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On the Perturbation Series in Large Order of Anharmonic Oscillators

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Abstract. We present a mathematically rigorous proof of the Bender-Wu (Banks-Bender) formula based on exact WKB analysis.

I Introduction

We will consider the large n asymptotics of a perturbation series of an eigenvalue

$$E^K = K + \frac{1}{2} + \sum_{n=1}^{\infty} A_n^K \lambda^n, \tag{1}$$

of a generalized anharmonic oscillator

$$\left(-\frac{d^2}{dx^2} + \frac{1}{4}\left(x^2 + \lambda x^{2N+2}\right)\right)\psi = E\psi, \tag{2}$$

where $\lambda > 0$, $K = 0, 1, 2, \cdots$ and $N = 1, 2, \cdots$.

In [3](for $N = 1$) and [2](for $N \geq 2$) the following interesting formula is presented:

$$\begin{aligned} A_n^K &= (-1)^{n+1} \frac{N 4^{(K+\frac{1}{2})/N}}{K! \sqrt{2\pi^3}} \left(\frac{B(\frac{3}{2}, \frac{1}{N})}{2N}\right)^{-K-\frac{1}{2}-nN} \\ &\quad \times \Gamma\left(K + \frac{1}{2} + nN\right)\left(1 + O\left(\frac{1}{n}\right)\right), \qquad (n \to \infty), \end{aligned} \tag{3}$$

where $B(x, y)$ and $\Gamma(x)$ respectively denote the beta and the gamma function.

For $N = 1$ there are some rigorous proofs of the above formula; [8] in connection with the resonance problem, and [5] from the resurgent theory. Generalization for a higher dimensional case was considered in [9].

The purpose of this article is to give another proof of the above formula for general N using the exact WKB analysis ([1], [4], [5]) along the original idea in [3]. By the help of Stokes geometry, our argument becomes simpler and more transparent; it clearly explains why complex turning points do not give any effects on the above formula (cf. [2]).

I would like to thank Prof.Kawai and Prof.Takei for giving me many valuable advise and comments. I also thank Prof.Aoki for providing me with a computer program for drawing Stokes curves, some of which I used in this article.

II Calculation of eigenvalues

Following Bender-Wu, we start our argument with the relation ((2.7) in [3])

$$A_n^K = \frac{1}{2\pi i}\int_{-\infty}^{0}\frac{\Delta E^K(\lambda)}{\lambda^{n+1}}d\lambda, \tag{4}$$

where $\Delta E^K(\lambda)$ denotes the difference of $E^K\big|_{\arg\lambda=\pi}$ and $E^K\big|_{\arg\lambda=-\pi}$. This relation was derived from the analyticity of an eigenvalue. (See [6],[8].) By this relation, our problem is reduced to the determination of an eigenvalue for small λ and for $\arg\lambda=\pm\pi$. Keeping this fact in mind, we introduce a large parameter η in (2) by setting $\lambda\mapsto\eta^{-N}e^{i\theta}$ ($\theta=\arg\lambda$) and $x\mapsto\sqrt{\eta}x$ so that (2) becomes

$$\left(-\frac{d^2}{dx^2}+\eta^2\left(Q(x)-\eta^{-1}E\right)\right)\psi=0, \tag{5}$$

where

$$Q(x)=\frac{1}{4}\left(x^2+e^{i\theta}x^{2N+2}\right). \tag{6}$$

One important point is, however, Stokes geometry is degenerate for θ ($=\arg\lambda$) $=\pm\pi$: that is, there exists a Stokes curve connecting turning points. (See the left of Fig. 1, for example.) To resolve this degeneracy, we rotate the contour in (4) in the following way:

$$A_n^K = \frac{1}{2\pi i}\int_{-\infty\, e^{i\epsilon}}^{0}\frac{\Delta E^K(\lambda)}{\lambda^{n+1}}d\lambda, \tag{7}$$

where ϵ is a sufficiently small positive number. (See the right of Fig. 1. We can show this modified relation in a similar way as that of the original one. See also §V in [2].) Hence we determine an eigenvalue of (5) for $\theta=\pm\pi+\epsilon$.

There are one double turning point at $x=0$ and $2N$ simple turning points at $x=\exp(i((2k+1)\pi-\theta)/2N)$ ($k=0,1,2,\cdots,2N-1$).

For $\theta=-\pi+\epsilon$

First we compute $E^K(\lambda)$ when $\theta\,(=\arg\lambda)=-\pi+\epsilon$. The examples of Stokes curves are given in the right of Fig. 1 (for $N=1$) and Fig. 2 (for $N=2$). (The left of Fig. 1, 2 are that for $\epsilon=0$.)

We can determine analytically continued eigenvalues by the rotating sector condition (cf.[3], see also [8]); we require that the solutions of (5) tend to 0 in the sectors

$$\Sigma_\pm(\theta)=\left\{x\in\mathbf{C}\,;\,\left|\arg(\pm x)+\frac{\theta}{2(N+2)}\right|<\frac{\pi}{2(N+2)}\right\}. \tag{8}$$

As a path of the analytic continuation of solutions connecting these sectors $\Sigma_\pm(\theta)$ near ∞, we can choose and fix the following path Γ throughout this article: We define Γ with the help of Stokes curves for $\epsilon=0$. If ϵ is equal to 0, $x=\pm1$ are

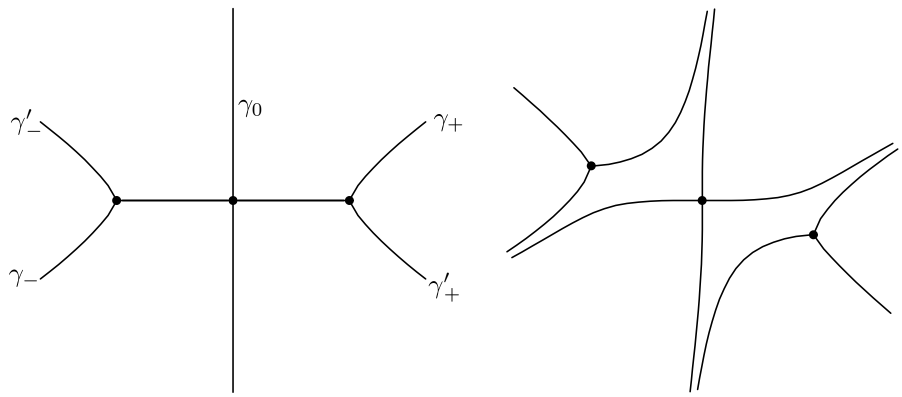

Figure 1: Stokes curves ($N = 1$) for $\epsilon = 0$ (left) and for $\epsilon > 0$ (right).

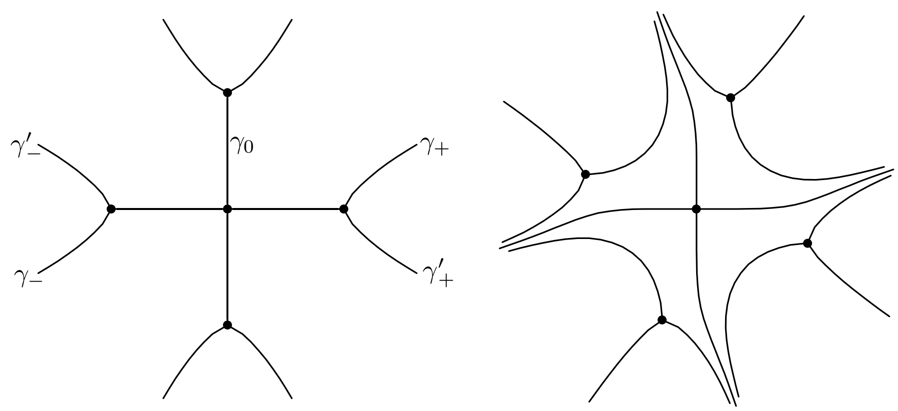

Figure 2: Stokes curves ($N = 2$) for $\epsilon = 0$ (left) and for $\epsilon > 0$ (right).

turning points, and $x = \pm i$ also become turning points if N is even. Let γ_0 be a Stokes curve emanating from $x = 0$ in the direction of positive imaginary axis, γ_+ (resp. γ_-) a Stokes curve emanating from $x = 1$ (resp. $x = -1$) and penetrating into the upper (resp. lower) half plane (cf. Fig. 1, 2). Then we take Γ as a path which is sufficiently close to $\gamma_+ \cup \{x; -1 < x < 1\} \cup \gamma_-$, crosses two Stokes curves $\{x; -1 < x < 0\}$ and γ_0 just once, and never crosses any other Stokes curves. The examples of Γ are given in Fig. 3. For $\epsilon > 0$, we find that this Γ crosses two Stokes curves when N is odd, and four Stokes curves when N is even. We denote them by γ_1 and γ_2 (or $\gamma_1, \cdots, \gamma_4$) as is shown in Fig. 3.

In the following we use WKB solutions which are normalized as

$$\psi_\pm = \frac{1}{\sqrt{S_{\mathrm{odd}}}} \exp\left(\pm\eta \int_0^x \sqrt{Q(x)}dx\right) \exp\left(\pm \int_\infty^x \left(S_{\mathrm{odd}} - \eta\sqrt{Q(x)}\right) dx\right), \quad (9)$$

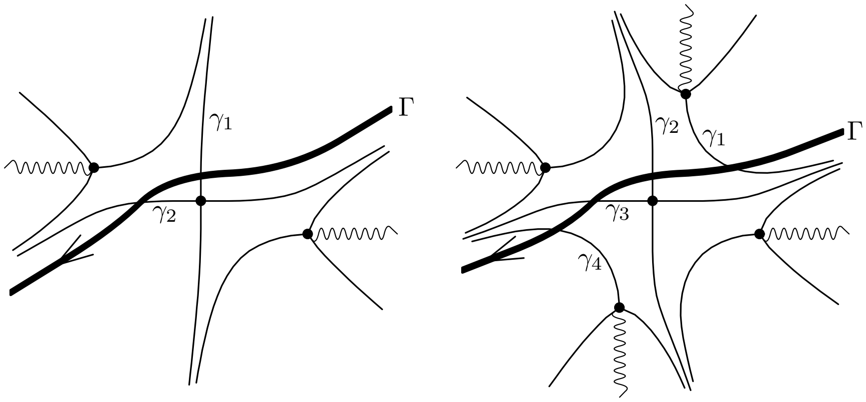

Figure 3: Path of analytic continuation for $\theta = -\pi + \epsilon$. (left: $N = 1$, right: $N = 2$. Wiggly lines indicate cuts.)

where

$$S_{\rm odd} = \sum_{j=-1}^{\infty} \eta^{-j} S_{{\rm odd},j} = \eta\sqrt{Q(x)} - \frac{E}{2\sqrt{Q(x)}} + \cdots \tag{10}$$

is the odd degree part of a solution S of the Riccati equation associated with (5). (See [1].)

Here we choose the branch of $\sqrt{Q(x)}$ so that

- we place cuts from each simple turning point to ∞ (See Fig. 3.);
- $\sqrt{Q(x)} = \frac{1}{2}x + O(x^2)$.

Then ψ_- satisfies the boundary condition in $\Sigma_\pm(\theta)$. We find that ψ_- is dominant on γ_1 (resp. γ_2) and subdominant on γ_2 (resp. γ_1 , γ_3 , γ_4), which implies that only γ_1 (resp. γ_2) is relevant for the calculation of the Jost function. As a matter of fact, using the connection formula near a double turning point, we find that ψ_- in $\Sigma_+(\theta)$ becomes $J_+(\eta, E)\psi_+ + J_-(\eta, E)\psi_-$ in $\Sigma_-(\theta)$ after the analytic continuation along Γ, where

$$J_+(\eta, E) = iC\frac{\sqrt{2\pi}}{\Gamma(F + \frac{1}{2})} e^{-i\pi F} \eta^F. \tag{11}$$

Here

$$C = \sum_{j=0}^{\infty} \eta^{-j} C_j = e^{E(\log 4 - i\pi)} + \cdots, \tag{12}$$

$$F = \operatorname*{Res}_{x=0} S_{\rm odd} = -E + \cdots, \tag{13}$$

are infinite series which are determined by the connection formula near a double turning point(cf.[5], [7], [12]. In the above "$\cdots$" means the negative degree part with respect to η.) Hence the eigenvalue E must satisfy $J_+(\eta, E) = 0$. We solve this equation by the help of the implicit resurgent function theorem[10]. Since this equation is equivalent to

$$\frac{1}{\Gamma(F+\frac{1}{2})} = 0 \Leftrightarrow F + \frac{1}{2} = -K \quad (K = 0, 1, 2, \cdots .), \tag{14}$$

we obtain

$$E^K = K + \frac{1}{2} + \sum_{j=1}^{\infty} E_j^K \eta^{-j}. \tag{15}$$

Remark. Many terms of $\{E_j^K\}$ actually vanish; we can show by induction that

$$E^K = K + \frac{1}{2} + \sum_{j=1}^{\infty} \widetilde{E}_j^K \left(e^{i\theta}\eta^{-N}\right)^j, \tag{16}$$

where each $\widetilde{E}_j^K$ is a constant independent of θ (cf. $\lambda = e^{i\theta}\eta^{-N}$).

For $\theta = \pi + \epsilon$

Stokes curves are the same as in the case of $\theta = -\pi + \epsilon$ (cf. Fig. 1, 2). We define a path Γ' of the analytic continuation in a similar way as that for $\theta = -\pi + \epsilon$, that is, let γ'_+ (resp. γ'_-) be the Stokes curve emanating from $x = 1$ (resp. $x = -1$) and penetrating into the lower (resp. upper) half plane. (See the left of Fig. 2.) Then we take Γ' as a path which is sufficiently close to $\gamma'_+ \cup \{x; -1 < x < 1\} \cup \gamma'_-$ and never crosses any Stokes curves except for $\{x; 0 < x < 1\}$ and γ_0. In this case Γ' cross 4 Stokes curves for $\epsilon > 0$. We denote them by $\gamma_1, \cdots, \gamma_4$ as is shown in Fig. 4. We can verify that ψ_- is dominant on γ_1, γ_3 and γ_4 and subdominant on γ_2.

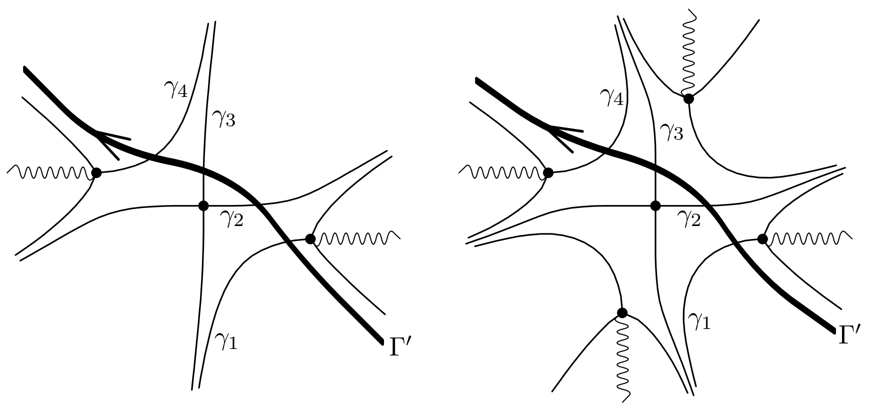

Figure 4: Path of analytic continuation for $\theta = \pi + \epsilon$. (left: $N = 1$, right: $N = 2$. Wiggly lines indicate cuts.)

We now consider the analytic continuation of ψ_- along Γ'. By using the connection formula near a simple turning point (cf. [1], [4], [5]) together with the connection formula near a double turning point, we obtain the following expression of the Jost function:

$$J'_+(\eta, E) = i\left(D_2 + A_- - A_+ + A_+ D_1 D_2 + A_+ A_- D_1\right), \tag{17}$$

where

$$\begin{aligned} A_\pm &= \exp\left(-2\eta \int_0^{\pm a} \sqrt{Q(x)}dx\right) \exp\left(-2\int_\infty^{\pm a} \left(S_{\text{odd}} - \eta\sqrt{Q(x)}\right) dx\right), \\ D_1 &= \frac{1}{C}\frac{\sqrt{2\pi}}{\Gamma(-F+1/2)}\eta^{-F}, \\ D_2 &= C\frac{\sqrt{2\pi}}{\Gamma(F+1/2)}e^{-i\pi F}\eta^F. \end{aligned}$$

Here $a = e^{-i\epsilon/N}$, and C, F are the same infinite series as in (12) and (13). Let ω denote $\int_0^a \sqrt{Q(x)}dx$. (A simple calculation shows

$$\omega = \frac{e^{-i\epsilon/N}}{2N} B\left(\frac{3}{2}, \frac{1}{N}\right), \tag{18}$$

where $B(x, y)$ denotes the beta function.) Then we find that $J'_+(\eta, E)$ has the following form:

$$J'_+(\eta, E) = J'_{+,0}(\eta, E) + J'_{+,1}(\eta, E)e^{-\eta\omega} + J'_{+,2}(\eta, E)e^{-2\eta\omega}, \tag{19}$$

where $J'_{+,l}(\eta, E)$ contains no exponential term. Let us suppose E has the form

$$E = E_0(\eta) + E_1(\eta)e^{-\eta\omega} + E_2(\eta)e^{-2\eta\omega} + \cdots. \tag{20}$$

Then by substituting (20) into (19) and comparing the same exponential terms, we find the following:

- Since $E_0(\eta)$ must satisfy $D_2(\eta, E_0) = 0$, or equivalently $F + 1/2 = -K$ $(K = 0, 1, 2, \cdots)$, its solution

$$E_0^K = K + \frac{1}{2} + \cdots, \tag{21}$$

  is equal to (15).
- Next $E_1(\eta)$ must satisfy

$$\left.\left(\frac{\partial D_2}{\partial E}E_1 + \widetilde{A}_- - \widetilde{A}_+ + D_1 D_2 \widetilde{A}_+\right)\right|_{E=E_0^K} = 0, \tag{22}$$

where

$$\widetilde{A}_{\pm} = \exp\left(-2\int_{\infty}^{\pm a}\left(S_{\rm odd} - \eta\sqrt{Q(x)}\right)dx\right). \tag{23}$$

Let us first note that the identity

$$\frac{\partial}{\partial z}\frac{1}{\Gamma(z)}\bigg|_{z=-n} = (-1)^n\, n! \quad (n = 0, 1, 2, \cdots) \tag{24}$$

implies

$$\frac{\partial D_2}{\partial E} = -i\sqrt{2\pi}C\eta^{-K-\frac{1}{2}}K! + \cdots. \tag{25}$$

Secondly, since

$$\int_{\infty}^{a} S_{{\rm odd},0}dx = -\frac{iE\pi}{2N} \tag{26}$$

and

$$\begin{aligned}\widetilde{A}_{-}\widetilde{A}_{+}^{-1} &= \exp\left(-2\pi i \operatorname*{Res}_{x=0} S_{\rm odd}\Big|_{\theta=\pi+\epsilon-0}\right) \\ &= e^{-2\pi i F}\end{aligned} \tag{27}$$

(which can be verified by deforming the contour appropriately), we obtain

$$\begin{aligned}\widetilde{A}_{-} - \widetilde{A}_{+} &= \widetilde{A}_{+}\left(\widetilde{A}_{-}\widetilde{A}_{+}^{-1} - 1\right) \\ &= \left(e^{iE\pi/N} + \cdots\right)\left(e^{-2\pi i F} - 1\right).\end{aligned} \tag{28}$$

That is,

$$\left(\widetilde{A}_{-} - \widetilde{A}_{+}\right)\Big|_{E=E_0^K} = -2\exp\left(\frac{i\pi}{N}\left(K + \frac{1}{2}\right)\right) + \cdots. \tag{29}$$

Thirdly we have

$$\begin{aligned}D_1 D_2|_{E=E_0^K} &= \frac{2\pi}{\Gamma(F+\frac{1}{2})\Gamma(-F+\frac{1}{2})}e^{-i\pi F}\bigg|_{E=E_0^K} \\ &= 0.\end{aligned} \tag{30}$$

Through (22), (25), (29) and (30), we obtain

$$E_1^K = \frac{2i}{\sqrt{2\pi^3}\,K!}\eta^{K+\frac{1}{2}}4^{\left(K+\frac{1}{2}\right)/N}(1 + \cdots). \tag{31}$$

Combining the above results for $\theta = \pm\pi + \epsilon$, we conclude that the leading term of ΔE^K is

$$\frac{2i}{\sqrt{2\pi^3}\,K!}4^{\left(K+\frac{1}{2}\right)/N}\eta^{K+\frac{1}{2}}e^{-\eta\omega}. \tag{32}$$

Hence by using (7) we have the desired result.

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