)|+|\alpha(F)|+|\underline{\alpha}(F)|+|r\nabla\!\!\!/\alpha(F)|+|r\nabla\!\!\!/\underline{\alpha}(F)|\Big)|\alpha(L_S\, L_{\mathcal{O}}F)|\Big]\\
&+\tau_-^2\Big[\Big(|r\nabla\!\!\!/\underline{\alpha}(F)|+|r^2\nabla\!\!\!/^2\underline{\alpha}(F)|+|r\nabla\!\!\!/\rho(F)|\Big)|\rho(L_S\, L_{\mathcal{O}}F)|\\
&+\Big(|r^2\nabla\!\!\!/^2\sigma(F)|+|\alpha(F)|+|\underline{\alpha}(F)|+|r\nabla\!\!\!/\alpha(F)|+|r\nabla\!\!\!/\underline{\alpha}(F)|\Big)|\underline{\alpha}(L_S\, L_{\mathcal{O}}F)|\Big]\Big\}
\end{aligned}
$$

To estimate $\int_{V_{\geq\bar{t}}(u,\underline{u})}|(\mathbf{Div}Q(L_S\, L_{\mathcal{O}}F))_\alpha\bar{K}^\alpha|$ we have to control the large family of integrals on $V_{\geq\bar{t}}(u,\underline{u})$ composing it. We divide them in two sets whose integrals have to be estimated in a different way.

set (A):

$$
\begin{aligned}
&\int_{V_{\geq\bar{t}}(u,\underline{u})}\frac{\mu_{(S)}(r)}{r}\tau_+|r^2\nabla\!\!\!/^2\alpha(F)|\tau_+|\rho(L_S\, L_{\mathcal{O}}F)|\\
&\int_{V_{\geq\bar{t}}(u,\underline{u})}\frac{\mu_{(S)}(r)}{r}\tau_+|r\nabla\!\!\!/\alpha(F)|\tau_+|\rho(L_S\, L_{\mathcal{O}}F)|\\
&\int_{V_{\geq\bar{t}}(u,\underline{u})}\frac{\mu_{(S)}(r)}{r}\tau_+|r\nabla\!\!\!/\rho(F)|\tau_+|\rho(L_S\, L_{\mathcal{O}}F)|\\
&\int_{V_{\geq\bar{t}}(u,\underline{u})}\frac{\mu_{(S)}(r)}{r}\tau_+|\alpha(F)|\tau_+|\alpha(L_S\, L_{\mathcal{O}}F)|\\
&\int_{V_{\geq\bar{t}}(u,\underline{u})}\frac{\mu_{(S)}(r)}{r}\tau_+|r\nabla\!\!\!/\alpha(F)|\tau_+|\alpha(L_S\, L_{\mathcal{O}}F)|
\end{aligned}
$$

$$\int_{V_{\geq\bar{t}}(u,\underline{u})} \frac{\mu_{(S)}(r)}{r}\tau_+|r^2\nabla\!\!\!/^2\sigma(F)|\tau_+|\alpha(L_S\, L_{\mathcal{O}}F)|$$
$$\int_{V_{\geq\bar{t}}(u,\underline{u})} \frac{\mu_{(S)}(r)}{r}\tau_-|r^2\nabla\!\!\!/^2\underline{\alpha}(F)|\tau_-|\rho(L_S\, L_{\mathcal{O}}F)| \tag{4.96}$$
$$\int_{V_{\geq\bar{t}}(u,\underline{u})} \frac{\mu_{(S)}(r)}{r}\tau_-|r\nabla\!\!\!/\rho(F)|\tau_-|\rho(L_S\, L_{\mathcal{O}}F)|$$
$$\int_{V_{\geq\bar{t}}(u,\underline{u})} \frac{\mu_{(S)}(r)}{r}\tau_-|\alpha(F)|\tau_-|\underline{\alpha}(L_S\, L_{\mathcal{O}}F)|$$
$$\int_{V_{\geq\bar{t}}(u,\underline{u})} \frac{\mu_{(S)}(r)}{r}\tau_-|r\nabla\!\!\!/\alpha(F)|\tau_-|\underline{\alpha}(L_S\, L_{\mathcal{O}}F)|$$
$$\int_{V_{\geq\bar{t}}(u,\underline{u})} \frac{\mu_{(S)}(r)}{r}\tau_-|\underline{\alpha}(F)|\tau_-|\underline{\alpha}(L_S\, L_{\mathcal{O}}F)|$$
$$\int_{V_{\geq\bar{t}}(u,\underline{u})} \frac{\mu_{(S)}(r)}{r}\tau_-|r\nabla\!\!\!/\underline{\alpha}(F)|\tau_-|\rho(L_S\, L_{\mathcal{O}}F)|$$
$$\int_{V_{\geq\bar{t}}(u,\underline{u})} \frac{\mu_{(S)}(r)}{r}\tau_-|r\nabla\!\!\!/\underline{\alpha}(F)|\tau_-|\underline{\alpha}(L_S\, L_{\mathcal{O}}F)|$$
$$\int_{V_{\geq\bar{t}}(u,\underline{u})} \frac{\mu_{(S)}(r)}{r}\tau_-|r^2\nabla\!\!\!/^2\sigma(F)|\tau_-|\underline{\alpha}(L_S\, L_{\mathcal{O}}F)|$$

set (B):
$$\int_{V_{\geq\bar{t}}(u,\underline{u})} \frac{\mu_{(S)}(r)}{r}\tau_+|\underline{\alpha}(F)|\tau_+|\alpha(L_S\, L_{\mathcal{O}}F)|$$
$$\int_{V_{\geq\bar{t}}(u,\underline{u})} \frac{\mu_{(S)}(r)}{r}\tau_+|r\nabla\!\!\!/\underline{\alpha}(F)|\tau_+|\alpha(L_S\, L_{\mathcal{O}}F)| \tag{4.97}$$

Every integral of the first group satisfies the following inequality

$$\begin{aligned}
[(A)] \leq & \left[\int_{\underline{u}_0(\bar{t})}^{\underline{u}} \frac{\mu_{(S)}(r(u,\underline{u}'))}{r(u,\underline{u}')}\left(\sum_{1\leq a\leq 2}\int_{\underline{C}(\underline{u}')\cap V_{\geq\bar{t}}(u,\underline{u})} Q(\, L^a_{\mathcal{O}}F)(\bar{K},e_3)\right)\right.\\
+ & \left.\int_{u_0(\bar{t})}^{u} \frac{\mu_{(S)}(r(u',\underline{u}))}{r(u',\underline{u})}\left(\sum_{1\leq a\leq 2}\int_{C(u')\cap V_{\geq\bar{t}}(u,\underline{u})} Q(\, L^a_{\mathcal{O}}F)(\bar{K},e_4)\right)\right]^{\frac{1}{2}}\\
& \cdot\left[\int_{\underline{u}_0(\bar{t})}^{\underline{u}} \frac{\mu_{(S)}(r(u,\underline{u}'))}{r(u,\underline{u}')}\left(\int_{\underline{C}(\underline{u}')\cap V_{\geq\bar{t}}(u,\underline{u})} Q(L_S\, L_{\mathcal{O}}F)(\bar{K},e_3)\right)\right.\\
+ & \left.\int_{u_0(\bar{t})}^{u} \frac{\mu_{(S)}(r(u',\underline{u}))}{r(u',\underline{u})}\left(\int_{C(u')\cap V_{\geq\bar{t}}(u,\underline{u})} Q(L_S\, L_{\mathcal{O}}F)(\bar{K},e_4)\right)\right]^{\frac{1}{2}}
\end{aligned}$$

$$
\begin{aligned}
&\leq \quad cm\frac{\left(1+|\log(\frac{r(\bar{t})}{2m}-1)|\right)}{r(u,\underline{u}_0(\bar{t}))}\left[\sup_{(u',\underline{u}')\in V_{\geq\bar{t}}(u,\underline{u})}\left(\mathcal{I}_0^{\mathcal{O}}(\geq\bar{t};u',\underline{u}')+\right.\right.\\
&\qquad\left.\left.\mathcal{I}_0^{S}(\geq\bar{t};u',\underline{u}')+\underline{\mathcal{I}}_0^{\mathcal{O}}(\geq\bar{t};u',\underline{u}')+\underline{\mathcal{I}}_0^{S}(\geq\bar{t};u',\underline{u}')\right)\right]\\
&\leq \quad cm\frac{\left(1+|\log(\frac{r(\bar{t})}{2m}-1)|\right)}{r(u,\underline{u}_0(\bar{t}))}\mathcal{H}(\geq\bar{t};u,\underline{u})
\end{aligned}
\tag{4.98}
$$

Again we can choose $\bar{t}\geq\bar{t}_0$ sufficiently large such that

$$
cm\frac{\left(1+|\log(\frac{r(\bar{t})}{2m}-1)|\right)}{r(u,\underline{u}_0(\bar{t}))}\leq\frac{\epsilon_{0,2}}{28}
\tag{4.99}
$$

obtaining

$$
[(A)]\leq\frac{\epsilon_{0,2}}{2}\left(\mathcal{H}^{\mathcal{O}}(\geq\bar{t};u,\underline{u})+\mathcal{H}^{S}(\geq\bar{t};u,\underline{u})\right)\leq\frac{\epsilon_{0,2}}{2}\mathcal{H}(\geq\bar{t};u,\underline{u})
\tag{4.100}
$$

We estimate the first integral integral of the group (B)

$$
\int_{V_{\geq\bar{t}}(u,\underline{u})}\frac{\mu_{(S)}(r)}{r}\tau_+|\underline{\alpha}(F)|\tau_+|\alpha(L_S\,L_{\mathcal{O}}F)|\ .
$$

The estimate of the other one is done exactly in the same way and we omit it. We recall from the expressions of $Q(\bar{K},e_4)$ and $Q(\bar{K},e_3)$, see equations 3.38, that we can control the integrals of α only along the null cones $C(u)$ and those of $\underline{\alpha}$ along the $\underline{C}(\underline{u})$ ones. Using Lemma 4.7 we bound $\frac{\tau_+}{r}$ in $V_{\geq\bar{t}}(u,\underline{u})$ for $\bar{t}$ sufficiently large, with a constant c independent from m,δ_0. Therefore

$$
\begin{aligned}
&\int_{V_{\geq\bar{t}}(u,\underline{u})}\frac{\mu_{(S)}(r)}{r}\tau_+|\underline{\alpha}(F)|\tau_+|\alpha(L_S\,L_{\mathcal{O}}F)|\\
&\leq cm\left(\int_{V_{\geq\bar{t}}(u,\underline{u})}\left(1+|\log(\frac{r(u',\underline{u}')}{2m}-1)|\right)\frac{\tau_-^2}{\tau_+^2}|\underline{\alpha}(F)|^2\right)^{\frac{1}{2}}\cdot\\
&\qquad\cdot\left(\int_{V_{\geq\bar{t}}(u,\underline{u})}\left(1+|\log(\frac{r(u',\underline{u}')}{2m}-1)|\right)\frac{\tau_+^2}{\tau_-^2}|\alpha(L_S\,L_{\mathcal{O}}F)|^2\right)^{\frac{1}{2}}\\
&\leq cm\left(\int_{\underline{u}_0(\bar{t})}^{\underline{u}}d\underline{u}'\frac{\left(1+|\log(\frac{\tau_+}{2m}-1)|\right)}{\tau_+^2}\int_{\underline{C}(\underline{u}')\cap V_{\geq\bar{t}}(u,\underline{u})}Q(\,L_{\mathcal{O}}F)(\bar{K},e_3)\right)^{\frac{1}{2}}\cdot\\
&\qquad\cdot\left(\int_{u_0(\bar{t})}^{u}du'\frac{\left(1+|\log(\frac{r(u',\underline{u})}{2m}-1)|\right)}{\tau_-^2}\int_{C(u')\cap V_{\geq\bar{t}}(u,\underline{u})}Q(L_S\,L_{\mathcal{O}}F)(\bar{K},e_4)\right)^{\frac{1}{2}}
\end{aligned}
$$

$$\leq cm\left(\int_{\underline{u}_0(\bar t)}^{\underline{u}} d\underline{u}'\frac{\left(1+|\log(\frac{\tau_+}{2m}-1)|\right)}{\tau_+^2}\right)^{\frac12}\left(\int_{u_0(\bar t)}^{u} du'\frac{\left(1+|\log(\frac{r(u',\underline{u})}{2m}-1)|\right)}{\tau_-^2}\right)^{\frac12}\cdot$$
$$\sup_{[u_0,u]\times[\underline{u}_0,\underline{u}]}\left[\left(\int_{\underline{C}(\underline{u}')\cap V_{\geq\bar t}(u,\underline{u})} Q(\,L_{\mathcal{O}}F)(\bar K,e_3)\right)\right.$$
$$\left.\left(\int_{C(u')\cap V_{\geq\bar t}(u,\underline{u})} Q(L_S\,L_{\mathcal{O}}F)(\bar K,e_4)\right)\right]^{\frac12} \tag{4.101}$$

with a generic constant c independent from $u,\underline{u},\bar t$. For $\bar t$ sufficiently large the first integral of the factor [36]

$$\left(\int_{\underline{u}_0(\bar t)}^{\underline{u}} d\underline{u}'\frac{\left(1+|\log(\frac{\tau_+}{2m}-1)|\right)}{\tau_+^2}\right)^{\frac12}\left(\int_{u_0(\bar t)}^{u} du'\frac{\left(1+|\log(\frac{r(u',\underline{u})}{2m}-1)|\right)}{\tau_-^2}\right)^{\frac12}$$

which is $O(\frac{\log \underline{u}_0(\bar t)}{\underline{u}_0(\bar t)})$, can be made sufficiently small so that [37]

$$\left(\int_{\underline{u}_0(\bar t)}^{\underline{u}} d\underline{u}'\frac{\left(1+|\log(\frac{\tau_+}{2m}-1)|\right)}{\tau_+^2}\right)^{\frac12}\left(\int_{u_0(\bar t)}^{u} du'\frac{\left(1+|\log(\frac{r(u',\underline{u})}{2m}-1)|\right)}{\tau_-^2}\right)^{\frac12}\leq\frac{\epsilon_{0,2}}{4} \tag{4.102}$$

and the following inequality holds

$$\begin{aligned}
&\int_{V_{\geq\bar t}(u,\underline{u})}\frac{\mu_{(S)}(r)}{r}\tau_+|\underline{\alpha}(F)|\tau_+|\alpha(L_S\,L_{\mathcal{O}}F)|\\
\leq\ &\frac{\epsilon_{0,2}}{4}\left(\sup_{\underline{u}'\in[\underline{u}_0,\underline{u}]}\int_{\underline{C}(\underline{u}')\cap V_{\geq\bar t}(u,\underline{u})} Q(\,L_{\mathcal{O}}F)(\bar K,e_3)\right)^{\frac12}\cdot\\
&\cdot\left(\sup_{u'\in[u_0,u]}\int_{C(u')\cap V_{\geq\bar t}(u,\underline{u})} Q(L_S\,L_{\mathcal{O}}F)(\bar K,e_4)\right)^{\frac12}\\
\leq\ &\frac{\epsilon_{0,2}}{4}\left(\mathcal{H}^{\mathcal{O}}(\geq\bar t;u,\underline{u})+\mathcal{H}^{S}(\geq\bar t;u,\underline{u})\right)
\end{aligned} \tag{4.103}$$

In conclusion

$$[(B)]\leq\frac{\epsilon_{0,2}}{2}\left(\mathcal{H}^{\mathcal{O}}(\geq\bar t;u,\underline{u})+\mathcal{H}^{S}(\geq\bar t;u,\underline{u})\right)\leq\frac{\epsilon_{0,2}}{2}\mathcal{H}(\geq\bar t;u,\underline{u}) \tag{4.104}$$

[36] This $\bar t$ can be larger than the previous ones.

[37] The second factor $\int_{u_0(\bar t)}^{u} du'\frac{\left(1+|\log(\frac{r(u',\underline{u})}{2m}-1)|\right)}{\tau_-^2}$ is bounded by $c\log\frac{r(u,\underline{u})}{m}$, not necessarily small if $u=-r_*(\delta_0)$ with δ_0 sufficiently small.

Collecting all these estimates Proposition 4.8 is proved, which together with Proposition 4.6 completes the proof of Proposition 4.4.

Remark. It is due to the presence of the integrals of group (B) that the Σ_t energy-type norms cannot be bounded in terms of the initial data. In fact the analogous of the inequality 4.104 cannot be obtained using the Σ_t energy-type norms. An estimate in terms of these norms is, anyway, needed to prove Proposition 4.5. This is the reason the function $C_1(m,\delta_0;\bar{t})$ depends on $\bar{t}$. In Proposition 4.5 this is not a problem, as $\bar{t}$, although large, is fixed independently from u and $\underline{u}$.

4.2 Proof of Proposition 4.5

The proof is similar to the proof of Proposition 4.4, but much simpler. Defining $\mathcal{H}(t)\equiv\mathcal{H}(\Sigma_t(r\geq r(u,\underline{u}_0(t)))$ it follows that, see eq. 4.69,

$$\mathcal{H}(t)\geq\mathcal{H}(\Sigma_t\cap V(u,\underline{u}))=\mathcal{I}_0^{\mathcal{O}}(\Sigma_t\cap V(u,\underline{u}))+\mathcal{I}_0^{S}(\Sigma_t\cap V(u,\underline{u}))\tag{4.105}$$

Recalling Proposition 4.1 and Corollary 4.2, applying again Stokes theorem, see Lemma 4.3, we obtain

Lemma 4.11. *Let $P_\alpha=Q(G)_{\alpha\beta}X^\beta$ be defined as in Proposition 4.1 then the Stokes theorem implies*

$$\begin{aligned}&\left\{\int_{\Sigma_{t+\delta}(r\geq r(u,\underline{u}_0(t+\delta)))}Q(G)(X,\tilde{T}_0)-\int_{\Sigma_t(r\geq r(u,\underline{u}_0(t)))}Q(G)(X,\tilde{T}_0)\right.\\&\left.+\int_{C(u,[\underline{u}_0(t),\underline{u}_0(t+\delta)])}\Phi^{-1}Q(G)(X,e_4)\right\}\\=&\ -\int_{V([t,t+\delta];u)}\left[(\mathbf{Div}Q(G))_\beta X^\beta+\frac{1}{2}Q(G)^{\alpha\beta(X)}\pi_{\alpha\beta}\right]\end{aligned}\tag{4.106}$$

where $V([t,t+\delta];u)$ is the volume whose boundaries are:
$\Sigma_{t+\delta}(r\geq r(u,\underline{u}_0(t+\delta)))$, $\Sigma_t(r\geq r(u,\underline{u}_0(t)))$ *and* $C(u,[\underline{u}_0(t),\underline{u}_0(t+\delta)])$.

As $\int_{C(u,[\underline{u}_0(t),\underline{u}_0(t+\delta)])}\Phi^{-1}Q(G)(X,e_4)$ is non negative

$$\begin{aligned}\mathcal{H}(t+\delta)-\mathcal{H}(t)\ \leq\ &\int_{V([t,t+\delta];u)}\left\{\sum_{1\leq a\leq 2}|{}^{(\bar{K})}\pi^{\alpha\beta}Q(\,L_{\mathcal{O}}^aF)_{\alpha\beta}|+\right.\\&\left.|{}^{(\bar{K})}\pi^{\alpha\beta}Q(L_S\,L_{\mathcal{O}}F)_{\alpha\beta}|+|(\mathbf{Div}Q(L_S\,L_{\mathcal{O}}F))_\alpha\bar{K}^\alpha|\right\}\end{aligned}\tag{4.107}$$

and, taking the limit $\delta\to 0$, the following differential inequality holds:

$$\frac{d\mathcal{H}}{dt}\ \leq\ \int_{\Sigma_t(r\geq r(u,\underline{u}_0(t)))}\left\{\sum_{1\leq a\leq 2}|{}^{(\bar{K})}\pi^{\alpha\beta}Q(\,L_{\mathcal{O}}^aF)_{\alpha\beta}|+\right.$$

$$+|{}^{(\bar{K})}\pi^{\alpha\beta} Q(L_S\, L_{\mathcal{O}} F)_{\alpha\beta}| + |(\mathbf{Div} Q(L_S\, L_{\mathcal{O}} F))_\alpha \bar{K}^\alpha| \Big\} \quad (4.108)$$

which we use to estimate $\mathcal{H}$ for $t \in [0, \bar{t}]$. To achieve it we express the right hand side of 4.108 in terms of $\mathcal{H}$. The following lemma is proved in the Appendix:

Lemma 4.12. *For $t \leq \bar{t}_0 = 2|r_*(\delta_0)|$ we have the following estimate*

$$\begin{aligned}
&\int_{\Sigma_t(r \geq r(u, \underline{u}_0(t))} \left(\sum_{1 \leq a \leq 2} |{}^{(\bar{K})}\pi^{\alpha\beta} Q(\, L^a_{\mathcal{O}} F)_{\alpha\beta}| + |{}^{(\bar{K})}\pi^{\alpha\beta} Q(L_S\, L_{\mathcal{O}} F)_{\alpha\beta}| \right) \\
\leq \quad & c\left(1 + (r_*(\delta_0))^3\right) \left(1 + \frac{(1 + (r_*(\delta_0))^2)}{1 + t^2}\right) \mathcal{H}(t) \qquad (4.109)
\end{aligned}$$

$$\int_{\Sigma_t(r \geq r(u, \underline{u}_0(t))} |(\mathbf{Div} Q(L_S\, L_{\mathcal{O}} F))_\alpha \bar{K}^\alpha| \leq c\left(1 + (r_*(\delta_0))^2\right) \mathcal{H}(t) \qquad (4.110)$$

Using this Lemma we rewrite the differential inequality 4.108 for $t \leq \bar{t}_0$:

$$\frac{d\mathcal{H}}{dt} \quad \leq \quad c\left(1 + (r_*(\delta_0))^3\right) \left(1 + \frac{(1 + (r_*(\delta_0))^2)}{1 + t^2}\right) \mathcal{H}(t) \qquad (4.111)$$

and from it we conclude

$$\mathcal{H}(t) \leq \mathcal{H}(0) \exp\{(1 + (r_*(\delta_0))^3) t\} \leq \left(\frac{2m}{\delta_0}\right)^{|m \log \frac{\delta_0}{2m}|^3} \mathcal{H}(0) \qquad (4.112)$$

5 Appendix

5.1 Proof of Proposition 2.3

Proposition 2.3 is a direct consequence of Lemma 2.2 and of the following one:

Lemma 5.1. *Let G be a C^∞ tensor field tangent at each point to the corresponding $S(u, \underline{u})$, then the following Sobolev inequalities hold*

$$\begin{aligned}
\int_{S(u,\underline{u})} r^4 |G|^4 \leq & \int_{S(u_0,\underline{u})} r^4 |G|^4 \\
& + c\left(\int_{\underline{C}(\underline{u};[u_0,u])} |G|^2 + r^2 |\nabla\!\!\!/ G|^2 + r^2 |\mathbf{D}\!\!\!\!/_3 G|^2\right)^2 \\
\int_{S(u,\underline{u})} r^2 \tau_-^2 |G|^4 \leq & \int_{S(u_0,\underline{u})} r^2 \tau_-^2 |G|^4 \\
& + c\left(\int_{\underline{C}(\underline{u};[u_0,u])} |G|^2 + r^2 |\nabla\!\!\!/ G|^2 + \tau_-^2 |\mathbf{D}\!\!\!\!/_3 G|^2\right)^2
\end{aligned}$$

$$\int_{S(u,\underline{u})} r^4|G|^4 \leq \int_{S(u,\underline{u}_0)} r^4|G|^4 + c\left(\int_{C(u;[\underline{u}_0,\underline{u}])} |G|^2 + r^2|\nabla\!\!\!/ G|^2 + r^2|\mathbf{D}\!\!\!/_3 G|^2\right)^2 \tag{5.113}$$

$$\int_{S(u,\underline{u})} r^2\tau_-^2|G|^4 \leq \int_{S(u,\underline{u}_0)} r^2\tau_-^2|G|^4 + c\left(\int_{C(u;[\underline{u}_0,\underline{u}])} |G|^2 + r^2|\nabla\!\!\!/ G|^2 + \tau_-^2|\mathbf{D}\!\!\!/_3 G|^2\right)^2$$

Substituting in Lemma 2.2 G with rU or with $r^{\frac{1}{2}}\tau_-^{\frac{1}{2}}U$ and using in the r.h.s. the inequalities provided by Lemma 5.1 we obtain the estimates 2.21...2.25, proving the Proposition 2.3. Lemma 2.2 is discussed in [Ch-Kl2], page 64. We discuss here the proof of Lemma 5.1. We write the integral $\int_{S(u,\underline{u})} r^4|G|^4$ in the following way

$$\begin{aligned}
\int_{S(u,\underline{u})} r^4|G|^4 &= \int_{S(u_0,\underline{u})} r^4|G|^4 + \int_{u_0}^{u} du' \frac{\partial}{\partial u'} \int_{S(u',\underline{u})} r^4|G|^4 \\
&= \int_{S(u_0,\underline{u})} r^4|G|^4 + \int_{u_0}^{u} du' \int_{S_1} d\omega \left(6r^5\frac{\partial r}{\partial u'}|G|^4 + r^6\frac{\partial}{\partial u'}|G|^4\right) \\
&= \int_{S(u_0,\underline{u})} r^4|G|^4 + 2\int_{u_0}^{u} du' r^4 \int_{S(u',\underline{u})} \Phi|G|^2(G\cdot\mathbf{D}\!\!\!/_3 G) \\
&\quad +6\int_{u_0}^{u} du' \frac{\partial r}{\partial u'} r^3 \int_{S(u',\underline{u})} |G|^4
\end{aligned} \tag{5.114}$$

where $d\omega$ is the angular part of the measure on $S(u,\underline{u})$, $\frac{\partial}{\partial u} = \frac{\Phi}{2}\mathbf{D}_3$ on the scalar functions, $|G|$ is the norm of the tensor G with respect to the induced metric on S and $(G\cdot\mathbf{D}\!\!\!/_3 G)$ is the scalar product between two tensors tangent to S made with respect to the same metric. As

$$\frac{\partial r}{\partial u'} = \frac{\partial r}{\partial r_*}\frac{\partial r_*}{\partial u'} = -\frac{1}{2}\Phi^2 \leq 0$$

the last term is non positive and we neglect it obtaining the following upper bound

$$\int_{S(u,\underline{u})} r^4|G|^4 \leq \int_{S(u_0,\underline{u})} r^4|G|^4 + 2\int_{\underline{C}(\underline{u};[u_0,u])} r^4|G|^2|(G\cdot\mathbf{D}\!\!\!/_3 G)| \tag{5.115}$$

Applying the Schwartz inequality we obtain

$$\int_{S(u,\underline{u})} r^4|G|^4 \leq \int_{S(u_0,\underline{u})} r^4|G|^4 + 2\left(\int_{\underline{C}(\underline{u};[u_0,u])} r^6|G|^6\right)^{\frac{1}{2}}\left(\int_{\underline{C}(\underline{u};[u_0,u])} r^2|\mathbf{D}\!\!\!/_3 G|^2\right)^{\frac{1}{2}} \tag{5.116}$$

Moreover using the isoperimetric inequality [38] it follows

$$\int_{S(u',\underline{u})} r^6|G|^6 \leq c\left(\int_{S(u',\underline{u})} r^4|G|^4\right)\left(\int_{S(u',\underline{u})} |G|^2 + r^2|\nabla\!\!\!/ G|^2\right) \tag{5.117}$$

and from it

$$\int_{\underline{C}(\underline{u};[u_0,u])} r^6|G|^6 \leq c\left(\sup_{u'\in[u_0,u]}\int_{S(u',\underline{u})} r^4|G|^4\right)\left(\int_{\underline{C}(\underline{u};[u_0,u])} |G|^2 + r^2|\nabla\!\!\!/ G|^2\right) \tag{5.118}$$

Substituting this inequality in the previous one we obtain

$$\begin{aligned}\sup_{u'\in[u_0,u]}\int_{S(u',\underline{u})} r^4|G|^4 \leq \int_{S(u_0,\underline{u})} r^4|G|^4 + c\left(\sup_{u'\in[u_0,u]}\int_{S(u',\underline{u})} r^4|G|^4\right)^{\frac{1}{2}}\cdot\\ \cdot\left(\int_{\underline{C}(\underline{u};[u_0,u])} |G|^2 + r^2|\nabla\!\!\!/ G|^2\right)^{\frac{1}{2}}\left(\int_{\underline{C}(\underline{u};[u_0,u])} r^2|\mathbf{D}\!\!\!\!/_3 G|^2\right)^{\frac{1}{2}}\end{aligned} \tag{5.119}$$

and finally

$$\begin{aligned}\sup_{u'\in[u_0,u]}\int_{S(u',\underline{u})} r^4|G|^4 \leq &\int_{S(u_0,\underline{u})} r^4|G|^4\\ &+c\left(\int_{\underline{C}(\underline{u};[u_0,u])} |G|^2 + r^2|\nabla\!\!\!/ G|^2 + r^2|\mathbf{D}\!\!\!\!/_3 G|^2\right)^2\end{aligned} \tag{5.120}$$

proving the first line of 5.113. The only difference in proving the second line is that we start with the integral $\int_{S(u,\underline{u})} r^2\tau_-^2|G|^4$ and instead of 5.117 we derive, from the isoperimetric inequality,

$$\int_{S(u',\underline{u})} r^4\tau_-^2|G|^6 \leq c\left(\int_{S(u',\underline{u})} r^2\tau_-^2|G|^4\right)\left(\int_{S(u',\underline{u})} |G|^2 + r^2|\nabla\!\!\!/ G|^2\right) \tag{5.121}$$

so that we obtain

$$\begin{aligned}\sup_{u'\in[u_0,u]}\int_{S(u',\underline{u})} r^2\tau_-^2|G|^4 \leq &\int_{S(u_0,\underline{u})} r^2\tau_-^2|G|^4\\ &+c\left(\int_{\underline{C}(\underline{u};[u_0,u])} |G|^2 + r^2|\nabla\!\!\!/ G|^2 + \tau_-^2|\mathbf{D}\!\!\!\!/_3 G|^2\right)^2\end{aligned} \tag{5.122}$$

[38] See [Ch-Kl2], page 58.

To prove the last two inequalities of 5.113 all the previous computations have to be done again to express the sup-norms in terms of integrals along the $C(u;[\underline{u}_0,\underline{u}])$ cones. The only difference is that the term, corresponding to $6\int_{u_0}^{u}du'\int_{S(u',\underline{u})} r^3\frac{\partial r}{\partial u'}|G|^4$,

$$6\int_{\underline{u}_0}^{\underline{u}}d\underline{u}'\int_{S(u,\underline{u}')}r^3\frac{\partial r}{\partial \underline{u}'}|G|^4,$$

is non negative and, therefore, cannot be omitted. We bound it as

$$\begin{aligned}6\int_{\underline{u}_0}^{\underline{u}}d\underline{u}'\int_{S(u,\underline{u}')}r^3\frac{\partial r}{\partial \underline{u}'}|G|^4 &\leq 3\int_{C(u;[\underline{u}_0,\underline{u}])}r^3|G|^4\\ &\leq 3\left(\int_{C(u;[\underline{u}_0,\underline{u}])}r^6|G|^6\right)^{\frac{1}{2}}\left(\int_{C(u;[\underline{u}_0,\underline{u}])}|G|^2\right)^{\frac{1}{2}}\end{aligned} \tag{5.123}$$

and add this term to the other one. The final result is of the same type:

$$\begin{aligned}\int_{S(u,\underline{u}')}r^4|G|^4 &\leq \int_{S(u,\underline{u}_0)}r^4|G|^4\\ &\quad+c\left(\int_{C(u;[\underline{u}_0,\underline{u}])}|G|^2+r^2|\nabla\!\!\!/ G|^2+r^2|\mathbf{D}\!\!\!\!/_4G|^2\right)^2\\ \int_{S(u,\underline{u}')}r^2\tau_-^2|G|^4 &\leq \int_{S(u,\underline{u}_0)}r^2\tau_-^2|G|^4\\ &\quad+c\left(\int_{C(u;[\underline{u}_0,\underline{u}])}|G|^2+r^2|\nabla\!\!\!/ G|^2+\tau_-^2|\mathbf{D}\!\!\!\!/_4G|^2\right)^2\end{aligned} \tag{5.124}$$

5.2 Proof of Proposition 2.4

To prove Proposition 2.4 we write $r^4|G|^4$ in the following way [39]:

$$(r^4|G|^4)(r,\omega) \quad = \quad -\int_r^\infty dr'\frac{\partial}{\partial r'}(r'^4|G|^4) \tag{5.125}$$

which is true, provided the assumption $\lim_{r\to\infty}r^4|G|^4=0$ is satisfied. Then integrating both sides on $S(t,r)$ we obtain

$$\int_{S(t,r)}d\sigma(r^4|G|^4)(r,\omega) \quad = \quad -\int_r^\infty dr'\int_{S(t,r)}d\sigma\frac{\partial}{\partial r'}(r'^4|G|^4)$$

[39] The reason for not starting in this proof directly from $\int_{S(t,r)}d\sigma\ r^4|G|^4$ is due to the fact that we do not want to make the assumption

$$\lim_{r\to\infty}\int_{S(t,r)}d\sigma\ r^4|G|^4=0$$

as this limit is not true for the solutions of the Maxwell equations.

$$
\begin{aligned}
&= -\int_r^\infty dr' \int_{S(t,r')} d\sigma \Phi(r')^{-1}\left[\frac{r^2}{r'^2}\Phi(r')\frac{\partial}{\partial r'}(r'^4|G|^4)\right]\\
&= -\int_{\Sigma_t([r,\infty))} d\mu \left[\frac{r^2}{r'^2}\Phi(r')\frac{\partial}{\partial r'}(r'^4|G|^4)\right]\\
&\leq -4\int_{\Sigma_t([r,\infty))} d\mu\, r^2 r'\Phi(r')|G|^4\\
&\quad +4\int_{\Sigma_t([r,\infty))} d\mu\, r^2 r'^2|G|^2|(G\cdot \mathbf{D}_{\tilde{N}}G)|\\
&\leq 4\int_{\Sigma_t([r,\infty))} d\mu\, r'^4|G|^2|(G\cdot \mathbf{D}_{\tilde{N}}G)| \qquad (5.126)
\end{aligned}
$$

and the last inequality is nearly the same as in eq. 5.124, the main difference being the presence of $\mathbf{D}_{\tilde{N}} = \Phi\mathbf{D}_{\frac{\partial}{\partial r}}$ instead of $\not{\mathbf{D}}_4$ or $\not{\mathbf{D}}_3$. The remaining of the proof goes just mimicking all the previous steps of Proposition 2.3.

5.3 Proof of Proposition 3.3

The proof relies on the estimates of Proposition 2.3, where $C(u;[\underline{u}_0,\underline{u}])$ and $\underline{C}(\underline{u};[u_0,u])$ are now the portions of the null cones above $\Sigma_{\bar{t}}$. The strategy is to estimate the various integrals in the r.h.s. of Proposition 2.3, with U equal to the null components of the Maxwell field, in terms of the integrals, on the same cones, of the $Q(L_S^b L_{\mathcal{O}}^a F)(\bar{K}, e_{(3,4)})$ functions with $(a,b)\in\{(1,0),(2,0),(1,1)\}$. We divide the proof in various parts, each one relative to some null components in which the Maxwell tensor F is decomposed.

5.3.1 α and $\underline{\alpha}$

Choosing in eq. 2.21 $U = r\alpha(F)$ it follows that to control $\sup_{S(u,\underline{u})}|r^{\frac{5}{2}}\alpha(F)|$ we have to control the following integrals

$$
\begin{aligned}
&\int_{C(u;[\underline{u}_0,\underline{u}])} r^2|\alpha(F)|^2\ ,\ \int_{C(u;[\underline{u}_0,\underline{u}])} r^4|\not\nabla\alpha(F)|^2\\
&\int_{C(u;[\underline{u}_0,\underline{u}])} r^4|\not{\mathbf{D}}_4\alpha(F)|^2\ ,\ \int_{C(u;[\underline{u}_0,\underline{u}])} r^6|\not\nabla^2\alpha(F)|^2\\
&\int_{C(u;[\underline{u}_0,\underline{u}])} r^6|\not\nabla\not{\mathbf{D}}_4\alpha(F)|^2 \qquad (5.127)
\end{aligned}
$$

To control the first two integrals we observe that, due to the symmetries of the Schwarzschild spacetime, $L_{\Omega_{ij}}$ commutes with the null decomposition:

$$\alpha(L_{\Omega_{ij}}F) = L_{\Omega_{ij}}\alpha(F).$$

Moreover it is easy to prove by direct computation that

$$|\,L_{\mathcal{O}}\alpha|^2 = r^2|\not\nabla\alpha|^2 + |\alpha|^2$$

where $|L_{\mathcal{O}}\alpha|^2 \equiv \sum_{i<j}|L_{\Omega_{(ij)}}\alpha|^2$. Therefore

$$\begin{aligned}\int_{C(u;[\underline{u}_0,\underline{u}])} r^2|\alpha(F)|^2 \leq \Phi^{-1}(r(u,\underline{u}_0))\int_{C(u;[\underline{u}_0,\underline{u}])} Q(L_{\mathcal{O}}F)(\bar{K},e_4)\\ \int_{C(u;[\underline{u}_0,\underline{u}])} r^4|\nabla\!\!\!/\alpha(F)|^2 \leq \Phi^{-1}(r(u,\underline{u}_0))\int_{C(u;[\underline{u}_0,\underline{u}])} Q(L_{\mathcal{O}}F)(\bar{K},e_4)\end{aligned} \tag{5.128}$$

In the same way it is easy to prove that

$$\int_{C(u;[\underline{u}_0,\underline{u}])} r^6|\nabla\!\!\!/^2\alpha(F)|^2 \leq \Phi^{-1}(r(u,\underline{u}_0))\int_{C(u;[\underline{u}_0,\underline{u}])} Q(L^2_{\mathcal{O}}F)(\bar{K},e_4)$$

In order to estimate the fourth integral we recall that, as $S = \frac{\Phi}{2}(\underline{u}e_4 + ue_3)$,

$$r^2|\mathbf{D}\!\!\!/_4\alpha|^2 \leq c\left(\Phi^{-2}|\mathbf{D}\!\!\!/_S\alpha|^2 + \tau_-^2|\mathbf{D}\!\!\!/_3\alpha|^2\right) \tag{5.129}$$

so that we are reduced to control the integrals

$$\int_{C(u;[\underline{u}_0,\underline{u}])} r^2\Phi^{-2}|\mathbf{D}\!\!\!/_S\alpha(F)|^2\ ,\ \int_{C(u;[\underline{u}_0,\underline{u}])} r^2\tau_-^2|\mathbf{D}\!\!\!/_3\alpha(F)|^2\ .$$

The first integral can be easily bounded in the following way [40]

$$\begin{aligned}&\int_{C(u;[\underline{u}_0,\underline{u}])} r^2\Phi^{-2}|\mathbf{D}\!\!\!/_S\alpha(F)|^2 \leq \Phi^{-2}(r(u,\underline{u}_0))\int_{C(u;[\underline{u}_0,\underline{u}])} r^2|\mathbf{D}\!\!\!/_S\alpha(F)|^2\\ &\leq c\Phi^{-2}(r(u,\underline{u}_0))\left(\int_{C(u;[\underline{u}_0,\underline{u}])} r^2|\alpha(L_SF)|^2 + \Phi^{-\epsilon}(r(u,\underline{u}_0))\int_{C(u;[\underline{u}_0,\underline{u}])} r^2|\alpha(F)|^2\right)\\ &\leq c\Phi^{-(3+\epsilon)}(r(u,\underline{u}_0))\left(\int_{C(u;[\underline{u}_0,\underline{u}])} Q(L_S\,L_{\mathcal{O}}F)(\bar{K},e_4) + \int_{C(u;[\underline{u}_0,\underline{u}])} Q(\,L_{\mathcal{O}}F)(\bar{K},e_4)\right)\end{aligned} \tag{5.130}$$

for any $\epsilon > 0$.

Using the Maxwell equation

$$\mathbf{D}\!\!\!/_3\alpha - (\partial_r\Phi + \frac{\Phi}{r})\alpha - \nabla\!\!\!/\rho -^* \nabla\!\!\!/\sigma = 0,$$

[40] We use the relation

$$\begin{aligned}|\mathbf{D}\!\!\!/_S\alpha(F)| &\leq |L_S\alpha(F)| + |\mathbf{D}S||\alpha(F)|\\ &\leq |\alpha(L_SF)| + c|\alpha(F)|(1 + \frac{m}{r}|\log\Phi|)\ .\end{aligned}$$

the second integral satisfies

$$\begin{aligned}
&\int_{C(u;[\underline{u}_0,\underline{u}])} r^2\tau_-^2|\not{D}_3\alpha(F)|^2 \\
&\qquad \leq c\int_{C(u;[\underline{u}_0,\underline{u}])} \tau_-^2|\alpha(F)|^2 + r^2\tau_-^2|\not\nabla\rho(F)|^2 + r^2\tau_-^2|\not\nabla\sigma(F)|^2 \\
&\qquad \leq c\left(\Phi^{-1}(r(u,\underline{u}_0))\int_{C(u;[\underline{u}_0,\underline{u}])} Q(\,L_{\mathcal{O}}F)(\bar{K},e_4)\right. \\
&\qquad\quad \left. + \int_{C(u;[\underline{u}_0,\underline{u}])} r^2\tau_-^2|\not\nabla\rho(F)|^2 + r^2\tau_-^2|\not\nabla\sigma(F)|^2\right)
\end{aligned} \tag{5.131}$$

As in the Minkowski case, see eq. (3.59) of [Ch-Kl1], the following inequality holds

$$\begin{aligned}
&\int_{C(u;[\underline{u}_0,\underline{u}])} \tau_-^2(|r\not\nabla\rho(F)|^2 + |r\not\nabla\sigma(F)|^2) \\
&\qquad \leq c\int_{C(u;[\underline{u}_0,\underline{u}])} \tau_-^2|\rho(\,L_{\mathcal{O}}F)|^2 + \tau_-^2|\sigma(\,L_{\mathcal{O}}F)|^2 \\
&\qquad \leq c\Phi^{-1}(r(u,\underline{u}_0))\int_{C(u;[\underline{u}_0,\underline{u}])} Q(\,L_{\mathcal{O}}F)(\bar{K},e_4)
\end{aligned} \tag{5.132}$$

and from it

$$\int_{C(u;[\underline{u}_0,\underline{u}])} r^2\tau_-^2|\not{D}_3\alpha(F)|^2 \leq c\Phi^{-1}(r(u,\underline{u}_0))\int_{C(u;[\underline{u}_0,\underline{u}])} Q(\,L_{\mathcal{O}}F)(\bar{K},e_4) \tag{5.133}$$

Finally [41]

$$\begin{aligned}
&\int_{C(u;[\underline{u}_0,\underline{u}])} r^4|\not{D}_4\alpha(F)|^2 \\
&\leq c\Phi^{-1}(r(u,\underline{u}_0))\left(\Phi^{-(2+\epsilon)}(r(u,\underline{u}_0))\int_{C(u;[\underline{u}_0,\underline{u}])} Q(L_S\,L_{\mathcal{O}}F)(\bar{K},e_4)+\right. \\
&\qquad\qquad \left.\int_{C(u;[\underline{u}_0,\underline{u}])} Q(\,L_{\mathcal{O}}F)(\bar{K},e_4)\right) \\
&\leq c\Phi^{-4}(r(u,\underline{u}_0))\left(\int_{C(u;[\underline{u}_0,\underline{u}])} Q(L_S\,L_{\mathcal{O}}F)(\bar{K},e_4) + \int_{C(u;[\underline{u}_0,\underline{u}])} Q(\,L_{\mathcal{O}}F)(\bar{K},e_4)\right)
\end{aligned} \tag{5.134}$$

[41] $\Phi^{-(2+\epsilon)}$ could in fact be substituted by $\Phi^{-2}|\log\Phi|$.

The last integral of 5.127 is estimated in the same way, obtaining

$$\int_{C(u;[\underline{u}_0,\underline{u}])} r^6|\nabla\!\!\!/ \mathbf{D}\!\!\!/_4\alpha(F)|^2 \leq c\Phi^{-4}(r(u,\underline{u}_0))\left(\int_{C(u;[\underline{u}_0,\underline{u}])} Q(L_S\, L_{\mathcal{O}}F)(\bar{K},e_4)\right.$$
$$\left. + \int_{C(u;[\underline{u}_0,\underline{u}])} Q(\,L_{\mathcal{O}}F)(\bar{K},e_4) + \int_{C(u;[\underline{u}_0,\underline{u}])} Q(\,L^2_{\mathcal{O}}F)(\bar{K},e_4)\right) \quad (5.135)$$

Collecting all these estimates we have

$$\begin{aligned} |r^{\frac{5}{2}}\alpha(F)(u,\underline{u})| &\leq c\Phi^{-2}(r(u,\underline{u}_0))\left[\int_{C(u;[\underline{u}_0,\underline{u}])} Q(L_S\, L_{\mathcal{O}}F)(\bar{K},e_4)\right. \\ &+ \left.\int_{C(u;[\underline{u}_0,\underline{u}])} Q(\,L_{\mathcal{O}}F)(\bar{K},e_4) + \int_{C(u;[\underline{u}_0,\underline{u}])} Q(\,L^2_{\mathcal{O}}F)(\bar{K},e_4)\right]^{\frac{1}{2}} \\ &\leq c\Phi^{-2}(r(u,\underline{u}_0))\left(\mathcal{I}_0^{\mathcal{O}}(\geq \bar{t};u,\underline{u}) + \mathcal{I}_0^{S}(\geq \bar{t};u,\underline{u})\right)^{\frac{1}{2}} \end{aligned} \quad (5.136)$$

Substituting in eq. 2.25 U with $\tau_-\underline{\alpha}(F)$ it follows that to control $|\underline{\alpha}(F)(u,\underline{u})|$ we have to control the following integrals [42]:

$$\begin{aligned} &\Phi^{-2}(r(u,\underline{u}))\int_{\underline{C}(\underline{u};[u_0,u])}\tau_-^2|\underline{\alpha}(F)|^2 \quad , \int_{\underline{C}(\underline{u};[u_0,u])}\tau_-^2 r^2|\nabla\!\!\!/\underline{\alpha}(F)|^2 \\ &\int_{\underline{C}(\underline{u};[u_0,u])}\tau_-^4|\mathbf{D}\!\!\!/_3\underline{\alpha}(F)|^2 \quad , \int_{\underline{C}(\underline{u};[u_0,u])}\tau_-^2 r^4|\nabla\!\!\!/^2\underline{\alpha}(F)|^2 \\ &\int_{\underline{C}(\underline{u};[u_0,u])}\tau_-^4 r^2|\nabla\!\!\!/\mathbf{D}\!\!\!/_3\underline{\alpha}(F)|^2 \end{aligned} \quad (5.137)$$

These integrals are estimated as before, with the obvious substitutions, and the final result is

$$\begin{aligned} |\underline{\alpha}(F)(u,\underline{u})| &\leq cr^{-1}\tau_-^{-\frac{3}{2}}\Phi^{-2}(r(u,\underline{u}))\left[\int_{\underline{C}(\underline{u};[u_0,u])} Q(L_S\, L_{\mathcal{O}}F)(\bar{K},e_3)\right. \\ &+ \left.\int_{\underline{C}(\underline{u};[u_0,u])} Q(\,L_{\mathcal{O}}F)(\bar{K},e_3) + \int_{\underline{C}(\underline{u};[u_0,u])} Q(\,L^2_{\mathcal{O}}F)(\bar{K},e_3)\right]^{\frac{1}{2}} \\ &\leq c\Phi^{-2}(r(u,\underline{u}))\left(\underline{\mathcal{I}}_0^{\mathcal{O}}(\geq \bar{t};u,\underline{u}) + \underline{\mathcal{I}}_0^{S}(\geq \bar{t};u,\underline{u})\right)^{\frac{1}{2}} \end{aligned}$$

[42] The factor $\Phi^{-2}(r(u,\underline{u}))$ in front of the first integral arises from the inequality

$$\int_{\underline{C}(\underline{u};[u_0,u])}\tau_-^2|\mathbf{D}\!\!\!/_3\tau_-\underline{\alpha}(F)|^2 \leq \Phi^{-2}(r(u,\underline{u}))\int_{\underline{C}(\underline{u};[u_0,u])}\tau_-^2|\underline{\alpha}(F)|^2 + \int_{\underline{C}(\underline{u};[u_0,u])}\tau_-^4|\mathbf{D}\!\!\!/_3\underline{\alpha}(F)|^2 \;.$$

It is important to remark that the choice of estimating $|\underline{\alpha}|$ in terms of integrals along the backward cones $\underline{C}(\underline{u})$, instead that along the forward cones $C(u)$ as was done for $|\alpha|$, is obliged. In fact these integrals have to be estimated in terms of the norms built with the energy momentum tensor Q and, see eqs. 3.38, the integrals of Q along the $C(u)$ cones do not contain $\underline{\alpha}$. The null components ρ and σ, viceversa, can be estimated using both types of integrals.

5.3.2 ρ and σ

We discuss here only the bound for the scalar function ρ, as for σ the proof is exactly the same [43].
Differently from the one form α, ρ satisfies the following equation

$$\sum_{i<j}|\rho(L_{\Omega_{ij}}F)|^2=\sum_{i<j}|L_{\Omega_{ij}}\rho(F)|^2=|\,L_{\mathcal{O}}\rho(F)|^2=r^2|\nabla\!\!\!\!/\,\rho|^2,$$

therefore the integrals of $Q(\,L_{\mathcal{O}}F)$ are not sufficient to control ρ. On the other side from the Poincaré inequality

$$\int_{S(u,\underline{u})}r^2|\rho-\bar{\rho}|^2\leq c\int_{S(u,\underline{u})}r^4|\nabla\!\!\!\!/\,\rho|^2\leq c\int_{S(u,\underline{u})}r^2|\rho(\,L_{\mathcal{O}}F)|^2 \tag{5.138}$$

we expect, using the previous energy type norms, to be able to control $(\rho-\bar{\rho})$. We divide the problem in two parts writing: $|\rho|\leq|\rho-\bar{\rho}|+|\bar{\rho}|$.

5.3.3 $\rho-\bar{\rho}$

Substituting U with $r(\rho-\bar{\rho})$ in eq. 2.22, using the Poincaré inequality and the equation $\mathbf{D}_3\bar{\rho}=\overline{\mathbf{D}_3\rho}$ we obtain that $|r^2\tau_-^{\frac{1}{2}}(\rho-\bar{\rho})|$ is bounded by the sum of the following three integrals

$$\int_{\underline{C}(\underline{u},[u_0,u])}r^4|\nabla\!\!\!\!/\,\rho|^2\ ,\ \int_{\underline{C}(\underline{u},[u_0,u])}r^6|\nabla\!\!\!\!/^2\rho|^2\ ,\ \int_{\underline{C}(\underline{u},[u_0,u])}r^4\tau_-^2|\nabla\!\!\!\!/\,\mathbf{D}\!\!\!\!/_3\rho|^2$$

From eq. 3.38 the first two integrals are controlled by

$$\Phi^{-1}(r(u,\underline{u}))\int_{\underline{C}(\underline{u},[u_0,u])}Q(\,L_{\mathcal{O}}F)(\bar{K},e_3)\text{ and }\Phi^{-1}(r(u,\underline{u}))\int_{\underline{C}(\underline{u},[u_0,u])}Q(\,L_{\mathcal{O}}^2F)(\bar{K},e_3).$$

Using the Maxwell equation $\mathbf{D}_3\rho-2\frac{\Phi}{r}\rho+\mathrm{d}\!\!\!/\mathrm{iv}\,\underline{\alpha}=0$, the third integral is controlled by the following ones

$$\int_{\underline{C}(\underline{u},[u_0,u])}r^2\tau_-^2|\nabla\!\!\!\!/\,\rho|^2\leq\int_{\underline{C}(\underline{u},[u_0,u])}r^2\tau_+^2|\nabla\!\!\!\!/\,\rho|^2\ ,$$
$$\int_{\underline{C}(\underline{u},[u_0,u])}r^4\tau_-^2|\nabla\!\!\!\!/^2\underline{\alpha}|^2\ .$$

[43] The two functions will be treated differently only when we discuss their initial conditions as they have a different physical meaning.

These integrals are bounded by

$$\Phi^{-1}(r(u,\underline{u}))\int_{\underline{C}(\underline{u},[u_0,u])} Q(\mathcal{L}_{\mathcal{O}}F)(\bar{K},e_3)\ ,\ \Phi^{-1}(r(u,\underline{u}))\int_{\underline{C}(\underline{u},[u_0,u])} Q(\mathcal{L}_{\mathcal{O}}^2F)(\bar{K},e_3)$$

so that finally

$$\begin{aligned}|r^2\tau_-^{\frac{1}{2}}(\rho-\bar{\rho})(F)(u,\underline{u})| &\le c\Phi^{-1}(r(u,\underline{u}))\left[\int_{\underline{C}(\underline{u};[u_0,u])} Q(\mathcal{L}_{\mathcal{O}}F)(\bar{K},e_3)\right.\\ &\left.+\int_{\underline{C}(\underline{u};[u_0,u])} Q(\mathcal{L}_{\mathcal{O}}^2F)(\bar{K},e_3)\right]^{\frac{1}{2}} \le c\Phi^{-1}(r(u,\underline{u}))\left(\underline{\mathcal{I}}_0^{\mathcal{O}}(u,\underline{u})\right)^{\frac{1}{2}} \qquad (5.139)\end{aligned}$$

5.3.4 $\bar{\rho}$

We have to control $\bar{\rho}(u,\underline{u}) \equiv \frac{1}{|S(u,\underline{u})|}\int_{S(u,\underline{u})}\rho$, where $|S(u,\underline{u})| = 4\pi r^2(u,\underline{u})$ and $r(u,\underline{u})$ satisfies

$$r(u,\underline{u})+2m\log(\frac{r(u,\underline{u})}{2m}-1)=\frac{1}{2}(\underline{u}-u)=r_*(u,\underline{u}).$$

From $\frac{\partial}{\partial u}r=-\frac{\Phi^2}{2}$

$$\frac{\partial}{\partial u}|S(u,\underline{u})|=-|S(u,\underline{u})|\frac{\Phi^2}{r}$$

follows and

$$\frac{\partial}{\partial u}\int_{S(u,\underline{u})}\rho=\frac{-\Phi^2}{r}\int_{S(u,\underline{u})}\rho+\int_{S(u,\underline{u})}\frac{\partial}{\partial u}\rho \qquad (5.140)$$

so that finally

$$\begin{aligned}\frac{\partial\bar{\rho}}{\partial u} &= \frac{-1}{|S(u,\underline{u})|}(\frac{\partial}{\partial u}|S(u,\underline{u})|)\bar{\rho}+\frac{1}{|S(u,\underline{u})|}\frac{\partial}{\partial u}\int_{S(u,\underline{u})}\rho\\ &= \frac{1}{|S(u,\underline{u})|}\int_{S(u,\underline{u})}\frac{\partial\bar{\rho}}{\partial u} \qquad (5.141)\end{aligned}$$

As on the scalar functions $\frac{\partial}{\partial u}=\frac{\Phi}{2}\mathbf{D}_3$ using the Maxwell equations we obtain

$$\frac{\partial}{\partial u}\bar{\rho}=\frac{\Phi^2}{r}\bar{\rho}-\frac{\Phi}{2}\overline{\not{d}\mathrm{iv}\,\underline{\alpha}}=\frac{\Phi^2}{r}\bar{\rho}$$

so that $\frac{\partial}{\partial u}r^2\bar{\rho}=0$ and, finally,

$$|\bar{\rho}(u,\underline{u})|\le\frac{1}{r^2(u,\underline{u})}\sup_{\Sigma_{t=0}}|r^2\bar{\rho}|. \qquad (5.142)$$

5.4 Proof of Lemma 4.9

Let us define

$$\begin{aligned}\tilde{Q}_{\mu\nu} \equiv Q(L_S L_{\Omega_{ij}} F)_{\mu\nu} &= \tilde{F}_{\mu\rho}\tilde{F}_\nu{}^\rho + {}^*\tilde{F}_{\mu\rho}{}^*\tilde{F}_\nu{}^\rho \\ &= 2\tilde{F}_{\mu\rho}\tilde{F}_\nu{}^\rho - \frac{1}{2}g_{\mu\nu}\tilde{F}_{\rho\sigma}\tilde{F}^{\rho\sigma} \end{aligned} \tag{5.143}$$

where, for a generic couple (i,j) where $i,j \in \{1,2,3\}$, $\tilde{F}_{\mu\rho} \equiv L_S L_{\Omega_{ij}} F_{\mu\rho}$. It is easy to prove that

$$D^\mu \tilde{Q}_{\mu\nu} = 2(D^\mu \tilde{F}_{\mu\rho})\tilde{F}_\nu{}^\rho \tag{5.144}$$

We are therefore reduced to studying $D^\mu \tilde{F}_{\mu\rho} = D^\mu L_S\, L_{\mathcal{O}} F_{\mu\rho}$ where $L_{\mathcal{O}} F$ denotes the generic $L_{\Omega_{ij}} F$. As

$$\begin{aligned}(L_S(L_{\mathcal{O}} F))_{\mu\rho} &= (S^\lambda D_\lambda\, L_{\mathcal{O}} F)_{\mu\rho} + D_\mu\, S^\alpha(L_{\mathcal{O}} F)_{\alpha\rho} + D_\rho\, S^\beta(L_{\mathcal{O}} F)_{\mu\beta} \\ &= S^\lambda (D_\lambda\, L_{\mathcal{O}} F)_{\mu\rho} + D_\mu\, S^\alpha(L_{\mathcal{O}} F)_{\alpha\rho} + D_\rho\, S^\beta(L_{\mathcal{O}} F)_{\mu\beta}\end{aligned} \tag{5.145}$$

then

$$\begin{aligned}(D_\sigma\, \tilde{F})_{\mu\rho} &\equiv (D_\sigma\, L_S\, L_{\mathcal{O}} F)_{\mu\rho} \\ &= (D_\sigma\, S)^\lambda (D_\lambda\, L_{\mathcal{O}} F)_{\mu\rho} + S^\lambda (D_\sigma\, D_\lambda\, L_{\mathcal{O}} F)_{\mu\rho} \\ &+ (D_\sigma\, D_\mu\, S)^\alpha (L_{\mathcal{O}} F)_{\alpha\rho} + (D_\sigma\, D_\rho\, S)^\beta (L_{\mathcal{O}} F)_{\mu\beta} \\ &+ (D_\mu\, S)^\alpha (D_\sigma\, L_{\mathcal{O}} F)_{\alpha\rho} + (D_\rho\, S)^\beta (D_\sigma\, L_{\mathcal{O}} F)_{\mu\beta}\end{aligned} \tag{5.146}$$

Moreover

$$\begin{aligned}D_\sigma\, D_\mu\, S_\lambda &= [D_\sigma\, , D_\mu\,] S_\lambda + D_\mu\, D_\sigma\, S_\lambda \\ &= R_{\lambda\beta\sigma\mu}\, S^\beta + D_\mu\, {}^{(S)}\pi_{\sigma\lambda} - D_\mu\, D_\lambda\, S_\sigma \\ &= R_{\lambda\beta\sigma\mu}\, S^\beta - R_{\sigma\beta\mu\lambda}\, S^\beta + D_\mu\, {}^{(S)}\pi_{\sigma\lambda} - D_\lambda\, {}^{(S)}\pi_{\mu\sigma} + D_\lambda\, D_\sigma\, S_\mu \\ &= (\, R_{\lambda\beta\sigma\mu}\, - R_{\sigma\beta\mu\lambda}\, + \, R_{\mu\beta\lambda\sigma}\,) S^\beta \\ &+ D_\mu\, {}^{(S)}\pi_{\sigma\lambda} - D_\lambda\, {}^{(S)}\pi_{\mu\sigma} + D_\sigma\, {}^{(S)}\pi_{\lambda\mu} - D_\sigma\, D_\mu\, S_\lambda\end{aligned} \tag{5.147}$$

From it

$$\begin{aligned}D_\sigma\, D_\mu\, S_\lambda &= -\frac{1}{2} R_{\beta(\lambda\sigma\mu)} S^\beta + R_{\beta\sigma\mu\lambda}\, S^\beta + \frac{1}{2}(D_\mu\, {}^{(S)}\pi_{\sigma\lambda} - D_\lambda\, {}^{(S)}\pi_{\mu\sigma} + D_\sigma\, {}^{(S)}\pi_{\lambda\mu}) \\ &= R_{\lambda\mu\sigma\beta}\, S^\beta + {}^{(S)}\Gamma_{\sigma\mu\,\lambda}\end{aligned} \tag{5.148}$$

where we used the Bianchi identity $R_{\beta(\lambda\sigma\mu)} = 0$ and denoted

$${}^{(S)}\Gamma_{\sigma\mu\,\lambda} \equiv \frac{1}{2}\left[D_\mu\, ({}^{(S)}\pi_{\sigma\lambda}) - D_\lambda\, ({}^{(S)}\pi_{\mu\sigma}) + D_\sigma\, ({}^{(S)}\pi_{\lambda\mu}) \right].$$

From it

$$\begin{aligned}(D_\sigma L_S(\,L_{\mathcal{O}}F))_{\mu\rho} &= (D_\sigma S^\lambda)(D_\lambda\,L_{\mathcal{O}}F)_{\mu\rho}+S^\lambda(D_\sigma D_\lambda\,L_{\mathcal{O}}F)_{\mu\rho}\\ &+ (D_\sigma D_\mu\,S^\alpha)(\,L_{\mathcal{O}}F)_{\alpha\rho}+(D_\sigma D_\rho\,S^\alpha)(\,L_{\mathcal{O}}F)_{\mu\alpha}\\ &+ (D_\mu\,S^\alpha)(D_\sigma\,L_{\mathcal{O}}F)_{\alpha\rho}+(D_\rho\,S^\alpha)(D_\sigma\,L_{\mathcal{O}}F)_{\mu\alpha}\end{aligned} \tag{5.149}$$

and

$$\begin{aligned}(L_S D_\sigma(\,L_{\mathcal{O}}F))_{\mu\rho} &= S^\lambda(D_\lambda D_\sigma\,L_{\mathcal{O}}F)_{\mu\rho}+(D_\sigma\,S^\lambda)(D_\lambda\,L_{\mathcal{O}}F)_{\mu\rho}\\ &+ (D_\mu\,S^\alpha)(D_\sigma\,L_{\mathcal{O}}F)_{\lambda\rho}+(D_\rho\,S^\lambda)(D_\sigma\,L_{\mathcal{O}}F)_{\mu\lambda}\end{aligned} \tag{5.150}$$

Subtracting these expressions we obtain

$$\begin{aligned}[D_\sigma\,,L_S]\,L_{\mathcal{O}}F_{\mu\rho} &= S^\lambda(D_\sigma D_\lambda-D_\lambda D_\sigma)\,L_{\mathcal{O}}F_{\mu\rho}\\ &+ (D_\sigma D_\mu\,S^\alpha)(\,L_{\mathcal{O}}F)_{\alpha\rho}+(D_\sigma D_\rho\,S^\alpha)(\,L_{\mathcal{O}}F)_{\mu\alpha}\\ &= S^\lambda(\,R_{\mu\beta\sigma\lambda}\,L_{\mathcal{O}}F^\beta_\rho+\,R_{\rho\beta\sigma\lambda}\,L_{\mathcal{O}}F^\beta_\mu)\\ &+ (D_\sigma D_\mu\,S^\alpha)(\,L_{\mathcal{O}}F)_{\alpha\rho}+(D_\sigma D_\rho\,S^\alpha)(\,L_{\mathcal{O}}F)_{\mu\alpha}\\ &= S^\lambda(\,R_{\mu\beta\sigma\lambda}\,L_{\mathcal{O}}F^\beta_\rho+\,R_{\rho\beta\sigma\lambda}\,L_{\mathcal{O}}F^\beta_\mu)\\ &+ (\,R_{\lambda\mu\sigma\beta}\,S^\beta+\,{}^{(S)}\Gamma_{\sigma\mu\,\lambda}\,)\,L_{\mathcal{O}}F^\lambda{}_\rho\\ &+ (R_{\lambda\rho\sigma\beta}\,S^\beta+\,{}^{(S)}\Gamma_{\sigma\rho\,\lambda}\,)\,L_{\mathcal{O}}F^\lambda{}_\mu\\ &= {}^{(S)}\Gamma_{\sigma\mu\,\lambda}\,L_{\mathcal{O}}F^\lambda{}_\rho+{}^{(S)}\Gamma_{\sigma\rho\,\lambda}\,L_{\mathcal{O}}F^\lambda{}_\mu\end{aligned} \tag{5.151}$$

As from the Maxwell equations $D^\mu\,L_{\mathcal{O}}F_{\mu\rho}=0$ and as $L_S g^{\mu\sigma}=-{}^{(S)}\pi^{\mu\sigma}$, we obtain

$$\begin{aligned}D^\mu\tilde{F}_{\mu\rho} &= g^{\mu\sigma}D_\sigma\tilde{F}_{\mu\rho}=g^{\mu\sigma}D_\sigma L_S\,L_{\mathcal{O}}F_{\mu\rho}\\ &= g^{\mu\sigma}L_S D_\sigma\,L_{\mathcal{O}}F_{\mu\rho}+g^{\mu\sigma}(\,{}^{(S)}\Gamma_{\sigma\mu\,\lambda}\,L_{\mathcal{O}}F^\lambda{}_\rho+\,{}^{(S)}\Gamma_{\sigma\rho\,\lambda}\,L_{\mathcal{O}}F^\lambda{}_\mu)\\ &= {}^{(S)}\pi^{\mu\sigma}D_\sigma\,\,L_{\mathcal{O}}F_{\mu\rho}+g^{\mu\sigma}[\,{}^{(S)}\Gamma_{\sigma\mu\,\lambda}\,L_{\mathcal{O}}F^\lambda_\rho+\,{}^{(S)}\Gamma_{\sigma\rho\,\lambda}\,L_{\mathcal{O}}F^\lambda_\mu]\\ &= \left\{{}^{(S)}\hat{\pi}^{\mu\sigma}D_\sigma\,\,L_{\mathcal{O}}F_{\mu\rho}+\,{}^{(S)}\Gamma_\lambda\,\,L_{\mathcal{O}}F^\lambda_\rho+\,{}^{(S)}\Gamma_{\sigma\rho\,\lambda}\,\,L_{\mathcal{O}}F^{\sigma\lambda}\right\}\end{aligned} \tag{5.152}$$

where ${}^{(S)}\Gamma_\lambda\equiv g^{\mu\sigma}\,{}^{(S)}\Gamma_{\sigma\mu\,\lambda}$ and ${}^{(S)}\hat{\pi}^{\mu\sigma}\equiv{}^{(S)}\pi^{\mu\sigma}-(1/4)g^{\mu\sigma}tr^{(S)}\pi$.
Therefore from eq. 5.144

$$\begin{aligned}&D^\mu\tilde{Q}_{\mu\nu}=2(J^{(1)}_\rho+J^{(2)}_\rho+J^{(3)}_\rho)\tilde{F}_\nu{}^\rho\\ =\;&\sum_{l=1}^{3}\left[-J^{(l)}_0(e_3)e^\sigma_4-J^{(l)}_0(e_4)e^\sigma_3+2J^{(l)}_0(e_\theta)e^\sigma_\theta+2J^{(l)}_0(e_\phi)e^\sigma_\phi\right](L_S\,L_{\mathcal{O}}F)_{\nu\sigma}\end{aligned} \tag{5.153}$$

where

$$J^{(1)}_\rho ={}^{(S)}\, \hat{\pi}^{\mu\sigma} D_\sigma\, L_{\mathcal{O}} F_{\mu\rho}$$
$$J^{(2)}_\rho = {}^{(S)}\Gamma_\lambda\; L_{\mathcal{O}} F^{\lambda}_{\;\rho} \tag{5.154}$$
$$J^{(3)}_\rho = {}^{(S)}\Gamma_{\sigma\rho\,\lambda}\; L_{\mathcal{O}} F^{\sigma\lambda}$$

Recalling that

$$\begin{aligned}
&\bar{K}^\nu = \frac{\Phi}{2}(\tau_+^2 e_4^\nu + \tau_-^2 e_3^\nu)\\
&(L_S\, L_{\mathcal{O}} F)_{\nu\sigma} e_3^\nu e_4^\sigma = \rho(L_S\, L_{\mathcal{O}} F)\\
&(L_S\, L_{\mathcal{O}} F)_{\nu\sigma} e_a^\nu e_3^\sigma = \underline{\alpha}(L_S\, L_{\mathcal{O}} F)(e_a)\\
&(L_S\, L_{\mathcal{O}} F)_{\nu\sigma} e_a^\nu e_4^\sigma = \alpha(L_S\, L_{\mathcal{O}} F)(e_a)
\end{aligned} \tag{5.155}$$

we conclude that

$$\begin{aligned}
(\mathbf{Div} Q(L_S\, L_{\mathcal{O}} F))_\nu \bar{K}^\nu &= \sum_{l=1}^{3} \Big\{ [-J^{(l)}(e_3) e_4^\sigma - J^{(l)}(e_4) e_3^\sigma \\
+ \quad & 2J^{(l)}(e_\theta) e_\theta^\sigma + 2J^{(l)}(e_\phi) e_\phi^\sigma](L_S\, L_{\mathcal{O}} F)_{\nu\sigma} \Big\} (\Phi/2)(\tau_+^2 e_4^\nu + \tau_-^2 e_3^\nu) \\
= \quad & \sum_{l=1}^{3} \Phi \Big\{ \tau_+^2 [J^{(l)}(e_4) \rho(L_S\, L_{\mathcal{O}} F) - I^{(l)} \cdot \alpha(L_S\, L_{\mathcal{O}} F)] \\
- \quad & \tau_-^2 [J^{(l)}(e_3) \rho(L_S\, L_{\mathcal{O}} F) - I^{(l)} \cdot \underline{\alpha}(L_S\, L_{\mathcal{O}} F)] \Big\}
\end{aligned} \tag{5.156}$$

where $I^{(l)} \cdot \alpha(L_S\, L_{\mathcal{O}} F) \equiv \sum_a J^{(l)}(e_a) \cdot \alpha(L_S\, L_{\mathcal{O}} F)(e_a)$ and the similar expression for $I^{(l)} \cdot \underline{\alpha}(L_S\, L_{\mathcal{O}} F)$. The explicit expressions of ${}^{(S)}\Gamma_\lambda$ and ${}^{(S)}\Gamma_{\sigma\mu\,\lambda}$ are easily obtained by direct computation.

5.5 The currents $J^{(l)}(e_4), J^{(l)}(e_3), J^{(l)}(e_\theta), J^{(l)}(e_\phi)$

We start considering the currents $J^{(1)} \equiv J^{(1)}(\, L_{\mathcal{O}} F)$ whose explicit expression is

$$J^{(1)}_\rho ={}^{(S)}\, \hat{\pi}^{\mu\sigma} D_\sigma\, L_{\mathcal{O}} F_{\mu\rho} \tag{5.157}$$

From eqs. 2.11, 2.12 it follows that

$${}^{(S)}\hat{\pi}^{\mu\sigma} = \mathrm{sign}(\alpha) g_{\alpha\beta} \left(1 + r_*(\Phi\partial_r\Phi - \frac{\Phi^2}{r}) \right) \tag{5.158}$$

and the only components in the null frame different from zero are

$$\begin{aligned}
{}^{(S)}\hat{\pi}^{\mu\nu} e_3^\mu\, e_4^\nu \quad &\equiv \quad {}^{(S)}j = -2\mu_{(S)}\\
{}^{(S)}\hat{\pi}^{\mu\nu} e_a^\mu e_a^\nu \quad &\equiv \quad {}^{(S)}i_{ab} = -(\delta_a^\theta \delta_b^\theta + \delta_a^\phi \delta_b^\phi)\mu_{(S)}
\end{aligned} \tag{5.159}$$

Denoting $\hat{F} \equiv L_{\mathcal{O}} F$ we write

$$J^{(1)}_\rho \quad \equiv \quad {}^{(S)}\pi^{\mu\sigma} D_\sigma\, \hat{F}_{\mu\rho} ={}^{(S)}\, \hat{\pi}^{\mu\sigma} g^{\mu\mu'}\, g^{\sigma\sigma'}\, D_\sigma\, \hat{F}_{\mu\rho}$$

$$
\begin{aligned}
&= {}^{(S)}\hat{\pi}^{\mu\sigma}[(-1/2)(e_3^{\mu'}\, e_4^{\mu} + e_4^{\mu'}\, e_3^{\mu}) + e_\theta^{\mu'}\, e_\theta^{\mu} + e_\phi^{\mu'}\, e_\phi^{\mu}] \\
&\quad [(-1/2)(e_3^{\sigma'}\, e_4^{\sigma} + e_4^{\sigma'}\, e_3^{\sigma}) + e_\theta^{\sigma'}\, e_\theta^{\sigma} + e_\phi^{\sigma'}\, e_\phi^{\sigma}]D_\sigma \hat{F}_{\mu\rho} \\
&= -\frac{1}{2}\mu_{(S)}\left((\mathbf{D}_{e_4}\hat{F}_{\mu\rho}) + (\mathbf{D}_{e_3}\hat{F}_{\mu\rho})e_3^{\mu}\right) \\
&\quad -\mu_{(S)}\left((\mathbf{D}_{e_\theta}\hat{F}_{\mu\rho})e_\theta^{\mu} + (\mathbf{D}_{e_\phi}\hat{F}_{\mu\rho})e_\phi^{\mu}\right)
\end{aligned} \tag{5.160}
$$

Therefore

$$
\begin{aligned}
J^{(1)}(e_4) &= -\frac{1}{2}\mu_{(S)}\left[(\mathbf{D}_{e_3}\hat{F}_{\mu\rho})e_4^{\mu}e_4^{\rho} + (\mathbf{D}_{e_4}\hat{F}_{\mu\rho})e_3^{\mu}e_4^{\rho}\right] \\
&- \mu_{(S)}\left[(\mathbf{D}_{e_\theta}\hat{F}_{\mu\rho})e_\theta^{\mu}\, e_4^{\rho} + (\mathbf{D}_{e_\phi}\hat{F}_{\mu\rho})e_\phi^{\mu}\, e_4^{\rho}\right] \\
J^{(1)}(e_3) &= -\frac{1}{2}\mu_{(S)}\left[(\mathbf{D}_{e_3}\hat{F}_{\mu\rho})e_4^{\mu}e_3^{\rho} + (\mathbf{D}_{e_4}\hat{F}_{\mu\rho})e_3^{\mu}e_3^{\rho}\right] \\
&- \mu_{(S)}\left[(\mathbf{D}_{e_\theta}\hat{F}_{\mu\rho})e_\theta^{\mu}\, e_3^{\rho} + (\mathbf{D}_{e_\phi}\hat{F}_{\mu\rho})e_\phi^{\mu}\, e_3^{\rho}\right] \qquad (5.161) \\
J^{(1)}(e_\theta) &= -\frac{1}{2}\mu_{(S)}\left[(\mathbf{D}_{e_3}\hat{F}_{\mu\rho})e_4^{\mu}e_\theta^{\rho} + (\mathbf{D}_{e_4}\hat{F}_{\mu\rho})e_3^{\mu}e_\theta^{\rho}\right] \\
&- \mu_{(S)}\left[(\mathbf{D}_{e_\theta}\hat{F}_{\mu\rho})e_\theta^{\mu}\, e_\theta^{\rho} + (\mathbf{D}_{e_\phi}\hat{F}_{\mu\rho})e_\phi^{\mu}\, e_\theta^{\rho}\right] \\
J^{(1)}(e_\phi) &= -\frac{1}{2}\mu_{(S)}\left[(\mathbf{D}_{e_3}\hat{F}_{\mu\rho})e_4^{\mu}e_\phi^{\rho} + (\mathbf{D}_{e_4}\hat{F}_{\mu\rho})e_3^{\mu}e_\phi^{\rho}\right] \\
&- \mu_{(S)}\left[(\mathbf{D}_{e_\phi}\hat{F}_{\mu\rho})e_\theta^{\mu}\, e_\phi^{\rho} + (\mathbf{D}_{e_\phi}\hat{F}_{\mu\rho})e_\phi^{\mu}\, e_\phi^{\rho}\right].
\end{aligned}
$$

As easily

$$
\begin{aligned}
(\mathbf{D}_{e_4}\hat{F}_{\mu\rho})e_3^{\mu}\, e_4^{\rho} &= 2\partial_{e_4}\rho(\hat{F}) \\
(\mathbf{D}_{e_\theta}\hat{F}_{\mu\rho})e_\theta^{\mu}\, e_4^{\rho} &= \partial_{e_\theta}(\alpha(\hat{F})(e_\theta)) - (\Phi/r)\rho(\hat{F}) \\
(\mathbf{D}_{e_\phi}\hat{F}_{\mu\rho})e_\phi^{\mu}\, e_4^{\rho} &= \partial_{e_\phi}(\alpha(\hat{F})(e_\phi)) - (\Phi/r)\rho(\hat{F}) + (1/r)(\cot\theta)(\alpha(\hat{F})(e_\theta)) \\
(\mathbf{D}_{e_3}\hat{F}_{\mu\rho})e_4^{\mu}\, e_3^{\rho} &= -2\partial_{e_3}\rho(\hat{F}) \\
(\mathbf{D}_{e_\theta}\hat{F}_{\mu\rho})e_\theta^{\mu}\, e_3^{\rho} &= \partial_{e_\theta}(\underline{\alpha}(\hat{F})(e_\theta)) - (\Phi/r)\rho(\hat{F}) \\
(\mathbf{D}_{e_\phi}\hat{F}_{\mu\rho})e_\phi^{\mu}\, e_3^{\rho} &= \partial_{e_\phi}(\underline{\alpha}(\hat{F})(e_\phi)) - (\Phi/r)\rho(\hat{F}) + (1/r)(\cot\theta)(\underline{\alpha}(\hat{F})(e_\theta)) \\
(\mathbf{D}_{e_4}\hat{F}_{\mu\rho})e_3^{\mu}\, e_\theta^{\rho} &= -\partial_{e_4}(\underline{\alpha}(\hat{F})(e_\theta)) - \partial_r\Phi(\underline{\alpha}(\hat{F})(e_\theta)) \\
(\mathbf{D}_{e_3}\hat{F}_{\mu\rho})e_4^{\mu}\, e_\theta^{\rho} &= -\partial_{e_3}(\alpha(\hat{F})(e_\theta)) + \partial_r\Phi(\alpha(\hat{F})(e_\theta)) \\
(\mathbf{D}_{e_4}\hat{F}_{\mu\rho})e_3^{\mu}\, e_\phi^{\rho} &= -\partial_{e_4}(\underline{\alpha}(\hat{F})(e_\phi)) - \partial_r\Phi(\underline{\alpha}(\hat{F})(e_\phi)) \\
(\mathbf{D}_{e_3}\hat{F}_{\mu\rho})e_3^{\mu}\, e_\phi^{\rho} &= -\partial_{e_3}(\underline{\alpha}(\hat{F})(e_\phi)) + \partial_r\Phi(\underline{\alpha}(\hat{F})(e_\phi)) \\
(\mathbf{D}_{e_\phi}\hat{F}_{\mu\rho})e_\phi^{\mu}\, e_\theta^{\rho} &= -\partial_{e_\phi}\sigma(\hat{F}) - (\Phi/2r)[\alpha(\hat{F})(e_\theta) - \underline{\alpha}(\hat{F})(e_\theta)] \\
(\mathbf{D}_{e_\theta}\hat{F}_{\mu\rho})e_\theta{}^{\mu}e_\phi^{\rho} &= \partial_{e_\phi}\sigma(\hat{F}) - (\Phi/2r)[\alpha(\hat{F})(e_\phi) - \underline{\alpha}(\hat{F})(e_\phi)]
\end{aligned} \tag{5.162}
$$

substituting these expressions in the various components of the $J^{(1)}$ current and using the Maxwell equations we obtain

$$
\begin{aligned}
J^{(1)}(e_4) &= \mu_{(S)}(r)[4\frac{\Phi}{r}\rho - 2\text{d}\!\!/\text{iv}\,\alpha] \\
J^{(1)}(e_3) &= \mu_{(S)}(r)[4\frac{\Phi}{r}\rho - 2\text{d}\!\!/\text{iv}\,\underline{\alpha}] \\
J^{(1)}(e_\theta) &= \mu_{(S)}(r)[2\partial_{e_\phi}\sigma + \frac{\Phi}{r}(\alpha(e_\theta) - \underline{\alpha}(e_\theta))] \\
J^{(1)}(e_\phi) &= \mu_{(S)}(r)[-2\partial_{e_\theta}\sigma + \frac{\Phi}{r}(\alpha(e_\phi) - \underline{\alpha}(e_\phi))]
\end{aligned}
$$

where with $\alpha, \underline{\alpha}, \rho, \sigma$ we indicate $\alpha(\mathcal{L}_{\mathcal{O}}F), \underline{\alpha}(\mathcal{L}_{\mathcal{O}}F), \rho(\mathcal{L}_{\mathcal{O}}F), \sigma(\mathcal{L}_{\mathcal{O}}F)$.

5.6 Proof of Lemma 4.12

We start estimating the term $\int_{\Sigma_t\cap V(u,\underline{u})} |{}^{(\bar{K})}\pi^{\alpha\beta} Q(\mathcal{L}^a_{\mathcal{O}}F)_{\alpha\beta}|$. Using eq. 4.77

$$
\begin{aligned}
&\int_{\Sigma_t\cap V(u,\underline{u})} |{}^{(\bar{K})}\pi^{\alpha\beta} Q(\mathcal{L}^a_{\mathcal{O}}F)_{\alpha\beta}| \\
&\leq c\frac{m}{\Phi(r(u,\underline{u}_0(t)))}\int_{\Sigma_t\cap V(u,\underline{u})} \frac{t\left(1+|\log(\frac{r}{2m}-1)|\right)}{\tau_+^2 r} Q(\mathcal{L}^a_{\mathcal{O}}F)(\bar{K},e_3) \\
&\leq c\frac{m}{\Phi(r(u,\underline{u}_0(t)))}\int_{\Sigma_t(r\geq r(u,t))} \frac{t\left(1+|\log(\frac{r}{2m}-1)|\right)}{\tau_+^2 r} Q(\mathcal{L}^a_{\mathcal{O}}F)(\bar{K},e_3)
\end{aligned}
\tag{5.163}
$$

where $r(u,t)$ is the radius of the two dimensional surface $S(u,t) = \Sigma_t \cap C(u)$. To control the factor $\frac{t\left(1+|\log(\frac{r}{2m}-1)|\right)}{\tau_+^2 r}$ we observe that, for $r \leq 4m$, $t \leq 2|r_*(\delta_0)|$, $\frac{1}{\tau_+^2} \leq 1$ and

$$
\frac{t\left(1+|\log(\frac{r}{2m}-1)|\right)}{r} \leq \frac{1}{m}|r_*(\delta_0)|\left(1+|\log(\frac{\delta_0}{2m})|\right) \leq c\left(1+|\log(\frac{\delta_0}{2m})|\right)^2
\tag{5.164}
$$

Using these estimates we have for $t \leq |r_*(\delta_0)|$:

$$
\begin{aligned}
& c\frac{m}{\Phi}(r(u,\underline{u}_0(t)))\int_{\Sigma_t(r\geq r(u,t))} \frac{t\left(1+|\log(\frac{r}{2m}-1)|\right)}{\tau_+^2 r} Q(\mathcal{L}^a_{\mathcal{O}}F)(\bar{K},e_3) \\
\leq\;& c\frac{m}{\Phi}(r(u,\underline{u}_0(t)))\left(1+|\log(\frac{\delta_0}{2m})|\right)^2 \frac{1}{1+t^2}\int_{\Sigma_t(r\geq 4m)} Q(\mathcal{L}^a_{\mathcal{O}}F)(\bar{K},e_3) \\
+\;& c\frac{m}{\Phi}(r(u,\underline{u}_0(t)))\left(1+|\log(\frac{\delta_0}{2m})|\right)^2 \int_{\Sigma_t(r\leq 4m)} Q(\mathcal{L}^a_{\mathcal{O}}F)(\bar{K},e_3) \\
\leq\;& c\left(1+|\log(\frac{\delta_0}{2m})|\right)^3 \mathcal{H}(t)
\end{aligned}
\tag{5.165}
$$

The same estimate holds for the term $\int_{\Sigma_t \cap V(u,\underline{u})} |{}^{(\bar{K})}\pi^{\alpha\beta} Q(L_S\, L_{\mathcal{O}} F)_{\alpha\beta}|$.
The estimate of the terms in $\int_{\Sigma_t \cap V(u,\underline{u})} |(\mathbf{Div} Q(L_S\, L_{\mathcal{O}} F))_\alpha \bar{K}^\alpha|$ is done in the same way. The final result is

$$\int_{\Sigma_t \cap V(u,\underline{u})} |(\mathbf{Div} Q(L_S\, L_{\mathcal{O}} F))_\alpha \bar{K}^\alpha| \leq c \left(1 + |\log(\frac{\delta_0}{2m})|\right)^2 \mathcal{H}(t) \tag{5.166}$$

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