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Two Dimensional Magnetic Schrödinger Operators : Width of Mini Bands in the Tight Binding Approximation

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Abstract. The spectral properties of two dimensional magnetic Schrödinger operators are studied. It is shown in the tight-binding limit that when a nonzero constant magnetic field is perturbed by an infinite number of magnetic and scalar "wells", the essential spectrum continues to have gaps and moreover, it can be nonempty in between the Landau levels and is localized near the one well Hamiltonian eigenvalues which develop into mini-bands whose width is believed to be optimally controlled.

Résumé. On va étudier les propriétés spectrales de l'operateur de Schrödinger pour une particule bidimensionelle qui se trouve dans un champ magnétique, dans l'approximation tight-binding. On va montrer que, pour un champ magnétique constant, différent de zéro, perturbé par un nombre infini de puits magnétiques et éléctriques, le spectre essentiel continue de présenter des lacunes spectrales et qu'il peut être non vide entre les niveaux de Landau. Plus encore, chaque valeur propre de l'hamiltonien avec un seul puits se transforme dans une bande spectrale dont la largeur est controlée de manière précise.

I Introduction

In this paper we continue the study (begun in [C-N]) of the spectral properties of two dimensional magnetic Schrödinger operators. In [C-N] we considered the "one well problem" i.e.

$$H = (\mathbf{p} - \mathbf{a}_0 - \mathbf{a})^2 + V, \tag{1.1}$$

where $\mathbf{a}_0$ corresponds to a nonzero constant magnetic field, $\mathbf{B}_0$, the magnetic perturbation $\mathbf{B}'(\mathbf{x}) = \mathrm{curl}\, \mathbf{a}(\mathbf{x})$ is bounded in the sense that:

$$b \equiv max\{||D^\alpha B'||_\infty, |\alpha| \leq 1\} < \infty \tag{1.2}$$

and the scalar perturbation $V = V_1 + V_2$ obeys:

$$V_1 \in \mathbf{L}^2(\mathbf{R}^2), \qquad V_2 \in \mathbf{L}^\infty(\mathbf{R}^2) \tag{1.3}$$

It is known that if both the magnetic and the scalar perturbations are vanishing at infinity, then (see [I; H]):

$$\sigma_{ess}(H) = \sigma_L(B_0) = \{(2n+1)B_0 \mid n = 0, 1, ...\} \tag{1.4}$$

It was proved in [C-N] that if $\mathrm{dist}(z, \sigma_L(B_0)) = d > 0$, then for sufficiently small b, $||V_1||_2$ and $||V_2||_\infty$ we have that $z \notin \sigma(H)$ and $(H-z)^{-1}$ is an integral operator with a kernel which obeys:

$$|K(\mathbf{x}, \mathbf{x}')| \leq const(d) \exp\left(-\mu|\mathbf{x}-\mathbf{x}'|\right), \quad |\mathbf{x}-\mathbf{x}'| > 1 \tag{1.5}$$

where μ goes to infinity when b, $||V_1||_2$ and $||V_2||_\infty$ go to zero. (Actually, in [C-N] the above estimate was given in the absence of the scalar potential, but the extension is straightforward).

When the perturbations are vanishing at infinity, an important consequence of (1.5) (proved in [C-N]) is that if $E \in \sigma_{disc}(H)$, its corresponding eigenfunctions decay quicker than any exponential as $|\mathbf{x}| \to \infty$. Under more restrictive conditions imposed on V and $\mathbf{B}'$, a quicker (eventually Gaussian) decay can be proved (see [E; Na 2; S; C-N]). In particular, it is easy to see that if V and $\mathbf{B}'$ vanish outside a compact set, then the decay is Gaussian.

In this paper we shall deal with the multiple well case. The reason for considering this case is that when adding to a nonzero constant magnetic field a magnetic field perturbation and a scalar potential both having no decay at infinity a rich structure of the spectrum arise: the Landau spectrum suffers a radical change and one is expecting to find essential spectrum and gaps in between the Landau levels; moreover in the tight binding limit , there is a remarkable enhancement in the localization of the spectrum in comparison with a higher dimensional case (see Section 3 for precise formulation of our main result). The multiple well problem has been considered both in the zero and nonzero magnetic field case but (see [H-S 1,2; C; B-C-D; N-B; H-H; Na]) mainly below the essential spectrum of the "unperturbed" Hamiltonian; what we add to the existent results is that in the two dimensional nonzero magnetic field case the width of the "mini-bands" located below or in between the Landau levels shrinks Gaussian like in the limit when the inter well distance goes to infinity. Notice that the limit considered in [H-H; Na 1] is the strong field case i.e. the magnetic field outside the wells goes to infinity.

The contents of the paper is as follows:

Section 2 fixes some notations and gives a few results needed in the next section. Lemma 2.1 outlines the Gaussian decay of the kernel of the "free resolvent" (the magnetic field is constant here and the scalar potential is absent); in Lemma 2.2 the localization of eigenfunctions of magnetic Schrödinger operators is briefly discussed.

Propositions 2.1 and 2.2 give explicit examples of one well Hamiltonians with discrete eigenvalues in between the Landau levels.

Section 3 contains the main result of this paper (namely Theorem 3.1) and it is devoted to the multiple well case, when the wells are far apart one from each other. For simplicity, we shall consider only the case of identical wells (but not necessarily arranged in a periodic lattice). The heuristics behind the proofs is the same as in the zero magnetic field case: due to the "interactions" between wells, each eigenvalue of the one well Hamiltonian develops into a mini-band whose width shrinks to zero as the separation between wells tends to infinity. From the technical side our proof is in the spirit of the "geometric perturbation theory " in [B-C-D]. As in the zero magnetic field case, the size of the width of the "mini-bands" is dictated by the decay of the one well eigenfunctions and that's where the difference from higher dimensions appears: while in higher dimensions the width is shrinking exponentially with the inter-well distance, in our setting the width has a Gaussian decay (see Theorem 3.1 for the precise statement and the remark before its proof).

Finally, Corollary 3.1 gives the existence of essential spectrum in between the Landau levels provided the one well Hamiltonian has discrete eigenvalues there.

II Preliminaries

As already said, we shall consider only the two dimensional case (i.e. the particle is confined in the plane $x_3 = 0$ and the magnetic field is orthogonal to that plane). Let $B(\mathbf{x}) \in C^1(\mathbf{R}^2)$. We shall use the following family of vector potentials corresponding to $B(\mathbf{x})$:

$$\mathbf{a}(\mathbf{x}, \mathbf{x}') = \int_0^1 ds\, s\, \mathbf{B}(\mathbf{x}' + s(\mathbf{x} - \mathbf{x}')) \wedge (\mathbf{x} - \mathbf{x}') \tag{2.1}$$

For $\mathbf{x}' = 0$, this is nothing but the usual transversal gauge (see e.g. [T]):

$$\mathbf{a}(\mathbf{x}, 0) \equiv \mathbf{a}(\mathbf{x}) = \int_0^1 ds\, s\, \mathbf{B}(s\, \mathbf{x}) \wedge \mathbf{x} \tag{2.2}$$

If we define

$$\mathbf{f}(\mathbf{x}, \mathbf{x}') = \mathbf{a}(\mathbf{x}) - \mathbf{a}(\mathbf{x}, \mathbf{x}') \tag{2.3}$$

then there exists $\varphi(\mathbf{x}, \mathbf{x}')$ such that

$$\nabla_{\mathbf{x}} \varphi(\mathbf{x}, \mathbf{x}') = \mathbf{f}(\mathbf{x}, \mathbf{x}') \tag{2.4}$$

The additional requirement

$$\varphi(\mathbf{x}', \mathbf{x}') = 0 \tag{2.5}$$

gives

$$\varphi(\mathbf{x}, \mathbf{x}') \quad = \quad \int_{x_1'}^{x_1} dt\, f_1(t, x_2; \mathbf{x}') + \int_{x_2'}^{x_2} dt\, f_2(x_1', t; \mathbf{x}') \tag{2.6}$$

where x_i, x_i', f_i are the Cartesian components of $\mathbf{x}, \mathbf{x}'$ and $\mathbf{f}$ respectively. Performing the path integral of $\mathbf{f}(\mathbf{y}, \mathbf{x}')$ on the segment

$$\gamma(\mathbf{x}, \mathbf{x}') = \{\mathbf{y}(t) = \mathbf{x}' + t(\mathbf{x} - \mathbf{x}') | t \in [0, 1]\} \tag{2.7}$$

and because $\mathbf{a}(\mathbf{y}(t), \mathbf{x}') \cdot (\mathbf{x} - \mathbf{x}') = 0$ for all t, one obtains:

$$\varphi(\mathbf{x}, \mathbf{x}') = \int_{\gamma(\mathbf{x}, \mathbf{x}')} \mathbf{a}(\mathbf{y}) \cdot d\mathbf{y} \tag{2.8}$$

The last equation shows that $\varphi(\mathbf{x}, \mathbf{x}') = -\varphi(\mathbf{x}', \mathbf{x})$ and $\varphi(\mathbf{x}', 0) = \varphi(0, \mathbf{x}) = 0$, therefore

$$\varphi(\mathbf{x}, \mathbf{x}') = \varphi(\mathbf{x}', 0) + \varphi(\mathbf{x}, \mathbf{x}') + \varphi(0, \mathbf{x}) = \int_{\Delta} \mathbf{a}(\mathbf{y}) \cdot d\mathbf{y} \tag{2.9}$$

where Δ is the triangle $\gamma(\mathbf{x}', 0) \cup \gamma(\mathbf{x}, \mathbf{x}') \cup \gamma(0, \mathbf{x})$. The last equality says (via the Stokes theorem) that $-\varphi(\mathbf{x}, \mathbf{x}')$ equals the flux of the magnetic field through Δ.

Using (2.8), after a little calculation one obtains (and this is true in three dimensions, too):

$$\varphi(\mathbf{x}, \mathbf{x}') = -\left(\int_0^1 dt \int_0^1 ds\, s\, \mathbf{B}(s\, t\, (\mathbf{x} - \mathbf{x}') + s\, \mathbf{x}') \right) \cdot (\mathbf{x} \wedge \mathbf{x}') \tag{2.10}$$

If $B(\mathbf{x}) = B_0$ is constant, then

$$\begin{aligned} \varphi_0(\mathbf{x}, \mathbf{x}') &= -\frac{1}{2} B_0\, (x_1\, x_2' - x_1'\, x_2) \\ \mathbf{a}_0(\mathbf{x}, \mathbf{x}') &= \frac{1}{2} \mathbf{B}_0 \wedge (\mathbf{x} - \mathbf{x}') \end{aligned} \tag{2.11}$$

The Hamiltonian of a particle in the presence of the magnetic field and a scalar potential V is (in the transversal gauge):

$$\begin{aligned} H &= (\mathbf{p} - \mathbf{a}(\mathbf{x}))^2 + V(\mathbf{x}) \\ \mathbf{p} &= \left(-i \frac{\partial}{\partial x_1}, -i \frac{\partial}{\partial x_2} \right) \\ \mathbf{a}(\mathbf{x}) &= \left(-x_2 \int_0^1 ds\, s\, B(s\, \mathbf{x}), x_1 \int_0^1 ds\, s\, B(s\, \mathbf{x}) \right) \end{aligned} \tag{2.12}$$

In the case of a constant magnetic field, one has the Hamiltonian

$$H_0 = (\mathbf{p} - \mathbf{a}_0(\mathbf{x}))^2 \text{ where} \tag{2.13}$$

$$\mathbf{p} = -i \nabla_{\mathbf{x}} \quad \text{and} \quad \mathbf{a}_0(\mathbf{x}) = \left(-\frac{1}{2} B_0\, x_2, \frac{1}{2} B_0\, x_1 \right) \tag{2.14}$$

which is essentially self-adjoint on $C_0^\infty\left(\mathbf{R}^2\right)$ and its spectrum is the well known Landau spectrum

$$\sigma(H_0) = \sigma_{ess}(H_0) \equiv \sigma_L(B_0) = \{(2n+1)B_0 \mid n = 0, 1, 2, \ldots\} \tag{2.15}$$

For $z \notin \sigma(H_0)$ and $g \in L^2\left(\mathbf{R}^2\right)$, we write

$$\begin{aligned}\left[(H_0 - z)^{-1} g\right](\mathbf{x}) &= \int d\mathbf{x}'\, K_0(\mathbf{x}, \mathbf{x}'; z) g(\mathbf{x}') \\ (H_0 - z) K_0(\mathbf{x}, \mathbf{x}'; z) &= \delta(\mathbf{x} - \mathbf{x}')\end{aligned} \tag{2.16}$$

Then takes place (see e.g. [J-P]):

Lemma II.1 *Let*

$$\begin{aligned}\varphi_0(\mathbf{x}, \mathbf{x}') &= -\frac{B_0}{2}\,(x_1\, x_2' - x_2\, x_1') \\ \psi(\mathbf{x}, \mathbf{x}') &= \frac{B_0}{4}\,|\mathbf{x} - \mathbf{x}'|^2 \\ \alpha &= -\frac{1}{2}\left(\frac{z}{B_0} - 1\right) \neq -1, -2, \ldots\end{aligned}$$

Then

$$\begin{aligned}K_0(\mathbf{x}, \mathbf{x}'; z) &= e^{i\,\varphi_0(\mathbf{x}, \mathbf{x}')}\, G_0(\mathbf{x}, \mathbf{x}'; z) \equiv \\ &\equiv \frac{\Gamma(\alpha)}{4\,\pi} e^{i\,\varphi_0(\mathbf{x}, \mathbf{x}')}\, e^{-\psi(\mathbf{x}, \mathbf{x}')} U(\alpha, 1; 2\,\psi(\mathbf{x}, \mathbf{x}'))\end{aligned} \tag{2.17}$$

where Γ is the Euler function and $U(\alpha, \gamma; \xi)$ is the confluent hyper-geometric function [A-S].

From Lemma II.1 one sees that $K_0(\mathbf{x}, \mathbf{x}'; z)$ has a Gaussian decay as $|\mathbf{x} - \mathbf{x}'| \to \infty$. We shall use this in the following form:

Corollary II.1 *Let $\chi_1, \chi_2 \in \mathbf{L}^\infty(\mathbf{R}^2)$ such that*

$$|\chi_1|, |\chi_2| \leq M \quad \textit{and} \quad \mathrm{dist}\{\textit{supp}\, \chi_2, \textit{supp}\, \chi_1\} = d > 0.$$

Then for all $0 < \delta < \frac{B_0}{4}$ and $z \in \rho(H_0)$, one has that

$$||\chi_1\, (H_0 - z)^{-1}\, \chi_2|| \leq\; M^2\, \textit{const}(z)\; \exp(-\delta\, d^2) \tag{2.18}$$

Proof. Use the explicit form of K_0 and Young inequalities (see [C-N] for further discussions).

Remark. Since under a gauge transformation

$$\begin{aligned}(U_\chi f)(\mathbf{x}) &= e^{i\,\chi(\mathbf{x})} f(\mathbf{x}) \quad \text{and} \\ \left(U_\chi^*(H_0 - z)^{-1} U_\chi f\right)(\mathbf{x}) &= \int_{\mathbf{R}^2} d\mathbf{x}'\, K_\chi(\mathbf{x}, \mathbf{x}'; z) = \\ &= \int_{\mathbf{R}^2} d\mathbf{x}'\, e^{-i\,\chi(\mathbf{x})} K_0(\mathbf{x}, \mathbf{x}'; z) e^{i\,\chi(\mathbf{x}')} f(\mathbf{x}')\end{aligned} \tag{2.19}$$

one has

$$|K_\chi(\mathbf{x},\mathbf{x}';z)| = |K_0(\mathbf{x},\mathbf{x}';z)| \tag{2.20}$$

i.e. the Gaussian decay is valid for an arbitrary gauge.

Suppose now that the scalar potential V and the magnetic field which corresponds to $\mathbf{a}$ describe the one well case studied in [C-N] i.e. satisfy the following conditions :

$$\begin{aligned} & B = B_0 + B', \quad B_0 > 0 \\ & B' \in C^1\left(\mathbf{R}^2\right) \; ; \; \lim_{n\to\infty} ||B'||_{C^1(\mathbf{R}^2\setminus\{|\mathbf{x}|\leq n\})} = 0 \\ & V = V_1 + V_2 \; ; \; V_1 \in L^2\left(\mathbf{R}^2\right), \; V_2 \in L^\infty\left(\mathbf{R}^2\right) \\ & \lim_{n\to\infty} \sup_{|\mathbf{x}|\geq n} |V_2(\mathbf{x})| = 0 \end{aligned} \tag{2.21}$$

In particular, under these conditions H is essentially self-adjoint on $C_0^\infty\left(\mathbf{R}^2\right)$ (see e.g. [C-F-K-S]). Moreover, V is relatively compact with respect to $(\mathbf{p}-\mathbf{a})^2$ [C-F-K-S] which together with the results in [I, H] it implies that

$$\sigma_{ess}(H) = \sigma(H_0) = \{(2n+1)B_0 \mid n = 0,1,2,\ldots\} \tag{2.22}$$

In the rest of this section,

$$g \in C^\infty\left(\mathbf{R}^2;\mathbf{R}\right) \; ; \; ||g||_{C^2(\mathbf{R}^2)} = M < \infty \tag{2.23}$$

Let $E \in \sigma_{disc}(H)$ (the discrete spectrum of H) and let ψ be a normalized eigenfunction corresponding to E. We are interested now in controlling as good as possible the term $||[H,g]\psi||$.

Under the conditions (2.21), one has $D(H) = D((\mathbf{p}-\mathbf{a})^2)$ and $(p_j - a_j)(H+i)^{-1}$ is bounded, $j \in \{1,\ 2\}$. Moreover, because

$$[H,g] = -i\{(\mathbf{p}-\mathbf{a})\cdot\nabla g + \nabla g\cdot(\mathbf{p}-\mathbf{a})\} \tag{2.24}$$

it follows that

$$||[H,g](H+i)^{-1}|| \leq const(M) \tag{2.25}$$

which gives the following rough result:

$$||[H,g]\psi|| \leq const(M)\;(E^2+1)^{1/2} \tag{2.26}$$

In order to obtain a sharper estimate on this term, we use the following form of the I.S.M. localization lemma [C-F-K-S]:

Lemma II.2 *Let $\varphi \in D(H)$. Then:*

$$< \varphi, gHg\varphi >= \Re(< \varphi, g^2 H\varphi >)+ < \varphi, |\nabla g|^2\varphi > \tag{2.27}$$

and

$$< g\psi, (H-E)g\psi >=< \psi, |\nabla g|^2\psi > \tag{2.28}$$

Under the conditions (2.21), there exists a constant $c > 0$ independent of ψ such that:

$$| < g\psi, Vg\psi > | \leq \frac{1}{2} < g\psi, (\mathbf{p} - \mathbf{a})^2 g\psi > +c||g\psi||^2 \tag{2.29}$$

From (2.28) and (2.29) it follows that:

$$\begin{aligned} &||(\mathbf{p} - \mathbf{a})g\psi||^2 \equiv < g\psi, (\mathbf{p} - \mathbf{a})^2 g\psi > \leq \\ &\leq 2M^2[|E| + c + 1] \int_{supp\ g} d\mathbf{x}|\psi(\mathbf{x})|^2 \end{aligned} \tag{2.30}$$

After a little calculation, from (2.24) and (2.30) one obtains:

$$||[H, g]\psi||^2 \leq \ const(E, M) \left(\int_{supp\ |\nabla g|} d\mathbf{x}|\psi(\mathbf{x})|^2 \right) \tag{2.31}$$

which is the needed estimate.

We'll show now that there are many examples of one well Hamiltonians with discrete spectrum in between the Landau levels. We put this in the form of two propositions: the first one constructs a purely electric well which gives an eigenvalue located anywhere we want outside $\sigma_L(B_0)$ and the second one states that any sufficiently "small" purely magnetic well with definite sign creates eigenvalues *near* any Landau level we choose.

Proposition II.1 *Take $\lambda \in \mathbf{R}$, $\lambda \notin \sigma_L(B_0)$. Then there exists a bounded, compactly supported potential $V \in \mathbf{L}^\infty\left(\mathbf{R}^2\right)$ such that λ is a discrete eigenvalue for the operator sum $H = H_0 + V$.*

Proof. Fix λ as mentioned above. From Lemma II.1, one can easily see that

$$K_0(\mathbf{x}, 0; \lambda) = \overline{K_0(0, \mathbf{x}; \overline{\lambda})} = \overline{K_0(\mathbf{x}, 0; \lambda)} \tag{2.32}$$

where the over-line means complex conjugation. Because the confluent hypergeometric function $U(\alpha, 1, \xi)$ is analytic in $\{\xi \in \mathbf{C},\ \Re\xi > 0\}$ and together with the realty of $K_0(\mathbf{x}, 0; \lambda)$ one obtains the existence of $A > 0$, $0 < \epsilon < A$ and $0 < \delta < 1$ such that if $A - \epsilon \leq |\mathbf{x}| \leq A + \epsilon$, then $K_0(\mathbf{x}, 0; \lambda)$ is not changing sign and moreover, one can suppose without loss that

$$K_0(\mathbf{x}, 0; \lambda) \geq \delta \tag{2.33}$$

Define now:

$$\begin{aligned} &\eta_1 \in C_0^\infty(\mathbf{R}^2), \quad 0 \leq \eta_1 \leq 1 \text{ and} \\ &\eta_1(\mathbf{x}) = \left\{ \begin{array}{ll} 1 & \text{if} \quad |\mathbf{x}| \leq A \\ 0 & \text{if} \quad |\mathbf{x}| \geq A + \epsilon \end{array} \right. \end{aligned} \tag{2.34}$$

$$\eta_2 \in C^\infty(\mathbf{R}^2), \quad 0 \le \eta_2 \le 1 \text{ and}$$
$$\eta_2(\mathbf{x}) = \begin{cases} 0 & \text{if} \quad |\mathbf{x}| \le A - \epsilon \\ 1 & \text{if} \quad |\mathbf{x}| \ge A \end{cases} \tag{2.35}$$

$$\psi(\mathbf{x}) = \eta_1(\mathbf{x}) + \eta_2(\mathbf{x}) K_0(\mathbf{x}, 0; \lambda) \tag{2.36}$$

We also require η_1 and η_2 to be radially symmetric.

Using (2.33) and the definitions of the cut-off functions, one obtains that $\psi \in \mathbf{L}^2\left(\mathbf{R}^2\right)$ and $\psi(\mathbf{x}) \ge \delta$ if $|\mathbf{x}| \le A + \epsilon$.

Take now $\Phi \in \mathbf{L}^\infty\left(\mathbf{R}^2\right)$ such that:

$$\Phi(\mathbf{x}) = \begin{cases} \frac{1}{\psi(\mathbf{x})} & \text{if} \quad |\mathbf{x}| \le A + \epsilon \\ 1 & \text{if} \quad |\mathbf{x}| > A + \epsilon \end{cases} \tag{2.37}$$

Finally, the potential we are looking for will be:

$$V = -\Phi \cdot \{(H_0 - \lambda)\eta_1 + [H_0, \eta_2] K(\cdot, 0; \lambda)\} \tag{2.38}$$

Due to the fact that $\mathbf{a}_0$ is written in the transversal gauge (which implies $\mathbf{a}_0(\mathbf{x}) \cdot \mathbf{x} = 0$), it follows that H_0 maps radially symmetric functions into real functions, and that V is real, bounded and compactly supported. Moreover, $H\,\psi = \lambda\,\psi$.

Proposition II.2 *Let $B' \in C_0^1\left(\mathbf{R}^2; \mathbf{R}\right)$ be a nonnegative, compactly supported function and let $\mathbf{a}'(\mathbf{x})$ be the transversal gauge which gives B'.*

For $b > 0$, define $H_b = (\mathbf{p} - \mathbf{a}_0 - b\mathbf{a}')^2$. Let $E_n = (2n+1)B_0$ be the n-th Landau level. Then for b sufficiently small, H_b will have at least one eigenvalue near E_n.

Proof. Because B' has compact support, one has $|\mathbf{a}'(\mathbf{x})| \le$ *const*$\cdot < \mathbf{x} >^{-1}$ where $< \mathbf{x} > \equiv \left(1 + \mathbf{x}^2\right)^{\frac{1}{2}}$.

Denote with $W(b) = -b(\mathbf{p} - \mathbf{a}_0) \cdot \mathbf{a}' - b\mathbf{a}' \cdot (\mathbf{p} - \mathbf{a}_0) + b^2 \mathbf{a}'^2$ and with $V = -(\mathbf{p} - \mathbf{a}_0) \cdot \mathbf{a}' - \mathbf{a}' \cdot (\mathbf{p} - \mathbf{a}_0)$. It is easy to see that $W(b)$ is relatively bounded to H_0; one can then apply the analytic perturbation theory around E_n if b is kept small enough. The reduced operator defined in RanP_n (P_n being the projector associated with E_n) will have the form:

$$H_{eff}(b) = E_n P_n + P_n T(b) P_n$$
$$T(b) \equiv b P_n V P_n + \mathcal{O}(b^2). \tag{2.39}$$

The only thing we should check is that $P_n T(b) P_n$ is not zero; this would imply that $H_{eff}(b) - E_n$ has nonzero spectrum, therefore H_b will have (discrete) spectrum near E_n. To achieve that, one can compute $< f_n, V f_n >$ where f_n stands for the spherically symmetric, real eigenfunction of H_0, corresponding to E_n. This computation gives:

$$< f_n, V f_n >= 2 < f_n, \mathbf{a}_0 \cdot \mathbf{a}' f_n >= B_0 \int d\mathbf{x}\ f_n^2(\mathbf{x}) \int_0^{|\mathbf{x}|} d\rho\ \rho B'(\rho,\theta) \tag{2.40}$$

This quantity is not zero because B' is not changing sign; therefore, if b is small enough, $P_n T(b) P_n \neq 0$.

Remark. This type of argument also works in the case of a purely electric well; one only has to check that the term in (2.40) (where V stands now for the scalar potential) is different from zero. In conclusion, it is not difficult to give examples of one well Hamiltonians with discrete spectrum outside the Landau levels; the really hard problem is to study their behaviour near the essential spectrum.

III Gaps in the essential spectrum

Consider

$$\Gamma_N = \{\mathbf{x}(i)\}_{i=1}^N \subset \mathbf{R}^2, \quad N \leq \infty \tag{3.1}$$

Without loss of generality, one can always take $\mathbf{x}(1) = 0$. The main assumption about Γ_N is that:

$$\inf_{j\neq k} |\mathbf{x}(j) - \mathbf{x}(k)| = r > 0 \tag{3.2}$$

and since the limit to be considered is $r \to \infty$, we assume for technical reasons that r is sufficiently large, say $r \geq 1000$.

Concerning the magnetic field and the potential, we assume:

$$\begin{aligned} &B_0 > 0\,,\ B' \in C^1(\mathbf{R}^2) \text{ and } supp\, B' \subset \{|\mathbf{x}| \leq 1\} \\ &V \in L^2(\mathbf{R}^2),\ supp\, V \subset \{|\mathbf{x}| \leq 1\} \end{aligned} \tag{3.3}$$

Let

$$\begin{aligned} B_N(\mathbf{x}) &= B_0 + \sum_{j=1}^N B'(\mathbf{x} - \mathbf{x}(j)),\ V_N(\mathbf{x}) = \sum_{j=1}^N V(\mathbf{x} - \mathbf{x}(j)), \\ \mathbf{a}_N(\mathbf{x}) &= \int_0^1 ds\ s\, \mathbf{B}_N(s\,\mathbf{x}) \wedge \mathbf{x} \end{aligned} \tag{3.4}$$

Consider now for $N = 1, 2, \ldots, \infty$ the following family of Hamiltonians:

$$H_N = (\mathbf{p} - \mathbf{a}_N)^2 + V_N. \tag{3.5}$$

These operators are essentially self-adjoint on $C_0^\infty(\mathbf{R}^2)$ and for $N < \infty$

$$\sigma_{ess}(H_N) = \sigma_L(B_0) \tag{3.6}$$

In particular,

$$H_1 = (\mathbf{p} - \mathbf{a}_1)^2 + V_1 \tag{3.7}$$

is the "one well" Hamiltonian.

The main result of this section is contained in

Theorem III.1 *Let $1 \leq N < \infty$, $c < d$, $K \equiv [c, d]$, $K \cap \sigma_L(B_0) = \emptyset$ and suppose that*

$$\begin{aligned} \sigma(H_1) \cap K &= \{E_1 < \ldots < E_s\} \subset \sigma_{disc}(H_1),\ s \geq 1 \\ \mathrm{mult}(E_j) &= m_j,\ j \in \{1, \ldots s\} \end{aligned}$$

If c, d are not eigenvalues for H_1, then there exist $r_0(K, m_1, \ldots, m_s), \mathcal{C} < \infty$ and $u > 0$ independent of N *such that*

$$i)\ \sigma(H_N) \cap K \subset \bigcup_{j=1}^{s} [E_j - \delta, E_j + \delta],\ 0 \leq \delta \leq \mathcal{C} e^{-u\, r^2},\ \textit{for all}\, r \geq r_0 \tag{3.8}$$

$$ii)\ \dim\{\mathrm{Ran}\, \mathcal{P}_N[\sigma(H_N) \cap K]\} = N \sum_{j=1}^{s} m_j \tag{3.9}$$

where $\mathcal{P}_N$ is the spectral measure associated with H_N.

Remark. If one drops the compactness condition in (3.3) but imposes additional conditions to (2.21) in order to ensure the finiteness of the "total perturbations" in H_∞, such as:

$$\max\{|B'(\mathbf{x})|, |V(\mathbf{x})|\} \leq \textit{ const}\ (1 + |\mathbf{x}|)^{-\beta},\ \beta > 2 \tag{3.10}$$

then the proof of Theorem 3.1 can be adapted such that *ii)* remains true and *i)* is changing in the sense that instead of a Gaussian decay in r, we can only say that δ goes to zero when r goes to infinity and this comes from the fact that in this case, the wells are no longer well individualized.

Proof of i). Define :

$$\Sigma_N \equiv \sigma(H_N) \cap K \tag{3.11}$$

Because of (3.6), Σ_N is discrete if not empty. For simplicity, we suppose $s = 1$ and $m_1 = 1$; the proof in the general case is similar.

During the proof of this theorem, $E \in \Sigma_N$ will denote an eigenvalue of H_N and ψ a corresponding normalized eigenfunction.

We give first a few helpful technical lemmas and we start with some definitions. For $(p_1, p_2) \in \mathbf{Z}^2$ and $\delta > 0$, define:

$$K(p_1, p_2; \delta) = \left\{ \mathbf{x} \in \mathbf{R}^2 |\ |x_j - \frac{r}{100}\left(p_j + \frac{1}{2}\right)| \leq \frac{\delta}{2},\ j \in \{1, 2\} \right\} \tag{3.12}$$

It is easily seen that for any $\delta \geq \frac{r}{100}$,

$$\bigcup_{(p_1,p_2)\in \mathbf{Z}^2} K(p_1,p_2;\delta) = \mathbf{R}^2 \tag{3.13}$$

If $\mathbf{m} \in \mathbf{R}^2$, then the translation $t_{\mathbf{m}} : L^2 \to L^2$, $(t_{\mathbf{m}} f)(\mathbf{x}) = f(\mathbf{x}-\mathbf{m})$ is an unitary operator. Given any $j \in \{1,\ldots,N\}$, there exists $(p_1^j, p_2^j) \in \mathbf{Z}^2$ such that:

$$\mathbf{x}(j) \in K(p_1^j, p_2^j; r/100) \text{ and } \mathbf{x}(k) \notin K(p_1^j, p_2^j; r/100) \text{ if } j \neq k \tag{3.14}$$

If $\beta, \gamma \in \{-1,0,1\}$ then define

$$\mathcal{K}_j(\delta) = \bigcup_{\beta,\gamma} K(p_1^j+\beta, p_2^j+\gamma;\delta), \quad \delta > 0 \tag{3.15}$$

By construction,

$$\mathrm{dist}\{\mathbf{x}(j), \partial\mathcal{K}_j(r/100)\} \geq r/100. \tag{3.16}$$

Denote with

$$\mathcal{F}_N = \bigcup_{j=1}^{N} \mathcal{K}_j(r/100) \tag{3.17}$$

Lemma III.1 *Take*

$$K(p,q;r/100) \not\subset \mathcal{F}_N \text{ and } \eta \in C_0^\infty(\mathbf{R}^2),$$
$$\mathit{supp}\,\eta \subset K(p,q;r/98).$$

Denote with $\mathbf{m} = \left(\frac{r}{100}\left(p+\frac{1}{2}\right), \frac{r}{100}\left(q+\frac{1}{2}\right)\right)$. *Then:*

$$H_N\eta = e^{i\,\varphi_N(.,\mathbf{m})} t_{\mathbf{m}}\, H_0\, t_{-\mathbf{m}} e^{-i\,\varphi_N(.,\mathbf{m})}\eta \tag{3.18}$$

Proof. If $\mathbf{x} \in \mathit{supp}\,\eta$, then :

$$\begin{aligned} \mathbf{a}_N(\mathbf{x}) &= \mathbf{a}_N(\mathbf{x},\mathbf{m}) + \nabla\varphi_N(\mathbf{x},\mathbf{m}) \\ V_N(\mathbf{x}) &= 0 \\ \mathbf{a}_N(\mathbf{x},\mathbf{m}) &= \int_0^1 ds\; s\, \mathbf{B}_N(\mathbf{m}+s(\mathbf{x}-\mathbf{m})) \wedge (\mathbf{x}-\mathbf{m}) \end{aligned} \tag{3.19}$$

Because for all $y \in \{\mathbf{m}+s(\mathbf{x}-\mathbf{m}),\ 0 \leq s \leq 1\}$ one has

$$\mathbf{B}_N(y) = \mathbf{B}_0 \tag{3.20}$$

then

$$\begin{aligned} \mathbf{a}_N(\mathbf{x},\mathbf{m}) &= \mathbf{a}_0(\mathbf{x}-\mathbf{m}) \quad \text{and} \\ H_N\eta &= e^{i\,\varphi_N(\mathbf{x},\mathbf{m})}[\mathbf{p}-\mathbf{a}_0(\mathbf{x}-\mathbf{m})]^2 e^{-i\,\varphi_N(\mathbf{x},\mathbf{m})}\eta \\ &= e^{i\,\varphi_N(.,\mathbf{m})} t_{\mathbf{m}}\, H_0\, t_{-\mathbf{m}} e^{-i\,\varphi_N(.,\mathbf{m})}\eta \end{aligned}$$

Lemma III.2 *Fix* $j \in \{1,\dots,N\}$*. Take* $\eta_j \in C_0^\infty(\mathbf{R}^2)$ *and supp* $\eta_j \subset \mathcal{K}_j(r/98)$*. Then :*

$$H_N\eta_j = e^{i\,\varphi_N(.,\mathbf{x}(j))}t_{\mathbf{x}(j)}\;H_1\;t_{-\mathbf{x}(j)}e^{-i\,\varphi_N(.,\mathbf{x}(j))}\eta_j \tag{3.21}$$

Proof. As before,

$$\mathbf{a}_N(\mathbf{x}) = \mathbf{a}_N(\mathbf{x},\mathbf{x}(j)) + \nabla\varphi_N(\mathbf{x},\mathbf{x}(j)) \tag{3.22}$$

Since $r - \frac{3\sqrt{2}}{98}r > 1$ and $\mathbf{x} \in supp\,\eta_j$, then

$$|\mathbf{x}(j)-\mathbf{x}(k)+s(\mathbf{x}-\mathbf{x}(j))| \geq |\mathbf{x}(j)-\mathbf{x}(k)| - |\mathbf{x}-\mathbf{x}(j)| > 1, \quad j \neq k \tag{3.23}$$

therefore:

$$\begin{aligned} & \mathbf{a}_N(\mathbf{x},\mathbf{x}(j)) = \mathbf{a}_0(\mathbf{x}-\mathbf{x}(j)) + \\ + & \int_0^1 ds\, s \sum_{k=1}^N \mathbf{B}'(\mathbf{x}(j)-\mathbf{x}(k)+s(\mathbf{x}-\mathbf{x}(j))) \wedge (\mathbf{x}-\mathbf{x}(k)) \\ = & \ \mathbf{a}_1(\mathbf{x}-\mathbf{x}(j)) + \\ + & \int_0^1 ds\, s \sum_{k\neq j} \mathbf{B}'(\mathbf{x}(j)-\mathbf{x}(k)+s(\mathbf{x}-\mathbf{x}(j))) \wedge (\mathbf{x}-\mathbf{x}(k)) \\ = & \ \mathbf{a}_1(\mathbf{x}-\mathbf{x}(j)) \end{aligned} \tag{3.24}$$

If $\mathbf{x} \in supp\,\eta_j$, then $V_N(\mathbf{x}) = V(\mathbf{x}-\mathbf{x}(j))$; putting all these together, (3.21) follows.

Lemma III.3 *Under the same assumptions made in Lemma 3.1, suppose now that*

$$\begin{aligned} & 0 \leq \eta \leq 1 \quad \textit{and} \\ & \eta(\mathbf{x}) = \begin{cases} 1 & \textit{if} \quad \mathbf{x} \in K(p,q;r/99) \\ 0 & \textit{if} \quad \mathbf{x} \notin K(p,q;r/98) \end{cases} \end{aligned} \tag{3.25}$$

Then there exist $\mathcal{C}_1 > 0$ *and* $u > 0$ *(which are* independent *of* N*,* (p,q) *and* $E \in \Sigma_N$*) such that:*

$$\int_{K(p,q;r/100)} d\mathbf{x}\,|\psi(\mathbf{x})|^2 \leq e^{-ur^2}\mathcal{C}_1 \int_{\overline{K(p,q;r/98)\setminus K(p,q;r/99)}} d\mathbf{x}\,|\psi(\mathbf{x})|^2 \tag{3.26}$$

Proof. (3.18) implies that

$$[H_N,\eta]\psi = e^{i\,\varphi_N(.,\mathbf{m})}t_{\mathbf{m}}\,(H_0-E)\,t_{-\mathbf{m}}e^{-i\,\varphi_N(.,\mathbf{m})}\eta\psi \tag{3.27}$$

or

$$\eta\psi = e^{i\,\varphi_N(.,\mathbf{m})}t_{\mathbf{m}}\,(H_0-E)^{-1}\,t_{-\mathbf{m}}e^{-i\,\varphi_N(.,\mathbf{m})}[H_N,\eta]\psi \tag{3.28}$$

If

$$\mathbf{x} \in K(p,q;r/100) \text{ and } \mathbf{x}' \in \overline{K(p,q;r/98)\setminus K(p,q;r/99)} \tag{3.29}$$

then

$$|\mathbf{x}-\mathbf{x}'|^2 \geq \frac{r^2}{4}\left(\frac{1}{99}-\frac{1}{100}\right)^2 \tag{3.30}$$

and using Corollary 2.1, one has ($0 < u < \frac{B_0}{64}\left(\frac{1}{99}-\frac{1}{100}\right)^2$):

$$\int_{K(p,q;r/100)} d\mathbf{x}\,|\psi(\mathbf{x})|^2 \leq e^{-ur^2}\mathcal{C}_1\,||[H_N,\eta]\psi||^2 \tag{3.31}$$

Using (2.31) in (3.31), one obtains (3.26).

Lemma III.4 *There exist $u > 0$, $\mathcal{C} < \infty$ with the properties given in Lemma III.3 such that:*

$$\int_{(\mathcal{F}_N)^c} d\mathbf{x}\,|\psi(\mathbf{x})|^2 \leq \mathcal{C}\,e^{-ur^2} \tag{3.32}$$

Proof. Adding the contributions given by all $K(p,q;r/100) \not\subset \mathcal{F}_N$ in (3.26) and because

$$\sum_{(p,q)}\int_{\overline{K(p,q;r/98)\setminus K(p,q;r/99)}} d\mathbf{x}\,|\psi(\mathbf{x})|^2 \leq 4 \tag{3.33}$$

the result follows.

We are now able to prove the first affirmation of Theorem 3.1. For $j \in \{1,\ldots,N\}$ take

$$\begin{aligned}&\eta_j \in C_0^\infty(\mathbf{R}^2), \quad 0 \leq \eta_j \leq 1 \text{ and}\\ &\eta_j(\mathbf{x}) = \begin{cases} 1 & \text{if} \quad \mathbf{x} \in \mathcal{K}_j(r/99)\\ 0 & \text{if} \quad \mathbf{x} \notin \mathcal{K}_j(r/98)\end{cases}\end{aligned} \tag{3.34}$$

Let

$$\tilde{\psi} = \sum_{j=1}^N \eta_j\psi \tag{3.35}$$

then from Lemma 3.2 one has:

$$(H_N - E)\tilde{\psi} = \sum_{j=1}^N e^{i\,\varphi_N(.,\mathbf{x}(j))}t_{\mathbf{x}(j)}\,(H_1 - E)\,t_{-\mathbf{x}(j)}e^{-i\,\varphi_N(.,\mathbf{x}(j))}\eta_j\psi \tag{3.36}$$

or

$$\sum_{j=1}^N ||(H_1 - E)t_{-\mathbf{x}(j)}e^{-i\,\varphi_N(.,\mathbf{x}(j))}\eta_j\psi||^2 = \sum_{j=1}^N ||[H_N,\eta_j]\psi||^2 \tag{3.37}$$

But

$$||(H_1 - E)t_{-\mathbf{x}(j)}e^{-i\,\varphi_N(.,\mathbf{x}(j))}\eta_j\psi||^2 \geq \mathrm{dist}^2\{E,\sigma(H_1)\}||\eta_j\psi||^2 \tag{3.38}$$

therefore, together with (3.37) and (2.31) one obtains:

$$\operatorname{dist}^2\{E,\sigma(H_1)\}\sum_{j=1}^{N}||\eta_j\psi||^2 \le \mathcal{C}\int_{(\mathcal{F}_N)^c} d\mathbf{x}\,|\psi(\mathbf{x})|^2 \tag{3.39}$$

or

$$\operatorname{dist}^2\{E,\sigma(H_1)\}\left(1-\int_{(\mathcal{F}_N)^c} d\mathbf{x}|\psi(\mathbf{x})|^2\right) \le \mathcal{C}\int_{(\mathcal{F}_N)^c} d\mathbf{x}|\psi(\mathbf{x})|^2 \tag{3.40}$$

and together with (3.32), the affirmation stated in (3.8) follows.

Proof of ii). Let's show first that

$$\dim\{Ran[\mathcal{P}_N(\Sigma_N)]\} \ge N. \tag{3.41}$$

Denote with ψ_1 the normalized eigenvector of H_1 corresponding to E_1:

$$H_1\psi_1 = E_1\psi_1, \quad ||\psi_1|| = 1 \tag{3.42}$$

With the notations introduced in (3.34), let

$$\mathcal{V}_N = \left\{\tilde{\psi}_j = e^{i\,\varphi_N(.,\mathbf{x}(j))}\eta_j t_{\mathbf{x}(j)}\psi_1\right\}_{j=\overline{1,N}} \tag{3.43}$$

be an orthogonal system.

Because

$$||\tilde{\psi}_j||^2 = ||(t_{-\mathbf{x}(j)}\eta_j)\psi_1||^2 \ge 1-\int_{(\mathcal{F}_1)^c} d\mathbf{x}\,|\psi_1(\mathbf{x})|^2 \sim 1-\mathcal{C}e^{-ur^2} \tag{3.44}$$

then for r large enough, $\mathcal{V}_N$ is an "almost orthonormal system" and

$$\dim \mathcal{V}_N = N, \quad r \ge r_0 \tag{3.45}$$

(notice that r_0 does not depend upon N).

Suppose now that (3.41) were false; this would imply the existence of an $r > r_0$ such that

$$\dim\{\mathrm{Ran}[\mathcal{P}_N(\Sigma_N)]\} \le N-1. \tag{3.46}$$

(3.46) and (3.8) would imply then that:

$$\dim\{\mathrm{Ran}[\mathcal{P}_N(K)]\} \le N-1. \tag{3.47}$$

Then there exists $\tilde{\psi} \in \mathcal{V}_N$, $||\tilde{\psi}|| = 1$ such that:

$$\tilde{\psi} \in \{\mathrm{Ran}[\mathcal{P}_N(K)]\}^{\perp} \tag{3.48}$$

Because $<\tilde{\psi}_j, \tilde{\psi}_k>=0$ if $j \neq k$ and using (3.44), one has that

$$\tilde{\psi} = \sum_{j=1}^{N} C_j \tilde{\psi}_j, \quad \sum_{j=1}^{N} |C_j|^2 \sim 1 - \mathcal{C} e^{-ur_0^2} \tag{3.49}$$

Without loss, suppose that there exists $\delta > 0$ such that $[E_1 - \delta, E_1 + \delta] \subset K$. Then (3.48) implies

$$||(H_N - E_1)\tilde{\psi}|| \geq \delta. \tag{3.50}$$

But using Lemma 3.2, one obtains:

$$\begin{aligned} (H_N - E_1)\tilde{\psi} &= \sum_{j=1}^{N} C_j (H_N - E_1)\tilde{\psi}_j \\ &= \sum_{j=1}^{N} C_j e^{i\,\varphi_N(.,\mathbf{x}(j))} t_{\mathbf{x}(j)} (H_1 - E_1) t_{-\mathbf{x}(j)} \eta_j t_{\mathbf{x}(j)} \psi_1 \\ &= \sum_{j=1}^{N} C_j e^{i\,\varphi_N(.,\mathbf{x}(j))} t_{\mathbf{x}(j)} (H_1 - E_1)(t_{-\mathbf{x}(j)} \eta_j) \psi_1 \end{aligned} \tag{3.51}$$

Using that

$$< t_{\mathbf{x}(j)}(H_1 - E_1)(t_{-\mathbf{x}(j)}\eta_j)\psi_1, t_{\mathbf{x}(k)}(H_1 - E_1)(t_{-\mathbf{x}(k)}\eta_k)\psi_1 >= 0, \; j \neq k \tag{3.52}$$

one has

$$||(H_N - E_1)\tilde{\psi}||^2 = \sum_{j=1}^{N} |C_j|^2 ||[H_1, (t_{-\mathbf{x}(j)}\eta_j)]\psi_1||^2. \tag{3.53}$$

From (2.31), (3.32) and (3.49) it follows that

$$||(H_N - E_1)\tilde{\psi}||^2 \sim e^{-ur_0^2} \tag{3.54}$$

which can be made arbitrarily small and then contradicting (3.50).

Let's prove now that

$$\dim\{\mathrm{Ran}[\mathcal{P}_N(K)]\} \leq N \tag{3.55}$$

In order to prove (3.55), we shall construct a finite rank operator P'_N (not necessary an orthogonal projector) such that:

$$\begin{aligned} &\dim \mathrm{Ran}\, P'_N \leq N \quad \text{and} \\ &||\mathcal{P}_N(K) - P'_N|| < 1, \quad \text{for all} \quad r \geq r_0 \end{aligned} \tag{3.56}$$

Proposition III.1 *Suppose (3.56) fulfilled. Then (3.55) takes place.*

Proof. Assume that $\dim\{\mathrm{Ran}[\mathcal{P}_N(K)]\} \geq N+1$ for some $r \geq r_0$. Then there exists $\psi \in \mathrm{Ran}[\mathcal{P}_N(K)]$, $||\psi|| = 1$ such that:

$$\psi \in [\mathrm{Ran}(P'_N)]^{\perp} \tag{3.57}$$

But

$$|<\psi, (\mathcal{P}_N(K) - P'_N)\psi>| \leq ||\mathcal{P}_N(K) - P'_N|| < 1 \tag{3.58}$$

and

$$|<\psi, (\mathcal{P}_N(K) - P'_N)\psi>| = |<\psi, \mathcal{P}_N(K)\psi>| = 1 \tag{3.59}$$

which contradicts (3.58).

Let's construct now P'_N. Using (3.8), one obtains the existence of $r_0(\epsilon)$ such that

$$\{|z - E_1| = \epsilon\} \cap \sigma(H_N) = \emptyset \tag{3.60}$$

as soon as $r \geq r_0(\epsilon)$ (ϵ being chosen sufficiently small then kept fixed).

The idea consists (see for similar reasoning [B-C-D] and [Na 1]) in approximating the resolvent $(H_N - z)^{-1}$ for $|z - E_1| = \epsilon$ and then integrating over the contour.

Let

$$\mathbf{m}(p,q) = (r/100(p+1/2), r/100(q+1/2)) \text{ if } K(p,q;r/100) \not\subset \mathcal{F}_N \tag{3.61}$$

and

$$\Gamma_\infty = \Gamma_N \bigcup \{\mathbf{m}(p,q)\}_{(p,q)} \tag{3.62}$$

It is possible to construct a quadratic partition of unity which has the following properties (see [C-F-K-S]):

- $$\sum_{\mathbf{m}\in\Gamma_\infty} \eta_{\mathbf{m}}^2 = 1, \quad \eta_{\mathbf{m}} \in C_0^\infty(\mathbf{R}^2), \quad 0 \leq \eta_{\mathbf{m}} \leq 1 \tag{3.63}$$
- $$\eta_{\mathbf{m}}(\mathbf{x}) = 0 \quad \text{if} \quad \mathbf{x} \notin \mathcal{K}_{\mathbf{m}}(r/99) \quad \text{and} \quad \mathbf{m} \in \Gamma_N \tag{3.64}$$
- $$\eta_{\mathbf{m}}(\mathbf{x}) = 0 \quad \text{if} \quad \mathbf{x} \notin K(p,q;r/99) \quad \text{and} \quad \mathbf{m} \in \{\mathbf{m}(p,q)\}_{(p,q)} \tag{3.65}$$
- $$||(\partial\eta_{\mathbf{m}}/\partial x_i)||_{C^1} \leq \frac{const}{r}, \quad \mathbf{m} \in \Gamma^\infty, \quad i = \overline{1,2} \tag{3.66}$$

Lemma III.5 *The operator*

$$\begin{aligned} A_N(z) &= \sum_{\mathbf{m}\in\Gamma_N} e^{i\,\varphi_N(.,\mathbf{m})}\eta_{\mathbf{m}} t_{\mathbf{m}}(H_1 - z)^{-1} t_{-\mathbf{m}} \eta_{\mathbf{m}} e^{-i\,\varphi_N(.,\mathbf{m})} \\ &+ \sum_{\mathbf{m}\in\Gamma_\infty\setminus\Gamma_N} e^{i\,\varphi_N(.,\mathbf{m})}\eta_{\mathbf{m}} t_{\mathbf{m}}(H_0 - z)^{-1} t_{-\mathbf{m}} \eta_{\mathbf{m}} e^{-i\,\varphi_N(.,\mathbf{m})} \end{aligned} \tag{3.67}$$

is bounded and moreover

$$||A_N(z)|| \leq \frac{const}{\epsilon} \tag{3.68}$$

where the constant in (3.68) is independent upon N.

Proof. Take $f \in L^2(\mathbf{R}^2)$. Then, from (3.60) and (3.63), $A_N(z)f$ consists in a sum in which each term is bounded and from (3.64) and (3.65) there results that each term is orthogonal to all others except at most 16 "neighbours".

$A_N(z)$ is our approximation of the resolvent. From Lemma 3.1 and Lemma 3.2, one obtains:

$$\begin{aligned}
&(H_N - z)A_N(z) \\
&= \sum_{\mathbf{m}\in\Gamma_N} e^{i\,\varphi_N(.,\mathbf{m})} t_{\mathbf{m}}(H_1 - z)t_{-\mathbf{m}}\eta_{\mathbf{m}} t_{\mathbf{m}}(H_1 - z)^{-1} t_{-\mathbf{m}}\eta_{\mathbf{m}} e^{-i\,\varphi_N(.,\mathbf{m})} \\
&+ \sum_{\mathbf{m}\in\Gamma_\infty\setminus\Gamma_N} e^{i\,\varphi_N(.,\mathbf{m})} t_{\mathbf{m}}(H_0 - z)t_{-\mathbf{m}}\eta_{\mathbf{m}} t_{\mathbf{m}}(H_0 - z)^{-1} t_{-\mathbf{m}}\eta_{\mathbf{m}} e^{-i\,\varphi_N(.,\mathbf{m})} \\
&= 1 + \sum_{\mathbf{m}\in\Gamma_\infty} e^{i\,\varphi_N(.,\mathbf{m})} t_{\mathbf{m}}[H_{\mathbf{m}}, (t_{-\mathbf{m}}\eta_{\mathbf{m}})](H_{\mathbf{m}} - z)^{-1} t_{-\mathbf{m}}\eta_{\mathbf{m}} e^{-i\,\varphi_N(.,\mathbf{m})} \\
&\equiv 1 + T_N(z)
\end{aligned} \tag{3.69}$$

where

$$H_{\mathbf{m}} = \begin{cases} H_1 & \text{if} \quad \mathbf{m} \in \Gamma_N \\ H_0 & \text{if} \quad \mathbf{m} \in \Gamma_\infty \setminus \Gamma_N \end{cases} \tag{3.70}$$

By essentially the same argument used in deriving (2.25), one obtains:

$$||[H_{\mathbf{m}}, (t_{-\mathbf{m}}\eta_{\mathbf{m}})](H_{\mathbf{m}} - z)^{-1}|| \leq \frac{const(\epsilon)}{r}. \tag{3.71}$$

As in Lemma 3.5, one finally obtains

$$||T_N(z)|| \leq \frac{const(\epsilon)}{r}. \tag{3.72}$$

Take $r_1(\epsilon) \geq r_0(\epsilon)$ such that $||T_N(z)|| \leq \frac{1}{2}$ if $r \geq r_1(\epsilon)$; then one can write:

$$(H_N - z)^{-1} = A_N(z) - A_N(z)T_N(z)[1 + T_N(z)]^{-1} \tag{3.73}$$

Integrating over $\{|z - E_1| = \epsilon\}$, it follows that:

$$\begin{aligned}
&\mathcal{P}_N([E_1 - \epsilon, E_1 + \epsilon]) \\
&= \sum_{\mathbf{m}\in\Gamma_N} e^{i\,\varphi_N(.,\mathbf{m})} \eta_{\mathbf{m}} t_{\mathbf{m}} \mathcal{P}_1(\{E_1\}) t_{-\mathbf{m}}\eta_{\mathbf{m}} e^{-i\,\varphi_N(.,\mathbf{m})} + R_N \\
&\equiv P'_N + R_N
\end{aligned} \tag{3.74}$$

where

$$||R_N|| < 1, \qquad r \geq r_2(\epsilon) \geq r_1(\epsilon) \tag{3.75}$$

and the proof is completed, due to the fact that P'_N has its rank equal to N.

Corollary III.1 *Let $N = \infty$. Then Theorem III.1 remains true (with N formally replaced with ∞).*

Proof. Define

$$\Gamma_{N(R)} = \{\mathbf{x} \in \Gamma_\infty \mid |\mathbf{x}| \leq R\} \tag{3.76}$$

Then :

$$H_{N(R)} \to H_\infty \quad \text{in the strong sense} \tag{3.77}$$

From the essential self-adjoint-Ness on the same core and from (3.77), one obtains

$$H_{N(R)} \to H_\infty \quad \text{in the strong resolvent convergence sense} \tag{3.78}$$

(see e.g. [K]).

Then Theorem VIII.1.4 in [K] and (3.8) imply that (3.8) remains true for H_∞.

Finally, to show that

$$\dim\{\mathrm{Ran}\,\mathcal{P}_\infty[\sigma(H_\infty) \cap K]\} = \infty \tag{3.79}$$

one can use an ad-absurdum argument as that used in proving (3.41).

Remark. It is easy to show now that when increasing r, one still can find essential spectrum of H_∞ near a part of $\sigma_L(B_0)$.

Take for example the old eigenvalue B_0; an eigenfunction of H_0 which corresponds to it reads as:

$$\psi_0(\mathbf{x}) = \sqrt{\frac{B_0}{2\,\pi}} \exp\left(-\frac{B_0\,|\mathbf{x}|^2}{4}\right).$$

Take $\epsilon > 0$ sufficiently small and let's prove that ($K = [B_0 - \epsilon, B_0 + \epsilon]$):

$$\dim\{\mathrm{Ran}\,\mathcal{P}_\infty[\sigma(H_\infty) \cap K]\} = \infty$$

The ad-absurdum argument used in proving (3.41) can be applied again, replacing H_1, ψ_1, E_1 and $\mathcal{K}_j$ with H_0, ψ_0, B_0 and $K(p,q)$.

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