Ann. Henri Poincaré 1 (2000) 341 – 384

1424-0637/00/020341-44 $1.50+0.20/0

Annales Henri Poincaré

On the Initial Value Problem for the Ishimori System

C.E. Kenig, Gustavo Ponce and Luis Vega

I Introduction

In this paper we study Ishimori system

$$\begin{cases} \partial_t S = S \wedge (\partial_x^2 S \pm \partial_y^2 S) + b(\partial_x \phi \partial_y S + \partial_y \phi \partial_x S), \\ \qquad\qquad\qquad\qquad\qquad\qquad t \in \mathbb{R},\ x, y \in \mathbb{R} \\ \partial_x^2 \phi \mp \partial_y^2 \phi = \mp 2 S \cdot (\partial_x S \wedge \partial_y S), \end{cases} \tag{I.1}$$

where $S(\cdot, t) : \mathbb{R}^2 \to \mathbb{R}^3$ with $\|S\| = 1$, $S \to (0,0,1)$ as $\|(x,y)\| \to \infty$, and $\wedge$ denotes the wedge product in $\mathbb{R}^3$.

This model was proposed by Y. Ishimori in [Is] as a two dimensional generalization of the Heisenberg equation in ferromagnetism, which corresponds to the case $b = 0$ and signs $(-,+,+)$ in (1.1) and was studied in [SuSuBa].

For $b = 1$ the system (1.1) is completely integrable by inverse scattering, (see [AbHa],[BeCo],[KoMa],[Sn],[ZaKu] and references therein).

Using the stereographic variable $u : \mathbb{R}^2 \to \mathbb{C}$ one can get rid of the constraint $\|S\| = 1$. Thus, for

$$u = \frac{S_1 + iS_2}{1 + S_3}, \quad S = (S_1, S_2, S_3) = \frac{1}{1 + |u|^2}(u + \bar{u}, -i(u - \bar{u}), 1 - |u|^2), \tag{I.2}$$

the initial value problem (IVP) for (1.1) can be written as

$$\begin{cases} i\partial_t u + \partial_x^2 u \pm \partial_y^2 u = \frac{2\bar{u}}{1+|u|^2}((\partial_x u)^2 - (\partial_y u)^2) \\ \qquad\qquad\qquad\qquad\qquad\qquad + ib(\partial_x \phi \partial_y u + \partial_y \phi \partial_x u), \\ \partial_x^2 \phi \mp \partial_y^2 \phi = 4i\, \frac{\partial_x u \partial_y \bar{u} + \partial_x \bar{u} \partial_y u}{(1+|u|^2)^2}, \\ u(x,y,0) = u_0(x,y), \end{cases} \tag{I.3}$$

with the condition $u(x,y,t) \to 0$ as $\|(x,y)\| \to \infty$.

The case $(-,+)$ in (1.3), i.e. $-$ in the first equation and $+$ in the second, was studied by A. Souyer [So]. He obtained local well posedness and global existence of solution for small data in an appropriate Sobolev space. It was remarked in [So] that the arguments there do not extend to the case $(+,-)$ in (1.3).

The case, $(+,-)$ in (1.3), was first studied by Hayashi-Saut [HySa]. They consider the problem in a class of analytic functions which allowed them obtain local and global existence for small analytic data, thus overcoming the so called "loss of derivatives" introduced by the nonlinearity.

In [Hy], N. Hayashi removed the analyticity assumptions in [HySa] by establishing the local well posedness of the IVP (1.3), for the case $(+,-)$, with small data u_0 in the weighted Sobolev space $H^4(\mathbb{R}^2)\cap L^2((x^2+y^2)^4dxdy)$.

Our main result here, Theorem 1.1, removes the smallness assumptions in [Hy]. In particular we show the local well-posedness of the IVP (1.3) with $(+,-)$ sign, and data of arbitrary size in a weighted Sobolev space. Before stating our results we shall discuss the problem in a more general context.

By inverting the operator $\partial_x^2 \mp \partial_y^2$ one can rewrite the system in (1.3) as an scalar equation of Schrödinger type

$$i\partial_t u + \partial_x^2 u \mp \partial_y^2 u = F(u, \nabla_x u, \overline{u}, \nabla_x \overline{u}, Ku, ...) \tag{I.4}$$

where $F(\cdot)$ represents the nonlinearity and $K=\partial\partial(-\Delta)^{-1}$ for the $+$ sign and K an operator of "order one" for the $-$ sign.

The IVP for the equation in (1.4), without the operator K in the nonlinearity $F(\cdot)$ and in arbitrary dimension, i.e.

$$\begin{cases} \partial_t u = i\mathcal{L}u + F(u, \nabla_x u, \overline{u}, \nabla_x \overline{u}), & x \in \mathbb{R}^n \\ u(x,0) = u_0(x), \end{cases} \tag{I.5}$$

where $\nabla_x = (\partial_{x_1}, .., \partial_{x_n})$, $\mathcal{L}$ is a non-degenerate constant coefficient, second order operator

$$\mathcal{L} = \sum_{j\le k} \partial_{x_j}^2 - \sum_{j>k} \partial_{x_j}^2, \qquad \text{for some} \quad k \in \{1, .., n\}, \tag{I.6}$$

and $F(\cdot)$ is a polynomial, having no constant or linear terms, has been studied in recents works.

In [KePoVe1] we proved that (1.5) is locally well posed for "small" data, in some weighted Sobolev spaces. The proof in [KePoVe1] applies to the general form of $\mathcal{L}$ in (1.6). In [KePoVe1] the key estimates were

$$\begin{cases} (i) \;\; |||D^{1/2}e^{it\mathcal{L}}u_0|||_T \equiv \sup_{\mu\in\mathbb{Z}^n}(\int_0^T \int_{Q_\mu} |D^{1/2}e^{it\mathcal{L}}u_0|^2dxdt)^{1/2} \\ \hfill \le c\|u_0\|_2, \\ (ii) \; |||\nabla_x \int_0^t e^{i(t-t')\mathcal{L}}F(t')dt'|||_T \le c|||F|||'_T, \end{cases} \tag{I.7}$$

where $\{Q_\mu\}_{\mu\in\mathbb{Z}^n}$ is a family cubes of side one with disjoint interiors covering $\mathbb{R}^n$, and $D=(-\Delta)^{1/2}$. The local smoothing effect in (i), known as Kato smoothing effects, see [Kt], was proven by Constantin-Saut [CnSa], Sjölin [Sj], and Vega [Ve]. We proved the inhomogeneous version (ii) in [KePoVe1].

It is essential the gain of one derivative in (1.7) (ii). This allows to use the contraction principle in (1.5) and avoid the "loss of derivatives". However, the

$|||\cdot|||_T$ norm forces the use the following

$$\|G\|_{l^1_\mu(L^\infty(Q_\mu\times[0,T]))}=\sum_{\mu\in\mathbb{Z}^n}(Sup_{[0,T]}Sup_{Q_\mu}|G(x,t)|. \tag{I.8}$$

This factor cannot be made small by taking T small, except if $G(t)$ is small at $t=0$. It is here where the restriction on the size of the data appears.

In [HyOz] for the one dimensional case $n=1$, Hayashi-Ozawa removed the smallness assumption on the size of the data in [KePoVe1]. By introducing a change of variables they reduced the problem to a new one which can be treated by standard energy methods. This technique is similar to that used by A. Souyer in [So] in his study of (1.3) with signs $(-,+)$.

In [Ch] for the elliptic case $\mathcal{L}=\Delta$, H. Chihara was able to remove the size restriction on the data in [KePoVe1] in any dimension.

Finally in[KePoVe2] we showed how to remove the smallness assumptions in [KePoVe1] for the general dispersive operator $\mathcal{L}$ in (1.5), see (1.6).

The arguments in [Ch], [KePoVe2] are based in techniques involving ψ.d.o's. However, in some cases it is not clear how to extend them to treat specific models arising in both mathematics and physics. For example, consider the IVP for the Davey-Stewartson (D-S) system which arises in water waves problems, see [DS],[DjRe],[ZaSc], and inverse scattering see [AbHa],[BeCo],[KoMa],

$$\begin{cases} i\partial_t v+c_0\partial_x^2 v+\partial_y^2 v=c_1|v|^2v+c_2u\partial_x\varphi,\\ \partial_x^2\varphi+c_3\partial_y^2\varphi=\partial_x|v|^2\\ v(x,y,0)=v_0(x,y)\end{cases} \tag{I.9}$$

where $c_0,..,c_3$ are real parameters.

In [GhSa], Ghidaglia-Saut studied the existence problem for solutions of the IVP (1.9). They classified the system as elliptic-elliptic, elliptic-hyperbolic, hyperbolic-elliptic and hyperbolic-hyperbolic according to the respective sign of (c_0,c_3): $(+,+)$, $(+,-)$, $(-,+)$ and $(-,-)$. In [LiPo], Linares-Ponce adapted the results in [KePoVe1] to show that in this hyperbolic-hyperbolic case the IVP (1.9) is locally well posed for small data in weighted Sobolev spaces, (see also [HySa]). However, this smallness assumptions have yet to be removed. For the elliptic-elliptic, elliptic-hyperbolic, and hyperbolic-elliptic cases, where a more complete set of results are available, we refer to [GhSa],[Hy],[HySa],[LiPo], and references therein.

The necessity of the decay assumption on the data can be justified by the following result due to S. Mizohata [Mz]. Consider the linear IVP

$$\begin{cases} \partial_t v=i\Delta v+b(x)\cdot\nabla_x v+f(x,t), \quad t\in\mathbb{R},\ x\in\mathbb{R}^n,\\ v(x,0)=v_0(x)\in L^2(\mathbb{R}^n),\end{cases} \tag{I.10}$$

with $b(\cdot)$ and $f(\cdot)$ smooth enough functions. In [Mz] it was shown that the following condition is necessary for the L^2-local solvability of (1.8)

$$\sup_{x\in\mathbb{R}^n,\omega\in\mathbb{S}^{n-1},R>0} |\operatorname{Im} \int_0^R b(x+r\omega)\cdot\omega dr| < \infty. \tag{I.11}$$

We observe that the condition (1.11) may fail for $b \in H^s(\mathbb{R}^n)$. It holds for $b \in W^{s,1}(\mathbb{R}^n)$, s large, however this class is not preserve by the associated linear group $\{e^{it\Delta} : t \in \mathbb{R}\}$. Thus the use of weighted L^2 spaces seems natural.

Following [Hy] one extends the classification in for the DS system in (1.9) given in [GhSa] to the Ishimori system by considering its generalized form

$$\begin{cases} i\partial_t u + \partial_x^2 u + c_0\partial_y^2 u = c_1 \frac{\bar{u}\ ((\partial_x u)^2 + c_2(\partial_y u)^2)}{1+|u|^2} \\ \qquad\qquad\qquad\qquad\qquad +c_3(\partial_x\phi\partial_y u + \partial_y\phi\partial_x u), \\ \partial_x^2\phi + c_4\partial_y^2\phi = c_5\ \frac{\partial_x u\ \partial_y\bar{u} + \partial_x\bar{u}\ \partial_y u}{(1+|u|^2)^2}, \\ u(x,y,0) = u_0(x,y), \end{cases} \tag{I.12}$$

where $c_0, c_4 \in \mathbb{R} - \{0\}$, and $c_1, c_2, c_3, c_5 \in \mathbb{C}$.

As was remarked in [Hy] the local existence results for arbitrary size data in [So] applies to the cases $(c_0, c_2, c_4) = (c, c, b)$ in (1.12) with $c \in \mathbb{R} - \{0\}$, and $b > 0$. The small data results in [Hy] corresponds to the elliptic-hyperbolic case, i.e. $c_0 > 0$, $c_4 < 0$, and as was mentioned there it does not extend to the hyperbolic-hyperbolic case i.e. $c_0 < 0$, $c_4 < 0$. In fact for this case no existence results are known besides those in [HySa] for "small" analytic data.

In [Sn], L. Y. Sung using the gauge equivalence between the integrable case of the Ishimori system, i.e. $b = 1$ in (1.3), and the DS system in (1.6) with $(c_0, c_1, c_2, c_3) = (-1, 2, -1, 1)$ proved global existence of solution of (1.3), with $b = 1$ with "small" data.

Our main result is the following Theorem.

Theorem 1.1 *Given $N \geq 1$ there exist $s, m \in \mathbb{Z}^+$ such that for any $u_0 \in H^s \cap L^2(|x|^m dx)$ the IVP (1.12) has a unique solution $u(\cdot)$ defined in the time interval $[0, T]$ satisfying that*

$$u \in C([0,T] : H^s(\mathbb{R}^2) \cap L^2(\mathbb{R}^2 : |x|^m dx)), \tag{I.13}$$

and

$$\|\lambda_N(x) J_x^{s+1/2} u\|_{L^2_{x,y,T}} + \|\lambda_N(y) J_y^{s+1/2} u\|_{L^2_{x,y,T}} < \infty, \tag{I.14}$$

where

$$\lambda_N(x) = (1+x^2)^{-N/2}, \quad \lambda_N(y) = (1+y^2)^{-N/2}. \tag{I.15}$$

Moreover, the map data → solution from $H^s(\mathbb{R}^2) \cap L^2(\mathbb{R}^2 : |x|^m dx)$ into the class in (1.13)-(1.14) is locally continuous.

If $s' > s$ *, then the above results hold, with* s' *instead of* s*, in the same time interval* $[0,T]$.

To explain our method of proof we assume without loss of generality $c_0 = 1$, $c_4 = -1$, $c_2 = 1$. Then we rotate coordinates in the xy-plane and rewrite the equations in (1.12) as

$$\begin{cases} i\partial_t u + \Delta u = c_1 \frac{\bar{u}}{1+|u|^2}\partial_x u \partial_y u + c_3(\partial_x\phi\partial_x u + \partial_y\phi\partial_y u), \\ \\ \partial_x\partial_y\phi = c_5 \frac{\partial_x u\partial_y\bar{u} - \partial_x\bar{u}\partial_y u}{(1+|u|^2)^2}, \end{cases} \tag{I.16}$$

and assuming, without loss of generality, a trivial radiation condition at infinity, we write the IVP (1.16) as an scalar equation

$$\begin{cases} i\partial_t u + \Delta u = c_1 \frac{\bar{u}}{1+|u|^2}\partial_x u \partial_y u \\ \\ +c_6\partial_x u \partial_y^{-1}\left(\frac{\partial_x u\partial_y\bar{u} - \partial_x\bar{u}\partial_y u}{(1+|u|^2)^2}\right) - c_7\partial_y u\partial_x^{-1}\left(\frac{\partial_x u\partial_y\bar{u} + \partial_x\bar{u}\partial_y u}{(1+|u|^2)^2}\right), \end{cases} \tag{I.17}$$

where

$$\partial_x^{-1} f(x,y) = \int_{-\infty}^{x} f(x',y)dx', \qquad (\text{resp. } \partial_y^{-1}). \tag{I.18}$$

First we observe that ∂_x^{-1} is not a ψ.d.o. However by adding some decay from the coefficients we get that for large M

$$\tilde{\partial}_x^{-1} f(x,y) = \frac{1}{(1+x^2)^M}\partial_x^{-1}\left(\frac{1}{(1+x^2)^M} f(\cdot,y)\right) \tag{I.19}$$

defines a ψ.d.o. of order -1 in the x-variable. However $\tilde{\partial}_x^{-1}$ is not a ψ.d.o. in both variables. Thus the techniques in [Ch],[KePoVe2] and in recent related works [CrKaSt], [Do] can not be carried out. One has to work in each variable separated and when results in both variables are required one uses operator valued version of some of the techniques. For example, to establish the local smoothing effect in its homogeneous and inhomogeneous versions, see (1.7), we shall use the operator valued version of the sharp Gårding inequality, see [Ho]. Another feature of our approach is that for the linearized system associated to (1.17) the coefficients of the first order terms do not decay in both variables. More precisely, in a simplified setting, our linearized IVP is as that in (1.8) with

$$b(x)\cdot\nabla_x = b_1(x,y)\partial_x + b_2(x,y)\partial_y \tag{I.20}$$

where b_1 is a smooth function with decay in x, uniformly in y, and b_2 is a smooth function with decay in y, uniformly in x. Under these assumptions is clear that Mizohata's condition in (1.11) for the IVP (1.10) holds. However, we consider the operator $\partial_x^2 - \partial_y^2$ under the above decay assumptions on the coefficient $b(\cdot)$ then the IVP (1.10) is, in general, ill-posed since in this case Mizohata's condition reads

$$\sup_{x\in\mathbb{R}^2,\omega\in\mathbb{S}^1,R>0} |\,\mathrm{Im}\int_0^R b(x+r\omega)\cdot\tilde{\omega}dr| < \infty, \quad \tilde{\omega}=(\omega_2,-\omega_1). \tag{I.21}$$

This may explain why for the hyperbolic-hyperbolic Ishimori system no existence results are available besides those in [HySa] for "small" analytic data.

In fact, our main step in the proof of Theorem 1.1 is the following regularity result for the linearized IVP associated to (1.17)

$$\begin{cases} \partial_t z = i\Delta z + r_1\partial_x z + r_2\partial_y z + \varphi_1\partial_x\partial_y^{-1}\varphi_2 z + \varphi_3\partial_y\partial_x^{-1}\varphi_4 z \\ \quad +\varphi_5\partial_x\partial_y^{-1}\varphi_6\bar{z} + \varphi_7\partial_y\partial_x^{-1}\varphi_8\bar{z} + p_1\partial_x f_1 + p_2\partial_y f_2 \\ \quad +\phi_1\partial_x\partial_y^{-1}\phi_2 f_3 + \phi_3\partial_y\partial_x^{-1}\phi_4 f_4 + f_5, \\ z(x,y,0) = z_0(x,y), \end{cases} \tag{I.22}$$

where $r_j = r_j(x,y)$, $j=1,2$ are smooth functions, r_1 with decay in x, uniformly in y, r_2 with decay in y, uniformly in x, $\varphi_j = \varphi_j(x,y)$, $j=1,..,8$ are smooth with decay in both variables, $p_1 = p_1(x,y,t)$, $p_2 = p_2(x,y,t)$ behave like r_1, r_2 respectively uniformly in $t\in[0,T]$, and ϕ_j, $j=1,..,4$ are like the φ_j's uniformly in $t\in[0,T]$.

Theorem 1.2 *Under the above hypothesis on the coefficients given $N>1$ there exist $M>0$, $k\in\mathbb{Z}^+$ and $T>0$ small enough such that the solution of the IVP (1.22) with $c_0>0$, $c_4<0$ satisfies $u\in C([0,T]:L^2(\mathbb{R}^2))$ with*

$$\begin{aligned} \sup_{0\le t\le T} \quad & \|z(t)\|^2_{L^2_{x,y}} + \|\lambda_N(x)J_x^{1/2}z\|^2_{L^2_{x,y,T}} + \|\lambda_N(y)J_y^{1/2}z\|^2_{L^2_{x,y,T}} \\ & \le c\|z_0\|^2_{L^2_{x,y}} + cA\ \textstyle\sum_{j=1}^4 \sup_{0\le t\le T}\|f_j\|^2_{L^2_{x,y}} \\ & +cA(\|\lambda_N(x)J_x^{1/2}f_1\|^2_{L^2_{x,y,T}} + \|\lambda_N(y)J_y^{1/2}f_2\|^2_{L^2_{x,y,T}}) \\ & +cA(\|\lambda_N(x)J_x^{1/2}f_3\|^2_{L^2_{x,y,T}} + \|\lambda_N(y)J_y^{1/2}f_4\|^2_{L^2_{x,y,T}}) \\ & +cT^{1/2}\|f_5\|^2_{L^2_{x,y,T}}, \end{aligned} \tag{I.23}$$

where $c = c(N)$ *and*

$$\begin{aligned}
A \le T\ \textstyle\sum_{j=1}^{2} \|p_j\|^2_{L^2_T C^k_b(\mathbb{R}^2_{x,y})} \ +\ \sum_{j=1}^{2} \|\lambda_N^{-2}(x_j)p_j\|^2_{L^\infty_{x,y,T}} \\
+T\textstyle\sum_{j=1}^{4} \|\tilde\phi_j\|^4_{L^2 C^k_b(\mathbb{R}^2_{x,y})} \ +\ \|\tilde\phi_1\|^2_{L^2_T L^\infty_{x,y}}\ \|\partial_x\tilde\phi_2\|^2_{L^2_T L^\infty_{x,y}} \\
+\|\tilde\phi_3\|^2_{L^2_T L^\infty_{x,y}}\ \|\partial_y\tilde\phi_4\|^2_{L^2_T L^\infty_{x,y}} \\
+\|\lambda_N^{-1}(x)\tilde\phi_1\|^2_{L^\infty_{x,y,T}}\ \|\lambda_N^{-1}(x)\tilde\phi_2\|^2_{L^\infty_{x,y,T}} \\
+\|\lambda_N^{-1}(y)\tilde\phi_3\|^2_{L^\infty_{x,y,T}}\ \|\lambda_N^{-1}(y)\tilde\phi_4\|^2_{L^\infty_{x,y,T}},
\end{aligned} \tag{I.24}$$

where $x_1 = x,\ x_2 = y,$

$$\begin{aligned}
\tilde\phi_j(x,y,t) = (1+y^2)^{M/2}\phi_j(x,y,t), \quad j = 1,2, \\
\tilde\phi_j(x,y,t) = (1+x^2)^{M/2}\phi_j(x,y,t), \quad j = 3,4.
\end{aligned} \tag{I.25}$$

Moreover there is a continuous dependence of the solution with respect to the coefficients in the norms appearing in (1.24). We observe that the result of Theorem 1.2 holds for solutions of the IVP (1.22) with $i\Delta + \epsilon\Delta$ instead of $i\Delta$, uniformly for $\epsilon \in (0,1]$, (see Corollary 4.1 at the end of Section 4). This provides a slightly weaker version of Theorem 1.1.

Theorem 1.3 *Given* $N > 1$ *here exist* $s, m \in \mathbb{Z}^+$ *such that for any* $u_0 \in H^s \cap L^2(|x|^m dx)$ *the IVP (1.12)with* $c_0 > 0$, $c_4 < 0$ *has a unique solution* $u(\cdot)$ *defined in the time interval* $[0,T]$ *satisfying that*

$$u \in C([0,T] : H^{s-1} \cap L^2(|x|^{m-1}dx)) \cap L^\infty([0,T] : H^s \cap L^2(|x|^m dx)), \tag{I.26}$$

and

$$\|\lambda_N(x)J_x^{s+1/2}u\|_{L^2_{x,y,T}} + \|\lambda_N(y)J_y^{s+1/2}u\|_{L^2_{x,y,T}} < \infty, \tag{I.27}$$

where

$$\lambda_N(x) = (1+x^2)^{-N/2}, \quad \lambda_N(y) = (1+y^2)^{-N/2}. \tag{I.28}$$

If $s' > s$ *, then the above results hold, with* s' *instead of* s*, in the same time interval* $[0,T]$.-

Once Theorem 1.3 has been established the proof of Theorem 1.1 follows by combining Theorem 1.3 and Theorem 1.2 and their proofs.

II Preliminary estimates

This section contains some estimates to be used in the coming sections. We start by recalling some results on ψ.d.o's

Definition 2.1 *The symbol class* $S^k(\mathbb{R}^{2n})$ *consists of the set of* $a \in C^\infty(\mathbb{R}^{2n})$ *such that*

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \le c_{\alpha,\beta}(1+|\xi|)^{k-|\beta|}, \quad x,\xi \in \mathbb{R}^n, \tag{II.1}$$

for all multi-indices $\alpha, \beta \in \mathbb{N}^n$.

We say that $a \in S_M^k(\mathbb{R}^{2n})$ *if it is of class* $C^M(\mathbb{R}^{2n})$ *and (2.1) holds for* $|\alpha|, |\beta| \le M$.

Theorem 2.2 *Let* $A,\ B \in S_M^0(\mathbb{R}^{2n})$ *with* M *large. Then* $A(x,D)B(x,D) = C(x,D)$, *where*

$$c(x,\xi) = a(x,\xi)b(x,\xi) + \sum_{|\gamma|=1} \int_0^1 q_{\gamma,\theta}(x,\xi)d\theta, \tag{II.2}$$

with

$$q_{\gamma,\theta}(x,\xi) = \ Os \int\int e^{-iy\cdot\eta} \partial_\xi^{(\gamma)} a(x,\xi+\theta\eta)\partial_x^{(\gamma)} b(x+y,\xi)dyd\eta. \tag{II.3}$$

Moreover, the $S_M^{-1}(\mathbb{R}^{2n})$ *seminorms of* $Q_{\gamma,\theta}$ *are bounded by products of seminorms of* $\partial_\xi^{(\gamma)}a,\ \partial_x^{(\gamma)}b$, *uniformly in* $\theta \in [0,1]$.

Also $A^*(x,D)$ *has symbol*

$$a^*(x,\xi) = \overline{a(x,\xi)} + \sum_{|\gamma|=1} \int_0^1 q^*_{\gamma,\theta}(x,\xi)d\theta, \tag{II.4}$$

where

$$q^*_{\gamma,\theta}(x,\xi) = \ Os \int\int e^{-iy\cdot\eta} \partial_\xi^{(\gamma)} \partial_x^{(\gamma)} \overline{a(x+y,\xi+\theta\eta)} dyd\eta. \tag{II.5}$$

Moreover, the $S_M^{-1}(\mathbb{R}^{2n})$ *seminorms of* $Q^*_{\gamma,\theta}$ *are bounded by seminorms of* $\partial_\xi^{(\gamma)}\partial_x^{(\gamma)}\overline{a}$, *uniformly in* $\theta \in [0,1]$.

Proposition 2.3 *Given* $M > 0$ *there exists* $N > 0$ *such that*

$$\frac{1}{(1+|x|^2)^N} \partial_x^{-1} \left(\frac{1}{(1+|x'|^2))^N} f \right) = a(x,D)f = \tilde{\partial}_x^{-1} f \tag{II.6}$$

with $a \in S_M^{-1}(\mathbb{R}\times\mathbb{R})$.

Remark. This proposition allows to rewrite (1.22) substituting, after changing the ϕ's and φ's, ∂_x^{-1} by $\tilde{\partial}_x^{-1}$, ∂_y^{-1} by $\tilde{\partial}_y^{-1}$ everywhere.

Proof of Proposition 2.3. We use that

$$\partial_x^{-1} g(x) = \int_{-\infty}^{x} g(x')dx' = \int_{-\infty}^{\infty} \chi_{[0,\infty)}(x-x')g(x')dx', \tag{II.7}$$

to obtain

$$\begin{aligned} &\tfrac{1}{(1+|x|^2)^N}\partial_x^{-1}\left(\tfrac{f(x')}{(1+|x'|^2)^N}\right) \\ &= \tfrac{1}{(1+|x|^2)^N}\int_{-\infty}^{\infty}\chi_{[0,\infty)}(x-x')\tfrac{f(x')}{(1+|x'|^2)^N}dx' \\ &= \tfrac{1}{(1+|x|^2)^N}\int_{-\infty}^{\infty}\tfrac{\chi_{[0,\infty)}(x-x')}{(1+|x-x'|^2)^N}\tfrac{(1+|x-x'|^2)^N}{(1+|x'|^2)^N}f(x')dx'. \end{aligned} \tag{II.8}$$

Now we observe that

$$\begin{aligned} &\tfrac{1}{(1+|x|^2)^N}\tfrac{1}{(1+|x'|^2)^N}(1+|x-x'|^2)^N \\ &= \textstyle\sum_{j=0}^{N} c_{N,j}\tfrac{|x-x'|^{2(N-j)}}{(1+|x|^2)^N(1+|x'|^2)^N} \\ &= \textstyle\sum_{j=0}^{N}\sum_{\substack{a,b\geq 0\\ a+b=N-j}} c_{N,j,a,b}\tfrac{x^{2a}x'^{2b}}{(1+|x|^2)(1+|x'|^2)^N}. \end{aligned} \tag{II.9}$$

Thus since

$$\frac{x^{2a}}{(1+|x|^2)^N} \quad , \quad \frac{x'^{2b}}{(1+|x'|^2)^N} \tag{II.10}$$

are bounded functions, together with all their derivatives, to establish the claim we just need to show that if

$$K_N(x) = \frac{\chi_{[0,\infty)}(x)}{(1+|x|^2)^N}, \tag{II.11}$$

then

$$\widehat{K}_N(\xi) = a(\xi) \in S_M^{-1}. \tag{II.12}$$

To prove (2.12) we write

$$a(\xi) = \int_0^{\infty} e^{ix\xi}\frac{1}{(1+|x|^2)^N}dx. \tag{II.13}$$

Clearly $a \in L^{\infty}(\mathbb{R})$. Next by integrating by parts it follows that

$$a(\xi) = -\frac{1}{i\xi} - \frac{N}{i\xi}\int_0^{\infty} e^{ix\xi}\frac{2x}{(1+|x|^2)^{N+1}}dx, \tag{II.14}$$

which shows that

$$|a(\xi)| \le \frac{c}{|\xi|}, \quad \text{for large } \xi. \tag{II.15}$$

Also, (2.14) and integration by parts lead to

$$\begin{aligned} a'(\xi) \quad &= \tfrac{1}{i\xi^2} + \tfrac{N}{i\xi^2}\int_0^\infty e^{ix\xi}\tfrac{2x}{(1+|x|^2)^{N+1}}dx \\ &\quad -\tfrac{iN}{i\xi}\int_0^\infty e^{ix\xi}\tfrac{2x^2}{(1+|x|^2)^{N+1}}dx \\ &= \tfrac{c}{\xi^2} + \tfrac{N}{i\xi^2}\int_0^\infty e^{ix\xi}\tfrac{2x}{(1+|x|^2)^{N+1}}dx \\ &\quad +\tfrac{iN}{(i\xi)^2}\int_0^\infty e^{ix\xi}\tfrac{\partial}{\partial x}\left(\tfrac{2ix^2}{(1+|x|^2)^{N+1}}\right)dx \end{aligned} \tag{II.16}$$

which shows that

$$|a'(\xi)| \le \frac{c}{|\xi|^2}, \quad \text{for large } |\xi|. \tag{II.17}$$

The proof for the higher derivatives is similar. Thus we have established the claim (2.12) and completed the proof of Proposition 2.2.

Let us now consider the action of ψ.d.o. in $S^0_M(\mathbb{R}^{2n})$, on weighted L^2 spaces. We recall the notation

$$\lambda_N(x) = \langle x\rangle^{-N} = \frac{1}{(1+|x|^2)^{N/2}}, \quad x \in \mathbb{R}^n. \tag{II.18}$$

Lemma 2.4 *Given $N \ge 0$, there exists $M = M(n,N) > 0$ such that, if $a \in S^0_M(\mathbb{R}^{2n})$, then*

$$a(x,D) : L^2(\mathbb{R}^n : \lambda_N(x)dx) \to L^2(\mathbb{R}^n : \lambda_N(x)dx), \tag{II.19}$$

with norm depending only on n, N, $c_{\alpha,\beta}$, $|\alpha|, |\beta| \le M$.

Proof. (see [KePoVe2], Lemma 2.3).

Next we recall some fact of the theory of vector valued ψ.d.o's of classical type, (as reference see [Ho], vol. 3, section 18.1, in particular Remark 2, page 79).

Let $H = L^2(\mathbb{R} : dy)$, and consider operators of the form

$$Bf(x,y) = \int e^{ix\cdot\xi_1} b(x,\xi_1)\hat{f}^x(\xi_1,-)d\xi_1, \tag{II.20}$$

where for each $(x,\xi_1) \in \mathbb{R}^2$, $b(x,\xi_1)$ is the symbol of an operator in H.

In this case the class S^k_M is defined by the inequality

$$|||\partial_x^\alpha\partial_{\xi_1}^\beta b(x,\xi_1)||| \le c_{\alpha,\beta}(1+|\xi_1|)^{k-|\beta|}, \tag{II.21}$$

for $|\alpha|, |\beta| \le M$, where $|||\partial_x^\alpha\partial_{\xi_1}^\beta b(x,\xi_1)|||$ denotes the operator norm in H.

Thus the calculus of ψ.d.o, L^2-boundedness, etc., have corresponding version in this context.

We also need the operator valued version of the sharp Gårding inequality:

Theorem 2.5 *Let $b(x,\xi_1)$ be the symbol of the operator B defined in (2.20). Assume that $b(x,\xi_1)$ satisfies (2.21) with $k=1$ for $|\alpha|+|\beta|\le M$, M large, (i.e., $B\in S^1_M$). If*

$$\langle (b(x,\xi_1)) + b(x,\xi_1)^*)h, h\rangle_H \ge 0, \qquad \forall h\in H, \tag{II.22}$$

for $x\in\mathbb{R}$, and $|\xi_1|\ge N$, then

$$Re\langle Bf,f\rangle \ge -c\|f\|^2_{L^2(H)} = -c\|f\|^2_{L^2(\mathbb{R}^2:dxdy)}, \tag{II.23}$$

where

$$\langle Bf,f\rangle = \int\left(\int Bf(x,y)\overline{f(x,y)}dy\right)dx = \int\langle Bf,f\rangle_H\,dx. \tag{II.24}$$

Clearly a corresponding theory holds if we interchange x and y.

III Proof of Theorem 1.2 (Diagonalization)

We split the proof of Theorem 1.2 in two steps. This section contains the first step, i.e. the diagonalization reduction.

By possibly changing $\varphi's$ and $\phi's$ we can rewrite the IVP (1.22) replacing $\partial_x^{-1},\partial_y^{-1}$ by $\widetilde{\partial}_x^{-1},\widetilde{\partial}_y^{-1}$ respectively, (see the remark after the statement of Proposition 2.3). Also we introduce the following notations:

$$R = r_1\partial_x + r_2\partial y \quad ; \quad \bar R = \bar r_1\partial_x + \bar r_2\partial_y, \tag{III.1}$$

$$\mathcal{L}_1 = \varphi_1\partial_x\widetilde{\partial}_y^{-1}\varphi_2 + \varphi_3\partial y\widetilde{\partial}_x^{-1}\varphi_4 \quad ; \quad \overline{\mathcal{L}}_1 = \overline{\varphi}_1\partial_x\widetilde{\partial}_y^{-1}\overline{\varphi}_2 + \overline{\varphi}_3\partial_y\widetilde{\partial}_x^{-1}\overline{\varphi}_4, \tag{III.2}$$

$$\mathcal{L}_2 = \varphi_5\partial_x\widetilde{\partial}_y^{-1}\varphi_6 + \varphi_7\partial y\widetilde{\partial}_x^{-1}\varphi_8 \quad ; \quad \overline{\mathcal{L}}_2 = \overline{\varphi}_5\partial_x\widetilde{\partial}_y^{-1}\overline{\varphi}_6 + \overline{\varphi}_7\partial_y\widetilde{\partial}_x^{-1}\overline{\varphi}_8, \tag{III.3}$$

$$F_1 = p_1\partial_x f_1 + p_2\partial_y f_2 \quad ; \quad \overline{F}_1 = \overline{p}_1\partial_x f_1 + \overline{p}_2\partial_y f_2, \tag{III.4}$$

$$\begin{aligned} F_2 &= \phi_1\partial_x\widetilde{\partial}_y^{-1}\phi_2 f_3 + \phi_3\partial y\widetilde{\partial}_x^{-1}\phi_4 f_4 \\ F_2 &= \overline{\phi}_1\partial_x\widetilde{\partial}_y^{-1}\overline{\phi}_2\overline{f}_3 + \overline{\phi}_3\partial_y\widetilde{\partial}_x^{-1}\overline{\phi}_4\overline{f}_4. \end{aligned} \tag{III.5}$$

Using (3.1)–(3.5) we rewrite the equation (1.22) as a system in $\vec{w} = (z \;\; \bar{z})^T$, with the notations

$$H = \begin{pmatrix} \Delta & 0 \\ 0 & -\Delta \end{pmatrix} \quad , \quad B = \begin{pmatrix} R & 0 \\ 0 & \bar{R} \end{pmatrix} \tag{III.6}$$

$$\beta_1 = \begin{pmatrix} \mathcal{L}_1 & 0 \\ 0 & \bar{\mathcal{L}}_1 \end{pmatrix} \; , \; \beta_2 = \begin{pmatrix} 0 & \mathcal{L}_2 \\ \bar{\mathcal{L}}_2 & 0 \end{pmatrix} \; , \; \vec{F} = \begin{pmatrix} F_1 + F_2 + f_5 \\ \bar{F}_1 + \bar{F}_2 + \bar{f}_5 \end{pmatrix} \tag{III.7}$$

as

$$\partial_t \vec{w} = iH\vec{w} + B\vec{w} + \beta_1\vec{w} + \beta_2\vec{w} + \vec{F}. \tag{III.8}$$

Our goal in this step is to "eliminate β_2" by accepting "semilinear errors."

We introduce the operator

$$\Lambda = I - S, \quad S = \begin{pmatrix} 0 & \mathcal{S}_1 \\ \mathcal{S}_2 & 0 \end{pmatrix}, \tag{III.9}$$

where $\mathcal{S}_i, \; i = 1, 2$ are to be determined, and write the system for $\vec{v} = \Lambda\vec{w}$. We shall see that modulo "semilinear terms", i.e. bounded L^2-terms, one has

$$\begin{aligned} \Lambda H - H\Lambda = HS - SH &= \begin{pmatrix} 0 & \Delta\mathcal{S}_1 \\ -\Delta\mathcal{S}_2 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -\mathcal{S}_1\Delta \\ \mathcal{S}_2\Delta & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & \Delta\mathcal{S}_1 + \mathcal{S}_1\Delta \\ -S_2\Delta - \Delta\mathcal{S}_2 & 0 \end{pmatrix} \end{aligned} \tag{III.10}$$

similarly, modulo bdd-L^2, term we have

$$\Lambda B - B\Lambda = BS - SB = \begin{pmatrix} 0 & R\mathcal{S}_1 - \mathcal{S}_1\bar{R} \\ \bar{R}\mathcal{S}_2 - \mathcal{S}_2 R & 0 \end{pmatrix}, \tag{III.11}$$

$$\Lambda\beta_1 - \beta_1\Lambda = \begin{pmatrix} 0 & \mathcal{L}_1\mathcal{S}_1 - \mathcal{S}_1\bar{\mathcal{L}}_1 \\ \bar{\mathcal{L}}_1\mathcal{S}_2 - \mathcal{S}_2\mathcal{L}_1 & 0 \end{pmatrix}, \tag{III.12}$$

$$\Lambda\beta_2 = \begin{pmatrix} 0 & \mathcal{L}_2 \\ \bar{\mathcal{L}}_2 & 0 \end{pmatrix} - \begin{pmatrix} \mathcal{S}_1\bar{\mathcal{L}}_2 & 0 \\ 0 & \mathcal{S}_2\mathcal{L}_2 \end{pmatrix}, \tag{III.13}$$

and

$$\Lambda\vec{F} = \vec{F} - \begin{pmatrix} \mathcal{S}_1(\bar{F}_1 + \bar{F}_2 + \bar{f}_5) \\ \mathcal{S}_1(F_1 + F_2 + f_5) \end{pmatrix}. \tag{III.14}$$

Indeed we shall show the following four statements (3.15(i))-(3.15(iv)):

$$S \text{ can be chosen such that } \Lambda \text{ is invertible in various spaces,} \tag{III.15(i)}$$

i.e. for $\vec{v} = \Lambda \vec{w}$

$$\sup_{|t|\le T} \|\vec{w}(t)\|_{H^s} \le c \sup_{|t|\le T} \|\vec{v}(t)\|_{H^s}, \tag{III.16}$$

and for $N > 0$

$$\begin{aligned} &\|\lambda_N(x) J_x^{s+1/2} \vec{w}\|_{L^2_{x,y,t}} + \|\lambda_N(y) J_y^{s+1/2} \vec{w}\|_{L^2_{x,y,t}} \\ &\le c\{\|\lambda_N(x) J_x^{s+1/2} \vec{v}\|_{L^2_{x,y,t}} + \|\lambda_N(y) J_y^{s+1/2} \vec{v}\|_{L^2_{x,y,t}}\}, \end{aligned} \tag{III.17}$$

$$\begin{pmatrix} 0 & i(\Delta\mathcal{S}_1 + \mathcal{S}_1\Delta) \\ -i(\Delta\mathcal{S}_2 + \mathcal{S}_2\Delta) & 0 \end{pmatrix} + \begin{pmatrix} 0 & \mathcal{L}_2 \\ \bar{\mathcal{L}}_2 & 0 \end{pmatrix} \text{ is } L^2\text{-bounded,} \tag{III.15(ii)}$$

$$\Lambda\beta_1 - \beta_1\Lambda \ , \ \begin{pmatrix} \mathcal{S}_1\bar{\mathcal{L}}_2 & 0 \\ 0 & \mathcal{S}_2\mathcal{L}_2 \end{pmatrix} \ , \ \Lambda B - B\Lambda \text{ are } L^2\text{-bounded,} \tag{III.15(iii)}$$

$$\begin{pmatrix} \mathcal{S}_1(\bar{F}_1 + \bar{F}_2 + \bar{f}_5) \\ \mathcal{S}_2(F_1 + F_2 + f_5) \end{pmatrix} \text{ has “semilinear control,”} \tag{III.15(iv)}$$

i.e.

$$\begin{aligned} &\|\mathcal{S}_1(\bar{F}_1 + \bar{F}_2 + \bar{f}_5)\|_{L^2_{x,y,t}} + \|\mathcal{S}_2(F_1 + F_2 + f_5)\|_{L^2_{x,y,t}} \\ &\le c \sum_{j=1}^{4} (\|p_j\|_{L^2_T L^\infty_{xy}} + \|\nabla p_j\|_{L^2_T L^\infty_{xy}} + \|\phi_j\|_{L^2_T L^\infty_{xy}} + \|\nabla \phi_j\|_{L^2_T L^\infty_{xy}}) \cdot \sum_{k=1}^{4} \|f_4\|_{L^\infty_T L^2_{xy}} \\ &\qquad \sum_{j=1}^{2} \|p_j\|_{L^2_T L^\infty_{xy}} + c\|f_5\|_{L^2_{x,y,t}} \equiv \Phi. \end{aligned} \tag{III.18}$$

Our choice for S is given as follows. Let $\theta \in \mathbb{C}^\infty(\mathbb{R})$, even, and

$$\theta(x) = \begin{cases} 1, & |x| \ge 2, \\ 0, & |x| \le 1, \end{cases} \tag{III.19}$$

and

$$(\Delta_R^{-1} f)^\wedge(\xi_1,\xi_2) = -\frac{1}{\xi_1^2+\xi_2^2}\,\theta\left(\frac{|(\xi_1,\xi_2)|}{R}\right)\hat f(\xi_1,\xi_2), \tag{III.20}$$

and define

$$\mathcal{S}_1 = \frac{1}{2i}\mathcal{L}_2\Delta_R^{-1} \quad , \quad \mathcal{S}_2 = \frac{1}{2i}\bar{\mathcal{L}}_2\Delta_R^{-1}, \tag{III.21}$$

where R is going to be chosen sufficiently large.

Verification of the Properties (3.15(i))–(3.15(iv))

We shall start with (3.15(iv)), estimating $\mathcal{S}_i(F_i)$ and obtain the "semilinear control" for it, see (3.18). For $\mathcal{S}_1(\bar F_1) = \mathcal{S}_1(\bar p_1\partial_x \bar f_1 + \bar p_2 \partial_y \bar f_2)$ we consider first

$$\begin{aligned}\mathcal{S}_1(\bar p_1\partial_x\bar f_1) &= -\frac{1}{2i}\mathcal{L}_2\Delta_R^{-1}(\bar p_1\partial_x\bar f_1)\\ &= -\frac{1}{2i}(\mathcal{L}_2\Delta_R^{-1}\partial_x)(\bar p_1\bar f_1) + \frac{1}{2i}\mathcal{L}_2\Delta_R^{-1}((\partial_x\bar p_1)\bar f_1).\end{aligned} \tag{III.22}$$

For the first term we write

$$\mathcal{L}_2\Delta_R^{-1}\partial_x = \varphi_5\partial_x\widetilde{\partial}_y^{-1}\varphi_6\Delta_R^{-1}\partial_x + \varphi_7\partial_y\widetilde{\partial}_x^{-1}\varphi_8\Delta_R^{-1}\partial_x = \mathrm{I}+\mathrm{II}. \tag{III.23}$$

We claim that both I and II are L^2-bounded. For I we use that

$$\mathrm{I} = \varphi_5\widetilde{\partial}_y^{-1}[\varphi_6;\partial_x]\Delta_R^{-1}\partial_x + \varphi_5\widetilde{\partial}_y^{-1}\varphi_6\partial_x^2\Delta_R^{-1}. \tag{III.24}$$

Since $[\varphi_6;\partial_x]$, $\Delta_R^{-1}\partial_x$, $\widetilde{\partial}_y^{-1}$ are L^2-bdd, the first term in (3.24) is L^2-bounded. For the second term we observe that $\partial_x^2\Delta_R^{-1}$ is L^2-bdd.

For II in (3.23) we proceed similarly using that $\partial_y\partial_x\Delta_R^{-1}$ is L^2-bdd.

Arguing in a similar manner for $\mathcal{S}_1(\bar p_2\partial_y\bar f_2)$ we see that

$$\|\mathcal{S}_1(\bar F_1)\|_{L^2_T L^2_{xy}} \le c\sum_{j=1}^{2}(\|p_j\|_{L^2_T L^\infty_{xy}} + \|\nabla p_j\|_{L^2_T L^\infty_{xy}})\,\|f_j\|_{L^\infty_T L^2_{xy}}, \tag{III.25}$$

which is the desired "semilinear estimate", see (3.18)).

We next estimate $\mathcal{S}_1(\bar F_2)$

$$\begin{aligned}\mathcal{S}_1(\bar F_2) &= \mathcal{S}_1(\bar\phi_1\partial_x\widetilde{\partial}_y^{-1}\bar\phi_2\bar f_3)\\ &= \varphi_5\partial_x\widetilde{\partial}_y^{-1}\varphi_6\Delta_R^{-1}(\bar\phi_1\partial_x\widetilde{\partial}_y^{-1}\bar\phi_2\bar f_3)\\ &\quad+\varphi_7\partial_y\widetilde{\partial}_x^{-1}\varphi_8\Delta_R^{-1}(\bar\phi_3\partial_x\widetilde{\partial}_y^{-1}\bar\phi_4\bar f_3).\end{aligned} \tag{III.26}$$

Since $\widetilde{\partial}_y^{-1}$ is bounded in L^2, we get the same bound as in (3.25) for $\mathcal{S}_1(\bar{F}_1)$. The bound for $\mathcal{S}_1(\bar{F}_2)$ and $\mathcal{S}_1(f_5)$ are similar.

Next we want to verify the property in (3.15(ii)). We look at

$$\mathcal{L}_2 + i(\Delta\mathcal{S}_1 + \mathcal{S}_1\Delta) = \mathcal{L}_2 - \frac{1}{2}\Delta\mathcal{L}_2\Delta_R^{-1} - \frac{1}{2}\mathcal{L}_2\Delta_R^{-1}\Delta, \tag{III.27}$$

and observe that $\Delta_R^{-1}\Delta$ has multiplier

$$\theta\left(\frac{|(\xi_1,\xi_2)|}{R}\right) = 1 - \psi_R(\xi_1,\xi_2). \tag{III.28}$$

Since $\mathcal{L}_2 = \varphi_5\partial_x\widetilde{\partial}_y^{-1}\varphi_6 + \varphi_7\partial_y\widetilde{\partial}_x^{-1}\varphi_8$ we get that $\mathcal{L}_2\psi_R$ is L^2-bounded ($\varphi_6\psi_R$, $\varphi_8\psi_R$ are $S^{-\infty}$ in both variables). Thus

$$-\frac{1}{2}\mathcal{L}_2\Delta_R^{-1}\Delta = -\frac{1}{2}\mathcal{L}_2 + L^2\text{-bdd}. \tag{III.29}$$

Now we consider

$$\frac{1}{2}\mathcal{L}_2 - \frac{1}{2}\Delta\mathcal{L}_2\Delta_R^{-1} = \frac{1}{2}(\mathcal{L}_2 - \Delta\mathcal{L}_2\Delta_R^{-1}). \tag{III.30}$$

We have that

$$\Delta\mathcal{L}_2 = (\partial_x^2 + \partial_y^2)\mathcal{L}_2 = (\partial_x^2 + \partial_y^2)(\varphi_5\partial_x\widetilde{\partial}_y^{-1}\varphi_6 + \varphi_7\partial_y\widetilde{\partial}_x^{-1}\varphi_8). \tag{III.31}$$

The first term in the r.h.s. above can be written as

$$\begin{aligned}\partial_x^2(\varphi_5\partial_x\widetilde{\partial}_y^{-1}\varphi_6) &= (\partial_x^2\varphi_5)\partial_x\widetilde{\partial}_y^{-1}\varphi_6 + \varphi_5\partial_x\widetilde{\partial}_y^{-1}(\partial_x^2\varphi_6)\\ &+2\varphi_5\partial_x\widetilde{\partial}_y^{-1}(\partial_x\varphi_6)\partial_x + \varphi_5\partial_x\widetilde{\partial}_y^{-1}\varphi_6\partial_x^2 + 2(\partial_x\varphi_5)\partial_x^2\widetilde{\partial}_y^{-1}\varphi_6.\end{aligned} \tag{III.32}$$

When we compose on the right with Δ_R^{-1}, all terms except the next to last give bounded operators in L^2. Similarly for the second term in the r.h.s. of (3.31)

$$\partial_x^2(\varphi_7\partial_y\widetilde{\partial}_x^{-1}\varphi_8) = \varphi_7\partial_y\widetilde{\partial}_x^{-1}\varphi_8\partial_x^2 + \text{o.w.c.r. } \Delta_R^{-1} - L^2\text{-bdd}, \tag{III.33}$$

where o.w.c.r. $\Delta_R^{-1} - L^2$ bdd means operators which composed on the right with Δ_R^{-1} are L^2-bdd.

Then, one sees that

$$\Delta\mathcal{L}_2\Delta_R^{-1} = \mathcal{L}_2\Delta\Delta_R^{-1} + L^2\text{-bdd} = \mathcal{L}_2 + L^2\text{-bdd}, \tag{III.34}$$

and thus

$$i(\Delta\mathcal{S}_1 + \mathcal{S}_1\Delta) + \mathcal{L}_2 = L^2\text{-bdd}, \tag{III.35}$$

and similarly

$$-i(\Delta\mathcal{S}_2+\mathcal{S}_2\Delta)+\bar{\mathcal{L}}_2=L^2\text{-bdd}. \tag{III.36}$$

This proves (3.15(ii)).

Next we shall verify (3.15(iii)). First we shall check that $\Lambda\beta_1-\beta_1\Lambda$ is L^2-bdd, by proving that $\mathcal{L}_1\mathcal{S}_1,\ \mathcal{S}_1\bar{\mathcal{L}}_1,\ \bar{\mathcal{L}}_1\mathcal{S}_2$ and $\mathcal{S}_2\mathcal{L}_1$ are L^2-bdd. We recall that

$$\Lambda\beta_1-\beta_1\Lambda=\begin{pmatrix} 0 & \mathcal{L}_1\mathcal{S}_1-\mathcal{S}_1\bar{\mathcal{L}}_1\\ \bar{\mathcal{L}}_1\mathcal{S}_2-\mathcal{S}_2\mathcal{L}_1 & 0\end{pmatrix} \tag{III.37}$$

with $\mathcal{L}_1=\varphi_1\partial_x\widetilde{\partial}_y^{-1}\varphi_2+\varphi_3\partial_y\widetilde{\partial}_x^{-1}\varphi_4,\quad \mathcal{L}_2=\varphi_5\partial_x\widetilde{\partial}_y^{-1}\varphi_6+\varphi_7\partial_y\widetilde{\partial}_x^{-1}\varphi_8$, and

$$\mathcal{S}_1=\frac{-1}{2i}\mathcal{L}_2\Delta_R^{-1}\ ,\ \text{with } \Delta_R^{-1} \text{ defined in (3.20)}. \tag{III.38}$$

We first consider $\mathcal{L}_1\mathcal{S}_1$,

$$\begin{aligned}&-2i\mathcal{L}_1\mathcal{S}_1=\\ &\quad(\varphi_1\partial_x\widetilde{\partial}_y^{-1}\varphi_2+\varphi_3\partial_y\widetilde{\partial}_x^{-1}\varphi_4)(\varphi_5\partial_x\widetilde{\partial}_y^{-1}\varphi_6+\varphi_7\partial_y\widetilde{\partial}_x^{-1}\varphi_8)\Delta_R^{-1}\\ &\quad=\varphi_1\partial_x\widetilde{\partial}_y^{-1}\varphi_2\varphi_5\partial_x\widetilde{\partial}_y^{-1}\varphi_6\Delta_R^{-1}+\varphi_1\partial_x\widetilde{\partial}_y^{-1}\varphi_2\varphi_7\partial_y\widetilde{\partial}_x^{-1}\varphi_8\Delta_R^{-1}\\ &\quad+\varphi_3\partial_y\widetilde{\partial}_x^{-1}\varphi_4\varphi_5\partial_x\widetilde{\partial}_y^{-1}\varphi_6\Delta_R^{-1}+\varphi_3\partial_y\widetilde{\partial}_x^{-1}\varphi_4\varphi_7\partial_y\widetilde{\partial}_x^{-1}\varphi_8\Delta_R^{-1}.\end{aligned} \tag{III.39}$$

We take the first term on the r.h.s. of (3.39).

$$\begin{aligned}&\varphi_1\partial_x\widetilde{\partial}_y^{-1}\varphi_2\varphi_5\partial_x\widetilde{\partial}_y^{-1}\varphi_6\Delta_R^{-1}\ =\ \varphi_1\widetilde{\partial}_y^{-1}(\partial_x\varphi_2)\varphi_5\partial_x\widetilde{\partial}_y^{-1}\varphi_6\Delta_R^{-1}\\ &+\varphi_1\widetilde{\partial}_y^{-1}\varphi_2(\partial_x\varphi_5)\partial_x\widetilde{\partial}_y^{-1}\varphi_6\Delta_R^{-1}+\varphi_1\widetilde{\partial}_y^{-1}\varphi_2\varphi_5\partial_x^2\widetilde{\partial}_y^{-1}\varphi_6\Delta_R^{-1}\\ &=\varphi_1\widetilde{\partial}_y^{-1}(\partial_x\varphi_2)\varphi_5\widetilde{\partial}_y^{-1}(\partial_x\varphi_6)\Delta_R^{-1}+\varphi_1\widetilde{\partial}_y^{-1}(\partial_x\varphi_2)\varphi_5\widetilde{\partial}_y^{-1}\varphi_6\partial_x\Delta_R^{-1}\\ &+\varphi_1\widetilde{\partial}_y^{-1}\varphi_2(\partial_x\varphi_5)\widetilde{\partial}_y^{-1}(\partial_x\varphi_6)\Delta_R^{-1}+\varphi_1\widetilde{\partial}_y^{-1}\varphi_2(\partial_x\varphi_5)\widetilde{\partial}_y^{-1}\varphi_6\partial_x\Delta_R^{-1}\\ &+\varphi_1\widetilde{\partial}_y^{-1}\varphi_2\varphi_5\partial_x^2\widetilde{\partial}_y^{-1}\varphi_6\Delta_R^{-1}.\end{aligned} \tag{III.40}$$

The first four terms in the right hand side of (3.40) are clearly L^2-bdd. For the fifth one we write

$$\begin{aligned}\varphi_1\widetilde{\partial}_y^{-1}\varphi_2\varphi_5\partial_x^2\widetilde{\partial}_y^{-1}\varphi_6\Delta_R^{-1}&=\varphi_1\widetilde{\partial}_y^{-1}\varphi_2\varphi_5\widetilde{\partial}_y^{-1}(\partial_x^2\varphi_6)\Delta_R^{-1}\\ &+2\varphi_1\widetilde{\partial}_y^{-1}\varphi_2\varphi_5\widetilde{\partial}_y^{-1}(\partial_x\varphi_6)\partial_x\Delta_R^{-1}+\varphi_1\widetilde{\partial}_y^{-1}\varphi_2\varphi_5\widetilde{\partial}_y^{-1}\varphi_6\partial_x^2\Delta_R^{-1},\end{aligned} \tag{III.41}$$

which are all L^2-bdd.

The second term in the right-hand side of (3.39) is slightly better because $\partial_x\widetilde{\partial}_x^{-1}$ is L^2-bdd. The third one is like the second one and the fourth like the first one. Thus collecting this information we find that $\mathcal{L}_1\mathcal{S}_1$ is L^2-bdd.

The proof of the L^2-boundedness of $\mathcal{S}_1\bar{\mathcal{L}}_1$, $\bar{\mathcal{L}}_1\mathcal{S}_2$ and $\mathcal{S}_2\mathcal{L}_1$ is similar.

Thus we have completed the proof of the first part of (3.15(iii)) i.e., $\Lambda\beta_1-\beta_1\Lambda$ is L^2 bounded. The proofs of the L^2-boundedness of $\mathcal{S}_1\bar{\mathcal{L}}_2$ and $\mathcal{S}_2\mathcal{L}_2$ are similar. Thus we study

$$\Lambda B - B\Lambda = \begin{pmatrix} 0 & R\mathcal{S}_1 - \mathcal{S}_1 R \\ \bar{R}\mathcal{S}_2 - \mathcal{S}_2 R & 0 \end{pmatrix}. \tag{III.42}$$

It will be shown that $R\mathcal{S}_1,\ \mathcal{S}_1\bar{R},\ \bar{R}\mathcal{S}_2,\ \mathcal{S}_2R$ are L^2-bdd. Thus since

$$\begin{aligned} 2iR\mathcal{S}_1 &= r_1\partial_x((\varphi_5\partial_x\widetilde{\partial}_y^{-1}\varphi_6 + \varphi_7\partial_y\widetilde{\partial}_x^{-1}\partial_8)\Delta_R^{-1}) \\ &\quad + r_2\partial_y((\varphi_5\partial_x\widetilde{\partial}_y^{-1}\varphi_6 + \varphi_7\partial_y\widetilde{\partial}_x^{-1}\varphi_3)\Delta_R^{-1}), \end{aligned} \tag{III.43}$$

the previous argument provides the result. A similar conclusion applies to $\mathcal{S}_1\bar{R}$, $\bar{R}\mathcal{S}_2$ and $\mathcal{S}_2R$.

Finally we shall prove (3.15(i)), i.e., the invertibility of Λ. First we shall see that $\mathcal{S}_1,\ \mathcal{S}_2$ have operator norm on L^2, which tends to zero as $R\uparrow\infty$.

Thus, we consider

$$\mathcal{S}_1 = \frac{1}{2i}(\varphi_5\partial_x\widetilde{\partial}_y^{-1}\varphi_6 + \varphi_7\partial_y\widetilde{\partial}_x^{-1}\varphi_8)\Delta_R^{-1}. \tag{III.44}$$

We take the first term in the r.h.s. of (3.44) and remark that the proof for the second one is similar. Then

$$\varphi_5\partial_x\widetilde{\partial}_y^{-1}\varphi_6\Delta_R^{-1} = \varphi_5\widetilde{\partial}_y^{-1}(\partial_x\varphi_6)\Delta_R^{-1} + \varphi_5\widetilde{\partial}_y^{-1}\varphi_6\partial_x\Delta_R^{-1}. \tag{III.45}$$

Now Δ_R^{-1} and $\partial_x\Delta_R^{-1}$ have norms on L^2 which tend to zero as $R\uparrow\infty$, see (3.20). This proves the invertibility in L^2 of $\Lambda = I - S$.

Next we shall show that for $N>1$, and $s\geq 0$ the operator norm of

$$\lambda_N(x)J_x^s\mathcal{S}_jJ_x^{-s} \quad \text{and} \quad \lambda_N(y)J_y^s\mathcal{S}_jJ_y^{-s}, \quad j=1,2, \tag{III.46}$$

in $L^2([0,T]\times\mathbb{R}_x\times\mathbb{R}_y)$ tend to zero as $R\uparrow\infty$. We start out with

$$\begin{aligned} J_x^s\varphi_5\partial_x\widetilde{\partial}_y^{-1}\varphi_6\Delta_R^{-1}J_x^{-s} &= [J_x^s;\varphi_5]\partial_x\widetilde{\partial}_y^{-1}\varphi_6\Delta_R^{-1}J_x^{-s} \\ &\quad + \varphi_5J_x^s\partial_x\widetilde{\partial}_y^{-1}\varphi_6\Delta_R^{-1}J_x^{-s} = \mathrm{I}+\mathrm{II}\ . \end{aligned} \tag{III.47}$$

For I we observe that $[J_x^s;\varphi_5]\partial_x = L_1$ is an operator of order s in x, uniformly in y so $L_1 = (L_1J_x^{-s})J_x^s$, where $L_1J_x^{-s}$ is an operator of order zero in x uniformly in y, i.e.,

$$(L_1J_x^{-s})f(x,y) = T_yf(\cdot,y)(x), \tag{III.48}$$

where T_y is a classical zero order ψ.d.o. in x with seminorms bounded uniformly in y.

Returning to (3.47), by the continuity of the ψ.d.o. of order zero in $L^2(\lambda_N(x)\,dx)$ (see Lemma 2.4) we have that

$$\begin{aligned}
&\|I(f)\|_{L^2(\mathbb{R}^2\times[0,T]:\lambda_N(x)dxdydt)}\\
&= \|(L_1 J_x^{-s})(J_x^s \widetilde{\partial}_y^{-1}\varphi_6 \Delta_R^{-1} J_x^{-s}) f\|_{L^2(\lambda_N(x)dxdydt)} \qquad \text{(III.49)}\\
&\le C\|(J_x^s \widetilde{\partial}_y^{-1}\varphi_6 \Delta_R^{-1} J_x^{-s}) f\|_{L^2(\lambda_N(x))} = C\|\widetilde{\partial}_y^{-1} J_x^s \varphi_6 \Delta_R^{-1} J_x^{-s}(f)\|_{L^2(\lambda_N(x))}\\
&\le C\|J_x^s \varphi_6 \Delta_R^{-1} J_x^{-s}(f)\|_{L^2(\lambda_N(x))} = C\|J_x^s \varphi_6 J_x^{-s} \Delta_R^{-1}(f)\|_{L^2(\lambda_N(x))}\\
&\le C\|\Delta_R^{-1}(f)\|_{L^2(\lambda_N(x))},
\end{aligned}$$

since $\widetilde{\partial}_y^{-1}$ is bdd in $L^2(dy)$ and $J_x^s \varphi_6 J_x^{-s}$ is a ψ.d.o. of order 0 in x, uniformly in y.

We are now going to prove that for $N \ge 1$

$$\Delta_R^{-1} : L^2(\lambda_N(x)dx\,dy\,dt) \to L^2(\lambda_N(x)dx\,dy\,dt) \qquad \text{(III.50)}$$

is bounded with norm tending to zero as $R \uparrow \infty$ uniformly in $N \le N_0$.

It suffices to see that for $f = f(x,y)$

$$\frac{1}{1+x^2}\Delta_R^{-1}((1+x^2)f) \text{ is bdd in } L^2(dx\,dy), \qquad \text{(III.51)}$$

with norm tending to 0 as $R \uparrow \infty$ (the proof for general $N \in \mathbb{Z}^+$ is similar).

Taking Fourier transform it follows that

$$\begin{aligned}
&\frac{1}{1+x^2}\Delta_R^{-1}((1+x^2)f)\\
&= \frac{1}{1+x^2}\iint e^{i(x\xi_1+y\xi_2)} \frac{\theta\left(\frac{|(\xi_1,\xi_2)|}{R}\right)}{\xi_1^2+\xi_2^2}((1+x^2)f)^\wedge(\xi_1,\xi_2)d\xi_1 d\xi_2\\
&= \frac{1}{1+x^2}\iint e^{i(x\xi_1+y\xi_2)} \frac{\theta\left(\frac{|(\xi_1,\xi_2)|}{R}\right)}{\xi_1^2+\xi_2^2}(I-\partial_{\xi_1}^2)\hat{f}(\xi_1,\xi_2)d\xi_1 d\xi_2\\
&= \frac{1}{1+x^2}\iint (I-\partial_{\xi_1}^2)\left\{e^{i(x\xi_1+y\xi_2)} \frac{\theta\left(\frac{|(\xi_1,\xi_2)|}{R}\right)}{\xi_1^2+\xi_2^2}\right\}\hat{f}(\xi_1,\xi_2)d\xi_1 d\xi_2 \qquad \text{(III.52)}\\
&= \frac{1}{1+x^2}\iint e^{i(x\xi_1+y\xi_2)}(I-\partial_{\xi_1}^2)\left\{\frac{\theta\left(\frac{|(\xi_1,\xi_2)|}{R}\right)}{\xi_1^2+\xi_2^2}\right\}\hat{f}(\xi_1,\xi_2)d\xi_1 d\xi_2\\
&\quad + \frac{x^2}{1+x^2}\iint e^{i(x\xi_1+y\xi_2)} \frac{\theta\left(\frac{|(\xi_1,\xi_2)|}{R}\right)}{\xi_1^2+\xi_2^2}\hat{f}(\xi_1,\xi_2)d\xi_1 d\xi_2\\
&\quad - \frac{2ix}{1+x^2}\iint e^{i(x\xi_1+y\xi_2)}\partial_{\xi_1}\left\{\frac{\theta\left(\frac{|(\xi_1,\xi_2)|}{R}\right)}{\xi_1^2+\xi_2^2}\right\}\hat{f}(\xi_1,\xi_2)d\xi_1 d\xi_2,
\end{aligned}$$

from which the result follows.

Now for II in (3.47) we have

$$\begin{aligned}\mathrm{II}(f) &= \varphi_5 J_x^s \partial_x \widetilde{\partial}_y^{-1} \varphi_6 \Delta_R^{-1} J_x^{-s}(f)\\ &= \varphi_5 \widetilde{\partial}_y^{-1}[J_x^s\partial_x;\varphi_6] J_x^{-s}\Delta_R^{-1}(f) + \varphi_5\widetilde{\partial}_y^{-1}\varphi_6 J_x^s\partial_x J_s^{-s}\Delta_R^{-1}(f)\\ &= \mathrm{II}_1(f)+\mathrm{II}_2(f).\end{aligned} \tag{III.53}$$

In $\mathrm{II}_1(\cdot)$, $[J_x^s\partial_x;\varphi_6]J_x^{-s}$ is a ψ.d.o. of order 0 in x, uniformly in y and we proceed as for $\mathrm{I}(\cdot)$. For $\mathrm{II}_2(\cdot)$, we combine that $J_x^s\partial_x J_x^{-s}\Delta_R^{-1} = \partial_x\Delta_R^{-1}$ and that $\partial_x\Delta_R^{-1}$ is $L^2(\lambda_N(x)dx\,dy\,dt)$ is bounded with norm tending to 0 as $R\uparrow\infty$. The proof of this last fact is similar to that in (3.52).

The other piece of $\mathcal{S}_1$ in (3.44) corresponds to

$$\begin{aligned}&J_x^s\varphi_7\partial_y\widetilde{\partial}_x^{-1}\varphi_8\Delta_R^{-1}J^{-s}\\ &= [J_x^s;\varphi_7]\partial_y\widetilde{\partial}_x^{-1}\varphi_8\Delta_R^{-1}J_x^{-s} + \varphi_7 J_x^s\partial_y\widetilde{\partial}_x^{-1}\varphi_8\Delta_R^{-1}J_x^{-s}\\ &= ([J_x^s;\varphi_7]J_x^{-s})J_x^s\partial_y\widetilde{\partial}_x^{-1}\varphi_8\Delta_R^{-1}J_x^{-s} + \varphi_7 J_x^s\partial_y\widetilde{\partial}_x^{-1}\varphi_8\Delta_R^{-1}J_x^{-s}\\ &= \mathrm{I}'+\mathrm{II}'.\end{aligned} \tag{III.54}$$

Now for I' we write

$$\begin{aligned}\mathrm{I}' &= [J_x^s;\varphi_7]J_x^{-s}\partial_y J_x^s\widetilde{\partial}_x^{-1}\varphi_8 J_x^{-s}\Delta_R^{-1}\\ &= ([J_x^s;\varphi_7]J_x^{-s})J_x^s\widetilde{\partial}_x^{-1}(\partial_y\varphi_8)J_x^{-s}\Delta_R^{-1}\\ &\quad+([J_x^s;\varphi_7]J_x^{-s})J_x^s\widetilde{\partial}_x^{-1}\varphi_8 J_x^{-s}\partial_y\Delta_R^{-1}.\end{aligned} \tag{III.55}$$

Since $[J_x^s;\varphi_7]\widetilde{\partial}_x^{-1}(\partial_y\varphi_8)J_x^{-s}$ is a ψ.d.o. of order 0 (in fact, of order -2) in x, uniformly in y we can handle the bound of the first term in the r.h.s. of (3.55) as that for I in the previous case. For the second term in the r.h.s. of (3.55) we see that $[J_x^s;\varphi_7]\widetilde{\partial}_x^{-1}\varphi_8 J_x^{-s}$ is a ψ.d.o of order 0 (in fact, of order -2) in x uniformly in y and $\partial_y\Delta_R^{-1}$ is $L^2(\lambda_N(x)dxdydt)$-bounded with norm tending to zero as $R\uparrow\infty$.

Finally we look at

$$\begin{aligned}\mathrm{II}' &= \varphi_7 J_x^s\partial_y\widetilde{\partial}_x^{-1}\varphi_8 J_x^{-s}\Delta_R^{-1}\\ &= \varphi_7 J_x^s\widetilde{\partial}_x^{-1}(\partial_y\varphi_8)J_x^{-s}\Delta_R^{-1} + \varphi_7 J_x^s\widetilde{\partial}_x^{-1}\varphi_8 J_x^{-s}\partial_y\Delta_R^{-1}.\end{aligned} \tag{III.56}$$

For the first term in the right hand side of (3.56) we use that $\varphi_7 J_x^s\widetilde{\partial}_x^{-1}(\partial_y\varphi_8)$ J_x^{-s} is of order 0 in x (in fact, or order -1) uniformly in y, and for the second one we use that $\varphi_7 J_x^s\widetilde{\partial}_x^{-1}\varphi_8 J^{-s}$ is of order 0 in x uniformly in y, together with a previous argument. By symmetry we have finished the proof of (3.15(i))–(3.15(iv)) and concluded the diagonalization.

Thus we have that $\vec{v} = \Lambda\vec{w} = \Lambda(z\ \ \bar{z})^T$ verifies the system

$$\partial_t\vec{v} = iH\vec{v} + B\vec{v} + \beta_1\vec{v} + C_1\vec{v} + \vec{G}, \tag{III.57}$$

with H, B, β_1 as in (3.6)–(3.7), C an operator in the (x,y)-variables, which is L^2-bounded (indeed H^2-bdd) and

$$\vec{G} = \vec{F} + \vec{E}, \tag{III.58}$$

where

$$\|\vec{E}\|_{L^2_{x,y,t}} \leq \Phi \text{ defined in (3.18)}, \tag{III.59}$$

and where

$$F_1 = p_1\partial_x f_1 + p_2\partial_y f_2, \quad F_2 = \phi_1\partial_x\widetilde{\partial}_y^{-1}\phi_2 f_3 + \phi_3\partial_y\widetilde{\partial}_x^{-1}\phi_4 f_4, \tag{III.60}$$

and

$$\vec{F} = \begin{pmatrix} F_1 + F_2 + f_5 \\ \bar{F}_1 + \bar{F}_2 + \bar{f}_5 \end{pmatrix}. \tag{III.61}$$

Thus we are reduced to proving the estimates for $\vec{v}$, which solves the "diagonal system" (3.57).

IV Proof of Theorem 1.2 (Conclusion)

In this section we complete the proof of Theorem 1.2. From the results in the previous section for all practical purpose to work with "diagonal system" (3.57) is equivalent to work with the single equation

$$\begin{aligned} \partial_t z &= i\Delta z + r_1\partial_x z + r_2\partial_y z + \varphi_1\partial_x\widetilde{\partial}_y^{-1}\varphi_2 z + \varphi_3\partial_y\widetilde{\partial}_x^{-1}\varphi_4 z \\ &\quad + c_1 z + \phi_1\partial_x f_1 + \phi_2\partial_y f_2 + \phi_3\partial_x\widetilde{\partial}_y^{-1}\phi_4 f_3 + \phi_5\partial_y\widetilde{\partial}_x^{-1}\phi_6 f_4 + f_5 \\ &= i\Delta z + \vec{r}\cdot\nabla z + \varphi_1\partial_x\widetilde{\partial}_y^{-1}\varphi_2 z + \varphi_3\partial_y\widetilde{\partial}_x^{-1}\varphi_4 z + C_1 z + \Gamma, \end{aligned} \tag{IV.1}$$

with C_1 bounded in L^2, $f_5 \in L^2_T L^2_{xy}$ and $z(0) = z_0(x)$.

We introduce classical ψ.d.o. in each variable. First we have $C_x(x, D_x)$, whose symbol is

$$C_{M,R}(x,\xi_1) = \exp\left(\frac{-M}{2}\int_0^x \mu^2(s)ds\ \frac{\xi_1}{|\xi_1|}\theta^2\left(\frac{\xi_1}{R}\right)\right) \tag{IV.2}$$

with $\theta(\cdot)$ defined in (3.19), and $\mu(\cdot) \in C^\infty$, an even function, $\mu \in L^2([0,\infty))$, with a decay at infinity to be determined. Clearly $C_x \in S^0(\mathbb{R}^2)$. Similarly we define $C_y(y, D_y)$.

We observe that the symbol of $\partial_x^2 C_x = \sigma(\partial_x^2 C_x)$ is

$$\sigma(\partial_x^2 C_x) = -\xi_1^2 C_{M,R}(x,\xi_1) + 2i\xi_1\partial_x C_{M,R}(x,\xi_1) + \partial_x^2 C_{M,R}(x,\xi_1), \tag{IV.3}$$

and

$$\sigma(C_x\partial_x^2) = -\xi_1^2 C_{M,R}(x,\xi_1). \tag{IV.4}$$

Therefore

$$\begin{aligned}\sigma(i[C_x\partial_x^2 - \partial_x^2 C_x]) &= 2\xi_1\partial_x C_{M,R}(x,\xi_1) + L^2\text{-bdd}\\ &= -M\mu^2(x)|\xi_1|\theta^2\left(\tfrac{\xi_1}{R}\right)C_{M,R}(x,\xi_1) + L^2\text{-bdd}.\end{aligned} \tag{IV.5}$$

Define the (self adjoint) operator C as

$$C = (C_xC_y)^* C_xC_y = C_x^*C_xC_y^*C_y, \tag{IV.6}$$

and

$$A = C^2 = C^*C = C_x^*C_xC_x^*C_xC_y^*C_yC_y^*C_y = A_xA_y. \tag{IV.7}$$

We shall compute

$$\begin{aligned}\partial_t\langle Az,z\rangle &= \langle A\partial_t z,z\rangle + \langle Az,\partial_t z\rangle\\ &= \langle iA\Delta z,z\rangle + \langle A\vec{r}\cdot\nabla z,z\rangle + \langle A(\varphi_1\partial_x\widetilde{\partial}_y^{-1}\varphi_2)z,z\rangle\\ &\quad+\langle A\varphi_3\partial_y\widetilde{\partial}_x^{-1}\varphi_4 z,z\rangle + \langle AC_1z,z\rangle + \langle A\Gamma,z\rangle\\ &\quad+\langle Az,i\Delta z\rangle + \langle Az,\vec{r}\cdot\nabla z\rangle + \langle Az,\varphi_1\partial_x\widetilde{\partial}_y^{-1}\varphi_2 z\rangle\\ &\quad+\langle Az,\varphi_3\partial_y\widetilde{\partial}_x^{-1}\varphi_4 z\rangle + \langle Az,C_1z\rangle + \langle Az,\Gamma\rangle.\end{aligned} \tag{IV.8}$$

We shall use that

$$\langle iA\Delta z,z\rangle + \langle Az,i\Delta z\rangle = \langle i[A\Delta - \Delta A]z,z\rangle, \tag{IV.9}$$

$$i[A\Delta - \Delta A] = i(A\partial_x^2 - \partial_x^2 A) + i(A\partial_y^2 - \partial_y^2 A), \tag{IV.10}$$

$$\sigma(i(A\partial_x^2 - \partial_x^2 A)) = \left(-4M\mu^2(x)|\xi_1|\theta^2\left(\frac{\xi_1}{R}\right)\right)a_xa_y + \text{bdd-}L^2, \tag{IV.11}$$

$$\sigma(i(A\partial_y^2 - \partial_y^2 A)) = \left(-4M\mu^2(y)|\xi_2|\theta^2\left(\frac{\xi_2}{R}\right)\right)a_xa_y + \text{bdd-}L^2, \tag{IV.12}$$

$$\begin{cases}\sigma(A_x) = a_x(x,\xi_1) = C_{4M,R}(x,\xi_1) + S_x^{-1},\\ \sigma(A_y) = a_y(y,\xi_2) = C_{4M,R}(y,\xi_2) + S_y^{-1},\end{cases} \tag{IV.13}$$

with $S_x^{-1}, S_y^{-1} \in S^{-1}(\mathbb{R}^2)$ (see Theorem 2.2). Also

$$\langle Az, z\rangle = \langle C^*Cz, z\rangle = \langle Cz, Cz\rangle, \tag{IV.14}$$

and

$$\begin{aligned}\langle A\vec{r}\cdot\nabla z, z\rangle + \langle Az, \vec{r}\cdot\nabla z\rangle &= \langle A\vec{r}\cdot\nabla z, z\rangle + \langle z, A\vec{r}\cdot\nabla z\rangle \\ &= 2\,\mathrm{Re}\langle A\vec{r}\cdot\nabla z, z\rangle. \end{aligned}\tag{IV.15}$$

The sum of the third and ninth terms in the right hand side of (4.8) gives $2\,\mathrm{Re}\langle A\varphi_1\partial_x\widetilde{\partial}_y^{-1}\varphi_2 z, z\rangle$, the sum of the 4th and 10th gives $2\,\mathrm{Re}\langle A\varphi_3\partial_y\widetilde{\partial}_x^{-1}\varphi_4 z, z\rangle$, the sum of the 5th and 11th gives $2\,\mathrm{Re}\langle AC_1 z, z\rangle$, and finally the sum of 6th and 12th gives $2\,\mathrm{Re}\langle A\Gamma, z\rangle$.

In order to apply the vector value sharp Gårding inequality (Theorem 2.5) we write out operators in a vector valued form. Thus from (4.11)-(4.13)

$$\sigma(i[A\partial_x^2 - \partial_x^2 A]) = -4M\mu^2(x)|\xi_1|\theta^2\left(\frac{\xi_1}{R}\right)a_x a_y + \text{bdd-}L^2, \tag{IV.16}$$

and

$$-4M\mu^2(x)|\xi_1|\theta^2\left(\frac{\xi_1}{R}\right)a_x = -4M\mu^2(x)|\xi_1|\theta^2\left(\frac{\xi_1}{R}\right)C_{4M,R}(x,\xi_1) + L_0, \tag{IV.17}$$

where $L_0 = L_0(x,\xi_1)$ is the symbol of a bdd operator in $L^2(dx)$. Thus, modulo L^2-bdd operator (L_0) we have

$$\begin{aligned} &i[A\partial_x^2 - \partial_x^2 A]f(x,y) \\ &= \int e^{ix\xi_1}(-4M\mu^2(x)|\xi_1|\theta^2\left(\tfrac{\xi_1}{R}\right)C_{4M,R}(x,\xi_1)(A_y f(x,-))^{\wedge_x}(\xi_1))d\xi_1. \end{aligned}\tag{IV.18}$$

Now

$$\begin{aligned}(A_y f(x,-))^{\wedge_x}(\xi_1) &= \textstyle\int e^{-ix\xi_1}A_y f(x,-)(y)dx \\ &= A_y\left(\textstyle\int e^{-ix\xi_1}f(x,-)dx\right)(y).\end{aligned}\tag{IV.19}$$

Thus, $i[A\partial_x^2 - \partial_x^2 A]$ has vector valued symbol (modulo L^2_{xy}-bdd operator)

$$-4M\mu^2(x)|\xi_1|\theta^2\left(\frac{\xi_1}{R}\right)C_{4M,R}(x,\xi_1)A_y. \tag{IV.20}$$

Next we look at the term $\langle Ar_1\partial_x z, z\rangle = \langle A_xA_y r_1\partial_x z, z\rangle$. We recall that $r_1 = r_1(x,y)$ decays in x, uniformly in y, then we write

$$r_1(x,y) = \lambda_N^2(x)\lambda_N^{-2}(x)r_1(x,y) = \lambda_N(x)\tilde{r}_1(x,y), \tag{IV.21}$$

and

$$\begin{aligned} A_xA_yr_1\partial_x &= A_xA_y\lambda_N^2(x)\tilde{r}_1\partial_x = A_x\lambda_N^2(x)A_y\tilde{r}_1\partial_x \\ &= A_x\lambda_N^2(x)\partial_xA_y\tilde{r}_1 - A_x\lambda_N^2(x)A_y(\partial_x\tilde{r}_1) \\ &= \lambda_N^2(x)\partial_xA_xA_y\tilde{r}_1 + L^2_{xy}\text{-bdd.} \end{aligned} \tag{IV.22}$$

Similarly

$$\begin{aligned} &A\varphi_1\partial_x\widetilde{\partial}_y^{-1}\varphi_2 = A_xA_y\varphi_1\partial_x\widetilde{\partial}_y^{-1}\varphi_2 \\ &= A_xA_y\lambda_N^2(x)\widetilde{\varphi}_1\partial_x\widetilde{\partial}_y^{-1}\varphi_2 = A_x\lambda_N^2(x)A_y\widetilde{\varphi}_1\partial_x\widetilde{\partial}_y^{-1}\varphi_2 \\ &= A_x\lambda_N^2(x)\partial_xA_y\widetilde{\varphi}_1\widetilde{\partial}_y^{-1}\varphi_2 + L^2\text{-bdd} \\ &= \lambda_N^2(x)\partial_xA_xA_y\widetilde{\varphi}_1\widetilde{\partial}_y^{-1}\varphi_2 + L^2\text{-bdd.} \end{aligned} \tag{IV.23}$$

Thus modulo L^2_{xy}-bdd operators we have the vector value symbols

$$\sigma(Ar_1\partial_x) = \lambda_N^2(x)(i\xi_1)C_{4M,R}(x,\xi_1)A_y\tilde{r}_1 + L^2\text{-bdd,} \tag{IV.24}$$

and

$$\sigma(A\varphi_1\partial_x\widetilde{\partial}_y^{-1}\varphi_2) = \lambda_N^2(x)(i\xi_1)C_{4M,R}(x,\xi_1)A_y\widetilde{\varphi}_1\widetilde{\partial}_y^{-1}\varphi_2 + L^2\text{-bdd.} \tag{IV.25}$$

Note that all these operators are of order 1.

In order to apply the vector valued sharp Gårding inequality (Theorem 2.5) we make the following claims :

Claim 1 *We can choose M, R, μ so that for $|\xi_1|$ large*

$$\begin{aligned} &-M\mu^2(x)|\xi_1|\theta^2\left(\tfrac{\xi_1}{R}\right)C_{4M,R}(x,\xi_1)\{A_y + A_y^*\} \\ &\le \lambda_N^2(x)i\xi_1C_{4M,R}(x,\xi_1)A_y\tilde{r}_1 - \lambda_N^2(x)i\xi_1C_{4M,R}(x,\xi_1)\overline{\tilde{r}_1}A_y^*, \end{aligned} \tag{IV.26}$$

as operators on $L^2(\mathbb{R} : dy)$, and

$$\begin{aligned} &-M\mu^2(x)|\xi_1|\theta^2\left(\tfrac{\xi_1}{R}\right)C_{4M,R}(x,\xi_1)\{A_y + A_y^*\} \\ &\le i\xi_1\lambda_N^2(x)C_{4M,R}(x,\xi_1)A_y\widetilde{\varphi}_1\widetilde{\partial}_y^{-1}\varphi_2 \\ &\quad -i\xi_1\lambda_N^2(x)C_{4M,R}(x,\xi_1)\{\bar{\varphi}_2(\widetilde{\partial}_y^{-1})^*\overline{\widetilde{\varphi}_1}A_y^*\}, \end{aligned} \tag{IV.27}$$

as operators on $L^2(\mathbb{R} : dy)$.

Claim 2 *With M, R, μ chosen as in Claim 1, we can choose R even larger so that C_x is invertible in $L^2(dx)$.*

Proof of Claim 1. Since

$$A_y = C_y^*C_yC_y^*C_y \quad \text{then} \quad A_y^* = A_y. \tag{IV.28}$$

Thus, we take $f \in L^2(\mathbb{R} : dy)$ and want to show that for ξ_1 large

$$\begin{aligned} &-2M\mu^2(x)|\xi_1|\theta^2\left(\tfrac{\xi}{R}\right)C_{4M,R}(x,\xi_1)\langle A_y f, f\rangle \\ &\le 2\,\mathrm{Re}\langle -\lambda_N^2(x) i\xi_1 C_{4M,R}(x,\xi_1)A_y\tilde{r}_1 f, f\rangle \\ &= 2\,\mathrm{Re}\{-\lambda_N^2(x) i\xi_1 C_{4M,R}(x,\xi_1)\langle A_y\tilde{r}_1 f, f\rangle\}. \end{aligned} \tag{IV.29}$$

It suffices to show that for ξ_1 large

$$\begin{aligned} &\left|\lambda_N^2(x)|\xi_1|C_{4M,R}(x,\xi_1)\langle A_y\tilde{r}_1 f, f\rangle\right| \\ &\le M\mu^2(x)|\xi_1|\theta^2\left(\tfrac{\xi_1}{R}\right)C_{4M,R}(x,\xi_1)\langle A_y f, f\rangle \end{aligned} \tag{IV.30}$$

or

$$\lambda_N^2(x)|\langle A_y\tilde{r}_1 f, f\rangle| \le M\mu^2(x)\theta^2\left(\frac{\xi_1}{R}\right)\langle A_y f, f\rangle \tag{IV.31}$$

for ξ_1 large. Thus, for ξ_1 such that $|\xi_1| \ge 2R$ (4.31) becomes (see (3.19))

$$\lambda_N^2(x)|\langle A_y\tilde{r}_1 f, f\rangle| \le M\mu^2(x)\langle A_y f, f\rangle. \tag{IV.32}$$

Choosing

$$\mu^2(x) = \lambda_N^2(x), \qquad N > 1, \tag{IV.33}$$

we reduce the proof of (4.26) to see that

$$|\langle A_y\tilde{r}_1 f, f\rangle| \le M\langle A_y f, f\rangle. \tag{IV.34}$$

We recall that $A_y = C_y^* C_y C_y^* C_y$, and so

$$\langle A_y f, f\rangle = \langle C_y^* C_y f, C_y^* C_y f\rangle = \|C_y^* C_y f\|^2_{L^2_y}. \tag{IV.35}$$

Now

$$|\langle A_y\tilde{r}_1 f, f\rangle| = |\langle C_y^* C_y\tilde{r}_1 f, C_y^* C_y f\rangle| \le \|C_y^* C_y\tilde{r}_1 f\|_{L^2_y}\|C_y^* C_y f\|_{L^2_y}. \tag{IV.36}$$

Thus we just need to establish the inequality

$$\|C_y^* C_y\tilde{r}_1 f\|_{L^2_y} \le M\|C_y^* C_y f\|_{L^2_y}, \quad \text{uniformly in } x. \tag{IV.37}$$

Letting $g = C_y^* C_y f$, it suffices to show that there exist M, R so that

$$\|C_y^* C_y\tilde{r}_1(C_y^* C_y)^{-1}g\|_{L^2_y} \le M\|g\|_{L^2_y}, \tag{IV.38}$$

uniformly in x. We recall that

$$\sigma(C_y) = C_{M,R}(y,\xi_2) = \exp\left(\frac{-M}{2}\int_0^y \mu^2(s)ds\,\frac{\xi_2}{|\xi_2|}\theta^2\left(\frac{\xi_2}{R}\right)\right). \tag{IV.39}$$

We first choose M such that

$$|\tilde{r}_1(x,y)| \le \frac{M}{10}. \tag{IV.40}$$

With M so chosen, we will choose R. First we compute $C_y^* C_y$. From Theorem 2.2 it follows that

$$\sigma(C_y^*) = C_{M,R}(y,\xi_2) + q_1(y,\xi_2), \tag{IV.41}$$

where $q_1(y,\xi_2)$ involves

$$\begin{aligned}\partial_{\xi_2}\partial_y C_{M,R}(y,\xi_2) &= \partial_{\xi_2}\partial_y \left(\exp\left(\tfrac{-M}{2}\int_0^y \mu^2(s)ds\ \tfrac{\xi_2}{|\xi_2|}\theta^2\left(\tfrac{\xi_2}{R}\right)\right)\right)\\ &= -M\mu^2(y)\partial_{\xi_2}\left\{\tfrac{\xi_2}{|\xi_2|}\theta^2\left(\tfrac{\xi_2}{R}\right)\right\} C_{M,R}(y,\xi_2)\\ &= -2M\mu^2(y)\tfrac{\xi_2}{|\xi_2|}\tfrac{1}{R}\theta\left(\tfrac{\xi_2}{R}\right)\theta'\left(\tfrac{\xi_2}{R}\right) C_{M,R}(y,\xi_2).\end{aligned} \tag{IV.42}$$

Note that the S^0 seminorms of $C_{M,R}(y,\xi_2)$ are uniformly bounded, depending only on M for $R \ge 1$. Thus the S^0 seminorms of $q_1(y,\xi_2)$ are c_M/R, for $R \ge 1$. Hence

$$C_y^* = C_y + E_0, \quad \text{with} \quad |||E_0||| \le \frac{c_M}{R}, \tag{IV.43}$$

with $|||\cdot|||$ denoting the operator norm in $L^2(\mathbb{R} : dy)$. Thus

$$C_y^* C_y = C_y C_y + E C_y = C_y C_y + E_0', \tag{IV.44}$$

where $E_0' = E_0 C_y$ inherits the boundedness property of E_0 in (4.43). Now we compute $C_y C_y$

$$\sigma(C_y C_y) = C_{2M,R}(y,\xi_2) + q_2(y,\xi_2), \tag{IV.45}$$

where $q_2(\cdot,\cdot)$ depends on $\partial_{\xi_2} C_{M,R}$ and $\partial_y C_{M,R}$, (see Theorem 2.2). A computation similar to that in (4.42) gives

$$|\partial_y C_{M,R}(y,\xi_2)| \le c_M, \qquad |\partial_{\xi_2} C_{M,R}(y,\xi_2)| \le c_M/R, \tag{IV.46}$$

for $R \ge 1$. Therefore combining (4.44)-(4.46) it follows that

$$\sigma(C_y^* C_y) = \exp\left(-M\int_0^y \mu^2(s)ds\ \frac{\xi_2}{|\xi_2|}\theta^2\left(\frac{\xi_2}{R}\right)\right) + e(y,\xi_2), \tag{IV.47}$$

where the operator $E(y,D_y)$ with symbol $e(y,\xi_2)$ has operator norm $|||\cdot|||$ satisfying

$$|||E||| \le \frac{C_M}{R} \quad \text{in} \quad L^2(dy). \tag{IV.48}$$

Now we define the operator $S = S(y, D_y)$ by its symbol $s(y, \xi_2)$ as

$$s(y,\xi_2) = \exp\left(M \int_0^y \mu^2(s)ds \, \frac{\xi_2}{|\xi_2|}\theta^2\left(\frac{\xi_2}{R}\right)\right). \tag{IV.49}$$

As before one sees that its operator norm $|||S|||$ in $L^2(dy)$ satisfies

$$|||S||| \le C_M, \quad \text{for} \quad R > 1. \tag{IV.50}$$

The same argument gives that

$$C_y^* C_y S = I + E_1, \quad \text{and} \quad S C_y^* C_y = I + E_2, \tag{IV.51}$$

with

$$|||E_j||| \le \frac{C_M}{R}, \quad j = 1, 2, \quad \text{and} \quad R > 1. \tag{IV.52}$$

For M fixed we choose R large enough such that

$$\frac{C_M}{R} \le 1/2, \tag{IV.53}$$

and get that for

$$T_1 = (I + E_1)^{-1}, \qquad T_2 = (I + E_2)^{-1}, \tag{IV.54}$$

one has

$$|||T_j||| \le 2, \quad j = 1, 2, \qquad T_2 S = S T_1 = (C_y^* C_y)^{-1}. \tag{IV.55}$$

Also

$$T_1 = I + E_3, \quad \text{with} \quad |||E_3||| < \frac{C_M}{R}, \tag{IV.56}$$

and

$$S T_1 = S + S E_3 \quad \text{with} \quad |||E_3'||| \le \frac{C_M}{R}. \tag{IV.57}$$

Thus

$$(C_y^* C_y)^{-1} = S + E_3'. \tag{IV.58}$$

Now we proceed to estimate the norm of

$$C_y^* C_y \tilde{r}_1 (C_y^* C_y)^{-1} = C_y^* C_y \tilde{r}_1 S + C_y^* C_y \tilde{r}_1 E_3'. \tag{IV.59}$$

We notice that

$$|||C_y^* C_y \tilde{r}_1 E_3'||| \le \frac{C_M}{R}, \text{ as operator in } L^2(dy), \text{ uniformly in } x, \tag{IV.60}$$

and

$$\sigma(\tilde{r}_1 S) = \tilde{r}_1(x,y)\exp\left(M\int_0^y \mu^2(s)ds\ \frac{\xi_2}{|\xi_2|}\theta^2\left(\frac{\xi_2}{R}\right)\right). \tag{IV.61}$$

Since modulo L^2-bdd operator with norm bounded by C_M/R, $C_y^*C_y\tilde{r}_1S$ is the product of the symbols $C_{M,2R}(y,\xi_2)$ and $\tilde{r}_1(x,y)C_{M,-2R}(y,\xi_2)$ we get

$$C_y^*C_y\tilde{r}_1(C_y^*C_y)^{-1} = \tilde{r}_1 + C_y^*C_y\tilde{r}_1E_3' + E_4, \tag{IV.62}$$

where the seminorms of the symbol of E_4 are controlled by products of the one of $\partial_{\xi_2}C_{2M,R}$ and those of $\partial_y(C_{-2M,R}(\cdot)\tilde{r}_1(x,-))$, uniformly in x. Hence

$$|||E_4||| \leq \frac{C_M}{R}, \quad \text{as operator in } L^2(dy) \text{ uniformly in } x, \tag{IV.63}$$

and consequently form (4.60), (4.63)

$$C_y^*C_y\tilde{r}_1(C_y^*C_y)^{-1} = \tilde{r}_1 + E_5, \tag{IV.64}$$

with

$$|||E_5||| \leq \frac{C_M}{R}, \quad \text{as operator in } L^2(dy), \text{ uniformly in } x. \tag{IV.65}$$

Since

$$|\tilde{r}_1(x,y)| \leq \frac{M}{10}, \tag{IV.66}$$

we obtain (4.38) by fixing R such that $C_M/R \leq M/10$. This proves the first part of Claim1, i.e. the inequality (4.26).

We turn to the proof of (4.27), the second part of Claim 1, i.e. for $f \in L^2(\mathbb{R} : dy)$

$$\begin{aligned}\mathrm{Re}\{-M\mu^2(x)|\xi_1|\theta^2\left(\tfrac{\xi_1}{R}\right)C_{4M,R}(x,\xi_1)\langle A_yf,f\rangle\}\\ \leq \mathrm{Re}\{i\xi_1\lambda_N^2(x)C_{4M,R}\langle A_y\widetilde{\varphi}_1\widetilde{\partial}_y^{-1}\varphi_2f,f\rangle\}.\end{aligned} \tag{IV.67}$$

We proceed as in (4.28)–(4.34). By taking ξ_1 such that $|\xi_1| \geq 2R$ we need to show that

$$|\langle A_y\widetilde{\varphi}_1\widetilde{\partial}_y^{-1}\varphi_2f,f\rangle| \leq M\langle A_yf,f\rangle \tag{IV.68}$$

or

$$|\langle C_y^*C_y\widetilde{\varphi}_1\widetilde{\partial}_y^{-1}\varphi_2f, C_y^*C_yf\rangle| \leq M\|C_y^*C_yf\|_{L_y^2}^2. \tag{IV.69}$$

Thus, it suffices to show that

$$|||C_y^* C_y \widetilde{\varphi}_1 \widetilde{\partial}_y^{-1} \varphi_2 (C_y^* C_y)^{-1}||| \le M, \quad \text{in } L^2(dy) \text{ uniformly in } x. \tag{IV.70}$$

We saw in (4.56)-(4.58) that

$$(C_y^* C_y)^{-1} = S + E_3', \quad \text{with } |||E_3'||| \le C_M/R, \tag{IV.71}$$

so we shall show that M can be chosen so that for R large enough

$$|||C_y^* C_y \widetilde{\varphi}_1 \widetilde{\partial}_y^{-1} \varphi_2 S||| \le M/2, \text{ as operator in } L^2(dy), \text{ uniformly in } x. \tag{IV.72}$$

From (4.47) one has that

$$\sigma(C_y^* C_y) = \exp\left(-M \int_0^y \mu^2(s) ds \, \frac{\xi_2}{|\xi_2|} \theta^2\left(\frac{\xi_2}{R}\right)\right) + e(y, \xi_2), \tag{IV.73}$$

with $\sigma(E(y, D_y)) = e(y, \xi_2)$ and $|||E||| \le C_M/R$. We notice that

$$\varphi_2 S = S \varphi_2 + E_6, \quad \text{with } |||E_6||| \le C_M/R, \tag{IV.74}$$

and

$$\begin{aligned} \sigma(C_y^* C_y \widetilde{\varphi}_1) &= \widetilde{\varphi}_1(x, y) \exp\left(-M \textstyle\int_0^y \mu^2(x) ds \, \frac{\xi_2}{|\xi_2|} \theta^2\left(\frac{\xi_2}{R}\right)\right) + e_7(x, y, \xi_2) \\ &= \widetilde{\varphi}_1(x, y) C_{2M,R}(y, \xi_2) + e_7(x, y, \xi_2), \end{aligned} \tag{IV.75}$$

with $\sigma(E_7^x(y, D_y)) = e_7(x, y, \xi_2)$ satisfying that

$$|||E_7||| \le \frac{C_M}{R}, \text{ as operator in } L^2(dy), \text{ uniformly in } x. \tag{IV.76}$$

Thus the problem has been reduced to show that

$$|||\widetilde{\varphi}_1 C_{2M,R}(y, D_y) \widetilde{\partial}_y^{-1} C_{-2M,R}(y, D_y) \varphi_2||| \le M/4, \tag{IV.77}$$

as operator in $L^2(dy)$ uniformly in x. But we can choose $M > 1$ such that

$$\|\widetilde{\varphi}_1\|_{L^\infty(\mathbb{R}^2)}, \|\varphi_2\|_{L^\infty(\mathbb{R}^2)} \le \frac{M^{1/4}}{10}. \tag{IV.78}$$

Thus we see that it suffices to show that

$$|||C_{-2M,R}(y, D_y) \widetilde{\partial}_y^{-1} C_{-2M,R}(y, D_y)||| \le \frac{M^{1/4}}{10}, \tag{IV.79}$$

with M large and R chosen after M. We recall that

$$\sigma(\widetilde{\partial}_y^{-1}) = \sum_{j=0}^{N_0} \psi_j^1(\cdot) m(\xi_2) \psi_j^2(\cdot), \tag{IV.80}$$

where $\psi_j^1,\ \psi_j^2$'s are multiplication operators by bounded smooth functions and in $m(\cdot)$ is a multiplier in S_M^{-1}. So we can reduce ourselves to show that

$$|||C_{2M,R}(y,D_y)m(D_y)C_{-2M,R}(y,D_y)||| \leq \frac{M^{1/4}}{10}. \tag{IV.81}$$

From Theorem 2.2 we have

$$\sigma(m(D_y)C_{-2M,R}(y,D_y)) = a(y,\xi_2) + q(y,\xi_2), \tag{IV.82}$$

where

$$a(y,\xi_2) = m(\xi_2)\exp\left(-M\int_0^y \mu^2(s)ds\frac{\xi_2}{|\xi_2|}\theta^2\left(\frac{\xi_2}{R}\right)\right), \tag{IV.83}$$

and

$$q(y,\xi_2) = \int_0^1 q_\delta(y,\xi_2)d\theta, \tag{IV.84}$$

with

$$q_\delta(y,\xi_2) = \iint e^{-iz\eta}m'(\xi_2+\delta\eta)\partial_y C_{-2M,R}(y+z,\xi_2)dz\,d\eta. \tag{IV.85}$$

Now

$$\partial_y C_{-2M,R}(y,\xi_2) = C_{-2M,R}(y,\xi_2)2M\mu^2(y)\frac{\xi_2}{|\xi_2|}\theta^2\left(\frac{\xi_2}{R}\right), \tag{IV.86}$$

therefore

$$\begin{aligned} q_\delta(y,\xi_2) &= \tfrac{\xi_2}{|\xi_2|}\theta^2\left(\tfrac{\xi_2}{R}\right)\iint e^{-iz\eta}m'(\xi_2+\delta\eta)b_1(y+z,\xi_2)dz\,d\eta \\ &= \tfrac{\xi_2}{|\xi_2|}\theta^2\left(\tfrac{\xi_2}{R}\right)q_{1,\delta}(y,\xi_2), \end{aligned} \tag{IV.87}$$

with

$$b_1(y,\xi_2) = M\mu^2(y)C_{-2M,R}(y,\xi_2). \tag{IV.88}$$

Note that $m' \in S^{-2}$, and $b_1 \in S^0$ with semi-norms depending only on M and not on $R \geq 1$. Thus $q_{1,\delta} \in S^{-2}$ with bounds depending only on M. But then the S^0 seminorms of q_δ are C_M/R^2, uniformly in δ. Hence

$$m(D_y)C_{-2M,R}(y,D_y) = C_{-2M,R}(y,D_y)m(D_y) + E_8, \tag{IV.89}$$

with

$$|||E_8||| \leq C_M/R, \text{ as operator in } L^2(dy). \tag{IV.90}$$

Finally

$$C_{2M,R}(y,D_y)C_{-2M,R}(y,D_y)m(D_y) = m(D_y) + E_9, \quad |||E_9||| \le C_M/R. \tag{IV.91}$$

We now take M so large that

$$\|m\|_{L^\infty(\mathbb{R})} \le \frac{M^{1/4}}{100^{10}}, \tag{IV.92}$$

and then choose R large, and (4.27), and consequently Claim 1, has been proved.

Note that Claim 2, and the symmetric of Claim 1 (a),(b) follow in the same manner.

We now fix λ, M and R as in all the claims, to obtain using the vector valued sharp Gårding inequality that

$$\begin{aligned} &2\,\mathrm{Re}\langle i[A\partial_x^2 - A\partial_x^2]z,z\rangle + 2\,\mathrm{Re}\langle Ar_1\partial_x z,z\rangle + 2\,\mathrm{Re}\langle A\varphi_1\partial_x\widetilde{\partial}_y^{-1}\varphi_2 z,z\rangle \\ &\qquad \le -M\langle \mu^2(x)|\xi_1|\theta^2\left(\frac{\xi_1}{R}\right)A_xA_y z,z\rangle + c_{(M,R)}\|z\|_{L^2}^2, \end{aligned} \tag{IV.93}$$

and

$$\begin{aligned} &2\,\mathrm{Re}\langle i[A\partial_y^2 - A\partial_y^2]z,z\rangle + 2\,\mathrm{Re}\langle Ar_2\partial_x z,z\rangle + 2\mathrm{Re}\langle A\varphi_3\partial_y\widetilde{\partial}_x^{-1}\varphi_4 z,z\rangle \\ &\qquad \le -M\langle \mu^2(y)|\xi_2|\theta^2\left(\frac{\xi_2}{R}\right)A_xA_y z,z\rangle + c_{(M,R)}\|z\|_{L^2}^2. \end{aligned} \tag{IV.94}$$

Thus upon integration between 0 and T in (4.8) we get

$$\begin{aligned} \|Cz(t)\|_{L^2}^2 \le \|Cz_0\|_{L^2}^2 &- M\textstyle\int_0^T\langle\mu^2(x)|\xi_1|\theta^2\left(\frac{\xi_1}{R}\right)A_xA_y z,z\rangle dt \\ &- M\textstyle\int_0^T\langle\mu^2(y)|\xi_2|\theta^2\left(\frac{\xi_2}{R}\right)A_xA_y z,z\rangle dt \\ &+ c_{(M,R)}T\sup_{0\le t\le T}\|z(t)\|_{L^2}^2 + 2\,\mathrm{Re}\textstyle\int_0^t\langle A\Gamma,z\rangle(t')dt'. \end{aligned} \tag{IV.95}$$

Hence if we denote by $Q_x = Q(x,D_1)$ (resp. $Q_y = Q(y,D_2)$) the operator defined by its symbol

$$\mu^2(x)|\xi_1|\theta^2\left(\frac{\xi_1}{R}\right), \qquad \left(\text{resp. } \mu^2(y)|\xi_2|\theta^2\left(\frac{\xi_2}{R}\right)\right), \tag{IV.96}$$

it follows that

$$\begin{aligned} M\int_0^T(\langle Q_xA_xA_y z,z\rangle + \langle Q_yA_xA_y z,z\rangle)dt + c_M\sup_{0\le t\le T}\|z(t)\|_{L^2}^2 \\ \le C(M)\|z_0\|_{L^2}^2 + Tc_{(M,R)}\sup_{0\le t\le T}\|z(t)\|_{L^2}^2 \\ + \sup_{0\le t\le T} 2\left|\mathrm{Re}\int_0^t\langle A\Gamma,z\rangle dt'\right|. \end{aligned} \tag{IV.97}$$

We now work with the term

$$\begin{aligned}\langle Q_x A_x A_y z, z\rangle &= \langle Q_x A_x C_y^* C_y C_y^* C_y z, z\rangle \\ &= \langle Q_x A_x C_y^* C_y z, C_y^* C_y z\rangle \equiv \iint Q_x A_x C_y^* C_y z \overline{C_y^* C_y z}\, dx\, dy. \end{aligned} \tag{IV.98}$$

We remark that $Q_x A_x$ is a ψ.d.o., in the x variable, of order 1, and that the symbol of $A_x \geq c_{(M,R)}$ modulo terms of order -1. By the sharp Gårding inequality in the x variable, we have, uniformly in y that

$$\begin{aligned}\int Q_x A_x C_y^* C_y z \overline{C_y^* C_y z}\, dx \quad &\geq c_{(M,R)} \int Q_x C_y^* C_y z \overline{C_y^* C_y z}\, dx \\ &-c_{(M,R)} \int |z(x,y)|^2\, dx. \end{aligned} \tag{IV.99}$$

Thus from (4.98)-(4.99), upon y-integration, it follows that

$$\langle Q_x A_x A_y z, z\rangle \geq c_{(M,R)} \langle Q_x C_y^* C_y z, C_y^* C_y z\rangle - c_{(M,R)} \|z\|^2_{L^2_{x,y}}. \tag{IV.100}$$

We remark that $\mu^2(x)|\xi_1|\theta^2\left(\frac{\xi_1}{R}\right) A_x$ is the symbol of a ψ.d.o., in the x-variable, of order 1, and that the symbol of $A_x \geq c_{(M,R)}$ modulo terms of order -1. By the sharp Gårding inequality in the x variable, we have, uniformly in y that

$$\begin{aligned}&\int \mu^2(x)|\xi_1|\theta^2\left(\tfrac{\xi_1}{R}\right) A_x C_y^* C_y z \overline{C_y^* C_y z}\, dx \\ &\geq c_{(M,R)} \int \mu^2(x)|\xi_1|\theta^2\left(\tfrac{\xi_1}{R}\right) C_y^* C_y z \overline{C_y^* C_y z}\, dx - c_{(M,R)} \int_x |z(x,y)|^2\, dx. \end{aligned} \tag{IV.101}$$

Thus upon y-integration it follows that

$$\begin{aligned}&\langle \mu^2(x)|\xi_1|\theta^2\left(\tfrac{\xi_1}{R}\right) A_x A_y z, z\rangle \\ &\quad \geq c_{(M,R)} \langle \mu^2(x)|\xi_1|\theta^2\left(\tfrac{\xi_1}{R}\right) C_y^* C_y z, C_y^* C_y z\rangle - c_{(M,R)} \|z\|^2_{L^2}. \end{aligned} \tag{IV.102}$$

Next we observe that

$$Q_x = \text{ Os}\left(\mu^2(x)|\xi_1|\theta^2\left(\frac{\xi_1}{R}\right)\right) = (\mu(x) J_x^{1/2})(\mu(x) J_x^{1/2}) + L_0, \tag{IV.103}$$

where L_0 is L^2-bdd. Hence, from (4.100)-(4.103) and (4.33) we have

$$\begin{aligned}\langle Q_x A_x A_y z, z\rangle \quad &\geq c_{(M,R)} \|\mu(x) J_x^{1/2} C_y^* C_y z\|^2_{L^2} - c_{(M,R)} \|z\|^2_{L^2} \\ &= c_{(M,R)} \|C_y^* C_y \mu(x) J_x^{1/2} z\|^2_2 - c_{(M,R)} \|z\|^2_{L^2} \\ &\geq c_{(M,R)} \|\lambda_N(x) J_x^{1/2} z\|^2_2 - c_{(M,R)} \|z\|^2_{L^2}, \end{aligned} \tag{IV.104}$$

since $C_y^*C_y$ is invertible in $L^2_{x,y}$. Gathering this information we end up with

$$\begin{aligned}\int_0^T(\|\lambda_N(x)J_x^{1/2}z\|_{L^2}^2+\|\lambda_N(y)J_y^{1/2}z\|_{L^2}^2)dt+\sup_{0\le t\le T}\|z(t)\|_{L^2}^2\\ \le c_{(M,R)}\|z_0\|_2^2+c_{(M,R)}T\sup_{0\le t\le T}\|z(t)\|_{L^2}^2\\ +c_{(M,R)}\sup_{0\le t\le T}\left|\int_0^t\langle A\Gamma,z\rangle dt'\right|.\end{aligned}\tag{IV.105}$$

It remains to study $\langle A\Gamma,z\rangle=\langle\Gamma,Az\rangle$. We recall that from (4.1)

$$\Gamma=\phi_1\partial_xf_1+\phi_2\partial_yf_2+\phi_3\partial_x\widetilde{\partial}_y^{-1}\phi_4f_3+\phi_5\partial_y\widetilde{\partial}_x^{-1}\phi_6f_4+f_5.\tag{IV.106}$$

First we have for $t\in[0,T]$

$$\begin{aligned}\left|\int_0^t\langle f_5,Az\rangle dt\right| &\le\int_0^T\|f_5\|_{L^2_{xy}}\|Az\|_{L^2_{xy}}dt\le C\int_0^T\|f_5\|_{L^2_{xy}}\|z\|_{L^2_{xy}}dt\\ &\le CT^{1/2}\|f_5\|_{L^2_{T,x,y}}\sup_{0\le t\le T}\|z(t)\|_{L^2_{xy}}\\ &\le\tfrac{1}{2c_{(M,R)}}\sup_{0\le t\le T}\|z(t)\|_{L^2_{xy}}^2+c_{(M,R)}T\|f_5\|_{L^2_{T,x,y}}^2.\end{aligned}\tag{IV.107}$$

Next we consider

$$\left|\int_0^t\langle\phi_1\partial_xf_1,Az\rangle dt\right|=\left|\int_0^t\langle\phi_1\partial_xf_1,A_xA_yz\rangle dt\right|.\tag{IV.108}$$

Writing $\partial_x=R_xJ_x^{1/2}J_x^{1/2}$ with R_x a ψ.d.o. of order zero in x, one sees that

$$\begin{aligned}&\langle\phi_1\partial_xf_1,A_xA_yz\rangle=\langle\phi_1R_xJ_x^{1/2}J_x^{1/2}f_1,A_xA_yz\rangle\\ &=\langle[\phi_1R_x;J_x^{1/2}]J_x^{1/2}f_1,A_xA_yz\rangle+\langle J_x^{1/2}\phi_1R_xJ_x^{1/2}f_1,A_xA_yz\rangle\\ &=\langle[\phi_1R_x;J_x^{1/2}]J_x^{1/2}f_1,A_xA_yz\rangle+\langle J_x^{1/2}[R_x;\phi_1]J_x^{1/2}f_1,A_xA_yz\rangle\\ &\quad+\langle J_x^{1/2}R_x\phi_1J_x^{1/2}f_1;A_xA_yz\rangle.\end{aligned}\tag{IV.109}$$

Since $[\phi_1R_x;J_x^{1/2}]J_x^{1/2}$, and $J_x^{1/2}[R_x;\phi_1]J_x^{1/2}$ are ψ.d.o's of order zero in x, uniformly in y and t, the first and second term in the r.h.s. of (4.109) are bounded, after integration in time, by

$$\begin{aligned}C\int_0^T\|z\|_{L^2_{xy}}\|\phi_1\|_{C_b^N(\mathbb{R}^2_{x,y})}\|f_1\|_{L^2_{xy}}dt\\ \le CT^{1/2}\|\phi_1\|_{L^2_T(C_b^k(\mathbb{R}^2_{x,y}))}\sup_{0\le t\le T}\|z(t)\|_{L^2_{xy}}\sup_{0\le t\le T}\|f_1\|_{L^2_{xy}}.\end{aligned}\tag{IV.110}$$

For the third term in the r.h.s. of (4.109) we have that

$$\begin{aligned}\langle J_x^{1/2}R_x\phi_1J_x^{1/2}f_1,A_xA_yz\rangle&=\langle\phi_1J_x^{1/2}f_1,R_x^*J_x^{1/2}A_xA_yz\rangle\\ &=\langle\phi_1J_x^{1/2}f_1,R_x^*J_x^{1/2}A_xJ_x^{-1/2}A_yJ_x^{1/2}z\rangle.\end{aligned}\tag{IV.111}$$

Using that $\mathcal{P} = R_x^* J_x^{1/2} A_x J_x^{-1/2}$ is a ψ.d.o. of order zero in x we get from (2.19), Lemma 2.4, that

$$\begin{aligned}
&|\langle R_x^* J_x^{1/2} A_x J_x^{-1/2} A_y J_x^{1/2} z, \phi_1 J_x^{1/2} f_1\rangle| \\
&= \left|\iint \mathcal{P} A_y J_x^{1/2} z \overline{\phi_1 J_x^{1/2} f} dx\, dy\right| \\
&\le \int \left(\int |\mathcal{P} A_y J_x^{1/2} z|^2 \lambda_N^2(x) dx\right)^{1/2} \left(\int |\phi_1 J_x^{1/2} f_1|^2 \tfrac{dx}{\lambda_N^2(x)}\right)^{1/2} dy \\
&\le C \int_y \left(\int |A_y J_x^{1/2} z|^2 \lambda_N^2(x) dx\right)^{1/2} \left(\int |\phi_1 J_x^{1/2} f_1|^2 \tfrac{dx}{\lambda_N^2(x)}\right)^{1/2} dy \qquad \text{(IV.112)} \\
&\le \tfrac{1}{c_{(M,R)}} \iint |A_y J_x^{1/2} z|^2 \lambda_N^2(x) dx\, dy + c_{(M,R)} \iint |\phi_1 J_x^{1/2} f_1|^2 \tfrac{dx\, dy}{\lambda_N^2(x)} \\
&\le \tfrac{1}{c_{(M,R)}} \iint |J_x^{1/2} z|^2 \lambda_N^2(x) dx\, dy \\
&+ c_{(M,R)} \iint |J_x^{1/2} f_1|^2 \lambda_N(x) dx\, dy \cdot \left\|\tfrac{\phi_1(\cdot,t)}{\lambda_N^2(x)}\right\|^2_{L^\infty_{x,y}} .
\end{aligned}$$

Hence fixing $t \in [0,T]$ one gets

$$\begin{aligned}
&c_{(M,R)} \left|\int_0^t \langle \phi_1 \partial_x f_1, Az\rangle dt\right| \\
&\quad \le \tfrac{1}{2} \int_0^T \iint |J_x^{1/2} z|^2 \lambda_N^2(x) dx\, dy\, dt + \tfrac{1}{2} \sup_{0\le t\le T} \|z(t)\|^2_{L^2_{x,y}} \\
&\quad + c_{(M,R)} T \sup_{0\le t\le T} \|f_1(t)\|^2_{L^2_{x,y}} \|\phi_1\|^2_{L^2_T(C_b^N(\mathbb{R}^2_{x,y}))} \qquad \text{(IV.113)} \\
&\quad + c_{(M,R)} \left\|\tfrac{\phi_1}{\lambda_N^2(x)}\right\|^2_{L^\infty_{x,y,T}} \|\lambda(x) J_x^{1/2} f_1\|^2_{L^2_{T,x,y}}.
\end{aligned}$$

The bound for the term

$$\left|\int_0^t \langle \phi_2 \partial_y f_2, Az\rangle dt\right| \qquad \text{(IV.114)}$$

is similar. We next turn to the estimate for

$$\begin{aligned}
&\langle \phi_3 \partial_x \widetilde{\partial}_y^{-1} \phi_4 f_3, A_x A_y z\rangle \\
&\quad = \langle \phi_3 \widetilde{\partial}_y^{-1} (\partial_x \phi_4) f_3, A_x A_y z\rangle + \langle \phi_3 \widetilde{\partial}_y^{-1} \phi_4 \partial_x f_3, A_x A_y z\rangle. \qquad \text{(IV.115)}
\end{aligned}$$

For the first term in the r.h.s. of (4.115) one has, after integration in time, the bound

$$\begin{aligned}
&c_{(M,R)} \int_0^t \|z\|_{L^2_{x,y}} \|\phi_3\|_{L^\infty_{x,y}} \|\partial_x \phi_4\|_{L^\infty_{x,y}} \|f_3\|_{L^2_{x,y}} dt \qquad \text{(IV.116)} \\
&\quad \le \tfrac{1}{4} \sup_{0\le t\le T} \|z(t)\|^2_{L^2_{x,y}} + c_{(M,R)} \|f_3\|^2_{L^\infty_T L^2_{x,y}} \|\phi_3\|^2_{L^2_T L^\infty_{x,y}} \|\partial_x \phi_4\|^2_{L^2_T L^\infty_{x,y}}
\end{aligned}$$

For the second term in (4.115) we write $\partial_x = R_x J_x^{1/2} J_x^{1/2}$ to have

$$\begin{aligned}
\langle \phi_3 \widetilde{\partial}_y^{-1} \phi_4 \partial_x f_3, A_x A_y z\rangle &= \langle \phi_3 \widetilde{\partial}_y^{-1} R_x J_x^{1/2} \phi_4 J_x^{1/2} f_3, A_x A_y z\rangle \\
&+ \langle \phi_3 \widetilde{\partial}_y^{-1} [\phi_4; R_x J_x^{1/2}] J_x^{1/2} f_3, A_x A_y z\rangle \qquad \text{(IV.117)}
\end{aligned}$$

where $[\phi_4; R_x J_x^{1/2}]J_x^{1/2}$ is a ψ.d.o. of order zero in x, uniform in y. Thus, the second term in the r.h.s. in (4.117) can be estimated as in (4.112). For the first one in we write

$$\begin{aligned}
&\langle \phi_3 \widetilde{\partial}_y^{-1} R_x J_x^{1/2} \phi_4 J_x^{1/2} f_3, A_x A_y z\rangle \\
&= \langle R_x J_x^{1/2} \phi_3 \widetilde{\partial}_y^{-1} \phi_4 J_x^{1/2} f_3, A_x A_y z\rangle \\
&+ \langle [\phi_3; R_x J_x^{1/2}] \widetilde{\partial}_y^{-1} \phi_4 J_x^{1/2} f_3, A_x A_y z\rangle .
\end{aligned} \tag{IV.118}$$

Now

$$\begin{aligned}
[\phi_3; R_x J_x^{1/2}] \widetilde{\partial}_y^{-1} \phi_4 J_x^{1/2} &= [\phi_3; R_x J_x^{1/2}] J_x^{1/2} J_x^{-1/2} \widetilde{\partial}_y^{-1} \phi_4 J_x^{1/2} \\
&= [\phi_3; R_x J_x^{1/2}] J_x^{1/2} \widetilde{\partial}_y^{-1} J_x^{-1/2} \phi_4 J_x^{1/2}.
\end{aligned} \tag{IV.119}$$

Since $[\phi_3; R_x J_x^{1/2}] J_x^{1/2}$ and $J_x^{-1/2} \phi_4 J_x^{1/2}$ are ψ.d.o. of order zero in x uniformly in y the bound of the second term in (4.118) follows the argument in (4.112). For the first term in (4.118) using that $J_x^{1/2} R_x^* A_x J_x^{-1/2}$ is a ψ.d.o. of order zero in x, and A_y is a ψ.d.o. of order zero in y we obtain, using the notation $\mathcal{A} = J_x^{1/2} R_x^* A_x J_x^{-1/2} A_y$ and Theorem 2.4 that

$$\begin{aligned}
&\langle R_x J_x^{1/2} \phi_3 \widetilde{\partial}_y^{-1} \phi_4 J_x^{1/2} f_3, A_x A_y z\rangle \\
&= \langle \phi_3 \widetilde{\partial}_y^{-1} \phi_4 J_x^{1/2} f_3, J_x^{1/2} R_x^* A_x A_y z\rangle \\
&= \langle \phi_3 \widetilde{\partial}_y^{-1} \phi_4 J_x^{1/2} f_3, J_x^{1/2} R_x^* A_x J_x^{-1/2} A_y J_x^{1/2} z\rangle \\
&= \langle \phi_3 \widetilde{\partial}_y^{-1} \phi_4 J_x^{1/2} f_3, \mathcal{A} J_x^{1/2} z\rangle \\
&\le \left(\iint |\mathcal{A} J_x^{1/2} z|^2 \lambda_N^2(x) dx\, dy \right)^{1/2} \left(\iint |\phi_3 \widetilde{\partial}_y^{-1} \phi_4 J_x^{1/2} f_3|^2 \tfrac{dx\, dy}{\lambda_N^2(x)} \right)^{1/2} \\
&\le c \left(\iint |J_x^{1/2} z|^2 \lambda_N^2(x) dx\, dy \right)^{1/2} \left(\iint |\phi_3 \widetilde{\partial}_y^{-1} \phi_4 J_4^{1/2} f_3|^2 \tfrac{dx\, dy}{\lambda_N^2(x)} \right)^{1/2}.
\end{aligned} \tag{IV.120}$$

Thus after integrating in time we get the bound

$$\begin{aligned}
&\tfrac{1}{4} \|\lambda_N(x) J_x^{1/2} z\|_{L^2_{xyT}} \\
&+ c_{(M,R)} \left\| \tfrac{\phi_3}{\lambda_N(x)} \right\|^2_{L^\infty_{Txy}} \left\| \tfrac{\phi_4}{\lambda_N(x)} \right\|^2_{L^\infty_{Txy}} \|\lambda_N(x) J_x^{1/2} f_3\|^2_{L^2_{x,y,T}}.
\end{aligned} \tag{IV.121}$$

Gathering this information we get the desired bound for the term in (4.115).

In the same approach shows that the term

$$\int_0^T \langle \phi_5 \partial_y \widetilde{\partial}_x^{-1} \phi_6 f_4, A_x A_y z\rangle dt, \tag{IV.122}$$

is bounded by

$$\begin{aligned}&\tfrac{1}{4}\ \sup_{0\le t\le T}\|z(t)\|^2_{L^2_{x,y}}+\tfrac{1}{4}\|\lambda_N(y)J_y^{1/2}z\|^2_{L^2_{x,y}}\\&+c_{(M,R)}\|f_4\|^2_{L^\infty_T L^2_{x,y}}\|\phi_5\|^2_{L^2_T L^\infty_{x,y}}\|\partial_y\phi_6\|^2_{L^2_T L^\infty_{x,y}}\\&+c_{(M,R)}\left\|\frac{\phi_5}{\lambda_N(y)}\right\|^2_{L^\infty_{x,y,T}}\left\|\frac{\phi_6}{\lambda_N(y)}\right\|^2_{L^\infty_{x,y,T}}\left\|\lambda_N(y)J_y^{1/2}f_4\right\|^2_{L^2_{x,y,T}}.\end{aligned} \tag{IV.123}$$

Finally, collecting the information in (4.105)-(4.123) we complete the proof of Theorem 1.2.

Corollary 4.1 Under the hypothesis of Theorem 1.2 the same results hold for solutions of the IVP (1.22) with $i\Delta+\epsilon\Delta$ instead of $i\Delta$, uniformly for $\epsilon\in(0,1]$.

Proof of Corollary 4.1. With the notation in (4.7) it suffices to see that (see (4.8))

$$\epsilon\langle A\Delta z,z\rangle+\epsilon\langle Az,\Delta z\rangle=2Re\epsilon\langle A\Delta z,z\rangle\le c\|z\|^2_{L^2_{x,y}},$$

with c independent of $\epsilon\in(0,1]$. Thus, we write

$$\langle A\partial_x^2 z,z\rangle=\langle A_yA_x\partial_x^2z,z\rangle=\langle A_y\partial_xA_x\partial_xz,z\rangle+\langle A_y[A_x;\partial_x]\partial_xz,z\rangle.$$

Since $C_x^*C_x$, $C_y^*C_y$ are invertible (see Claim 2 after (4.27)) it follows that

$$\begin{aligned}\langle A_y\partial_xA_x\partial_xz,z\rangle&=-\langle A_yA_x\partial_xz,\partial_xz\rangle\\&=-\|C_x^*C_xC_y^*C_y\partial_xz\|^2_{L^2_{x,y}}\le -c\|\partial_xz\|^2_{L^2_{x,y}},\end{aligned} \tag{IV.124}$$

which combined with

$$|\langle A_y[A_x;\partial_x]\partial_xz,z\rangle|\le c\|\partial_xz\|_{L^2_{x,y}}\|z\|_{L^2_{x,y}} \tag{IV.125}$$

yields the result.

V Proof of Theorem 1.3

We split the proof in three steps.

STEP 1. Existence of a local solution u^ϵ of (5.1) in a time interval $[0,T_\epsilon]$.

The proof of Theorem 1.3 is based on the viscosity method. Thus, for $\epsilon\in(0,1]$ we consider the IVP

$$\begin{cases}\partial_tu-i\Delta u-\epsilon\Delta u=G(u,\nabla_xu,\bar u,\nabla_x\bar u),\\ u(x,y,0)=u_0(x,y),\end{cases} \tag{V.1}$$

where

$$G(u,\nabla_x u,\bar u,\nabla_x\bar u) = c_1\tfrac{\bar u}{1+|u|^2}\partial_x u\partial_y u$$
$$+c_6\partial_x u\partial_y^{-1}\left(\tfrac{\partial_x u\partial_y\bar u-\partial_x\bar u\partial_y u}{(1+|u|^2)^2}\right)+c_7\partial_y u\partial_x^{-1}\left(\tfrac{\partial_x u\partial_y\bar u-\partial_x\bar u\partial_y u}{(1+|u|^2)^2}\right). \tag{V.2}$$

We write (5.1) in the intergal equation form

$$u(t)=e^{(\epsilon+i)t\Delta}u_0+\int_0^t e^{(\epsilon+i)(t-t')\Delta}G(u,\nabla_x u,\bar u,\nabla_x\bar u)(t')dt', \tag{V.3}$$

and defines the operator $\Phi=\Phi_{u_0}$ as

$$\Phi(v)(t)=e^{(\epsilon+i)t\Delta}u_0+\int_0^t e^{(\epsilon+i)(t-t')\Delta}G(v,\nabla_x v,\bar v,\nabla_x\bar v)(t')dt', \tag{V.4}$$

for

$$v\in X_{s,a}^T=\{v\in C([0,T]:H^s)\ :\ \sup_{0\le t\le T}\|u(t)\|_{H^s}\le a\}. \tag{V.5}$$

We shall use that if $f\in H^s,\ s\in\mathbb{R}$, then for $\epsilon\in(0,1]$

$$\begin{cases}\|e^{(\epsilon+i)t\Delta}f\|_{H^s}\le\|f\|_{H^s},\\ \|\nabla_x e^{(\epsilon+i)t\Delta}f\|_{H^s}\le\frac{c_s}{\epsilon\sqrt{t}}\ \|f\|_{H^{s-1}},\end{cases} \tag{V.6}$$

where $\nabla_x=(\partial_x,\partial_y)$.

Using that $H^s(\mathbb{R}^2)$, with $s>1$, is an algebra respect to the pointwise product of function, it follows that for $s\ge 3$ and $T>0$

$$\sup_{0\le t\le T}\|G(v,..)(t)\|_{H^{s-1}}\le c_s\sup_{0\le t\le T}\|v(t)\|_{H^s}^3. \tag{V.7}$$

Thus, inserting (5.6)-(5.7) in (5.4) we get that for $s\ge 3$

$$\begin{aligned}&\sup\nolimits_{0\le t\le T}\|\Phi(v)(t\|_{H^s}\\ &\le c_s\|u_0\|_{H^s}+c_s\,\epsilon^{-1/2}\ \textstyle\int_0^T\frac{1}{\sqrt{t-t'}}\|G(v,..)(t')\|_{H^{s-1}}dt'\\ &\le c_s\|u_0\|_{H^s}+c_s\,\epsilon^{-1/2}\,T^{1/2}\sup\nolimits_{0\le t\le T}\|v(t)\|_{H^s}^3.\end{aligned} \tag{V.8}$$

Therefore, fixing

$$a=2c_s\|u_0\|_{H^s}\qquad\text{and}\qquad T_\epsilon=(20\ c_2^4\ \epsilon^{-1/4}\ \|u_0\|_{H^s}^2)^{-4}, \tag{V.9}$$

in (5.5) it follows that $\Phi(X_{s,a}^{T_\epsilon})\subset X_{s,a}^{T_\epsilon}$. A similar argument shows that

$$\sup_{0\le t\le T_\epsilon}\|\Phi(v)-\Phi(w)\|_{H^s}\le\frac{1}{2}\sup_{0\le t\le T_\epsilon}\|v-w\|_{H^s}, \tag{V.10}$$

for any $v,\ w \in X^{T_\epsilon}_{s,a}$. Hence, (5.3), and consequently (5.1) has a unique solution $u^\epsilon \in C([0,T_\epsilon] : H^s)$.

STEP 2. *A priori* estimates for the u^ϵ's in $C([0,T] : H^s \cap L^2((x^2+y^2)^{m/2}dxdy))$, with $T>0$ independent of $\epsilon \in (0,1]$.

In this step, one of the key in the proof, we will use Theorem 1.2.

From (5.3) one has that

$$\sup_{0\le t\le T_\epsilon} \|u^\epsilon(t)\|_{L^2} \le c_0\|u_0\|_{L^2} + c_0 T^{1/2} \sup_{0\le t\le T_\epsilon} \|u^\epsilon(t)\|^3_{H^3}. \tag{V.11}$$

Next, we apply the operator ∂_x^s to the equation in (5.1) and write the result using the notation

$$v_1 = \partial_x^s u, \tag{V.12}$$

to get that

$$\begin{cases} \partial_t v_1 - i\Delta v_1 - \epsilon\Delta v_1 = r_1(u_0)\partial_x v_1 + r_2(u_0)\partial_y v_1 \\ +\varphi_1(u_0)\partial_x\partial_y^{-1}\varphi_2(u_0)v_1 + \varphi_5(u_0)\partial_x\partial_y^{-1}\varphi_6(u_0)\bar v_1 \\ +p_1\partial_x f_1 + p_2\partial_y f_2 + \phi_{1,1}\partial_x\partial_y^{-1}\phi_{1,2}f_1 + \phi_{2,1}\partial_x\partial_y^{-1}\phi_{2,2}f_1 \\ +\phi_{5,1}\partial_x\partial_y^{-1}\phi_{5,2}f_2 + \phi_{6,1}\partial_x\partial_y^{-1}\phi_{6,2}f_2 + f_5, \\ v(x,y,0) = \partial_x^s u_0(x,y), \end{cases} \tag{V.13}$$

where

$$\begin{aligned} r_1(u_0) &= r_{1,1}(u_0) + r_{1,2}(u_0) \\ &= \tfrac{c_1\bar u_0}{1+|u_0|^2}\partial_y u_0 + c_6\partial_y^{-1}\left(\tfrac{\partial_x u_0\partial_y\bar u_0 - \partial_x\bar u_0\partial_y u_0}{(1+|u_0|^2)^2}\right), \end{aligned} \tag{V.14}$$

$$\begin{aligned} r_2(u_0) &= r_{2,1}(u_0) + r_{2,2}(u_0) \\ &= \tfrac{c_1\bar u_0}{1+|u_0|^2}\partial_x u_0 + c_7\partial_x^{-1}\left(\tfrac{\partial_x u_0\partial_y\bar u_0 - \partial_x\bar u_0\partial_y u_0}{(1+|u_0|^2)^2}\right), \end{aligned} \tag{V.15}$$

$$\varphi_1(u_0) = c_6\ \partial_x u_0, \qquad \varphi_2(u_0) = \frac{\partial_y\bar u_0}{(1+|u_0|^2)^2} \tag{V.16}$$

$$\varphi_5(u_0) = -c_6\ \partial_x u_0, \qquad \varphi_6(u_0) = \frac{\partial_y u_0}{(1+|u_0|^2)^2} \tag{V.17}$$

$$p_1 = r_1(u^\epsilon(t)) - r_1(u_0), \qquad p_2 = r_2(u^\epsilon(t)) - r_2(u_0), \tag{V.18}$$

$$\phi_{1,1} = \varphi_1(u^\epsilon(t)) - \varphi_1(u_0), \qquad \phi_{1,2} = \varphi_2((u^\epsilon(t)), \tag{V.19}$$

$$\phi_{2,1} = \varphi_1(u_0), \qquad \phi_{2,2} = \varphi_2(u^\epsilon(t)) - \varphi_2(u_0), \tag{V.20}$$

$$\phi_{5,1} = \varphi_5(u^\epsilon(t)) - \varphi_5(u_0), \qquad \phi_{5,2} = \varphi_2((u^\epsilon(t)), \tag{V.21}$$

$$\phi_{6,1} = \varphi_5(u_0), \qquad \phi_{6,2} = \varphi_2(u^\epsilon(t)) - \varphi_2(u_0), \tag{V.22}$$

$$f_1 = \partial_x^s u = v_1, \qquad f_2 = \bar{\partial}_x^s u = \bar{v}_1, \tag{V.23}$$

and f_5 contains all the lower order term and satisfies

$$\sup_{0\le t\le T} \|f_5\|_{L^2_{x,y}} \le c_s \sup_{0\le t\le T} \|u^\epsilon\|^3_{H^s}. \tag{V.24}$$

To obtain the desired *a priori* estimate we apply Theorem 1.2 to the IVP (5.13). Thus, we need to show that the constant A in (1.24) can be made as small as we please by taking T sufficiently small, uniformly in $\epsilon \in (0,1]$.

We shall only be concerned with the term in (1.24) not having the factor T on it, i.e. the second, sixth and seventh. We observe from (5.14)-(5.22) that in each of these factor there is a term of the form

$$g(u^\epsilon(t)) - g(u_0) \equiv \int_0^t \frac{d}{dt} g(u^\epsilon(t'))dt'. \tag{V.25}$$

Thus, using the equation (5.13) we obtain an appropriate bound with a factor T on it. Thus, for the second terms in (1.24) we have

$$\begin{aligned} &\|\lambda_N^{-2}(x)p_1\|^2_{L^\infty_{x,y,T}} = \|\lambda_N^{-2}(x)\textstyle\int_0^t \frac{d}{dt} r_1(u^\epsilon(\cdot,t')dt'\|^2_{L^\infty_{x,y,T}} \\ &= \|\lambda_N^{-2}(x)\textstyle\int_0^t \frac{d}{dt}\big(\frac{c_1\bar{u}^\epsilon\partial_y u^\epsilon}{1+|u^\epsilon|^2} + c_6\partial_y^{-1}\big(\frac{\partial_x u^\epsilon\partial_y\bar{u}^\epsilon - \partial_x\bar{u}^\epsilon\partial_y u^\epsilon}{(1+|u^\epsilon|^2)^2}\big)\big)dt'\|^2_{L^\infty_{x,y,T}} \\ &\le CT\sup_{0\le t\le T}(\|u^\epsilon(t)\|^2_{H^3} + \|u^\epsilon(t)\|^6_{H^3})\ \textstyle\sum_{|\alpha|\le 4}\|\lambda_N^{-2}(x)\partial^\alpha u^\epsilon\|^2_{L^\infty_T L^2_{x,y}}). \end{aligned} \tag{V.26}$$

Next, we have the estimate for the sixth term in (1.24)

$$\begin{aligned} &\|\lambda_N^{-1}(x)\tilde{\phi}_1\|^2_{L^\infty_{x,y,T}} = \|\lambda_N^{-1}(x)\lambda_M^{-1}(y)\phi_{1,1}\|^2_{L^\infty_{x,y,T}} \\ &= \|\lambda_N^{-1}(x)\lambda_M^{-1}(y)(\varphi_1(u^\epsilon(t)) - \varphi_1(u_0))\|^2_{L^\infty_{x,y,T}} \\ &= \|\lambda_N^{-1}(x)\lambda_M^{-1}(y)\textstyle\int_0^t \frac{d}{dt}\varphi_1(u^\epsilon(t')dt'\|^2_{L^\infty_{x,y,T}} \\ &\le cT\|\lambda_N^{-1}(x)\lambda_M^{-1}(y)\partial_x\partial_t u^\epsilon(t)\|^2_{L^\infty_{x,y,T}} \\ &\quad \le cT(1+\sup_{0\le t\le T}\|u^\epsilon(t)\|^4_{H^4})\ \textstyle\sum_{|\alpha|\le 6}\|\lambda_N^{-1}(x)\lambda_M^{-1}(y)\partial^\alpha u^\epsilon\|^2_{L^\infty_T L^2_{x,y}}. \end{aligned} \tag{V.27}$$

The other terms in (1.24) can be bounded in a similar manner. Thus to close these estimates we need to bound terms of the form

$$\sum_{|\alpha|\le 6} \|\lambda_L^{-1}(x)\lambda_L^{-1}(y)\partial^\alpha u^\epsilon\|^2_{L^\infty_T L^2_{x,y}} \tag{V.28}$$

with $L = max\{N, M\}$. To achieve this we use the operators

$$\Gamma_x = x + 2it\partial_x, \quad \Gamma_y = y + 2it\partial_y, \tag{V.29}$$

and the following commutative relations and identities

$$\begin{cases} [\Gamma_x; \partial_t - i\Delta] = [\Gamma_y; \partial_t - i\Delta] = 0, \\ [\Gamma_x; \Delta] = -2\partial_x, \quad [\Gamma_y; \Delta] = -2\partial_y, \\ [\Gamma_x, \partial_x] = [\Gamma_y; \partial_y] = -1, \quad \Gamma(fg) = f\Gamma(g) + 2itg\partial f. \end{cases} \tag{V.30}$$

We shall estimate

$$\Gamma u^\epsilon,\ \Gamma\partial u^\epsilon,\ \Gamma^2 u^\epsilon, .., \ (\Gamma^\beta \partial^\alpha u^\epsilon)_{|\beta|\le 2L,\ |\alpha|\le 6}, \tag{V.31}$$

in this order. For the first step we apply Γ_x in (5.1) and use the notation $w_1 = \Gamma_x u^\epsilon$ to get

$$\begin{cases} (\partial_t - i\Delta - \epsilon\Delta)w_1 = c_1 \frac{\bar{u}}{1+|u|^2}\partial_y u \partial_x w_1 \\ \\ +c_6 \partial_x w_1 \partial_y^{-1}(\frac{\partial_x u \partial_y \bar{u} - \partial_x \bar{u} \partial_y u}{(1+|u|^2)^2}) + c_7 \partial_y w_1 \partial_x^{-1} u(\frac{\partial_x u \partial_y \bar{u} - \partial_x \bar{u} \partial_y u}{(1+|u|^2)^2}) + f_{5,1}, \end{cases} \tag{V.32}$$

where $f_{5,1}$ satisfies an estimate similar to that (5.24). To apply Theorem 1.2 we rewrite the IVP (5.31) using the notation in (5.14)-(5.22) as

$$\begin{cases} (\partial_t - i\Delta - \epsilon\Delta)w_1 = r_1(u_0)\partial_x w_1 + r_{2,2}(u_0)\partial_y w_1 \\ +p_1 \partial_x f_1 + (r_{2,2}(u^\epsilon(t)) - r_{2,2}(u_0))\partial_y f_1 + f_{5,1}, \end{cases} \tag{V.33}$$

with $f_1 = w_1$.

We observe that the coefficients in the equation in (5.33) are basically the same as those in (5.13). In fact, this is the case for all the equation for the terms in (5.31) except that in each case $\|f_{5,\cdot}\|_{L^2}$ can be bounded using the previous terms.

Hence, defining for $T > 0$

$$\|u\|_T \equiv \sup_{[0,T]} \left(\|u(t)\|_{H^s} + \|\lambda_m^{-1}(x^2+y^2)u(t)\|_{L^2} \right), \tag{V.34}$$

with $m \ge L,\ s > 2m$ and using that

$$x = \Gamma_x - 2it\partial_x; \ \ x^2 = \Gamma_x^2 + 4it\Gamma_x\partial_x + 4t^2\partial_x^2 + 2it; \ \ x^3 = \Gamma^3 + 2t(...), \tag{V.35}$$

from the above argument we get that

$$\begin{aligned} \|u^\epsilon\|_T \le c(\|u_0\|_{H^s} + \|\lambda_m^{-1}(x^2+y^2)u_0\|_{L^2}) \\ +cT(1+T^2)\left(\|u^\epsilon\|_T^3 + \|u^\epsilon\|_T^6\right). \end{aligned} \tag{V.36}$$

Then we conclude that there exists $T_0 = T_0(\|u_0\|_{H^s} + \|\lambda_m^{-1}(x^2+y^2)u_0\|_{L^2}) > 0$ such that the solution u^ϵ's can be extended to the interval $[0,T]$ such that $u^\epsilon \in C([0,T] : H^s \cap L^2((x^2+y^2)^m dxdy))$, with

$$\|u^\epsilon\|_{T_0} \le 2c_{s,m}\left(\|u_0\|_{H^s} + \|\lambda_m^{-1}(x^2+y^2)u_0\|_{L^2}\right) \equiv \delta. \tag{V.37}$$

<u>STEP 3</u>. Convergence of u^ϵ's in $L^\infty([0,T] : L^2)$-norm as $\epsilon \downarrow 0$.

In this step we shall use again Theorem 1.2.

For $\epsilon > \epsilon' > 0$ we define $\omega = \omega^{\epsilon,\epsilon'} = u^\epsilon - u^{\epsilon'}$ which satisfies the IVP

$$\begin{cases} \partial_t\omega - i\Delta\omega - \epsilon'\Delta\omega - (\epsilon-\epsilon')\Delta u^\epsilon \\ = c_1\frac{\bar u^{\epsilon'}}{1+|u^{\epsilon'}|^2}\partial_y u^\epsilon\partial_x\omega + c_1\frac{\bar u^{\epsilon'}}{1+|u^{\epsilon'}|^2}\partial_x u^\epsilon\partial_y\omega \\ +r_{1,2}(u^\epsilon(t))\partial_x\omega \ + c_6\partial_x u^{\epsilon'}\partial_y^{-1}\partial_x\left(\frac{\omega\partial_y\bar u^\epsilon - \bar\omega\partial_y u^\epsilon}{(1+|u^\epsilon|^2)^2}\right) \\ +r_{2,2}(u^\epsilon(t))\partial_y\omega \ + c_7\partial_x u^{\epsilon'}\partial_x^{-1}\partial_y\left(\frac{\omega\partial_x\bar u^\epsilon - \bar\omega\partial_x u^\epsilon}{(1+|u^\epsilon|^2)^2}\right) \\ +Q(\partial^\alpha u^\epsilon_{|\alpha|\le 1}, \partial^\beta u^{\epsilon'}_{|\beta|\le 1}, \omega)(t), \end{cases} \tag{V.38}$$

where (see (5.37))

$$\|Q(\partial^\alpha u^\epsilon_{|\alpha|\le 1}, \partial^\beta u^{\epsilon'}_{|\beta|\le 1}, \omega)(t)\|_{L^2_{x,y}} \le c\delta\|\omega(t)\|_{L^2_{x,y}}. \tag{V.39}$$

To apply Theorem 1.2 we rewrite (5.38) using the notation in (5.14)-(5.22) as

$$\begin{cases} \partial_t\omega - i\Delta\omega - \epsilon'\Delta\omega - (\epsilon-\epsilon')\Delta u^\epsilon \\ = r_1(u_0)\partial_x\omega + r_2(u_0)\partial_y\omega \\ +\varphi_1(u_0)\partial_x\partial_y^{-1}\varphi_2(u_0)\omega + \varphi_3(u_0)\partial_y\partial_x^{-1}\varphi_4(u_0)\omega \\ +\varphi_5(u_0)\partial_x\partial_y^{-1}\varphi_6(u_0)\bar\omega + \varphi_7(u_0)\partial_y\partial_x^{-1}\varphi_8(u_0)\bar\omega \\ +p_{1,1}\partial_x f + p_{2,1}\partial_y f \\ +\phi_{1,1}\partial_x\partial_y^{-1}\phi_{1,2}f_1 + \phi_{2,1}\partial_x\partial_y^{-1}\phi_{2,2}f_1 + \phi_{3,1}\partial_y\partial_x^{-1}\phi_{3,2}f_1 \\ +\phi_{4,1}\partial_y\partial_x^{-1}\phi_{4,2}f_1 + \phi_{5,1}\partial_x\partial_y^{-1}\phi_{5,2}f_2 + \phi_{6,1}\partial_x\partial_y^{-1}\phi_{6,2}f_2 \\ +\phi_{7,1}\partial_y\partial_x^{-1}\phi_{7,2}f_2 + \phi_{8,1}\partial_y\partial_x^{-1}\phi_{8,2}f_2 + Q(\partial^\alpha u^\epsilon_{|\alpha|\le 1}, \partial^\beta u^{\epsilon'}_{|\beta|\le 1}, \omega)(t), \end{cases} \tag{V.40}$$

where $\varphi_1, \varphi_2, \varphi_5, \varphi_6$ were defined in (5.16)-(5.17) and

$$\varphi_3(u_0) = -c_7\partial_y u_0, \qquad \varphi_4(u_0) = \frac{\partial_x \bar u_0}{(1+|u_0|^2)^2}, \tag{V.41}$$

$$\varphi_3(u_0) = c_7\partial_y u_0, \qquad \varphi_4(u_0) = \frac{\partial_x u_0}{(1+|u_0|^2)^2}, \tag{V.42}$$

$$p_{1,1} = c_1 \left(\frac{u^{\epsilon'} \partial_y u^{\epsilon}}{(1 + |u^{\epsilon'}|^2)^2} - \frac{u_0 \partial_y u_0}{(1 + |u_0|^2)^2} \right) + (r_{1,2}(u^{\epsilon}(t)) - r_{1,2}(u_0)), \tag{V.43}$$

$$p_{2,1} = c_1 \left(\frac{u^{\epsilon'} \partial_x u^{\epsilon}}{(1 + |u^{\epsilon'}|^2)^2} - \frac{u_0 \partial_x u_0}{(1 + |u_0|^2)^2} \right) + (r_{2,2}(u^{\epsilon}(t)) - r_{2,2}(u_0)) \tag{V.44}$$

$$\phi_{1,1} = \partial_x u^{\epsilon'} - \partial_x u_0, \qquad \phi_{1,2} = \frac{\partial_y \bar{u}^{\epsilon}}{(1 + |u^{\epsilon}|^2)^2}, \tag{V.45}$$

$$\phi_{2,1} = \partial_x u_0, \quad \phi_{2,2} = \frac{\partial_y \bar{u}^{\epsilon}}{(1 + |u^{\epsilon}|^2)^2} - \frac{\partial_y \bar{u}_0}{(1 + |u_0|^2)^2}, \tag{V.46}$$

$$\phi_{3,1} = \partial_y u^{\epsilon'} - \partial_y u_0, \qquad \phi_{3,2} = \frac{\partial_x \bar{u}^{\epsilon}}{(1 + |u^{\epsilon}|^2)^2}, \tag{V.47}$$

$$\phi_{4,1} = \partial_y u_0, \quad \phi_{4,2} = \frac{\partial_x \bar{u}^{\epsilon}}{(1 + |u^{\epsilon}|^2)^2} - \frac{\partial_x \bar{u}_0}{(1 + |u_0|^2)^2}, \tag{V.48}$$

$$\phi_{5,1} = -(\partial_x u^{\epsilon} - \partial_x u_0), \qquad \phi_{5,2} = \frac{\partial_y \bar{u}^{\epsilon}}{(1 + |u^{\epsilon}|^2)^2} \tag{V.49}$$

$$\phi_{6,1} = -\partial_x u_0, \qquad \phi_{6,2} = -\left(\frac{\partial_x \bar{u}^{\epsilon}}{(1 + |u^{\epsilon}|^2)^2} - \frac{\partial_x \bar{u}_0}{(1 + |u_0|^2)^2} \right) \tag{V.50}$$

$$\phi_{7,1} = \partial_y u^{\epsilon'} - \partial_y u_0, \qquad \phi_{7,2} = \frac{\partial_x \bar{u}^{\epsilon}}{(1 + |u^{\epsilon}|^2)^2} \tag{V.51}$$

$$\phi_{8,1} = \partial_y u_0, \qquad \phi_{8,2} = \frac{\partial_x \bar{u}^{\epsilon}}{(1 + |u^{\epsilon}|^2)^2} - \frac{\partial_x \bar{u}_0}{(1 + |u_0|^2)^2}, \tag{V.52}$$

$$f_1 = \omega, \qquad f_2 = \bar{\omega}. \tag{V.53}$$

To apply Theorem 1.2 to the IVP (5.40) we observe that as in the previous cases all the terms in (1.24) involved a factor T or a factor of the form described in (5.25), which can be bounded with a bound having a factor T on it. Thus, for T sufficiently small we get

$$\lim_{\epsilon, \epsilon' \to 0} \sup_{[0,T]} \|\omega^{\epsilon, \epsilon'}(t)\|_{L^2_{x,y}} = 0. \tag{V.54}$$

This proves the convergence of the u^ϵ's to a function u. By interpolation we get that the u^ϵ's converges to u in $C([0,T] : H^{s-1} \cap L^2(|x|^{m-1}dx))$. Using $weak*$-compactness and Fatou's lemma it follows that $u \in \cap L^\infty([0,T] : H^s)$, and $u \in \cap L^\infty([0,T] : L^2(|x|^m dx))$ respectively. It is clear that in the time interval $[0,T]$ u is a solution of the IVP (5.1).

Finally we remark that the proof of the uniqueness of the solution u in its class is similar to the argument described in (5.38)-(5.53), therefore it will be omitted.

References

[AbHa] Ablowitz, M. J., and Haberman, R., Nonlinear evolution equations in two and three dimensions, Phys. Rev. Lett. **35** 1185–1188 (1975).

[BeCo] Beals, R., and Coifmann, R. R., The spectral problem for the Davey-Stewartson and Ishimori hierarchies, Proc. Conf. on Nonlinear Evolution Equations: Integrability and Spectral Methods, Manchester, U.K. (1988)

[Ch] Chihara, H., The initial value problem semilinear Schrödinger equations, Publ. RIMS. Kyoto Univ. **32**, 445–471 (1996).

[CnSa] Constantin, P. and Saut, J. C., Local smoothing properties of dispersive equations, J. Amer. Math. Soc. **1**, 413–446 (1989).

[CrKaSt] Craig, W., Kappeler T., and Strauss, W., Microlocal dispersive smoothing for the Schrödinger equation, Comm. Pure Appl. Math. **48**, 769–860 (1995).

[DS] Davey, A., and Stewartson, K., On three dimensional packets of surface waves, Proc. R. Soc. A **338**, 101–110 (1974).

[DjRe] Djordjevic, V. D., and Redekopp, L. P., On two-dimensional packets of capillarity-gravity waves, J. Fluid Mech. **79**, 703–714 (1977).

[Do] Doi, S., Remarks on the Cauchy problem for Schrödinger type equations, Comm. P.D.E. **21**, 163–178 (1996).

[GhSa] Ghidaglia, J. M., and Saut, J. C., On the initial value problem for the Davey- Stewartson System, Nonlinearity **3**, 475–506 (1990).

[Hy] Hayashi, N., Local existence in time of small solutions to the Ishimori system, preprint

[HyOz] Hayashi, N., and Ozawa, T., Remarks on nonlinear Schrödinger equations in one space dimension, Diff. Integral Eqs **2**, 453–461 (1994).

[HySa] Hayashi, N., and Saut, J-C., Global existence of small solutions to the Davey-Stewartson and the Ishimori systems, Diff. Integral Eqs **8**, 1657–1675 (1995).

[Ho] Hörmander, L., The Analysis of Linear Partial Differential Operators, III, Springer-Verlag (1985).

[Is] Ishimori, Y., Multi vortex solutions of a two dimensional nonlinear wave equation, Progr. Theor. Phys. **72**, 33–37 (1984).

[Kt] Kato, T., On the Cauchy problem for the (generalized) Korteweg-de Vries equation, Advances in Math. Supp. Studies, Studies in Applied Math. **8**, 93–128 (1983).

[KePoVe1] Kenig, C. E., Ponce, G., and Vega, L., Small solutions to nonlinear Schrödinger equations, Annales de l'I.H.P. **10**, 255–288 (1993).

[KePoVe2] Kenig, C. E., Ponce, G. and Vega, L., Smoothing effects and local existence theory for the generalized nonlinear Schrödinger equations, to appear in Inventiones Math

[KoMa] Konopelchenko, B. G., and Matkarimov, B. T., Inverse spectral transform for the nonlinear evolution equation generating the Davey-Stewartson and Ishimori equations, Stud. Appl. Math. **82**, 319–359 (1990).

[LiPo] Linares, F., and Ponce, G., On the Davey-Stewartson systems, Annales de l'I.H.P. Analyse non linéaire **10**, 523–548 (1993).

[Mz] Mizohata, S., On the Cauchy problem, Notes and Reports in Math. in Science and Engineering, Science Press & Academic Press **3** (1985).

[Sj] Sjölin, P., Regularity of solutions to the Schrödinger equations, Duke Math. J. **55**, 699–715 (1987).

[So] Souyer, A., The Cauchy problem for the Ishimori equations, J. Funct. Anal. **105**, 233–255 (1992).

[Sn] Sung, L.-Y., The Cauchy problem for the Ishimori equation, J. Funct. Anal. **139**, 29–67 (1996).

[SuSuBa] Sulem, P.L., Sulem, C., and Bardos, C., On the continuous limit for a system of classical spins, Comm. Math. Phys. **107**, 431–454 (1986).

[Ve] Vega, L., The Schrödinger equation: pointwise convergence to the initial data, Proc. Amer. Math. Soc. **102** , 874–878 (1988).

[ZaKu] Zakharov, V. E., and Kuznetson, E. A., Multi-scale expansions in the theory of systems integrable by the inverse scattering method, Physica D **18**, 455–463 (1986).

[ZaSc] Zakharov, V. E., and Schulman, E. I., Degenerated dispersion laws, motion invariant and kinetic equations, Physica **1D**, 185–250 (1980).

C.E. Kenig
Dept. of Mathematics
Univ. of Chicago
Chicago, IL 60637

G. Ponce
Dept. of Mathematics
Univ. of California
Santa Barbara, CA 93106

L. Vega
Dep. de Matematicas
Univ. del Pais Vasco
Apartado 644
e-mail:mtpvegol@lg.ehu.es

Communicated by J. Bellissard
submitted 25/09/98, accepted 11/01/99