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Solution of Certain Integrable Dynamical Systems of Ruijsenaars-Schneider Type with Completely Periodic Trajectories

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Abstract. The first main result of this paper is the solution of the (complex) equations of motion $\ddot{z}_j + \mathrm{i}\Omega\dot{z}_j = \sum_{k=1,k\neq j}^{n} \dot{z}_j\dot{z}_k f(z_j - z_k)$ with $f(z) = 2a\,\mathrm{cotgh}\,(az)/[1+r^2\sinh^2(az)]$, and the consequent confirmation of the conjecture that all the trajectories of this dynamical system are completely periodic with period (at most) $T' = Tn!, T = 2\pi/\Omega$. We also discuss a symplectic reduction scheme which features new Lie-theoretic aspects for these systems. These developments are introduced here in the perspective of applying them in future studies to implement geometric quantization techniques.

I Introduction

It was recently pointed out [1] that the dynamical systems of Ruijsenaars-Schneider (RS) type characterized by the equations of motion

$$\ddot{z}_j + \mathrm{i}\Omega\dot{z}_j = \sum_{k=1,k\neq j}^{n} \dot{z}_j\dot{z}_k f(z_j - z_k), \quad j = 1, \ldots, n, \tag{1.1}$$

are “integrable” or “solvable” [2], if

$$f(z) = 2/z \qquad \text{“case (i)”}, \tag{1.2a}$$

$$f(z) = 2/[z(1 + r^2z^2)] \qquad \text{“case (ii)”}, \tag{1.2b}$$

$$f(z) = 2a\mathrm{cotgh}(az) \qquad \text{“case (iii)”}, \tag{1.2c}$$

$$f(z) = 2a/\sinh(az) \qquad \text{“case (iv)”}, \tag{1.2d}$$

$$f(z) = 2a\mathrm{cotgh}(az)/[1 + r^2\sinh^2(az)] \qquad \text{“case (v)”}, \tag{1.2e}$$

$$f(z) = -a\mathcal{P}'(az)/[\mathcal{P}(az) - \mathcal{P}(ab)] \qquad \text{“case (vi)”}, \tag{1.2f}$$

and it was conjectured that, if and only if the constant Ω is real and nonvanishing, *all* their trajectories $z_j(t), j = 1, \ldots, n$, are periodic with period (at most)

$$T' = Tn! \ \ , \ T = 2\pi/\Omega. \tag{1.3}$$

This conjecture was proven in Ref. [1] (on the basis of previous findings [3]) for *cases (i)* and *(iii)*, by solving the corresponding equations of motion: $z_j(t)$ are given by (in *case (i)*; in *case (iii)*, they are closely related to) the n (complex) zeros of a (monic) polynomial of degree n in z whose n coefficients are explicitly known functions of time, all periodic with period T. The availability of this method of solution suggests calling these two models "solvable" [2].

The main contribution of this paper is to also solve, rather explicitly, the equations of motion (1) for all the other cases listed above, except *case (vi)*: the validity of the conjecture [1] is thereby validated for these cases. The technique of solution is based on the possibility to put these equations of motion in "Lax pair" form [4,1] (for this reason these systems are called "integrable" [2]): the solutions $z_j(t)$ are then given by (or are closely related to) the n eigenvalues of an (explicitly known) $(n \times n)$-matrix which is periodic in time with period T.

Of course the solutions $z_j(t)$ of (1.1) move in the complex plane; and indeed all the constants appearing in (1.2), namely r, a and b, as well as the constants ω and ω' implicit in the definition of the Weierstrass function $\mathcal{P}(z) \equiv \mathcal{P}(z|\omega, \omega')$, might be complex. Note that in *case (i)* it is justified to identify the complex plane with the physical plane, obtaining thereby a (real) *rotation-invariant* model which describes the motion of particles in the plane, and which is in fact the special case of a more general solvable model of this type [5]. More generally, a reinterpretation of the complex model (1.1) as a real *rotation-invariant* model describing motions in the real ("physical") plane is also possible for *any* choice of the function $f(z)$ [6].

In the following Section II we describe a simple trick [2], namely a change of the dependent variable ("time"), which allows to relate equations of type (1.1) to analogous equations of motion, but with $\Omega = 0$.

In Section III we prove our main result for *case (v)*, and then by specialization for *cases (i)–(iv)* as well: these are indeed all special subcases of *case (v)*, see below (while *case (v)* is itself a subcase of *case (vi)*). We also discuss tersely the connection between "solvable" and "integrable" [2] models and the corresponding techniques of solution, for the solvable *cases (i)* and *(iii)*.

In Section IV we analize the possible usefulness of "fake Lax pairs" [1], which exist for systems of type(1.1) for arbitrary odd $f(z)$, to solve the corresponding equations of motion (which are all Hamiltonian [1]).

The results reported herein draw heavily on previous findings; indeed the techniques of solution we employ are not new, neither for solvable [3] nor for integrable [7] systems, although they had not been previously applied to solve the equations of motion (1.1) (but they have been certainly applied to analogous systems, indeed we like to record explicitly, for the integrable cases, the pioneering work by S.N.M. Ruijsenaars and his school [8]).

The elliptic system (namely, *case (vi)*) requires a separate treatment. Indeed, even in the simpler ("nonrelativistic") context (namely, for the so-called Calogero-Moser (CM) systems [7]), no *finite-dimensional* framework is available to solve the elliptic case: the only method known so far is due to I.M. Krichever and it is based on the study of solutions of the Kadomtsev-Petviashvili (KP) nonlinear partial differential equation in 2+1 dimensions [9]. Similar techniques were recently developed by I.M. Krichever and A.V. Zabrodin for RS systems [10]. Also, an *infinite-dimensional* Hamiltonian reduction scheme was recently introduced for the elliptic CM system [11, 12]. In the present paper we have restricted our investigation to the most general RS system which can be treated by *finite-dimensional* techniques, including its integrable deformation characterized by the remarkable property to feature only completely periodic trajectories (namely, *case (v)* as defined above, see (1.1) with (1.2e)).

Under this limitation, we also provide below, in Section V, a Hamiltonian reduction scheme analogous to the Kazhdan-Kostant-Sternberg (KKS) [13] construction for CM systems. Such a scheme was used by one of us (JPF) to demonstrate the existence of a symplectic action of the torus associated with the *rational* CM system with an external *quadratic* potential [14] (see also [15], where the KKS framework was used to demonstrate the complete integrability of the *rational* CM system with an external *quartic* potential). More recently this approach has featured in several papers. While the quick pace of development in this area entails that we cannot provide an exhaustive list of contributions, we would like to mention that this Hamiltonian reduction has been discussed by H.W. Braden and R. Sasaki for *case (iv)* (with $\Omega = 0$) [16]. To the best of our knowledge, it was not yet considered for *case (v)* (even for $\Omega = 0$); and of course the treatment given here, of the case $\Omega \neq 0$ featuring completely periodic orbits, is new.

The Hamiltonian reduction framework provides the tools to demonstrate the existence of a symplectic action of the torus which leaves the system invariant. We provide this proof for the rational case in Section VI. Hamiltonians which are invariant under a symplectic action of the torus feature specially interesting properties; in particular, contributions by M. Atiyah, Y. Colin de Verdiere, V. Guillemin and S. Sternberg [17] are then available to investigate the quantized systems. Hence the findings reported in Sections V and VI provide the foundations for an analytic approach, which we plan to provide in a subsequent paper, to the quantized systems corresponding to (1.1) with (1.2).

We do not consider here the R-matrix approach, which provides an alternative route to deal with quantization. For this approach we refer to several recent papers [18–23], none of which however considers the completely periodic case introduced in [1], on whose treatment the present paper is instead focussed.

II A simple trick

In this section we report a simple trick [2] that relates the solutions of (1.1) to the solutions of the same equations of motion, but with $\Omega = 0$.

Consider the (more general) equations of motion

$$\ddot{z}_j(t) - \alpha(t)\dot{z}_j(t) = \sum_{k,m=1}^{n} \dot{z}_k(t)\dot{z}_m(t) f_{jkm}[\mathbf{z}(t)], j = 1, \ldots, n, \tag{2.1}$$

where of course $\mathbf{z}$ is the n-vector of components z_j. Now set

$$z_j(t) = \zeta_j(\tau), \tau = \tau(t). \tag{2.2}$$

Then clearly

$$\zeta_j''(\tau) + [\dot\tau(t)]^{-2}[\ddot\tau(t) - \alpha(t)\dot\tau(t)]\zeta_j'(\tau) = \sum_{k,m=1}^{n} \zeta_k'(\tau)\zeta_m'(\tau) f_{jkm}[\boldsymbol{\imath}(\tau)], j = 1, \ldots, n. \tag{2.3}$$

Hence the choice of a function $\tau(t)$ which satisfies the (linear) equation

$$\ddot\tau(t) = \alpha(t)\dot\tau(t), \tag{2.4}$$

transforms the equations of motion (2.1) for $z_j(t)$ into the same equations of motion for $\zeta_j(\tau)$ (of course with $\dot z_j(t)$ replaced by $\zeta_j'(t) \equiv d\zeta_j(\tau)/d\tau$, and so on), but without the second term in the left-hand side. Note that this conclusion holds for arbitrary $f_{jkm}(\mathbf{z})$.

In particular the position

$$\tau(t) = (\mathrm{i}/\Omega)[\exp(-\mathrm{i}\Omega t) - 1] \tag{2.5}$$

transforms (1.1) into

$$\zeta_j''(\tau) = \sum_{k=1, k\neq j}^{n} \zeta_j'(\tau)\zeta_k'(\tau) f[\zeta_j(\tau) - \zeta_k(\tau)], j = 1, \ldots, n. \tag{2.6}$$

Hereafter we will therefore firstly solve (1.1) with $\Omega = 0$, since the solution of (1.1) with $\Omega \neq 0$ can then be recovered by replacing t with τ, see (2.5). Note that the finding described in this section reinforces the plausibility of the conjecture proffered in [1], but does not quite prove its validity.

III Solution of the equations of motion

In this section we obtain the (rather explicit) solution of the equations of motion

$$\ddot z_j = \sum_{k=1, k\neq j}^{n} \dot z_j \dot z_k f(z_j - z_k), \quad j = 1, \ldots, n \quad , \tag{3.1}$$

with $f(z)$ given by (1.2e) ("*case (v)*"). The solution of (1.1) with $f(z)$ given by (1.2a,b,c,d,e) is then obtained by using the trick of the preceding Section II and by appropriate specialization of the constants, as we indicate below.

The starting point of the analysis is the observation [4,1] that (3.1) with (1.2e) is equivalent to the following "Lax-type" $(n\times n)$-matrix equation:

$$\dot L = [L, M]_-, \tag{3.2}$$

with

$$L_{jk} = \delta_{jk}\dot{z}_j + (1 - \delta_{jk})(\dot{z}_j\dot{z}_k)^{1/2}\alpha(z_j - z_k), \tag{3.3}$$

$$M_{jk} = \delta_{jk}\sum_{m=1,m\neq j}^{n}\dot{z}_m\beta(z_j - z_m) + (1 - \delta_{jk})(\dot{z}_j\dot{z}_k)^{1/2}\gamma(z_j - z_k), \tag{3.4}$$

and

$$\alpha(z) = \sinh(a\mu)/\sinh[a(z + \mu)], \tag{3.5a}$$

$$\beta(z) = -a\mathrm{cotgh}(a\mu)/[1 + r^2\sinh^2(az)], \tag{3.5b}$$

$$\gamma(z) = -a\mathrm{cotgh}(az)\alpha(z), \tag{3.5c}$$

where

$$\sinh(a\mu) = \mathrm{i}/r. \tag{3.5d}$$

We then introduce the diagonal matrix

$$E(t) = \mathrm{diag}\{\exp[2az_j(t)]\}, \tag{3.6}$$

and we note that there holds the matrix formula

$$\dot{E} = [E, M]_- + a[E, L]_+, \tag{3.7}$$

whose validity can be very easily verified by explicit computation using (only!) (3.6), (3.3) and (3.4) with (3.5c). Above and throughout of course $[A, B]_- \equiv AB - BA$ and $[A, B]_+ \equiv AB + BA$.

One can now use a technique introduced, in the CM context, by M.A. Olshanetsky and A.M. Perelomov [7]. Set

$$\tilde{L} = ULU^{-1}, \tilde{M} = UMU^{-1}, \tilde{E} = UEU^{-1}, \tag{3.8}$$

and note that the last of these equations, together with (3.6), entails that *the quantities* $\exp[2az_j(t)]$ *are the* n *eigenvalues of the matrix* $\tilde{E}(t)$.

Define now the $(n\times n)$-matrix $U(t)$ via the equations

$$U(0) = I,\ \ U_{jk}(0) = \delta_{jk}\ , \tag{3.9a}$$

$$\dot{U}(t) = \tilde{M}(t)U(t), \quad \tilde{M}(t) = \dot{U}(t)[U(t)]^{-1}. \tag{3.9b}$$

These two equations define $U(t)$ uniquely; the fact that we do not know how to compute this matrix, since we neither know the matrix $\tilde{M}(t)$ nor how to solve (3.9b), is immaterial. Indeed (3.9b), together with (3.8) and with (3.2) and (3.7), clearly entail the equations

$$\dot{\tilde{L}}(t) = 0, \tag{3.10a}$$

$$\dot{\tilde{E}}(t) = a[\tilde{E}(t), \tilde{L}(t)]_+. \tag{3.10b}$$

Hence, from (3.10a) and (3.9a),

$$\tilde{L}(t) = \tilde{L}(0) = L(0), \tag{3.11}$$

and, from (3.10b), (3.11) and (3.9a),

$$\tilde{E}(t) = \exp[aL(0)t]E(0)\exp[aL(0)t]. \tag{3.12}$$

The matrices $E(0)$ and $L(0)$ are explicitly given, in terms of the initial data $z_j(0), \dot{z}_j(0), j = 1, \ldots, n$, by (3.6) and (3.3) with (3.5a). Hence the matrix $\tilde{E}(t)$ is rather explicitly given by this formula, (3.12), and its eigenvalues provide the quantities $\exp[2az_j(t)]$ (see the remark after equation (3.8)).

Replacement of t by τ, see (2.5), in the right-hand side of (3.12) entails that the matrix $\tilde{E}(t)$ becomes periodic in t with period T, see (1.3); hence its eigenvalues $\exp[2az_j(t)]$ are also periodic in time, with period (at most) T', see (1.3). And the periodicity of the quantities $\exp[2az_j(t)]$ entails periodicity, with the same period, of the quantities $z_j(t)$. [From the explicit time-evolution (3.12) of $\tilde{E}(t)$ it is clear that its determinant does not vanish over time: hence none of its eigenvalues vanishes, and their logarithms are uniquely defined by continuity.]

We have thereby proved the conjecture of Ref.[1] for *case* (v), namely for the equations of motion (1.1) with (1.2e). It is easily seen that this proof holds equally for the 4 *cases (i)–(iv)*, which are in fact all subcases of *case (v)*. Indeed *cases (iv)* respectively (iii) obtain from *case (v)* by setting $r^2 = 1$ (and changing a into $a/2$) respectively $r = 0$ (no change in a); *case (ii)* obtains by replacing r with r/a and then letting $a \to 0$, which entails the replacement of (3.12) by the formula

$$\tilde{Z}(t) = Z(0) + L(0)t \tag{3.13}$$

with

$$Z = \mathrm{diag}(z_j), \tag{3.14}$$

$$L_{jk} = \delta_{jk}\dot{z}_j + (1 - \delta_{jk})(\dot{z}_j\dot{z}_k)^{1/2}/[1 + \mathrm{i}r(z_j - z_k)], \tag{3.15}$$

and of course the property that *the n quantities $z_j(t)$ are the n eigenvalues of the matrix $\tilde{Z}(t)$.*

Finally, *case (i)* is merely the special case of the results we just detailed for *case (ii)*, corresponding to $r = 0$ (see (3.15)). Two remarks are appropriate in this connection (somewhat analogous observations apply to *case (iii)*; we leave their elaboration as a task for the diligent reader; but see also the treatment given in the following Section IV).

Firstly we observe that, in the $r = 0$ case, the matrix L, see (3.15), is highly degenerate (separable of rank 1); this corresponds to the fact that this case is not only "integrable" but also "solvable" [2]. Indeed the quantities $z_j(t)$, being the eigenvalues of the matrix $\tilde{Z}(t)$, are the roots of the following polynomial of degree n in z:

$$\det[zI - \tilde{Z}(t)] = \prod_{j=1}^{n}[z - z_j(t)] = z^n + \sum_{m=1}^{n} c_m(t)z^{n-m}. \tag{3.16}$$

This polynomial is linear in t (see below), hence

$$\ddot{c}_m(t) = 0, \quad m = 1, \ldots, n. \tag{3.17}$$

But this last equation, together with the nonlinear one-to-one mapping between the z_j's and the c_m's entailed by the second relation (3.16), is precisely the basis of the technique of solution of *case (i)* [3].

To complete this argument there remains to show that the polynomial (3.16) is linear in t. This is a consequence of the special structure of the matrix L in this case,

$$L_{jk} = (\dot{z}_j \dot{z}_k)^{1/2} \tag{3.18}$$

(see (3.15) with $r = 0$). Indeed (see (3.13) and (3.14))

$$\det[zI - \tilde{Z}(t)] = \prod_{m=1}^{n} (\mathbf{v}^{(m)}, \{\mathrm{diag}[z - z_j(0)] - L(0)t\}\mathbf{v}^{(m)}), \tag{3.19}$$

where the n-vectors $\mathbf{v}^{(m)}$ constitute an (arbitrary) *orthonormal* vector basis. Now choose a t-independent basis such that

$$v_j^{(1)} = [\dot{z}_j(0)]^{1/2}/[V(0)]^{1/2}, \; V(0) = \sum_{j=1}^{n} \dot{z}_j^{(0)}, \tag{3.20a}$$

(assuming for simplicity the quantity $V(0)$ does not vanish), so that

$$(\mathbf{v}^{(1)}, \mathbf{v}^{(m)}) = \delta_{1m}, \quad m = 1, \ldots, n \;, \tag{3.20b}$$

hence (see (3.18) and (3.20), (3.21))

$$L(0)\mathbf{v}^{(m)} = \delta_{1m} V(0)\mathbf{v}^{(m)}, \quad m = 1, \ldots, n \;, \tag{3.21}$$

hence (see (3.19))

$$\det[zI - \tilde{Z}(t)] = [(\mathbf{v}^{(1)}, \mathrm{diag}[z - z_j(0)]\mathbf{v}^{(1)}) - V(0)t] \prod_{m=2}^{n} (\mathbf{v}^{(m)}, \mathrm{diag}[z - z_j(0)]\mathbf{v}^{(m)}). \tag{3.22}$$

This completes the argument, except for the special case in which $V(0)$ vanishes ("center of mass at rest"). We leave the analysis of this exceptional case as an exercise for the diligent reader, who will also note the special behavior of the system in this case, as discussed in Section 4.F of Ref.[5].

The second remark notes that, in the $r = 0$ case, the matrix L, see (3.15), corresponds to a "fake Lax pair" [1], yet the technique of solution we have just exhibited works in this case as well (it does yield the solution to the equations of motion). This fact is sufficiently intriguing to justify a detailed discussion in the following Section IV.

But before ending this section, let us return to the dynamical system (3.1) with (1.2e) (*case* (v)), to report the following

Remark. The explicit expressions (3.3) with (3.5a) of L, and (3.6) of E, entail that the matrix

$$S \equiv \cosh(a\mu)[E, L]_- + \sinh(a\mu)[E, L]_+ \tag{3.23}$$

is symmetrical and has rank 1:

$$S_{jk} = s_j s_k \tag{3.24}$$

$$s_j = [2\sinh(a\mu)\ \dot{z}_j]^{1/2} \exp(az_j). \tag{3.25}$$

This suggests an alternative route to treat this dynamical system. One starts from the evolution equations (3.10a,b) for the two $(n{\times}n)$-matrices $\tilde{L}(t)$ and $\tilde{E}(t)$, which of course entail (3.11) and (3.12). Then one introduces the matrix $U(t)$ as the one that diagonalizes $\tilde{E}(t)$ (see the third equation (3.8), and (3.6)), and then the matrix $L(t)$ via the first of the 3 equations (3.8). Then one imposes the validity of (3.23) and (3.24) (subject to a final consistency check). It is then easily seen that the explicit expressions of s_j, see (3.25), and of L, see (3.3) and (3.5a), can be derived, as well as the equations of motion (3.1) with (1.2e). [This can be achieved via the following steps: introduce $\tilde{M}$ via (3.9b) and then M via the second of the (3.8); then (3.10 a,b) entail (3.7) and (3.2); then the diagonal part of (3.7), together with (3.6), yield the diagonal part of L, see (3.3); then equate the right-hand sides of (3.23) and (3.24), and thereby get firstly (3.25) (from the diagonal part, using (3.6) and the already evaluated diagonal part of L), and then the off-diagonal part of L (from the off-diagonal part); then, from the off-diagonal part of (3.7), evaluate the off-diagonal part of M (using (3.6) and the now known off-diagonal part of L); and finally, from the diagonal part of (3.2), get the equations of motion (3.1) with (1.2e) (using the now known off diagonal parts of L and M)].

This method of obtaining the equations of motion (3.1) with (1.2e), as well as their Lax-pair structure and their solution, might appear tortuous. But it shows that one can obtain this dynamical system starting from the extremely simple $(n{\times}n)$-matrix evolution equations (3.10a,b), and supplementing them with the (*a posteriori* compatible) constraint (3.23) with (3.24) (with E and L defined via (3.8) and (3.6)). This provides the framework for a reduction procedure which is discussed in some detail in Sections V and VI.

IV Natural ansatz for finding the "angle" equation

Complete integration of a Hamiltonian system needs finding a Lax matrix (which provides the conserved quantities) and an extra-equation (cf. 3.7) which may be called the "Angle" equation (in reference to the classical terminology of the Action-Angles coordinates here adopted in its broad sense). It seems quite interesting to note that in the examples we consider here, the Lax equation may be "fake" (in the sense that it does not provide the expected number of conserved quantities) and nevertheless it may be implemented with an extra-equation (the "Angle" equation) which still allows the full integration of the system. In this sense, even a "fake" Lax pair may be useful. We provide here a complete discussion of the ansatz for finding the "Angle" equation. We derive the system (1.2e) as only solution. This shows clearly that the elliptic system (1.2f) cannot be reached with our techniques and that a separated study for finding the "Angle" equation should be developed (perhaps based on the ideas of [9] and [1]).

It is easy to verify that the equations of motion

$$\ddot{z}_j = \sum_{k=1}^{n} \dot{z}_j \dot{z}_k f_{jk}(z_j - z_k) \quad , j = 1, \ldots, n \ , \tag{4.1}$$

correspond to the $(n{\times}n)$-matrix "Lax equation"

$$\dot{L} = [L, M]_-, \tag{4.2a}$$

with [1]

$$L_{jk} = \delta_{jk}\dot{z}_j + (1 - \delta_{jk})(\dot{z}_j \dot{z}_k)^{1/2} = (\dot{z}_j \dot{z}_k)^{1/2}, \tag{4.2b}$$

$$M_{jk} = \frac{1}{2}(\delta_{jk} - 1)(\dot{z}_j \dot{z}_k)^{1/2} f_{jk}(z_j - z_k). \tag{4.2c}$$

Note that the only condition required for the equivalence of (4.1) and (4.2) is that the matrix-valued function $f_{jk}(z)$ be "odd," in the following sense:

$$f_{kj}(-z) = -f_{jk}(z). \tag{4.3}$$

Incidentally, this condition is also sufficient to guarantee that the equations of motion (4.1) are Hamiltonian [1]; indeed a Hamiltonian whose equations of motion,

$$\dot{q}_j = \partial H/\partial p_j, \quad \dot{p}_j = -\partial H/\partial q_j \ , \tag{4.4a}$$

yield (4.1) (with $z_j = q_j$) reads as follows:

$$H = \sum_{j=1}^{n} h_j(sp_j, \mathbf{q}) \ , \quad s \ = \ \text{arbitrary nonvanishing constant}, \tag{4.4b}$$

$$h_j(p_j, \mathbf{q}) = \exp\{p_j + \sum_{k=1, k\neq j}^{n} F_{jk}(q_j - q_k)\} \ , \tag{4.4c}$$

$$f_{jk}(q) = F'_{kj}(-q) - F'_{jk}(q) \ . \tag{4.4d}$$

Note that the last equation entails (4.3).

Moreover, the condition (4.3) is clearly also sufficient to guarantee that the velocity of the center of mass, V/n,

$$V = \sum_{j=1}^{n} \dot{z}_j, \tag{4.5a}$$

is a constant of motion,

$$\dot{V} = 0 \ . \tag{4.5b}$$

The "Lax pair" (4.2b,c) is a "fake Lax pair:" indeed the time-independence of the eigenvalues of L implied by (4.2a) only entails (4.5b), due to the highly degenerate character of the matrix L, see (4.2b), which is clearly separable of rank 1.

But in the preceding Section III we have seen that a Lax pair of this kind may be instrumental to *solve* the corresponding equations of motion. It is therefore natural to investigate whether the technique of solution described in the preceding section is more generally applicable to a system of type (4.1).

Two ingredients play a crucial role in the technique of solution described in the preceding section. One is the Lax equation, see (3.2); we now have an analogous formula, see (4.2). The other is the matrix equation (3.7). Let us therefore see whether we can now manufacture an equation analogous to (3.7). To this end we introduce the matrix

$$G = \mathrm{diag}[g(z_j)], \tag{4.6}$$

with $g(z)$ a function to be determined, and we require that it satisfy the equation

$$\dot{G} = [G, M]_- + a[G, L]_+ + b[L, G]_- + cG + dL + h \tag{4.7}$$

with a, b, c, d and h five arbitrary (scalar) constants (the justification for this *ansatz* for the right-hand side of (4.7) is that, as it can be easily seen, it allows to perform all subsequent steps in the technique of solution described in the preceding Section III).

The compatibility of (4.7) with (4.6) and (4.2b,c) is easily seen to imply the following results: firstly, from the diagonal part of (4.7),

$$c = h = 0, \tag{4.8a}$$

$$g(z) = C \exp(2az) - d/(2a), C \text{ arbitrary}, \tag{4.8b}$$

and then, from the off-diagonal part of (4.7) (note the disappearance of C and d, as well as of the indices j and k, in the right-hand side)

$$f_{jk}(z) = 2a \ \mathrm{cotgh}(az) + 2b \ . \tag{4.8c}$$

But the condition (4.3) then entails

$$b = 0. \tag{4.8d}$$

Hence we have merely reobtained the "solvable" [1,3] *case (iii)*, see (1.2c); and of course, via the limit $a \to 0$, *case (i)* can be reobtained as well, see (3.13).

[The diligent reader may repeat this calculation starting from the more general *ansatz* for G that results from the replacement of $g(z_j)$ with $g_j(z_j)$ in (4.6). The more general result obtained in this manner corresponds merely to the freedom to perform, in the equations of motion (see (4.1) or (3.1)), the translations $z_j \to z_j + w_j$, the n quantities w_j being arbitrary *constants*].

V The Hamiltonian reduction procedure

The general idea of the "Hamiltonian reduction procedure" is to start with a "large" initial phase space and a "simple" Hamiltonian possessing a symmetry group. Factorizing the corresponding motion by this symmetry yields a nontrivial dynamical system defined on a reduced phase space. Let us tersely outline how it works here.

For the models treated in this paper, namely *cases (i)* to *(v)* (see (1.1) with (1.2a,b,c,d,e)), one can use the cotangent bundle T^*G over the Lie group $G = Gl(n, C)$. The space T^*G is naturally isomorphic to $G \times \mathcal{G}^*$ where $\mathcal{G}^*$ is dual to the Lie algebra $\mathcal{G} = Mat(n, C)$. Let $(\mathcal{E}, \mathcal{L})$ be an element of T^*G, where $\mathcal{E}$ belongs to the image of the exponential mapping: $\exp : \mathcal{G} \rightarrow G$. The group G acts on itself and this action gets lifted into a Hamiltonian action on T^*G. Write

$$\mathcal{E} = \exp(2a\mathcal{Z}), \tag{5.1}$$

where $\mathcal{Z}$ belongs to an orbit of maximal dimension of the adjoint action of G on $\mathcal{G}$, and it is diagonalizable,

$$\mathcal{Z} = WZW^{-1}, \tag{5.2a}$$

$$\mathcal{Z} = \mathrm{diag}(z_j). \tag{5.2b}$$

Note that one is now generalizing the Kazhdan-Kostant-Sternberg reduction techniques [13], by replacing the lifted action of G to T^*G by an action "weighted on the left and on the right", of the image of the exponential mapping: Write $U = \exp(T)$ then

$$U.\mathcal{E} = \exp[(\alpha + \beta)/2]T\mathcal{E}\exp[(-\alpha + \beta)/2]T, \tag{5.3a}$$

so that the corresponding momentum map is

$$(\mathcal{E}, \mathcal{L}) \longmapsto \alpha[\mathcal{E}, \mathcal{L}]_- + \beta[\mathcal{E}, \mathcal{L}]_+. \tag{5.3b}$$

Note that one is assuming here that $\mathcal{G}$ is not only a Lie algebra but also an associative algebra; namely that not only the commutator $[\mathcal{E}, \mathcal{L}]_- = \mathcal{E}\mathcal{L} - \mathcal{L}\mathcal{E}$ is defined but the anticommutator $[\mathcal{E}, \mathcal{L}]_+ = \mathcal{E}\mathcal{L} + \mathcal{L}\mathcal{E}$ as well. The two complex parameters α and β are fixed and determine the weights of the action.

Let us remind the reader that for the case of rational and trigonometric CM systems, the fiber of the momentum map corresponds to the specific rank-one matrix $c \otimes c^\dagger$, with $c^\dagger = (1, \ldots, 1)$ [13]. Here the analogous role is played by the more general rank-one matrices

$$\Sigma = \sigma \otimes \sigma', \tag{5.4}$$

where σ and σ' are two (appropriately chosen, see below) complex n-vectors. The manifold M on which the dynamics unfolds is the inverse image by the momentum map of the set of matrices Σ. Of course, we factorize this inverse image by the action of the subgroup that leaves this set invariant. Indeed, let us consider the simple dynamical system defined on T^*G by the differential equations

$$\dot{\mathcal{E}} = a[\mathcal{E}, \mathcal{L}]_+, \tag{5.5a}$$

$$\dot{\mathcal{L}} = \mathrm{i}\Omega\mathcal{L}, \tag{5.5b}$$

where a is a complex parameter and Ω is real. This differential matrix system can be easily integrated:

$$\mathcal{L} = \mathcal{L}(0)\exp(\mathrm{i}\Omega t), \tag{5.6a}$$

$$\mathcal{E} = \exp\{(-\mathrm{i}a/\Omega)[\exp(\mathrm{i}\Omega t)-1]\mathcal{L}(0)\}\mathcal{E}(0)\exp\{(-\mathrm{i}a/\Omega)[\exp(\mathrm{i}\Omega t)-1]\mathcal{L}(0)\}. \tag{5.6b}$$

Note that if

$$\alpha[\mathcal{E}(0), \mathcal{L}(0)]_- + \beta[\mathcal{E}(0), \mathcal{L}(0)]_+ = \sigma(0) \otimes \sigma'(0), \tag{5.7}$$

then (see (5.6a,b))

$$\alpha[\mathcal{E}(t), \mathcal{L}(t)]_- + \beta[\mathcal{E}(t), \mathcal{L}(t)]_+ = \sigma(t)\otimes\sigma'(t), \tag{5.8}$$

with

$$\sigma(t) = \exp(\mathrm{i}\Omega t/2)\exp\{(-\mathrm{i}a/\Omega)[\exp(\mathrm{i}\Omega t) - 1]\mathcal{L}(0)\}\sigma(0), \tag{5.9a}$$

$$\sigma'(t) = \sigma'(0)\exp(\mathrm{i}\Omega t/2)\exp\{(-\mathrm{i}a/\Omega)[\exp(\mathrm{i}\Omega t) - 1]\mathcal{L}(0)\}. \tag{5.9b}$$

Hence we see that the dynamics leaves globally invariant the manifold M defined as the inverse image of the set of rank-one matrices Σ, see (5.4).

We need finally to factorize the manifold M by the action of the group. We describe a parametrization of the reduced space which yields precisely the Lax matrix of Ref. [4]. To this end, we first diagonalize the matrix $\mathcal{Z}$ (see (5.2)) and introduce (see (5.1))

$$E = W^{-1}\mathcal{E}W = \mathrm{diag}[\exp(2az_j)]. \tag{5.10}$$

The entries of the matrix Σ, see (5.4), are $\Sigma_{jk} = \sigma_j\sigma'_k$. For generic values of σ and σ', there are a diagonal matrix D and a vector s such that

$$D^{-1}\Sigma D = s\otimes s. \tag{5.11}$$

Indeed, if $\sigma_j \neq 0$ for all $j = 1, \ldots, n$, the elements of the diagonal matrix D are $d_j = (\sigma'_j/\sigma_j)^{1/2}$, which yields $s_j = (\sigma_j\sigma'_j)^{1/2}$. Then we can set

$$E = U^{-1}\mathcal{E}U = \mathrm{diag}[\exp(2az_j)], \tag{5.12a}$$

with $U = DW$, and we also define

$$L = U^{-1}\mathcal{L}U. \tag{5.12b}$$

From here on, one can proceed as described in Section III, see (3.23) and the discussion following it; note that (3.23) and (5.6) entail the identification

$$\alpha = \cosh(a\mu), \tag{5.13a}$$

$$\beta = \sinh(a\mu). \tag{5.13b}$$

VI The associated Poisson action of the torus for the rational case

In this section we demonstrate, in the simpler case characterized by a rational $f(z)$, see (1.2a) and (1.2b), how one can introduce a Poisson action of the torus which leaves the motion invariant, and thereby identify explicitly n constants of the motion. We moreover show that, for Ω real and nonvanishing, *all* the flows of the "particle coordinates" $z_j(t)$ induced by these constants (considered as Hamiltonians) are completely periodic with period (at most) T', see (1.3).

Let us first of all obtain, via the $a\to 0$ limit, the equations relevant to the rational case. They read, in place of (5.3),

$$\dot{\mathcal{Z}} = \mathcal{L}, \tag{6.1a}$$

$$\dot{\mathcal{L}} = \mathrm{i}\Omega\mathcal{L}, \tag{6.1b}$$

entailing

$$\ddot{\mathcal{Z}} = \mathrm{i}\Omega\dot{\mathcal{Z}}. \tag{6.2}$$

The equations (6.1) are susceptible of Hamiltonian interpretation, with the matrix $\mathcal{Z}$ as canonical variable and the matrix $\mathcal{P}$, defined by

$$\mathcal{L} = \exp(\mathcal{P}), \tag{6.3}$$

as conjugated canonical momentum. Here we are of course assuming that $\mathcal{L}$ belongs to the image of the exponential mapping, which is consistent with (5.6a) or (6.1b).

The corresponding Hamiltonian reads

$$H = \mathrm{tr}[\exp(\mathcal{P}) - \mathrm{i}\Omega\mathcal{Z}], \tag{6.4}$$

and the symplectic form is

$$\omega = \mathrm{tr}[d\mathcal{Z}\wedge d\mathcal{P}] = \mathrm{tr}[d\mathcal{Z}\wedge\mathcal{L}^{-1}d\mathcal{L}], \tag{6.5a}$$

entailing

$$\mathrm{tr}[\dot{\mathcal{Z}}\mathcal{L}^{-1}d\mathcal{L} - d\mathcal{Z}\mathcal{L}^{-1}\dot{\mathcal{L}}] = dH = \mathrm{tr}[d\mathcal{L} - \mathrm{i}\Omega d\mathcal{Z}], \tag{6.5b}$$

so that Hamilton's equations read

$$\dot{\mathcal{Z}}\mathcal{L}^{-1} = I, \tag{6.6a}$$

$$\mathcal{L}^{-1}\dot{\mathcal{L}} = \mathrm{i}\Omega I, \tag{6.6b}$$

namely they reproduce (6.1). Note that the Hamiltonian (6.3) coincides with that considered in Ref.[1].

Before proceeding, let us review what was done in Ref.[14] for the rational CM system with an external quadratic potential, characterized by the equations of motion

$$\dot{x}_j = y_j, \tag{6.7a}$$

$$\dot{y}_j = -\Omega^2 x_j + \sum_{k=1, k\neq j}^{n} (x_j - x_k)^{-3}. \tag{6.7b}$$

This system was obtained as Hamiltonian reduction of the matrix "pure harmonic oscillator" system

$$\dot{\mathcal{X}} = \mathcal{Y}, \tag{6.8a}$$

$$\dot{\mathcal{Y}} = -\Omega^2 \mathcal{Y}. \tag{6.8b}$$

We may now compare this system with (6.1). Both systems obviously display periodic solutions. One of the specific features of (6.1) is that, in contrast with (6.8), it only makes sense for complex matrices $\mathcal{Z}$ and $\mathcal{L}$, so that in fact the real system associated with (6.8) is obtained by considering the time evolution of $Re(\mathcal{Z})$ and $Im(\mathcal{Z})$. Hamiltonian reductions of (6.8) yield systems of particles on the *line* while Hamiltonian reductions of (6.1) correspond to systems of particles on the *plane* [2, 6]. The quantization of the harmonic oscillator is completely classical in the framework of geometric quantization theory. The quantization of (6.1) is less standard and we plan to discuss it in a future paper.

Let us now pursue the analogy between the two systems. In Ref.[14] a new matrix variable was introduced,

$$\Xi = \mathcal{X} + \mathrm{i}\Omega\mathcal{Y}, \tag{6.9a}$$

$$\Xi^* = \mathcal{X} - \mathrm{i}\Omega\mathcal{Y}, \tag{6.9b}$$

with the Hamiltonian

$$H = \mathrm{tr}[\Xi\Xi^*], \tag{6.10}$$

and the KKS [13] symplectic form

$$\omega = \mathrm{tr}\{-[\mathrm{i}/(2\Omega)]d\Xi\wedge d\Xi^*\}. \tag{6.11}$$

It was then noted that not only is H a constant of the motion, but in fact the whole matrix $\mathcal{A} = \Xi\Xi^*$ is conserved by the flow (6.8). An analogous phenomenon occurs for (6.1). Indeed, setting

$$\mathcal{B} = \mathcal{L} - \mathrm{i}\Omega\mathcal{Z}, \tag{6.12}$$

one immediately sees that (6.1) entails

$$\dot{\mathcal{B}} = 0. \tag{6.13}$$

One of the main results obtained in Ref.[14] was that all the eigenvalues of $\mathcal{A}$, seen as functions of the matrix variables, generate via the symplectic form (6.11) commuting Hamiltonian systems, all of which possess only completely periodic orbits, with period $T = 2\pi/\Omega$. This set of commuting Hamiltonians defines a symplectic action of the torus, which leaves invariant the CM system with external quadratic potential (and eventually explains why the corresponding quantum spectrum coincides with its semi-classical approximation).

We prove here the following analogous result: the set of eigenvalues β_m of the matrix $\mathcal{B}$, see (6.12), generate, via the symplectic form (6.5), commuting Hamiltonian flows for the matrix variables, all of which are completely periodic, with period $T = 2\pi/\Omega$. The proof of this statement goes as follows. Let β_m be an eigenvalue of the matrix $\mathcal{B}$. Let $\Psi^{(m)}$ be the corresponding eigenvector and let $N^{(m)}$ be the projector on $\Psi^{(m)}$. The dynamical system associated with β_m is then characterized by the matrix differential equations

$$\dot{\mathcal{Z}} = \mathcal{L}N^{(m)}, \tag{6.14a}$$

$$\dot{\mathcal{L}} = \mathrm{i}\mathcal{L}N^{(m)}. \tag{6.14b}$$

Hence again, under *all* these flows, the matrix $\mathcal{B}$, see (6.12), is constant:

$$\dot{\mathcal{B}} = \mathrm{i}\Omega\mathcal{L}N^{(m)} - \mathrm{i}\Omega\mathcal{L}N^{(m)} = 0. \tag{6.15}$$

This clearly entails that all the eigenvalues of $\mathcal{B}$ are constants of the motion under *all* these flows; hence different eigenvalues Poisson commute pairwise, and define commuting flows.

Finally, let us prove that all these matrix flows are in fact periodic with period T, see (1.3). Let U be a (*time-independent*) matrix which diagonalizes the (*time-independent*) matrix $\mathcal{B}$, and let us define

$$\mathcal{L}' = U\mathcal{L}U^{-1}, \mathcal{Z}' = U\mathcal{Z}U^{-1}, \mathcal{B}' = U\mathcal{B}U^{-1}, N'^{(m)} = UN^{(m)}U^{-1}. \tag{6.16}$$

Here the matrix $\mathcal{B}'$ is by definition diagonal, and clearly

$$N'^{(m)}_{jk} = \delta_{jm}\delta_{km}. \tag{6.17}$$

Then (4.16b) yields

$$\dot{\mathcal{L}}' = \mathrm{i}\Omega\mathcal{L}'N'^{(m)}, \tag{6.18a}$$

which entails

$$\mathcal{L}'(t) = \mathcal{L}'(0)\exp(\mathrm{i}\Omega tN'^{(m)}). \tag{6.18b}$$

Hence (6.14a) yields

$$\dot{\mathcal{Z}}'(t) = \mathcal{L}'(0)\exp(\mathrm{i}\Omega tN'^{m})N'^{(m)}, \tag{6.19a}$$

which admits the solution:

$$\mathcal{Z}'(t) = (\mathrm{i}\Omega)^{-1}\mathcal{Z}'(0)\mathcal{L}'(0)\exp(\mathrm{i}\Omega tN'^{(m)})N'^{(m)}, \tag{6.19b}$$

entailing (see (6.17))

$$[\mathcal{Z}'(t)]_{jk} = [\mathcal{Z}'(0)]_{jk} + (\mathrm{i}\Omega)^{-1}\exp(\mathrm{i}\Omega t)[\mathcal{L}'(0)]_{jm}\delta_{km}, \tag{6.20}$$

which displays the explicit time-evolution of the matrix $\mathcal{Z}'(t)$ under the flow generated by the m-th eigenvalue of $\mathcal{B}$. Obviously this evolution is periodic, with period $T = 2\pi/\Omega$. The coordinates $z_j(t)$ are just the eigenvalues of this matrix $\mathcal{Z}'(t)$ (see (6.16) and (5.2)), hence they are all periodic with period (at most) T', see (1.3).

VII Outlook

This is meant to be the first paper of a series. In subsequent papers we plan to treat *case (vi)*, to report analogous results for systems with "nearest neighbor" interactions, to focus on formulations of these models that can be properly interpreted as (real) many-body problems *in the plane* featuring only completely periodic trajectories [6], and to discuss quantum versions of such many-body problems. The quantization of the Hamiltonian system (6.4) is nontrivial even in the simplest case $n = 1$. In this case, the system is defined via the complex coordinates (z, p) or equivalently via the real ones:

$$(x, y; p_x, p_y) : z = x + \mathrm{i}y, p = p_x - \mathrm{i}p_y \qquad (7.1).$$

The Hamiltonian H reads then:

$$H(z, p) = \exp(p) - \mathrm{i}\Omega z = \exp(y)[\cos(p_y) - \mathrm{i}\sin(p_y)] - \mathrm{i}\Omega(x + \mathrm{i}y) \qquad (7.2)$$

and this yields (see [6]) the 2-dimensional Hamiltonian system characterized by the real Hamiltonian:

$$H_1(x, p_x; y, p_y) = exp(y)cos(p_y) + \Omega y. \qquad (7.3)$$

All the trajectories of this dynamical system are completely periodic with period $T = \Omega/2\pi$ as it is easily seen from the second order differential equation associated with the Hamiltonian (7.2):

$$\ddot{z} = \mathrm{i}\Omega\dot{z} \qquad (7.4)$$

It is expected that the corresponding quantum model feature an equispaced energy spectrum (to be properly defined). But the system (7.4) is not trivially equivalent to a harmonic oscillator. Thus the usual schemes of geometric quantization shall have to be appropriately revisited.

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