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Invariant Tori, Effective Stability, and Quasimodes with Exponentially Small Error Terms II - Quantum Birkhoff Normal Forms

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Abstract. The aim of this paper is to obtain quasimodes for a Schrödinger type operator P_h in a semi-classical limit ($h \searrow 0$) with exponentially small error terms which are associated with Gevrey families of KAM tori of its principal symbol H. To do this we construct a Gevrey quantum Birkhoff normal form of P_h around the union Λ of the KAM tori starting from a suitable Birkhoff normal form of H around Λ. As an application we prove sharp lower bounds for the number of resonances of P_h defined by complex scaling which are exponentially close to the real axis. Applications to the discrete spectrum are also obtained.

Let M be either $\mathbf{R}^n$ or a compact real analytic manifold of dimension $n \geq 2$ and let

$$P_h \ = \ \sum_{j=0}^{J} P_j(x, hD)h^j \ , \ 0 < h \leq h_0 \, , \tag{.1}$$

be a formally selfadjoint h-differential operator acting on half densities in $C^\infty(M, \Omega^{\frac{1}{2}})$, where $P_j(x, \xi)$ are polynomials of ξ with analytic coefficients, and $D = (D_1, \ldots, D_n)$, $D_j = -i\partial/\partial x_j$. We denote the principal symbol of P_h by $H(x, \xi) = P_0(x, \xi)$, $(x, \xi) \in T^*(M)$, and suppose that its subprincipal symbol is zero. Our main example will be the Schrödinger operator

$$P_h \ = \ -h^2\Delta \, + \, V(x) \, ,$$

where Δ is the Laplace-Beltrami operator on M, associated with a real analytic Riemannian metric and $V(x)$ is a real analytic potential on M bounded from below.

Given $\varrho > 1$, we define a G^ϱ (Gevrey) quasimode $\mathcal{Q}$ of P_h as follows:

$$\mathcal{Q} = \{(u_m(\cdot, h), \lambda_m(h)) : \, m \in \mathcal{M}_h\},$$

where $u_m(\cdot, h) \in C_0^\infty(M)$ has a support in a fixed bounded domain independent of h, $\lambda_m(h)$ are real valued functions of $h \in (0, h_0]$, $\mathcal{M}_h$ is a finite index set for each fixed h, and

$$(i) \quad ||P_h u_m \ - \ \lambda_m(h) u_m||_{L^2} \ \leq \ C\, e^{-c/h^{1/\varrho}} \ , \ m \in \mathcal{M}_h \, ,$$

$$(ii) \quad |\langle u_m, u_l\rangle_{L^2} \ - \ \delta_{m,l}| \ \leq \ C\, e^{-c/h^{1/\varrho}} \ , \ m, l \in \mathcal{M}_h \, ,$$

for $0 < h \leq h_0$. Here C and c are positive constants, and $\delta_{m,l}$ is the Kronecker index. Recall that for any C^∞ quasimode $\mathcal{Q}$ the right hand side in (i) and (ii) is $O_N(h^N)$ for each $N \geq 0$. We define the G^ϱ micro-support $MS^\varrho(\mathcal{Q}) \subset T^*(M)$ of $\mathcal{Q}$ as follows: $(x_0, \xi_0) \notin MS^\varrho(\mathcal{Q})$ if there exist compact neighborhoods U of x_0 and V of ξ_0 in a given local chart such that for any G^ϱ function v with support in U

$$\int e^{-i\langle x,\xi\rangle/h} v(x) u_m(x,h) dx \ = \ O\left(e^{-c/h^{1/\varrho}}\right) \ , \text{ as } h \searrow 0 \, ,$$

uniformly with respect to $m \in \mathcal{M}_h$ and $\xi \in V$.

We are going to find a Gevrey quasimode $\mathcal{Q}$ of P_h, the Gevrey micro-support of which coincides with the union Λ of a suitable Gevrey family of Kolmogorov-Arnold-Moser (KAM) invariant tori Λ_ω, $\omega \in \Omega_\kappa$, of H, obtained in [19]. For this aim we construct a Quantum Birkhoff Normal Form (QBNF) of P_h around Λ in suitable Gevrey classes starting from the Birkhoff Normal Form (BNF) of its principal symbol H obtained in Theorem 1.1 [19]. In other words, conjugating P_h with an unitary h-Fourier Integral Operator (h-FIO) we transform it to a suitable h-pseudodifferential operator (h-PDO) P_h^0 acting on sections in $C^\infty(\mathbf{T}^n; \mathbf{L})$, where $\mathbf{L}$ is a flat Hermitian linear bundle of $\mathbf{T}^n = (\mathbf{R}/2\pi\mathbf{Z})^n$ associated to the Maslov class of the invariant tori. The operator P_h has a Gevrey symbol

$$p^0(\varphi, I, h) \ \sim \ \sum_{j=0}^{\infty} (K_j(I) + R_j(\varphi, I))h^j \ , \ (\varphi, I) \in \mathbf{T}^n \times D \, ,$$

such that each R_j is flat at the Cantor set

$$\mathbf{T}^n \times E_\kappa \ , \ E_\kappa = \{I \in D : \, \omega(I) \in \Omega_\kappa\} \, ,$$

where $\omega : D \to \Omega$ is a Gevrey diffeomorphism, Ω is a neighborhood of Ω_κ, $\omega(I) - \nabla K_0(I)$ is flat at E_κ, and $K_0(I) + R_0(\varphi, I)$ is just the BNF of H around Λ obtained in [19]. Then R_j turn out to be exponentially small around $\mathbf{T}^n \times E_\kappa$ and we obtain a Gevrey quasimode of P_h (see Corollary 1.2). In the C^∞ case a similar QBNF was first obtained by Colin de Verdière [7] for the Laplace-Beltrami operator Δ on a compact Riemannian manifold M. As a consequence, C^∞ quasimodes for Δ were obtained in [7].

Quasimodes provide information about the spectrum of P_h. If P_h has discrete spectrum, we can find eigenvalues of P_h exponentially close to the quasi-eigenvalues $\lambda_j(h)$. Moreover, the total multiplicity of the part of the spectrum of P_h approximated by $\mathcal{Q}$ modulo an exponentially small error term is given asymptotically by $(2\pi h)^{-n}\mathrm{Vol}\,(\Lambda)$ as $h \searrow 0$. The notion "total multiplicity" will be explained in Sect. 1.2. In the case of "scattering", using a result of Stefanov [25], we shall find a large set of resonances of P_h (defined by complex scaling) which are exponentially close to the real axis (see Sect. 1.3).

Quasimodes associated to a Cantor family of invariant tori were first obtained by Lazutkin [14] for the Laplace operator in strictly convex bounded domains in $\mathbf{R}^2$ (see also [15] and references there) and for $n \geq 2$ by Colin de Verdière [7] who also constructed a QBNF around a family of invariant tori in the C^∞ case. Quasimodes associated with invariant tori of the classical Hamiltonian have been obtained also in [5], [6], [8], [18]. An extension of Nekhoroshev's theorem in quantum mechanics is proposed by Bellissard and Vittot [3]. They investigate the rate of divergence in the Rayleigh-Schrödinger perturbation series when the unperturbed Hamiltonian is given by a family of harmonic oscillators whose frequencies satisfy a small divisor condition. If 0 is a nondegenerate minimum of V, Sjöstrand [22] obtained a quantization formula for all eigenvalues of P_h in an interval $[0, h^\delta]$, where $\delta > 0$ is fixed. Stronger result has been proved recently for Gevrey smooth potentials $V(x)$ by Bambusi, Graffi and Paul [1]. They obtained a quantization formula modulo $O(h^\infty)$ for all eigenvalues of P_h in an interval $[0, \varphi(h)]$ where $\varphi(h)^b \ln h \to 0$ as $h \searrow 0$ and b is an explicitly determined constant. A link between Nekhoroshev's stability for the classical system and the semi-classical asymptotics with exponentially small error term of the low lying eigenvalues of the corresponding Schrödinger operator is suggested by Sjöstrand [22].

The techniques developed in the present paper could be used to obtain quasimodes with exponentially small error terms for the Laplace operator $-\Delta$ with Dirichlet (Neumann) boundary conditions in a domain $\Omega \subset \mathbf{R}^n$ with a compact analytic boundary which are associated to Gevrey families of invariant tori of the broken bicharacteristic flow.

The paper is organized as follows: The main results are formulated in Sect. 1. In Sect. 2 we define suitable classes of Gevrey symbols, h-PDOs and h-FIOs. We conjugate P_h with an elliptic h-FIO T_h to a h-PDO $\tilde{P}_h$ of Gevrey class acting on sections in $C^\infty(\mathbf{T}^n; \mathbf{L})$, the principal symbol of which is just the BNF of H and the subprincipal symbol is 0. In Sect. 3 we obtain a QBNF of $\tilde{P}_h$ conjugating it with an elliptic h-PDO A_h. We first find the full symbol of A_h on the Cantor set $\mathbf{T}^n \times E_\kappa$ and then use a suitable Whitney extension theorem in Gevrey classes. To obtain the full Gevrey symbol of A_h on $\mathbf{T}^n \times E_\kappa$ we have to solve the homological equation

$$\langle \nabla K_0(I), \partial_\varphi \rangle f(\varphi, I) \ = \ g(\varphi, I) \ , \ \varphi \in \mathbf{T}^n \ ,$$

uniformly with respect to $I \in E_\kappa$ and to provide the corresponding Gevrey estimates for the solution. Here $g(\varphi, I)$ is a Gevrey function in $\mathbf{T}^n \times E_\kappa$ in the sense of Whitney, and

$$\int_{\mathbf{T}^n} g(\varphi, I)\, d\varphi \ = \ 0 \, .$$

The analysis of the solution of the homological equation is done in Sect. 4. In Sect. 5 we complete the construction of the normal form of P_h near Λ.

I QBNF around KAM tori and quasimodes

1.1 Main results. We are going to formulate the main assumptions on the principal symbol H of P_h. Fix $\kappa > 0$ and τ such that $\tau > n-1$ when $n \geq 3$ and $\tau > 3/2$ when $n = 2$. Given a bounded domain $\Omega \subset \mathbf{R}^n$ we consider the set Ξ_κ of all

$\omega \in \Omega$ having distance $\geq \kappa$ to the boundary of Ω and satisfying the Diophantine condition

$$|\langle \omega, k\rangle| \;\geq\; \frac{\kappa}{|k|^{\tau}}\,, \quad \text{for all } 0 \neq k \in \mathbf{Z}^n\,, \tag{I.1}$$

where $|k| = |k_1| + \cdots + |k_n|$. We denote by Ω_κ the set of points of a positive Lebesgue density in Ξ_κ, namely, $\omega \in \Omega_\kappa$ iff for any neighborhood U of ω the Lebesgue measure of $U \cap \Omega_\kappa$ is positive. Fix $s = \tau' + 2$ with $\tau' > max\{\tau, 5/2\}$. We suppose that there exists a real analytic exact symplectic diffeomorphism

$$\chi_1 \;:\; \mathbf{T}^n \times D \;\longrightarrow\; U \;\subset\; T^*(M)\,,$$

where D is a domain in $\mathbf{R}^n$ such that the Hamiltonian

$$\widetilde{H}(\phi, I) \;\stackrel{\text{def}}{=}\; (H \circ \chi_1)(\varphi, I) \tag{I.2}$$

admits a G^s-BNF around a family of invariant tori with frequencies in a suitable Ω_κ. In other words, we assume that the Hamiltonian $\widetilde{H}(\phi, I)$ satisfies :

(BF) There exists a domain Ω, a G^s-diffeomorphism $\omega : D \to \Omega$, and an exact symplectic transformation $\chi_0 \in G^{1,s}(\mathbf{T}^n \times D, \mathbf{T}^n \times D)$ such that $\widetilde{H}(\chi_0(\varphi, I)) = K_0(I) + R_0(\varphi, I)$ in $\mathbf{T}^n \times D$, where $K_0 \in G^s(D)$ and $R_0 \in G^{1,s}(\mathbf{T}^n \times D)$ satisfy $D_I^\alpha R_0(\varphi, I) = 0$ and $D_I^\alpha(\nabla K_0(I) - \omega(I)) = 0$ for any $(\varphi, I) \in \mathbf{T}^n \times \omega^{-1}(\Omega_\kappa)$ and $\alpha \in \mathbf{Z}_+^n$. Moreover, there exists a generating function $\Phi \in G^{1,s}(\mathbf{T}^n \times D)$ of χ_0 such that $||\mathrm{Id} - \Phi_{\varphi I}(\varphi, I)|| \leq \varepsilon$ in $\mathbf{T}^n \times D$ for some $0 < \varepsilon < 1$.

Here $||\cdot||$ is the usual sup-norm in the space of $n \times n$ matrices. Recall that Φ is a generating function of χ_0 if $\chi_0(\nabla_I \Phi(\varphi, I), I) = (\varphi, \nabla_\varphi \Phi(\varphi, I))$ for any $(\varphi, I) \in \mathbf{T}^n \times D$. Theorem 1.1 in [19] shows that any real analytic Hamiltonian $\widetilde{H}(\varphi, I)$, $(\varphi, I) \in \mathbf{T}^n \times D$, which is a sufficiently small perturbation of a non-degenerate real analytic completely integrable Hamiltonian $H^0(I)$, satisfies (BF) with $\Omega = \nabla H^0(D)$. The map χ_1 provides "action-angle" coordinates for the completely integrable Hamiltonian H^0 and it can be constructed by the Liouville-Arnold theorem. For example we can take $M = \mathbf{R}^n$ and suppose that V has a non-degenerate minimum $E_0 = V(0)$ and that there are no resonances of order 4 (see (0.3), [19]). Then Corollary 1.3, [19], holds. In this case χ_1 transforms H to its Birkhoff normal form.

Set $\chi = \chi_1 \circ \chi_0 : \mathbf{T}^n \times D \longrightarrow U \subset T^*(M)$. Let Λ be the union of the invariant tori $\Lambda_\omega = \chi(\mathbf{T}^n \times \{I(\omega)\})$ of H with frequencies $\omega \in \Omega_\kappa$, where $\Omega \ni \omega \to I(\omega) \in D$ is the inverse to the frequency map $D \ni I \to \omega(I) \in \Omega$. The Maslov class of Λ_ω, $\omega \in \Omega_\kappa$, can be identified to an element ϑ of $H^1(\mathbf{T}^n; \mathbf{Z}) = \mathbf{Z}^n$ via the symplectic map χ. Notice that $\vartheta = (2, \ldots, 2)$ in the case when V has a nondegenerate minimum $E_0 = V(0)$. As in [7] we consider the flat Hermitian line bundle $\mathbf{L}$ over $\mathbf{T}^n$ which is associated to the class ϑ. The sections f in $\mathbf{L}$ can be identified canonically with functions $\widetilde{f} : \mathbf{R}^n \to \mathbf{C}$ so that

$$\widetilde{f}(x + 2\pi p) \;=\; e^{i\frac{\pi}{2}\langle \vartheta, p\rangle}\widetilde{f}(x) \tag{I.3}$$

for each $x \in \mathbf{R}^n$ and $p \in \mathbf{Z}^n$. It is easy to see that an orthonormal basis of $L^2(\mathbf{T}^n; \mathbf{L})$ is given by $e_m,\ m \in \mathbf{Z}^n$, where

$$\widetilde{e}_m(x) \ = \ \exp\left(i\langle m + \vartheta/4, x\rangle\right).$$

Set $\nu = \tau + n + 1$ and fix τ' such that $\tau + n - 1 \ > \tau' \ > \ \max(\tau, 5/2)$. Then fix $\nu \ > \ \mu \ > \ \tau' + 2$, choose $\sigma > 1$ sufficiently close to 1 such that

$$\nu \ > \ \mu \ > \ \sigma(\tau' + 1) + 1\,, \tag{I.4}$$

and set $\varrho = \sigma\nu$. Thus ϱ could be any number bigger than ν and sufficiently close to ν. Set $\ell = (\sigma, \mu, \varrho)$ and consider the corresponding class of Gevrey symbols $S_\ell(\mathbf{T}^n \times D)$ (see Sect. 2). Starting from the $G^{\tau'+1}$-BNF of H around Λ given by (BH), we are going to find a QBNF of P_h around Λ in the class of h-PDOs in $L^2(\mathbf{T}^n; \mathbf{L})$ with a symbol in $S_\ell(\mathbf{T}^n \times D)$, conjugating P_h with a suitable h-FIO with canonical relation $C = \mathrm{graph}\,(\chi)$. Recall that P_h is a selfadjoint h-differential operator acting on half densities in $C^\infty(M, \Omega^{\frac{1}{2}})$ of the form (.1) with analytic coefficients in M and with a subprincipal symbol equal to zero.

Theorem I.1 *Suppose that there exists a real analytic exact symplectic map* $\chi_1 : \mathbf{T}^n \times D \ \to \ U \ \subset \ T^*(M)$ *such that the Hamiltonian* $\widetilde{H}(\varphi, I) \ = \ H(\chi_1(\varphi, I))$, $(\varphi, I) \in \mathbf{T}^n \times D$, *satisfies* (BF) *for* $s = \tau' + 2$. *Then there exist a family of uniformly bounded* h*-FIOs* $U_h : L^2(\mathbf{T}^n; \mathbf{L}) \to L^2(M)$, $0 < h \le h_0$, *associated with the canonical relation* C *such that the following holds:*

(i) $U_h^* U_h - \mathrm{Id}$ *is a pseudodifferential operator with a symbol in the Gevrey class* $S_\ell(\mathbf{T}^n \times D)$ *which is equivalent to 0 on* $\mathbf{T}^n \times D_0$, *where* D_0 *is a subdomain of* D *containing the union* Λ *of the invariant tori*

(ii) $P_h \circ U_h \ = \ U_h \circ P_h^0$, *and the full symbol* $p^0(\varphi, I, h)$ *of* P_h^0 *has the form* $p^0(\varphi, I, h) \ = \ K^0(I, h) \ + \ R^0(\varphi, I, h)$, *where the symbols*

$$K^0(I,h) \ = \sum_{0 \le j \le \eta h^{-1/\varrho}} K_j(I) h^j \quad \text{and} \quad R^0(\varphi, I, h) \ = \sum_{0 \le j \le \eta h^{-1/\varrho}} R_j(\varphi, I) h^j$$

belong to the Gevrey class $S_\ell(T^*(\mathbf{T}^n))$, $\eta > 0$ *is a constant,* K^0 *is real valued, and* R^0 *is equal to zero to infinite order on the Cantor set* $\mathbf{T}^n \times E_\kappa$.

As a consequence we obtain a G^ϱ - quasimode $\mathcal{Q}$ of P_h with an index set

$$\mathcal{M}_h \ = \ \{m \in \mathbf{Z}^n : \ |\, E_\kappa \ - \ h(m + \vartheta/4)| \ \le \ h^\varepsilon\}$$

where $\varepsilon = \varepsilon(\mu) \in (0,1)$. It is easy to see that

$$\begin{aligned}
\#\{m \in \mathcal{M}_h\} \ &= \ \frac{1}{(2\pi h)^n}\,\mathrm{Vol}\,(\mathbf{T}^n \times E_\kappa)(1 + o(1)) \\
&= \ \frac{1}{(2\pi h)^n}\,\mathrm{Vol}\,(\Lambda)(1 + o(1))\,, \ h \searrow 0\,,
\end{aligned} \tag{I.5}$$

where $\mathrm{Vol}\,(\Lambda)$ stands for the Lebesgue measure of the union Λ of the invariant tori in $T^*(M)$.

Corollary I.2 *Let* $u_m(x,h) = U_h(e_m)(x)$, *and* $\lambda_m(h) = K^0(h(m+\frac{1}{4}\vartheta),h)$, *for* $m \in \mathcal{M}_h$. *Then*

$$\mathcal{Q} = \{(u_m(x,h), \lambda_m(h)) : m \in \mathcal{M}_h\}$$

is a G^ϱ*-quasimode of* P_h. *Moreover,*

$$MS^\varrho(\mathcal{Q}) = \Lambda. \tag{I.6}$$

To prove Corollary 1.2 we write $P_h^0 = K_h^0 + R_h^0$, where the symbols of K_h^0 and R_h^0 are $K^0(I,h)$ and $R^0(\varphi, I, h)$ respectively. It is easy to see that

$$P_h^0(e_m)(\varphi) = (\lambda_m(h) + R^0(\varphi, h(m+\vartheta/4), h))\, e_m(\varphi)$$

for any $m \in \mathcal{M}_h$. On the other hand,

$$|D_\varphi^\beta D_I^\alpha R^0(\varphi, I, h)| \le C^{|\alpha|+|\beta|+1} \beta!^{\,\sigma} \alpha!^{\,\mu}\ ,\ \forall\,(\varphi, I, h) \in \mathbf{T}^n \times D \times (0, h_0]\,,$$

because of (II.3). Then there exist two positive constants C_1 and c depending only on the constant C such that for every $\alpha, \beta \in \mathbf{Z}_+^n$ the following estimate holds

$$|\partial_\varphi^\beta \partial_I^\alpha R^0(\varphi, I, h)| \le C_1^{|\alpha|+|\beta|+1} \beta!^{\,\sigma} \alpha!^{\,\mu} \exp\left(-c\,|E_\kappa - I|^{\,-\frac{1}{\mu-1}}\right)\,,$$

for each $(\varphi, I, h) \in \mathbf{T}^n \times D \times (0, h_0]$, $I \notin E_\kappa$, where $|E_\kappa - I| = \inf_{I' \in E_\kappa} |I' - I|$ is the distance to the compact set E_κ (see [19], (1.3)). Using the inequality $\mu < \nu < \varrho$, and choosing appropriately ε we prove that $\mathcal{Q}$ satisfies (i) in the introduction. On the other hand (ii) and (I.6) follow directly from the definition of the index set $\mathcal{M}_h$, the orthogonality of e_m, and (i) in Theorem 1.1. □

1.2 Applications to the discrete spectrum. Consider now the Schrödinger operator $P_h = -h^2\Delta + V(x)$ in M, where Δ is the Laplace-Beltrami operator associated with a real analytic Riemannian metric on M which coincides with the Euclidean metric when $M = \mathbf{R}^n$. Suppose that P_h satisfies the assumptions of Theorem 1.1 in a bounded subdomain of $T^*(M)$. Set $E_1 = \max\{H(x,\xi) : (x,\xi) \in \Lambda\}$. Suppose that $H^{-1}((-\infty, E_2])$ is compact for some $E_2 > E_1$ and fix $E \in (E_1, E_2)$ and $E_0 < \min\{H(x,\xi) : (x,\xi) \in \Lambda\}$. We need that assumption only when $M = \mathbf{R}^n$. Then P_h, $0 < h \le h_0$, has only a discrete spectrum in $[E_0, E]$. Hereafter $h_0 > 0$ is chosen sufficiently small. Fix $c \ge \varepsilon \ge 0$ and $C' > C$, where c and C are the constants in the definition of $\mathcal{Q}$. Denote by Π^h the spectral projector of P_h and for each $0 < h \le h_0$ and $m \in \mathcal{M}_h$ set

$$\Delta_{\varepsilon,m}^h = \left[\lambda_m(h) - C'\, e^{-(c-\varepsilon)/h^{1/\varrho}}\,,\ \lambda_m(h) + C'\, e^{-(c-\varepsilon)/h^{1/\varrho}}\right].$$

Then there exists at least one eigenvalue of P_h in $\Delta_{0,m}^h$, and we have

$$||\Pi^h(\Delta_{\varepsilon,m}^h)u_m - u_m|| \le e^{-\varepsilon/h^{1/\varrho}}\,,\ 0 < h \le h_0 \ll 1\,, m \in \mathcal{M}_h$$

(see [15], Proposition 32.1 and (32.2)). Set

$$\mathcal{I}^h = \cup\{\Delta^h_{\varepsilon,m} : m \in \mathcal{M}_h\}$$

and fix $A > 2(2\pi)^{-n}\operatorname{Vol}(\Lambda)$. Taking into account (I.5) we obtain that $\mathcal{I}^h \subset [E_0, E]$ is a finite union of disjoint intervals $\mathcal{I}^h_j$ of length

$$\left|\mathcal{I}^h_j\right| \le AC' h^{-n} e^{-(c-\varepsilon)/h^{1/\varrho}} .$$

Denote by $\mathcal{L}^h_j$ the span of all $u_m(\cdot, h)$ such that $m \in \mathcal{M}_h$ and $\lambda_m(h) \in \mathcal{I}^h_j$. Then

$$||\Pi^h(\mathcal{I}^h_j)v - v|| \le A' h^{-n} e^{-\varepsilon/h^{1/\varrho}} ||v||, \ 0 < h \le h_0 \ll 1 ,$$

for each $v \in \mathcal{L}^h_j$ and some constant $A' > 0$. Then it is natural to call

$$N^*_h(\mathcal{I}^h) = \sum_j \dim \Pi^h(\mathcal{I}^h_j)\mathcal{L}^h_j$$

total multiplicity of the part of spectrum of P_h in $\mathcal{I}^h$ which is approximated by the quasimode $\mathcal{Q}$ modulo an exponentially small error term (for C^∞ quasimodes see [15]). Moreover, we have

$$\dim \Pi^h(\mathcal{I}^h_j)\mathcal{L}^h_j = \dim \mathcal{L}^h_j , \ 0 < h \le h_0 \ll 1 ,$$

hence,

$$N^*_h(\mathcal{I}^h) = \#\{m \in \mathcal{M}_h\} = \frac{1}{(2\pi h)^n} \operatorname{Vol}(\Lambda)(1 + o(1)), \ h \searrow 0 . \qquad \text{(I.7)}$$

Recall that the function $N_h([E_0, E])$ counting with multiplicities the eigenvalues of P_h in $[E_0, E]$ has a semiclassical asymptotic $N_h([E_0, E]) = (2\pi h)^{-n} C_1(1 + o(1))$, where $C_1 = \operatorname{Vol}(H^{-1}([E_0, E])$ is the Lebesgue measure of $H^{-1}([E_0, E])$ in $T^*(M)$.

1.3 Applications to resonances. Consider a selfadjoint second order differential operator in $\mathbf{R}^n$

$$P_h = \sum_{|\alpha|+j \le 2} a_\alpha(x)(hD)^\alpha h^j .$$

As in [26] we impose the following hypothesis:

(H_1) The coefficients $a_\alpha(x)$ are real analytic and they can be extended holomorphically to

$$\{r\omega : \omega \in \mathbf{C}^n, \operatorname{dist}(\omega, \mathbf{S}^n) < \varepsilon, r \in \mathbf{C}, |r| > R, \arg r \in [-\varepsilon, \theta_0 - \varepsilon]\}$$

for some $\varepsilon > 0$ and $\theta_0 > 0$ and the coefficients of $-h^2\Delta - P_h$ tend to zero as $|x| \to \infty$ in that set uniformly with respect to h.

(H_2) For some $C > 0$ we have

$$\sum_{|\alpha|=2} a_\alpha(x)\xi^\alpha \;\geq\; C\,|\xi|^2\,,\;(x,\xi)\in T^*(\mathbf{R}^n)\,.$$

Then the resonances $\operatorname{Res} P_h$ of P_h close to the real axis can be defined in a conic neighborhood Γ of the positive half axis in the lower half plain by the method of complex scaling (see [23] and [24]). They coincide in Γ with the poles of the meromorphic continuation of the resolvent

$$(P_h - z)^{-1} : \,L^2_{\mathrm{comp}}(\mathbf{R}^n) \,\to\, H^2_{\mathrm{loc}}(\mathbf{R}^n)\,,\;\operatorname{Im} z \,>\, 0\,.$$

Thang and Zworski [26] obtained a result which implies lower bounds of the the resonances $\operatorname{Res} P_h$ of P_h close to the real axis for any $h \in (0, h_0]$, provided that there exists a quasimode $\mathcal{Q}$ for P_h. Stefanov [25] obtained sharp lower bounds, he showed that for each $h \in (0, h_0]$ the number of the resonances of P_h close to the real axis is not less than the cardinality of the index set $\mathcal{M}_h$ of the quasimode $\mathcal{Q}$. We set

$$N_h \;=\; \#\{\lambda \in \operatorname{Res} P_h : \,\operatorname{Re}\lambda \in [E_0, E],\, 0 \,<\, -\operatorname{Im}\lambda \,\leq\, h^{-n-2}e^{-c/h^{1/\varrho}}\,\}\,,$$

where the resonances are counted with multiplicities, $c > 0$ is the constant in the definition of $\mathcal{Q}$ and $E_0 < E$ are as in *1.2.* Burq [4] showed that there exists $\varepsilon > 0$ and $C > 0$ such that there are no resonances of P_h, $0 < h \leq h_0$, in

$$\{\lambda \in \mathbf{C} : \,\operatorname{Re}\lambda \in [E_0, E],\, 0 \,<\, -\operatorname{Im}\lambda \,\leq\, \varepsilon e^{-C/h}\,\}\,.$$

Combining Corollary 1.2 with Theorem 1.1 in [25] (which holds also for non-compactly supported perturbations of $-h^2\Delta$ satisfying (H_1) and (H_2)), and using (I.5), we obtain the following:

Theorem I.3 *Suppose that P_h satisfies (H_1), (H_2), and the assumptions of Theorem 1.1. Then*

$$N_h \;\geq\; \frac{1}{(2\pi h)^n}\operatorname{Vol}(\Lambda)(1 + o(1))\,,\;h \searrow 0\,.$$

II Gevrey symbols h-PDOs and h-FIOs

2.1. Gevrey symbols. We are going to put the operator P_h in a QBNF around the union of the invariant tori Λ conjugating it by an elliptic h-FIO with a suitable Gevrey symbol. The resulting operator will be a h-PDO with a Gevrey symbol. First we define the class of Gevrey symbols that we need. Denote by D a bounded domain in $\mathbf{R}^n$. Let X be either $\mathbf{T}^n$ or a bounded domain in $\mathbf{R}^m$, $m \geq 1$. Fix $\sigma, \mu > 1$, $\varrho \geq \sigma + \mu - 1$, and set $\ell = (\sigma, \mu, \varrho)$. We introduce a class of formal Gevrey symbols $FS_\ell(X \times D)$ as follows. Consider a sequence of smooth functions

$p_j \in C_0^\infty(X \times D)$, $j \in \mathbf{Z}_+$ such that $\operatorname{supp} p_j$ is contained in a fixed compact subset of $X \times D$. We say that

$$\sum_{j=0}^{\infty} p_j(\varphi, I)\, h^j \tag{II.1}$$

is a formal Gevrey symbol in $FS_\ell(X \times D)$ if there exists a positive constant C such that p_j satisfies the estimates

$$\sup_{X\times D} |\partial_\varphi^\beta \partial_I^\alpha p_j(\varphi, I)| \ \le\ C^{j+|\alpha|+|\beta|+1}\, \beta!^{\,\sigma}\, \alpha!^{\,\mu}\, j!^{\,\varrho} \tag{II.2}$$

for any α, β and j.

The function $p(\varphi, I; h)$, $(\varphi, I) \in X \times \mathbf{R}^n$, is called a realization of the formal symbol (II.1) in $X \times D$ if for each $0 < h \le h_0$ it is smooth with respect to (φ, I) and has compact support in $X \times D$, and if there exists a positive constant C_1 such that

$$\begin{aligned} &\sup_{\mathbf{Q}} \big|\partial_\varphi^\beta \partial_I^\alpha (p(\varphi, I, h) - \sum_{j=0}^{N} p_j(\varphi, I) h^j)\big| \\ &\qquad \le\ h^{N+1}\, C_1^{N+|\alpha|+|\beta|+2}\, \beta!^{\,\sigma}\, \alpha!^{\,\mu}\, (N+1)!^{\,\varrho} \end{aligned} \tag{II.3}$$

for any multi-indices α, β and $N \in \mathbf{Z}_+$, where $\mathbf{Q} = X \times D \times (0, h_0]$. For example, one can take

$$p(\varphi, I, h) \ =\ \sum_{j \le \varepsilon h^{-1/\varrho}} p_j(\varphi, I)\, h^j\ ,$$

where $\varepsilon > 0$ depends only on the constant C_1 and the dimension n (for $\sigma = \mu = 1$ see [22], Sect. 1). We denote by $S_\ell(X \times D)$ the corresponding class of symbols.

Given $g \in S_\ell(X \times D)$, we say that $g \in S_\ell^{-\infty}(X \times D)$ if

$$\sup_{\mathbf{Q}} |\partial_\varphi^\beta \partial_I^\alpha\, g(\varphi, I; h)| \ \le\ h^N\, C^{N+|\alpha+\beta|+1}\, \beta!^{\,\sigma}\, \alpha!^{\,\mu}\, N!^{\,\varrho}$$

for $0 < h \le h_0$, $\forall N \in \mathbf{Z}_+$, and for any multi-indices $\alpha, \beta \in \mathbf{Z}_+^N$, or equivalently

$$\sup_{\mathbf{Q}} |\partial_\varphi^\beta \partial_I^\alpha\, g(\varphi, I; h)| \ \le\ C_1^{|\alpha+\beta|+1}\, \beta!^{\,\sigma}\, \alpha!^{\,\mu}\, \exp(-ch^{-1/\varrho})$$

for some $C_1,\ c > 0$, and any $h \in (0, h_0]$, $\alpha,\ \beta\ \in \mathbf{Z}_+^n$. Moreover, given $f, g \in S_\ell(X \times D)$, we say that f is equivalent to g ($f \sim g$) if $f - g \in S_\ell^{-\infty}(X \times D)$. It is not hard to prove that any two realizations of $\sum_{j=0}^{\infty} p_j h^j$ in $S_\ell(X \times D)$ are equivalent. When $\sigma = \mu$ and $\varrho = 2\sigma - 1$, we set $S^\sigma = S_\ell$ and $S^{\sigma,-\infty} = S_\ell^{-\infty}$.

Having two symbols $p, q \in S_\ell(X \times D)$ we denote their composition by $p \circ q \in S_\ell(X \times D)$ which is the realization of

$$\sum_{j=0}^{\infty} c_j\, h^j \ \in\ FS_\ell(X \times D),$$

where

$$c_j(\varphi, I) \;=\; \sum_{r+s+|\gamma|=j} \frac{1}{\gamma!} D_I^\gamma p_r(\varphi, I)\, \partial_\varphi^\gamma q_s(\varphi, I). \tag{II.4}$$

In particular, S_ℓ becomes an algebra under this composition. Having a symbol $p \in S_\ell(X \times D)$ associated to the formal symbol (II.1), we define its conjugate p^* as the realization of the formal symbol

$$\sum_{j=0}^{\infty} c_j\, h^j \;\in\; FS_\ell(X \times D),$$

where

$$c_j(\varphi, I) \;=\; \sum_{r+|\gamma|=j} \frac{1}{\gamma!} D_I^\gamma \partial_\varphi^\gamma p_r(\varphi, I).$$

To each symbol $p \in S_\ell(X \times D)$ we associate an h-pseudodifferential operator (h-PDO) by

$$P_h u(x) \;=\; (2\pi\, h)^{-n} \int_{\mathbf{R}^{2n}} e^{i\langle x-y,\xi\rangle/h}\, p(x,\xi,h)\, u(y) d\xi dy,\;\; u \in C_0^\infty(X).$$

It is well defined modulo $\exp(-ch^{-1/\varrho})$. Indeed, for any $p \in S_\ell^{-\infty}$ we have

$$||P_h u||_{L^2} \;\le\; C \exp(-ch^{-1/\varrho}) ||u||_{L^2}\,,\;\; u \in C_0^\infty(X),$$

with some positive constants c and C. Then the composition of two h-PDOs P_h and Q_h with symbols $p, q \in S_\ell(X \times D)$ is a h-PDO of the same class with a symbol $p \circ q$, and the L^2-adjoint of P_h has a symbol p^*. Moreover, h-PDOs with symbols of the class $S^\sigma = S_\ell$, $\ell = (\sigma, \sigma, 2\sigma - 1)$, $\sigma > 1$, remain in the same class after a G^σ change of the x variables, and they can be defined as well on any G^σ compact manifold (see Theorem 2.3 [9]).

Let $u(x,h)$ be a family of smooth functions in M for $0 < h \le h_0$. The G^ϱ micro-support $MS^\varrho(u) \subset T^*(M)$ of u is defined as follows: $(x_0, \xi_0) \notin MS^\varrho(u)$ if there exists $c > 0$ and compact neighborhoods U of x_0 and V of ξ_0 in a given local chart such that for any G^ϱ function v with compact support in U

$$\int e^{-i\langle x,\xi\rangle/h}\, v(x) u(x,h) dx \;=\; O\left(e^{-ch^{-1/\varrho}}\right)\;,\;\text{as } h \searrow 0\,,$$

uniformly with respect to $\xi \in V$. Obviously, $(x_0, \xi_0, x_0, -\xi_0)$ does not belong to the G^ϱ microsupport of the distribution kernel of the h-PDO P_h above if its amplitude $p \in S_\ell$ belongs to $S_\ell^{-\infty}$ in a neighborhood of (x_0, ξ_0).

2.2. Quantization of χ_1. We are going to quantize the real analytic symplectic transformation $(x,\xi) = \chi_1(y,\eta)$ defined by (I.2). Set

$$C_1 \;=\; \{(\chi_1(y,\eta), y, \eta) :\, (y,\eta) \in \mathbf{T}^n \times D\}$$

and denote

$$C_1' \;=\; \{(x,y,\xi,\eta):\,(x,\xi,y,-\eta)\in C_1\}.$$

Recall that $\chi_1 : \mathbf{T}^n \times D \to T^*(M)$, $D \subset \mathbf{R}^n$, is exact symplectic , hence C_1' is an exact Lagrangian submanifold of $T^*(M \times \mathbf{T}^n)$. In other words, the pull-back $\imath^*\alpha$ of the canonical one-form α of $T^*(M\times\mathbf{T}^n)$ via the inclusion map is an exact form,

$$\imath^*\alpha = df \tag{II.5}$$

for some analytic function f on C_1'. This means that the Liouville class $[\imath^*\alpha]$ of C_1' is trivial in the first cohomology group $H^1(C_1';\mathbf{R})$ which allows us to quantize χ_1. Given $\sigma > 1$, we are going to define a class of h-FIOs the distribution kernels of which are oscillatory integrals in the sense of Duistermaat [5] associated with C_1' and having Gevrey symbols in S^σ.

Locally C_1' can be defined by a nondegenerate real analytic phase function as follows. Let us fix some $\zeta_0 = (x_0, y_0, \xi_0, \eta_0)$ in C_1'. Choosing suitable analytic local coordinates x in a neighborhood U_0 of x_0, we can parameterize (locally) the Lagrangian manifold C_1 by $(y,\xi)\in U_1\times U_2$, where U_1 is a local chart of $\mathbf{T}^n$ and U_2 is a neighborhood of ξ^0 in $\mathbf{R}^n$. Then there exists a real analytic function $\phi(y,\xi)$ in $U_1\times U_2$ such that $C_1 \;=\; \{(\phi'_\xi,\xi,y,\phi'_y)\}$ and $\det \partial^2\phi/\partial y\partial\xi \;\neq\; 0$ in $U_1\times U_2$ (see [12], Proposition 25.3.3). It is uniquely defined up to a constant, and we fix it by $\phi(y_0,\xi_0) \;=\; \langle x_0,\xi_0\rangle - f(\zeta_0)$, where f is given by (II.5). The real analytic phase function $\Psi(x,y,\xi) = \langle x,\xi\rangle - \phi(y,\xi)$ defines locally the Lagrangian manifold C_1', namely, $\operatorname{rank} d_{(x,y,\xi)} d_\xi \Psi = n$ on $O_\Psi = \{(x,y,\xi):\, d_\xi\Psi = 0\}$, and the map

$$\imath_\Psi : O_\Psi \ni (x,y,\xi) \longrightarrow (x,y,\Psi'_x,\Psi'_y)\in C'_\Psi$$

is a local diffeomorphism in C_1'. Moreover, we have

$$\Psi(x_0,y_0,\xi_0) \;=\; f(\zeta_0). \tag{II.6}$$

We are ready to define h-FIOs associated to C_1 and mapping $C^\infty(\mathbf{T}^n;\Omega^{\frac12}\times\mathbf{L})$ to $C_0^\infty(M,\Omega^{\frac12})$, where $\Omega^{\frac12}$ is the corresponding half density bundle and the sections in $\mathbf{L}$ are defined by (I.3). Fix $\sigma > 1$ and choose a symbol $a \in S_\ell(U\times U_2) = S^\sigma(U\times U_2)$, $\ell = (\sigma,\sigma,2\sigma-1)$, where $U = U_0\times U_1$. We extend a for $y\in\mathbf{R}^n$ by

$$\widetilde{a}(x,y+2\pi p,\xi,h) \;=\; e^{-i\frac{\pi}{2}\langle\vartheta,p\rangle}\, a(x,y,\xi,h)\,,\ (x,y,\xi)\in U\times\mathbf{R}^n\,,\ p\in\mathbf{Z}^n\,,$$

and we extend ϕ as a 2π periodic function with respect to y in $U_0\times(U_1+2\pi\mathbf{Z}^n)\times U_2$. Then given a section $u\in C^\infty(\mathbf{T}^n;\mathbf{L})$ of the linear bundle $\mathbf{L}$ we set

$$T_h u(x) \;=\; (2\pi\, h)^{-n}\int_{\mathbf{R}^n}\int_{U_1} e^{i\Psi(x,y,\xi)/h}\,\widetilde{a}(x,y,\xi,h)\,\widetilde{u}(y)d\xi dy, \tag{II.7}$$

where $\widetilde{u}$ satisfies (I.3). Notice that $\widetilde{a}(x,y,\xi,h)\widetilde{u}(y)$ is 2π periodic with respect to y in $\mathbf{R}^n$ and we can replace U_1 by $U_1 + 2\pi p$ for any $p\in\mathbf{Z}^n$. Denote by $K_h(\Psi,a)$ the distribution kernel of T_h. We define a class of h-FIOs

$$T_h :\, C^\infty(\mathbf{T}^n;\Omega^{\frac12}\otimes\mathbf{L}) \;\to\; C_0^\infty(M,\Omega^{\frac12})$$

with G^σ Gevrey symbols as a finite sum of operators given microlocally by (II.7), where the half density bundles have been trivialized by dividing with the corresponding canonical half densities.

We denote the class of the distribution kernels K_h of T_h by $I_\sigma(M \times \mathbf{T}^n, C_1'; \Omega^{\frac{1}{2}} \otimes \mathbf{L}', h)$, where $\mathbf{L}'$ is the dual bundle to $\mathbf{L}$. One can show that the definition does not depend on the choice of the phase functions. Indeed, fix $\zeta_0 \in C_1'$ as above and choose as above a real analytic nondegenerate phase function $\Phi(x, y, \xi)$ such that $C_1' = C_\Phi$ locally near ζ_0 and such that (II.6) holds. It can be proved that there exists a symbol $g \in S^\sigma$ such that $\zeta_0 \notin MS^{2\sigma-1}(K_h - K_h(\Phi, g))$, where $K_h(\Psi, a)$ denotes the distribution kernel of (II.7) (one can also take more general phase functions as in [5], Proposition 1.3.1). Here we use the following stationary phase lemma:

Lemma II.1 *Let $\Phi(x, y)$ be a real analytic function in a neighborhood of $(0, 0)$ in $\mathbf{R}^{m_1+m_2}$. Assume that $\Phi_x'(0, 0) = 0$ and that $\Phi_{xx}''(0, 0)$ is non-singular. Denote by $x(y)$ the solution of the equation $\Phi_x'(x, y) = 0$ with $x(0) = 0$ given by the implicit function theorem. Then for any $g \in S^\sigma(U)$, where U is a suitable neighborhood of $(0, 0)$ we have*

$$\int e^{i\Phi(x,y)/h}\, g(x, y, h)dx \;=\; e^{i\Phi(x(y),y)/h}\, G(y, h),$$

where $G \in S^\sigma$.

To prove the lemma we first use the Morse lemma with parameters for real analytic functions which can be proved as in [12], Lemma C.6.1, and then we follow the proof of Lemma 7.7.3 in [12] (see also [9]). Actually it could be proved that Lemma 2.1 holds also when $\Phi \in G^\sigma$.

The principal symbol of T_h (see [5], [16]) is of the form $e^{if(\zeta)/h}\Upsilon(\zeta)$, where Υ is a smooth section in $\Omega^{\frac{1}{2}}(C_1') \otimes M_C \otimes \pi_C^*(\mathbf{L}')$. Here $\Omega^{\frac{1}{2}}(C_1')$ is the half density bundle of C_1', M_C is the Maslov bundle of C_1', and $\pi_C^*(\mathbf{L}')$ is the pull-back of $\mathbf{L}'$ via the canonical projection $\pi_C : C_1' \to \mathbf{T}^n$. The bundle $\Omega^{\frac{1}{2}}(C_1')$ is trivialized by the pull-back of the canonical half density of $\mathbf{T}^n \times D$ via the canonical projection $\pi_2 : C_1' \to \mathbf{T}^n \times D$. As in the proof of Theorem 2.5, [7], $\pi_C^*(\mathbf{L}')$ can be canonically identified with the dual M_C' of the Maslov bundle. Hence, the principal symbol of T_h can be canonically identified with a smooth function b on C_1'. Moreover, for any T_h of the form (II.7) we have

$$b(\phi_\xi'(y, \xi), y, \xi, -\phi_y'(y, \xi)) = a_0(\phi_\xi'(y, \xi), y, \xi)|\det \partial^2\phi/\partial y \partial\xi(y, \xi)|^{-1/2},$$

where a_0 is the leading term of the amplitude a.

We choose an operator T_{1h} as above with a principal symbol equal to one in a neighborhood of the pull-back via π_2 of the union of the invariant tori Λ of $H \circ \chi_1$, given by (BF).

Using Lemma 2.1 it can be proved that $Q_h = T_{1h}^* T_{1h}$ is a h-PDO in $C^\infty(\mathbf{T}^n, \mathbf{L})$, with a symbol $q(x, \xi) = \sum_{j=0}^\infty q_j(x, \xi)h^j$ in $S^\sigma(\mathbf{T}^n \times D)$. Moreover, its principal symbol is equal to 1 in a neighborhood U of Λ and we can assume that $q_1(x,\xi)=0$ in U. To do this we write $T_{1h}=A_h+hB_h$, where the principal symbol of A_h is equal to 1 in U, and then we solve a linear equation for the real part of the principal symbol of B. Let us conjugate P_h by an operator T_{1h} defined as above. Using Lemma 2.1 it can be proved that $P_h^1=T_{1h}^*P_hT_{1h}$ is a h-PDO in $C^\infty(\mathbf{T}^n,\mathbf{L})$, with a symbol in $S^\sigma(\mathbf{T}^n\times D)$. Moreover, we have $P_h^1=T_{1h}^{-1}P_hT_{1h}+h^2R_h$, where R_h is a h-PDO. As in Lemma 2.9, [7], we obtain that the principal symbol of P_h^1 is equal to $H\circ\chi_1$ and that its subprincipal symbol is zero.

2.3. Quantization of χ_0. We are going to conjugate P_h^1 with a h-FIO T_{2h} : $L^2(\mathbf{T}^n;\mathbf{L})\to L^2(\mathbf{T}^n;\mathbf{L})$ associated to the canonical relation graph(χ_0), where $(x,\xi)=\chi_0(y,I)$ is given by (BF). The distribution kernel of T_{2h} has the form

$$(2\pi\, h)^{-n}\int e^{i(\langle x-y,I\rangle+\phi(x,I))/h}\, b(x,I,h)\, dI,$$

where $\phi(x,I)=\Phi(x,I)-\langle x,I\rangle$, and $\Phi\in G^{1,s}(\mathbf{T}^n\times D)$ is given by (BF), while b is a symbol of Gevrey class $S_{\widetilde{\ell}}(\mathbf{T}^n\times D)$ with $\widetilde{\ell}=(\sigma,\mu,\sigma+\mu-1)$ and $\mu>s=\tau'+1>\sigma>1$ is fixed in (I.4). We suppose that the principal symbol of T_{2h} is equal to 1 in a neighborhood of $\mathbf{T}^n\times D$. Set $T_h=T_{1h}\circ T_{2h}$.

Proposition II.2 *The operator $\widetilde{P}_h=T_h^*\circ P_h\circ T_h$ is a h-PDO with a symbol in the class $S_{\widetilde{\ell}}$, where $\widetilde{\ell}=(\sigma,\mu,\sigma+\mu-1)$. Moreover, the principal symbol of $\widetilde{P}_h$ equals $\widetilde{H}=H\circ\chi$, and its sub-principal symbol is zero.*

Proof. We are going to show that $\widetilde{P}_h=T_{2h}^*\circ P_h^1\circ T_{2h}$ is a h-PDO with a symbol in $S_{\widetilde{\ell}}$. Denote by $a\in S^\sigma=S_{(\sigma,\sigma,2\sigma-1)}(\mathbf{T}^n\times D)$ the amplitude of P_h^1 and recall that $b\in S_{\widetilde{\ell}}(\mathbf{T}^n\times D)$. Choosing a suitable partition of the unity in $\mathbf{T}^n$, we suppose that the support of $b(z,\eta,h)$ with respect to z is contained in a fixed local chart of $\mathbf{T}^n$. Then the Schwartz kernel of the operator $P_h^1\circ T_{2h}$ can be written in the form

$$\begin{aligned}&(2\pi\, h)^{-n}\int_{\mathbf{R}^n} e^{i(\langle x-y,\eta\rangle+\phi(x,\eta))/h}\\ &\times\left((2\pi h)^{-n}\int_{\mathbf{R}^n\times D} e^{i\psi(x,z,\xi,\eta)/h}\, q(x,z,\xi,\eta,h)\, dzd\xi\right)\, d\eta\,,\end{aligned} \tag{II.8}$$

where $q(x,z,\xi,\eta,h)=a(x,\xi,h)b(z,\eta,h)$, and

$$\begin{aligned}\psi(x,z,\xi,\eta) &= \langle x-z,\xi-\eta\rangle+\phi(z,\eta)-\phi(x,\eta)\\ &= \langle x-z,\xi-\eta+\widetilde{\phi}_z(x,z,\eta)\rangle\,.\end{aligned}$$

Setting $\widetilde{x}=(x,z)$ and $\widetilde{\xi}=(\xi,\eta)$ we obtain that $q(\widetilde{x},\widetilde{\xi},h)$ belongs to the symbol class $S_{\widetilde{\ell}}$. Consider the inner integral $u(x,\eta,h)$ in (II.8). Changing the variables in it we obtain

$$u(x,\eta,h) \;=\; (2\pi h)^{-n}\int e^{i\langle z,\xi\rangle/h}\, Q(x,z,\xi,\eta,h)\, dzd\xi \quad (\operatorname{mod} S^{\sigma,-\infty}\,)\,,$$

where Q is again in $S_{\widetilde{\ell}}$. Applying the Taylor formula at $z = 0$ and then integrating by parts with respect to ξ we obtain that u belongs to $S_{\widetilde{\ell}}$. Now we can write the Schwartz kernel of the operator $\widetilde{P}_h$ in the form

$$(2\pi\, h)^{-n} \int_{\mathbf{R}^n} e^{i\langle x-y\,,\,\xi\rangle/h}$$

$$\times \left((2\pi\, h)^{-n} \int_{\mathbf{R}^n \times D} e^{i\Psi(y,z,\xi,\eta)/h}\, \overline{b(z,\xi,h)} u(z,\eta,h) dz d\eta \right) d\xi. \tag{II.9}$$

The phase function Ψ can be written as follows

$$\begin{aligned} \Psi(x,y,z,\xi,\eta) \;&=\; \langle y - z\,,\,\xi - \eta\rangle + \phi(z,\eta) - \phi(z,\xi) \\ &=\; \langle y - z - \widetilde{\phi}_\xi(z,\xi,\eta)\,,\,\xi - \eta\rangle \end{aligned}$$

where

$$\widetilde{\phi}_\xi(z,\xi,\eta) \;=\; \int_0^1 \partial\phi/\partial\xi(z,\xi + t(\eta - \xi))\, dt$$

is analytic with respect to z and G^μ with respect to (ξ,η), i.e. $\widetilde{\phi}_\xi \in G^{1,\mu}$ with respect to (z,ζ), $\zeta = (\xi,\eta)$. The stationary points with respect to (z,η) are $\eta = \xi$ and $z = y - \widetilde{\phi}_\xi(z,\xi,\xi)$ in view of (BF). Integrating by parts with respect to (z,η) in the inner integral we can suppose that

$$|z - y + \widetilde{\phi}_\xi(z,\xi,\eta)|\;,\;\; |\eta - \xi| \;\ll 1$$

on the support of $\overline{b(z,\xi,h)}u(z,\eta,h)$. On the other hand $d_z\widetilde{\phi}_\xi(0,\xi,\xi)$ is nondegenerate in view of (BF) and there exists $z = \widetilde{z}(y,\varphi,\xi,I)$ given by the implicit function theorem such that $\varphi = z - y + \widetilde{\phi}_\xi(\widetilde{z},\xi,\eta)$. Moreover, one can show that $\widetilde{z}(y,\varphi,\xi,\eta)$ is real analytic with respect to (y,φ) and G^μ with respect to (ξ,η) and that for any function $g(z,\xi,\eta)$ of class $G^{\sigma,\mu}$ with respect to (z,ζ), $\zeta = (\xi,\eta)$, the function $g(\widetilde{z}(y,\varphi,\xi,\eta),\xi,\eta)$ is $G^{\sigma,\mu}$ with respect to $((y,\varphi),(\xi,\eta))$ (see Appendix A.2). We make a change of the variables in the inner integral in (II.9) setting $\varphi = z - y + \widetilde{\phi}_\xi(z,\xi,\eta)$ and $I = \eta - \xi$. Then the inner integral becomes

$$v(y,\xi,h) \;=\; (2\pi h)^{-n} \int e^{i\langle\varphi,I\rangle/h}\, R(\varphi,y,\xi,I,h)\, d\varphi dI\,,$$

where $R \in S_{\widetilde{\ell}}$. Using the Taylor formula at $\varphi = 0$ and integrating by parts with respect to I we obtain that v belongs to $S_{\widetilde{l}}$. Moreover, choosing the subprincipal symbol of T_{2h} so that $T^*_{2h} = T^{-1}_{2h} + O(h^2)$, we obtain that the subprincipal symbol of $\widetilde{P}_h$ is 0. □

III Normal form of $\widetilde{P}_h$

We can suppose now that $\widetilde{P}_h$ is a selfadjoint pseudodifferential operator with a symbol $p \in S_{\widetilde{\ell}}(\mathbf{T}^n \times \Gamma)$, $\widetilde{\ell} = (\sigma, \mu, \sigma + \mu - 1)$, where

$$p(\varphi, I; h) \sim \sum_{j=0}^{\infty} p_j(\varphi, I) h^j \ ,$$

and

$$p_0(\varphi, I) = p_0(I) = K_0(I)\,, \quad p_1(\varphi, I) = 0, \quad \forall (\varphi, I) \in \mathbf{T}^n \times E_\kappa.$$

Recall that E_κ is a Cantor set in a bounded domain D such that each $I \in E_\kappa$ is of a positive Lebesgue density, i.e. the Lebesgue measure of $E_\kappa \cap U$ is positive for any neighborhood U of I. Then given $\ell = (\sigma, \mu, \varrho)$, we can define $S_\ell(\mathbf{T}^n \times E_\kappa)$ as above, where the derivatives with respect to I in E_κ are taken in the sense of Whitney. On the other hand, having a (formal) symbol

$$p(\varphi, I) \ = \ \sum_{j=0}^{\infty} p_j(\varphi, I) h^j \ \in \ FS_\ell(\mathbf{T}^n \times E_\kappa)\,,$$

we can extend it to a formal symbol $\widetilde{p} \in FS_\ell(\mathbf{T}^n \times D)$ using a suitable Whitney extension theorem in Gevrey classes (see [19], Theorem 4.1). In other words, using that theorem we can extend simultaneously all p_j to Gevrey functions of the same class in $\mathbf{T}^n \times D$ with a Gevrey constant C independent on j. Recall that for any $f \in C^\infty(\mathbf{T}^n \times D)$ with $f(\varphi, I) = 0$ for $(\varphi, I) \in \mathbf{T}^n \times E_\kappa$, we have

$$\partial_I^\alpha \partial_\varphi^\beta f(\varphi, I) = 0, \quad \text{for all} \quad (\varphi, I) \in \mathbf{T}^n \times E_\kappa,$$

for any multi-indices $\alpha, \beta \in \mathbf{Z}_+^n$. Hence, if

$$p^{(k)} = \sum_{0 \le j \le \eta h^{-1/\varrho}} p_j^{(k)} h^j \ , \ k = 1, 2\,, \ \eta > 0\,,$$

are two extensions of the formal Gevrey symbol p in $FS_\ell(\mathbf{T}^n \times D)$, then $p^{(1)} - p^{(2)}$ is a flat function on $\mathbf{T}^n \times E_\kappa$ for each $0 < h \le h_0$.

We are going to transform $\widetilde{P}_h$ to a normal formal P_h^0 conjugating it with an elliptic pseudodifferential operator A_h with a symbol $a(\varphi, I, h)$ in $S_\ell(\mathbf{T}^n \times \Gamma)$ where $\ell = (\sigma, \mu, \varrho)$, $\varrho = \sigma\nu$ and $\nu = \tau + n + 1$. To this end we consider $p(\varphi, I; h)$ as a symbol of the class $S_{\widetilde{\ell}}(\mathbf{T}^n \times E_\kappa)$, where $\widetilde{\ell} = (\sigma, \mu, \sigma + \mu - 1)$. The main technical part in the proof is the following:

Theorem III.1 *There exist symbols a and p^0 in $S_\ell(\mathbf{T}^n \times E_\kappa)$, $\ell = (\sigma, \mu, \varrho)$, given by*

$$a(\varphi, I, h) \ \sim \ \sum_{j=0}^{\infty} a_j(\varphi, I) h^j \ , \ p^0(I, h) \ \sim \ \sum_{j=0}^{\infty} p_j^0(I) h^j \,,$$

with $a_0 = 1$, $p_0^0 = K_0$, $p_1^0 = 0$, *such that*

$$p \circ a - a \circ p^0 \sim 0$$

in $S_\ell(\mathbf{T}^n \times E_\kappa)$.

Theorem 1.1 follows from the result above. First, using [19], Theorem 4.1, we extend a to a symbol of a pseudodifferential operator A_h in $S_\ell(\mathbf{T}^n \times \Gamma)$, $\ell = (\sigma, \mu, \varrho)$ so that $a_0 = 1$, and set $V_h = T_h \circ A_h$. Then we have $P_h \circ V_h = V_h \circ (P_h^0 + R_h)$, where P_h^0 and R_h have the desired properties. Unfortunately, V_h may not be an unitary operator. For this reason we consider the pseudodifferential operator $W_h = V_h^* \circ V_h$ with a symbol $w(\varphi, I, h) = \sum_{j=0}^{\infty} w_j(\varphi, I) h^j$ in $S_\ell(\mathbf{T}^n \times \Gamma)$. Then $w_0 = 1$ and we have:

Lemma III.2 *For each j the function $p_j^0(I)$ is real valued on E_κ and $w_j(\varphi, I)$ does not depend on φ for each $I \in E_\kappa$.*

Proof. We have $w_0 = 1$, $p_0^0(I) = K_0(I)$, $p_1^0 = 0$. Moreover, it is easy to see that

$$W_h \circ (P_h^0 + R_h) \ = \ (P_h^{0*} + R_h^*) \circ W_h \,,$$

since P_h is selfadjoint. Then we have $\overline{p^0} \circ w = w \circ p^0$ on $\mathbf{T}^n \times E_\kappa$. This equality implies

$$\frac{1}{i} \mathcal{L}_\omega w_1(\varphi, I) + \overline{p_2^0(I)} - p_2^0(I) = 0, \ I \in E_\kappa,$$

where $\mathcal{L}_\omega$ stands for the derivative along the vector field $\omega(I) = \nabla K_0(I)$, namely,

$$\mathcal{L}_w \ \stackrel{\text{def}}{=} \ \sum_{j=1}^{n} \omega_j(I) \partial_{\varphi_j}. \tag{III.1}$$

Integrating in $\varphi \in \mathbf{T}^n$ we obtain that the imaginary part $\Im p_2^0 = 0$ and $w_1(\varphi, I) = w_1(0, I)$. In the same way we get by induction

$$\frac{1}{i} \mathcal{L}_\omega w_j(\varphi, I) + \overline{p_{j+1}^0(I)} - p_{j+1}^0(I) = 0, \ I \in E_\kappa,$$

and as above we prove that p_j^0 is real valued and that w_{j+1} does not depend on $I \in E_\kappa$. □

The symbol $q(\varphi, I, h)$ of $Q_h = (V_h^* \circ V_h)^{-1/2}$ belongs to $S_\ell(\mathbf{T}^n \times \Gamma)$, $\ell = (\sigma, \mu, \varrho)$, and $q(\varphi, I, h) - q(0, I, h)$ has a zero of infinite order at $\mathbf{T}^n \times E_\kappa$ in view of Lemma 3.2. Now $U_h = V_h \circ Q_h$ is the desired unitary operator. □

IV On the homological equation in Gevrey classes

The aim of this Section is to solve the equation $\mathcal{L}_\omega u = f$ in Gevrey classes in E_κ.

Lemma IV.1 *Let $\omega \in C^\infty(E_\kappa; \mathbf{R}^n)$ satisfy the following Gevrey type estimates:*

$$|D^\alpha \omega(I)| \ \leq \ C_1^{|\alpha|}\, \alpha!^{\,\tau'+2}\,, \ \forall\, I \in E_\kappa,\ \alpha \in \mathbf{Z}_+^n \setminus \{0\}, \tag{IV.1}$$

$$|\langle \omega(I), k\rangle| \ \geq \ \kappa\, |k|^{-\tau}\,, \ \forall\, I \in E_\kappa,\ k \in \mathbf{Z}^n \setminus \{0\}\,. \tag{IV.2}$$

Then there exists a positive constant C_0 depending only on n, κ, τ', and C_1, such that

$$\left| D_I^\alpha \left(\langle \omega(I), k\rangle^{-1} \right) \right| \ \leq \ C_0^{|\alpha|+1}\, \alpha! \max_{0 \leq j \leq |\alpha|} \left(|k|^{\tau j+\tau+j} (|\alpha|-j)!^{\,\tau'+1} \right), \tag{IV.3}$$

for any $I \in E_\kappa$, $0 \neq k \in \mathbf{Z}^n$ and $\alpha \in \mathbf{Z}_+^n$.

Proof. Set $g_k(I) = \langle \omega(I), k\rangle$ for $0 \neq k \in \mathbf{Z}^n$. Applying the Leibnitz rule to the identity $D_I^\alpha(g_k g_k^{-1}) = 0$, $|\alpha| \geq 1$, we get

$$D_I^\alpha \left(g_k(I)^{-1} \right) \ = \ -\, g_k(I)^{-1} \sum_{0<\beta\leq\alpha} \binom{\alpha}{\beta} D_I^\beta g_k(I)\, D_I^{\alpha-\beta} \left(g_k(I)^{-1} \right)\,.$$

Assuming that (IV.3) is valid for $|\alpha| < m$, we shall prove it for $|\alpha| = m$. In view of (IV.1) there exists $C_2 > 0$ depending only on C_1 and τ' such that

$$|D_I^\alpha \omega(I)| \ \leq \ C_2^{|\alpha|} \left(\frac{\alpha!}{|\alpha|} \right)^{\tau'+2}, \ \forall\, I \in E_\kappa\,,\ \alpha \in \mathbf{Z}_+^n \setminus \{0\}.$$

Set $C_0 = \varepsilon^{-1} C_2$ with some $\varepsilon > 0$ which will be determined later. Then using the above inequality, (IV.2), as well as the estimate $x!y! \leq (x+y)!$, we obtain

$$\begin{aligned}
\left| D_I^\alpha \left(g_k(I)^{-1} \right) \right| \ &\leq \ \kappa^{-1} |k|^{\tau+1} \alpha! \sum_{0<\beta\leq\alpha} \left(\frac{\beta!}{|\beta|} \right)^{\tau'+1} C_2^{|\beta|} C_0^{|\alpha-\beta|+1} \\
&\quad \times \max_{0\leq j\leq|\alpha-\beta|} \left((|\alpha-\beta|-j)!^{\,\tau'+1} |k|^{\tau j+\tau+j} \right) \\
&\leq \ d_\varepsilon\, C_0^{|\alpha|+1}\, \alpha\,! \max_{0\leq j\leq|\alpha|-1} \left((|\alpha|-j-1)!^{\,\tau'+1} |k|^{\tau(j+1)+\tau+j+1} \right) \\
&\leq d_\varepsilon\, C_0^{|\alpha|+1} \alpha! \max_{0\leq j\leq|\alpha|} \left((|\alpha|-j)!^{\,\tau'+1} |k|^{\tau j+\tau+j} \right),
\end{aligned}$$

where

$$d_\varepsilon \ = \ \kappa^{-1} \sum_{0<\beta} \varepsilon^{|\beta|} \ < \ \varepsilon\, \kappa^{-1} \sum_{s=2}^{\infty} s^n \varepsilon^{s-2} \ < \ 1$$

choosing ε sufficiently small. $\square$

For any $m > 0$ we set $\langle k\rangle_m = 1 + |k_1|^m + \cdots + |k_n|^m,\ k \in \mathbf{Z}^n$. Next, for any $j \in \mathbf{Z}_+$ we denote

$$m(j) \ = \ [(\tau' + 1)j + \tau] + n + 1\,,$$

where $[p]$ stands for the integer part of $p \in \mathbf{R}$. Set $W(k) = (1 + |k|)^{n+\varepsilon}$ with

$$0 \ < \ \varepsilon \ \leq \ \min\{\tau' - \tau, [\tau] - \tau + 1\}.$$

Then

$$\tau j + j + \tau + n + \varepsilon \ < \ m(j) \ \leq \ (\tau' + 1)j + \nu\,,\ \forall j \in \mathbf{Z}^+\,,$$

with $\nu = \tau + n + 1$. Taking into account the inequality

$$|k|^m \ \leq \ n^m \langle k\rangle_m\,,\ m > 0,\ k \in \mathbf{Z}^n,$$

and using Lemma 4.1, we get

$$W(k)\left|D_I^\alpha\left(\langle\omega(I), k\rangle^{-1}\right)\right| \ \leq \ C_0^{|\alpha|+1}\,\alpha!\, \max_{0\leq j\leq|\alpha|}\left((|\alpha| - j)!^{\,\tau'+1}\langle k\rangle_{m(j)}\right)\,, \tag{IV.4}$$

for any $I \in E_\kappa,\ \alpha \in \mathbf{Z}_+^n$, and $0 \neq k \in \mathbf{Z}^n$, with a constant $C_0 > 0$ depending only on $n,\ \kappa$, and C_1.

Suppose that $f \in C^\infty(\mathbf{T}^n \times E_\kappa)$ satisfies

$$\left|D_I^\alpha D_\varphi^\beta f(\varphi, I)\right| \ \leq \ d_0\, C^{\mu|\alpha|+|\beta|}\,\Gamma(\mu|\alpha| + \sigma|\beta| + q) \tag{IV.5}$$

for any $I \in E_\kappa,\ \alpha,\ \beta \in \mathbf{Z}_+^n$, and some $q > 0$, where $\Gamma(x),\ x > 0$, is the Gamma function and σ and μ are suitable positive constants. Let

$$\int_{\mathbf{T}^N} f(\varphi, I)d\varphi \ = \ 0\,. \tag{IV.6}$$

We are going to solve the equation

$$\mathcal{L}_\omega u(\varphi, I) \ = \ f(\varphi, I)\,,\ u(0, I) = 0\,, \tag{IV.7}$$

and provide the corresponding estimates for the derivatives of u, where $\mathcal{L}_\omega$ is defined in (III.1) and $\omega(I)$ satisfies (IV.1), (IV.2) and (IV.4) on E_κ.

Proposition IV.2 *Let $f \in C^\infty(\mathbf{T}^n \times E_\kappa)$ satisfy (IV.5) and (IV.6), where $\sigma > 1$ and $\mu - 1 > \sigma(\tau' + 1)$. Then the equation (IV.7) has a unique solution $u \in C^\infty(\mathbf{T}^n \times E_\kappa)$ and there is $c_0 = c_0(n, C_0) > 1$, C_0 being the constant in (IV.4), such that if $C > c_0$, then the solution u of (IV.7) satisfies the estimate*

$$\left|D_I^\alpha D_\varphi^\gamma u(\varphi, I)\right| \ \leq \ R\, d_0\, C^{\mu|\alpha|+|\gamma|+\nu}\,\Gamma\left(\mu|\alpha| + \sigma|\gamma| + \sigma\nu + q\right)\,, \tag{IV.8}$$

for any $I \in E_\kappa$, and $\alpha,\ \gamma \in \mathbf{Z}_+^n$, where $R > 0$ depends only on n, τ, τ' and C_0.

Proof. Consider the Fourier expansions of f and u

$$f(\varphi, I) \;=\; \sum_{k\in\mathbf{Z}^n} e^{i\langle k,\varphi\rangle}\, f_k(I),$$

$$u(\varphi, I) \;=\; \sum_{k\in\mathbf{Z}^n} e^{i\langle k,\varphi\rangle}\, u_k(I),$$

where

$$f_k(I) \;=\; (2\pi)^{-n}\int_{\mathbf{T}^n} e^{-i\langle k,\varphi\rangle}\, f(\varphi, I)d\varphi,$$

and $u_k(I)$ is defined in the same way. Now, $u_0 = 0$ in view of (IV.6), and

$$u_k(I) \;=\; \langle\omega(I), k\rangle^{-1}\, f_k(I),\ I\in E_\kappa,\ 0\neq k\in\mathbf{Z}^n\,.$$

Integrating by parts, and using (IV.5) we get for any $\gamma\in\mathbf{Z}^n_+$ and $m\in\mathbf{Z}_+$ the following estimate for the Fourier coefficients of f:

$$|k^\gamma\langle k\rangle_m D_I^\alpha f_k(I)| \;\le\; (n+1)d_0\, C^{\mu|\alpha|+|\gamma|+m}\,\Gamma(\mu|\alpha|+\sigma|\gamma|+\sigma m+q)\,,$$

for any $I\in E_\kappa,\ k\in\mathbf{Z}^n$, and any $\alpha,\ \gamma\in\mathbf{Z}^n_+$. Now, taking into account (IV.4) we estimate the quantity

$$\begin{aligned} A_k \;&=\; W(k)\,|k^\gamma D_I^\alpha u_k(I)| \\ &\le \sum_{0\le\beta\le\alpha}\binom{\alpha}{\beta}\,\beta\,!\,C_0^{|\beta|+1}\max_{0\le j\le|\beta|}\left|(|\beta|-j)!^{\,\tau'+1}\,k^\gamma\,\langle k\rangle_{m(j)}\,D_I^{\alpha-\beta}f_k(I)\right| \\ &\le\; (n+1)d_0\sum_{0\le\beta\le\alpha}\frac{\alpha!}{(\alpha-\beta)!}\,C_0^{|\beta|+1}\max_{0\le j\le|\beta|}\left|(|\beta|-j)!^{\,\tau'+1}\,\Gamma(s)\,C^t\right|\,. \end{aligned}$$

Here, $t=\mu|\alpha-\beta|+|\gamma|+m(j)$ and we write

$$s \;\stackrel{\text{def}}{=}\; \mu|\alpha-\beta|+\sigma|\gamma|+\sigma m(j)+q.$$

Using the inequality $\mu-1>\sigma(\tau'+1)$ we get

$$\begin{aligned} s \;&\le\; \mu|\alpha-\beta|+\sigma(\tau'+1))j+\sigma|\gamma|+\sigma\nu+q \\ &\le\; \mu|\alpha|-|\beta|-\sigma(\tau'+1))(|\beta|-j)+\sigma|\gamma|+\sigma\nu+q\,. \end{aligned}$$

On the other hand, by Stirling's formula we have

$$(x\,!)^{\,\tau'+1} \;\le\; C_2^x\,\Gamma((\tau'+1)x)\,,\ x\ge 1,$$

with some constant $C_2>0$. Using the relations

$$\Gamma(s+1) \;=\; s\,\Gamma(s)\,,\ \Gamma(s)\Gamma(u) \;\le\; \Gamma(s+u)\,,\ \forall\, s\,,\, u\,\ge\,1\,,$$

and the inequalities $\sigma > 1$ and $s \geq 1$, we obtain for each $0 \leq j < |\beta|$

$$\frac{\alpha!}{(\alpha-\beta)!}\,(|\beta|-j)!^{\,\tau'+1}\,\Gamma(s)$$

$$\leq\ C_2^{|\beta|}\,(s+|\beta|-1)\cdots s\,\Gamma(s)\,\Gamma((\tau'+1)(|\beta|-j))$$

$$\leq\ C_2^{|\beta|}\,\Gamma(s+|\beta|+\sigma(\tau'+1)(|\beta|-j))$$

$$\leq\ C_2^{|\beta|}\,\Gamma(\mu|\alpha|+\sigma|\gamma|+\sigma\nu+q)\,.$$

Obviously, the same inequality holds for $j = |\beta|$. Moreover,

$$t\ \leq\ \mu|\alpha|+|\gamma|-|\beta|+\nu.$$

Hence,

$$A_k\ \leq\ (n+1)d_0\,C_0\sum_{0\leq\beta\leq\alpha}(C_0C_2C^{-1})^{|\beta|}\,C^{\mu|\alpha|+|\gamma|+\nu}\,\Gamma(\mu|\alpha|+\sigma|\gamma|+\sigma\nu+q)\,.$$

We choose $c_0 > C_0C_2 > 1$ and set $\varepsilon = C_0C_2c_0^{-1}$. Then for any $C > c_0$ we obtain

$$A_k\ \leq\ d_0\,R_1\,C^{\mu|\alpha|+|\gamma|+\nu}\,\Gamma(\mu|\alpha|+\sigma|\gamma|+\sigma\nu+q)\,,$$

where

$$R_1\ =\ (n+1)C_0\sum_{s=1}^{\infty}s^n\varepsilon^{s-1}\,.$$

Finally, we obtain

$$\left|D_I^\alpha D_\varphi^\gamma u(\varphi,I)\right|\ \leq\ \sum_{k\in\mathbf{Z}^n\setminus 0}W(k)^{-1}A_k$$

$$\leq\ d_0\,R\,C^{\mu|\alpha|+|\gamma|+\nu}\,\Gamma(\mu|\alpha|+\sigma|\gamma|+\sigma\nu+q)\,,$$

where

$$R\ =\ R_1\sum_{k\in\mathbf{Z}^n}W(k)^{-1}\,.$$

The proof of the proposition is complete. □

V Proof of Theorem 3.1

Set $\ell = (\sigma, \mu, \varrho)$, where $\varrho = \sigma\nu$. We are looking for symbols a and p^0 in $S_\ell(\mathbf{T}^n \times E_\kappa)$ of the form

$$a \sim \sum_{j=0}^{\infty} a_j(\varphi, I)h^j, \quad p^0 \sim \sum_{j=0}^{\infty} p_j^0(I)h^j,$$

where $a_j \in C^\infty(\mathbf{T}^n \times E_\kappa)$ and $p_j^0 \in C^\infty(E_\kappa)$. Consider the symbol

$$r = p \circ a - a \circ p^0 \sim \sum_{j=0}^{\infty} r_j(\varphi, I)h^j.$$

We have $a_0 = 1$, $p_0^0(I) = p_0(I) = K_0(I)$, and $p_1^0 = p_1 = 0$ in $\mathbf{T}^n \times E_\kappa$. Then $r_0 = r_1 = 0$ and for any $j \geq 2$ we get

$$r_j(\varphi, I) = \frac{1}{i}(\mathcal{L}_w\, a_{j-1})(\varphi, I) + p_j(\varphi, I) - p_j^0(I) + F_j(\varphi, I).$$

Here $F_2(\varphi, I) = 0$, and for $j \geq 3$, we have

$$F_j(\varphi, I) = F_{j1}(\varphi, I) - F_{j2}(\varphi, I),$$

$$F_{j1}(\varphi, I) = \sum_{s=1}^{j-2} \sum_{r+|\gamma|=j-s} \frac{1}{\gamma!} D_I^\gamma p_r(\varphi, I)\, \partial_\varphi^\gamma\, a_s(\varphi, I),$$

$$F_{j2}(\varphi, I) = \sum_{s=1}^{j-2} a_s(\varphi, I)\, p_{j-s}^0(I).$$

We solve the equations $r_j = 0$, $j \geq 2$, as follows: First we put

$$p_j^0(I) = (2\pi)^{-n} \int_{\mathbf{T}^n} (p_j(\varphi, I) + F_j(\varphi, I))\, d\varphi, \tag{V.1}$$

then, using Proposition 4.2, we find a_{j-1} from the equations

$$\frac{1}{i}\mathcal{L}_w a_{j-1}(\varphi, I) = f_j(\varphi, I), \tag{V.2}$$

$$\int_{\mathbf{T}^n} a_{j-1}(\varphi, I) d\varphi = 0, \tag{V.3}$$

where $f_j(\varphi, I) = p_j^0(I) - p_j(\varphi, I) - F_j(\varphi, I)$. For $j = 2$ we obtain

$$p_2^0(I) = (2\pi)^{-n} \int_{\mathbf{T}^n} p_2(\varphi, I)\, d\varphi,$$

and

$$\frac{1}{i}\mathcal{L}_w a_1(\varphi, I) \;=\; p_2^0(I) \,-\, p_2(\varphi, I)\,, \quad \int_{\mathbf{T}^n} a_1(\varphi, I)\, d\varphi \;=\; 0. \tag{V.4}$$

On the other hand, we can suppose that p_j, $j \in \mathbf{Z}_+$, satisfy the estimates

$$|\partial_I^\alpha \partial_\varphi^\beta p_j(\varphi, I)| \;\le\; C_1^{j+|\alpha|+|\beta|+1}\, \alpha!^\mu \beta!^\sigma\, (j!)^{\sigma+\mu-1}$$

$$\le\; C_0^{j+|\alpha+\beta|+1}\, \alpha!\, \beta!\, \Gamma_+((\mu-1)|\alpha| + (\sigma-1)|\beta| + (\sigma+\mu-1)(j-1))\,, \tag{V.5}$$

for any multi-indices $\alpha, \beta \in \mathbf{Z}_+^n$, $j \in \mathbf{Z}_+$, where $\Gamma_+(x) = \Gamma(x)$ for $x \ge 1$ and $\Gamma_+(x) = 1$ for $x \le 1$. In particular, using Proposition 4.2 we find a solution a_1 of (V.4) such that

$$|\partial_I^\alpha \partial_\varphi^\beta a_1(\varphi, I)| \;\le\; 2RC_0\, C^{\mu|\alpha|+|\beta|}\, \Gamma(\mu|\alpha| + \sigma|\beta| + \varrho)\,,$$

choosing $C > c_0$. Fix $j \ge 3$ and suppose that there exist $p_k^0(I)$, $2 \le k \le j-1$, satisfying (V.1) and $a_k(\varphi, I)$, $1 \le k \le j-2$, satisfying (V.2) and (V.3), and such that

$$|\partial_I^\alpha p_k^0(I)| \;\le\; d^{k-3/2} C^{\mu|\alpha|} \Gamma(\mu|\alpha| + (k-1)\varrho)\,,\; 2 \le k \le j-1\,, \tag{V.6}$$

$$|\partial_I^\alpha \partial_\varphi^\beta a_k(\varphi, I)| \;\le\; d^k C^{\mu|\alpha|+|\beta|}\, \Gamma(\mu|\alpha| + \sigma|\beta| + k\varrho)\,,\; 1 \le k \le j-2\,, \tag{V.7}$$

for any $(\varphi, I) \in \mathbf{T}^n \times E_\kappa$ and $\alpha, \beta \in \mathbf{Z}_+^n$, where $d \ge 2RC_0$. Choosing appropriately d as a function of $n, \tau, \mu, \sigma, C_0$ and C only, we shall prove that p_j^0 and a_{j-1} satisfy the same estimates. First we estimate the derivatives of F_j.

Lemma V.1 *Let $C > 4C_0$. Then for any α and β in $\mathbf{Z}_+^n$ we have*

$$|D_I^\alpha D_\varphi^\beta F_{j1}(\varphi, I)| \;\le\; R_1 d^{j-2} C^{\mu|\alpha|+|\beta|} \Gamma(\mu|\alpha| + \sigma|\beta| + (j-1)\varrho)\,,\; (\varphi, I) \in \mathbf{T}^n \times E_\kappa\,,$$

where R_1 depends only on $n, \tau, \mu, \sigma, C_0$ and C.

Proof. Set

$$B_{r,s,\gamma}(\varphi, I) = \frac{1}{\gamma!}\partial_I^\gamma\, p_r(\varphi, I) \partial_\varphi^\gamma a_s(\varphi, I), \tag{V.8}$$

where

$$3 \le r + s + |\gamma| = j\;, \quad 1 \le s \le j-2. \tag{V.9}$$

Then $|\gamma| + r \ge 2$, and by (I.4) we have

$$(\mu-1)|\gamma| + (\sigma+\mu-1)(r-1) \ge (\mu-1)(|\gamma|+r-1) - \sigma \ge \mu-\sigma-1 > \sigma\tau' > 1\,. \tag{V.10}$$

Taking into account the above inequality, (V.5) and (V.7) we obtain

$$|\partial_I^\alpha \partial_\varphi^\beta B_{r,s,\gamma}(\varphi, I)|$$

$$\leq \sum_{\alpha_1\leq\alpha}\sum_{\beta_1\leq\beta}\frac{1}{\gamma!}\binom{\alpha}{\alpha_1}\binom{\beta}{\beta_1}|\partial_I^{\gamma+\alpha_1}\partial_\varphi^{\beta_1}p_r(\varphi,I)|\,|\partial_I^{\alpha-\alpha_1}\partial_\varphi^{\gamma+\beta-\beta_1}a_s(\varphi,I)|$$

$$\leq\ d^s\sum_{\alpha_1\leq\alpha}\sum_{\beta_1\leq\beta}(\gamma+\alpha_1)!\,\frac{\beta_1!}{\gamma!}\binom{\alpha}{\alpha_1}\binom{\beta}{\beta_1}$$

$$\times\Gamma((\mu-1)|\gamma+\alpha_1|+(\sigma-1)|\beta_1|+(\sigma+\mu-1)(r-1))$$

$$\leq\ \Gamma(\mu|\alpha-\alpha_1|+\sigma|\gamma+\beta-\beta_1|+s\varrho)\,C_0^{|\gamma+\alpha_1|+|\beta_1|+r+1}\,C^{\mu|\alpha-\alpha_1|+|\gamma+\beta-\beta_1|}\,.$$

Now Lemma A.2 yields

$$|\partial_I^\alpha\partial_\varphi^\beta B_{r,s,\gamma}(\varphi,I)|\leq$$

$$d^s\,C^{\mu|\alpha|+|\beta|}\sum_{\alpha_1\leq\alpha}\sum_{\beta_1\leq\beta}\Gamma(\mu|\alpha|+\sigma|\beta|+(\sigma+\mu-1)(|\gamma|+r-1)+s\varrho)$$

$$\times(2C_0/C)^{|\alpha_1+\beta_1|}\,C_0^{|\gamma|+r+1}\,(2C)^{|\gamma|}\,.$$

Set $\delta=\varrho-\sigma-\mu+1$. Since $\nu>\mu$ we have

$$\delta\ =\ \sigma\nu-\mu-\sigma+1\ >\ (\mu-1)(\sigma-1)\ >\ 0\,.$$

On the other hand,

$$(j-1)\varrho-\delta(|\gamma|+r-1)\ =\ (\sigma+\mu-1)(|\gamma|+r-1)+s\varrho\ \geq\ 1\,.$$

Hence, using Lemma A.1 we get

$$\Gamma(\mu|\alpha|+\sigma|\beta|+(\sigma+\mu-1)(|\gamma|+r-1)+s\varrho)$$

$$=\Gamma(\mu|\alpha|+\sigma|\beta|+(j-1)\varrho-\delta(|\gamma|+r-1))$$

$$\leq\frac{\Gamma(\mu|\alpha|+\sigma|\beta|+(j-1)\varrho)}{\delta\,\Gamma(\delta(|\gamma|+r-1))}\,.$$

Suppose that $C>4C_0$. Then, for any r,s,γ satisfying (V.9) we obtain

$$|\partial_I^\alpha\partial_\varphi^\beta B_{r,s,\gamma}(\varphi,I)|\quad\leq\quad R_0\,d^{j-2}\,C^{\mu|\alpha|+|\beta|}\,\Gamma(\mu|\alpha|+\sigma|\beta|+(j-1)\nu)$$

$$\times\quad\frac{C_0C^{2|\gamma|+2r}}{\delta\,\Gamma(\delta(|\gamma|+r-1))}\,,$$

where

$$R_0^{1/2}\ =\ \sum_{\alpha\in\mathbf{Z}_+^n}2^{-|\alpha|}\,.$$

Hence, we obtain

$$\begin{aligned}|D_I^\alpha D_\varphi^\beta F_{j1}(\varphi,I)| &\le \sum_{s=1}^{j-2} \sum_{|\gamma|+r=j-s} |D_I^\alpha D_\varphi^\beta B_{r,s,\gamma}(\varphi,I)| \\ &\le R_1\, d^{j-2}\, C^{\mu|\alpha|+|\beta|}\, \Gamma(\mu|\alpha|+\sigma|\beta|+(j-1)\varrho)\,, \end{aligned} \tag{V.11}$$

where

$$R_1 \;=\; \frac{R_0 C_0}{\delta} \sum_{p\in \mathbf{Z}_+^{n+1}} \frac{C^{2|p|+2}}{\Gamma(\delta(|p|+1))} \;<\; \infty\,.$$

We have proved the lemma. □

Now we can estimate $p_j^0(I),\ j\ge 3$, given by (V.1). Notice that

$$\int_{\mathbf{T}^n} F_{j2}(\varphi,I)d\varphi \;=\; \sum_{s=1}^{j-2} p_{j-s}^0(I) \int_{\mathbf{T}^n} a_s(\varphi,I)d\varphi \;=\; 0$$

in view of (V.3). Hence,

$$p_j^0(I) \;=\; (2\pi)^{-n} \int_{\mathbf{T}^n} (p_j(\varphi,I) + F_{j1}(\varphi,I))d\varphi\,,$$

and taking into account (V.5) and (V.11) we obtain for any $j\ge 2$ the following inequality:

$$\begin{aligned}|\partial_I^\alpha p_j^0(I)| &\le R_1 d^{j-2} C^{\mu|\alpha|}\Gamma(\mu|\alpha|+(j-1)\varrho) \\ &+ C_0^{|\alpha|+j+1}\,\Gamma(\mu|\alpha|+(j-1)(\sigma+\mu-1)) \\ &\le d^{j-3/2} C^{\mu|\alpha|}\Gamma(\mu|\alpha|+(j-1)\varrho)\,,\end{aligned}$$

since $\varrho = \sigma\nu > \mu + (\sigma-1)\nu > \sigma+\mu-1$. Here we choose d sufficiently large as a function of n, τ, μ, σ, C_0 and C. This proves (V.6). It remains to estimate $F_{j2}(\varphi,I)$ and $a_{j-1}(\varphi,I)$.

Lemma V.2 *For any α and β in $\mathbf{Z}_+^n$ we have*

$$|D_I^\alpha D_\varphi^\beta F_{j,2}(\varphi,I)| \;\le\; M_2\, d^{j-\frac{3}{2}}\, C^{\mu|\alpha|+|\beta|}\Gamma(\mu|\alpha|+\sigma|\beta|+(j-1)\varrho)\,,\ (\varphi,I)\in \mathbf{T}^n\times E_\kappa\,,$$

where M_2 depends only on n,τ,μ,σ,C_0 and C.

Proof. In view of (V.6) and (V.7) we have

$$|D_I^\alpha D_\varphi^\beta(a_s(\varphi,I)p^0_{j-s}(I))| \;\le\; \sum_{\gamma\le\alpha}\binom{\alpha}{\gamma}|D_I^\gamma D_\varphi^\beta a_s(\varphi,I)||D_I^{\alpha-\gamma}p^0_{j-s}(I)|$$
$$\le\; d^{j-\frac{3}{2}}C^{\mu|\alpha|+|\beta|}\sum_{\gamma\le\alpha}\binom{\alpha}{\gamma}\Gamma(\mu|\gamma|+\sigma|\beta|+s\varrho)$$
$$\times\Gamma(\mu|\alpha-\gamma|+(j-s-1)\varrho).$$

Recall that $1\le s\le j-2$ and $\mu>\tau'+2>9/2$. Using Lemma A.3 and the inequalities

$$B(\sigma|\beta|+s\varrho,(j-s-1)\varrho) \;<\; B(s,j-s-1) \;<\; \binom{j-2}{s-1}^{-1},$$

we obtain

$$|D_I^\alpha D_\varphi^\beta(a_s(\varphi,I)p^0_{j-s}(\varphi,I))|$$

$$\le\; M\,d^{j-\frac{3}{2}}C^{\mu|\alpha|+|\beta|}\sum_{\gamma\le\alpha}\binom{|\alpha|}{|\gamma|}^{-1/6}B(\sigma|\beta|+s\varrho,(j-s-1)\varrho)^{1/3}$$
$$\times\Gamma(\mu|\alpha|+\sigma|\beta|+(j-1)\varrho) \;<\; M_1\,d^{j-\frac{3}{2}}C^{\mu|\alpha|+|\beta|}\binom{j-2}{s-1}^{-1/3}$$
$$\times\Gamma(\mu|\alpha|+\sigma|\beta|+(j-1)\varrho),$$

where $M_1=2M\sum_{\gamma\in\mathbf{Z}_+^n}2^{-|\gamma|/6}$. On the other hand

$$\sum_{s=1}^{j-2}\binom{j-2}{s-1}^{-1/3} \;\le\; 2\sum_{p=0}^{+\infty}2^{-p/3} \;<\; \infty\,.$$

Then we get

$$|D_I^\alpha D_\varphi^\beta F_{j,2}(\varphi,I)| \;\le\; M_2\,d^{j-\frac{3}{2}}\,C^{\mu|\alpha|+|\beta|}\,\Gamma(\mu|\alpha|+\sigma|\beta|+(j-1)\varrho)\,,$$

which proves the lemma. □

Finally, combining Lemma 5.1 and Lemma 5.2 we estimate the right hand side of (V.2) as follows:

$$|\partial_I^\alpha\partial_\varphi^\beta f_j(\varphi,I)| \;\le\; M_3\,d^{j-\frac{3}{2}}\,C^{\mu|\alpha|+|\beta|}\,\Gamma(\mu|\alpha|+\sigma|\beta|+(j-1)\varrho)\,,\ \forall\alpha,\beta\in\mathbf{Z}_+^n\,,$$

where M_3 depends only on n, τ, μ, σ, C_0 and C. Now applying Proposition 4.2 we find a solution a_{j-1} of (V.2) and (V.3) which satisfies (V.7) for $k=j-1$, choosing $d=d(n,\tau,\mu,\sigma,C_0,C)$ sufficiently large. □

Appendix

A.1. We are going to recall certain properties of the Gamma function

$$\Gamma(x) \ = \ \int_0^\infty e^{-t}\, t^{x-1}\ dt \ , x > 0.$$

We have the following relation

$$\Gamma(x)\Gamma(y) \ = \ \Gamma(x+y)B(x,y)\,,\ \ x,y>0\,,$$

(see [2]), where

$$B(x,y) \ = \ \int_0^1 (1-t)^{x-1}\, t^{y-1}\, dt\,.$$

In particular, $B(x,y) \le y^{-1}$ for any $x \ge 1$ and $y > 0$, and we obtain

Lemma A. 1 *For any $x \ge 1$ and $y > 0$ we have*

$$\Gamma(x)\,\Gamma(y) \ \le \frac{1}{y}\,\Gamma(x+y)\,.$$

For any $0 \le y \le x$, $x, y \in \mathbf{Z}_+^n$ we set $\begin{pmatrix} x \\ y \end{pmatrix} = \frac{x!}{y!(x-y)!}$ where $0! = 1$ by convention.

Lemma A. 2 *For any $\alpha_1 \le \alpha$, $\beta_1 \le \beta$, and $\gamma \in \mathbf{Z}_+^n$ and for any $s \ge 1$, $r \ge 0$ with $|\gamma| + r \ge 2$, we have*

$$\begin{aligned}
(\gamma+\alpha_1)!\,\frac{\beta_1!}{\gamma!}\begin{pmatrix} \alpha \\ \alpha_1 \end{pmatrix}\begin{pmatrix} \beta \\ \beta_1 \end{pmatrix}\Gamma((\mu-1)|\gamma+\alpha_1| + (\sigma-1)|\beta_1| + (\sigma+\mu-1)(r-1))& \\
\times\,\Gamma(\mu|\alpha-\alpha_1| + \sigma|\gamma+\beta-\beta_1| + s\varrho)& \\
\le\ 2^{|\gamma+\alpha_1|}\,\Gamma(\mu|\alpha| + \sigma|\beta| + (\sigma+\mu-1)(|\gamma|+r-1) + s\varrho)\,.&
\end{aligned}$$

Proof. Using the equality $x\,\Gamma(x) = \Gamma(x+1)$, $x > 0$, we obtain

$$\begin{aligned}
&(\gamma+\alpha_1)!\frac{\beta_1!}{\gamma!}\begin{pmatrix} \alpha \\ \alpha_1 \end{pmatrix}\begin{pmatrix} \beta \\ \beta_1 \end{pmatrix}\Gamma(\mu|\alpha-\alpha_1| + \sigma|\gamma+\beta-\beta_1| + s\varrho) \\
&\le 2^{|\gamma+\alpha_1|}\,\frac{|\alpha|!}{|\alpha-\alpha_1|!}\,\frac{|\beta|!}{|\beta-\beta_1|!}\,\Gamma(\mu|\alpha-\alpha_1| + \sigma|\beta-\beta_1| + \sigma|\gamma| + s\varrho) \\
&\le\ 2^{|\gamma+\alpha_1|}\,\Gamma(|\alpha| + |\beta| + (\mu-1)|\alpha-\alpha_1| + (\sigma-1)|\beta-\beta_1| + \sigma|\gamma| + s\varrho) \\
&=\ 2^{|\gamma+\alpha_1|}\,\Gamma(\mu|\alpha| + \sigma|\beta| - (\mu-1)|\alpha_1| - (\sigma-1)|\beta_1| + \sigma|\gamma| + s\varrho)\,.
\end{aligned}$$

On the other hand, $s\varrho > 1$ and by (V.10)

$$(\mu-1)|\gamma+\alpha_1| + (\sigma+\mu-1)(r-1) \ > \ 1\,,$$

and applying Lemma A.1 we complete the proof of the assertion. □

Lemma A. 3 *Let $\mu \geq 9/2$. Then there exists a positive constant M such that for any $x, y \in \mathbf{Z}_+$ and $p \geq 1,\ q \geq 1$, we have*

$$\binom{x+y}{x}^{7/6} \Gamma(\mu x+p)\,\Gamma(\mu y+q) \ \leq\ M\,\Gamma(\mu(x+y)+p+q)\,B(p,q)^{1/3}.$$

Proof. Suppose that $x \geq 1$ and $y \geq 1$. We have

$$\Gamma(\mu x+p)\Gamma(\mu y+q) \ =\ \Gamma(\mu(x+y)+p+q)B(\mu x+p, \mu y+q).$$

On the other hand

$$B(\mu x+p, \mu y+q) \ =\ \int_0^1 t^{\mu x+p-1}(1-t)^{\mu y+q-1}dt \ \leq B(\mu x, \mu y),$$

and in the same way we get

$$B(\mu x+p, \mu y+q) \ \leq\ B(p,q).$$

Hence

$$\Gamma(\mu x+p)\,\Gamma(\mu y+q) \ \leq\ \Gamma(\mu(x+y)+p+q)\,B(\mu x, \mu y)^{2/3}\,B(p,q)^{1/3}\,. \tag{A.1}$$

By Stirling's formula there exists $L > 0$ such that for any $x \geq 1$ we have

$$L^{-1} \ \leq\ \Gamma(x)(2\pi)^{-1/2}x^{\frac{1}{2}-x}e^{x} \ \leq\ L\,.$$

Then

$$\begin{aligned}\Gamma(\mu x) \ &\leq\ L(2\pi)^{1/2}x^{\mu x-\frac{1}{2}}e^{-\mu x}\mu^{\mu x-\frac{1}{2}} \\ &\leq\ L^{\mu+1}\Gamma(x)^{\mu}\left(\frac{x}{2\pi}\right)^{\frac{\mu-1}{2}}\mu^{\mu x-\frac{1}{2}}.\end{aligned}$$

In the same way we get

$$\Gamma(\mu y) \ \leq\ L^{\mu+1}\Gamma(y)^{\mu}\left(\frac{y}{2\pi}\right)^{\frac{\mu-1}{2}}\mu^{\mu y-\frac{1}{2}},$$

$$\Gamma(\mu(x+y))^{-1} \ \leq\ L^{\mu+1}\Gamma(x+y)^{-\mu}\left(\frac{2\pi}{x+y}\right)^{\frac{\mu-1}{2}}\mu^{-\mu(x+y)+\frac{1}{2}}.$$

Hence

$$\begin{aligned}B(\mu x, \mu y) \ &\leq\ L^{3\mu+3}(2\pi)^{\frac{1-\mu}{2}}\mu^{-1/2}\left(\frac{xy}{x+y}\right)^{\frac{\mu-1}{2}} B(x,y)^{\mu} \ = \\ &\leq\ M\left(\frac{xy}{x+y}\right)^{\frac{\mu-1}{2}} B(x,y)^{\frac{\mu-1}{2}}\end{aligned}$$

$$= \; M \begin{pmatrix} x+y \\ x \end{pmatrix}^{(1-\mu)/2} \;\leq\; M \begin{pmatrix} x+y \\ x \end{pmatrix}^{-7/4} ,$$

since $\mu \geq 9/2$. This proves the assertion for $x\,, y \geq 1$. On the other hand, if $x = 0$ and $y \geq 0$, we have

$$\begin{aligned} \Gamma(p)\Gamma(\mu y + q) \;&=\; \Gamma(\mu y + p + q) B(p, \mu y + q) \;\leq \\ &\leq\; \Gamma(\mu y + p + q) B(p, q) \;\leq\; \Gamma(\mu y + p + q) B(p, q)^{1/3} , \end{aligned}$$

which completes the proof of Lemma A.3. □

A.2. At the end of this section, we collect some more or less known facts about the composition of Gevrey functions. Fix $\mu \geq \sigma \geq 1$. Let $f \in G^\sigma$ in a neighborhood of $0 \in \mathbf{R}^{n_1}$ and $g = (g_1, \ldots, g_{n_1}) \in G^{\sigma,\mu}$ with respect to $(x, y) \in \mathbf{R}^{n_2} \times \mathbf{R}^{n_3}$ in a neighborhood of $(0,0)$, $g(0,0) = 0$. Following an argument in [11] (see also [9]), we shall show that $h(x,y) = f(g(x,y))$ belongs to $G^{\sigma,\mu}$ in a neighborhood of $(0,0) \in \mathbf{R}^{n_2} \times \mathbf{R}^{n_3}$.

Set $F(z,x,y) = f(z)$ and denote $L = (L_1, \ldots, L_{n_2})$ and $K = (K_1, \ldots, K_{n_3})$, where

$$L_j \;=\; \partial/\partial x_j \,+\, \langle \partial g/\partial x_j, \partial/\partial z \rangle \quad , \quad K_j \;=\; \partial/\partial y_j \,+\, \langle \partial g/\partial y_j, \partial/\partial z \rangle \,.$$

Then given $(\alpha, \beta) \in \mathbf{Z}_+^{n_2} \times \mathbf{Z}_+^{n_3}$, we obtain

$$(\partial/\partial x)^\alpha (\partial/\partial y)^\beta h(x,y) \;=\; \left(L^\alpha K^\beta F \right) (g(x,y), x, y) \,. \tag{A.2}$$

Set $n = n_1 + n_2$, $m = n_3$, and $t = (z, x)$, and denote by U a compact neighborhood of $(0,0)$ in $\mathbf{R}^n \times \mathbf{R}^m$. Consider g_k as functions in U and denote by $\mathcal{A}$ the finite set of functions $a = 1$, $\partial g_k/\partial x_j$, and $\partial g_k/\partial y_j$ defined in U. Fix $C > 0$ such that

$$\left| (\partial/\partial t)^\alpha (\partial/\partial y)^\beta a(t,y) \right| \leq \; C^{|\alpha+\beta|+1} \alpha!^{\,\sigma} \beta!^{\,\mu}$$

in U for any $a \in \mathcal{A}$ and any $(\alpha, \beta) \in \mathbf{Z}_+^n \times \mathbf{Z}_+^m$. We suppose that $F(t,y)$ satisfies the same inequalities in U. Notice that the right hand side of (A.2) is a sum of at most $(n+m)^N$, $N = |\alpha + \beta|$, terms of the form

$$D^{\gamma,\delta}(t,y) \;=\; P_1^{\gamma_1} Q_1^{\delta_1} \cdots P_N^{\gamma_N} Q_N^{\delta_N} F(t,y) \,,$$

where

$$\gamma_j, \delta_j \in \{0,1\}\,,\;\; \gamma_j + \delta_j = 1\,,\;\; |\gamma| = \sum_{j=1}^N \gamma_j \geq |\alpha|\,,\;\; |\delta| = \sum_{j=1}^N \delta_j \leq |\beta|\,,\;\; |\gamma| + |\delta| = N\,,$$

and $P_j = a_j(t,y)\partial/\partial t_{k_j}$, $Q_j = b_j(t,y)\partial/\partial y_{m_j}$, with a_j and b_j in $\mathcal{A}$. We use the convention $P_j^0 = Q_k^0 = 1$. Then $|\gamma|!^{\,\sigma} |\delta|!^{\,\mu} \leq (|\alpha| + |\beta|)!^{\,\sigma} |\beta|!^{\,\mu-\sigma} \leq C_0^{N+1} |\alpha|!^{\,\sigma} |\beta|!^{\,\mu}$, and the statement follows from the following lemma, which is a variant of [11], Lemma 5.3 and [9], Lemma 3.1.

Lemma A. 4 *There exists a constant $C_1 > 0$ independent of γ and δ such that*

$$\left|D^{\gamma,\delta}(t,y)\right| \;\leq\; (C_1 C)^{N+1} |\gamma|!^{\,\sigma} |\delta|!^{\,\mu}\,. \tag{A.3}$$

To prove (A.3), we notice that

$$\left|D^{\gamma,\delta}(t,y)\right| \;\leq C^{N+1} |\gamma|!^{\,\sigma} |\delta|!^{\,\mu} \#\mathcal{B}_N\,,$$

where

$$\mathcal{B}_N \;=\; \{u \in \mathbf{Z}_+^{N-1}:\; u_1 + \cdots + u_j \leq j,\, 1 \leq j \leq N-1\}\,,$$

and $\#\mathcal{B}_N$ stands for its cardinality. Setting $w_1 = u_1$ and $w_j = u_1 + \cdots + u_j$, $2 \leq j \leq N-1$, we obtain $0 \leq w_1 \leq 1$ and $0 \leq w_j \leq w_{j+1} \leq j+1$ for $1 \leq j \leq N-2$. Assigning to any such $w = (w_1, \ldots, w_{N-1})$ the unit cube $[w_1, w_1+1] \times \cdots \times [w_{N-1}, w_{N-1}+1]$ in $\mathbf{R}^{N-1}$, we estimate $\#\mathcal{B}_N$ from above by the volume of

$$W_N \;=\; \{s = (s_1, \ldots, s_{N-1}) \in \mathbf{R}^{N-1}:\; 0 \leq s_j \leq s_{j+1} + 1 \leq N+1,\, 1 \leq j \leq N-2\}\,.$$

On the other hand,

$$\operatorname{vol} W_N \;\leq\; 2^{N-1}(N-1)^{(N-1)}/(N-1)! \;\leq\; C_1^{N+1}\,,$$

and we obtain the desired inequality.

In the same way one can prove that $h(x,y) = f(x, g(y))$ is a $G^{\sigma,\mu}$ function if $f \in G^{\sigma,\mu}$ and $g \in G^{\mu}$. Using a similar argument one can prove also the implicit function theorem in Gevrey classes (see also [13]). More precisely, let $f = (f_1, \ldots, f_{n_1}) \in G^{\sigma,\sigma,\mu}$, $\mu \geq \sigma \geq 1$, with respect to $(z,x,y) \in \mathbf{R}^{n_1} \times \mathbf{R}^{n_2} \times \mathbf{R}^{n_3}$ in a neighborhood of $(0,0,0)$. Suppose that $f(0,0,0) = 0$ and that $d_z f(0,0,0)$ is nondegenerate. Let $z = \widetilde{z}(x,y)$, $\widetilde{z}(0,0) = 0$, be the function given by the implicit function theorem. Then we obtain $\widetilde{z} \in G^{\sigma,\mu}$ in a neighborhood of $(0,0)$.

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