Ann. Henri Poincaré 1 (2000) 1 – 57

1424-0637/00/010001-57 $ 1.50+0.20/0

Annales Henri Poincaré

Continuous Constructive Fermionic Renormalization

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Abstract. We build the two dimensional Gross-Neveu model by a new method which requires neither cluster expansion nor discretization of phase-space. It simply reorganizes the perturbative series in terms of trees. With this method we can define non perturbatively the renormalization group differential equations of the model and at the same time construct explicitly their solution.

I Introduction

The popular versions of renormalization and the renormalization group in field theory are based on differential equations (among which the most famous one is the Callan-Symanzik equation). However no non-perturbative version of these differential equations has been given until now.

On the other hand the renormalization group in statistical mechanics, for instance for spin systems after the works of Kadanoff and Wilson, relies on closely related but discretized equations. When block spinning or other discretization of momentum space is used, the result is a discretized evolution of the effective action step by step. This point of view, in contrast with the first one, has led to rigorous non perturbative constructions for various models which have renormalizable power counting. In particular the two dimensional Gross Neveu model has been built by two groups [FMRS][GK]; also the infrared limit of the ϕ^4_4, a bosonic theory, has been controlled (see [R] and references therein). In all these cases the methods always involved some discretization of phase space and the outcome is a discrete (not differential) flow equation. Furthermore, the rigorous discretization of phase space came with a price, namely the use of some technical tools such as cluster or Mayer expansions which are neither popular among theoretical physicists nor among mathematicians.

The proposal of Manfred Salmhofer to build a continuous version of the renormalization group for Fermionic theories [S] is therefore very interesting and welcome. Indeed Fermionic series with cutoffs are convergent (in contrast with Bosonic ones, which are Borel summable at best), and the continuous version of renormalization group which works so well at the perturbative level should therefore apply to them[1].

In this paper we realize the Salmhofer proposal on the particular example of the two-dimensional Gross-Neveu model. We rearrange Fermionic perturbation

[1] The continuous limit of the discretized non perturbative RG equations has been also studied for a certain many fermion system in 1+1 dimension in reference [C].

theory according to trees, an idea first developed in [AR2], perform subtractions only when necessary according to the relative scales of the subgraphs, and obtain (to our own surprise, quite easily) an explicit convergent representation of the model without any discretization or cluster or Mayer expansion. To prove the convergence requires only some well-known perturbative techniques of parametric representations ("Hepp's sectors"), Gram's bound on determinants and a crucial but rather natural concatenation of some intervals of integration for loop lines.

Therefore we can now consider that constructive theory for Fermions has been "reduced" to perturbation theory. Remark also that since the representation we use is an "effective" representation in the sense of [R], hence with subtractions performed only when necessary according to the relative scales of the subgraphs, we never meet the so-called problem of "overlapping divergences" or classification of Zimmermann's forests. In this sense constructive renormalization is easier than ordinary perturbative renormalization (which, from the constructive point of view, is flawed anyway because it generates renormalons; these renormalons are the reason for which the ordinary renormalized perturbation theory of this model is only Borel summable [FMRS], as stated in Theorem 1).

Having an explicit convergent representation of the theory with a continuously moving cutoff, it is trivial both to define the continuous renormalization group equations which correspond to the variation of this cutoff and, at the same time, to check that our explicit representation is a solution of these equations.

Remark however we have not yet found the way to short-circuit our representation and to prove that the equations and their solutions exist by a purely inductive argument à la Polchinski [P] which would avoid an explicit formula for the solution. This is presumably possible but this question as well as the extension to other models, in particular to interacting Fermions models of condensed matter physics, is left for future investigation. It is also important to recall that we do not see at the moment how to extend this method to Bosons, since there are no determinant and Gram's bound for them.

II Model and main result

We consider the massive Gross-Neveu model GN_2, which describes N types of Fermions. These Fermions interact through a quartic term. Actually, the GN_2 action also requires a quadratic mass counterterm and a wave function counterterm in order for the ultraviolet limit to be finite. Therefore the bare action in a finite volume V is (using the notations of [FMRS]):

$$\begin{aligned} \mathcal{S}_V &= \frac{\lambda}{N}\int_V d^2x\,[\sum_a \bar\psi_a(x)\psi_a(x)]^2 \\ &\quad +\delta m\int_V d^2x\,[\sum_a \bar\psi_a(x)\psi_a(x)] + \delta\zeta\int_V d^2x\,[\sum_a \bar\psi_a(x) i\,\partial\!\!\!/\psi_a(x)] \end{aligned} \tag{II.1}$$

where λ is the bare coupling constant, δm and $\delta\zeta$ are the bare mass and wave function counterterms, and a is the color index: $a = 1, \dots, N$. The action (II.1) and the power counting of the GN_2 model are like the ones for the Bosonic ϕ_4^4 theory, except that, unlike the latter, the GN_2 theory is asymptotically free for $N \geq 2$, a condition which we assume from now on. The free covariance in momentum space is

$$C_{ab}^{\gamma\delta}(p) = \delta_{a,b}\left(\frac{1}{-\not p + m}\right)_{\gamma,\delta} = \delta_{a,b}\left(\frac{\not p + m}{p^2 + m^2}\right)_{\gamma,\delta} \tag{II.2}$$

where γ, δ are the spin indices, and a, b are the color indices. Most of the time we skip the inessential spin indices to simplify notation. The mass m is the renormalized mass. To avoid divergences, according to the notations of [KKS] we introduce an *ultraviolet cut-off* Λ_0 and (for later study of the renormalization group flow) a *scale parameter* Λ which plays the role of an infrared cutoff:

$$C_\Lambda^{\Lambda_0}(p) \quad = \quad C(p)\left[\eta\left(\frac{(p^2+m^2)}{\Lambda_0^2}\right) - \eta\left(\frac{(p^2+m^2)}{\Lambda^2}\right)\right]. \tag{II.3}$$

The cutoff function η might be any function which satisfies $\eta(0) = 1$, which is smooth, monotone and rapidly decreasing at infinity (this means faster than any fixed power). For simplicity in this paper we restrict ourselves to the most standard case $\eta(x) = e^{-x}$. In this case both $C_\Lambda^{\Lambda_0}$ and its Fourier transform have explicit so-called parametric representations:

$$\begin{aligned} C_\Lambda^{\Lambda_0}(p) &= \int_{\Lambda_0^{-2}}^{\Lambda^{-2}} (\not p + m)\, e^{-\alpha(p^2+m^2)} d\alpha \\ C_\Lambda^{\Lambda_0}(x-y) &= \pi \int_{\Lambda_0^{-2}}^{\Lambda^{-2}} \left(i\frac{(\not x - \not y)}{2\alpha^2} + \frac{m}{\alpha}\right) e^{-\alpha m^2 - |x-y|^2/4\alpha} d\alpha \end{aligned} \tag{II.4}$$

We define now the connected truncated Green functions, also called vertex functions, which are the coefficients of the *effective action*. The partition function with external fields $\xi, \bar\xi$ is

$$\begin{aligned} Z_V^{\Lambda\Lambda_0}(\xi, \bar\xi) &= \int d\mu_{C_\Lambda^{\Lambda_0}}(\psi, \bar\psi) e^{-\mathcal{S}_V(\psi,\bar\psi) + <\psi,\xi> + <\xi,\psi>} \\ <\psi, \xi> &:= \int_V d^2x\, \bar\psi(x)\xi(x). \end{aligned} \tag{II.5}$$

The vertex function with $2p$ external points is:

$$\begin{aligned} \Gamma_{2p}^{\Lambda\Lambda_0}(\{y\},\{z\}) :&= \Gamma_{2p}^{\Lambda\Lambda_0}(y_1, \dots, y_p, z_1, \dots, z_p) \\ &= \lim_{V\to\infty} \frac{\delta^{2p}}{\delta\xi(z_1)\dots\delta\xi(z_p)\delta\bar\xi(y_1)\dots\delta\bar\xi(y_p)} \left((\ln Z_V^{\Lambda\Lambda_0} - F)(C_\Lambda^{\Lambda_0})^{-1}(\xi)\right)\Big|_{\xi=0} \end{aligned} \tag{II.6}$$

where $F(\xi) = <\xi, C_\Lambda^{\Lambda_0}\xi>$ is the propagator, and color indices are implicit. Please do not confuse these vertex functions, which are connected, with the one-particle irreducible functions, since the latter are usually also called Γ in the literature. These functions (in fact distributions) form the coefficients of the effective action (expanded in powers of the external fields) at energy Λ with UV cutoff Λ_0. Developing the exponential in Z and attributing prime and double prime indices respectively to the mass and wave function counterterms we have:

$$Z_V^{\Lambda\Lambda_0}(\xi) = \sum_{p=0}^{\infty} \frac{1}{p!^2} \sum_{n,n',n''=0}^{\infty} \frac{(-1)^{n+n'+n''}}{n!n'!n''!} \sum_{a_i b_i c_i d_i} \left(\tfrac{\lambda}{N}\right)^n (\delta m)^{n'} (\delta\zeta)^{n''} \tag{II.7}$$

$$\int_V d^2y_1 \ldots d^2y_p d^2z_1 \ldots d^2z_p d^2x_1 \ldots d^2x_n d^2x'_1 \ldots d^2x'_{n'} d^2x''_1 \ldots d^2x''_{n''} \prod_{i=1}^{p} \xi_{d_i}(z_i)\bar{\xi}_{c_i}(y_i)$$

$$\left\{ \begin{array}{ccccccccccc} y_{1,c_1} & \ldots & y_{p,c_p} & x_{1,a_1} & x_{1,b_1} & \ldots & x_{n,a_n} & x_{n,b_n} & x'_{1,a'_1} & \ldots & x''_{n'',a''_{n''}} \\ z_{1,d_1} & \ldots & z_{p,d_p} & x_{1,a_1} & x_{1,b_1} & \ldots & x_{n,a_n} & x_{n,b_n} & x'_{1,b'_1} & \ldots & x''_{n'',b''_{n''}} \end{array} \right\}$$

where we used Cayley's notation for the determinants:

$$\left\{ \begin{array}{c} u_{i,a} \\ v_{j,b} \end{array} \right\} = \det(D_{ab}(u_i - v_j)) \tag{II.8}$$

and $a_i, b_i, a'_i, b'_i, a''_i, b''_i, c_i, d_i$ are the color indices. By convention

$$D_{ab}(u_i - v_j) := C_{ab}(u_i - v_j) \tag{II.9}$$

except when the second index is the one of a ψ field hooked to a $\delta\zeta$ vertex. In this particular case the vertex has a so-called derivative coupling, and therefore the propagator D bears a derivation, namely $D_{ab''}(u_i - v_j) := i \not\partial_{v_j} C_{ab''}(u_i - v_j) :=$ $C'_{ab''}(u_i - v_j)$. This derived propagator is explicitly

$$C'^{\Lambda_0}_{\Lambda}(x-y) = \pi \int_{\Lambda_0^{-2}}^{\Lambda^{-2}} \left(\frac{|x-y|^2}{4\alpha^3} + \frac{im(\not x - \not y)}{2\alpha^2} - \frac{1}{\alpha^2} \right) e^{-\alpha m^2 - |x-y|^2/4\alpha} d\alpha \tag{II.10}$$

Expanding the determinant in (II) one obtains the usual perturbation theory in terms of Feynman graphs with the three types of vertices corresponding to the three terms of the action (II.1), and the logarithm is simply the sum over connected graphs. To see if a graph is connected, it is not necessary to know its whole structure but only a tree in it. Based on this remark the logarithm of (II) was computed in [AR2] using an expansion which is intermediate between the determinant form (II) and the fully expanded Feynman graphs. This expansion is based on a *forest formula*. Such formulas, discussed in [AR1], are Taylor expansions with integral remainders. They test the coupling or links (here the propagators) between $n \geq 1$ points (here the vertices) and stop as soon as the final connected components are built. The result is therefore a sum over forests, which are simply defined as union of disjoint trees. A forest is therefore a (pedantic, but poetic) word for a Feynman graph without loops, and our point of view is that these are the natural objects to express Fermionic perturbation theory.

Here we use the most symmetric forest formula, the *ordered Brydges-Kennedy Taylor formula*, which states [AR1] that for any smooth function H of the $n(n-1)/2$ variables u_l, $l \in P_n = \{(i,j)|i,j \in \{1,\ldots,n\}, i \neq j\}$,

$$H|_{u_l=1} = \sum_{o-\mathcal{F}} \left(\int_{0 \le w_1 \le \ldots \le w_k \le 1} \prod_{q=1}^{k} dw_q \right) \left(\prod_{q=1}^{k} \frac{\partial}{\partial u_{l_q}} H \right) (w_l^{\mathcal{F}}(w_q), l \in P_n) \quad \text{(II.11)}$$

where $o-\mathcal{F}$ is any ordered forest, made of $0 \le k \le n-1$ links $l_1, \ldots, l_k$ over the n points. To each link l_q $q = 1, \ldots, k$ of $\mathcal{F}$ is associated the parameter w_q, and to each pair $l = (i,j)$ is associated the weakening factor $w_l^{\mathcal{F}}(w_q)$. These factors replace the variables u_l as arguments of the derived function $\prod_{q=1}^{k} \frac{\partial}{\partial u_{l_q}} H$ in (II.11). These weakening factors $w_l^{\mathcal{F}}(w)$ are themselves functions of the parameters w_q, $q = 1, \ldots, k$ through the formulas

$$\begin{aligned} w_{i,i}^{\mathcal{F}}(w) &= 1 \\ w_{i,j}^{\mathcal{F}}(w) &= \inf_{l_q \in P_{i,j}^{\mathcal{F}}} w_q, \qquad \text{if } i \text{ and } j \text{ are connected by } \mathcal{F} \\ & \quad\ \text{where } P_{i,j}^{\mathcal{F}} \text{ is the unique path in the forest } \mathcal{F} \text{ connecting } i \text{ to } j \\ w_{i,j}^{\mathcal{F}}(w) &= 0 \qquad \text{if } i \text{ and } j \text{ are not connected by } \mathcal{F}. \end{aligned} \quad \text{(II.12)}$$

We apply this formula to the determinant in (II), inserting the interpolation parameter u_l in the cut-off (but only between distinct vertices, so not for the "tadpole" lines):

$$\begin{aligned} C_\Lambda^{\Lambda_0}(x,y,u) &= \delta(x-y) C_\Lambda^{\Lambda_0}(x,x) + [1-\delta(x-y)] C_\Lambda^{\Lambda_0}(x,y,u) \\ &:= C_\Lambda^{\Lambda_0(u)}(x,y) \\ &:= \pi \int_{\Lambda_0^{-2}(u)}^{\Lambda^{-2}} \left(i \frac{(\not x - \not y)}{2\alpha^2} + \frac{m}{\alpha} \right) e^{-\alpha m^2 - |x-y|^2/4\alpha} d\alpha \end{aligned} \quad \text{(II.13)}$$

where

$$\Lambda_0^{-2}(u) = \Lambda^{-2} + u(\Lambda_0^{-2} - \Lambda^{-2}). \quad \text{(II.14)}$$

We use similar interpolation for the C' propagators. When u grows from 0 to 1, the ultraviolet cut-off of the interpolated propagator (between distinct vertices) grows therefore from Λ to Λ_0.

We define

$$\begin{aligned} C_\Lambda^{\Lambda_0,u}(x,y) &:= \frac{\partial}{\partial u} C_\Lambda^{\Lambda_0}(x,y,u) = \pi \left(i \frac{(\not x - \not y)\Lambda_0^4(u)}{2} + m\Lambda_0^2(u) \right) \\ &\quad \cdot (\Lambda^{-2} - \Lambda_0^{-2}) e^{-m^2 \Lambda_0^{-2}(u) - |x-y|^2 \Lambda_0^2(u)/4} \\ C_\Lambda^{\prime\,\Lambda_0,u}(x,y) &:= \frac{\partial}{\partial u} C_\Lambda^{\prime\,\Lambda_0}(x,y,u) = \pi \Bigg(\frac{|x-y|^2 \Lambda_0^6(u)}{4} \\ &\quad + \frac{im(\not x - \not y)\Lambda_0^4(u)}{2} - \Lambda_0^4(u) \Bigg) (\Lambda^{-2} - \Lambda_0^{-2}) e^{-m^2 \Lambda_0^{-2}(u) - |x-y|^2 \Lambda_0^2(u)/4} \end{aligned} \quad \text{(II.15)}$$

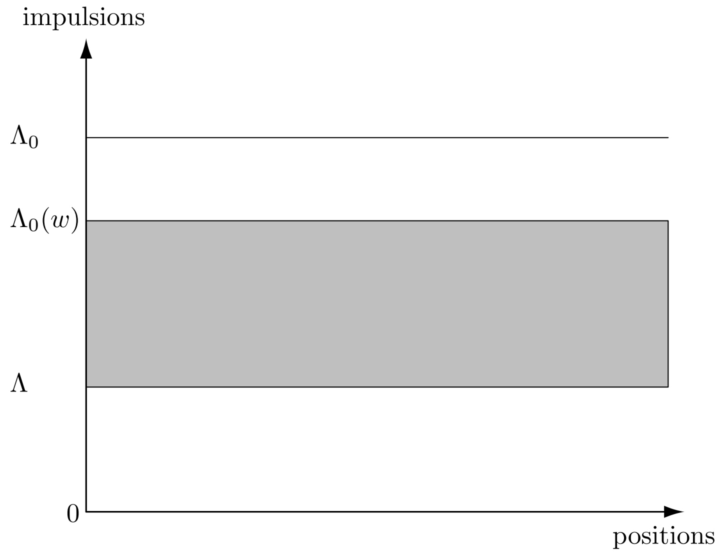


Figure 1

The derivative of η fixes C^u at an energy near $\Lambda_0(u)$. We observe that for any fixed ϵ' we have the scaled decay:

$$|C^u(x,y)| \le K\Lambda_0^3(u)(\Lambda^{-2} - \Lambda_0^{-2})e^{-|x-y|(1-\epsilon')\Lambda_0^m(u) - \epsilon' m^2 \Lambda_0^{-2}(u)/2} \tag{II.16}$$

$$|C'^{\,u}(x,y)| \le K\Lambda_0^4(u)(\Lambda^{-2} - \Lambda_0^{-2})e^{-|x-y|(1-\epsilon')\Lambda_0^m(u) - \epsilon' m^2 \Lambda_0^{-2}(u)/2} \tag{II.17}$$

where K is a constant depending only on ϵ' and

$$\Lambda_0^m(u) := \sup[m, \Lambda_0(u)]. \tag{II.18}$$

Applying this interpolation and the ordered forest formula (II.11) to the propagators in the determinant of (II) we obtain

$$Z_V^{\Lambda\Lambda_0}(\xi) = \sum_{p=0}^{\infty} \frac{1}{p!^2} \sum_{n,n',n''=0}^{\infty} \frac{1}{n!n'!n''!} \sum_{o-\mathcal{F}} \sum_{\mathcal{C}ol} \sum_{\Omega} \left(\tfrac{\lambda}{N}\right)^n (\delta m)^{n'} (\delta\zeta)^{n''} \epsilon(\mathcal{F},\Omega)$$
$$\int_V d^2y_1 \ldots d^2y_p d^2z_1 \ldots d^2z_p d^2x_1 \ldots d^2x_{\bar{n}} \prod_{r=1}^{p} \xi_{d_r}(z_r)\bar{\xi}_{c_r}(y_r)$$
$$\int_{0\le w_1 \le \ldots \le w_k \le 1} \left[\prod_{q=1}^{k} D_{\Lambda}^{\Lambda_0, w_q}(\bar{x}_{l_q}, x_{l_q}) dw_q\right] [\det]_{\text{left}}(w^{\mathcal{F}}(w)) \tag{II.19}$$

where for simplicity the position of any vertex is simply denoted by the letter x and $\bar{n} := n + n' + n''$. $\bar{x}_{l_q}$ and x_{l_q} are the ends of line l_q. $[\det]_{\text{left}}$ is the remaining determinant. Its entries correspond to the remaining fields necessary to complete each vertex of the forest into a quartic or quadratic vertex, according to its type (interaction or counterterm). For this model, remark that the sum $\sum_{o-\mathcal{F}}$ is performed only over the ordered forests that have, for each point x_i coordination number $n(i) \leq 4$ or $n(i) \leq 2$ depending of the type of the vertex (all other terms being zero). The additional sums over $\mathcal{C}ol$ and Ω correspond to coloring choices at each vertex and "fields versus antifields" choices at each line and vertex [AR2]. The sign $\epsilon(\mathcal{F}, \Omega)$ comes in from the antisymmetry of Fermions and is computed in [AR2]: here we only need to know that it factorizes over the connected components of $\mathcal{F}$. To find the expression for $\ln Z$ we write Z as an exponential. In equation (II), the determinant factorizes over the ordered trees $\mathcal{T}_1 \ldots \mathcal{T}_j$ forming the forest. Indeed one can resum all orderings of the ordered forest $\mathcal{F}$ compatible with fixed orderings of its connected components, the trees $\mathcal{T}_1 \ldots \mathcal{T}_j$. Furthermore the "weakening factor" $w^{\mathcal{F}}$ vanishes between vertices belonging to different connected components. Hence:

$$
\begin{aligned}
& Z_V^{\Lambda\Lambda_0}(\xi) \\
&= \sum_{p=0}^{\infty} \frac{1}{p!^2} \sum_{n,n',n''=0}^{\infty} \frac{1}{n!n'!n''!} \sum_{j=0}^{n} \frac{1}{j!} \sum_{\substack{n_1,\ldots n_j \\ n_1+\cdots+n_j=n}} \sum_{\substack{n'_1,\ldots n'_j \\ n'_1+\cdots+n'_j=n'}} \sum_{\substack{n''_1,\ldots n''_j \\ n''_1+\cdots+n''_j=n''}} \\
& \sum_{\substack{p_1,\ldots p_j,p' \\ p_1+\cdots+p_j+p'=p}} \frac{n!n'!n''!}{n_1!\ldots n_j!n'_1!\ldots n'_j!n''_1!\ldots n''_j!} \frac{p!^2}{p_1!^2\ldots p_j!^2 p'!^2} p'! \\
& (\xi, C_\Lambda^{\Lambda_0}\xi)^{p'} \prod_{i=1}^{j} \left[\left(\tfrac{\lambda}{N}\right)^{n_i} (\delta m)^{n'_i} (\delta\zeta)^{n''_i} A(n_i, n'_i, n''_i, p_i) \right]
\end{aligned}
\tag{II.20}
$$

where

$$
\begin{aligned}
& A(n_i, n'_i, n''_i, p_i) \\
&= \sum_{\mathcal{T}_i} \sum_{\mathcal{C}ol_i, \Omega_i} \epsilon(\mathcal{T}_i, \Omega_i) \int d^2y_1 \ldots d^2y_{p_i} d^2z_1 \ldots d^2z_{p_i} d^2x_1 \ldots d^2x_{\bar{n}_i} \\
& \prod_{r=1}^{p} \xi_{d_r}(z_r) \bar{\xi}_{c_r}(y_r) \int_{0 \leq w_1 \leq \ldots \leq w_{\bar{n}_i - 1} \leq 1} \left[\prod_{q=1}^{\bar{n}_i - 1} D_\Lambda^{\Lambda_0, w_q}(\bar{x}_{l_q}, x_{l_q}) dw_q \right] \\
& [\det]_{\text{left,i}}(w^{\mathcal{T}_i}(w))
\end{aligned}
\tag{II.21}
$$

where $\bar{n}_i$ is the number of vertices in the ordered tree $\mathcal{T}_i$, which has therefore $\bar{n}_i - 1$ lines, p_i is the number of external fields of type y (and z) attached to the $\mathcal{T}_i$, and p' is the number of free external propagators (not connected to any vertex) in the forest. This can be written as an exponential, hence

$$
\begin{aligned}
&\ln Z_V^{\Lambda\Lambda_0}(\xi) \\
&= (\xi, C_\Lambda^{\Lambda_0}\xi) + \sum_{p=0}^{\infty}\frac{1}{p!^2}\sum_{n,n',n''=0}^{\infty}\frac{1}{n!n'!n''!}\left(\tfrac{\lambda}{N}\right)^n(\delta m)^{n'}(\delta\zeta)^{n''} \\
&\sum_{o-\mathcal{T}}\sum_{\mathcal{C}ol,\Omega}\epsilon(\mathcal{T},\Omega)\int_V d^2y_1\dots d^2y_pd^2z_1\dots d^2z_pd^2x_1\dots d^2x_{\bar n}\prod_{r=1}^{p}\xi_{d_r}(z_r)\bar\xi_{c_r}(y_r) \\
&\int_{0\le w_1\le\dots\le w_{\bar n-1}\le 1}\left[\prod_{q=1}^{\bar n-1}D_\Lambda^{\Lambda_0,w_q}(\bar x_{l_q},x_{l_q})dw_q\right][\det]_{\text{left}}(w^{\mathcal{T}}(w))
\end{aligned}
\tag{II.22}
$$

where $\mathcal{T}$ is an ordered tree over $\bar n$ points, and the external points are all connected to the tree. Now, applying the definition (II.6), we obtain the vertex functions, for which the limit $V\to\infty$ can be performed (because the external points hooked to the tree ensure convergence). The set

$$
E = \{(i_1,\dots i_p,j_1,\dots,j_p)|i_1,\dots i_p,j_1,\dots j_p\in\{1,\dots,\bar n\}\} \tag{II.23}
$$

fixes the internal points to which the $2p$ external lines hook.

We recall the well-known fact that the vertex functions in x-space are in fact distributions. For instance it is easy to see that when some of the external points i_k, j_k in the previous sum coincide, one has to factor out the product of the corresponding delta functions of the external arguments to obtain smooth functions. This little difficulty can be treated either by considering the vertex functions in momentum space (they are then ordinary functions of external momenta, after factorization of global momentum conservation), or by smearing the vertex functions with test functions. Here we adopt this last point of view. The quantity under study is then $\Gamma_{2p}^{\Lambda\Lambda_0}$ smeared with smooth test functions $\phi_1(y_1)$, …, $\phi_p(y_p)$, $\phi_{p+1}(z_1),\dots,\phi_{2p}(z_p)$:

$$
\begin{aligned}
&\Gamma_{2p}^{\Lambda\Lambda_0}(\phi_1,\dots\phi_{2p}) \\
&= \int d^2y_1\dots d^2y_pd^2z_1\dots d^2z_p \\
&\Gamma_{2p}^{\Lambda\Lambda_0}(y_1,\dots,y_p,z_1,\dots,z_p)\phi_1(y_1)\dots\phi_p(y_p)\phi_{p+1}(z_1)\dots\phi_{2p}(z_p).
\end{aligned}
\tag{II.24}
$$

where we asked the test functions to have compact support: $\phi\in\mathcal{D}(\mathbf{R}^2)$.

Remark that when some external antifield hooks to a $\delta\zeta$ vertex, the amputation by C instead of C' leaves a δ' distribution, which means a derivative acting on the corresponding test function.

We obtain the formula:

$$\Gamma_{2p}^{\Lambda\Lambda_0}(\phi_1,\dots\phi_{2p}) = \sum_{n,n',n''=0}^{\infty} \left(\tfrac{\lambda}{N}\right)^n (\delta m)^{n'} (\delta\zeta)^{n''} \frac{1}{n!n'!n''!} \tag{II.25}$$
$$\sum_{o-\mathcal{T}} \sum_{E} \sum_{\mathcal{C}ol,\Omega} \epsilon(\mathcal{T},\Omega) \int d^2x_1 \dots d^2x_{\bar n} \phi_1(x_{i_1}) \dots \phi_{2p}(x_{j_p})$$
$$\int_{0\le w_1\le\dots\le w_{\bar n -1}\le 1} \left[\prod_{q=1}^{\bar n -1} D_\Lambda^{\Lambda_0,w_q}(\bar x_{l_q}, x_{l_q}) dw_q\right] [\det]_{\text{left}}(w^{\mathcal{T}}(w), E)$$

where the propagator D is now C or C' according to the discussion above.

When renormalization is introduced, it will be convenient to use the BPHZ subtraction prescription at 0 external momenta, which corresponds to integrate the vertex functions over all arguments except one. In this prescription one defines the renormalized coupling constant as the 4-vertex function of the full theory at zero external momenta:

$$\frac{\lambda_{ren}}{N} \quad := \quad \widehat\Gamma_4^{\Lambda\Lambda_0}(0,0,0,0) = \int d^2x_2 d^2x_3 d^2x_4 \; \Gamma_4^{\Lambda\Lambda_0}(0,x_2,x_3,x_4) \tag{II.26}$$

Moreover we want the renormalized mass and wave function constant to be respectively m and 1. This means that we impose the additional renormalization conditions:

$$\delta m_{ren} := \widehat\Gamma_2^{\Lambda\Lambda_0}(0,0) = \int d^2x_2 \Gamma_2^{\Lambda\Lambda_0}(0,x_2) = 0 \tag{II.27}$$

$$\delta\zeta_{ren} := \not\partial \widehat\Gamma_2^{\Lambda\Lambda_0}(0,0) = \int d^2x_2 i \not x_2 \Gamma_2^{\Lambda\Lambda_0}(0,x_2) = 0 \tag{II.28}$$

With these conditions the whole theory (at fixed renormalized mass m) becomes parametrized only by λ_{ren}, hence not only λ but also δm and $\delta\zeta$ in (II.1) become functions of λ_{ren}. This of course has a precise meaning only if we can construct the theory and solve the renormalization group flows, which is precisely what we are going to do. We can express the main result of this paper as a theorem on the existence of the ultraviolet limit of the vertex functions and of the renormalization group flows. Recall that the theory is not directly the sum but the Borel sum of the renormalized perturbation theory. In summary

Theorem 1 *The limit $\Lambda_0 \to \infty$ of $\Gamma_{2p}^{\Lambda\Lambda_0}(\phi_1,\dots\phi_{2p})$ exists and is Borel summable in the renormalized coupling constant λ_{ren}, uniformly in N (where N is the number of colors). Since the parameter Λ varies continuously, the continuous renormalization group equations and in particular the β function are also well defined in the limit $\Lambda_0 \to \infty$.*

The first part of the theorem is similar to [FMRS], but the second part (the existence of the continuous renormalization group equations) is new. The rest of the paper is devoted to the proof of this theorem.

The precise bounds on the smeared vertex functions are given in Theorem 3 below. They are uniform in N (and in fact proportional to N^{1-p}). Let us discuss also briefly the dependence in m, the renormalized mass. For $m \neq 0$ fixed, we can define the physical scale of the system by putting $m = 1$. The theorem is then uniform in the infrared cutoff Λ, including the point $\Lambda = 0$. In the case $m = 0$ our method requires a nonzero infrared cutoff $\Lambda \neq 0$. Since this cutoff is the only scale of the problem, we can then put it to 1: $\Lambda = 1$. In this last case, improperly called the "massless theory", we know that there should be a non-perturbative mass generation [GN]. This mass generation has been proved rigorously for the model with fixed ultraviolet cutoff and large number N of components in [KMR], using the Matthews-Salam formalism of an intermediate Bosonic field and a cluster expansion with a small/large field expansion. Our result in the massless case $m = 0$ with a finite infrared cutoff Λ should therefore glue with the method and results of [KMR] to obtain at large N the mass generation of the full model without ultra-violet cutoff.

III The expansion

III.1 The continuous band structure

Remark that in (II.25)

$$w_1 \leq w_2 \leq \cdots \leq w_{\bar{n}-1} \Longrightarrow \Lambda_0(w_1) \leq \Lambda_0(w_2) \cdots \leq \Lambda_0(w_{\bar{n}-1}). \tag{III.1}$$

This naturally cuts the space of momenta into $\bar{n}$ bands $B = \{1, \ldots, \bar{n}\}$ (see Figure 2).

Looking at equation (II.4), we see that the covariance can be written as a sum of propagators restricted to single bands:

$$C_{\Lambda}^{\Lambda_0}(p) = \sum_{k=1}^{\bar{n}} C_{\Lambda_0(w_{k-1})}^{\Lambda_0(w_k)}(p) = \sum_{k=1}^{\bar{n}} \int_{\Lambda_0^{-2}(w_k)}^{\Lambda_0^{-2}(w_{k-1})} (\not{p} + m) e^{-\alpha(p^2+m^2)} d\alpha = C(p) \sum_{k=1}^{\bar{n}} \eta^k \tag{III.2}$$

where we defined

$$\eta^k \quad := \quad e^{-\Lambda_0^{-2}(w_k)(p^2+m^2)} - e^{-\Lambda_0^{-2}(w_{k-1})(p^2+m^2)} \tag{III.3}$$

and we adopted the convention

$$w_0 = 0 \Rightarrow \Lambda_0^{-2}(w_0) = \Lambda^{-2} \quad w_{\bar{n}} = 1 \Rightarrow \Lambda_0^{-2}(w_{\bar{n}}) = \Lambda_0^{-2}. \tag{III.4}$$

Similar formulas hold for C' but with an additional $\not{p}$.

To cure the ultraviolet divergences we have to combine the divergent local parts of some subgraphs with counterterms and reexpress the series for G as an effective series in the sense of [R]. For that purpose we use the band structure to distinguish the divergent subgraphs from the convergent ones and hence decide where renormalization is necessary. Please recall that in contrast with usual

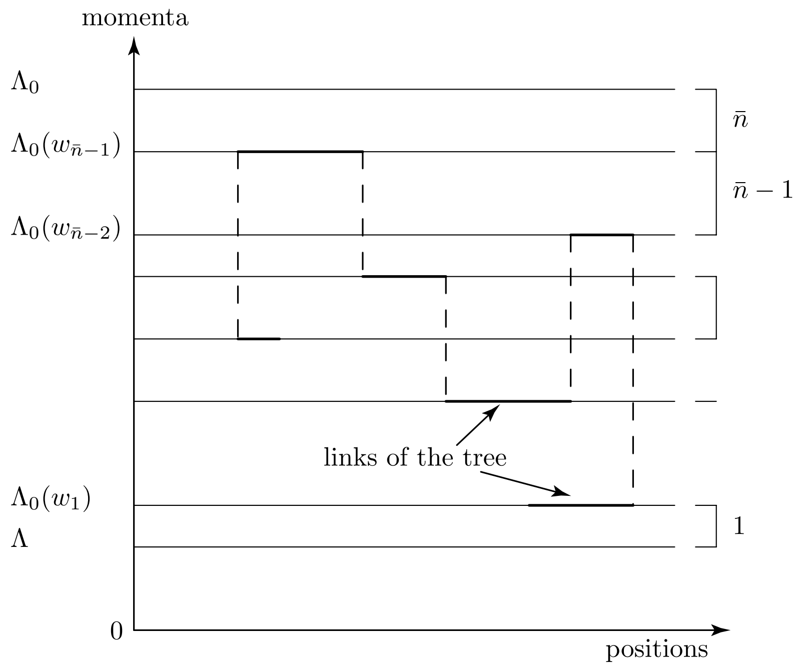


Figure 2

perturbation theory we never develop explicitly the loop lines of these subgraphs. Contrary to naive expectation, one does not need to know the particular loop structure to perform renormalization!

III.2 Notations

Now we fix some notations. It is convenient to give indices to the fields variables or the half-lines which correspond to these fields after Grassmann integration. We observe that there are several types of such variables, the half-lines which form the lines of the tree, the external variables (which correspond to amputated lines) and the entries (rows or columns) in the determinant $\det_{\text{left}}$. These entries will be called "loop fields" or "loop half-lines" since they form the usual loop lines of the Feynman graphs if one expands the determinant. We define E and L as the set of all external and loop half-lines. For each level i there is a tree-line l_i with two ends corresponding to two half-lines called f_i and g_i (to fix ideas let's say that f_i is the end corresponding to the field and g_i the end corresponding to the anti-field, as decided by the index Ω in (II)). The loop fields $\psi(x_g)$ and $\bar{\psi}(x_f)$ are called h_f and h_g, and (when expanded) the loop line $\bar{\psi}(x_f)\psi(x_g)$ is called l_{fg} (it corresponds to a particular coefficient in the determinant $\det_{\text{left}}$). Each tree half-line f_i or g_i, each loop field h_f or h_g is hooked to a vertex called v_{f_i} or v_{g_i} or v_f or v_g. We need also to care about the set S of special fields (or antifields) which are hooked to the $\delta\zeta$ vertices and correspond to propagators C' which have different "power counting" than C. Finally the index of the highest tree-line hooked to a vertex v is called i_v.

Now $[\det]_{\text{left}}$ is the determinant of a matrix $(n+1-p)\times(n+1-p)$. The corresponding loop fields can be labeled by an index $a = 1,\dots,2n+2-2p$. The matrix elements are $D(x_f, x_g, w^{\mathcal{T}}_{v_f,v_g}(w))$. Therefore in terms of bands the line l_{fg} is restricted by the weakening factor $w^{\mathcal{T}}_{v_f,v_g}(w)$ to belong to the bands from 1 to the lowest index in the path $P^{\mathcal{T}}_{f,g}$ (this path $P^{\mathcal{T}}_{f,g}$ is defined in equation (II.12)). We call $i^{\mathcal{T}}_{f,g}$ this index:

$$i^{\mathcal{T}}_{f,g} = \inf\ \{q \mid l_q \in P^{\mathcal{T}}_{f,g}\} \tag{III.5}$$

$$D(x_f, x_g, w^{\mathcal{T}}_{v_f,v_g}(w)) = D(p) \sum_{k=1}^{i^{\mathcal{T}}_{f,g}} \eta^k(p) \tag{III.6}$$

By multilinearity one can expand the determinant in (II.25) according to the different bands in the sum (III.6) for each row and column.

$$[\det]_{\text{left}}(w^{\mathcal{T}}(w), E) = \sum_{\mu} \det \mathcal{M}(\mu) \tag{III.7}$$

where we define the attribution μ as a collection of band indices for each loop field a:

$$\mu = \{\mu(f_1), \dots \mu(f_{n+1-p}), \mu(g_1), \dots \mu(g_{n+1-p})\}\ ,\ \mu(a) \in B \ \text{ for } a = 1\dots 2n{+}2{-}2p. \tag{III.8}$$

Now, for each attribution μ we need to exploit power counting. This requires notations for the various types of fields or half-lines which form the analogs of the quasi local subgraphs of [R] in our formalism. We define:

$$\begin{aligned}
T_k &= \{l \in \mathcal{T} |\ i_{v_l} \geq k\}\\
IT_k &= \{f_i, g_i \in \mathcal{T} |\ i \geq k\}\\
IL_k &= \{a \in L |\ \mu(a) \geq k\}\\
EE_k &= \{f, g \in E | i_{v_f}, i_{v_g} \geq k\}\\
ET_k &= \{f_i | i_{v_{f_i}} \geq k, i < k\} \cup \{g_i | i_{v_{g_i}} \geq k, i < k\}\\
EL_k &= \{a \in L | i_{v_a} \geq k, \mu(a) < k\}\\
N_k &= \{v \text{ of type } \lambda\ | i_v \geq k\}\\
N'_k &= \{v \text{ of type } \delta m\ | i_v \geq k\}\\
N''_k &= \{v \text{ of type } \delta\zeta\ | i_v \geq k\}\\
\bar{N}_k &= N_k \cup N'_k \cup N''_k\\
G_k &= IT_k \cup IL_k\\
E_k &= EE_k \cup ET_k \cup EL_k\\
E''_k &= E_k \cap S\ ,\ T_k = \{l_i |\ i \geq k\}
\end{aligned} \tag{III.9}$$

where we recall that S in the last definition is the set of those fields hooked to a vertex of type $\delta\zeta$ which bear a derivation. We note $|A|$ the number of elements in the set A. For instance the reader can check that $|IT_1| = 2\bar{n} - 2$ and that $|T_1| = \bar{n} - 1$. Each G_k has $c(k)$ connected components G_k^j, $j = 1, \ldots, c(k)$.

To help the reader understand better these technical definitions, let's say that:

- T stands for "tree"
- IT stands for the set of "internal tree half lines" of a subgraph;
- IL stands for the set of "internal loop half lines" of a subgraph. Together IT_k and IL_k form the full subgraph G_k;
- EE stands for the set of "half lines which are external both for the subgraph and for the whole graph";
- ET stands for the set of "half lines which are external for the subgraph but not for the whole graph, and which belong to the tree";
- EL stands for the set of "half lines which are external for the subgraph but not for the whole graph, and which are loop lines in the full graph";
- N N' and N'' are used for the different types of vertices in a subgraph.

All the definitions in (III.9) can be restricted to each connected component. Applying power counting, the convergence degree for the subgraph G_i^k is

$$\omega(G_i^k) = \frac{1}{2}(|E_i^k| + 2|N_i^{'k}| - 4) \tag{III.10}$$

where we assumed that no external half-line hooked to a vertex of type $\delta\zeta$ bears a $i\, \not\partial$. To assure this for any G_i^k, we apply, for each vertex v'', the operator $i\, \not\partial$ (or $-i\, \not\partial$) to the highest tree half-line hooked to v'' (there is always at least one). In this way for all k $|E_k''| = 0$, and no loop line bears a gradient. Then $\mathcal{M}(\mu)$ is a matrix whose coefficients are

$$\mathcal{M}_{fg}(\mu)(x_f, x_g) = \delta_{\mu(f),\mu(g)} \int \frac{d^2p}{(2\pi)^2}\, e^{-ip(x_f - x_g)} C(p)\, \eta^{\mu(f)}(p) W^{\mu(f)}_{v_f, v_g} \tag{III.11}$$

where

$$\begin{aligned} W^k_{v,v'} &= 1 \quad \text{if } v \text{ and } v' \text{ are connected by } T_k \\ &= 0 \quad \text{otherwise} \end{aligned} \tag{III.12}$$

since we always have $D = C$ in the matrix $\mathcal{M}(\mu)$.

From (III.10) we see that there are three types of divergent subgraphs:

- for $|E_i^k| = 4$, $|N_i^{'k}| = 0$ we have logarithmic divergence ($\omega(G_i^k) = 0$);
- for $|E_i^k| = 2$, $|N_i^{'k}| = 0$ we have linear divergence ($\omega(G_i^k) = -1$);
- for $|E_i^k| = 2$, $|N_i^{'k}| = 1$ we have logarithmic divergence ($\omega(G_i^k) = 0$).

In fact the divergent graphs are only those for which the algebraic structure of the external legs is of one of the three types in (II.1). For instance not all four-point subgraphs are divergent, but only those for which the flow of spin indices follows the flow of color indices [GK][FMRS]. Using the invariance of $\mathcal{L}$ under parity and charge conjugation one finds that all counterterms which are not of the three types in (II.1) are zero (this means that the corresponding subgraphs have 0 local part). Then renormalizing these subgraphs we improve power counting without generating new counterterms. In what follows, for simplicity, "divergent subgraph" always means subgraph with two or four external legs (this means we will renormalize some subgraph which does not need it but this does not affect the convergence of the series). Also for simplicity we change the definition of convergence degree (III.10) in

$$\omega'(G_i^k) = \frac{1}{2}(|E_i^k| - 4) \ . \tag{III.13}$$

To cure divergences, we apply to the amplitude of each divergent subgraph g the operator $(1-\tau_g)+\tau_g$. In the momentum space τ_g is the Taylor expansion at order $-\omega(g)$ of the amplitude $\hat{g}(p)$ at $p=0$. The operator $1-\tau_g$ makes the amplitude convergent when the UV cut-off is sent to infinity. The remaining term $\tau_g\hat{g}$ gives a local counterterm for the coupling constant that depends on the energy of the external lines of g. At each vertex v, we can resum the series of all counterterms obtained applying τ_g to all divergent subgraphs (for different attributions μ) that have the same set of external lines as v itself. In this way we obtain an effective coupling constant which depends on the energy $\Lambda_0(w_{i_v})$ of the highest tree line hooked to the vertex v. This is true because *after applying the* $1-\tau_g$ *operators*, for each graph with nonzero amplitude the highest index at each vertex coincides with the highest *tree index* i_v at each vertex! Indeed at vertices v for which this is not true, there are loop fields with attribution μ higher than i_v. By (III.12) they must contract together forming tadpoles, which are set to zero by the $1-\tau_g$ operators. The corresponding graphs therefore disappear from the expansion.

For each attribution μ we define the set of divergent subgraphs as

$$D_\mu := \{\ G_i^k | \omega'(G_i^k) \leq 0\}. \tag{III.14}$$

The action of τ_g is

$$\tau_g\hat{g}(p_1,\ldots,p_k) = \sum_{j=0}^{-\omega'(g)} \frac{1}{j!}\frac{d^j}{dt^j}\hat{g}(tp_1,\ldots tp_k)_{|p=0} \quad k=2,4. \tag{III.15}$$

With this definition the effective constants λ_w, δm_w, $\delta\zeta_w$ turn out to be the vertex functions Γ_4, Γ_2 and $\partial\!\!\!/\Gamma_2$ for an effective theory with infrared parameter $\Lambda = \Lambda_0(w)$:

III.3 Effective constants

In the space of positions, the operator τ_g is computed by partial integration on the product of external propagators $a(x_1, \dots x_{v_e})$, as in [R]:

$$\tau^*_g(v_e) a(x_1, \dots x_{v_e}) = \sum_{j=0}^{-\omega'(g)} \frac{1}{j!} \frac{d^j}{dt^j} a(x_1(t), \dots, x_{v_e}(t))_{|t=0} \tag{III.16}$$

where $x_i(t) = x_{v_e} + t(x_i - x_{v_e})$, and v_e is an external vertex of g chosen as 'reference vertex'. This formula means that τ^*_g for each divergent tree subgraph g moves all external half-lines to a single reference vertex in the subgraph, hence computes a local couterterm. The choice of this reference vertex is given in Section IV.3.1. As announced we find the three possible counterterms of (II.1). For $|E_i| = 4$ we have

$$\tau^*(x_1) \prod_{i=1}^{4} C(x_i, y_i) = \prod_{i=1}^{4} C(x_1, y_i), \tag{III.17}$$

so the counterterm is

$$\begin{aligned} &\int d^2x_1 \prod_{i=1}^{4} C^{a_i a_i}_{\alpha_i \alpha'_i}(x_1, y_i) \int d^2x_2 d^2x_3 d^2x_4 \; g(0, x_2, x_3, x_4)^{a_1 a_2 a_3 a_4}_{\alpha_1 \alpha_2 \alpha_3 \alpha_4} \\ &= \int d^2x_1 \prod_{i=1}^{4} C^{a_i a_i}_{\alpha_i \alpha'_i}(x_1, y_i) \; \hat{g}(0, \dots, 0) \end{aligned} \tag{III.18}$$

This gives a coupling constant counterterm. For $|E_i| = 2$ we have

$$\tau^*(x_1)[C(x_1, y_1) C(x_2, y_2)] = C(x_1, y_1) \left[C(x_1, y_2) + (x_2 - x_1)^\mu \frac{\partial}{\partial x_1^\mu} C(x_1, y_2) \right]. \tag{III.19}$$

Integrating over internal points, we obtain a mass counterterm from the first term in the sum:

$$\begin{aligned} &\int d^2x_1 \prod_{i=1}^{2} C^{a_i a_i}_{\alpha_i \alpha'_i}(x_1, y_i) \int d^2x_2 \; g^{a_1 a_2}_{\alpha_1 \alpha_2}(0, x_2) \\ &= \int d^2x_1 \prod_{i=1}^{2} C^{a_i a_i}_{\alpha_i \alpha'_i}(x_1, y_i) \; \hat{g}_{\alpha_1 \alpha_2}(0) \delta_{a_1, a_2} \\ &= \int d^2x_1 \prod_{i=1}^{2} C^{a_i a_i}_{\alpha_i \alpha'_i}(x_1, y_i) \; \delta_{a_1, a_2} \; f_1(0) \end{aligned} \tag{III.20}$$

where we applied the development

$$\hat{g}(p) = f_1(p^2) + \gamma_5 f_2(p^2) + \not{p} f_3(p^2) + \gamma_5 \not{p} f_4(p^2) \tag{III.21}$$

and we adopted for the gamma matrices the conventions in [FMRS]. By invariance under charge conjugation and parity $f_2(0) = f_4(0) = 0$. For the second term in the sum we obtain a wave function counterterm:

$$\begin{aligned}
&\int d^2x_1 C^{a_1 a_1}_{\alpha_1 \alpha_1'}(x_1, y_1) \frac{\partial}{\partial x_1^\mu} C^{a_2 a_2}_{\alpha_2 \alpha_2'}(x_1, y_2) \int d^2x_2 (x_2 - x_1)^\mu g^{a_1 a_2}_{\alpha_1 \alpha_2}(x_1 - x_2) = \\
&= \int d^2x_1 C^{a_1 a_1}_{\alpha_1 \alpha_1'}(x_1, y_1) \frac{\partial}{\partial x_1^\mu} C^{a_2 a_2}_{\alpha_2 \alpha_2'}(x_1, y_2) i \frac{\partial}{\partial p^\mu} \hat{g}(p)_{|p=0} \delta_{a_1 a_2} \\
&= \int d^2x_1 C^{a_1 a_1}_{\alpha_1 \alpha_1'}(x_1, y_1) i \not{\partial} C^{a_1 a_1}_{\alpha_2 \alpha_2'}(x_1, y_2) f_3(0).
\end{aligned} \tag{III.22}$$

Theorem 2 *If we apply to each divergent subgraph $g \in D_\mu$, for any attribution μ, the operator $(1 - \tau_g) + \tau_g = R_g + \tau_g$, the function (II.25) can be written as*

$$\begin{aligned}
\Gamma^{\Lambda\Lambda_0}_{2p}(\phi_1, \dots \phi_{2p}) = &\sum_{n,n',n''=0}^{\infty} \frac{1}{n!n'!n''!} \sum_{o-\mathcal{T}} \sum_{E,\mu} \sum_{\mathcal{C}ol,\Omega} \epsilon(\mathcal{T}, \Omega) \int d^2x_1 \dots d^2x_{\bar{n}} \\
&\int_{0 \le w_1 \le \dots \le w_{\bar{n}-1} \le 1} \prod_{q=1}^{\bar{n}-1} dw_q \left[\prod_v \left(\frac{\lambda_{w(v)}}{N} \right) \right] \left[\prod_{v'} \delta m_{w(v')} \right] \left[\prod_{v''} \delta\zeta_{w(v'')} \right] \\
&\prod_{G_i^k \in D_\mu} R_{G_i^k} \left[\prod_{q=1}^{\bar{n}-1} D^{\Lambda_0, w_q}_{\Lambda}(\bar{x}_{l_q}, x_{l_q}) \det \mathcal{M}(\mu) \phi_1(x_{i_1}) \dots \phi_{2p}(x_{j_p}) \right]
\end{aligned} \tag{III.23}$$

where the constants λ_w, δm_w, $\delta\zeta_w$ are the 'effective constants', defined as:

$$\begin{aligned}
\frac{\lambda_w}{N} &= \hat{\Gamma}^{\Lambda_0(w),\Lambda_0}_4(0,0,0,0) = \int d^2x_2 d^2x_3 d^2x_4 \; \Gamma^{\Lambda_0(w),\Lambda_0}_4(0, x_2, x_3, x_4) \\
\delta m_w &= \hat{\Gamma}^{\Lambda_0(w),\Lambda_0}_2(0,0) = \int d^2x_2 \; \Gamma^{\Lambda_0(w),\Lambda_0}_2(0, x_2) \\
\delta\zeta_w &= \not{\partial}\hat{\Gamma}^{\Lambda_0(w),\Lambda_0}_2(p)_{|p=0} = \int d^2x_2 \; i \not{x}_2 \Gamma^{\Lambda_0(w),\Lambda_0}_2(0, x_2)
\end{aligned} \tag{III.24}$$

The effective constants are the vertex functions Γ_4, Γ_2 and $\not{\partial}\Gamma_2$ for an effective theory with infrared parameter $\Lambda_0(w)$, and the renormalized constants correspond to the effective ones at the energy Λ. (For the massive theory recall that we can use $\Lambda = 0$.)

$$\begin{aligned}
\lambda_{w=0} &= \lambda_r \\
\delta m_{w=0} &= \delta m_r = 0 \\
\delta\zeta_{w=0} &= \delta\zeta_r = 0
\end{aligned} \tag{III.25}$$

The reshuffling of perturbation theory performed by Theorem II can be proved by standard combinatorial arguments as in [R] (the only difficulty was discussed above, when we remarked that the parameter w of the effective constants always corresponds to the highest tree line of the vertex. Otherwise the effective vertex generates a tadpole graph whose later renormalization gives 0). This reshuffling is similar to the reorganization of renormalized perturbation theory according to the formalism of Gallavotti and coworkers [GN].

IV Convergence of the series

Theorem 3 *Let $\epsilon > 0$ be fixed. Suppose Λ^m (defined below) belongs to some fixed compact X of $]0,+\infty)$. The series (III.23) is absolutely convergent for $|\lambda_w|$, $|\delta m_w|$, $|\delta\zeta_w| \leq c$, c small enough. This convergence is uniform in Λ_0 and N (actually Γ_{2p} is proportional to N^{1-p}). The ultraviolet limit $\Gamma^{\Lambda}_{2p} = \lim_{\Lambda_0\to\infty} \Gamma^{\Lambda\Lambda_0}_{2p}$ exists and satisfies the bound:*

$$
|\Gamma^{\Lambda}_{2p}(\phi_1,\dots\phi_{2p})| \leq (p!)^{5/2}[K(c,\epsilon,X)]^p\ (\Lambda^m)^{2-p}\ N^{1-p} \qquad \text{(IV.1)}
$$
$$
||\phi_1||_1 \prod_{i=2}^{2p} ||\phi_i||_{\infty,2}\ e^{-(1-\epsilon)\Lambda^m d_T(\Omega_1,\dots\Omega_{2p})}
$$

where

$$
\Lambda^m := \sup\,[\Lambda, m]\ , \qquad \text{(IV.2)}
$$
$$
||\phi_i||_{\infty,2} := \left(||\phi_i||_\infty + ||\phi_i'||_\infty + ||\phi_i''||_\infty \right)\ , \qquad \text{(IV.3)}
$$

Ω_i is the compact support of ϕ_i, $K(c,\epsilon,X)$ is some function of c ϵ and X, which tends to zero when c tends to 0, $||\phi_i||_\infty = \sup_{x\in\Omega_i}|\phi_i(x)|$, $||\phi_1||_1 = \int d^2x|\phi_1(x)|$, and

$$
\begin{aligned}
d_T(\Omega_1,\dots\Omega_{2p}) &:= \inf_{x_i\in\Omega_i} d_T(x_1,\dots x_{2p}) \\
d_T(x_1,\dots x_{2p}) &:= \inf_{u-\mathcal{T}} \sum_{l\in\mathcal{T}} |\bar{x}_l - x_l|. \qquad \text{(IV.4)}
\end{aligned}
$$

where in the definition of $d_T(x_1,\dots x_{2p})$, called the "tree distance of $x_1,\dots x_{2p}$", the infimum over $u-\mathcal{T}$ is taken over all unordered trees (with any number of internal vertices) connecting $x_1,\dots x_{2p}$.

This bound means that one can construct in a non perturbative sense the ultraviolet limit of either the massive theory with any infrared cutoff Λ including $\Lambda = 0$, or the weakly coupled massless theory with nonzero infrared cutoff Λ. To complete Theorem 1 from Theorem 3, one needs only to check Borel summability by expanding explicitly at finite order n in λ_{ren} and controlling the Taylor remainder. This additional expansion generates a finite number of Taylor operators τ_g for a finite number of non quasi-local subgraphs, which are responsible for the $n!$ of Borel summability [R]. Since this is rather standard we will not include this additional argument here. Finally the renormalization group equations are discussed in Section V. The rest of the section is devoted to the proof of this theorem.

IV.1 Plan of the proof

To prove the theorem we show that the absolute value of the term (n, n', n'') in the sum (excluding the effective constants) is bounded by $K^{\bar{n}}$. The strategy for the proof consists in moving the absolute value inside all sums and integrals, bounding the product of effective constants,

$$\left[\prod_{v} |\lambda_{w(v)}|\right] \left[\prod_{v'} |\delta m_{w(v')}|\right] \left[\prod_{v''} |\delta\zeta_{w(v'')}|\right] \le c^{\bar{n}} \ , \tag{IV.1}$$

then taking $c < K^{-1}$.

The loop determinant will be bounded by a Gram inequality, and we shall use the tree lines decay to bound the spatial integrals. Actually, we cannot move the absolute value directly inside the sum over attributions because $\#\{\mu\} \simeq \bar{n}!$. In other words fixing the band index for each single half-line develops too much the determinant. The way to overcome this difficulty is to remark that the attributions contain much more information than necessary. We can in fact group the attributions into packets to reduce the number of determinants to bound. We observe that, if for the level i a connected component G_i^k has $|EE_i^k| + |ET_i^k| \ge 5$, the subgraph is convergent and we do not need to know the band indices for the loop lines in that connected component. So for each convergent G_i^k:

- if $|EE_i^k| + |ET_i^k| \ge 5$, we do not want to know anything on loop lines;
- if $|EE_i^k| + |ET_i^k| < 5$, we just want to fix $5 - |EE_i^k| - |ET_i^k|$ half-lines with energy lower than i, but we are not interested in knowing the band index ($\simeq$ energy) of the other half-lines because we need to know whether a subgraph is convergent or not (i.e. has more than 4 external half lines or not), but we do not care about its exact number of external half lines; whether it has 10 or 25 does not matter, and it is in fact precisely the extraction of this information that could be dangerous for convergence of the expansion.

Instead of expanding the loop determinant over lines and columns as a sum over all attributions

$$\det \mathcal{M} = \sum_{\mu} \det \mathcal{M}(\mu, E) \tag{IV.2}$$

we write it as a sum over a smaller set $\mathcal{P}$ (called the set of packets). These packets are defined by means of the function

$$\begin{array}{rcl} \phi : \{\mu\} & \longrightarrow & \mathcal{P} \\ \mu & \mapsto & \mathcal{C} = \phi(\mu) \end{array} \tag{IV.3}$$

but this function must respect some constraints related to the future use of Gram's inequality. This motivates the following definition:

Definition 1 *The pair $(\mathcal{P}, \phi)$ is called a "Gram-compatible pair" if*

$$\forall\, \mathcal{C} \in \mathcal{P}, \forall a,\ \exists\, J_a(\mathcal{C}) \subset B \tag{IV.4}$$

with the property $\phi^{-1}(\mathcal{C}) = \{\mu | \mu(a) \in J_a(\mathcal{C})\ \forall\, a\}$.

This definition means that for any packet $\mathcal{C}$ the attributions in the packet exactly correspond to a particular set of band indices allowed for each loop line. It ensures that there exists a matrix $\mathcal{M}'$ such that

$$\sum_{\mu \in \phi^{-1}(\mathcal{C})} \det \mathcal{M}(\mu) = \det \mathcal{M}'(\mathcal{C}) \tag{IV.5}$$

because each loop line a is a matrix entry. This in turn ensures that Gram's inequality can be applied to $\det \mathcal{M}'(\mathcal{C})$, as shown in Lemma 4.

IV.2 Construction of P

We build first the *partition* $\mathcal{P}$ of the set of attributions into packets. These packets should contain the informations we need over $|E^j_k|$. In contrast with attributions there should be few of them; more precisely they should satisfy $\#\mathcal{P} \leq K^{\bar{n}}$. Finally, together with the function ϕ, they should form a Gram-compatible pair. To define $\mathcal{P}$ we introduce some preliminary definitions and notations.

To each ordered tree $o-\mathcal{T}$ we can associate a rooted tree $R_\mathcal{T}$, which pictures the inclusion relation of the G^j_k [R]. We can picture this tree with two types of vertices: crosses and dots. We recall that the leaves of a rooted tree are the vertices of the rooted tree with coordination number one. The leaves in our case are the dot-vertices and correspond exactly to the vertices v, v' or v'' of the initial ordered tree $\mathcal{T}$. The other vertices of $R_\mathcal{T}$ are crosses. Each cross i corresponds to a line l_i of the initial ordered tree $\mathcal{T}$, and has coordination number three, except the root which has coordination number two. To build $R_\mathcal{T}$ we take the lowest line in $\mathcal{T}$, l_1, as root 1.

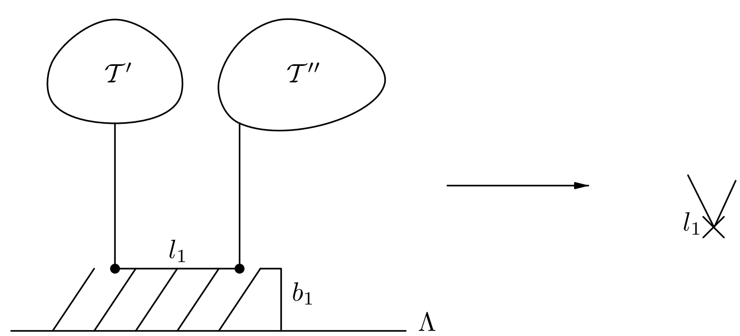


Figure 3

This line l_1, or root, separates $\mathcal{T}$ into two connected components $\mathcal{T}'$ and $\mathcal{T}''$ possibly reduced to a single vertex. When $\mathcal{T}'$ or $\mathcal{T}''$ is a single vertex, it gives a dot connected to 1. Otherwise it gives a cross, which is the lowest line of $\mathcal{T}$ in it. This procedure is repeated at each cross-vertex obtained, and generates $R_{\mathcal{T}}$.

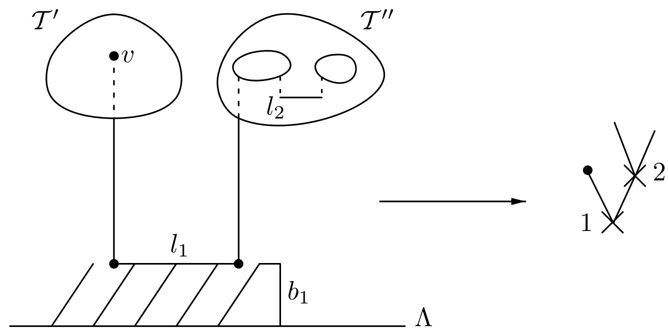


Figure 4

Finally to complete the picture to each dot of $R_{\mathcal{T}}$ we hook all loop half-lines hooked to the corresponding vertex (there could be none). We define the *ancestor of* i $\mathcal{A}(i)$ as the cross-vertex just under i in $R_{\mathcal{T}}$ and we call v_a, the dot-vertex to which the half-line a is hooked and i_a the cross-vertex connected to v_a (which represents a line of the initial tree!). For each cross-vertex i we define

$$t_i := \{l_j \in \mathcal{T} | j \geq i, l_j \text{ connected to } l_i \text{ by } T_i\} \tag{IV.6}$$

This is the spanning tree in the connected component of G_i containing the line l_i.

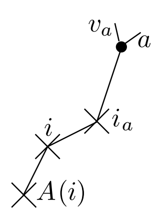


Figure 5

An example of a tree with its associated $R_{\mathcal{T}}$ is given in Figure 6:

For each tree line (cross-vertex) i and each connected component G_i^k, no new line connects to t_i in the interval between i and $\mathcal{A}(i)$. Hence $\omega'(G_{i'}^{k'}) \geq \omega'(G_{\mathcal{A}(i)+1}^{k''})$ $\forall i \geq i' > \mathcal{A}(i)$ and $T_{i'}^{k'} \subset G_{i'}^{k'} \subset G_i^k$. Therefore we can neglect what happens in this interval and generalize the definitions of (III.9) for the internal lines of a subgraph G_i^k.

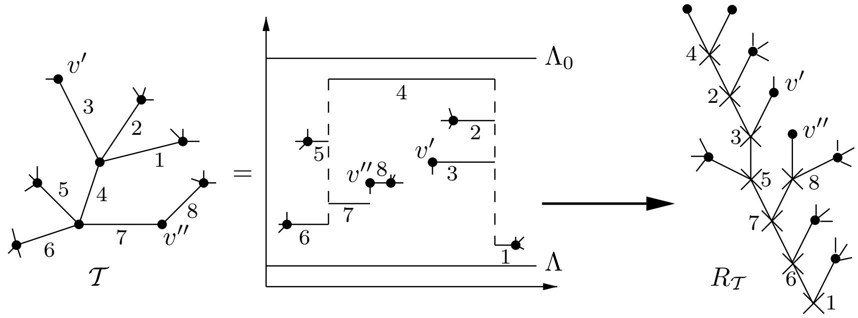


Figure 6

We define:

$$
\begin{aligned}
g_i &:= t_i \cup \{a \in L | i_a \geq i, v_a \in t_i, \mu(a) \geq \mathcal{A}(i) + 1\} \\
eg_i &:= et_i \cup ee_i \cup el_i \\
et_i &:= \{f_{i'} | i_{v_f} \geq i, v_f \in t_i, i' < i\} \cup \{g_{i'} | i_{v_g} \geq i, v_g \in t_i, i' < i\} \\
ee_i &:= \{f, g \in E | i_{v_f}, i_{v_g} \geq i, v_f, v_g \in t_i\} \\
el_i &:= \{a \in L | i_a \geq i, v_a \in t_i, \mu(a) \leq \mathcal{A}(i)\}
\end{aligned}
\tag{IV.7}
$$

This set of definitions (IV.7) concerns the connected component g_i above line i. Remark that we defined as loop internal lines of g_i, all loop lines higher than $\mathcal{A}(i)$. We also need some additional definitions concerning the other connected components:

$$
\begin{aligned}
i(k) &:= \inf_{\{j \geq i, v_j \in T_i^k\}} j \\
g_i^k &:= g_{i(k)} \\
eg_i^k &:= et_i^k \cup ee_i^k \cup el_i^k \\
et_i^k &:= et_{i(k)} \quad ee_i^k := ee_{i(k)} \quad el_i^k := el_{i(k)}
\end{aligned}
\tag{IV.8}
$$

This second set of definitions is used only much later in the bounds when all connected components are considered at once.

Definition 2 *A <u>chain</u> $C_{a,i}$ is the unique path in $R_{\mathcal{T}}$ from the half-line a to the cross-vertex i with $i_a \geq_T i$:*

$$
C_{ai} := \{i' | i \leq_T i' \leq_T i_a\} \cup \{a\} \tag{IV.9}
$$

In the following, we write $i_a \geq_T i$ to specify that v_a and v_i are connected by t_i.

Definition 3 *A <u>class</u> $\mathcal{C}$ is a set of chains over $R_{\mathcal{T}}$ with the properties:*

$$
\begin{aligned}
&\forall C_{ai} \in \mathcal{C},\ \forall C_{a'i'} \in \mathcal{C} \text{ one has } a \neq a' \\
&\forall\, i \quad c_i \leq \max[0; 5 - |ee_i| - |et_i| - c_i']
\end{aligned}
\tag{IV.10}
$$

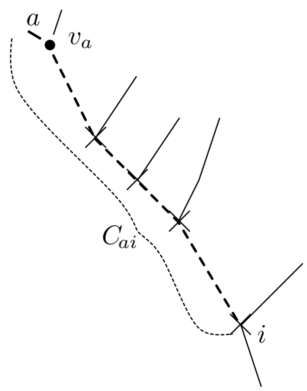


Figure 7

where we defined:

$$\begin{aligned} c_i &= \#\{C_{ai} \in \mathcal{C} |\ i \text{ fixed}\} \\ c'_i &= \#\{C_{a'i'} \in \mathcal{C} |\ i_{a'} \geq_T i,\ i' < i\} \end{aligned} \tag{IV.11}$$

So c_i is the total number of chains arriving at i and c'_i is the total number of chains passing through i and continuing further below. This definition ensures therefore that there are at most five chains passing through each cross i.

Definition 4 *The partition $\mathcal{P}$ is the set of all possible classes $\mathcal{C}$ over $R_{\mathcal{T}}$.*

To verify that this is a good definition, we have to prove three lemmas.

Lemma 1 *The cardinal of $\mathcal{P}$ is bounded by $K^{\bar{n}}$.*

Proof. We prove that $\mathcal{P} \subseteq \mathcal{P}'$ and $\#\mathcal{P}' \leq K^{\bar{n}}$. We define $\mathcal{P}'$ as the set of all sets of chains $\mathcal{D}$, that are unions of five subsets (possibly empty) Y_j, where Y_j is a set of completely disjoint chains (this means they have no cross and no dot in common).

$$\mathcal{P}' := \{\mathcal{D}\} \qquad \mathcal{D} := \cup_{j=1}^{5} Y_j. \tag{IV.12}$$

To build a set of disjoint chains Y_j, we have at most three possible choices for each vertex: at each cross-vertex we can have no chain passing, a chain going right or left; at each dot-vertex touched by a chain, we have to choose among three (at most) loop half-lines. Putting all this together we have:

$$\#\mathcal{P}' \leq (3^5)^{\bar{n}-1}(3^5)^{2n+2-2p} \leq K^{\bar{n}} \tag{IV.13}$$

where the number 5 comes because each element of $\mathcal{C}$ is made of five sets Y_j.

Figure 8 shows an example of disjoint sets built in this way.

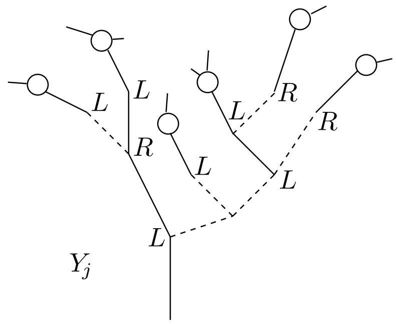


Figure 8

Now we prove that $\mathcal{P} \subseteq \mathcal{P}'$ by induction on i. For each $\mathcal{C} \in \mathcal{P}$ we define $\mathcal{C}(i)$ as the subset of $\mathcal{C}$ that contains only chains ending in some point (cross-vertex) of the unique path connecting i to the root.

$$\mathcal{C}(i) := \{C_{ai'} \in \mathcal{C}|\ i' \leq_T i\} \tag{IV.14}$$

This set satisfies the following induction law: if, for $\mathcal{A}(i)$ there are five sets (eventually empty) of disjoint chains $Y_1(\mathcal{A}(i))\ldots Y_5(\mathcal{A}(i))$ with

$$\mathcal{C}(\mathcal{A}(i)) = \cup_{j=1}^{5} Y_j(\mathcal{A}(i)), \tag{IV.15}$$

then there are five sets $Y_1(i),\ldots,Y_5(i)$ with $\mathcal{C}(i) = \cup_j Y_j(i)$. This can be seen observing that $\mathcal{C}(i)$ can be written as

$$\mathcal{C}(i) = \mathcal{C}(\mathcal{A}(i)) \cup \{C_{ai'} \in \mathcal{C}|i' = i\}. \tag{IV.16}$$

Among the five sets Y_j forming $\mathcal{C}(\mathcal{A}(i))$ there are c_i' ones containing chains passing through i: $Y_1(\mathcal{A}(i)),\ldots,\ Y_{c_i'}(\mathcal{A}(i))$. If $c_i' + |ee_i| + |et_i| \geq 5$, there are no chains ending at i so $\mathcal{C}(i) = \mathcal{C}(\mathcal{A}(i)) \subset \mathcal{P}'$. If $c_i' + |ee_i| + |et_i| < 5$ there are c_i chains ending at i $C_{a_1,i},\ldots C_{a_{c_i},i}$, with $c_i \leq 5 - c_i'$, so we can define

$$\begin{aligned} Y_j(i) &= Y_j(\mathcal{A}(i)) \quad \text{for } j \leq c_i', \\ Y_{c_i'+j}(i) &= Y_{c_i'+j}(\mathcal{A}(i)) \cup \{C_{a_j i}\} \quad j = 1,\ldots,c_i, \\ Y_j(i) &= Y_j(\mathcal{A}(i)) \quad \text{for } j > c_i' + c_i. \end{aligned} \tag{IV.17}$$

With these definitions we have

$$\mathcal{C}(i) = \cup_{j=1}^{5} Y_j \subset \mathcal{P}' \tag{IV.18}$$

Now, the hypothesis (IV.15) is true for the root r. In fact, by construction, we have at most five chains ending at r: $C_{a_1,r},\ldots C_{a_5,r}$. If we define:

$$Y_1(r) = \{C_{a_1 r}\},\ldots,Y_5(r) = \{C_{a_5 r}\} \tag{IV.19}$$

we have $\mathcal{C}(r) = Y_1 \cup \cdots \cup Y_5 \subset \mathcal{P}'$. Working the induction up to the leaves of $R_{\mathcal{T}}$ completes the proof of the lemma. □

Lemma 2 *There exists a function $\phi : \{\mu\} \longrightarrow \mathcal{P}$ which associates to each attribution $\mu = (\mu(1), \mu(2), \dots)$ a class $\mathcal{C}$ in $\mathcal{P}$.*

To define ϕ we fix an order over the half-lines and the lines of $R_{\mathcal{T}}$. We do it turning around $R_{\mathcal{T}}$ clockwise and we call $n(a)$ the index of a in the ordering and s_i the index of the line in $R_{\mathcal{T}}$ connecting i to $\mathcal{A}(i)$.

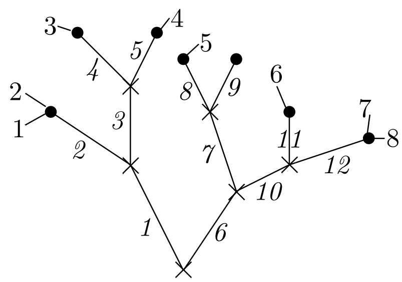


Figure 9

We build the class $\phi(\mu)$ as a union of chains by induction, defining first the chains in $\phi(\mu)$ ending at the root, then the ones ending at the cross connected to the root by the line 1, and so on, following the ordering s_i. Therefore for each i we consider the set $A_i = \{a \in el_i | \ \nexists C_{ai'} \in \phi(\mu) \text{ with } \ i' < i\}$ which is the set of loop half-lines that are external lines for g_i and are not connected to a chain in $\phi(\mu)$ ending lower than i.

- If $[5-|ee_i|-|et_i|-c'_i] > 0$ and $\#A_i < [5-|ee_i|-|et_i|-c'_i]$ we have a divergent subgraph, and we add to the part already built of $\phi(\mu)$ all the chains starting at an element of A_i and ending at i, so
$$c_i = \#A_i. \tag{IV.20}$$
- If $\#A_i \geq \max[0, 5 - |ee_i| - |et_i| - c'_i]$, we have a convergent subgraph, so we put
$$c_i = \max[0, 5 - |ee_i| - |et_i| - c'_i] \tag{IV.21}$$
and we add to the part already built of $\phi(\mu)$ the c_i chains $\mathcal{C}_{a',i}$, with $a' = a_i^j, \ j = 1 \dots, c_i$, which start at the c_i elements in A_i that have the lowest values of $n(a)$, and end at i.

In this way we obtain a set of chains with the two properties (IV.10). For each μ, $\phi(\mu)$ is an element of $\mathcal{P}$ and $\{\phi^{-1}(\mathcal{C})\}_{\mathcal{C}\in\mathcal{P}}$ is a partition of the set of attributions. □

We call B_i the set of half-lines in A_i which are the starting points of chains in $\phi(\mu)$ ending at i (see Figure 10). Therefore in the divergent case $B_i = A_i$ and in the convergent case $B_i = \{a_i^j, \ j = 1 \dots, c_i\}$. We also define

$$eg_i(\mathcal{C}) := et_i \cup ee_i \cup \{a | i_a \geq_T i \text{ and } a \in B_{i'} \text{ for some } i' \leq_T i\} \tag{IV.22}$$

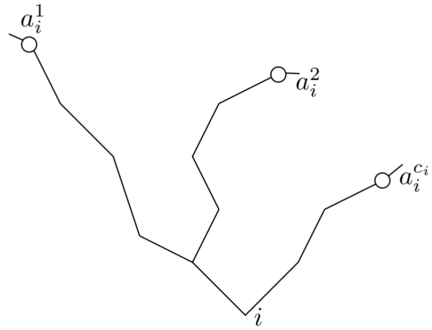


Figure 10

With this definition we have $|eg_i(\mathcal{C})| = c_i + c'_i + |et_i| + |ee_i|$. Remark that in the divergent case $|eg_i| \leq 4$, one has $|eg_i| = |eg_i(\mathcal{C})|$, and in the convergent case one has $|eg_i| \geq |eg_i(\mathcal{C})| \geq 5$. The next lemma describes the structure of the classes $\mathcal{C}$.

Lemma 3 *For each class $\mathcal{C} \in \mathcal{P}$ and each half-line $a = 1, \ldots, 2n+2-2p$ there exists a subset of band indices $J_a(\mathcal{C}) \subseteq B$ such that*

$$\phi^{-1}(\mathcal{C}) = \{\mu | \mu(a) \in J_a(\mathcal{C}) \; \forall \; a\}. \tag{IV.23}$$

Proof. The existence of the c_i chains C_{ai} for $a \in B_i$ ending at i implies a certain set of constraints on attributions. We distinguish two situations.

1) If $|eg_i(\mathcal{C})| \leq 4$ (divergent case)
- $\forall a \in B_i, \; \underline{\mu(a) \leq \mathcal{A}(i)}$;
- $\forall \; a \notin B_i$ with $i_a \geq_T i$, $\underline{\mu(a) > \mathcal{A}(i)}$.

2) If $|eg_i(\mathcal{C})| \geq 5$ (convergent case)
- $\forall a \in B_i, \; \underline{\mu(a) \leq \mathcal{A}(i)}$;
- $\forall a \notin B_i$ with $i_a \geq_T i$, and $n(a) < \max_{a' \in B_{i'}} n(a')$, $\underline{\mu(a) > \mathcal{A}(i)}$.

In any other case, there is no particular constraint. We observe that the underlined constraints for $\mu(a)$ are therefore determined by the chain structure and the ordering, but the crucial point is that they are independent from each other. Hence $J_a(\mathcal{C})$ is an interval in terms of band indices. Remark that if some chain in $\mathcal{C}$ starts from a, it must end at some unique i, called i'_a. In this case we define $M(a, \mathcal{C}) = \mathcal{A}(i'_a)$. Otherwise we define $M(a, \mathcal{C}) = i_a$. Moreover for each i' such that $a \notin B_{i'}$ we have two different lower bounds on $\mu(a)$, depending whether $g_{i'}$ is divergent or convergent. So the constraints in cases 1 and 2 simply mean $m(a, \mathcal{C}) \leq \mu(a) \leq M(a, \mathcal{C})$, where

$$M(a, \mathcal{C}) \quad = \quad \mathcal{A}(i'_a) \text{ if } a \in B_{i'_a} \quad , \quad M(a, \mathcal{C}) = i_a \text{ otherwise}$$

$$\begin{aligned} m(a,\mathcal{C}) &= \sup_{i'\in I(a,\mathcal{C})} [\mathcal{A}(i')+1] \\ m(a,\mathcal{C}) &= 1 \quad \text{if } I(a,\mathcal{C})=\emptyset \end{aligned} \tag{IV.24}$$

and

$$I(a,\mathcal{C}) := \{i' \,|i_a \geq_T i',\ a\notin B_{i'},\ \text{and, if } |eg_{i'}(\mathcal{C})|\geq 5,\ n(a) < \max_{a'\in B_{i'}} n(a')\} \tag{IV.25}$$

In summary the constraints are expressed by

$$\begin{aligned} \phi^{-1}(\mathcal{C}) &= \{\mu|\ \mu(a)\in J_a(\mathcal{C})\ \forall\, a\} \\ J_a(\mathcal{C}) &= [m(a,\mathcal{C}), M(a,\mathcal{C})] \end{aligned} \tag{IV.26}$$

□

Example. As the definition of the interval $J_a(\mathcal{C}) = [m(a,\mathcal{C}), M(a,\mathcal{C})]$ is certainly hard to grasp, let us give an example. In Figure 11 we pictured a class $\mathcal{C}$ made of two chains C_{a_1,i_1} and C_{a_2,i_2} with $a_1 \leq a_2$ in the clockwise ordering. The allowed interval for a_1 has maximum $M(a_1,\mathcal{C}) = M_1$, the cross just below i_1, since the presence of the chain forces the half line a_1 to be an external leg strictly below i_1. The minimum is $m(a_1,\mathcal{C}) = m_1$, the cross where the second chain ends. Indeed, since $a_2 \geq a_1$, the attribution for a_1 cannot go below m_1, otherwise a longer chain C_{a_1,j_1} with $j_1 \leq m_1$ would have been chosen earlier, lower in the tree.

Finally suppose a_3 is a loop line with index bigger than a_2 in the clockwise ordering, and suppose that the cross m_1 corresponds to a divergent subgraph G_1, for which the number of external legs is fixed. Then $m(a_3,\mathcal{C}) = m_1$, since the leg a_3 cannot go below m_1; this would add a forbidden external leg to the divergent subgraph G_1. We invite the reader to check his understanding on further examples.

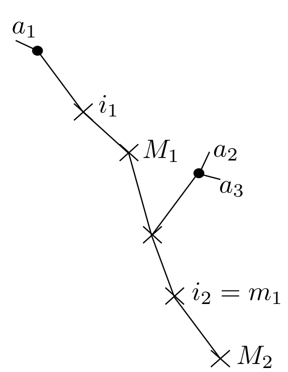


Figure 11

We observe that, after packing the attributions into classes, the sets T_i, t_i, ee_i, et_i are still well defined, but we can no longer define g_i and el_i. We already defined $eg_i(\mathcal{C})$ in (IV.22). We add further definitions

$$\begin{aligned} g_i(\mathcal{C}) &:= t_i \cup il_i(\mathcal{C}) \\ il_i(\mathcal{C}) &:= \{a \in L | i_a \geq_T i, M(a,\mathcal{C}) \geq \mathcal{A}(i)+1\} \\ el_i(\mathcal{C}) &:= \{a \in L | i_a \geq_T i, M(a,\mathcal{C}) \leq \mathcal{A}(i)\} \end{aligned} \tag{IV.27}$$

which generalize the notions of internal and external loop lines. Remark that $eg_i(\mathcal{C}) = et_i \cup ee_i \cup el_i(\mathcal{C})$, and $|el_i(\mathcal{C})| = c_i + c'_i$. In the same way we extend these definitions to the other connected components

$$g_i^k(\mathcal{C}) := g_{i(k)}(\mathcal{C}) \ , \ il_i^k(\mathcal{C}) := il_{i(k)}(\mathcal{C}) \ , \ el_i^k(\mathcal{C}) := el_{i(k)}(\mathcal{C}) \tag{IV.28}$$

Furthermore the generalized definitions for the convergence degree and the set of divergent subgraphs after packing the attributions into classes become:

$$\begin{aligned} \omega(g_i(\mathcal{C})) &:= (|eg_i(\mathcal{C})| - 4)/2. \\ D_{\mathcal{C}} &:= \{g_i(\mathcal{C}) \mid \omega(g_i(\mathcal{C})) \leq 0\}. \end{aligned} \tag{IV.29}$$

We return now to the loop determinant in (III.23). Lemma 3 ensures that

$$\sum_{\mu \in \phi^{-1}(\mathcal{C})} det\mathcal{M}(\mu) = det\mathcal{M}'(\mathcal{C}) \tag{IV.30}$$

and that for each loop half-line a there exists a characteristic function

$$\chi_a(\mathcal{C}) : k \in B \to \{0,1\} \quad \chi_a^k(\mathcal{C}) = \begin{cases} 0 & \text{if } k \notin J_a(\mathcal{C}) \\ 1 & \text{if } k \in J_a(\mathcal{C}) \ . \end{cases} \tag{IV.31}$$

Therefore the matrix elements for $\mathcal{M}'(\mathcal{C})$ can be written

$$\begin{aligned} \mathcal{M}'_{fg}(x_f,x_g) &= \int \frac{d^2p}{(2\pi)^2}\, e^{-ip(x_f-x_g)} C(p) \sum_{k\in B} \chi^k_{a(f)} \chi^k_{a(g)} \eta^k(p) W^k_{v_f,v_g} \\ &= \int \frac{d^2p}{(2\pi)^2}\, F_f^*(p) G_g(p) \sum_k \chi^k_{a(f)} \chi^k_{a(g)} \eta^k(p) W^k_{v_f,v_g} \end{aligned}$$

where we omit for simplicity to write the dependence in $\mathcal{C}$, and we defined:

$$F_f(p) = e^{ix_f p} \frac{1}{(p^2+m^2)^{\frac{1}{4}}} \qquad G_g(p) = e^{ix_g p} \frac{(-\not p + m)}{(p^2+m^2)^{\frac{3}{4}}}. \tag{IV.32}$$

v_a is the vertex to which the half-line a is hooked and η^k is the cutoff restricted to the band k (see equation (III.3)). Finally W^k is the $\bar{n} \times \bar{n}$ matrix defined in equation (III.12). Our next lemma is crucial since it bounds the loop determinant without generating any factorial.

Lemma 4 *The matrix $\mathcal{M}'(\mathcal{C})$ satisfies the following Gram inequality:*

$$|\det \mathcal{M}'(\mathcal{C})| \leq \prod_f \left[\int \frac{d^2p}{(2\pi)^2}\ \eta^f_{\mathcal{C}}(p) |F_f(p)|^2 \right]^{\frac{1}{2}} \prod_g \left[\int \frac{d^2p}{(2\pi)^2}\ \eta^g_{\mathcal{C}}(p) |G_g(p)|^2 \right]^{\frac{1}{2}} \tag{IV.33}$$

where the cutoff functions $\eta^f_{\mathcal{C}}(p)$ and $\eta^g_{\mathcal{C}}(p)$ corresponding to fields f and g are defined in equation (IV.44) below.

Proof. The Gram inequality states:
If M is a $n \times n$ matrix with elements $M_{ij} =< f_i, g_j >$ and f_i, g_j are vectors in a Hilbert space, we have $|\det M| \leq \prod_{i=1}^n ||f_i||\ \prod_{j=1}^n ||g_j||$.

To apply Gram's inequality, the matrix elements must be written as scalar products. We introduce the $q \times q$ matrix 1_q which is not the identity, but the matrix with all coefficients equal to 1. It is obviously a non-negative symmetric matrix. We observe that the matrix $W^k_{v,v'}$ is block diagonal with diagonal blocks of type 1_{q_j}, and $\sum q_j = \bar{n}$. Each block corresponds to all the vertices in a given connected component of T_k. Therefore W itself is non-negative symmetric. We can define the symmetric matrix $(2n+2-2p) \times (2n+2-2p)$:

$$R^k_{ab} := \chi^k_a \chi^k_b \tag{IV.34}$$

where a and b are the indices for the loop half-lines. By a permutation of field indices, we can list first the q half-lines for which $\chi^k_a(\mathcal{C}) = 1$. In this way the matrix R becomes 2×2 block diagonal non-negative of the type

$$\begin{pmatrix} 1_q & 0 \\ 0 & 0 \end{pmatrix}. \tag{IV.35}$$

Now we can group W and R in a unique matrix (tensor product)

$$\mathcal{W}^k_{v,a;v',b} := \chi^k_a \chi^k_b W^k_{v,v'} \tag{IV.36}$$

that is still non-negative as we can diagonalize separately W and R. Hence the matrix

$$\sum_k \eta^k \mathcal{W}^k_{v,a;v',b} = T_{v,a;v',b} \tag{IV.37}$$

is non-negative symmetric, as it is a linear combination (with positive coefficients η^k) of non-negative symmetric matrices; therefore we can take its square root (which is also non-negative symmetric):

$$T_{v,a;v',b} = \sum_{w,c} U_{v,a;w,c} U_{w,c;v',b}. \tag{IV.38}$$

Now, we can write $\mathcal{M}'_{fg}$ as

$$\begin{aligned}\mathcal{M}'_{fg} &= \int \frac{d^2p}{(2\pi)^2}\; F^*_f(p) G_g(p) T_{v_f,a(f);v_g,a(g)} \\ &= \int \frac{d^2p}{(2\pi)^2}\; F^*_f(p) G_g(p) \sum_{v's} U_{v_f,a(f);v',s} U_{v',s;v_g,a(g)}\end{aligned} \tag{IV.39}$$

If we introduce the vectors

$$\mathcal{F}^f_{v's}(p) = F_f(p) U_{v',s;v(f),a(f)} \qquad \mathcal{G}^g_{v's}(p) = G_g(p) U_{v',s;v(g),a(g)} \tag{IV.40}$$

we can write $\mathcal{M}'_{fg}$ as

$$\mathcal{M}'_{fg} = \int \frac{d^2p}{(2\pi)^2}\; \sum_{v',s} \mathcal{F}^{f*}_{v's}\;\; \mathcal{G}^g_{v's}\;\; =< \vec{\mathcal{F}}^f, \vec{\mathcal{G}}^g > . \tag{IV.41}$$

Now we can apply Gram's inequality:

$$|\det \mathcal{M}'_{fg}| \le \prod_{f=1}^{n+1-p} ||\vec{\mathcal{F}}^f|| \prod_{g=1}^{n+1-p} ||\vec{\mathcal{G}}^g|| \tag{IV.42}$$

where

$$\begin{aligned}&||\vec{\mathcal{F}}^f||^2 = \int \frac{d^2p}{(2\pi)^2}\; \sum_{v',s} (\mathcal{F}^f_{v's})^t (\mathcal{F}^f_{v's}) \\ &= \int \frac{d^2p}{(2\pi)^2}\; \sum_{v's} U_{v(f),a(f);v',s} U_{v',s;v(f),a(f)} |F_f|^2 \\ &= \int \frac{d^2p}{(2\pi)^2}\; T_{v(f),a(f);v(f),a(f)} |F_f|^2 = \int \frac{d^2p}{(2\pi)^2} \sum_k \chi^k_{a(f)} \chi^k_{a(f)} W^k_{v(f),v(f)} \eta^k |F_f|^2 \\ &= \int \frac{d^2p}{(2\pi)^2}\; (\sum_k \eta^k(p) \chi^k_{a(f)}) |F_f(p)|^2 = \int \frac{d^2p}{(2\pi)^2}\; \eta^{a(f)}(p)\; |F_f(p)|^2\end{aligned} \tag{IV.43}$$

as $(\chi^k_{a(f)})^2 = \chi^k_{a(f)}$, and, as the bands in χ_a are adjacents, the cut-offs sum up (using equations (III.2–III.3) to give

$$\eta^a_{\mathcal{C}}(p) := \left[\eta \left(\frac{p^2+m^2}{\Lambda_0(w_{M(a,\mathcal{C})})} \right) - \eta \left(\frac{p^2+m^2}{\Lambda_0(w_{m(a,\mathcal{C})-1})} \right) \right] \tag{IV.44}$$

We can treat in the same way G and this achieves the proof of (IV.33). $\square$

IV.3 Bound on the series

We are now in the position to bound the series (III.23). After packing the attributions into packets we can put the absolute value inside the integrals and the sums and bound the product of effective constants by $c^{\bar{n}}$. Moreover, we observe that the two sums $\sum_{\mathcal{C}ol,\Omega}$ in (III.23) are bounded by taking the supremum over $\mathcal{C}ol$ and Ω and multiplying by the number of elements. We have

$$\begin{aligned} &\#\{\Omega\} \leq 2^{2n+n'+n''-p} < 4^{\bar{n}} 2^{-p} \\ &\#\{\mathcal{C}ol\} \leq N^{n+1-p} \end{aligned} \tag{IV.45}$$

Indeed to estimate $\#\{\mathcal{C}ol\}$ remark that, once $\mathcal{T}$ and Ω are known, the circulation of color indices is determined. If there are no external color indices fixed (vacuum graph), the attribution of color indices costs N^2 at the first four-point vertex (taken as root) and climbing inductively into the tree layer by layer a factor N for each of the remaining four-point vertices of the tree (see [AR2]). The two-point vertices do not contribute as color is conserved at them. When we have fixed the p independent external colors for the $2p$ external fields only N^{n+1-p} choices remain.

We introduce some notations. Recalling the definitions (IV.27) and (IV.29) we say that a divergent subgraph $g_i(\mathcal{C}) \in D_{\mathcal{C}}$ is 'D1PR' ('dangerous one particle reducible') if, by cutting a single tree line, we can cut it into two subgraphs $g_j(\mathcal{C})$ and $g_{j'}(\mathcal{C})$, one of them, say $g_j(\mathcal{C})$, being a two legged subgraph. The line to cut is then the tree line $l_{\mathcal{A}(j)}$. In Figure 12 we show some examples of D1PR subgraphs, where tree lines are solid lines and loop half-lines are wavy.

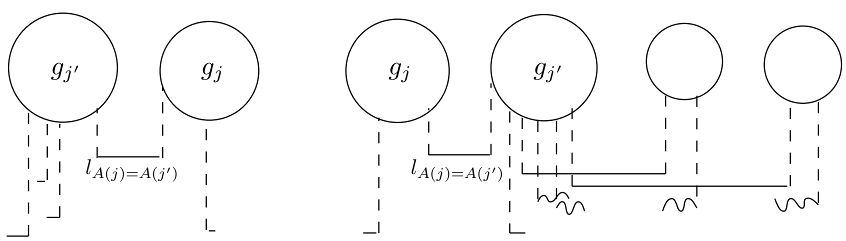


Figure 12

All subgraphs that cannot be cut in this way are called D1PI ('dangerous one particle irreducible'). We say that a four-point D1PI subgraph $g_i(\mathcal{C})$ is 'open' (as in [R]) if there exists a two-point subgraph $g_j(\mathcal{C}) \in D_{\mathcal{C}}$ (called its closure) with $j \leq_T i$ (then $g_i(\mathcal{C}) \subset g_j(\mathcal{C})$) and they have two external lines in common (see Figure 13).

A four-point subgraph is called 'closed' if it is D1PI but not open. A two-point D1PI subgraph is always closed by definition. This classification of subgraphs is useful, as only closed subgraphs contribute in the product $\prod_{g \in D_{\mathcal{C}}}(1-\tau^*_g)$. Applying the definition of τ_g in the momentum space one can see that:

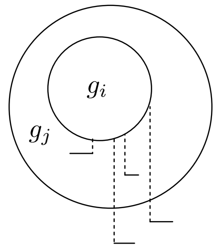


Figure 13

- if $g_i(\mathcal{C})$ is D1PR and $g_j(\mathcal{C})$ is the corresponding divergent subgraph, then

$$\tau_{g_i(\mathcal{C})}(1-\tau_{g_j(\mathcal{C})}) = 0 \tag{IV.46}$$

so the renormalization of $g_i(\mathcal{C})$ is ensured by that of $g_j(\mathcal{C})$;
- if $g_i(\mathcal{C})$ is four-point and open, and $g_j(\mathcal{C})$ is the associated two-point subgraph containing it, then

$$(1-\tau_{g_j(\mathcal{C})})(\tau_{g_i(\mathcal{C})}) = 0. \tag{IV.47}$$

For any $g_i(\mathcal{C}) \in D_{\mathcal{C}}$ we know exactly which loop half-lines are external lines, therefore we can still apply the operator $1-\tau^*_g = R^*_g$ to the external propagators, and distinguish closed subgraphs. Hence we define

$$D^c_{\mathcal{C}} := \{g_i(\mathcal{C}) \in D_{\mathcal{C}} | g_i(\mathcal{C}) \text{ closed}\} \tag{IV.48}$$

and we apply R^*_g only to $g \in D^c_{\mathcal{C}}$. By the relation of partial order in $R_{\mathcal{T}}$ we see that for each pair $g_i(\mathcal{C})$,$g_{i'}(\mathcal{C}) \in D^c_{\mathcal{C}}$ we can only have that $g_i(\mathcal{C}) \cap g_{i'}(\mathcal{C}) = \emptyset$, or $g_i(\mathcal{C}) \subseteq g_{i'}(\mathcal{C})$ (if $i' \leq i$). Hence $D^c_{\mathcal{C}}$ has a forest structure. Following [R] we define the 'ancestor' of $g_i(\mathcal{C}) \in D^c_{\mathcal{C}}$, called $B(g_i(\mathcal{C}))$, as the smallest subgraph in $D^c_{\mathcal{C}}$ containing $g_i(\mathcal{C})$:

$$B(g_i(\mathcal{C})) := g_{i'}(\mathcal{C}), \quad i' = \max_{g_{i''}(\mathcal{C}) \in D^c_{\mathcal{C}},\ g_i(\mathcal{C}) \subseteq g_{i''}(\mathcal{C})} i''. \tag{IV.49}$$

With all these bounds and definitions, the sum (III.23) becomes:

$$|\Gamma^{\Lambda\Lambda_0}_{2p}(\phi_1,\ldots,\phi_{2p})| \leq \sum_{n,n',n''=0}^{\infty} N^{1-p}\ (cK)^{\bar n}\ \frac{1}{n!n'!n''!} \sum_{o-\mathcal{T}} \sum_{E,\mathcal{C}} \tag{IV.50}$$
$$\int d^2x_1 \ldots d^2x_{\bar n} \int_{0 \leq w_1 \leq \ldots \leq w_{\bar n -1} \leq 1} \prod_{q=1}^{\bar n -1} dw_q$$
$$\left| \prod_{g \in D^c_{\mathcal{C}}} R^*_g \left[\prod_{q=1}^{\bar n -1} D^{\Lambda_0, w_q}_{\Lambda}(\bar x_{l_q}, x_{l_q})\ \det \mathcal{M}'(\mathcal{C})\ \phi_1(x_{i_1}) \ldots \phi_{2p}(x_{j_p}) \right] \right|$$

Before performing any bound we must study the action of $\prod_{g\in D^c_{\mathcal{C}}} R^*_g$ on the tree propagators, the loop determinant and the external test functions. As the external half-lines for any subgraph cannot be of type C' we will write C instead of D in the formulas. We distinguish two situations.

1) If $|eg(\mathcal{C})| = 4$ then $\omega(g_i(\mathcal{C})) = 0$ and the action of R^*_g is:

$$\begin{aligned}
R^*_g(x_1)\prod_{i=1}^4 C(x_i,y_i) &:= \sum_{i=2}^4 R^0_{gi}(x_1)[\prod_{i=1}^4 C(x_i,y_i)] \qquad \text{(IV.51)}\\
&= C(x_1,y_1)\left[\sum_{i=2}^4 \prod_{2\le j<i} C(x_j,y_j)[C(x_i,y_i)-C(x_1,y_i)]\prod_{i<j\le 4} C(x_1,y_j)\right]\\
&= C(x_1,y_1)\left[\sum_{i=2}^4 \prod_{2\le j<i} C(x_j,y_j)[\delta_0 C(x_i,x_1,y_i)]\prod_{i<j\le 4} C(x_1,y_j)\right]
\end{aligned}$$

where we took as reference vertex x_1 and we defined R^0_{gi} as the operator that moves the external line with i on the reference vertex x_1, and applies a difference $\delta_0 C(x_i,x_1,y_i)$ between two covariances on the line i.

2) If $|eg_i(\mathcal{C})| = 2$ then $\omega(g_i(\mathcal{C})) = -1$ and the action of R^*_g is:

$$\begin{aligned}
&R^*_g(x_1)C(x_1,y_1)C(x_2,y_2) := R^1_g(x_1)C(x_1,y_1)C(x_2,y_2)\\
&= C(x_1,y_1)\left[C(x_2,y_2)-C(x_1,y_2)-(x_2-x_1)^\mu\frac{\partial}{\partial x_1^\mu}C(x_1,y_2)\right]\\
&= C(x_1,y_1)\left[\delta_1 C(x_2,x_1,y_2)\right] \qquad \text{(IV.52)}
\end{aligned}$$

where we took as reference vertex x_1.

IV.3.1 Choice of the reference vertex

Now, for each $g_i \in D^c_{\mathcal{C}}$ we call the reference vertex $v_e(g_i(\mathcal{C}))$. In this paper the choice of this vertex is adapted to the tree $\mathcal{T}$, and is different from previous rules such as [R], chap.II. We adopt the following rule. We call the first external vertex of the graph, the one with position x_{i_1} the root of the tree. We define $D^{0c}_{\mathcal{C}}$ and $D^{1c}_{\mathcal{C}}$ as the subsets of four-point and two-point subgraphs in $D^c_{\mathcal{C}}$.

For every subgraph $g \in D^{0c}_{\mathcal{C}}$ and any vertex of g there is a single path in $\mathcal{T}$ joining this vertex to the root. This path must contain a single well defined external line of g. The vertex to which this external line hooks is by definition our reference vertex for g (see Figure 14).

This rule leaves us free of choosing in a different way the reference vertex for any two-point D1PI subgraph $D^{1c}_{\mathcal{C}}$. The rule must ensure that open subgraphs and D1PR subgraphs are automatically renormalized by renormalization of their closure or proper parts. We decide to take as border vertex of any subgraph $g \in D^{1c}_{\mathcal{C}}$

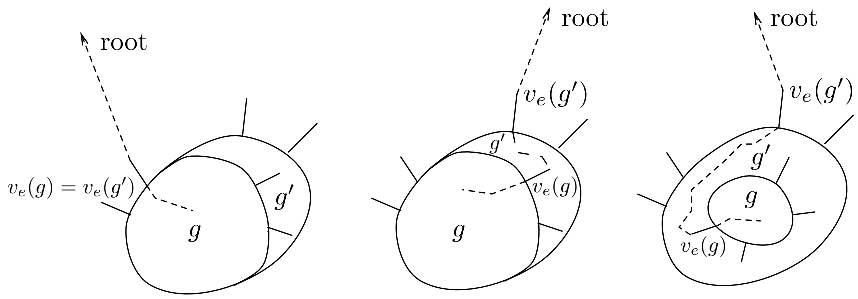


Figure 14

the one to which the highest of the two external half-lines of g hooks. Remark that this external half-line is always a tree half-line in $\mathcal{T}$, so we know its scale. This rule, shown in Figure 15, fulfills the desired requirement, as will be shown below.

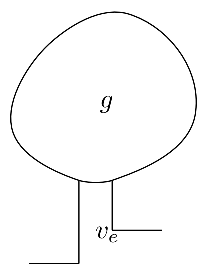


Figure 15

Finally we add a rule which is not strictly speaking necessary but simplifies the discussion: to compute the action of the renormalization operator we perform first all operations corresponding to two point subgraphs, then all operations corresponding to four point subgraphs, starting from the smallest graphs towards the largest. This rule ensures that any external half-line of a subgraph g bearing one or two gradients because of the action of the Taylor operator for g, cannot bear additional gradients from the later action of another Taylor operator for a different subgraph g'.

IV.3.2 Processes

Returning to equations (IV.51) and (IV.52), we start the renormalization for the two point subgraphs from the leaves of $R_{\mathcal{T}}$ (hence from the smallest subgraphs at highest energy) and go down. Then we perform the renormalization of four point subgraphs. Some of them may be already convergent due to renormalization of two point subgraphs. Also, even after fixing the reference vertex for each closed subgraph, there remains an arbitrary ordering of the other external lines for each four-point graph and a sum over three possible terms as shown in IV.51. Again some of these terms may themselves renormalize some lower four point subgraphs, so the outcome of the renormalization is difficult to capture in a single formula. We index the terms finally obtained by an index P, called the process, which summarize all these choices made for the four point subgraphs. Hence

$$\prod_{g\in D^c_{\mathcal{C}}} (R_g) = \left(\sum_P \prod_{g\in D^{0c}_{\mathcal{C}}(P)} [R_g(P)] \right) \prod_{g\in D^{1c}_{\mathcal{C}}} R^1_g \tag{IV.53}$$

where $D^{0c}_{\mathcal{C}}(P)$ is the subset of $D^{0c}_{\mathcal{C}}$ made of the subgraphs for which $R^*_{g(\mathcal{C})} \neq 1$, hence for which there is a non-trivial renormalization.

$$R_g(P) \quad = \quad R^0_{g\, i(P)},\ i(P) \in \{2,3,4\} \quad \text{if } g \in D^{0c}_{\mathcal{C}}(P) \tag{IV.54}$$

Hence in equation (IV.50), the absolute value inside the integrals can be bounded by:

$$\left| \prod_{g\in D^c_{\mathcal{C}}} R_g \left[\prod_{q=1}^{\bar{n}-1} D^{\Lambda_0,w_q}_{\Lambda}(\bar{x}_{l_q}, x_{l_q})\ \det \mathcal{M}'(\mathcal{C})\ \phi_1(x_{i_1}) \dots \phi_{2p}(x_{j_p}) \right] \right| \tag{IV.55}$$

$$\leq \quad \sum_P \prod_{q=1}^{\bar{n}-1} |D^{r,\Lambda_0,w_q}_{\Lambda}(\bar{x}_{l_q}, x_{l_q})|\, |\det \mathcal{M}'^r(\mathcal{C})|\ |\phi^r_1(x_{i_1}) \dots \phi^r_{2p}(x_{j_p})|$$

where we defined D^r, $\mathcal{M}'^r$, ϕ^r as the functions obtained after the application of $\prod_{g\in D^c_{\mathcal{C}}(P)} R_g(P)$. Again we bound the sum over processes P by the supremum times the number of possible processes. This number is bounded by 3^{n-1}. Indeed we recall that for any forest $\mathcal{F}$ of closed four-point subgraphs, we have $f(\mathcal{F}) \leq n-1$, where $f(\mathcal{F})$ is the number of four-point subgraphs in $\mathcal{F}$ [CR, Lemma C1]; this maximal number is not changed by adding $n' + n''$ two point vertices because of one particle irreducibility of the closed subgraphs.

From now on we work therefore with a fixed process P. We introduce some notations. We define $L^0(P)$ and L^1 as the set of loop half-lines which bear some single or double gradient respectively by some $R^0_{gi}(P)$ or R^1_g operator, $L^{r0}(P)$ as the set of loop half-lines moved to the reference vertex by some $R^0_{gi}(P)$ and $L^u(P)$ the loop half-lines left unchanged. In the same way we define the sets $T^0(P)$, T^1, $T^{r0}(P)$ and $T^u(P)$ for the tree half-lines, and $E^0(P)$, E^1, $E^{r0}(P)$ and $E^u(P)$ for the external half-lines. To avoid confusion, from now on we call f_i and $\bar{f}_i$ the two half-lines forming the tree-line l_i.

IV.3.3 Interpolations of the lines

For a four-point subgraph the difference $\delta_0 C$ is expressed by

$$\delta_0 C(x, x_v, y) \quad = \quad \int_0^1 dt \frac{d}{dt} C(x(t), y) \tag{IV.56}$$

For a two-point subgraph $\delta_1 C$ is expressed by

$$\delta_1 C(x, x_v, y) \quad = \quad \int_0^1 dt(1-t)\frac{d^2}{dt^2}C(x(t), y) \tag{IV.57}$$

This means that the external line hooked to x has been hooked to the point $x(t)$ on any piecewise differentiable path joining x to x_v and has now a propagator (see Figure 16)

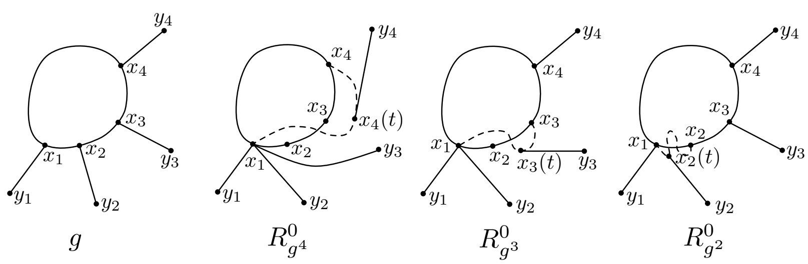


Figure 16

$$C^0(x(t), y) := \frac{d}{dt}C(x(t), y) \tag{IV.58}$$

or (see Figure 17)

$$C^1(x(t), y) := (1-t)\frac{d^2}{dt^2}C(x(t), y). \tag{IV.59}$$

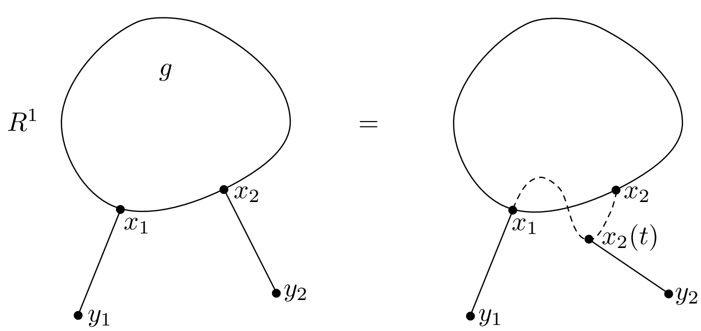


Figure 17

In previous perturbative or constructive works, this path $x(t)$ is always defined to be the linear segment connecting x to x_v hence is parametrized by

$$x(t) := x_v + t(x - x_v) \quad x(0) = x_v \;\; x(1) = x \tag{IV.60}$$

But with the continuous band structure this obvious choice when applied to tree half-lines leads to difficulties. It is therefore more convenient to treat differently the loop, tree and external half-lines. Loop lines and external half-lines (except the root) do not affect spatial integration (recall that this spatial integration is performed using the decay of the tree lines). So for them we can choose the obvious linear interpolation that makes easier to factorize the matrix elements of $\mathcal{M}'$ as scalar products and to apply Gram's inequality. For the tree lines it will be convenient to exploit the existence of $\mathcal{T}$ to choose a different path.

IV.3.2.A: Loop lines Now for each $h_a \in L^0(P) \cup L^1$ we define

$$x_a(t) := x_{v_e} + t(x_a - x_{v_e}) \tag{IV.61}$$

which is the obvious linear path.

The propagator for the line bearing the difference becomes respectively for a four-point and a two-point subgraph

$$C^0(x_a(t), y) := \frac{d}{dt} C(x_a(t), y) = (x_a - x_v)^\mu \frac{\partial}{\partial x^\mu} C(x_a(t), y) \tag{IV.62}$$

$$C^1(x_a(t), y) := (1-t)\frac{d^2}{dt^2} C(x_a(t), y) = (x_a - x_v)^\mu (x_a - x_v)^\nu \frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu} C(x_a(t), y). \tag{IV.63}$$

If the second end of the line y is also moved we just apply the same formulas to $C^0(x_a(t), y)$ and $C^1(x_a(t), y)$, interpolating y. Introducing these formulas into the matrix element ${\mathcal{M}'}^r_{fg}$ we can factorize it as the scalar product of F^r_f and G^r_g where

$$\begin{aligned} F^r_f &= F_f \quad \text{for} \quad h_f \notin L^0(P) \cup L^1 && \text{(IV.64)} \\ F^r_f &= \int dt (x_f - x_v)^\mu \frac{\partial}{\partial x^\mu} F_f(x_f(t)) \qquad \text{for} \qquad h_f \in L^0(P) \\ F^r_f &= \int dt (1-t)(x_f - x_v)^\mu (x_f - x_v)^\nu \frac{\partial}{\partial x^\nu}\frac{\partial}{\partial x^\mu} F_f(x_f(t)) \;\; \text{for } h_f \in L^1 \end{aligned}$$

where F_f, G_g are defined in (IV.32). The same definitions hold for G^r_g. With these definitions the determinant is bounded by

$$|\det {\mathcal{M}'}^r(\mathcal{C})| \leq \prod_f ||F^r_f|| \quad \prod_g ||G^r_g|| \tag{IV.65}$$

Now we can bound the norms using the following lemma.

Lemma 5 *The norms of F_f and G_g satisfy the bounds*

$$\begin{aligned} ||F_f||^2_{\mathcal{C}} &\leq K\ [\Lambda_0(w_{M(f,\mathcal{C})}) - \Lambda_0(w_{m(f,\mathcal{C})-1})] \\ ||G_g||^2_{\mathcal{C}} &\leq K\ [\Lambda_0(w_{M(g,\mathcal{C})}) - \Lambda_0(w_{m(g,\mathcal{C})-1})] \end{aligned} \tag{IV.66}$$

Proof. Applying the definition (IV.44), $||F_f||^2_{\mathcal{C}}$ is written

$$
\begin{aligned}
||F_f||^2_{\mathcal{C}} &= \int \frac{d^2p}{(2\pi)^2} \frac{1}{(p^2+m^2)^{\frac{1}{2}}} \left[\eta\left(\frac{p^2+m^2}{\Lambda_0^2(w_{M(f,\mathcal{C})})}\right) - \eta\left(\frac{p^2+m^2}{\Lambda_0^2(w_{m(f,\mathcal{C})-1})}\right)\right] \\
&= \int \frac{d^2p}{(2\pi)^2} (p^2+m^2)^{\frac{1}{2}} \int_{\Lambda_0^{-2}(w_{M(f,\mathcal{C})})}^{\Lambda_0^{-2}(w_{m(f,\mathcal{C})-1})} d\alpha \left(-\eta'[(p^2+m^2)\,\alpha]\right) \\
&\leq \int_{\Lambda_0^{-2}(w_{M(f,\mathcal{C})})}^{\Lambda_0^{-2}(w_{m(f,\mathcal{C})-1})} d\alpha\, \pi\, \alpha^{-\frac{3}{2}} \int_0^\infty dv \sqrt{v}[-\eta'(v)] \\
&\leq K\left[\Lambda_0(w_{M(f,\mathcal{C})}) - \Lambda_0(w_{m(f,\mathcal{C})-1})\right]
\end{aligned} \tag{IV.67}
$$

where, in the third line, we performed the change of variable $v = \alpha(p^2+m^2)$. The same result holds for $||G_g||^2_{\mathcal{C}}$. □

A similar argument can be performed for loop lines with some gradient applied. Each derivative adds a factor $\alpha^{-\frac{1}{2}}$ in the integral. With these definitions the determinant is bounded by

$$
\begin{aligned}
&\prod_{a\in L^u(P)\cup L^{r0}(P)} [\Lambda_0(w_{M(a,\mathcal{C})}) - \Lambda_0(w_{m(a,\mathcal{C})-1})]^{\frac{1}{2}} \\
&\prod_{a\in L^0(P)} |x_a - x_{v(a)}|[\Lambda_0^3(w_{M(a,\mathcal{C})}) - \Lambda_0^3(w_{m(a,\mathcal{C})-1})]^{\frac{1}{2}} \\
&\prod_{a\in L^1} |x_a - x_{v(a)}|^2[\Lambda_0^5(w_{M(a,\mathcal{C})}) - \Lambda_0^5(w_{m(a,\mathcal{C})-1})]^{\frac{1}{2}}
\end{aligned} \tag{IV.68}
$$

IV.3.2.B: External lines. The only external line essential in spatial integration is the root y_1, then we can choose this point as reference vertex for the whole graph so that it is never interpolated. For the other external lines we take again the easiest formula, the linear one. All gradients generated by moving the external lines in fact apply to the test functions. Therefore the product is bounded by

$$
\begin{aligned}
&\prod_{h_{i_e}\in E^u(P)\cup E^{r0}(P), i_e\neq 1} ||\phi_{i_e}||_\infty \prod_{h_{i_e}\in E^0(P)} |x_{i_e} - x_v|||\phi'_{i_e}||_\infty \\
&\qquad \prod_{h_{i_e}\in E^1} |x_{i_e} - x_v|^2||\phi''_{i_e}||_\infty
\end{aligned} \tag{IV.69}
$$

IV.3.2.C: Tree lines. Now we consider tree lines.

We observe that the set of $f_i, \bar{f}_i \in T^{r0}$ modifies the tree $\mathcal{T}$ but does not disconnect it in the sense that it simply changes the hooking vertices of some line. On the other hand, interpolating each $f_i, \bar{f}_i \in T^0(P)\cup T^1$ with the linear rule (equation (IV.60)) in an intuitive sense "disconnects" the tree, since the point $x(t)$ in general no longer hooks to some point on a segment corresponding to a tree line. This defect would lead to difficulties when integrating over spatial positions. To avoid it we express the differences $\delta_0 C$ and $\delta_1 C$ using the connection between external vertices of any subgraph which is provided by the tree $\mathcal{T}$ itself. But, as the tree $\mathcal{T}$ is itself modified by renormalization, this process has to be inductive, starting from the smallest two point subgraphs of $D^{1c}_{\mathcal{C}}$ and proceeding towards the biggest, then again from the smallest four point subgraphs of $D^{0c}_{\mathcal{C}}(P)$ and proceeding towards the biggest.

Our induction creates progressively a new tree $\mathcal{T}(P,J)$. To describe it, we number the subgraphs to renormalize in the order the operations are performed as $g_1, \dots g_r$. At the stage $1 \le p \le r$, before renormalization of g_p, the tree is called $\mathcal{T}(P, J_{p-1})$ (we put $\mathcal{T}(P, J_0) = \mathcal{T}$). If the renormalization of g_p as specified by the process P does not act on a tree half-line external to g_p, we neither modify J_{p-1} nor $\mathcal{T}(P, J_{p-1})$. If the renormalization of g_p results in some half-tree line f_i or $\bar f_i$ belonging to $T^{r0}(P)$, as shown in Figure 18, case 1), we do not modify J_{p-1}, so we put $J_p = J_{p-1}$, but we modify $\mathcal{T}(P, J_{p-1})$, that is we define $\mathcal{T}(P, J_p)$ as $\mathcal{T}(P, J_{p-1})$ but with the half line now hooked to the reference vertex of g_p.

Finally when the renormalization of g_p interpolates a tree half-line f_i or $\bar f_i$, we modify both J_{p-1} and $\mathcal{T}(P, J_{p-1})$. There exists a unique path $\mathcal{P}^{\mathcal{T}(P,J_{p-1})}_{x_f, x_v}$ joining the vertex x_f where the half-line hooked to the fixed vertex of g_p. This path is made of q lines and goes through $q+1$ vertices with positions $x_0 = x_v, x_1, \dots, x_q = x_f$. We interpolate the half line using this path instead of the linear path. This means that we write, if g_p is a four point subgraph,

$$\begin{aligned}
\delta_0 C(x_f, x_v, y) &= \sum_{j=1}^{q} \delta_0 C(x_j, x_{j-1}, y) \\
&= \sum_{j=1}^{q} (x_j - x_{j-1})^\mu \int_0^1 dt \frac{\partial}{\partial x_j(t)^\mu} C(x_j(t), y)
\end{aligned} \tag{IV.70}$$

and if g_p is a two point subgraph we write

$$\begin{aligned}
&\delta_1 C(x_f, x_v, y) = \delta_0 C(x_f, x_v, y) - (x_f - x_v)^\mu \frac{\partial}{\partial x_v^\mu} C(x_v, y) \\
&= \sum_{j=1}^{q} \delta_0 C(x_j, x_{j-1}, y) - \sum_{j=1}^{q} (x_j - x_{j-1})^\mu \left[\frac{\partial}{\partial x_{j-1}^\mu} C(x_{j-1}, y) - \right. \\
&\quad \left. - \left[\sum_{k=1}^{j-1} \frac{\partial}{\partial x_k^\mu} C(x_k, y) - \frac{\partial}{\partial x_{k-1}^\mu} C(x_{k-1}, y) \right] \right] \\
&= \int_0^1 dt \sum_{j=1}^{q} (1-t)(x_j - x_{j-1})^\mu (x_j - x_{j-1})^\nu \frac{\partial^2}{\partial^\mu \partial^\nu} C(x_j(t), y) \\
&\quad + \int_0^1 dt \sum_{j=1}^{q} \sum_{k=1}^{j-1} (x_j - x_{j-1})^\mu (x_k - x_{k-1})^\nu \frac{\partial^2}{\partial^\mu \partial^\nu} C(x_k(t), y)
\end{aligned} \tag{IV.71}$$

where for each j (and $k \leq j$) we defined

$$x_j(t) := x_{j-1} + t(x_j - x_{j-1}) \ , \ x_k(t) := x_{k-1} + t(x_k - x_{k-1}) \ . \tag{IV.72}$$

Then we update J and $\mathcal{T}$. In the first case, where g is a four point subgraph, we define $J_p = J_{p-1} \cup \{j\}$, where j is the index of the line of $\mathcal{T}(P, J_{p-1})$ chosen in IV.70. In the second case, where g is a two point subgraph, we define $J_p = J_{p-1} \cup \{j\} \cup \{k\}$, where j and k are the indices of the lines of $\mathcal{T}(P, J_{p-1})$ chosen in IV.71. Finally we update the tree according to Figure 18, case 2 and 3.

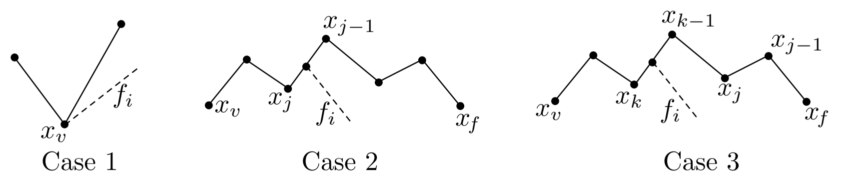


Figure 18

This means that, for g_p a four point subgraph, in $\mathcal{T}(P, J_p)$ the external line hooked to x_f is now hooked to the point $x_j(t)$ on the tree segment $[x_{j-1}, x_j]$ and has propagator

$$C^0(x_j(t), y) := (x_j - x_{j-1})^\mu \frac{\partial}{\partial^\mu} C(x_j(t), y). \tag{IV.73}$$

For g_p a two point subgraph, the external line previously hooked to x_f is now hooked to the point $x_k(t)$ on the tree segment $[x_{k-1}, x_k]$ (with $k \leq j$) and has propagator

$$C^1(x_k(t), y) := (x_j - x_{j-1})^\mu (x_k - x_{k-1})^\nu \frac{\partial^2}{\partial^\mu \partial^\nu} C(x_k(t), y) \quad k \neq j$$

$$C^1(x_j(t), y) := (1-t)(x_j - x_{j-1})^\mu (x_j - x_{j-1})^\nu \frac{\partial^2}{\partial^\mu \partial^\nu} C(x_j(t), y). \tag{IV.74}$$

Remark that the new tree $\mathcal{T}(P, J_p)$ has therefore one additional vertex and one line more than $\mathcal{T}(P, J_{p-1})$. The final tree built inductively in this way, $\mathcal{T}(P, J)$ is still a tree connecting all initial vertices, with at most $n-1$ new vertices and new lines (as $n-1$ is the maximal number of closed divergent subgraphs).

The treatment of two or four point subgraphs and the rules for their fixed vertex being different, we write $J = J^0 \cup J^1$, where J^0 is the set of indices j for the interpolations associated to the renormalization of four point graphs, and J^1 is the set of indices j_g and k_g for the interpolations associated to the renormalization of two point graphs.

IV.3.4 Bound on the sum over J^1 and on the associated distance factors

Now, before going on, in order to reproduce the spatial decay between supports of Theorem 3, we take out a fraction $(1-\epsilon)$ of the exponential decay of each tree line in (II.16) and (II.17): this factor is bounded by

$$\prod_{i=1}^{\bar{n}-1} e^{-(1-\epsilon)|\bar{x}_i - x_i|\Lambda_0^m(w_i)} \leq e^{-\Lambda^m(1-\epsilon)d_T(\Omega_1,\ldots\Omega_{2p})} \tag{IV.75}$$

and we keep the remaining decay $e^{-(\epsilon/2)|\bar{x}_i - x_i|\Lambda_0^m(w_i)}$ (adjusting $\epsilon' = \epsilon/2$) for two purposes: a fraction of this decay is used to perform spatial integration and the other to bound the sum over J^0 and the distance factors generated by equations (IV.62)–(IV.63) and (IV.73)–(IV.74). Therefore we have to bound, for a fixed process P

$$\sum_{J^1}\sum_{J^0}\int dx\; A(x,J,P,\mathcal{T})\; B(x,J,P,\mathcal{T}) \tag{IV.76}$$

where

$$A(x,J,P,\mathcal{T}) := \prod_{f_i,\bar{f}_i\in T^1} |x_j - x_{j-1}||x_k - x_{k-1}| \prod_{h_a\in L^1} |x_a - x_v|^2$$
$$\prod_{h_{i_e}\in E^1} |x_{i_e} - x_v|^2 \prod_{l\in\mathcal{T}(P,J)} e^{-(\epsilon/4)|\bar{x}_l - x_l|\Lambda_0^m(w_l)} \tag{IV.77}$$

$$B(x,J,P,\mathcal{T}) := \prod_{f_i,\bar{f}_i\in T^0(P)} |x_j - x_{j-1}| \prod_{h_a\in L^0(P)} |x_a - x_v|$$
$$\prod_{h_{i_e}\in E^0(P)} |x_{i_e} - x_v| \prod_{l\in\mathcal{T}(P,J)} e^{-(\epsilon/4)|\bar{x}_l - x_l|\Lambda_0^m(w_l)} \tag{IV.78}$$

where J specifies in particular the distance factors $|x_j - x_{j-1}|$ and $|x_k - x_{k-1}|$, as explained above.

The strategy of the bound is to write

$$\sum_{J^1}\sum_{J^0}\int dx\; A(x,J,P,\mathcal{T})\; B(x,J,P,\mathcal{T}) \leq \sum_{J^1}\sum_{J^0} A(J,P,\mathcal{T})\int dx\; B(x,J,P,\mathcal{T}) \tag{IV.79}$$

where $A(J,P,\mathcal{T}) := \sup_x A(x,J,P,\mathcal{T})$.

For each divergent subgraph $g_i \in \mathcal{D}^c_{\mathcal{C}}$ we define $t(i)$ as the index of the lowest tree line in the path of $\mathcal{T}(P,J)$ joining $x_{v(g_i)}$ to the interpolated half-line which renormalize it. The next lemma proves that $A(J,P,\mathcal{T})$ is bounded by something independent of J:

Lemma 6

$$A(J,P,\mathcal{T}) \leq K^n \prod_{g_i\in\mathcal{D}^{c1}_{\mathcal{C}}} [\Lambda_0(w_{t(i)}) - \Lambda_0(w_{\mathcal{A}(i)})]^{-2} \tag{IV.80}$$

where K is some ϵ dependent constant.

Proof. For each loop or external line the difference $|x-x_v|$ can be bounded, applying several triangular inequalities, by the sum over the tree lines on the unique path in $\mathcal{T}(P,J)$ connecting x to x_v.

We observe that the same tree line l_j can appear in several paths connecting different pairs of points x_v, x. Using the same fraction of its exponential decay many times might generate some unwanted factorials since $\sup_x x^n \exp(-x) = (n/e)^n$. To avoid this problem we define D_j as the set of subgraphs $g_i \in \mathcal{D}^{1c}_{\mathcal{C}} \cup \mathcal{D}^{0c}_{\mathcal{C}}(P)$ which use the tree distance $|\bar{x}_{l_j} - x_{l_j}|$ to bound the norm of $|x - x_{v(g_i)}|$ or its square and we apply the relation

$$e^{-\frac{\epsilon}{4}|\bar{x}_{l_j}-x_{l_j}|\Lambda_0^m(w_j)} \quad \leq \quad e^{-\frac{\epsilon}{4}|\bar{x}_{l_j}-x_{l_j}|\sum_{g_i\in D_j}[\Lambda_0(w_{t(i)})-\Lambda_0(w_{\mathcal{A}(i)})]} \tag{IV.81}$$

With this expression a different decay factor is used for each subgraph. Applying this result and the inequalities $xe^{-x} \leq 1$, $x^2e^{-x} \leq 1$ completes the proof of the lemma. □

It is proved in the next section that the sum and spatial integral

$$\sum_{J^0} \int dx \; B(x, J, P, \mathcal{T})$$

is in fact independent of P and J^1 (and of the interpolation parameters t that we omitted). Therefore the sum over J^1 will simply lead to the bound of the next section multiplied by the cardinal of the set J^1, that is by $|J^1|$. This is done thanks to the following lemma:

Lemma 7 *We have* $|J^1| \leq e^{2\bar{n}}$.

Proof. We consider the graphs $g_1, \dots g_{r_1}$ of $D^{1c}_{\mathcal{C}}$ in the order used for their renormalization in Subsection IV.3.2.C. We define, for each such two point subgraph $g \in D^{1c}_{\mathcal{C}}$ the set $\mathcal{A}(g)$ of maximal subgraphs g', $g' \in D^{1c}_{\mathcal{C}}$, $g' \subset g$, and the reduced graph $g/D^{1c}_{\mathcal{C}}$ where each $g' \in \mathcal{A}(g)$ has been reduced to a single point.

We also count the number L_g of lines on the unique path in the tree $\mathcal{T} \cap g/D^{1c}_{\mathcal{C}}$ which joins the two external vertices of g. Remark that this number L_g is independent of J and that $\sum_{g\in D^{1c}_{\mathcal{C}}} L_g \leq \bar{n} - 1$. Finally we define the subset $\mathcal{A}'(g)$ of $\mathcal{A}(g)$ made of those g' in $\mathcal{A}(g)$ which appear as reduced points on this unique path (see Figure 19).

By induction one can bound the number of choices in J^1 by

$$|J^1| \leq \prod_{g\in D^{1c}_{\mathcal{C}}} \sum_{j_g=1}^{L_g+\sum_{g'\in\mathcal{A}'(g)} k_{g'}} \sum_{k_g=1}^{j_g} 1 \tag{IV.82}$$

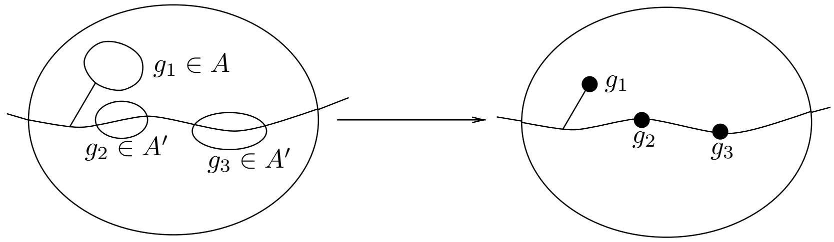


Figure 19

where this product is again ordered from the smallest to the largest graphs. Now for any positive increasing function f we have $\sum_{j=1}^{L} f(j) \le \int_1^{1+L} f(x)dx$ so that

$$|J^1| \le \prod_{g\in D_{\mathcal{C}}^{1c}} \int_1^{1+L_g+\sum_{g'\in\mathcal{A}'(g)} y_{g'}} dx_g \int_1^{1+x_g} dy_g$$
$$\le \prod_{g\in D_{\mathcal{C}}^{1c}} \int_0^{1+L_g+\sum_{g'\in\mathcal{A}'(g)} y_{g'}} dx_g \int_0^{1+x_g} dy_g \le e^{2r_1} \prod_{g\in D_{\mathcal{C}}^{1c}} e^{L_g} \le e^{2\bar{n}} \tag{IV.83}$$

where in the last inequality, we bounded the last integral $\int_0^{1+x_{g_{r_1}}} dy_{g_{r_1}}$ by $e^{1+x_{g_{r_1}}}$ and then effectuate each integral exactly and bound each difference $e^x - 1$ by e^x. Finally we used the fact that every subgraph g' is in $\mathcal{A}'(g)$ for at most one g, and the fact that $2r_1 \le \bar{n} - 1$ (again [CR, Lemma C1]). □

Remark. This lemma does not apply to J^0. For a counter-example, the reader can look at the following graph and tree, for which J^0 can be of order $K^n(n/5)!$.

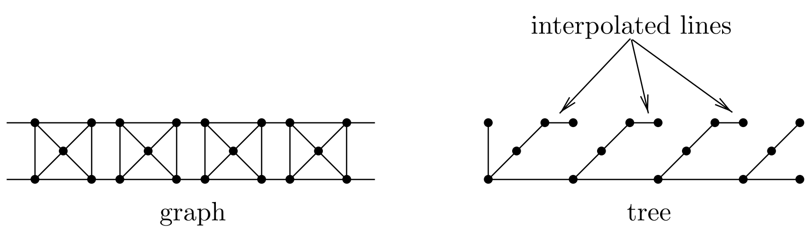


Figure 20

IV.3.5 Bound on the sum over J^0, on the associated distance factors, and spatial integration of the vertices

It remains now to perform the sum over J^0 and the integral over the position of internal vertices, using the remaining tree decay $\prod_{l\in\mathcal{T}(P,J)} e^{-(\epsilon/4)|\bar{x}_l - x_l|\Lambda_0^m(w_l)}$ in

B, and to check that the result, as announced, is independent of P, J and of the interpolation parameters t. We recall that $\Lambda_0^m(w)$ was defined in (II.18).

Lemma 8

$$\sum_{J^0}\int dx\; B(x,J,P,\mathcal{T}) \le K^{\bar n} \prod_{g_i\in\mathcal{D}_{\mathcal{C}}^{c0}(P)} [\Lambda_0(w_{t(i)})-\Lambda_0(w_{\mathcal{A}(i)})]^{-1}\prod_{q=1}^{\bar n-1}(\Lambda_0^m(w_q))^{-2} \tag{IV.84}$$

where K is some ϵ dependent constant.

Proof. First we divide one half of our remaining tree decay as in IV.81. This half will be used to bound each distance factor in $\prod_{h_a\in L^0(P)}|x_a-x_v|\prod_{h_{i_e}\in E^0(P)}|x_{i_e}-x_v|$ as in Lemma 6, the sum over J^0 and the distance factors $\prod_{f_i,\bar f_i\in T^0(P)}|x_j-x_{j-1}|$.

As in Lemma 6, each distance factor in $\prod_{h_a\in L^0(P)}|x_a-x_v|\prod_{h_{i_e}\in E^0(P)}|x_{i_e}-x_v|$ leads to a bound $K[\Lambda_0(w_{t(i)})-\Lambda_0(w_{\mathcal{A}(i)})]^{-1}$.

Then we perform the spatial integrals from the leaves of the tree $\mathcal{T}(P,J)$ towards the root x_1, using the other half of the tree decay. In this inductive process when we meet an interpolated line hooked at some interpolated point $x_j(t)$ or $x_k(t)$ two different situations can occur, as pictured in Figure 21. The second situation (interpolated point not towards the root) can occur only for interpolations of two point subgraphs, hence associated to the J^1 indices. The first situation (interpolated point towards the root) must occur for all interpolations associated to four point subgraphs plus possibly some interpolation associated to two point subgraphs. This is the consequence of our rule for the preferred vertices of four point subgraphs (the interpolations associated to four point subgraphs always bring nearer to the root).

The sum over J^0 is performed in pieces; the sum over each index j_g in J^0 is performed right after the spatial integration which has used the corresponding interpolated line. By the remark above, each sum over j_g in J^0 occurs in the first situation of Figure 21.

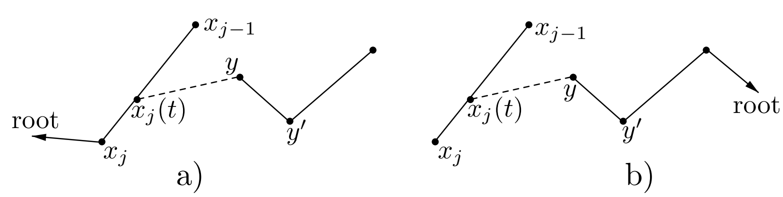


Figure 21

The decay of any tree line $l_i=[x,y]$ with both ends $f_i,\bar f_i\in T^u(P)\cup T^{r0}(P)$ gives, by translation invariance, a factor:

$$\int e^{-(\epsilon/8)|x-y|\Lambda_0^m(w_i)}d^2x = 128\pi\epsilon^{-2}[\Lambda_0^m(w_i)]^{-2} \tag{IV.85}$$

Surprisingly, the same result holds when one or both ends of the line have been interpolated, as we explain now[2].

• In the first situation of Figure 21, y, the other end of the interpolated line, is connected to the root through the interpolated point (see Figure 21a). We can integrate over y before integrating over x_j and x_{j-1}. We use translation invariance to cancel the dependence from $x_j(t)$ in the interpolated covariance. The integral over the variable t is bounded by 1. The spatial integral over y then gives the same factor as in (IV.85). Then we perform the corresponding sum over j in J^0 using

$$\sum_{j=1}^{q} |x_j - x_{j-1}| e^{-\frac{\epsilon}{8}[\sum_{j=1}^{q} |x_j - x_{j-1}|][\Lambda_0(w_{t(i)}) - \Lambda_0(w_{\mathcal{A}(i)})]} \leq K[\Lambda_0(w_{t(i)}) - \Lambda_0(w_{\mathcal{A}(i)})]^{-1} \tag{IV.86}$$

• When the point $x(t)$ is connected to the root through y (see Figure 21b) we have to compute the integral

$$I = \int_0^1 dt \int d^2x_j\, d^2x_{j-1}\, e^{-(\epsilon/8)|x_j - x_{j-1}|\Lambda_0^m(w_j)} e^{-(\epsilon/8)|x_j(t) - y|\Lambda_0^m(w_k)} \tag{IV.87}$$

where $x_j(t) = x_{j-1} + t(x_j - x_{j-1})$. Performing, for $t \neq 0$ the change of variables

$$v_1 = x_j(t) = tx_j + (1-t)x_{j-1} \quad v_2 = \frac{1}{t}(x_{j-1} - x_j(t)) = (x_{j-1} - x_j) \tag{IV.88}$$

the integral becomes

$$\begin{aligned} I &= \int_0^1 dt \int d^2v_1\, e^{-(\epsilon/8)|v_1|\Lambda_0^m(w_k)} \int d^2v_2\, e^{-(\epsilon/8)|v_2|\Lambda_0^m(w_j)} \\ &= [128\pi\epsilon^{-2}]^2 [\Lambda_0^m(w_j)]^{-2} [\Lambda_0^m(w_k)]^{-2} \end{aligned} \tag{IV.89}$$

This is again the same contribution as (IV.85), hence the same contribution as if the line had not been interpolated!

Following the tree from the leaves to the root we can perform the integrals over all positions in this way, except for x_{i_1}. This last point is integrated using the test function ϕ_1, which gives a factor $||\phi_1||_1$. Hence, the result of all spatial integrations is $K^n \prod_{q=1}^{\bar{n}-1} (\Lambda_0^m(w_q))^{-2}$ and the sum over J^0 has been performed at a cost of K^n (although $|J^0|$ can be very large). This completes the proof of Lemma 8. □

Since the result of Lemma 8 is independent of J^1, as announced, we can apply Lemma 7. Let us collect the result of Lemmas 6-8, together with the other factors remaining after spatial integration. The product over tree lines propagators, is

[2]We thank V. Mastropietro for explaining to us that this surprising fact was known to him and can be found in his work [BM]

bounded by

$$K^{\bar n}(\Lambda^{-2}-\Lambda_0^{-2})^{\bar n-1}\ \ e^{-(\epsilon/4)m^2\Lambda_0^{-2}(w_1)}$$
$$\cdot\prod_{q=1}^{\bar n-1}\Lambda_0^3(w_q)\prod_{f_q,\bar f_q\in T'\cup T^0(P)}\Lambda_0(w_q)\prod_{f_q,\bar f_q\in T^1}\Lambda_0^2(w_q) \tag{IV.90}$$

where K is some ϵ dependent constant,and we included the scaling prefactors in (II.16)–(II.17). These factors are

- a factor $\Lambda_0(w)^3$ for each tree line,
- a factor $\Lambda_0(w)$ for each half-line in $T^0(P)$,
- a factor $\Lambda_0(w)^2$ for each half-line in T^1,
- a factor $\Lambda_0(w)$ for each half-line in T', the set of half-lines hooked to a $\delta\zeta$ vertex which bears a derivative, hence has covariance C'.

Remark that we kept for only one line, the lowest one, the massive decay $e^{-(\epsilon/4)m^2\Lambda_0^{-2}(w_1)}$ from bounds (II.16)–(II.17). It will be useful only to conclude the bound in the massive case when $\Lambda < m$. In this case the n' vertices of type δm may create some infrared difficulties if we were to replace directly for them the factor $(\Lambda_0^m(w_q))^{-2}$ by $(\Lambda_0(w_q))^{-2}$. We introduce the set $\mathcal{T}'$ of the tree lines used for integration of the δm vertices. There are n' of them (or $n'-1$ if the root is of type δm, a case we will exclude for simplicity). Recall that by (II.18) we have

$$(\Lambda_0^m(w_q))^{-2}\le(\Lambda_0(w_q))^{-2} \tag{IV.91}$$
$$(\Lambda_0^m(w_q))^{-2}\le m^{-1}(\Lambda_0(w_q))^{-1} \tag{IV.92}$$

We use the bound (IV.92) only for the lines of $\mathcal{T}'$ when $\Lambda < m$. For all other cases we use the bound (IV.91).

IV.3.6 Integration over the parameters w_i

Now, putting everything together, we describe first the bound when $\Lambda > m$, hence $\Lambda^m = \Lambda$. Equation (IV.50) is bounded by

$$|\Gamma^{\Lambda}_{2p}(\phi_1,\dots\phi_{2p})|\le||\phi_1||_1\prod_{i=2}^{2p}||\phi_i||_{\infty,2}$$
$$N^{1-p}e^{-(1-\epsilon)\Lambda^m d_T(\Omega_1,\dots\Omega_{2p})}(\Lambda^{-2}-\Lambda_0^{-2})^{\bar n-1}$$
$$\sum_{n,n',n''=0}^{\infty}(cK)^{\bar n}\frac{1}{n!n'!n''!}\sum_{o-\mathcal{T}}\sum_{E,\mathcal{C}}\int_{0\le w_1\le\dots\le w_{\bar n-1}\le 1}\prod_{q=1}^{\bar n-1}dw_q$$
$$\prod_{q=1}^{\bar n-1}\Lambda_0(w_q)\prod_{f_q,\bar f_q\in T'\cup T^0(P)}\Lambda_0(w_q)\prod_{f_q,\bar f_q\in T^1}\Lambda_0^2(w_q)$$
$$\prod_{a\in L^u(P)\cup L^{r0}(P)}\Lambda_0^{\frac12}(w_{M(a,\mathcal{C})})\left[1-\frac{\Lambda_0(w_{m(a,\mathcal{C})-1})}{\Lambda_0(w_{M(a,\mathcal{C})})}\right]^{\frac12}$$

$$\prod_{a\in L^0(P)} \Lambda_0^{\frac{3}{2}}(w_{M(a,\mathcal{C})}) \left[1-\frac{\Lambda_0^3(w_{m(a,\mathcal{C})-1})}{\Lambda_0^3(w_{M(a,\mathcal{C})})}\right]^{\frac{1}{2}}$$
$$\prod_{a\in L^1} \Lambda_0^{\frac{5}{2}}(w_{M(a,\mathcal{C})}) \left[1-\frac{\Lambda_0^5(w_{m(a,\mathcal{C})-1})}{\Lambda_0^5(w_{M(a,\mathcal{C})})}\right]^{\frac{1}{2}}$$
$$\prod_{g_i\in\mathcal{D}_{\mathcal{C}}^{0c}} \Lambda_0^{-1}(w_{t(i)}) \left[1-\frac{\Lambda_0(w_{\mathcal{A}(i)})}{\Lambda_0(w_{t(i)})}\right]^{-1}$$
$$\prod_{g_i\in\mathcal{D}_{\mathcal{C}}^{1c}} \Lambda_0^{-2}(w_{t(i)}) \left[1-\frac{\Lambda_0(w_{\mathcal{A}(i)})}{\Lambda_0(w_{t(i)})}\right]^{-2} \tag{IV.93}$$

The differences $\left[1-\Lambda_0(w_{\mathcal{A}(i)})/\Lambda_0(w_{t(i)})\right]$ are dangerous as they appear with a negative exponent. They are the price to pay for implementing continuous renormalization group. Indeed, in this continuous formalism one has to perform renormalization even when the differences between internal and external energies of subgraphs are arbitrarily small. However, there is a natural solution to this problem: each subgraph to renormalize has necessarily loop lines and these loop lines, when evaluated in the continuous formalism by Gram's inequality, give small factors precisely when the differences between internal and external energies of subgraphs become arbitrarily small.

In other words, we can cancel the dangerous differences with a negative exponent against the analogous differences with a positive exponent given by the loop lines. This is the purpose of the next lemma.

Lemma 9 *If $g_i \in \mathcal{D}_{\mathcal{C}}^{c0}(P)$ there is at least one loop line internal to g_i which satisfies $\Lambda_0(w_{M(a,\mathcal{C})}) \leq \Lambda_0(w_{t(i)})$ and $\Lambda_0(w_{m(a,\mathcal{C})-1}) \geq \Lambda_0(w_{\mathcal{A}(i)})$. If $g_i \in \mathcal{D}_{\mathcal{C}}^{c1}$ there are at least two loop lines internal to g_i and which satisfy $\Lambda_0(w_{M(a,\mathcal{C})}) \leq \Lambda_0(w_{t(i)})$ and $\Lambda_0(w_{m(a,\mathcal{C})-1}) \geq \Lambda_0(w_{\mathcal{A}(i)})$.*

Assuming the lemma true, and using the relations $\sqrt{\frac{1-x^3}{1-x}} \leq \sqrt{3}$ and $\sqrt{\frac{1-x^5}{1-x}} \leq \sqrt{5}$, one obtains

$$\prod_{a\in L^u(P)\cup L^{r0}(P)} \left[1-\frac{\Lambda_0(w_{m(a,\mathcal{C})-1})}{\Lambda_0(w_{M(a,\mathcal{C})})}\right]^{\frac{1}{2}} \prod_{a\in L^0(P)} \left[1-\frac{\Lambda_0^3(w_{m(a,\mathcal{C})-1})}{\Lambda_0^3(w_{M(a,\mathcal{C})})}\right]^{\frac{1}{2}}$$
$$\prod_{a\in L^1} \left[1-\frac{\Lambda_0^5(w_{m(a,\mathcal{C})-1})}{\Lambda_0^5(w_{M(a,\mathcal{C})})}\right]^{\frac{1}{2}} \prod_{g_i\in\mathcal{D}_{\mathcal{C}}^{0c}} \left[1-\frac{\Lambda_0(w_{\mathcal{A}(i)})}{\Lambda_0(w_{t(i)})}\right]^{-1} \tag{IV.94}$$
$$\prod_{g_i\in\mathcal{D}_{\mathcal{C}}^{1c}} \left[1-\frac{\Lambda_0(w_{\mathcal{A}(i)})}{\Lambda_0(w_{t(i)})}\right]^{-2} \leq \quad 5^{n-1}$$

where we bounded by 1 the loop lines differences that were not used to compensate some $\left[1-\Lambda_0(w_{\mathcal{A}(i)})/\Lambda_0(w_{t(i)})\right]^{-1}$ factor.

Proof of Lemma 9. We observe that the lowest tree line $l_{t(i)}$ in $\mathcal{T}_i(J,P)$ joining the interpolated line and the reference vertex is external line for the two subgraphs of g_i, $g_{t(i)1}$ and $g_{t(i)2}$. One of these two subgraphs has for external line the reference external line of g_i and the other has for external line the interpolated line moved by the renormalization $R^*_{g_i}$. But $g_{t(i)1}$ and $g_{t(i)2}$ must both have at least some additional external lines, otherwise g_i would be D1PR. By parity $g_{t(i)1}$ and $g_{t(i)2}$ must both have at least *two* such additional external lines.

We distinguish two cases:

- If $g_i \in \mathcal{D}^{c0}_{\mathcal{C}}(P)$, since there are at most two additional external lines of g_i, we find that there must be at least *two* external half-lines of $g_{t(i)1} \cup g_{t(i)2}$ different from $l_{t(i)}$ which are internal in g_i. If they are both loop half-lines we are done. If some of them is a tree half-line, the other half is external line for another subgraph of g_i, g'. Repeating the argument for g' (as $|eg| = 1$ is forbidden) we finally must find an associated loop half-line (see Figure 12).
- If $g_i \in \mathcal{D}^{c1}_{\mathcal{C}}$, since there was no additional external line of g_i, then both $g_{t(i)1}$ and $g_{t(i)2}$ must have at least *two* external half-lines different from $l_{t(i)}$ which are internal in g_i. Either these four half-lines are loop half-lines and we are done, or some of them are tree lines, which we follow as above until we find the corresponding loop half-lines. $\square$

After applying the bound (IV.94) we can take the limit $\Lambda_0 \to \infty$. Performing the change of variable $u_i = 1 - w_i$ equation (IV.93) becomes:

$$
\begin{aligned}
&|\Gamma^{\Lambda}_{2p}(\phi_1, \dots \phi_{2p})| \leq ||\phi_1||_1 \prod_{i=2}^{2p} ||\phi_i||_{\infty,2} e^{-(1-\epsilon)\Lambda^m d_T(\Omega_1, \dots \Omega_{2p})} \\
&\quad N^{1-p} \sum_{n,n',n''=0}^{\infty} \Lambda^{2-p-n'} (cK)^{\bar{n}} \frac{1}{n!n'!n''!} \sum_{o-\mathcal{T}} \sum_{E,\mathcal{C}} \int_{0 \leq u_{\bar{n}-1} \leq \dots \leq u_1 \leq 1} \prod_{q=1}^{\bar{n}-1} du_q \\
&\quad \left[\prod_{q=1}^{\bar{n}-1} \frac{1}{\sqrt{u_q}}\right] \left[\prod_{f_q, \bar{f}_q \in T' \cup T^0(P)} \frac{1}{\sqrt{u_q}}\right] \left[\prod_{f_q, \bar{f}_q \in T^1} \frac{1}{u_q}\right] \\
&\quad \left[\prod_{a \in L^u(P) \cup L^{r0}(P)} (u_{M(a,\mathcal{C})})^{-\frac{1}{4}}\right] \left[\prod_{a \in L^0(P)} (u_{M(a,\mathcal{C})})^{-\frac{3}{4}}\right] \left[\prod_{a \in L^1} (u_{M(a,\mathcal{C})})^{-\frac{5}{4}}\right] \\
&\quad \left[\prod_{g_i \in \mathcal{D}^{0c}_{\mathcal{C}}} (u_{t(i)})^{\frac{1}{2}}\right] \left[\prod_{g_i \in \mathcal{D}^{1c}_{\mathcal{C}}} (u_{t(i)})\right] \qquad \text{(IV.95)}
\end{aligned}
$$

To factorize the integrals we perform the change of variable:

$$
u_i = \beta_i u_{i-1} \qquad \beta_i \in [0,1] \qquad \text{(IV.96)}
$$

where by convention $u_0 = 1$. The Jacobian of this transformation is the determinant of a triangular matrix hence it is given by:

$$J = \beta_1(\beta_1\beta_2)\dots(\beta_1\beta_2\dots\beta_{\bar n-2}) = \prod_{i=1}^{\bar n-1} \beta_i^{\bar n-1-i}. \tag{IV.97}$$

We absorb $\Lambda^{-n'}$ into the term $K^{\bar n}$ since we recall that $\Lambda > m$ hence that $\Lambda^{-n'} = (\Lambda^m)^{-n'}$, and that in Theorem 3 Λ^m remains in the compact X, hence is bounded away from 0. Then the integral (IV.95) becomes

$$\begin{aligned}
&|\Gamma^\Lambda_{2p}(\phi_1,\dots\phi_{2p})| \le ||\phi_1||_1 \prod_{i=2}^{2p} ||\phi_i||_{\infty,2} e^{-(1-\epsilon)\Lambda^m d_T(\Omega_1,\dots\Omega_{2p})} \qquad \text{(IV.98)}\\
&\quad (\Lambda^m)^{2-p}\, N^{1-p} \sum_{n,n',n''=0}^{\infty} (cK)^{\bar n} \frac{1}{n!n'!n''!} \sum_{o-\mathcal{T}} \sum_{E,\mathcal{C}} \int_0^1 \dots \int_0^1 \prod_{i=1}^{\bar n-1} d\beta_i \\
&\quad \prod_{i=1}^{\bar n-1} \beta_i^{-1+\frac12(\bar n-i)-\frac12|N_i''|} \left[\prod_{f_q,\bar f_q\in T^0(P)} \frac{1}{\sqrt{\beta_q\dots\beta_1}}\right] \left[\prod_{f_q,\bar f_q\in T^1} \frac{1}{\beta_q\dots\beta_1}\right] \\
&\quad \left[\prod_{a\in L^u(P)\cup L^{r0}(P)} (\beta_{M(a,\mathcal{C})}\dots\beta_1)^{-\frac14}\right] \left[\prod_{a\in L^0(P)} (\beta_{M(a,\mathcal{C})}\dots\beta_1)^{-\frac34})\right] \\
&\quad \left[\prod_{a\in L^1} (\beta_{M(a,\mathcal{C})}\dots\beta_1)^{-\frac54}\right] \left[\prod_{g_i\in\mathcal{D}^{0c}_{\mathcal{C}}} (\beta_{t(i)}\dots\beta_1)^{\frac12}\right] \left[\prod_{g_i\in\mathcal{D}^{1c}_{\mathcal{C}}} (\beta_{t(i)}\dots\beta_1)\right]
\end{aligned}$$

Each β_i appears with the exponent $-1 + x_i$.

$$\begin{aligned}
x_i \;&=\; \frac12[\bar n - i] - \frac12|N_i''| - \frac14|IL_i(\mathcal{C})| \\
&\quad + \frac12[|S_i^0(\mathcal{C})| - |IT_i^0| - |IL_i^0(\mathcal{C})|] + [|S_i^1(\mathcal{C})| - |IT_i^1| - |IL_i^1(\mathcal{C})|]
\end{aligned} \tag{IV.99}$$

where we defined

$$\begin{aligned}
IT_i^0 &:= \{f_j, \bar f_j \in T^0(P) | j \ge i\} \\
IT_i^1 &:= \{f_j, \bar f_j \in T^1 | j \ge i\} \\
IL_i(\mathcal{C}) &:= \{a \in L | M(a,\mathcal{C}) \ge i\} \\
IL_i^0(\mathcal{C}) &:= \{a \in L^0(P) | M(a,\mathcal{C}) \ge i\} \\
IL_i^1(\mathcal{C}) &:= \{a \in L^1 | M(a,\mathcal{C}) \ge i\} \\
S_i^0(\mathcal{C}) &:= \{g_j \in \mathcal{D}^{0c}_{\mathcal{C}}(P) | t(j) \ge i\} \\
S_i^1(\mathcal{C}) &:= \{g_j \in \mathcal{D}^{1c}_{\mathcal{C}} | t(j) \ge i\}.
\end{aligned} \tag{IV.100}$$

$c(i)$ is the number of connected components in $T_i(P)$. All these definitions can be restricted to the connected components: IT_i^{0k},IT_i^{1k} , $IL_i^k(\mathcal{C})$, $IL_i^{0k}(\mathcal{C})$ $IL_i^{1k}(\mathcal{C})$, $S_i^{0k}(\mathcal{C})$ and $S_i^{1k}(\mathcal{C})$. We observe that $IL_i(\mathcal{C})$ corresponds to the set of half-lines that could have, in the class $\mathcal{C}$, $\mu(a) \geq i$ and it is the equivalent of IL_i defined in (III.9). IT_i^0 (respectively IT_i^1) and $IL_i^0(\mathcal{C})$ (respectively $IL_i^1(\mathcal{C})$) are the set of tree half-lines and loop half-lines at a level higher or equal to i, which are the interpolated external lines for some divergent subgraph in $\mathcal{D}_{\mathcal{C}}^{0c}(P)$ (respectively in $\mathcal{D}_{\mathcal{C}}^{1c}$), $S_i^0(\mathcal{C})$ (respectively $S_i^1(\mathcal{C})$) is the set of subgraphs in $\mathcal{D}_{\mathcal{C}}^{0c}(P)$ (respectively in $\mathcal{D}_{\mathcal{C}}^{1c}$) that have the internal tree line $l_{t(j)}$ of a level higher or equal to i. In the same way, we can define the equivalent of EL_i and E_i as

$$EL_i(\mathcal{C}) := \cup_{k=1}^{c(i)} el_i^k(\mathcal{C}) \tag{IV.101}$$

which is the set of loop half-lines that are forced to have $\mu(a) \leq i$, and

$$E_i(\mathcal{C}) := \cup_{k=1}^{c(i)} eg_i^k(\mathcal{C}). \tag{IV.102}$$

The integral in the variable $d\beta_i$ can be performed only if the exponent of β_i is bigger than -1. Using the relations

$$\begin{aligned}
\bar{n} - i &= \sum_{k=1}^{c(i)} [|N_i^k| + |N_i^{'k}| + |N_i^{''k}| - 1] \\
|E_i(\mathcal{C})| &= |EL_i(\mathcal{C})| + |ET_i| + |EE_i| \\
&= 2|N_i| + 2 - |IL_i(\mathcal{C})| \ ,
\end{aligned} \tag{IV.103}$$

the exponent of β_i can be written as $-1 + \sum_{k=1}^{c(i)} x_i^k$, where

$$\begin{aligned}
x_i^k \ := \ & \frac{1}{2}\Bigg[\frac{1}{2}(|E_i^k(\mathcal{C})| + 2|N_i^{'k}| - 4) \\
& + [|S_i^{0k}(\mathcal{C})| - |IT_i^{0k}| - |IL_i^{0k}(\mathcal{C})|] + 2[|S_i^{1k}(\mathcal{C})| - |IT_i^{1k}| - |IL_i^{1k}(\mathcal{C})|]\Bigg]
\end{aligned} \tag{IV.104}$$

Remark that for any level i we have

$$[|S_i^0(\mathcal{C})| - |IT_i^0| - |IL_i^0(\mathcal{C})|] \geq 0 \quad [|S_i^1(\mathcal{C})| - |IT_i^1| - |IL_i^1(\mathcal{C})|] \geq 0 \tag{IV.105}$$

as each half-line in IT_i^0 (IT_i^1) or $IL_i^0(\mathcal{C})$ ($IL_i^1(\mathcal{C})$) is the external interpolated line for a subgraph g_j. This subgraph g_j must have $j > i$ hence have $t(j) > i$. Therefore for each half-line in one of these sets there is always at least one corresponding half-line in $S_i^0(\mathcal{C})$ ($S_i^1(\mathcal{C})$).

Lemma 10 *For any connected component in $\mathcal{T}_i^k$ we have $x_i^k \geq 1/2$.*

Proof. We distinguish three situations.

• If $|E_i^k(\mathcal{C})| \geq 5$, in fact, by parity of the number of external half-lines of any subgraph, $|E_i^k(\mathcal{C})| \geq 6$ and then

$$x_i^k \geq (1/4)(|E_i^k(\mathcal{C})| - 4) \geq 1/2. \tag{IV.106}$$

• If $|E_i^k(\mathcal{C})| = 4$, then there must be a subgraph $g_j \in \mathcal{D}_{\mathcal{C}}^{0c}(P)$ with $j \geq i$ ($j = i$ only if l_i belongs to the connected component $\mathcal{T}_i^k(J, P)$) and $\mathcal{A}(j) < i$. Hence the interpolated line for g_j does not belong to IT_i^{0k} or $IL_i^{0k}(\mathcal{C})$, but the corresponding internal line $l_{t(j)}$ belongs to S_i^{0k}. Then

$$|S_i^{0k}| - |IT_i^{0k}| - |IL_i^{0k}(\mathcal{C})| \geq 1 \tag{IV.107}$$

and

$$x_i^k = \frac{1}{2}[|N_i'^k| + [|S_i^{0k}| - |IT_i^{0k}| - |IL_i^{0k}(\mathcal{C})|] + 2[|S_i^{1k}| - |IT_i^{1k}| - |IL_i^{1k}(\mathcal{C})|] \geq \frac{1}{2}. \tag{IV.108}$$

• Finally if $|E_i^k(\mathcal{C})| = 2$ one can see, by the same arguments, that

$$|S_i^{1k}| - |IT_i^{1k}| - |IL_i^{1k}(\mathcal{C})| \geq 1 \tag{IV.109}$$

and

$$\begin{aligned} x_i^k = \frac{1}{2}&[-1 + |N_i^{'k}| + [|S_i^{0k}| - |IT_i^{0k}| - |IL_i^{0k}(\mathcal{C})|] + 2[|S_i^{1k}| - |IT_i^{1k}| - |IL_i^{1k}(\mathcal{C}|) \\ &\geq [-1 + 2[|S_i^{1k}| - |IT_i^{1k}| - |IL_i^{1k}(\mathcal{C})|] \geq 1/2. \end{aligned} \tag{IV.110}$$

□

Now we can perform the integrals in equation (IV.98) and we obtain

$$\begin{aligned} |\Gamma_{2p}^{\Lambda}(\phi_1, \dots \phi_{2p})| &\leq ||\phi_1||_1 \prod_{i=2}^{2p} ||\phi_i||_{\infty,2} e^{-(1-\epsilon)\Lambda^m d_T(\Omega_1,\dots\Omega_{2p})} \\ &(\Lambda^m)^{2-p} N^{1-p} \sum_{n,n',n''=0}^{\infty} (cK)^{\bar{n}} \frac{1}{n!n'!n''!} \sum_{u-\mathcal{T}} \sum_{\sigma} \sum_{E,\mathcal{C}} \prod_{i=1}^{\bar{n}-1} \frac{1}{\sum_{k=1}^{c(i)} x_i^k} \end{aligned} \tag{IV.111}$$

where we wrote the sum over ordered trees as sum over unordered trees and sum over all possible orderings σ of the tree. The sum $\sum_{\mathcal{C}}$ is over a set whose cardinal is bounded by $K^{\bar{n}}$ so it's sufficient to bound them with the supremum over this set, as we are interested in a theorem at weak coupling λ. However the sum over E to attribute the $2p$ external lines to particular vertices runs over a set of at most $\bar{n}^{2p}$ (this is an overestimate!). This will lead to the factorial in Theorem 3. We remark however that a better bound on the behaviour of the vertex functions at large p can presumably be obtained when the external points are sufficiently spread (not too closely packed), but we leave this improved estimate to a future study.

Moreover, we bound $\frac{1}{(\bar{n})!}\sum_{u-\mathcal{T}} f(\mathcal{T})$ by $\frac{\bar{n}^{\bar{n}-2}}{\bar{n}!}\sup_{u-\mathcal{T}}|f(\mathcal{T})|$ using Cayley's theorem. Therefore, by Stirling's formula, it's enough to consider the unordered tree $\mathcal{T}$ which gives the $\max_{u-\mathcal{T}}|f(\mathcal{T})|$. The sum that could still give some factorial is $\sum_\sigma$. To bound it we use the product of fractions obtained after integration on the β_i.

- if $|ET_i^k| \geq 5$ we have

$$(|ET_i^k| + |EE_i^k| + |EL_i^k(\mathcal{C})| - 4)/4 \geq (|ET_i^k| - 4)/4 \geq \frac{|ET_i^k|+1}{24} \quad \text{(IV.112)}$$

- if $|ET_i^k| < 5$ we have

$$x_i^k \geq 1/2 \geq \frac{|ET_i^k|+1}{24} \quad \text{(IV.113)}$$

Now $|ET_i|$ depends on the (now unordered) tree $\mathcal{T}$ *and on its ordering* σ. Therefore we call it from now on $|ET_i^\sigma|$. Recall that it is the total number of external tree half-lines of the subset $\mathcal{T}_i^\sigma$ of $\mathcal{T}$ made of the $\bar{n}-i$ highest lines in the permutation σ. Since $\sum_k(|ET_i^k|+1) \geq |ET_i^\sigma|+1$, equation (IV.111) becomes

$$|\Gamma_{2p}^\Lambda(\phi_1,\dots\phi_{2p})| \leq ||\phi_1||_1 \prod_{i=2}^{2p} ||\phi_i||_{\infty,2} e^{-(1-\epsilon)\Lambda^m d_T(\Omega_1,\dots\Omega_{2p})}$$
$$(\Lambda^m)^{2-p} N^{1-p} \sum_{n,n',n''=0}^{\infty} \bar{n}^{2p} (cK)^{n+n'+n''} \sum_\sigma \prod_{i=1}^{\bar{n}-1} \frac{1}{|ET_i^\sigma|+1} \quad \text{(IV.114)}$$

At this point we can apply a result of [CR] (Lemma A,1, B.3, B.4) which states that for any tree we have

$$\sum_\sigma \prod_{i=1}^{\bar{n}-1} \frac{1}{|ET_i^\sigma|+1} \leq 4^{\bar{n}}. \quad \text{(IV.115)}$$

For completeness let us recall the proof of this result. For each tree $\mathcal{T}$ we can define a mapping ξ of $\mathcal{T}$ on a chain-tree with the same number of vertices:

$$\xi : \mathcal{T} \to \xi_{\mathcal{T}} \quad \text{(IV.116)}$$

To define ξ, we turn around $\mathcal{T}$ starting from an arbitrary end line, and we number the lines in the order we meet them for the first time. The lines of $\xi_{\mathcal{T}}$ are numbered in the same way and $\xi_{\mathcal{T}}$ associates the lines with the same number.

Now we observe that the sum over the orders on $\mathcal{T}$ corresponds to the sum over all permutations of the indices in $\xi(\mathcal{T})$. Moreover Lemma B.3 in [CR] proves that for any connected or disconnected subgraph R of $\mathcal{T}$, we have

$$E\mathcal{T}(R) + 1 \geq c(\xi_{\mathcal{T}}(R)) \quad \text{(IV.117)}$$

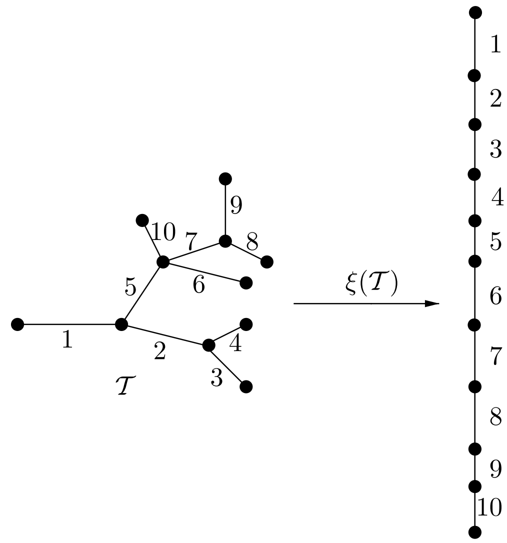


Figure 22

where $c(\xi_{\mathcal{T}}(R))$ is the number of connected components of the image of R $\xi_{\mathcal{T}}(R)$ and $E\mathcal{T}(R)$ is the number of external half-lines of R in $\mathcal{T}$. Finally we note that $\xi(T_i)$ is the set of lines with number $j \geq \bar{n} - i$ so we can write

$$\sum_{\sigma} \prod_{i=1}^{\bar{n}-1} \frac{1}{|ET_i^{\sigma}|+1} = \sum_{\sigma} \prod_{i=1}^{\bar{n}-1} \frac{1}{c(D_i^{\sigma})} \ := \ \Delta_{\bar{n}} \ , \tag{IV.118}$$

where D_i^{σ} is the set of lines in the chain-tree $\xi(\mathcal{T})$, that have $\sigma(j) \geq \bar{n} - i$ (after the permutation σ). Now, applying Lemma B.4. in [CR], we obtain

$$\Delta_{\bar{n}} \leq 4^{\bar{n}} \tag{IV.119}$$

We recall that this can be proved by remarking that $\Delta_{\bar{n}}$ satisfies the inductive equation

$$\Delta_{\bar{n}} = \sum_{k=1}^{\bar{n}-1} \Delta_p \Delta_{\bar{n}-k} \ , \tag{IV.120}$$

so that equation (IV.114) becomes

$$\begin{aligned} |\Gamma_{2p}^{\Lambda}(\phi_1, \ldots \phi_{2p})| &\leq ||\phi_1||_1 \prod_{i=2}^{2p} ||\phi_i||_{\infty,2} e^{-(1-\epsilon)\Lambda^m d_T(\Omega_1, \ldots \Omega_{2p})} \\ &(\Lambda^m)^{2-p} N^{1-p} \sum_{n,n',n''=0}^{\infty} \bar{n}^{2p} (4cK)^{\bar{n}} \end{aligned} \tag{IV.121}$$

where K depends only on ϵ. Taking c small enough completes the proof of the theorem in the case $\Lambda > m$, since $\sum_{\bar{n}} \bar{n}^{2p} e^{-\bar{n}} \leq K^p (p!)^2$.

In the case $\Lambda < m$, we have a few changes to perform. Replacing the lines of $\mathcal{T}'$ in (IV.90) by the bound (IV.92), keeping the massive decay factor $e^{-(\epsilon/4)m^2\Lambda_0^{-2}(w_1)}$ in (IV.90) and passing to the limit $\Lambda_0 \to \infty$ we have the following changes: in (IV.95) we add the factors

$$(\Lambda/m)^{n'} \big[\prod_{l_q \in \mathcal{T}'} (u_q)^{-1/2} \big] e^{-(\epsilon/4)u_1 m^2 \Lambda^{-2}} \tag{IV.122}$$

The factor $(\Lambda/m)^{n'}$ exactly changes $\Lambda^{2-p-n'}$ into $\Lambda^{2-p} m^{-n'} = \Lambda^{2-p} (\Lambda^m)^{-n'}$. The factor $(\Lambda^m)^{-n'}$ is absorbed in $K^{\bar{n}}$ since Λ^m in the hypothesis of Theorem 3 remains in the compact X. Passing to the variables β_i, the factor $\left[\prod_{l_q\in\mathcal{T}'} (u_q)^{-1/2}\right]$ is bounded by the factor $\prod_i \beta_i^{|N_i'^k|}$ in (IV.104), which was previously bounded by 1, hence not used at all. Finally the last integral over β_1 becomes bounded, for $p > 2$ by:

$$\Lambda^{2-p} \int_0^1 \beta_1^{(p-2)/2} \frac{d\beta_1}{\beta_1} e^{-(\epsilon/4)m^2\beta_1\Lambda^{-2}} \tag{IV.123}$$

Changing to the variable $v = (\epsilon/4)m^2\beta_1\Lambda^{-2}$ we obtain for the final bound a factor

$$(4/\epsilon m^2)^{(p-2)/2} \int_0^{\epsilon m^2 \Lambda^{-2}/4} v^{(p-2)/2} \frac{dv}{v} e^{-v} \leq (\Lambda^m)^{2-p} K^p \sqrt{p!} \tag{IV.124}$$

The case $p = 2$ is easy and left to the reader. Hence Theorem 3 holds in every case, by combining the factor $\sqrt{p!}$ with the factor $(p!)^2$ coming from the sum over E. Remark that in the case $m = 0$ we have never $\Lambda < m$, hence the factor $(p!)^{5/2}$ can be replaced by $(p!)^2$.

V The renormalization group equations

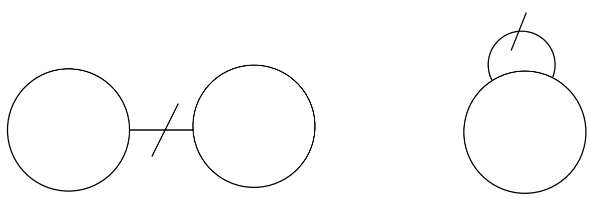
In this section we establish the renormalization group equations obtained when varying Λ and we check that for a fixed and small renormalized coupling constant, the effective constants remain bounded and small as predicted by the well known perturbative analysis of the model, which is asymptotically free in the ultraviolet regime [MW].

The derivative $\frac{\partial}{\partial\Lambda}\Gamma_{2p}^{\Lambda\Lambda_0}(\phi_1, \ldots \phi_{2p})$ can be written, using the expression (III.23), as:

$$\frac{\partial}{\partial\Lambda}\Gamma_{2p}^{\Lambda\Lambda_0}(\phi_1, \ldots, \phi_{2p}) = T\Gamma_{2p}^{\Lambda\Lambda_0}(\phi_1, \ldots, \phi_{2p}) + L\Gamma_{2p}^{\Lambda\Lambda_0}(\phi_1, \ldots, \phi_{2p}). \tag{V.125}$$

The first term $T\Gamma_{2p}^{\Lambda\Lambda_0}(\phi_1,\ldots,\phi_{2p})$ is the series obtained when the derivative falls on a tree line propagator (see Figure 23a):

$$
\begin{aligned}
T\Gamma_{2p}^{\Lambda\Lambda_0}(\phi_1,\ldots\phi_{2p}) = & \sum_{n,n',n''=0}^{\infty} \frac{1}{n!n'!n''!} \sum_{o-\mathcal{T}} \sum_{E,\mu} \sum_{\mathcal{C}ol,\Omega} \epsilon(\mathcal{T},\Omega) \int d^2x_1 \ldots d^2x_{\bar n} \\
& \int_{0\le w_1\le\cdots\le w_{\bar n-1}\le 1} \prod_{q=1}^{\bar n-1} dw_q \left[\prod_v \left(\frac{\lambda_{w(v)}}{N}\right)\right] \left[\prod_{v'} \delta m_{w(v')}\right] \left[\prod_{v''} \delta\zeta_{w(v'')}\right] \\
& \prod_{G_i^k\in D_\mu} R_{G_i^k} \Bigg[\sum_{q'=1}^{\bar n-1} \frac{\partial}{\partial\Lambda} D_\Lambda^{\Lambda_0,w_q}(\bar x_{l_{q'}}, x_{l_{q'}}) \\
& \prod_{q\neq q'} D_\Lambda^{\Lambda_0,w_q}(\bar x_{l_q}, x_{l_q}) \ \det\mathcal{M}(\mu) \ \phi_1(x_{i_1})\ldots\phi_{2p}(x_{j_p}) \Bigg]
\end{aligned}
\tag{V.126}
$$



a) b)

Figure 23

The second term $L\Gamma_{2p}^{\Lambda\Lambda_0}(\phi_1,\ldots,\phi_{2p})$ is the series obtained when the derivative falls on a loop line in the determinant (see Figure 23b):

$$
\begin{aligned}
L\Gamma_{2p}^{\Lambda\Lambda_0}(\phi_1,\ldots\phi_{2p}) = & \sum_{n,n',n''=0}^{\infty} \frac{1}{n!n'!n''!} \sum_{o-\mathcal{T}} \sum_{E,\mu} \sum_{\mathcal{C}ol,\Omega} \epsilon(\mathcal{T},\Omega) \int d^2x_1 \ldots d^2x_{\bar n} \\
& \int_{0\le w_1\le\cdots\le w_{\bar n-1}\le 1} \prod_{q=1}^{\bar n-1} dw_q \left[\prod_v \left(\frac{\lambda_{w(v)}}{N}\right)\right] \left[\prod_{v'} \delta m_{w(v')}\right] \left[\prod_{v''} \delta\zeta_{w(v'')}\right] \\
& \prod_{G_i^k\in D_\mu} R_{G_i^k} \Bigg[\prod_{q=1}^{\bar n-1} D_\Lambda^{\Lambda_0,w_q}(\bar x_{l_q}, x_{l_q}) \phi_1(x_{i_1})\ldots\phi_{2p}(x_{j_p}) \\
& \sum_{h_f,h_g|\mu(f)=\mu(g)} (-1)^{\epsilon(f,g)} \frac{\partial}{\partial\Lambda} C_{\Lambda_0(w_{\mu(f)})}^{\Lambda_0(w_{\mu(f)-1})}(x_f,x_g) \det{}_{left}\mathcal{M}(\mu) \Bigg]
\end{aligned}
\tag{V.127}
$$

where $\epsilon(f,g)$ is a sign coming from the development of the determinant. The convergence proofs of course extend to both terms of equation (V.125). Indeed, in the first one, the sum over the tree lines is bounded by a factor $\bar n$, and in the second one the sum is over the set of loop half-lines which is bounded by a factor $\bar{n}^2$. Therefore these sums cannot generate any factorial. Then we obtain the same bound as in (IV.114), with an additional factor $1/\Lambda$. This factor disappears when, as usual, the renormalization group equations are written as derivatives with respect to $\log\Lambda$ rather than Λ.

From these equations one can derive also equations for the flow of the effective constants defined in (III.24). For instance to obtain the flow of the effective coupling constant λ which is the four-point vertex function at zero external momenta, we can use equations (V.125)–(V.127) in which we let $\phi_1 \to \delta(0)$, ϕ_2, ϕ_3, $\phi_4 \to 1$. This is compatible with our L_1-L_∞ bounds, so that everything remains bounded. We obtain in this way the famous *continuous* flow equation which gives the derivative of the coupling constant with respect to $\log\Lambda$:

$$\frac{\partial}{\partial \log\Lambda} N\widehat{\Gamma}_4^{\Lambda}(0,0,0,0) = \frac{\partial}{\partial \log\Lambda}\lambda_\Lambda = \beta_2\lambda_\Lambda^2 + O(c^3) + \lambda_\Lambda^2 O(\Lambda^{-\alpha}) \tag{V.128}$$

where

$$\beta_2 = -2(N-1)/\pi \tag{V.129}$$

is the first non trivial term corresponding to the four-point graph with one tree line and one loop line, and the last term $\lambda_\Lambda^2 O(\Lambda^{-\alpha})$ is an infrared correction to the asymptotic flow (see [FMRS]). The negative sign of β_2 is responsible for the asymptotic freedom of the model. Similar equations hold for the flow of δm and $\delta\zeta$ (which remain bounded). For these equations up to one loop, see [MW] [GN] [GK] [FMRS]. For the computation up to two loops, we refer to [W].

From these renormalization group equations one can control the behavior of the effective constants and check that they remained bounded (until now this was assumed). The reader might be afraid that there is something circular in this argument. In fact this is not the case. Let us discuss for simplicity the massless case where the renormalized coupling $\widehat{\Gamma}_4^{\Lambda\Lambda_0}(0,0,0,0)$ is only a function of Λ_0/Λ and of the bare coupling λ. We know that it is analytic at the origin as function of the bare coupling λ [AR2]. Therefore from (V.125)–(V.128) it is for small bare λ and Λ_0/Λ a monotone increasing function of the ratio Λ_0/Λ (although this function might blow up in finite time).

Inverting the map from bare to renormalized couplings, one can prove that conversely for small renormalized coupling $\widehat{\Gamma}_4^{\Lambda\Lambda_0}(0,0,0,0)$ all the effective constants λ_w remain bounded by the renormalized one. Therefore one can pass to the ultraviolet limit $\Lambda_0 \to \infty$, in analogy with the completeness of flows of vector fields on compact manifolds. Furthermore one can compute the asymptotic behavior of the bare coupling which tends to 0 as $1/(|\beta_2|\log(\Lambda_0/\Lambda)))$. Similar arguments hold for the mass and wave function effective constants and achieve the proof of Theorem 1.

We recall for completeness that it is easy to build the Schwinger functions from the vertex functions and that the Osterwalder-Schrader axioms of continuous Euclidean Fermionic theories hold for the massive Gross-Neveu model at $\Lambda = 0$.

The simplest proof is to remark that being the Borel sum of the renormalized expansion, the Schwinger functions we build are unique. Building them as limits of theories with different kinds of cutoffs prove the axioms since different sets of cutoffs violate different axioms [FMRS].

Acknowledgements

It is a pleasure to thank D. Brydges, C. de Calan, H. Knörrer, C. Kopper, G. Poirot and M. Salmhofer for discussions related to this paper. In particular we thank C. Kopper for his careful reading of this preprint, which lead us to correct the initial version of Subsections IV.3.4 and IV.3.5.

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Communicated by J. Bellissard
submitted 02/03/98, revised 08/09/98; accepted 18/11/98