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Lifshitz Tail for 2D Discrete Schrödinger Operator with Random Magnetic Field

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Abstract. Lifshitz tail for 2 dimensional discrete Schrödinger operator with Anderson-type random magnetic field is proved. We first prove local energy estimates for deterministic discrete magnetic Schrödinger operators, and then follow the large deviation argument of Simon [6].

1 Introduction

We first define our Hamiltonian. Let $\mathcal{F}$ be the set of unit squares with the vertices in $\mathbb{Z}^2$, and let $\mathcal{E}$ be the set of edges, i.e.,

$$\mathcal{F} = \big\{[x_1, x_1+1] \times [x_2, x_2+1] \;\big|\; x_1, x_2 \in \mathbb{Z}\big\},$$
$$\mathcal{E} = \big\{(x, y) \;\big|\; x, y \in \mathbb{Z}^2, |x-y| = 1\big\}.$$

For $e = (x, y) \in \mathcal{E}$, we write $\overline{e} = (y, x)$. For $f \in \mathcal{F}$, we denote the boundary of f by $\partial f \subset \mathcal{E}$, i.e., if $f = [x_1, x_1+1] \times [x_2, x_2+1]$ then

$$\partial f = \big\{(x, x+\delta_1), (x+\delta_1, x+\delta_1+\delta_2), (x+\delta_1+\delta_2, x+\delta_2), (x+\delta_2, x)\big\},$$

where $\delta_1 = (1, 0)$ and $\delta_2 = (0, 1) \in \mathbb{Z}^2$.

Let $A(e)$ be a function of $e \in \mathcal{E}$ with values in $\mathbb{R}/(2\pi\mathbb{Z})$ such that

$$A(\overline{e}) = -A(e), \qquad e \in \mathcal{E}.$$

The discrete magnetic Schrödinger operator H on $\mathbb{Z}^2$ with a vector potential A is defined by

$$H\psi(x) = \sum_{|x-y|=1} \big(\psi(x) - e^{iA((x,y))}\psi(y)\big), \qquad x \in \mathbb{Z}^2,$$

for $\psi \in \ell^2(\mathbb{Z}^2)$. H is a bounded self-adjoint operator on $\ell^2(\mathbb{Z}^2)$, and $0 \le H \le 8$. For a given vector potential A, the magnetic field is defined by

$$B(f) = \sum_{e \in \partial f} A(e), \qquad f \in \mathcal{F},$$

which is a function on $\mathcal{F}$ with values in $\mathbb{R}/(2\pi\mathbb{Z})$. It is well-known that the spectral properties of H depends only on B, not A. Namely, if A and $\tilde{A}$ induces the same magnetic field B, then the corresponding Schrödinger operators H and $\tilde{H}$ are unitarily equivalent by a gauge transform (i.e., a unimodular multiplication operator on $\mathbb{Z}^2$).

We suppose that $B = B_w(f)$, $w \in \Omega$, is an identically distributed independent random variables (i.i.d.) with distribution $d\mu$ (on $\mathbb{R}/(2\pi\mathbb{Z})$). We denote the probability space by Ω. Our principal example is the uniform distribution on $\mathbb{R}/(2\pi\mathbb{Z})$. Then H is an ergodic operator, and the spectrum is independent of w almost surely (a.s.). See, e.g., [2]. Moreover, the integrated density of states:

$$k(E) = \lim_{L\to\infty} \frac{1}{|\Lambda_L|} \#\{\text{eigenvalues of } H^{\Lambda_L} \le E\}$$

exists a.s., where

$$\Lambda_L = \bigl\{x = (x_1, x_2) \in \mathbb{Z}^2 \bigm| |x_1|, |x_2| \le L\bigr\},$$

$|\Lambda_L| = (2L+1)^2$, and H^{Λ_L} is the Hamiltonian restricted on $\ell^2(\Lambda_L)$. (See Appendix C for the proof.) We suppose the distribution $d\mu$ satisfies the following assumptions:

Assumption A. (1) $d\mu$ is not point measure at 0, i.e., B_w is not identically zero.
(2) There is C and $a > 0$ such that

$$d\mu([-\varepsilon, \varepsilon]) \ge C\varepsilon^a, \qquad 0 \le \varepsilon \le \pi,$$

where we identify $\mathbb{R}/(2\pi\mathbb{Z})$ with $[-\pi, \pi)$.

Theorem 1. *Let $k(E)$ be the integrated density of states for $H = H_w$, and suppose Assumption A. Then*

$$\lim_{E\downarrow 0} \log(-\log k(E))/\log E = -1.$$

Remark. (1) This is a natural analogue of the Lifshitz singularity of the integrated density of states for Schrödinger operator with random potential (see, e.g., [6], [2], [4] and references therein). Lifshitz tail for magnetic Schrödinger operator is recently studied by Ueki [8], where the random magnetic field is supposed to be Gaussian random field. Note that the space dimension d is 2 in our setting and the right hand side of the statement is $-d/2$.
(2) Under our assumption, $\sigma(H_w) = [0, 8]$ a.s., and hence Theorem 1 describe the behavior of $k(E)$ at the lower edge of the spectrum. Similar result holds at $8 = \sup \sigma(H)$ with little modifications.

In Section 2, we prove a simple local energy estimate for discrete magnetic Schrödinger operators. It is an analogue of the Avron-Herbst-Simon estimate [1] for continuous magnetic Schrödinger operators, and we also show that it is optimal. In the rest of the paper, we mimic the argument of Simon [6] to prove the Lifshitz tail. We define the Dirichlet-Neumann decoupling in Section 3 and prove Theorem 1 in

Section 4. In Appendix A, we discuss a generalization of the local energy estimate of Section 2. It is not necessary in this paper, but is interesting in itself and maybe useful in the analysis of discrete magnetic Schrödinger operators. In Appendix B, we give a proof of the spectral property of an example of Section 2. We give a proof of the existence of the integrated density of states in Appendix C.

Notations. We denote the inner product of $\ell^2(\mathbb{Z}^2)$ by

$$\langle\varphi|\psi\rangle = \sum_{x\in\mathbb{Z}^2} \overline{\varphi(x)}\psi(x), \qquad \varphi,\psi\in\ell^2(\mathbb{Z}^2).$$

$\mathbb{P}(\cdot)$ and $\mathbb{E}(\cdot)$ denote the probability and the expectation with respect to $w\in\Omega$, respectively. For $e\in\mathcal{E}$, $i(e)$ and $t(e)$ denote the initial point and the terminal point of e, respectively, i.e.,

$$i(e)=x, \quad t(e)=y, \qquad \text{if } e=(x,y).$$

We write $\mathbb{Z}_+ = \{0,1,2\dots\}$.

2 Local energy estimates for discrete magnetic Schrödinger operators

In this section, we consider deterministic magnetic Schrödinger operators on $\mathbb{Z}^2$. For a given magnetic field B, we set

$$W_B(x) = \sum_{x\in f}(1-\cos(B(f)/4)), \qquad x\in\mathbb{Z}^2.$$

Here we identify $\mathbb{R}/(2\pi\mathbb{Z})$ with $[-\pi,\pi)$, and hence $B(f)/4\in[-\pi/4,\pi/4)$. Thus, in particular,

$$0\le W_B(x)\le 4(1-1/\sqrt{2}), \qquad x\in\mathbb{Z}^2.$$

Theorem 2. *Let H and W_B as above. Then $H\ge W_B$, i.e.,*

$$\langle\psi|H\psi\rangle \ge \langle\psi|W_B\psi\rangle = \sum_{x\in\mathbb{Z}^2} W_B(x)|\psi(x)|^2, \qquad \psi\in\ell^2(\mathbb{Z}^2).$$

Remark. The motivation of this estimate comes from recent works by Higuchi and Shirai [3]. (See also Sunada [7] for related results on generalized Harper operators.)

Proof. We note

$$\begin{aligned}\langle\psi|H\psi\rangle &= \sum_{|x-y|=1} \overline{\psi(x)}\Big(\psi(x)-e^{iA((x,y))}\psi(y)\Big)\\ &= \sum_{|x-y|=1} \overline{\psi(y)}\Big(\psi(y)-e^{-iA((x,y))}\psi(x)\Big)\end{aligned}$$

since $A((y,x)) = -A((x,y))$. We take the average of these expressions, and obtain

$$\begin{aligned}\langle\psi|H\psi\rangle &= \frac{1}{2}\sum_{|x-y|=1}\Big|\psi(x) - e^{iA((x,y))}\psi(y)\Big|^2\\ &= \frac{1}{2}\sum_{e\in\mathcal{E}}\Big|\psi(i(e)) - e^{iA(e)}\psi(t(e))\Big|^2.\end{aligned}$$

Since each $e \in \mathcal{E}$ is an element of ∂f for only one $f \in \mathcal{F}$, we may write

$$\langle\psi|H\psi\rangle = \frac{1}{2}\sum_{f\in\mathcal{F}}\sum_{e\in\partial f}\Big|\psi(i(e)) - e^{iA(e)}\psi(t(e))\Big|^2.$$

Lemma 3.

$$\sum_{e\in\partial f}\Big|\psi(i(e)) - e^{iA(e)}\psi(t(e))\Big|^2 \geq 2\Big(1 - \cos\Big(\frac{B(f)}{4}\Big)\Big)\sum_{x\in f}|\psi(x)|^2.$$

Now Theorem 2 follows immediately.

Proof of Lemma 3. Let

$$f = [x_1, x_1+1] \times [x_2, x_2+1], \qquad x = (x_1, x_2) \in \mathbb{Z}^2,$$

and we write

$$\begin{aligned}&u_1 = \psi(x), \quad u_2 = \psi(x+\delta_1), \quad u_3 = \psi(x+\delta_1+\delta_2), \quad u_4 = \psi(x+\delta_2),\\ &\theta_1 = A((x, x+\delta_1)), \quad \theta_2 = A((x+\delta_1, x+\delta_1+\delta_2)),\\ &\theta_3 = A((x+\delta_1+\delta_2, x+\delta_2)), \quad \theta_4 = A((x+\delta_2, x)).\end{aligned}$$

Then we have

$$\begin{aligned}&\sum_{e\in\partial f}\Big|\psi(i(e)) - e^{iA(e)}\psi(t(e))\Big|^2\\ &= (\overline{u_1}, \overline{u_2}, \overline{u_3}, \overline{u_4})\begin{pmatrix} 2 & -e^{i\theta_1} & 0 & -e^{-i\theta_4}\\ -e^{-i\theta_1} & 2 & -e^{i\theta_2} & 0\\ 0 & -e^{-i\theta_2} & 2 & -e^{i\theta_3}\\ -e^{i\theta_4} & 0 & -e^{-i\theta_3} & 2\end{pmatrix}\begin{pmatrix}u_1\\u_2\\u_3\\u_4\end{pmatrix} \equiv \mathbf{u}^* h_\theta \mathbf{u}.\end{aligned}$$

We set $\alpha_1 = \theta_1$, $\alpha_2 = \theta_1 + \theta_2$, $\alpha_3 = \theta_1 + \theta_2 + \theta_3$, and

$$A = \begin{pmatrix}1 & & & \\ & e^{i\alpha_1} & & \\ & & e^{i\alpha_2} & \\ & & & e^{i\alpha_3}\end{pmatrix}.$$

Then we have

$$Ah_\theta A^* = \begin{pmatrix} 2 & -1 & 0 & -e^{-iB} \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -e^{iB} & 0 & -1 & 2 \end{pmatrix} \equiv A_B$$

since $B = B(f) = \theta_1 + \theta_2 + \theta_3 + \theta_4$. Note that A_B is the Hamiltonian of the free discrete Schrödinger operator on the closed chain $\{0, 1, 2, 3\}$ with the periodic boundary condition with an additional phase e^{iB}. Thus the eigenvectors are given by

$$\mathbf{v}^j = (1, e^{i\mu_j}, e^{2i\mu_j}, e^{3i\mu_j}), \qquad \mu_j = (B + 2\pi j)/4, \qquad j = 0, 1, 2, 3,$$

and the eigenvalues are $\lambda_j = 2(1 - \cos \mu_j)$. In particular, the lowest eigenvalue is $\lambda_0 = 2(1 - \cos(B/4))$ since $B \in [-\pi, \pi)$. This implies $A_B \geq 2(1 - \cos(B/4))$, and hence $h_\theta \geq 2(1 - \cos(B/4))$ to conclude the assertion.

Example 1. Let $b \in (-\pi, \pi)$ and let

$$B(f) = \begin{cases} b, & \text{if } x_1 + x_2 \text{ is even,} \\ -b, & \text{if } x_1 + x_2 \text{ is odd,} \end{cases}$$

where $f = [x_1, x_1 + 1] \times [x_2, x_2 + 1]$. In this case, $W_B(x) = 4(1 - \cos(b/4))$ for any $x \in \mathbb{Z}^2$. H is solvable and we can show

$$\sigma(H) = [4(1 - \cos(b/4)), 4(1 + \cos(b/4))].$$

This example shows that Theorem 1 is optimal. See Appendix B for the computation of this example.

We will use the following (almost trivial) analogue of the Kato inequality:

Lemma 4.

$$\langle \psi | H\psi \rangle \geq \langle |\psi| \mid H_0 |\psi| \rangle, \qquad \psi \in \ell^2(\mathbb{Z}^2),$$

where H_0 denotes the free discrete Schrödinger operator on $\mathbb{Z}^2$, i.e., H with $A \equiv 0$.

Proof. We have

$$\begin{aligned} \langle \psi | H\psi \rangle &= \frac{1}{2} \sum_e \left| \psi(i(e)) - e^{iA(e)} \psi(t(e)) \right|^2 \\ &\geq \frac{1}{2} \sum_e \big| |\psi(i(e))| - |\psi(t(e))| \big|^2 = \langle |\psi| \mid H_0 |\psi| \rangle. \end{aligned}$$ □

This implies, for example,

$$\begin{aligned} \langle \psi | H\psi \rangle &\geq \frac{1}{2} \langle |\psi| \mid H_0 \, |\psi| \rangle + \frac{1}{2} \langle \psi | W_B \psi \rangle \\ &= \frac{1}{2} \langle |\psi| \mid (H_0 + W_B) |\psi| \rangle. \end{aligned}$$

By the min-max principle, we obtain the following lemma.

Lemma 5.

$$\inf \sigma(H) \geq \frac{1}{2} \inf \sigma(H_0 + W_B).$$

Note that $H_0 + W_B$ is a usual discrete Schrödinger operator. We will use (a modified version of) Lemma 5 in the proof of Theorem 1.

3 Dirichlet-Neumann bracketing

In this section, we generalize the Dirichlet and Neumann decoupling of Simon [6, Section 2] to discrete magnetic Schrödinger operators. Let $\mathbb{Z}^2 = \sum_\alpha S_\alpha$ be a disjoint decomposition of $\mathbb{Z}^2$, and let

$$\Sigma = \left\{ e \in \mathcal{E} \;\middle|\; e \not\subset S_\alpha \text{ for any } S_\alpha \right\}$$

be the boundary set of the decomposition. Then for a given magnetic Schrödinger operator H, we construct operators $H^{\Sigma;N}$ and $H^{\Sigma;D}$ such that

$$H^{\Sigma;N} \leq H \leq H^{\Sigma;D}$$

and they commute with the direct decomposition: $\ell^2(\mathbb{Z}^2) = \bigoplus_\alpha \ell^2(S_\alpha)$, i.e., they act on each $\ell^2(S_\alpha)$.

For each $e \in \mathcal{E}$, we set

$$L_e\psi(x) = \begin{cases} \frac{1}{2}\Big(\psi(x) - e^{iA(e)}\psi(t(e))\Big), & \text{if } x = i(e), \\ \frac{1}{2}\Big(\psi(x) - e^{-iA(e)}\psi(i(e))\Big), & \text{if } x = t(e), \\ 0, & \text{otherwise.} \end{cases}$$

L_e corresponds to 2×2-matrix

$$\frac{1}{2}\begin{pmatrix} 1 & -e^{iA(\varepsilon)} \\ -e^{-iA(e)} & 1 \end{pmatrix} \qquad \text{on } \ell^2(\{i(e), t(e)\}),$$

and hence L_e has eigenvalues 1 and 0. In particular, $L_e \geq 0$. We note

$$H = \sum_{e \in \mathcal{E}} L_e.$$

On the other hand, we also set

$$M_e\psi(x) = \begin{cases} \frac{1}{2}\Big(\psi(x) + e^{iA(e)}\psi(t(e))\Big), & \text{if } x = i(e), \\ \frac{1}{2}\Big(\psi(x) + e^{-iA(e)}\psi(i(e))\Big), & \text{if } x = t(e), \\ 0, & \text{otherwise.} \end{cases}$$

Then, similarly, we learn $M_e \geq 0$. Now we set

$$H^{\Sigma;N} = H - \sum_{e\in\Sigma} L_e = \sum_{e\notin\Sigma} L_e,$$
$$H^{\Sigma;D} = H + \sum_{e\in\Sigma} M_e.$$

Then, clearly, we have

$$0 \leq H^{\Sigma;N} \leq H \leq H^{\Sigma;D},$$

and $H^{\Sigma;\#}$ has no off-diagonal elements corresponding to $e \in \Sigma$, where $\# = N$ or D. Hence $H^{\Sigma;\#}$ acts on each component of the direct decomposition: $\ell^2(\mathbb{Z}^2) = \bigoplus_\alpha \ell^2(S_\alpha)$.

For $L \in \mathbb{Z}_+$, we set

$$S_\alpha^{(L)} = \bigl\{x \in \mathbb{Z}^2 \bigm| L\alpha_j \leq x_j < L(\alpha_j+1), j = 1,2\bigr\} \qquad \text{for } \alpha \in \mathbb{Z}^2.$$

Let $H^{L;\#}$ be the operator $H^{\Sigma;\#}$ restricted to $\ell^2(S_0^{(L)})$. As in [6, Section 2], we set

$$k_L^\#(E) = L^{-2}\mathbb{E}\bigl[\#\{\text{eigenvalues of } H^{L;\#} \leq E\}\bigr]$$

with $\# = N$ or D. Then we have

$$k_L^D(E) \leq k(E) \leq k_L^N(E), \qquad E \in \mathbb{R},$$

for each $L \in \mathbb{Z}_+$. We will estimate $k_D^N(E)$ and $k_L^N(E)$ to obtain lower and upper bounds of $k(E)$, respectively.

4 Proof of Theorem 1

The proof of Theorem 1 is similar to [6], and we mainly discuss the necessary modifications.

4.1 Lower bound

Recall that the lowest eigenvalue of $H_0^{L;D}$ is given by $e_0^{L;D} = 2(1-\cos(\pi/L))$ ([6, Theorem 2.4]). For $E > 0$, we set $L \in \mathbb{Z}_+$ such that

$$\sqrt{4\pi^2/E} < L \leq \sqrt{4\pi^2/E} + 1$$

so that $e_0^{L;D} \leq 2\pi^2/L^2 < E/2$. If we suppose

$$|B(f)| \leq E/8L \qquad \text{for any } f \subset S_0^{(L)},$$

then we can find a vector potential A such that

$$|A(e)| \leq E/8 \qquad \text{for any } e \subset S_0^{(L)}.$$

This implies $\left\| H_0^{L;D} - H^{L;D} \right\| \leq E/2$, and hence the lowest eigenvalue of $H^{L;D}$ is smaller than E. Thus we have

$$\begin{aligned} k_L^D(E) &\geq L^{-2}\mathbb{P}\big(|B(f)| \leq E/8L \text{ for any } f \subset S_0^{(L)}\big) \\ &= L^{-2}\mathbb{P}\big(|B(f)| \leq E/8L\big)^{L^2} \geq L^{-2}C^{L^2}(cE^{3/2})^{aL^2} \\ &= \exp\big[-2\log L + L^2(\log C + a\log c + (3a/2)\log E)\big] \end{aligned}$$

This implies

$$\liminf_{E\to 0} \log(-\log k_L^D(E))/\log E \geq -1$$

since $L^2 \sim 4\pi^2 E^{-1}$ as $E \to 0$, and hence

$$\liminf_{E\to 0} \log(-\log k(E))/\log E \geq -1.$$

4.2 Upper bound

We first note an analogue of Lemma 5.

Lemma 6. *Let*

$$W_B^L(x) = \sum_{x\in f\subset S_0^{(L)}} \Big(1 - \cos\Big(\frac{B(f)}{4}\Big)\Big).$$

Then

$$\inf \sigma(H^{L,N}) \geq \frac{1}{2} \inf \sigma(H_0^{L,N} + W_B^L).$$

The proof is almost the same, and we omit it.

We fix ε_0 and $f_0 > 0$ such that

$$\mathbb{P}(1 - \cos(B(f)/4) \geq \varepsilon_0) = f_0$$

Now the main lemmas of [6], i.e., Theorems 4.1 and 4.2 are rewritten as follows in our setting:

Proposition 7. *There exist L_0 and α_0 such that if $L \geq L_0$ and if*

$$L^{-2}\#\big\{x \in S_0^{(L)} \bigm| W_B^L(x) \geq \varepsilon_0\big\} \geq f_0/3,$$

then

$$\inf \sigma(H_0^{L;N} + W_B^L) \geq \alpha_0 L^{-2}.$$

Proposition 8.

$$\mathbb{P}\Big((L-1)^{-2}\#\Big\{f\in\mathcal{F}\;\Big|\;f\subset S_0^{(L)},1-\cos\Big(\frac{B(f)}{4}\Big)\ge\varepsilon_0\Big\}<\frac{1}{2}f_0\Big)$$
$$\le\exp\Big(-\frac{1}{2}f_0^2(L-1)^2\Big).$$

Proposition 8 implies

$$\mathbb{P}\Big((L-1)^{-2}\#\Big\{x\in S_0^{(L)}\;\Big|\;W_B^L(x)\ge\varepsilon_0\Big\}<\frac{1}{2}f_0\Big)\le\exp\Big(-\frac{1}{2}f_0^2(L-1)^2\Big).$$

by the definition of W_B^L. Combining this with Proposition 7, we observe

$$\mathbb{P}\Big(\inf\sigma(H_0^{L;N}+W_B^L)<\alpha_0L^{-2}\Big)\le\exp\Big(-\frac{1}{3}f_0^2L^2\Big)$$

if $L>L_0$. Hence we have

$$k_L^N(E)\le\mathbb{P}(\inf\sigma(H^{L,N})\le E)$$
$$\le\mathbb{P}\Big(\inf\sigma(H_0^{L;N}+W_B^L)\le 2E\Big)\le\exp\Big(-\frac{1}{3}f_0^2L^2\Big)$$

if we choose L so that $2E<\alpha_0L^{-2}$, i.e., if

$$L<\sqrt{\alpha_0/2E}.$$

We set L be the largest integer satisfying the above condition, so that $L\sim\sqrt{\alpha_0/2}\cdot E^{-1/2}$ as $E\to 0$. Then we learn

$$k_L^N(E)\le\exp\Big(-\frac{1}{2}f_0^2\Big(\sqrt{\alpha_0/2}\cdot E^{-1/2}-1\Big)^2\Big)\sim\exp\Big(-\frac{\alpha_0}{6}f_0^2E^{-1}\Big)$$

as $E\to 0$, and we conclude

$$\limsup_{E\to 0}\log(-\log k(E))/\log E\le -1.$$

Appendix A Generalization of local energy estimate

Let $m\in\mathbb{N}$ and let $\mathcal{F}_m$ be the set of squares of size m with the vertices at $\mathbb{Z}^2$, i.e.,

$$\mathcal{F}_m=\big\{[x_1,x_1+m]\times[x_2,x_2+m]\;\big|\;x_1,x_2\in\mathbb{Z}\big\}.$$

Let $A(e)$ be a given vector potential. Then for $f\in\mathcal{F}_m$, we denote the magnetic flux through f by

$$B(f)=\sum_{e\in\partial f}A(e),$$

where ∂f is the boundary of f. Here we again identify $\mathbb{R}/(2\pi\mathbb{Z})$ with $[-\pi,\pi)$ and we always suppose $B(f)\in[-\pi,\pi)$. Then we set

$$W_B^{(m)}(x)=\frac{1}{m}\sum_{x\in f\in\mathcal{F}_m}\Big(1-\cos\Big(\frac{B(f)}{4m}\Big)\Big).$$

Theorem 9. *Under the above notations, $H\geq W_B^{(m)}$ for any $m\in\mathbb{N}$.*

Proof. We note that each $e\in\mathcal{E}$ is an element of ∂f for m different $f\in\mathcal{F}_m$. Hence, as in the proof of Theorem 2, we have

$$\langle\psi|H\psi\rangle=\frac{1}{2m}\sum_{f\in\mathcal{F}_m}\Big|\psi(i(e))-e^{iA(e)}\psi(t(e))\Big|^2.$$

Then we use the following lemma:

Lemma 10. *For each $f\in\mathcal{F}_m$,*

$$\sum_{e\in\partial f}\Big|\psi(i(e))-e^{iA(e)}\psi(t(e))\Big|^2\geq 2\Big(1-\cos\Big(\frac{B(f)}{4m}\Big)\Big)\sum_{x\in f}|\psi(x)|^2.$$

The lemma is proved similarly as Lemma 3, since the left hand side of the above formula is the energy function for the free discrete Schrödinger operator on the closed chain of size $4m$ with the periodic boundary condition with the phase $e^{iB(f)}$. Now the theorem follows immediately. □

Theorem 9 may imply better estimate if $B(f)$ does not change the sign frequently. Let us consider the constant magnetic case to observe this.

Example 2. Suppose $B(f)=b$ for all $f\in\mathcal{F}$. Then for $f\in\mathcal{F}_m$, $B(f)=m^2b$ (modulo $2\pi\mathbb{Z}$). We choose $m\in\mathbb{N}$ so that

$$\frac{\pi}{(m+1)^2}<|b|\leq\frac{\pi}{m^2}$$

when $|b|$ is small enough. Then $m\sim|\pi/b|^{1/2}$ and $|m^2b|=\pi+O(|b|^{1/2})$ when $b\sim 0$. By simple computations, we learn

$$H\geq W_B^m(x)=\frac{\pi}{4}|b|+O(|b|^{3/2})$$

as $b\to 0$. Probably this lower bound is not optimal, but it is better than the estimate which follows from Theorem 2 for small b.

Appendix B Spectrum of Example 1

We set $\theta = b/2$,

$$A((x, x+\delta_1)) = \begin{cases} \theta, & \text{if } x_1 + x_2 \text{ is even,} \\ -\theta, & \text{if } x_1 + x_2 \text{ is odd,} \end{cases}$$

and $A((x, x+\delta_2)) = 0$ for any $x \in \mathbb{Z}^2$. Then A defines a magnetic Schrödinger operator of Example 1. Note that this operator is invariant under the following change of coordinates:

$$T_1\psi(x) = \psi(x + \delta_1 + \delta_2), \qquad T_2\psi(x) = \psi(x + \delta_1 - \delta_2)$$

where $\psi \in \ell^2(\mathbb{Z}^2)$. Hence we can apply the Floquet-Bloch theory. Namely, we compute the (generalized) eigenfunction and eigenvalues of H under conditions

$$T_1\psi = e^{i\alpha}\psi, \qquad T_2\psi = e^{i\beta}\psi$$

with fixed $\alpha, \beta \in \mathbb{R}/(2\pi\mathbb{Z})$. These lead to a system of equations:

$$\begin{cases} H\psi(0) = 4\psi(0) - \left\{e^{i\theta} + e^{i(\theta-\alpha-\beta)} + e^{-i\alpha} + e^{-i\beta}\right\}\psi(\delta_1), \\ H\psi(\delta_1) = -\left\{e^{-i\theta} + e^{i(-\theta+\alpha+\beta)} + e^{i\alpha} + e^{i\beta}\right\}\psi(0) + 4\psi(\delta_1). \end{cases}$$

Thus the eigenvalues are the characteristic roots of the matrix $\begin{pmatrix} 4 & \lambda \\ \bar{\lambda} & 4 \end{pmatrix}$ with $\lambda = -\{e^{i\theta} + e^{i(\theta-\alpha-\beta)} + e^{-i\alpha} + e^{-i\beta}\}$. They are $4 \pm |\lambda|$, and by simple computations we obtain

$$|\lambda|^2 = 4\left\{1 + \cos\alpha\cos\beta + \cos\theta(\cos\alpha + \cos\beta)\right\}.$$

It is then easy to see

$$\sup_{\alpha,\beta} |\lambda|^2 = 4(2 + 2\cos\theta) = (4\cos(b/4))^2,$$

and the range of $|\lambda|$ (as a function of α and β) is $[0, 4\cos(b/4)]$. Thus the spectrum of H is given by

$$\sigma(H) = \left\{4 \pm |\lambda| \;\middle|\; \alpha, \beta \in \mathbb{R}/(2\pi\mathbb{Z})\right\} = [4 - 4\cos(b/4), 4 + 4\cos(b/4)]$$

and it is absolutely continuous.

Appendix C Existence of the integrated density of states

Theorem 11. *The integrated density of states*

$$k(E) = \lim_{L\to\infty} \frac{1}{|\Lambda_L|} \#\{\text{e.v. of } H^{\Lambda_L} \le E\}$$

exists for $E \in \mathbb{R}$ almost surely. Moreover,

$$k(E) = \lim_{L\to\infty} k_L^D(E) = \sup_L k_L^D(E)$$
$$= \lim_{L\to\infty} k_L^N(E) = \inf_L k_L^N(E),$$

where k_L^D and k_L^N are defined in Section 3.

Proof. Let $\Lambda \subset \mathbb{Z}^2$ be a box in $\mathbb{Z}^2$, and let $H^{\Lambda;D}$ and $H^{\Lambda;N}$ be defined similarly as $H^{L;D}$ and $H^{L;N}$ in Section 3. Namely, they are defined by

$$\langle\psi|H^{\Lambda;N}\psi\rangle = \frac{1}{2}\sum_{\substack{|x-y|=1\\ x,y\in\Lambda}} \left|\psi(x) - e^{iA((x,y))}\psi(y)\right|^2,$$
$$\langle\psi|H^{\Lambda;D}\psi\rangle = \frac{1}{2}\sum_{\substack{|x-y|=1\\ x,y\in\Lambda}} \left|\psi(x) - e^{iA((x,y))}\psi(y)\right|^2 + \sum_{\substack{|x-y|=1\\ x\in\Lambda, y\in\Lambda^c}} |\psi(x)|^2$$

for $\psi \in \ell^2(\Lambda)$. We set

$$k_\Lambda^*(E) = \frac{1}{|\Lambda|}\#\{\text{e.v. of } H^{\Lambda;*} \le E\}$$

for $E \in \mathbb{R}$ with $* = D$ or N. Note that $k_{\Lambda_L}^*(E) = k_{2L+1}^*(E)$. We fix E. Now it is a standard procedure to see that $k_\Lambda^N(E)$ and $k_\Lambda^D(E)$ are subadditive process and superadditive process, respectively, in the sense of [2] Definition VI.1.6. Then by Theorem VI.1.7 of [2], we learn

$$k^N(E) = \lim_{L\to\infty} k_L^N(E) = \inf_L k_L^N(E),$$
$$k^D(E) = \lim_{L\to\infty} k_L^D(E) = \sup_L k_L^D(E)$$

exist. Since

$$k_{2L+1}^D(E) \le \frac{1}{|\Lambda|}\#\{\text{e.v. of } H^{\Lambda_L} \le E\} \le k_{2L+1}^N(E),$$

it remains only to show

$$k^N(E) = k^D(E). \tag{N-D}$$

By the definition, $H^{\Lambda_L;D} - H^{\Lambda_L;N}$ is an operator of rank $\#(\partial\Lambda_L) = 4(2L+1)$. Hence, by the min-max principle ([5] Section XIII.1), we have

$$0 \le \#\{\text{e.v. of } H^{\Lambda_L;N} \le E\} - \#\{\text{e.v. of } H^{\Lambda_L;D} \le E\} \le 4(2L+1).$$

This implies

$$0 \le k_{\Lambda_L}^N(E) - k_{\Lambda_L}^D(E) \le \frac{4}{2L+1}$$

and (N-D) follows immediately from this. $\square$

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