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Strong Magnetic Field Asymptotics of the Integrated Density of States for a Random 3D Schrödinger Operator

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Abstract. We consider the three-dimensional Schrödinger operator with constant magnetic field and bounded random electric potential. We investigate the asymptotic behaviour of the integrated density of states for this operator as the norm of the magnetic field tends to infinity.

Résumé On considère l'opérateur de Schrödinger tridimensionnel avec un champ magnétique constant et un potentiel électrique aléatoire borné. On étudie le comportement asymptotique de la densité d'états pour cet opérateur-ci lorsque la norme du champ magnétique tend vers l'infini.

1 Introduction

Let $\mathbf{b} := (0,0,b)$, $b > 0$, $\mathbf{x} = (x,y,z) \in \mathbb{R}^3$. Introduce the unperturbed selfadjoint Schrödinger operator

$$H_0(b) := \left(i\nabla + \frac{\mathbf{b}\wedge\mathbf{x}}{2}\right)^2 \equiv \left(i\frac{\partial}{\partial x} - \frac{by}{2}\right)^2 + \left(i\frac{\partial}{\partial y} + \frac{bx}{2}\right)^2 - \frac{\partial^2}{\partial z^2}, \tag{1.1}$$

defined originally on $C_0^\infty(\mathbb{R}^3)$, and then closed in $L^2(\mathbb{R}^3)$. It is well-known that for each $b > 0$ we have

$$\sigma(H_0(b)) = [b, +\infty) \tag{1.2}$$

where $\sigma(H_0(b))$ denotes the spectrum of the operator $H_0(b)$ (see e.g. [A.H.S]).

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $V_\omega(\mathbf{x})$, $\omega \in \Omega$, $\mathbf{x} \in \mathbb{R}^3$, be a real random field. We assume that V_ω is $\mathbb{G}^3$-ergodic with $\mathbb{G} = \mathbb{Z}$ or $\mathbb{G} = \mathbb{R}$ (see [K, Section 3.1] or [P.Fi, Section 1C]). In other words, there exists an ergodic group of measure preserving automorphisms $\mathcal{T}_{\mathbf{k}} : \Omega \to \Omega$, $\mathbf{k} \in \mathbb{G}^3$, such that

$$V_\omega(\mathbf{x} + \mathbf{k}) = V_{\mathcal{T}_{\mathbf{k}}\omega}(\mathbf{x}),\ \mathbf{x} \in \mathbb{R}^3,\ \omega \in \Omega,\ \mathbf{k} \in \mathbb{G}^3. \tag{1.3}$$

We recall that ergodicity of a group G of automorphisms of Ω means that the invariance of a given set $\mathcal{A} \in \mathcal{F}$ under the action of G (i.e. $g\mathcal{A} = \mathcal{A}$ for each $g \in G$) implies either $\mathbb{P}(\mathcal{A}) = 1$ or $\mathbb{P}(\mathcal{A}) = 0$.

Let $\mathbf{x} \in \mathbb{R}^3$. We shall write occasionally $\mathbf{x} = (X, z)$ with $X \in \mathbb{R}^2$ and $z \in \mathbb{R}$. We suppose that V_ω is $\mathbb{G}$-ergodic with $\mathbb{G} = \mathbb{Z}$ or $\mathbb{G} = \mathbb{R}$ in the direction of the magnetic field (or, in brief, in the z-direction), i.e. that the subgroup $\{\mathcal{T}_{\mathbf{k}} | \mathbf{k} = (0,0,k),\ k \in \mathbb{G}\}$ is ergodic.

Further, we assume that the realizations of V_ω are almost surely uniformly bounded, i.e. we have

$$c_0 := \text{ess} - \sup_{\omega \in \Omega} \sup_{\mathbf{x} \in \mathbb{R}^3} |V_\omega(\mathbf{x})| < \infty. \tag{1.4}$$

Finally, for simplicity we suppose that the realizations of V_ω are almost surely continuous.

Examples : (i) Let $\alpha_{\mathbf{j}} : \Omega \to \mathbb{R}$, $\mathbf{j} \in \mathbb{Z}^3$, be independent identically distributed almost surely uniformly bounded random variables. Assume that $v : \mathbb{R}^3 \to \mathbb{R}$ is a continuous function satisfying

$$|v(\mathbf{x})| \leq c(1 + |\mathbf{x}|)^{-\beta}, \quad c > 0, \quad \beta > 3.$$

Set

$$V_\omega(\mathbf{x}) := \sum_{\mathbf{j} \in \mathbb{Z}^3} \alpha_{\mathbf{j}}(\omega)\, v(\mathbf{x} - \mathbf{j}),\ \omega \in \Omega,\ \mathbf{x} \in \mathbb{R}^3.$$

Then the random field V_ω is $\mathbb{Z}^3$-ergodic (see [K, Model I, Section 3.3] or [P.Fi, Example 1.15a, p.23]), $\mathbb{Z}$-ergodic in the direction of the magnetic field (as a matter of fact, in all directions; see [E.K.Sch.S, Example 2, p.615]). Moreover, it is obvious that almost surely the realizations of V_ω are uniformly bounded and continuous.
(ii) Let $\xi_\omega(\mathbf{x})$, $\omega \in \Omega$, $\mathbf{x} \in \mathbb{R}^3$, be a real-valued homogeneous Gaussian field whose correlation function is continuous at the origin and decays at infinity (see [P.Fi, Example 1.15c, p.26] and [E.K.Sch.S, Example 3, p.615]). Assume that $F : \mathbb{R} \to \mathbb{R}$ is a bounded continuous function. Set $V_\omega(\mathbf{x}) := F(\xi_\omega(\mathbf{x}))$, $\omega \in \Omega$, $\mathbf{x} \in \mathbb{R}^3$. Then the random field V_ω is $\mathbb{R}^3$-ergodic, $\mathbb{R}$-ergodic in the direction of the magnetic field (and all other directions) whose realizations are almost surely uniformly bounded and continuous.

On $D(H_0(b))$ define the perturbed Schrödinger operator

$$H(b, \omega) := H_0(b) + V_\omega,\ b > 0,\ \omega \in \Omega.$$

It follows from (1.2) and (1.4) that almost surely we have

$$\sigma(H(b, \omega)) \subseteq [b - c_0, +\infty). \tag{1.5}$$

The aim of this paper is to study the asymptotic behaviour as $b \to \infty$ of the integrated density of states (IDOS) for the operator $H(b, \omega)$. In order to recall the definition of the IDOS, we need several auxiliary concepts.
Let φ_r, $r \in \mathbb{R}_+$, and φ be non-decreasing functions defined on a common domain $I \subseteq \mathbb{R}$. We shall write

$$v - \lim_{r \to \infty} \varphi_r = \varphi$$

if we have $\lim_{r \to \infty} \varphi_r(t) = \varphi(t)$ at all continuity points t of the function φ. In this case the function φ is called the vague limit as $r \to \infty$ of the family φ_r (cf. [K, p.313]).

Further, let $T = T^*$ be a selfadjoint operator in a Hilbert space. Denote by $P_{\mathcal{I}}(T)$ its spectral projection corresponding to the interval $\mathcal{I} \subset \mathbb{R}$. Set

$$N(\lambda; T) := \operatorname{rank} P_{(-\infty,\lambda)}(T), \ \lambda \in \mathbb{R},$$

$$n_{\pm}(s; T) := \operatorname{rank} P_{(s,+\infty)}(\pm T), \ s > 0. \tag{1.6}$$

On the Sobolev space $\mathrm{H}^2\left(\left(-\frac{R}{2}, \frac{R}{2}\right)^3\right)$, $R > 0$, with Dirichlet boundary conditions define the operator $H_{0,R}(b) := \left(i\nabla + \frac{\mathbf{b}\wedge\mathbf{x}}{2}\right)^2$. Put

$$\mathcal{D}_b(.) := v - \lim_{R\to\infty} R^{-3}\, N(\,.\,; H_{0,R}(b) + V_\omega). \tag{1.7}$$

Any non-decreasing function $\mathcal{D}_b(\mu)$, $\mu \in \mathbb{R}$, satisfying (1.7), is called IDOS for the operator $H(b,\omega)$. It is well-known that almost surely the vague limit (1.7) exists and the quantity $\mathcal{D}_b(\mu)$ is non-random (see e.g. [Bro.H.L], [Ma], [U], and the references cited there). Since $\mathcal{D}_b$ is non-decreasing, the set of its eventual discontinuity points is not more than countable.
Note that (1.5) implies that almost surely $\inf \sigma(H_{0,R}(b) + V_\omega) \geq b - c_0$ for all $R > 0$. Therefore,

$$\mathcal{D}_b(\mu) = 0, \quad \mu < b - c_0. \tag{1.8}$$

For $\mu \in \mathbb{R}$ set

$$D_b(\mu) := \frac{b}{2\pi^2}\sum_{q=1}^{\infty}(\mu - (2q-1)b)_+^{1/2}.$$

By [CdV, Theorem 3.1] the estimates

$$(R - R_0)^3 D_b(\mu - CR_0^{-2} - c_0) \leq N(\mu; H_{0,R}(b) + V_\omega) \leq$$

$$R^3 D_b(\mu + c_0), \ \mu \in \mathbb{R}, \ R > 0, \ R_0 \in (0, R),$$

hold with C independent of μ, R, and R_0. Then it follows easily from (1.7) that

$$D_b(\mu - c_0) \leq \mathcal{D}_b(\mu) \leq D_b(\mu + c_0), \ \mu \in \mathbb{R}. \tag{1.9}$$

In this paper we study the asymptotic behaviour as $b \to \infty$ of $\mathcal{D}_b(\lambda + b)$, the parameter $\lambda \in \mathbb{R}$ being fixed.

2 Statement of the main result

On $\mathrm{H}^2\left(\left(-\frac{R}{2}, \frac{R}{2}\right)\right)$ with Dirichlet boundary conditions define the operator $h_{0,R} := -\frac{d^2}{dz^2}$.

Proposition 2.1. *Let $\mathbb{G} = \mathbb{Z}$ or $\mathbb{G} = \mathbb{R}$. Let $f_\omega(z)$, $\omega \in \Omega$, $z \in \mathbb{R}$, be a real $\mathbb{G}$-ergodic random field whose realizations are almost surely uniformly bounded and continuous. Then for each $\lambda \in \mathbb{R}$ the limit*

$$\varrho(\lambda; f) := \lim_{R\to\infty} R^{-1}\, N(\lambda; h_{0,R} + f_\omega) \tag{2.1}$$

exists almost surely. Moreover, the function $\varrho(\lambda; f)$ is non-random, and continuous with respect to $\lambda \in \mathbb{R}$.

The proof of the existence and the non-randomness of $\varrho(\lambda; f)$ for much more general ergodic fields f_ω can be found in [K, Chapter 7]. The continuity of $\varrho(\lambda; f)$ which is guaranteed by the fact that $h_{0,R} + f_\omega$ is an *ordinary* differential operator, is discussed in [P.Fi, Chapter III].

Lemma 2.1. *Assume that the hypotheses of Proposition* 2.1 *hold. Let $\mathcal{T} : \Omega \to \Omega$ be a measure preserving automorphism. Then we have*

$$\lim_{R\to\infty} R^{-1}\, N(\lambda; h_{0,R} + f_{\mathcal{T}\omega}) = \lim_{R\to\infty} R^{-1}\, N(\lambda; h_{0,R} + f_\omega). \tag{2.2}$$

Proof. By [K, Theorem 6, Chapter 7] we have

$$\lim_{R\to\infty} R^{-1}\, N(\lambda; h_{0,R} + f_\omega) = \sup_{R>0} R^{-1}\mathbb{E}\left(N(\lambda; h_{0,R} + f_\omega)\right) \equiv$$

$$\sup_{R>0} R^{-1} \int_\Omega N(\lambda; h_{0,R} + f_\omega)\, d\mathbb{P}(\omega)$$

where $\mathbb{E}$ is used as the symbol of the mathematical expectation. Analogously,

$$\lim_{R\to\infty} R^{-1}\, N(\lambda; h_{0,R} + f_{\mathcal{T}\omega}) = \sup_{R>0} R^{-1} \int_\Omega N(\lambda; h_{0,R} + f_{\mathcal{T}\omega})\, d\mathbb{P}(\omega).$$

Since $\mathcal{T}$ is a measure preserving automorphism, we get

$$\sup_{R>0} R^{-1} \int_\Omega N(\lambda; h_{0,R} + f_{\mathcal{T}\omega})\, d\mathbb{P}(\omega) = \sup_{R>0} R^{-1} \int_\Omega N(\lambda; h_{0,R} + f_\omega)\, d\mathbb{P}(\omega)$$

which yields (2.2). □

Our assumptions concerning V_ω guarantee that the random field $f_\omega = V_\omega(X, .)$ depending on the parameter $X \in \mathbb{R}^2$, satisfies the hypotheses of Proposition 2.1. Moreover, if $\mathbb{G} = \mathbb{Z}$, then the function $\varrho(\lambda; V(X, .))$ is $\mathbb{Z}^2$-periodic with respect to $X \in \mathbb{R}^2$. In order to see this, one may apply Lemma 2.1 for $\mathcal{T} = \mathcal{T}_{\mathbf{k}_0}$ (see (1.3)) with $\mathbf{k}_0 = (K, 0)$, $K \in \mathbb{Z}^2$, and conclude that

$$\varrho(\lambda; V(X + K, .)) = \varrho(\lambda; V(X, .)),\ \lambda \in \mathbb{R},\ X \in \mathbb{R}^2,\ K \in \mathbb{Z}^2.$$

Note that the continuity of $V_\omega(\mathbf{x})$ with respect to $\mathbf{x} \in \mathbb{R}^3$, and the continuity of $\varrho(\lambda; f)$ with respect to $\lambda \in \mathbb{R}$, imply the continuity of $\varrho(\lambda; V(X,.))$ with respect to $X \in \mathbb{R}^2$. Taking into account also its periodicity, we find that $\varrho(\lambda; V(X,.))$, $X \in \mathbb{R}^2$, is uniquely determined by its values for $X \in \left(-\frac{1}{2}, \frac{1}{2}\right)^2$.
Similarly, if $\mathbb{G} = \mathbb{R}$, the quantity $\varrho(\lambda; V(X,.))$ is independent of $X \in \mathbb{R}^2$. In order to see this, one may apply Lemma 2.1 for $\mathcal{T} = \mathcal{T}_{\mathbf{x}_0}$ (see (1.3)) with $\mathbf{x}_0 = (X, 0)$, $X \in \mathbb{R}^2$, and conclude that

$$\varrho(\lambda; V(X,.)) = \varrho(\lambda; V(0,.)), \ \lambda \in \mathbb{R}, \ X \in \mathbb{R}^2.$$

Finally, using the elementary estimate

$$N(\lambda; h_{0,R} + V_\omega(X,.)) \leq \frac{R}{\pi} (\lambda + c_0)_+^{1/2}, \ X \in \left(-\frac{1}{2}, \frac{1}{2}\right)^2, \ R > 0,$$

we get

$$\varrho(\lambda; v(X,.)) \leq \frac{1}{\pi} (\lambda + c_0)_+^{1/2}, \ X \in \left(-\frac{1}{2}, \frac{1}{2}\right)^2.$$

For $\lambda \in \mathbb{R}$ set

$$k(\lambda) := \begin{cases} \int_{(-\frac{1}{2}, \frac{1}{2})^2} \varrho(\lambda, V(X,.))\, dX & \text{if} \quad \mathbb{G} = \mathbb{Z}, \\ \varrho(\lambda, V(0,.)) \quad \text{if} \quad \mathbb{G} = \mathbb{R}. \end{cases}$$

Obviously, $k(\lambda)$ is continuous with respect to λ.

Theorem 2.1. *Let $\mathbb{G} = \mathbb{Z}$ or $\mathbb{G} = \mathbb{R}$. Let V_ω be a real $\mathbb{G}^3$-ergodic random field whose realizations are almost surely uniformly bounded and continuous. Assume in addition that V_ω is $\mathbb{G}$-ergodic in the direction of the magnetic field. Then we have*

$$\lim_{b \to \infty} b^{-1} \mathcal{D}_b(\lambda + b) = \frac{1}{2\pi} k(\lambda), \ \lambda \in \mathbb{R}. \tag{2.3}$$

Remark: For definiteness, we shall prove Theorem 2.1 in the case $\mathbb{G} = \mathbb{Z}$. The proof in the case $\mathbb{G} = \mathbb{R}$ is quite similar and only simpler.

The asymptotics as $b \to \infty$ of the IDOS for the two-dimensional Schrödinger operator with constant magnetic field has been extensively investigated during the last two decades (see e.g. [Br.G.I], [Bro.H.L], [M.Pu], [Pu.Sc], [W]). As far as the authors are informed, no results concerning the strong magnetic field asymptotics of the IDOS for the three-dimensional Schrödinger operator considered in this paper, are known.
Besides the quantity $\mathcal{D}_b(\lambda + b)$ whose main asymptotic term is obtained in (2.3), we could consider more general quantities $\mathcal{D}_b(\lambda_2 + \epsilon b) - \mathcal{D}_b(\lambda_1 + \epsilon b)$ with $\lambda_j \in \mathbb{R}$, $j = 1, 2$, $\lambda_1 < \lambda_2$, and $\epsilon \in \mathbb{R}$. Recall that the numbers $\{(2q-1)b)\}_{q \geq 1}$ are called Landau levels. For this reason we shall refer to the asymptotics as $b \to \infty$ of

$\mathcal{D}_b(\lambda_2 + (2q-1)b) - \mathcal{D}_b(\lambda_1 + (2q-1)b)$, $q \in \mathbb{Z}$, $q \geq 1$, as the asymptotics of the IDOS near the kth Landau level. Analogously, if $\epsilon > 1$ is not an odd integer, we shall refer to the asymptotics as $b \to \infty$ of $\mathcal{D}_b(\lambda_2 + \epsilon b) - \mathcal{D}_b(\lambda_1 + \epsilon b)$ as the asymptotics of the IDOS far from the Landau levels. Note that the case $\epsilon < 1$ is trivial since (1.8) implies $\mathcal{D}_b(\lambda + \epsilon b) = 0$, $\lambda \in \mathbb{R}$, $\epsilon < 1$, provided that b is large enough.

Since (2.3) entails

$$\lim_{b\to\infty} b^{-1} \left(\mathcal{D}_b(\lambda_2 + b) - \mathcal{D}_b(\lambda_1 + b)\right) = \frac{1}{2\pi} \left(k(\lambda_2) - k(\lambda_1)\right),$$

we can say that Theorem 2.1 concerns the asymptotics of the IDOS near the first Landau level. The problems of obtaining the first asymptotic term of the IDOS near the higher Landau levels and far from the Landau levels remain open as far as the methods used in this paper are not directly applicable to them. We hope to solve these problems in a future work. Here we would like to note that

$$\lim_{b\to\infty} b^{-1} \left(D_b(\lambda_2 + \epsilon b) - D_b(\lambda_1 + \epsilon b)\right) = \frac{1}{2\pi^2} \left((\lambda_2)_+^{1/2} - (\lambda_1)_+^{1/2}\right), \tag{2.4}$$

if $\epsilon > 1$ is an odd integer, and

$$\lim_{b\to\infty} b^{-1/2} \left(D_b(\lambda_2 + \epsilon b) - D_b(\lambda_1 + \epsilon b)\right) = \frac{1}{4\pi^2} (\lambda_2 - \lambda_1) \sum_{1\leq q<(\epsilon+1)/2} (\epsilon - (2q-1))^{-1/2}, \tag{2.5}$$

if $\epsilon > 1$ is not an odd integer. Combining (2.4) (respectively, (2.5)) with (1.9) we obtain generically the correct asymptotic order of the IDOS near the higher Landau levels (respectively, far from the Landau levels).

Finally, note that (1.9) implies immediately

$$\lim_{b\to\infty} b^{-3/2} \mathcal{D}_b(\lambda + \epsilon b) = D_1(\epsilon), \ \lambda \in \mathbb{R}, \ \epsilon > 1,$$

which however yields only the rough estimate

$$\mathcal{D}_b(\lambda_2 + \epsilon b) - \mathcal{D}_b(\lambda_1 + \epsilon b) = o(b^{3/2}), \quad b \to \infty.$$

The methods we apply are relatively simple. First of all, we give an equivalent representation of $\mathcal{D}_b(\lambda + b)$ which is more convenient for our purposes. Namely, on $D(H_0(b))$ we define the operator

$$\tilde{H}(b, \omega, \lambda, R) := H_0(b) - b + (V_\omega - \lambda - 1)\mathbf{1}_{\left(-\frac{R}{2}, \frac{R}{2}\right)^3}, \ b > 0, \ \lambda \in \mathbb{R}, \ \omega \in \Omega, \tag{2.6}$$

(see (1.1) for the definition of $H_0(b)$), which has purely discrete negative spectrum, and show that almost surely we have

$$\mathcal{D}_b(\lambda + b) = \lim_{R\to\infty} R^{-3} N(-1; \tilde{H}(b, \omega, \lambda, R)), \tag{2.7}$$

provided that $\lambda + b$ is a continuity point of $\mathcal{D}_b$ (see Proposition 4.1 below).

Moreover, we apply the Birman-Schwinger principle (see [B]), and similarly to [R 1 − 3] we employ the Kac-Murdock-Szegö theorem in order to reduce the study of the asymptotics as $b \to \infty$ of $\mathcal{D}_b(\lambda + b)$ to the asymptotic analysis as $R \to \infty$ and $b \to \infty$ of the traces of the powers of certain trace-class operators $t_{b,R}$ (see (3.4) below). The Birkhoff-Khintchine ergodic theorem plays a crucial rôle in this analysis.
The paper is organized as follows. In Section 3 we investigate the asymptotics of $R \to \infty$ and $b \to \infty$ of Tr $t_{b,R}^l$, $l \geq 1$, and some related traces. Section 4 contains auxiliary results. In particular, we prove (2.7) as well as an analogous formula concerning $\varrho(\lambda, V(X,.))$ (see (2.1)). Finally, the proof of Theorem 2.1 can be found in Section 5.

3 Trace asymptotics

Let $\mathcal{H}_0(b) := \left(i\frac{\partial}{\partial x} - \frac{by}{2}\right)^2 + \left(i\frac{\partial}{\partial y} + \frac{bx}{2}\right)^2$ be the selfadjoint operator defined originally on $C_0^\infty(\mathbb{R}^2)$ and then closed in $L^2(\mathbb{R}^2)$. The spectrum of $\mathcal{H}_0(b)$ coincides with the set of the Landau levels, i.e. $\sigma(\mathcal{H}_0(b)) = \bigcup_{q=1}^{\infty}\{(2q-1)b\}$, and the multiplicity of each eigenvalue $(2q-1)b$, $q \geq 1$, is infinite. Denote by $p_b : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ the orthogonal projection onto the eigenspace of $\mathcal{H}_0(b)$ associated with the first Landau level b. In other words, $p_b w = w$ implies $w \in D(\mathcal{H}_0(b))$ and $\mathcal{H}_0(b)w = bw$. It is well-known that

$$(p_b w)(x,y) = \int_{\mathbb{R}^2} \mathcal{P}_b(x,y;x',y')w(x',y')\,dx'dy', \quad w \in L^2(\mathbb{R}^2),$$

with

$$\mathcal{P}_b(x,y;x',y') := \frac{b}{2\pi}\exp\left\{-\frac{b}{4}\left[(x-x')^2 + (y-y')^2 + 2i(xy' - yx')\right]\right\}. \tag{3.1}$$

Set

$$P_b := \int_{\mathbb{R}}^{\oplus} p_b\,dz.$$

Evidently, $P_b : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ is an orthogonal projection,

$$(P_b u)(x,y,z) = \int_{\mathbb{R}^2} \mathcal{P}_b(x,y;x',y')u(x',y',z)\,dx'dy', \quad u \in L^2(\mathbb{R}^3),$$

and P_b commutes with $H_0(b)$ and $\frac{\partial}{\partial z}$. Moreover, we have

$$H_0(b)P_b u = \left(-\frac{\partial^2}{\partial z^2} + b\right)P_b u, \quad u \in D(H_0(b)), \tag{3.2}$$

(see (1.1)). Define the operator $r := \left(-\frac{\partial^2}{\partial z^2}+1\right)^{-1/2}$, bounded and selfadjoint in $L^2(\mathbb{R}^3)$. Evidently,

$$(r^2u)(x,y,z) = \frac{1}{2}\int_{\mathbb{R}} e^{-|z-z'|}u(x,y,z')\,dz', \quad u \in L^2(\mathbb{R}^3).$$

Moreover, the operators P_b and r commute.

Fix $\lambda \in \mathbb{R}$ and for brevity set

$$Q_\omega(\mathbf{x}) \equiv Q_\omega(\mathbf{x};\lambda) := V_\omega(\mathbf{x}) - \lambda - 1, \quad \mathbf{x} \in \mathbb{R}^3, \quad \omega \in \Omega.$$

Almost surely we have

$$|Q_\omega(\mathbf{x})| \le c_1,\ \mathbf{x} \in \mathbb{R}^3, \tag{3.3}$$

with $c_1 := c_0 + |\lambda+1|$ (see (1.4)). Put

$$\tilde{Q}_{\omega,R}(\mathbf{x}) := Q_\omega(\mathbf{x})\mathbf{1}_{\left(-\frac{R}{2},\frac{R}{2}\right)^3}(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^3.$$

Define the operator

$$t_{b,R}(Q_\omega) := P_b r\tilde{Q}_{\omega,R} r P_b, \tag{3.4}$$

compact and selfadjoint in $L^2(\mathbb{R}^3)$. It is easy to check that we have

$$\|P_b r|\tilde{Q}_{\omega,R}|^{1/2}\|_2^2 = \frac{b}{4\pi}\int_{\left(-\frac{R}{2},\frac{R}{2}\right)^3}|Q_\omega(\mathbf{x})|\,d\mathbf{x} \le \frac{bR^3}{4\pi}c_1\ , \tag{3.5}$$

where $\|.\|_2$ denotes the Hilbert-Schmidt norm. Therefore, $t_{b,R}(Q_\omega)$ is a trace-class operator. Set

$$M_1(b) := \frac{b}{4\pi}\mathbb{E}\left(\int_{\left(-\frac{1}{2},\frac{1}{2}\right)^3} Q_\omega(X,z)\,dX dz\right). \tag{3.6}$$

Let $l \ge 2$. Put

$$M_l(b) :=$$

$$\frac{b}{2\pi}\mathbb{E}\left(\int_{\left(-\frac{1}{2},\frac{1}{2}\right)^3} Q_\omega(X_1,z_1)\int_{\mathbb{R}^{3(l-1)}}\Pi_{s=2}^{l}Q_\omega(X_1+b^{-1/2}X_s, z_1+z_s)\right.$$
$$\left.\psi_l(z_2,\dots,z_l)\Psi_l(X_2,\dots,X_l)dX_2\dots dX_l\,dz_2\dots dz_l\ dX_1dz_1\right), \tag{3.7}$$

where

$$\psi_l(z_2,\dots,z_l) := \frac{1}{2^l}\exp\left\{-|z_2|-|z_l|-\sum_{s=2}^{l-1}|z_{s+1}-z_s|\right\},$$

$$\Psi_l(X_2,\dots,X_l) \equiv \Psi_l(x_2,y_2,\dots,x_l,y_l) := \frac{1}{(2\pi)^{l-1}}\exp\{-\Phi_l(x_2,y_2,\dots,x_l,y_l)\},$$

and

$$\Phi_l(x_2, y_2, \ldots, x_l, y_l) := \frac{1}{4}\left\{{x_2}^2 + {y_2}^2 + {x_l}^2 + {y_l}^2 + \sum_{s=2}^{l-1}\left((x_{s+1} - x_s)^2 + (y_{s+1} - y_s)^2 + 2i(x_{s+1}y_s - y_{s+1}x_s)\right)\right\} ;$$

if $l = 2$, then the sums with respect to s in the formulae defining ψ_l and Φ_l, should be omitted.
Note that $\psi_l \in L^1(\mathbb{R}^{l-1})$, $\Psi_l \in L^1(\mathbb{R}^{2(l-1)})$. Hence, (3.3) implies that the integral defining $M_l(b)$ is absolutely convergent.

Proposition 3.1. *Almost surely we have*

$$\lim_{R\to\infty} R^{-3}\mathrm{Tr}\ t_{b,R}(Q_\omega)^l = M_l(b), \quad l \geq 1, \tag{3.8}$$

the operator $t_{b,R}(Q_\omega)$ *being defined in* (3.4).

Proof. We shall prove (3.8) in the generic case $l \geq 2$.
It is easy to verify that $\mathrm{Tr}\ t_{b,R}(Q_\omega)^l = \mathrm{Tr}\left(P_b r^2 \tilde{Q}_{\omega,R}\right)^l$, and that $\mathrm{Tr}\left(P_b r^2 \tilde{Q}_{\omega,R}\right)^l$ can be written in a standard way as an integral over $\mathbb{R}^{3l}$ of the diagonal value of the integral kernel of the operator $\left(P_b r^2 \tilde{Q}_{\omega,R}\right)^l$. Hence, we have

$$\mathrm{Tr}\ t_{b,R}(Q_\omega)^l = \int_{\mathbb{R}^{3l}} {\Pi_{s=1}^l}' \left(\frac{1}{2}\mathcal{P}_b(X_{s+1}, X_s)e^{-|z_{s+1}-z_s|}\right) \Pi_{s=1}^l \tilde{Q}_{\omega,R}(X_s, z_s) dX_1 \ldots dX_l\, dz_1 \ldots dz_l$$

where the notation ${\Pi_{s=1}^l}'$ means that in the product of l factors the variables X_{l+1} and z_{l+1}, should be set equal respectively to X_1 and z_1. Changing the variables

$$X_1 = X_1',\ X_s = X_1' + b^{-1/2}X_s',\ s = 2, \ldots, l,$$

$$z_1 = z_1',\ z_s = z_1' + z_s',\ s = 2, \ldots, l,$$

we get

$$\mathrm{Tr}\ t_{b,R}(Q_\omega)^l = \frac{b}{2\pi}\int_{\left(-\frac{R}{2},\frac{R}{2}\right)^3} Q_\omega(X_1', z_1') \int_{\mathbb{R}^{3(l-1)}} \Pi_{s=2}^l \tilde{Q}_{\omega,R}(X_1' + b^{-1/2}X_s', z_1' + z_s')$$
$$\psi_l(z_2', \ldots, z_l')\Psi_l(X_2', \ldots, X_l')dX_2' \ldots dX_l'\, dz_2' \ldots dz_l'\, dX_1' dz_1'. \tag{3.9}$$

Our next step is to show that if we replace in (3.9) all the functions $\tilde{Q}_{\omega,R}$ by Q_ω, the error will be of order $o(R^3)$ as $R \to \infty$. More precisely, we set

$$\tilde{M}_{l,R}(b) :=$$

$$\frac{b}{2\pi}\int_{\left(-\frac{R}{2},\frac{R}{2}\right)^3} Q_\omega(X_1,z_1)\int_{\mathbb{R}^{3(l-1)}} \Pi_{s=2}^l Q_\omega(X_1+b^{-1/2}X_s, z_1+z_s)$$
$$\psi_l(z_2,\dots,z_l)\Psi_l(X_2,\dots,X_l)dX_2\dots dX_l\, dz_2\dots dz_l\; dX_1 dz_1, \quad l\geq 2,$$

write

$$\mathrm{Tr}\,(t_{b,R}(Q_\omega))^l = \tilde{M}_{l,R}(b)+\mathcal{E}_l(R,b,\omega),\ l\geq 2, \tag{3.10}$$

and shall demonstrate that almost surely we have

$$\lim_{R\to\infty} R^{-3}\mathcal{E}_l(R,b,\omega)=0. \tag{3.11}$$

Evidently, $\mathcal{E}_l(R,b,\omega)$ admits the estimate

$$|\mathcal{E}_l(R,b,\omega)|\leq$$
$$\frac{b}{2\pi}c_1^l R^3\int_{\left(-\frac{1}{2},\frac{1}{2}\right)^3}\int_{\mathbb{R}^{3(l-1)}}\Pi_{s=2}^l E_{R,b}(X_1,\dots,X_l,z_1,\dots,z_l)$$
$$\psi_l(z_2,\dots,z_l)|\Psi_l(X_2,\dots,X_l)|dX_2\dots dX_l\, dz_2\dots dz_l\; dX_1dz_1$$

where

$$E_{R,b}(X_1,\dots,X_l,z_1,\dots,z_l):=$$
$$\left(1-\Pi_{s=2}^l\mathbf{1}_{\left(-\frac{R}{2},\frac{R}{2}\right)^3}(RX_1+b^{-1/2}X_s,Rz_1+z_s)\right).$$

Since $\psi_l\Psi_l\in L^1(\mathbb{R}^{3(l-1)})$, $\|E_{R,b}\|_{L^\infty\left(\left(-\frac{1}{2},\frac{1}{2}\right)^3\times\mathbb{R}^{3(l-1)}\right)}=1$ for every $b>0$ and $R>0$, and

$$\lim_{R\to\infty}E_{R,b}(X_1,\dots,X_l,z_1,\dots,z_l)=0$$

for almost every $(X_1,\dots,X_l,z_1,\dots,z_l)\in\left(-\frac{1}{2},\frac{1}{2}\right)^3\times\mathbb{R}^{3(l-1)}$, the dominated convergence theorem yields (3.11).
Set $L=L(R)=\mathrm{ent}\left(\frac{R}{2}\right)$ where $\mathrm{ent}(x)$ denotes the integer part of $x\in\mathbb{R}$. Obviously,

$$R^{-3}\tilde{M}_{l,R}(b)=(2L+1)^{-3}\tilde{M}_{l,2L+1}(b)+o(1),\ R\to\infty. \tag{3.12}$$

Let $\mathbf{j}=(J,j)\in\mathbb{Z}^3$, $J\in\mathbb{Z}^2$, $j\in\mathbb{Z}$. Introduce the random variables

$$\gamma_{l,\mathbf{j}}(\omega,b):=\frac{b}{2\pi}\int_{\left(-\frac{1}{2},\frac{1}{2}\right)^3}Q_\omega(X_1+J,z_1+j)\int_{\mathbb{R}^{3(l-1)}}\Pi_{s=2}^l Q_\omega(X_1+J+b^{-1/2}X_s,z_1+j+z_s)$$
$$\psi_l(z_2,\dots,z_l)\Psi_l(X_2,\dots,X_l)dX_2\dots dX_l\, dz_2\dots dz_l\; dX_1dz_1,\ l\geq 2.$$

Set

$$\Gamma_L=\left\{\mathbf{j}=(j_1,j_2,j_3)\in\mathbb{Z}^3 \,\middle|\, |j_s|\leq L,\ s=1,2,3\right\},\ L\in\mathbb{N}, L\geq 1.$$

It is easy to verify that

$$\tilde{M}_{l,2L+1}=\sum_{\mathbf{j}\in\Gamma_L}\gamma_{l,\mathbf{j}}(\omega,b). \tag{3.13}$$

On the other hand, it is obvious that for each $\mathbf{k} \in \mathbb{Z}^3$ we have

$$\gamma_{l,\mathbf{j}+\mathbf{k}}(\omega, b) = \gamma_{l,\mathbf{j}}(\mathcal{T}_{\mathbf{k}}\omega, b),\ \mathbf{j} \in \mathbb{Z}^3,\ \omega \in \Omega,$$

(see (1.3)). Hence, the sequence $\{\gamma_{l,\mathbf{j}}\}_{\mathbf{j}\in\mathbb{Z}^3}$ is a $\mathbb{Z}^3$-ergodic random field. Therefore, we can apply the Birkhoff-Khintchine ergodic theorem (see e.g. [K, Theorem 2, Section 3.2] or [P.Fi, Proposition 1.7, p.18]). As a result we get almost surely

$$\lim_{L\to\infty} (2L+1)^{-3} \sum_{\mathbf{j}\in\Gamma_L} \gamma_{l,\mathbf{j}}(\omega, b) = \mathbb{E}(\gamma_{l,\mathbf{0}}) \equiv M_l(b), \quad l \geq 2. \tag{3.14}$$

Now the combination of (3.10)-(3.14) yields (3.8) with $l \geq 2$.
The proof in the case $l = 1$ is similar but much simpler. □

Fix $X \in \left(-\frac{1}{2}, \frac{1}{2}\right)^2$. Introduce the operator

$$\tau_R(Q_\omega(X)) := \left(-\frac{d^2}{dz^2} + 1\right)^{-1/2} Q_\omega(X, z)\mathbf{1}_{\left(-\frac{R}{2}, \frac{R}{2}\right)}(z) \left(-\frac{d^2}{dz^2} + 1\right)^{-1/2} \tag{3.15}$$

which is compact and selfadjoint in $L^2(\mathbb{R}_z)$, and depends on the parameters $X \in \left(-\frac{1}{2}, \frac{1}{2}\right)^2$ and $\omega \in \Omega$ (see (3.4) in order to compare $\tau_R(Q_\omega(X))$ with the operator $t_{b,R}(Q_\omega)$). It is easy to check that $\tau_R(Q_\omega(X))$ is a trace-class operator.
For $X \in \left(-\frac{1}{2}, \frac{1}{2}\right)^2$ set

$$\mu_1(X) := \frac{1}{2}\mathbb{E}\left(\int_{\left(-\frac{1}{2}, \frac{1}{2}\right)} Q_\omega(X, z)\, dz\right),$$

$$\mu_l(X) :=$$

$$\mathbb{E}\left(\int_{\left(-\frac{1}{2}, \frac{1}{2}\right)} Q_\omega(X, z_1) \int_{\mathbb{R}^{l-1}} \Pi_{s=2}^l Q_\omega(X, z_1 + z_s)\psi_l(z_2, \ldots, z_l) dz_2 \ldots z_l\, dz_1\right),\ l \geq 2,$$

(see (3.6)–(3.7) in order to compare $\mu_l(X)$ with the quantities $M_l(b)$, $l \geq 1$). By analogy with Proposition 3.1 we can demonstrate the following

Proposition 3.2. *Almost surely we have*

$$\lim_{R\to\infty} R^{-1}\mathrm{Tr}\, \tau_R(Q_\omega(X))^l = \mu_l(X),\ l \geq 1,\ X \in \left(-\frac{1}{2}, \frac{1}{2}\right)^2.$$

Set $m_l := \int_{\left(-\frac{1}{2}, \frac{1}{2}\right)^2} \mu_l(X) dX,\ l \geq 1.$

Proposition 3.3. *We have*

$$\lim_{b\to\infty} b^{-1} M_l(b) = \frac{1}{2\pi} m_l,\ l \geq 1. \tag{3.16}$$

Proof. Obviously $M_1(b) = \frac{b}{2\pi} m_1$ which yields immediately (3.16) with $l = 1$.
Let $l \geq 2$. Using the fact that p_1 is an orthogonal projection, and taking into account the explicit form of its kernel $\mathcal{P}_1$ (see (3.1)), we get

$$\int_{\mathbb{R}^{2(l-1)}} \Psi_l(X_2, \dots, X_l) dX_2 \dots dX_l =$$

$$2\pi \int_{\mathbb{R}^{2(l-1)}} \mathcal{P}_1(0; X_2)\mathcal{P}_1(X_2; X_3) \dots \mathcal{P}_1(X_{l-1}; X_l)\mathcal{P}_1(X_l; 0) dX_2 \dots dX_l =$$

$$2\pi \mathcal{P}_1(0; 0) = 1.$$

(cf. [R 2, pp.16-17]). Hence, we have

$$m_l = \mathbb{E}\left(\int_{\left(-\frac{1}{2},\frac{1}{2}\right)^3} Q_\omega(X_1, z_1) \int_{\mathbb{R}^{3(l-1)}} \psi_l(z_2, \dots, z_l)\Psi_l(X_2, \dots, X_l) \right.$$

$$\left. \Pi_{s=2}^{l} Q_\omega(X_1, z_1 + z_s) dX_2 \dots dX_l \, dz_2 \dots dz_l \; dX_1 dz_1 \right).$$

Then, obviously,

$$b^{-1}M_l(b) - \frac{1}{2\pi} m_l = \frac{1}{2\pi}\mathbb{E}\left(\int_{\left(-\frac{1}{2},\frac{1}{2}\right)^3} Q_\omega(X_1, z_1) \int_{\mathbb{R}^{3(l-1)}} \psi_l(z_2, \dots, z_l)\Psi_l(X_2, \dots, X_l) \right.$$

$$\left(\Pi_{s=2}^{l} Q_\omega(X_1 + b^{-1/2}X_s, z_1 + z_s) - \Pi_{s=2}^{l} Q_\omega(X_1, z_1 + z_s) \right)$$

$$\left. dX_2 \dots dX_l \, dz_2 \dots dz_l \; dX_1 dz_1 \right). \tag{3.17}$$

Since Q_ω is almost surely uniformly bounded and continuous, while $\psi_l \in L^1(\mathbb{R}^{l-1})$ and $\Psi_l \in L^1(\mathbb{R}^{2(l-1)})$, we find that it follows from the dominated convergence theorem that (3.17) implies (3.16). □

Combining Propositions 3.1, 3.2, and 3.3, we get the following

Corollary 3.1. *For each* $l \geq 1$ *almost surely the limits*

$$\lim_{b\to\infty} \lim_{R\to\infty} b^{-1}R^{-3}\mathrm{Tr}\; t_{b,R}(Q_\omega)^l$$

and

$$\frac{1}{2\pi} \lim_{R\to\infty} R^{-1} \int_{\left(-\frac{1}{2},\frac{1}{2}\right)^2} \mathrm{Tr}\; \tau_R(Q_\omega(X))^l dX$$

exist, coincide, and are non-random.

4 Auxiliary results

Proposition 4.1. *Let $\tilde{H}(b,\omega,\lambda,R)$ be the operator defined in* (2.6)*. Let $\lambda+b$ with $\lambda\in\mathbb{R}$ and $b>0$, be a continuity point of $\mathcal{D}_b$. Then* (2.7) *is valid.*

Proof. First, note that we have

$$N(\lambda+b;H_{0,R}(b)+V_\omega)=N(-1;H_{0,R}(b)-b+V_\omega-\lambda-1).$$

The minimax principle implies

$$N(-1;H_{0,R}(b)-b+V_\omega-\lambda-1)\le N(-1;\tilde{H}(b,\omega,\lambda,R)).$$

Therefore,

$$\liminf_{R\to\infty}R^{-3}N(-1;\tilde{H}(b,\omega,\lambda,R))\ge\liminf_{R\to\infty}R^{-3}N(\lambda+b;H_{0,R}(b)+V_\omega)=$$

$$\lim_{R\to\infty}R^{-3}N(\lambda+b;H_{0,R}(b)+V_\omega)=\mathcal{D}_b(\lambda+b). \tag{4.1}$$

Further, fix $R>0$, $R_0\in(0,R)$, put

$$\mathcal{O}_1=\mathcal{O}_{1,R}=\left(-\frac{R}{2},\frac{R}{2}\right)^3,\quad \mathcal{O}_2=\mathcal{O}_{2,R,R_0}=\mathbb{R}^3\setminus\left[-\frac{R-R_0}{2},\frac{R-R_0}{2}\right]^3,$$

and pick two functions φ_1 and φ_2 satisfying the following properties :

i) $\varphi_j\in C^\infty(\mathbb{R}^3)$, $j=1,2$;
ii) supp $\varphi_j\subseteq\mathcal{O}_j$, $j=1,2$;
iii) $\varphi_1^2(\mathbf{x})+\varphi_2^2(\mathbf{x})=1$ for every $\mathbf{x}\in\mathbb{R}^3$;
iv) $|\nabla\varphi_j(\mathbf{x})|\le c_2R_0^{-1}$ for every $\mathbf{x}\in\mathbb{R}^3$ with $c_2>0$ which is independent of $\mathbf{x}$, R, and R_0.

Introduce the selfadjoint operator

$$\tilde{H}_D(b,\omega,\lambda,R,R_0):=\left(i\nabla+\frac{\mathbf{b}\wedge\mathbf{x}}{2}\right)^2-b+(V_\omega-\lambda-1)\mathbf{1}_{(-\frac{R}{2},\frac{R}{2})^3}$$

whose quadratic form is defined originally on $C_0^\infty(\mathcal{O}_2)$, and then is closed in $L^2(\mathcal{O}_2)$ (cf. (1.1) and (2.6)). Then the "magnetic" version of the so-called ISM localization formula (see [C.F.K.S, Section 3.1]) yields

$$N(-1;\tilde{H}(b,\omega,\lambda,R))\le N(-1;H_{0,R}(b)-b+V_\omega-\lambda-1-\sum_{j=1,2}|\nabla\varphi_j|^2)+$$

$$N(-1;\tilde{H}_D(b,\omega,\lambda,R,R_0)-\sum_{j=1,2}|\nabla\varphi_j|^2). \tag{4.2}$$

Obviously,

$$N(-1;H_{0,R}(b)-b+V_\omega-\lambda-1-\sum_{j=1,2}|\nabla\varphi_j|^2)\leq N(\lambda+b+2c_2^2R_0^{-2};H_{0,R}(b)+V_\omega). \tag{4.3}$$

Choose a sequence $\{\varepsilon_r\}_{r\geq 1}$ such that $\varepsilon_r>0$, $r\geq 1$, $\lim_{r\to\infty}\varepsilon_r=0$, and $\lambda+b+\varepsilon_r$, $r\geq 1$, are continuity points of $\mathcal{D}_b$. Fix $r\geq 1$ and set $R_0=\sqrt{2}c_2/\sqrt{\varepsilon_r}$. Then we have

$$\limsup_{R\to\infty}R^{-3}N(\lambda+b+2c_2^2R_0^{-2};H_{0,R}(b)+V_\omega)=$$

$$\lim_{R\to\infty}R^{-3}N(\lambda+b+\varepsilon_r;H_{0,R}(b)+V_\omega)=\mathcal{D}_b(\lambda+b+\varepsilon_r),\ r\geq 1. \tag{4.4}$$

The combination of (4.3) and (4.4) yields

$$\limsup_{R\to\infty}R^{-3}N(-1;H_{0,R}(b)-b+V_\omega-\lambda-1-\sum_{j=1,2}|\nabla\varphi_j|^2)\leq\mathcal{D}_b(\lambda+b+\varepsilon_r),\ r\geq 1. \tag{4.5}$$

On the other hand, by the minimax principle we have

$$N(-1;\tilde{H}_D(b,\omega,\lambda,R,R_0)-\sum_{j=1,2}|\nabla\varphi_j|^2)\leq N(-1;H_0(b)-b+W) \tag{4.6}$$

where

$$W=W_{\omega,\lambda,R,R_0}=\left((V_\omega-\lambda-1)\mathbf{1}_{\left(-\frac{R}{2},\frac{R}{2}\right)^3}-\sum_{j=1,2}|\nabla\varphi_j|^2\right)\mathbf{1}_{\mathcal{O}_2}.$$

Arguing as in [R 1, Section 5], we deduce the estimate

$$N(-1;H_0(b)-b+W)\leq c_3\int_{\mathbb{R}^3}|W|^{3/2}d\mathbf{x}\leq c_4\left(R^3-(R-R_0)^3\right) \tag{4.7}$$

where the quantities c_3 and c_4 may depend on b, λ, and R_0, but are independent of R. The combination of (4.6) and (4.7) yields

$$\lim_{R\to\infty}R^{-3}N(-1;\tilde{H}_D(b,\omega,\lambda,R,R_0)-\sum_{j=1,2}|\nabla\varphi_j|^2)=0. \tag{4.8}$$

Putting together (4.2), (4.5), and (4.8), we obtain

$$\limsup_{R\to\infty}R^{-3}N(-1;\tilde{H}(b,\omega,\lambda,R))\leq\mathcal{D}_b(\lambda+b+\varepsilon_r),\ r\geq 1.$$

Letting $r \to \infty$ (hence, $\varepsilon_r \to 0$), and bearing in mind that $\lambda + b$ is a continuity point of $\mathcal{D}_b$, we get

$$\limsup_{R\to\infty} R^{-3} N(-1; \tilde{H}(b,\omega,\lambda,R)) \le \mathcal{D}_b(\lambda + b). \tag{4.9}$$

Now, the combination of (4.1) and (4.9) yields (2.7). □

For $b > 0$, $\omega \in \Omega$, $\lambda \in \mathbb{R}$, and $R > 0$, introduce the operator

$$T_{b,\omega,\lambda,R} := (H_0(b) - b + 1)^{-1/2}(V_\omega - \lambda - 1)\mathbf{1}_{(-\frac{R}{2},\frac{R}{2})^3}(H_0(b) - b + 1)^{-1/2}, \tag{4.10}$$

compact and selfadjoint in $L^2(\mathbb{R}^3)$.

Corollary 4.1. *Let $\lambda + b$ be a continuity point of $\mathcal{D}_b$. Then almost surely we have*

$$\mathcal{D}_b(\lambda + b) = \lim_{R\to\infty} R^{-3} n_-(1; T_{b,\omega,\lambda,R}),$$

the quantity $n_-(s;T)$ being defined in (1.6).

Proof. It suffices to note that the Birman-Schwinger principle (see [B, Lemma 1.1]) implies $N(-1; \tilde{H}(b,\omega,\lambda,R)) = n_-(1; T_{b,\omega,\lambda,R})$, and then to apply Proposition 4.1. □

Fix $X \in \left(-\frac{1}{2}, \frac{1}{2}\right)^3$ and $\lambda \in \mathbb{R}$. For $s \in \mathbb{R}$, $s \neq 0$, set

$$\tilde{\varrho}_\lambda(s; X) := -\mathrm{sign}(s)\ \varrho\left(-\frac{\lambda+1}{s} - 1; -\frac{1}{s}V(X,.)\right)$$

(see (2.1) for the definition of $\varrho(\lambda; f)$). Since $\varrho(\lambda, V(X,.))$ is a continuous function with respect to $\lambda \in \mathbb{R}$ (see Proposition 2.1), and V_ω is uniformly bounded (see (1.4)), we find that $\tilde{\varrho}_\lambda(s; X)$ is a continuous function with respect to $s \in \mathbb{R} \setminus \{0\}$ for any fixed $\lambda \in \mathbb{R}$. Moreover,

$$\tilde{\varrho}_\lambda(-1, X) = \varrho(\lambda, V(X,.)),\ X \in \left(-\frac{1}{2}, \frac{1}{2}\right)^2.$$

For $R > 0$, $\omega \in \Omega$, $\lambda \in \mathbb{R}$, $X \in \left(-\frac{1}{2}, \frac{1}{2}\right)^2$, and $s \in \mathbb{R} \setminus \{0\}$, set

$$\nu_{R,\omega,X,\lambda}(s) = \begin{cases} n_-(-s; \tau_R(V_\omega(X,.) - \lambda - 1)) & \text{if} \quad s < 0, \\ -n_+(s; \tau_R(V_\omega(X,.) - \lambda - 1)) & \text{if} \quad s > 0, \end{cases}$$

the operator τ_R being defined in (3.15).

Proposition 4.2. *For every $\lambda \in \mathbb{R}$, $X \in \left(-\frac{1}{2}, \frac{1}{2}\right)^2$, and $s \in \mathbb{R} \setminus \{0\}$, we have*

$$\tilde{\varrho}_\lambda(s; X) = \lim_{R\to\infty} R^{-1} \nu_{R,\omega,X,\lambda}(s) \tag{4.11}$$

almost surely.

Proof. We shall prove (4.11) in the case $s<0$. In this case, by Proposition 2.1

$$\tilde{\varrho}_\lambda(s,X) = \lim_{R\to\infty} R^{-1} N(-\frac{\lambda+1}{s} - 1; h_{0,R} - \frac{1}{s} V_\omega(X,.)) =$$

$$\lim_{R\to\infty} R^{-1} N(-1; h_{0,R} - \frac{1}{s}(V_\omega(X,.) - \lambda - 1)). \tag{4.12}$$

On $\mathrm{H}^2(\mathbb{R})$ define the operator

$$\tilde{h}_{s,\omega,X,\lambda,R} := -\frac{d^2}{dz^2} - \frac{1}{s}(V_\omega(X,.) - \lambda - 1)\mathbf{1}_{\left(-\frac{R}{2},\frac{R}{2}\right)},\ R>0.$$

Applying the minimax principle, and bearing in mind that $h_{0,R} - \frac{1}{s}(V_\omega(X,.) - \lambda - 1)$ and $\tilde{h}_{s,\omega,X,\lambda,R}$ are second-order ordinary differential operators, we get

$$0 \leq N(-1; \tilde{h}_{s,\omega,X,\lambda,R}) - N(-1; h_{0,R} - \frac{1}{s}(V_\omega(X,.) - \lambda - 1)) \leq 2.$$

Therefore, (4.12) implies

$$\tilde{\varrho}_\lambda(s,X) = \lim_{R\to\infty} R^{-1} N(-1; \tilde{h}_{s,\omega,X,\lambda,R}). \tag{4.13}$$

On the other hand, by the Birman-Schwinger principle we have

$$N(-1; \tilde{h}_{s,\omega,X,\lambda,R}) = n_-(-s; \tau_R(V_\omega(X,.) - \lambda - 1)) \equiv \nu_{R,\omega,X,\lambda}(s). \tag{4.14}$$

Putting together (4.13) and (4.14), we obtain (4.11) with $s<0$. The proof in the case $s>0$ is completely analogous. □

For $\lambda \in \mathbb{R}$ and $s \in \mathbb{R}\setminus\{0\}$ set

$$\tilde{k}_\lambda(s) := \int_{\left(-\frac{1}{2},\frac{1}{2}\right)^2} \tilde{\varrho}_\lambda(s;X)dX. \tag{4.15}$$

Obviously, $\tilde{k}_\lambda(s)$ is continuous with respect to s. Moreover,

$$\tilde{k}_\lambda(-1) = k(\lambda),\ \lambda\in\mathbb{R}. \tag{4.16}$$

Remark. The function $\nu_{R,\omega,X,\lambda}(s)$ of the variable $s\in\mathbb{R}\setminus\{0\}$ is non-negative on $(-\infty,0)$, non-positive on $(0,\infty)$, and non-decreasing on $(-\infty,0)$ and $(0,\infty)$. By (4.11) and (4.15), the functions $\tilde{\varrho}_\lambda(s;X)$ and $\tilde{k}_\lambda(s)$ have the same properties.

Corollary 4.2. *For each $\lambda\in\mathbb{R}$ almost surely we have*

$$\lim_{b\to\infty}\lim_{R\to\infty} b^{-1}R^{-3}\mathrm{Tr}\, t_{b,R}(V_\omega - \lambda - 1)^l = \frac{1}{2\pi}\int_{\mathbb{R}} s^l d\tilde{k}_\lambda(s),\ l\geq 1, \tag{4.17}$$

the operator $t_{b,R}$ being defined in (3.4).

Proof. By Corollary 3.1, we have

$$\lim_{b\to\infty}\lim_{R\to\infty} b^{-1}R^{-3}\mathrm{Tr}\, t_{b,R}(V_\omega-\lambda-1)^l =$$
$$\frac{1}{2\pi}\lim_{R\to\infty} R^{-1}\int_{(-\frac12,\frac12)^2}\mathrm{Tr}\,\tau_R(V_\omega(X,.)-\lambda-1)^l dX = \tag{4.18}$$
$$\frac{1}{2\pi}\lim_{R\to\infty} R^{-1}\int_{(-\frac12,\frac12)^2}\int_{\mathbb{R}} s^l d\nu_{R,\omega,X,\lambda}(s)\, dX.$$

Proposition 4.2 easily implies that

$$\lim_{R\to\infty} R^{-1}\int_{(-\frac12,\frac12)^2}\int_{\mathbb{R}} s^l d\nu_{R,\omega,X,\lambda}(s)\, dX =$$
$$\int_{(-\frac12,\frac12)^2}\int_{\mathbb{R}} s^l d\tilde{\varrho}_\lambda(s;X)\, dX = \int_{\mathbb{R}} s^l d\tilde{k}_\lambda(s), \tag{4.19}$$

and the combination of (4.18) and (4.19) yields (4.17). □

Corollary 4.3. *For each $\lambda \in R$ and $s<0$ we have*

$$\lim_{b\to\infty}\lim_{R\to\infty} b^{-1}R^{-3}n_-(-s;t_{b,R}(V_\omega-\lambda-1)) = \frac{1}{2\pi}\tilde{k}_\lambda(s). \tag{4.20}$$

Proof. We have $\|t_{R,b}(V_\omega-\lambda-1)\| \le c_1$ (see (3.3)). Moreover, $\tilde{k}_\lambda(s)=0$ if $|s|>c_1$. Hence we can apply the Kac-Murdock-Szegö theorem (see [R 1, Section 3]) and, taking into account the continuity of $\tilde{k}_\lambda(.)$, to conclude that (4.20) follows from (4.17). □

5 Proof of Theorem 2.1

In order to prove (2.3), it suffices to show that for each sequence $\{b_j\}_{j\ge1}$ such that $b_j\to\infty$ as $j\to\infty$, we have

$$\lim_{j\to\infty} b_j^{-1}\mathcal{D}_{b_j}(\lambda+b_j) = \frac{1}{2\pi}k(\lambda),\ \lambda\in\mathbb{R}. \tag{5.1}$$

Fix two sequences $\{\lambda_m^\pm\}_{m\ge1}$ such that $\lambda_m^- < \lambda < \lambda_m^+$, $m\ge1$, $\lim_{m\to\infty}\lambda_m^\pm=\lambda$, and $\lambda_m^\pm+b_j$ are continuity points of $\mathcal{D}_{b_j}$ for all $m\ge1$ and $j\ge1$. Then by Corollary 4.1 we have

$$\limsup_{j\to\infty} b_j^{-1}\mathcal{D}_{b_j}(\lambda+b_j) \le \limsup_{j\to\infty}\lim_{R\to\infty} b_j^{-1}R^{-3}n_-(1;T_{b_j,\omega,\lambda_m^+,R}), \tag{5.2}$$

$$\liminf_{j\to\infty} b_j^{-1}\mathcal{D}_{b_j}(\lambda+b_j) \ge \liminf_{j\to\infty}\lim_{R\to\infty} b_j^{-1}R^{-3}n_-(1;T_{b_j,\omega,\lambda_m^-,R}), \tag{5.3}$$

the operator $T_{b,\omega,\lambda,R}$ being defined in (4.10). By the minimax principle

$$\liminf_{j\to\infty} \lim_{R\to\infty} b_j^{-1} R^{-3} n_-(1; T_{b_j,\omega,\lambda_m^-,R}) \geq$$

$$\liminf_{j\to\infty} \liminf_{R\to\infty} b_j^{-1} R^{-3} n_-(1; P_{b_j} T_{b_j,\omega,\lambda_m^-,R}\, P_{b_j}). \tag{5.4}$$

Note that the operator $P_b T_{b,\omega,\lambda,R}\, P_b$ coincides with the operator $t_{b,R}(V_\omega - \lambda - 1)$ (see (4.10), (3.4), and (3.2)). Hence, Corollary 4.3 and (4.16) entail

$$\liminf_{j\to\infty} \liminf_{R\to\infty} b_j^{-1} R^{-3} n_-(1; P_{b_j} T_{b_j,\omega,\lambda_m^-,R} P_{b_j}) \geq$$

$$\lim_{b\to\infty} \lim_{R\to\infty} b^{-1} R^{-3} n_-(1; t_{R,b}(V_\omega - \lambda_m^- - 1)) = \frac{1}{2\pi} \tilde{k}_{\lambda_m^-}(-1) = \frac{1}{2\pi} k(\lambda_m^-). \tag{5.5}$$

The combination of (5.3)-(5.5) yields

$$\liminf_{j\to\infty} b_j^{-1} \mathcal{D}_{b_j}(\lambda + b_j) \geq \frac{1}{2\pi} k(\lambda_m^-). \tag{5.6}$$

On the other hand, we have

$$T_{b_j,\omega,\lambda_m^+,R} = P_{b_j} T_{b_j,\omega,\lambda_m^+,R}\, P_{b_j} + (\mathrm{Id} - P_{b_j}) T_{b_j,\omega,\lambda_m^+,R} (\mathrm{Id} - P_{b_j})$$

$$+ 2\mathrm{Re}\, P_{b_j} T_{b_j,\omega,\lambda_m^+,R} (\mathrm{Id} - P_{b_j}). \tag{5.7}$$

Set

$$\tilde{T}_{b,\omega,\lambda,R} :=$$

$$(\mathrm{Id} - P_b)(H_0(b) - b + 1)^{-1/2} |V_\omega - \lambda - 1| \mathbf{1}_{\left(-\frac{R}{2},\frac{R}{2}\right)^3} (H_0(b) - b + 1)^{-1/2} (\mathrm{Id} - P_b).$$

Applying the elementary operator inequalities

$$(\mathrm{Id} - P_{b_j}) T_{b_j,\omega,\lambda_m^+,R} (\mathrm{Id} - P_{b_j}) \geq -\tilde{T}_{b_j,\omega,\lambda_m^+,R},$$

$$2\mathrm{Re}\, P_{b_j} T_{b_j,\omega,\lambda_m^+,R} (\mathrm{Id} - P_{b_j}) \geq -\varepsilon^2 t_{b_j,R}(|V_\omega - \lambda_m^+ - 1|) - \varepsilon^{-2} \tilde{T}_{b_j,\omega,\lambda_m^+,R},\ \varepsilon > 0,$$

we find that (5.7) implies

$$n_-(1; T_{b_j,\omega,\lambda_m^+,R}) \leq n_-(1; t_{b_j,R}(V_\omega - \lambda_m^+ - 1) - \varepsilon^2 t_{b_j,R}(|V_\omega - \lambda_m^+ - 1|)) +$$

$$n_+(1; (1 + \varepsilon^{-2}) \tilde{T}_{b_j,\omega,\lambda_m^+,R}),\ \varepsilon > 0. \tag{5.8}$$

It is easy to verify the estimate

$$\|\tilde{T}_{b_j,\omega,\lambda_m^+,R}\| \leq (b_j + 1)^{-1} (c_0 + |\lambda_m^+ + 1|),\ R > 0. \tag{5.9}$$

Fix $\varepsilon > 0$ and assume that b_j is so large that $(1+\varepsilon^{-2})(b_j+1)^{-1}(c_0 + |\lambda_m^+| + 1) < 1$. Then (5.9) entails

$$n_+(1; (1 + \varepsilon^{-2}) \tilde{T}_{b_j,\omega,\lambda_m^+,R}) = 0. \tag{5.10}$$

By the Weyl inequalities for the eigenvalues of compact selfadjoint operators we have

$$n_-(1; t_{b_j,R}(V_\omega - \lambda_m^+ - 1) - \varepsilon^2 t_{b_j,R}(|V_\omega - \lambda_m^+ - 1|)) \leq$$

$$n_-(1-\varepsilon; t_{b_j,R}(V_\omega - \lambda_m^+ - 1)) + n_+(1; \varepsilon t_{b_j,R}(|V_\omega - \lambda_m^+ - 1|)), \varepsilon \in (0,1). \quad (5.11)$$

The estimate (3.5) implies

$$n_+(1; \varepsilon t_{b_j,R}(|V_\omega - \lambda_m^+ - 1|)) \leq c_5 \varepsilon b_j R^3 \quad (5.12)$$

with $c_5 := (c_0 + |\lambda_m^+ + 1|)/4\pi$. Now, the combination of (5.8), (5.10), (5.11), and (5.12) yields

$$\limsup_{j\to\infty} \lim_{R\to\infty} b_j^{-1} R^{-3} n_-(1; T_{b_j,\omega,\lambda_m^+,R}) \leq$$

$$\limsup_{j\to\infty} \limsup_{R\to\infty} b_j^{-1} R^{-3} n_-(1-\varepsilon; t_{b_j,R}(V_\omega - \lambda_m^+ - 1)) + c_5\varepsilon, \ \varepsilon \in (0,1). \quad (5.13)$$

By Corollary 4.3

$$\limsup_{j\to\infty} \limsup_{R\to\infty} b_j^{-1} R^{-3} n_-(1-\varepsilon; t_{b_j,R}(V_\omega - \lambda_m^+ - 1)) =$$

$$\lim_{b\to\infty} \lim_{R\to\infty} b^{-1} R^{-3} n_-(1-\varepsilon; t_{b,R}(V_\omega - \lambda_m^+ - 1)) = \frac{1}{2\pi} \tilde{k}_{\lambda_m^+}(-1+\varepsilon), \ \varepsilon \in (0,1). \quad (5.14)$$

The combination of (5.2), (5.13), and (5.14) yields

$$\limsup_{j\to\infty} b_j^{-1} \mathcal{D}_{b_j}(\lambda + b_j) \leq \frac{1}{2\pi} \tilde{k}_{\lambda_m^+}(-1+\varepsilon) + c_5\varepsilon, \ \varepsilon \in (0,1).$$

Letting $\varepsilon \downarrow 0$, we get

$$\limsup_{j\to\infty} b_j^{-1} \mathcal{D}_{b_j}(\lambda + b_j) \leq \frac{1}{2\pi} \tilde{k}_{\lambda_m^+}(-1) = \frac{1}{2\pi} k(\lambda_m^+). \quad (5.15)$$

Letting $m \to \infty$ (hence, $\lambda_m^\pm \to \lambda$) in (5.6) and (5.15), we arrive at (5.1).

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