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A Special Case of Mahler's Conjecture*

M. A. Lopez[1] and S. Reisner[1,2]

[1]Department of Mathematics and Computer Science, University of Denver,
Denver, CO 80208, USA
mlopez@cs.du.edu
sreisner@cs.du.edu

[2]Department of Mathematics and School of Education-Oranim,
University of Haifa, Haifa 31905, Israel
reisner@mathcs2.haifa.ac.il

Abstract. A special case of Mahler's conjecture on the volume-product of symmetric convex bodies in n-dimensional Euclidean space is treated here. This is the case of polytopes with at most $2n+2$ vertices (or facets). Mahler's conjecture is proved in this case for $n \le 8$ and the minimal bodies are characterized.

1. Introduction and Statement of the Results

A well-known problem in the theory of convex sets is the one of finding an exact lower bound for the product of n-dimensional volumes

$$\mathrm{volprod}(K) = \mathrm{vol}_n(K)\,\mathrm{vol}_n(K^*)$$

(the "volume-product" of K), where K is a convex body (i.e., a compact, convex set with nonempty interior) in $\mathbb{R}^n$, which is centrally symmetric about the origin, and K^* is the polar body of K with respect to the origin (another variant of the problem asks a similar question without the assumption of central symmetry, we do not consider that variant here). An old conjecture of Mahler (who proved it for $n = 2$) [10], [11] is

$$\mathrm{volprod}(K) \ge \frac{4^n}{n!}. \tag{1.1}$$

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The volume-product of K is clearly invariant under linear transformation acting on K. We note that an n-dimensional cube and its polar body—the cross-polytope—give equality in (1.1), but for every $n \geq 4$ there exist examples of symmetric convex bodies for which equality in (1.1) is obtained, which are not affinely isomorphic to a cube or to a cross-polytope. The general result in the direction of (1.1) currently known is the theorem of Bourgain and Milman [6] which states that (1.1) is valid, up to a factor c^n where c is a constant. The inequality (1.1) has been proved in the case that K is a zonoid, in that case equality is obtained only for cubes and their affinely isomorphic associates, the parallelotopes (Reisner [15], see also [9]). Another case where (1.1) is known to be true is when there exists an affinely isomorphic associate of K which is symmetric about the coordinate hyperplanes in $\mathbb{R}^n$ (Saint-Raymond [17], with characterization of the equality cases by Meyer [13] and by Reisner [16]). We also mention the upper bound, known as the Blaschke–Santaló inequality:

$$\mathrm{volprod}(K) \leq \chi_n^2,$$

where χ_n is the volume of the n-dimensional Euclidean unit ball.

In this paper we treat a special case of Mahler's conjecture, this case has recently been raised as a separate problem by Ball ([2], where some implications to other areas are discussed). We consider here the problem whether (1.1) is true if K is an n-dimensional polytope which is centrally symmetric about the origin and has at most $2n+2$ vertices (or, equivalently since the problem is self-dual, at most $2n+2$ facets). Notice that $2n+2$ is the minimal number of vertices of such polytopes if they are not affinely isomorphic to the cross-polytope.

By reducing the problem to a search over a finite set of polytopes for each fixed dimension, we managed, with the help of computer calculations, to prove (1.1) for $n \leq 8$ and to characterize the cases of equality in (1.1) in this restricted problem and for these dimensions (Theorems 1.1 and 1.2). It is probable that further investigation into the method may lead to a proof of (1.1) in this restricted case, for every n. An estimate which we bring in what follows shows that the amount of computing time which is needed to carry out the process in dimensions higher than 8 is enormous. At this stage we did not treat higher dimensions.

We now introduce some notations: A convex body in $\mathbb{R}^n$ is called here *symmetric* if it is centrally symmetric about the origin. The *polar body* K^* of a symmetric convex body K is defined as

$$K^* = \{x \in \mathbb{R}^n; |\langle x, y\rangle| \leq 1 \text{ for all } y \in K\},$$

where $\langle x, y\rangle = \sum_{i=1}^n x(i)y(i)$ is the standard scalar product in $\mathbb{R}^n$. ∂K is the boundary of K and a *facet* is an $(n-1)$-dimensional face of K, the *vertices* of K are its extreme points.

For a set $A \subset \mathbb{R}^n$ we denote $A^\perp = \{x \in \mathbb{R}^n; \langle x, y\rangle = 0 \text{ for all } y \in A\}$, if $A = \{u\}$ is a singleton, we write $A^\perp = u^\perp$; in this case (if $u \neq 0$) P_u denotes the orthogonal projection onto $u^\perp$.

If K is a symmetric convex body in $\mathbb{R}^n$ we denote by $\|\cdot\|_K$ the norm induced in $\mathbb{R}^n$ by K: $\|x\|_K = \inf\{\rho > 0; \rho^{-1}x \in K\}$. $(\mathbb{R}^n, \|\cdot\|_K)$ is a Banach space and $(\mathbb{R}^n, \|\cdot\|_{K^*})$ is its dual Banach space (duality defined via the scalar product). In the other direction, if $X = (\mathbb{R}^n, \|\cdot\|)$ is a Banach space, then the unit ball of X, $K = B(X)$ is a symmetric convex body and $\|\cdot\|_K = \|\cdot\|$, we allow an abuse of notation and in the last example write also $\|\cdot\|_K = \|\cdot\|_X$.

The norms $\|\cdot\|_p$ $(1 \le p \le \infty)$ on $\mathbb{R}^n$ are defined by

$$\|x\|_p = \left(\sum_{i=1}^{n} |x(i)|^p\right)^{1/p} \qquad (\max_{1\le i\le n} |x(i)| \text{ if } p = \infty).$$

The Banach space $(\mathbb{R}^n, \|\cdot\|_p)$ is denoted be ℓ_p^n and its unit ball by B_p^n. In particular, B_∞^n is the n-dimensional cube, B_1^n is the cross-polytope, and B_2^n is the Euclidean ball, the Euclidean sphere is S^{n-1}. If $X = (\mathbb{R}^n, \|\cdot\|_X)$ and $Y = (\mathbb{R}^m, \|\cdot\|_Y)$ are Banach spaces, then $X \oplus_p Y$ is the Banach space $X \oplus Y$ with the norm $\|(x, y)\|_{X\oplus_p Y} = \|(\|x\|_X, \|y\|_Y)\|_p$.

Finally, the closed convex hull of a set $A \subset \mathbb{R}^n$ is denoted by $\operatorname{conv}(A)$ and the unit coordinate vectors in $\mathbb{R}^n$ are denoted by e_i $(1 \le i \le n)$.

Theorem 1.1. *Let* $2 \le n \le 8$ *be an integer and let* K *be a symmetric convex body in* $\mathbb{R}^n$, *which is a polytope with at most* $2n+2$ *vertices (or at most* $2n+2$ *facets). Then*

$$\operatorname{vol}_n(K)\operatorname{vol}_n(K^*) \ge \frac{4^n}{n!}.$$

Theorem 1.2. *Let* $2 \le n \le 8$ *be an integer and let* K *be a symmetric convex body in* $\mathbb{R}^n$, *which is a polytope with at most* $2n+2$ *facets, the equality*

$$\operatorname{vol}_n(K)\operatorname{vol}_n(K^*) = \frac{4^n}{n!}$$

holds if and only if K *satisfies the following*:

If $n = 2$, K *is a parallelogram.*
If $n = 3$, K *is either a parallelotope or is affinely isomorphic to the cross-polytope* B_1^3.
If $4 \le n \le 8$, K *is either a parallelotope or is affinely isomorphic to the unit ball of the Banach space* $\ell_1^3 \oplus_\infty \ell_\infty^{n-3}$.

The proofs of the two theorems are structured along the following lines: In Sections 2 and 3 we reduce the problem to a search for the polytope which minimizes $\operatorname{volprod}(K)$, among a *subset* of the set of polytopes satisfying the right conditions, and this subset is *finite* for every fixed n. In this step the dimension is not restricted by 8. In Section 4 we present the results of the computations for the dimensions $2 \le n \le 8$. These results establish the truth of Theorem 1.1 for these dimensions. Some further observations are also presented in Section 4. In Section 5 we prove Theorem 1.2. We conjecture that Theorem 1.2 is true for all $n \ge 2$.

2. Transforming the Problem into a Finite Search

It is well known (see, e.g., [2]) that the problem of finding the minimum of

$$\operatorname{volprod}(K) = \operatorname{vol}_n(K)\operatorname{vol}_n(K^*),$$

where K ranges over all n-dimensional centrally symmetric convex bodies in $\mathbb{R}^n$ with at most $2n+2$ vertices, is equivalent to that of finding

$$\min \operatorname{vol}_n(P_u(B_1^{n+1}))\operatorname{vol}_n(u^\perp \cap B_\infty^{n+1}), \tag{2.1}$$

where u ranges over all nonzero vectors in $\mathbb{R}^{n+1}$.

Let $E = (\mathbb{R}^{n+1}, \|\cdot\|_E)$ be the $(n+1)$-dimensional Banach space whose unit ball is the polar body of the projection body of B_1^{n+1} (see, e.g., [8]). That is, for $0 \neq v \in \mathbb{R}^{n+1}$,

$$\|v\|_E = \|v\|_2 \operatorname{vol}_n(P_v(B_1^{n+1})) = \frac{1}{2n!}\sum_{\varepsilon\in\{-1,1\}^{n+1}} |\langle v, \varepsilon\rangle|. \tag{2.2}$$

Let $F = (\mathbb{R}^{n+1}, \|\cdot\|_F)$ be the $(n+1)$-dimensional Banach space whose unit ball is the intersection body of B_∞^{n+1}. That is, for $0 \neq v \in \mathbb{R}^{n+1}$,

$$\|v\|_F = \frac{\|v\|_2}{\operatorname{vol}_n(v^\perp \cap B_\infty^{n+1})}. \tag{2.3}$$

(It is well known that (2.2) and (2.3) define norms on $\mathbb{R}^{n+1}$. In fact, for (2.2) this is obvious, while for (2.3) this is a consequence of a result of Busemann [7] (see [8])).

Using these notations we may write the minimum (2.1) as

$$\min \frac{\|v\|_E}{\|v\|_F} = \frac{1}{\max(\|v\|_F/\|v\|_E)} = \frac{1}{\|I : E \to E\|}, \tag{2.4}$$

where in both minimum and maximum in the above expression, v ranges over all nonzero vectors in $\mathbb{R}^{n+1}$; $\|I\colon\ E \to F\|$ is the operator-norm of the identity on $\mathbb{R}^{n+1}$ acting from E into F.

Since a linear operator on a finite-dimensional Banach space E attains its norm at an extreme point of the unit ball of E we get:

Claim 2.1. *The minimum in* (2.1) *is attained at an extreme point u of the polar of the projection body of B_1^{n+1}.*

The minimum in (2.1) is independent of $\|u\|_2$, hence we can replace the polar of the projection body of B_1^{n+1} in Claim 2.1 by the symmetric convex body $C = n!\,B(E)$ which is the unit ball of the norm

$$\|v\|_C = \sum_{\varepsilon\in\mathcal{E}} |\langle v, \varepsilon\rangle|, \tag{2.5}$$

where $\mathcal{E} = \{\varepsilon = (\varepsilon(1), \ldots, \varepsilon(n+1)) \in \{-1, 1\}^{n+1};\ \varepsilon(1) = 1\}$. A use of the embedding

$$v \to (\langle v, \varepsilon\rangle)_{\varepsilon\in\mathcal{E}}$$

of $\mathbb{R}^{n+1}$ in $\mathbb{R}^{2^n}$ together with (2.5) enables us to consider C as the intersection of an $(n+1)$-dimensional subspace M of $\mathbb{R}^{2^n}$ with the ℓ_1-unit ball $B_1^{2^n}$. Under this embedding, an extreme point of C is an intersection point of M with a $(2^n-(n+1))$-dimensional face of $B_1^{2^n}$. We index the coordinates of $\mathbb{R}^{2^n}$ in the above embedding by $\varepsilon \in \mathcal{E}$, thus, using an obvious characterization of the faces of $B_1^{2^n}$, the above observation can be formulated as:

Claim 2.2. *If a vector $v \in \mathbb{R}^{n+1}$ is an extreme point of C, then*

$$\sum_{\varepsilon \in \mathcal{E}} |\langle v, \varepsilon \rangle| = 1 \tag{2.6}$$

and there exist $\varepsilon_1, \ldots, \varepsilon_n \in \mathcal{E}$ ($\varepsilon_i \neq \varepsilon_j$ if $i \neq j$) such that

$$\langle v, \varepsilon_i \rangle = 0 \tag{2.7}$$

for $i = 1, \ldots, n$.

In fact, $v \in \mathbb{R}^{n+1}$ ($\mathbb{R}^{n+1}$ being identified with $M \subset \mathbb{R}^{2^n}$) is an extreme point of C *if and only if* (2.6) is satisfied and there are $\varepsilon_1, \ldots, \varepsilon_n \in \mathcal{E}$ satisfying (2.7), such that v is, up to multiplication by a scalar, the only nonzero intersection point of M with the $(2^n - n)$-dimensional subspace N of $\mathbb{R}^{2^n}$ given by

$$N = \{x \in \mathbb{R}^{2^n}; x(\varepsilon_i) = 0, i = 1, \ldots, n\}.$$

This additional condition means that (2.7), considered as a system of n homogeneous linear equations in $n + 1$ unknowns has a one-dimensional solution space. We conclude:

Proposition 2.3. *A vector $v \in \mathbb{R}^{n+1}$ is an extreme point of C if and only if* (2.6) *is satisfied and there exist* linearly independent *vectors $\varepsilon_1, \ldots, \varepsilon_n \in \mathcal{E}$ for which* (2.7) *holds.*

Proposition 2.3 reduces the search for a minimum in (2.1) to a finite search. In fact, by Claim 2.1 and the remarks following it, we may consider in (2.1) only the vectors u which are direction vectors of extreme points of C, i.e., nontrivial solutions of systems of homogeneous linear equations of the form (2.7) with $\varepsilon_1, \ldots, \varepsilon_n \in \mathcal{E}$ linearly independent.

Formally, a finite algorithm could be formulated as follows:

(a) Find all the subsets $\{\varepsilon_1, \ldots, \varepsilon_n\} \subset \mathcal{E}$ which are linearly independent.
(b) For each subset found in (a), find a nontrivial solution $u \in \mathbb{R}^{n+1}$ for the system of equations

$$\langle u, \varepsilon_i \rangle = 0, \qquad i = 1, \ldots, n.$$

(c) For every u found in (b), evaluate

$$\mathrm{vol}_n(P_u(B_1^{n+1}))\,\mathrm{vol}_n(u^\perp \cap B_\infty^{n+1})$$

and take the minimum of these values.

This algorithm, especially step (a), is however quite costly. In the next section we describe the actual algorithm which we used.

3. The Finite-Search Algorithm

For the sake of completeness, we begin with the case $n = 2$, even though the truth of Mahler's conjecture in this case was proved by Mahler [10] (see [17] and [9] for alternative proofs), with characterization of the case of equality given in [15] (see also [9]).

It is easy to check that, up to a multiplication by a scalar, the only nonzero vectors v which are solutions of a system of linear homogeneous equations of the form (2.7) with $\varepsilon_i = (1, \pm 1, \pm 1)$, $i = 1, 2$, $\varepsilon_1, \varepsilon_2$ linearly independent, are $v = (1, 1, 0)$ and all the vectors obtained from it by permutations and sign-changes of the coordinates. The central sections of B^3_∞ orthogonal to all these vectors v are congruent to one another and all are rectangles whose vertices are vertices of the cube B^3_∞. Hence for all of them we get the volume-product $4^2/2! = 8$. Which proves the result for dimension 2.

We now investigate the case of arbitrary $n > 2$. The search is reduced using the following claim and two lemmas.

Claim 3.1. *Invariance of B^{n+1}_∞ and B^{n+1}_1 under sign-changes and permutations of the coordinates implies that it is sufficient to search only for vectors $v \in \mathbb{R}^{n+1}$ with nonnegative entries and such that their entries form a nonincreasing sequence.*

We call a vector $0 \neq v \in \mathbb{R}^{n+1}$ which satisfies (2.7) for linearly independent $\varepsilon_1, \dots, \varepsilon_n \in \mathcal{E}$ and is of the form specified in Claim 3.1 a *candidate*.

Lemma 3.2. *If the inequality* (1.1) *has been proven for all dimensions* $\leq (n-1)$, *then for dimension n we have to check only candidates with strictly positive entries.*

Proof. Let $v = (v(1), \dots, v(n+1))$ be a candidate with $v(k+2) = \cdots = v(n+1) = 0$. Let $\tilde{v} = (v(1), \dots, v(k+1)) \in \mathbb{R}^{k+1}$. Then $v^\perp = \tilde{v}^\perp \times \mathbb{R}^{n-k}$ and $v^\perp \cap B^{n+1}_\infty = (\tilde{v}^\perp \cap B^{k+1}_\infty) \times B^{n-k}_\infty$.

That is, if $\tilde{v}^\perp \cap B^{k+1}_\infty$ is considered to be a unit ball of a k-dimensional Banach space X, then the n-dimensional Banach space with unit ball $v^\perp \cap B^{n+1}_\infty$ is isometric to the ℓ_∞-sum $X \oplus_\infty \ell^{n-k}_\infty$ and its dual space is, via the same isometry, $X^* \oplus_1 \ell^{n-k}_1$.

It is well known and not hard to show (see, e.g., [17]) that the volume-product of the unit balls of the above two spaces is

$$\binom{n}{k}^{-1} \operatorname{volprod}(B(X)) \operatorname{volprod}(B(\ell^{n-k}_\infty)) = \frac{4^{n-k}k!}{n!} \operatorname{volprod}(B(X)) \geq \frac{4^n}{n!} \tag{3.1}$$

by the induction hypotheses. □

In view of Lemma 3.2 we give the name a *strong candidate* to a candidate with strictly positive entries.

Lemma 3.3. *Let $v \in \mathbb{R}^{n+1}$ be a strong candidate. Then v is parallel to a strong candidate $w \in \mathbb{R}^{n+1}$, $w = (w(1), \dots, w(n+1))$, such that $w(i)$ is an integer for all i and*

$$1 \leq w(i) \leq \frac{n^{n/2}}{2^{n-1}}, \qquad i = 1, \dots, n+1. \tag{3.2}$$

Proof. Let $\varepsilon_1, \dots, \varepsilon_n \in \mathcal{E}$ be linearly independent and such that $\langle v, \varepsilon_i \rangle = 0$ for $i = 1, \dots, n$. We may assume without loss of generality that $v(1) = 1$ and

$$D_1 = \det\begin{pmatrix} \varepsilon_1(2) & \cdots & \varepsilon_1(n+1) \\ \vdots & \ddots & \vdots \\ \varepsilon_n(2) & \cdots & \varepsilon_n(n+1) \end{pmatrix} \neq 0.$$

Since

$$\varepsilon_i(2)v_2 + \cdots + \varepsilon_i(n+1)v_{n+1} = -1 \qquad \text{for} \quad i = 1, \dots, n, \tag{3.3}$$

we get, by Cramer's rule,

$$v_j = \frac{D_j}{D_1}, \qquad j = 2, \dots, n+1,$$

where, for $j = 2, \dots, n+1$ (with obvious modifications for $j = 2$ or $n+1$),

$$D_j = \det\begin{pmatrix} \varepsilon_1(2) & \cdots & \varepsilon_1(j-1) & -1 & \varepsilon_1(j+1) & \cdots & \varepsilon_1(n+1) \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \varepsilon_n(2) & \cdots & \varepsilon_n(j-1) & -1 & \varepsilon_n(j+1) & \cdots & \varepsilon_n(n+1) \end{pmatrix}.$$

So the vector $u = (D_1, \dots, D_{n+1})$ has no zero entries and is parallel to v. By Hadamard's inequality we have

$$|D_j| \leq n^{n/2}.$$

Also, 2^{n-1} is a common divisor of D_j, $j = 1, \dots, n+1$. This can be seen by performing one step of the Gaussian elimination process on the matrix defining D_j, obtaining a matrix of the form

$$A = \begin{pmatrix} \pm 1 & \pm 1 & \cdots & \pm 1 \\ 0 & \eta_{22} & \cdots & \eta_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \eta_{n2} & \cdots & \eta_{nn} \end{pmatrix},$$

in which $\eta_{ij} = \pm 2$ or 0 for $2 \leq i, j \leq n$. Clearly, $D_j = \det(A)$ is divisible by 2^{n-1}. As u is parallel to v, it has entries with equal signs, whose absolute values form a nonincreasing sequence. By changing signs of the entries of u if necessary, and dividing them by 2^{n-1}, we obtain a vector w as stated. $\square$

It follows from Lemma 3.3 that we have to search only for strong candidates with entries which are integers between 1 and $\lfloor n^{n/2}/2^{n-1} \rfloor$.

The following algorithm creates, for any n, a set SC of all the strong candidates satisfying the conditions on w in Lemma 3.3, excluding cases of parallel candidates. Once the full set SC has been created, it is left to evaluate

$$\mathrm{vol}_n(P_{v^\perp}(B_1^{n+1}))\,\mathrm{vol}_n(v^\perp \cap B_\infty^{n+1})$$

for all $v \in SC$ and verify that this value is always $\geq 4^n/n!$. The algorithm begins with $SC = \emptyset$.

(a) Create all nonincreasing sequences $w = (w(1), \dots, w(n+1))$ of integers with $1 \leq w(i) \leq \lfloor n^{n/2}/2^{n-1} \rfloor$.
(b) In (a), for each w, if $w(1) + \dots + w(n+1) \equiv 1 (\mathrm{mod}\ 2)$, then drop this w.
(c) In (a), for each w, if the greatest common divisor of the entries of w is bigger than 1, then drop this w.
(d) In (a), if w has not been dropped in (b) or (c), let U be the matrix with $n+1$ columns, whose rows are all the vectors $\varepsilon \in \mathcal{E}$ such that $\langle w, \varepsilon \rangle = 0$. If $\mathrm{rank}(U) = n$, then join w to SC, else, drop w (clearly, $\mathrm{rank}(U) \leq n$).

Remark. The condition $w(1) + \dots + w(n+1) \equiv 0 (\mathrm{mod}\ 2)$ is necessary for $\langle w, \varepsilon \rangle = 0$, even for one $\varepsilon \in \mathcal{E}$.

4. The Computations

The Cauchy formula gives the volume of a projection of B_1^{n+1}:

$$\mathrm{vol}_n(P_v(B_1^{n+1})) = \frac{1}{2n!\,\|v\|_2} \sum_{\varepsilon \in \{-1,1\}^{n+1}} |\langle \varepsilon, v \rangle| \tag{4.1}$$

(see, e.g., [2]).

A formula for the n-dimensional volume of $v^\perp \cap B_\infty^{n+1}$, where $v = (v(1), \dots, v(n+1))$ satisfies $v(i) > 0$ for $i = 1, \dots, n+1$, can be derived as follows. It is known (see, e.g., [14] or [3]) that, for $t > 0$,

$$\begin{aligned} &\mathrm{vol}_{n+1}(\{x \in [0,1]^{n+1}; \langle x, v \rangle \leq t\}) \\ &\quad = \frac{1}{(n+1)! \prod_{i=1}^{n+1} v(i)} \left[\sum_{I \subset \{1,\dots,n+1\}} (-1)^{|I|} \left(t - \sum_{j \in I} v(j) \right)_+^{n+1} \right] \end{aligned} \tag{4.2}$$

($|I|$ denotes the cardinality of I).

Normalizing v, differentiating with respect to t, and then rescaling, we get

$$\mathrm{vol}_n(v^\perp \cap B_\infty^{n+1}) = \frac{\|v\|_2}{n! \prod_{i=1}^{n+1} v(i)} \left[\sum_{I \subset \{1,\dots,n+1\}} (-1)^{|I|} \left(\sum_{i \notin I} v(i) - \sum_{i \in I} v(i) \right)_+^n \right]. \tag{4.3}$$

Combining (4.1) and (4.3) together, we obtain the formula

$$\begin{aligned} &\mathrm{volprod}(P_v(B_1^{n+1})) \\ &\quad = \frac{1}{(n!)^2 \prod_{i=1}^{n+1} v(i)} \left(\sum_{\vartheta \in \{-1,1\}^{n+1}} \langle \vartheta, v \rangle_+ \right) \left(\sum_{\varepsilon \in \{-1,1\}^{n+1}} \left(\prod_{j=1}^{n+1} \varepsilon(j) \right) \langle \varepsilon, v \rangle_+^n \right). \end{aligned} \tag{4.4}$$

We have used (4.4) to compute the volume-product associated with a strong candidate v.

In all the cases which we have checked, i.e., the dimensions $n = 3, \dots, 8$, the minimal volume-product associated with strong candidates has been obtained for the strong candidate

$$v_0 = v_0^{n+1} = (n-2, 1, \dots, 1)$$

(of length $n+1$). v_0^{n+1} is, in fact, a strong candidate for any dimension n: it satisfies $\langle v_0^{n+1}, \varepsilon_i \rangle = 0$ where ε_i ($i = 1, \dots, n$) are the vectors $\varepsilon_i = (1, -1, \dots, -1, +1, -1, \dots, -1)$ ($+1$ in the $(i+1)$-coordinate), which are clearly linearly independent.

In fact, the n-dimensional convex body $(v_0^{n+1})^\perp \cap B_\infty^{n+1}$ has a nice description: it is affinely isomorphic to the convex body B in $\mathbb{R}^n$ which is obtained from the cube B_∞^n by truncating its "corner," the simplex

$$S = \operatorname{conv}(\{(1, 1, \dots, 1), (-1, 1, \dots, 1), (1, -1, \dots, 1) \dots\})$$

and its opposite "corner," $-S$. The volume-product of this body can be computed explicitly and we get

$$\operatorname{volprod}((v_0)^\perp \cap B_\infty^{n+1}) = \operatorname{vol}_n(B) \operatorname{vol}_n(B^*) = \frac{2^n}{n!}\left(2^n + \frac{4}{n-2}\right)\left(1 - \frac{2}{n!}\right). \quad (4.5)$$

For $n = 3$ (then $v_0 = (1, 1, 1, 1)$), B is affinely isomorphic to B_1^3 and indeed (4.5) gives in this case

$$\operatorname{volprod}((v_0)^\perp \cap B_\infty^4) = \frac{32}{3} = \frac{4^3}{3!}. \quad (4.6)$$

However, for $n \geq 4$, examination of (4.5) shows that

$$\operatorname{volprod}((v_0^{n+1})^\perp \cap B_\infty^{n+1}) > \frac{4^n}{n!}$$

$$\left(\text{but } \lim_{n \to \infty} \frac{\operatorname{volprod}((v_0^{n+1})^\perp \cap B_\infty^{n+1})}{4^n/n!} = 1\right).$$

Hence, the truth of the following conjecture would imply the truth of the result of this paper for *any n*:

Conjecture. *The minimum of*

$$\operatorname{volprod}(P_v(B_1^{n+1}))$$

for strong candidates v, is obtained for $v = v_0^{n+1}$.

It is worthwhile remarking that, for odd n, the vector $v_1^{n+1} = (1, \dots, 1)$ ($(n+1)$-entries) is always a strong candidate. However, for $n \geq 5$, $\operatorname{volprod}(P_{v_1^{n+1}}(B_1^{n+1}))$ is

Table 1. Partial results of the algorithm of Section 3.

n	$\lfloor n^{n/2}/2^{n-1} \rfloor$	Number of precandidates	Number of strong candidates	Computing time
3	1	1	1	Negligible
4	2	6	1	Negligible
5	3	28	4	10 mseconds
6	6	792	14	330 mseconds
7	14	203,490	122	129 seconds
8	32	273,438,880	3287	53.12 hours
9	76	3,129,162,672,636	—	> 155 years
10	195	512,362,040,342,757,150	—	>25×10^6 years

always bigger than $4^n/n!$ and, in fact,

$$\lim_{k\to\infty} \frac{\text{volprod}(P_{v_1^{2k}}(B_1^{2k}))}{4^{(2k-1)}/(2k-1)!} = \frac{\sqrt{12}}{\pi} = 1.1026\ldots,$$

this can be shown using the integral formula for $\text{vol}_n(v^\perp \cap B_\infty^n)$ (which is found, e.g., in [1]).

The algorithm of Section 3 was implemented in C++ and run on a Sun Sparc 20 workstation. In order to guarantee the accuracy of the results all operations were done using only integer arithmetic. Furthermore, computing the volume-product of candidates in dimensions higher than 6 requires dealing with integers larger than those supported by the hardware (a Sun Sparc long integer uses 32 bits). Therefore, all volume-product calculations (and only those) were done using the extended integer class available in the LEDA package [12]. Volume-products were stored and compared as rational numbers (ratios of LEDA integers). These products were converted to floating-point values only when producing the final output for the sake of presentation.

The results of the algorithm for different values of n are summarized in Tables 1 and 2. These results complete the proof of Theorem 1.1. The term *precandidates* in Table 1 refers to the vectors $v = (v(1), \ldots, v(n+1))$ with positive nonincreasing entries and with $v(1) \le \lfloor n^{n/2}/2^{n-1} \rfloor$. The running time for dimension 9 was estimated by sampling with 10^9 precandidates and that for dimension 10 by extrapolating from dimension 9. Clearly, due to its time complexity, it is not practical to run this algorithm for $n > 8$. The interested reader can obtain the software and the full program output by contacting one of the authors.

5. The Cases of Equality

For the proof of Theorem 1.2 we need the following description of the facial structure of the intersection body of an n-dimensional cube ($n \ge 3$). The Banach space F has been defined in (2.3) except that here we replace $\mathbb{R}^{n+1}$ by $\mathbb{R}^n$. For discussion of intersection bodies we refer the reader to [8].

Proposition 5.1. *Let $B(F)$ be the intersection body of B_∞^n ($n \ge 3$), i.e., $B(F)$ is the symmetric convex body whose radial function r is given by $r(u) = \text{vol}_{n-1}(u^\perp \cap B_\infty^n)$.*

Table 2. Volume product of the strong candidates found by the algorithm of Section 3. For $n \geq 7$ only the vector that produces the minimum volume product is shown.

n	$4^n/n!$	Strong candidates	Volume-product
3	10.6667...	$v_0^4 = (1,1,1,1)$	$\text{volprod}(P_v(B_1^4)) = 10.6667\ldots$
4	10.6667...	$v_0^5 = (2,1,1,1,1)$	$\text{volprod}(P_v(B_1^5)) = 11$
5	8.5333...	$v_0 = (3,1,1,1,1,1)$	$\text{volprod}(P_{v_0}(B_1^6)) = 8.7407\ldots$
		$v_1 = (1,1,1,1,1,1)$	$\text{volprod}(P_{v_1}(B_1^6)) = 8.8$
		$v_2 = (2,2,1,1,1,1)$	$\text{volprod}(P_{v_2}(B_1^6)) = 8.9955\ldots$
		$v_3 = (3,2,2,1,1,1)$	$\text{volprod}(P_{v_3}(B_1^6)) = 9.0651\ldots$
6	5.6888...	$v_0 = (4,1,1,1,1,1,1)$	$\text{volprod}(P_{v_0}(B_1^7)) = 5.7617\ldots$
		$v_1 = (5,2,2,2,1,1,1)$	$\text{volprod}(P_{v_1}(B_1^7)) = 5.9164\ldots$
		$v_2 = (4,2,2,2,1,1,1)$	$\text{volprod}(P_{v_2}(B_1^7)) = 5.9629\ldots$
		$v_3 = (2,1,1,1,1,1,1)$	$\text{volprod}(P_{v_3}(B_1^7)) = 5.9654\ldots$
		$v_4 = (3,2,1,1,1,1,1)$	$\text{volprod}(P_{v_4}(B_1^7)) = 5.9843\ldots$
		$v_5 = (5,3,3,2,1,1,1)$	$\text{volprod}(P_{v_5}(B_1^7)) = 6.0632\ldots$
		$v_6 = (2,2,2,1,1,1,1)$	$\text{volprod}(P_{v_6}(B_1^7)) = 6.0633\ldots$
		$v_7 = (4,3,2,2,1,1,1)$	$\text{volprod}(P_{v_7}(B_1^7)) = 6.0698\ldots$
		$v_8 = (3,2,2,2,1,1,1)$	$\text{volprod}(P_{v_8}(B_1^7)) = 6.0744\ldots$
		$v_9 = (3,3,2,1,1,1,1)$	$\text{volprod}(P_{v_9}(B_1^7)) = 6.0911\ldots$
		$v_{10} = (5,4,3,2,2,1,1)$	$\text{volprod}(P_{v_{10}}(B_1^7)) = 6.1111\ldots$
		$v_{11} = (4,3,3,2,2,1,1)$	$\text{volprod}(P_{v_{11}}(B_1^7)) = 6.1249\ldots$
		$v_{12} = (3,3,2,2,2,1,1)$	$\text{volprod}(P_{v_{12}}(B_1^7)) = 6.1334\ldots$
		$v_{13} = (4,3,3,1,1,1,1)$	$\text{volprod}(P_{v_{13}}(B_1^7)) = 6.1355\ldots$
7	3.2507...	$v_0 = (5,1,1,1,1,1,1,1)$	$\text{volprod}(P_{v_0}(B_1^8)) = 3.2698\ldots$
8	1.6254...	$v_0 = (6,1,1,1,1,1,1,1,1)$	$\text{volprod}(P_{v_0}(B_1^9)) = 1.6295\ldots$

Then

(a) *The following sets G_i as well as $-G_i$ $(i = 1, \ldots, n)$ are facets of $B(F)$:*

$$G_i = 2^{n-1}\operatorname{conv}(\{e_i + e_j, e_i - e_j;\ j \neq i, j = 1, \ldots, n\}).$$

(b) *If $v \in \partial B(F) \setminus \bigcup_{i=1}^n (\pm G_i)$, then $B(F)$ is strictly convex at v.*

Proof. (a) Notice that

$$\frac{1}{2^{n-1}} G_1 = \left\{ v = (1, v(2), \ldots, v(n));\ \sum_{j=2}^n |v(j)| \leq 1 \right\}.$$

Denote $K_1 = B_\infty^{n-1} \times \mathbb{R} = \{x \in \mathbb{R}^n; |x(i)| \leq 1 \text{ for } i = 2, \ldots, n\}$. It is not hard to check that if $v(1) = 1$, then $v \in (1/2^{n-1})G_1$ if and only if $v^\perp \cap B_\infty^n = v^\perp \cap K_1$. So, if $v \in (1/2^{n-1})G_1$, then

$$\|v\|_F = \frac{\|v\|_2}{\operatorname{vol}_{n-1}(v^\perp \cap K_1)} = \frac{\|v\|_2}{\operatorname{vol}_{n-1}(B_\infty^{n-1})/\cos(v, e_1)} = \frac{1}{2^{n-1}}.$$

This shows that

$$G_1 \subseteq \{v \in \mathbb{R}^n; x(1) = 2^{n-1}\} \cap B(F). \tag{5.1}$$

However, the same calculation shows that if $v(1) = 1$ and $v \notin (1/2^{n-1})G_1$, then $\|v\|_F > 1/2^{n-1}$. This shows that the inclusion (5.1) is in fact an equality and G_1 is a facet of $B(F)$.

(b) Denote $K = B_\infty^n$. Assume that $v_1, v_2 \in \partial B(F)$ are such that the nondegenerate line segment $[v_1, v_2]$ is contained in $\partial B(F)$. It may also be assumed that the angle between v_1 and v_2 is acute. Let S be the $(n-2)$-dimensional subspace $S = \{v_1, v_2\}^\perp$ and let $u_i \in S^{n-1} \cap S^\perp \cap v_i^\perp$, $i = 1, 2$, be chosen so that the angle between them is acute.

The proof of (b) is done by a close check of the proof of Busemann's theorem [7]. To avoid rewriting this proof, we refer to its presentation (with some improvements upon the original proof) in [8], pp. 276–278. We refer to this presentation of the proof as [7, 8], our above notation fits the notation of [7, 8].

Let $u_3 = (u_1 + u_2)/\|u_1 + u_2\|_2$. We have

$$r(u) = \mathrm{vol}_{n-1}(K \cap S_u),$$

where $S_u = v^\perp$ is the $(n-1)$-dimensional subspace spanned by S and u. The fact that $[v_1, v_2] \subset \partial B(F)$ implies that

$$\frac{r(u_3)}{\|u_1 + u_2\|_2} = (r(u_1)^{-1} + r(u_2)^{-1})^{-1}. \tag{5.2}$$

We define increasing functions $r_j(s)$, $0 \le s \le 1$, $j = 1, 2, 3$, as in [7, 8]. It is shown in [7, 8] that for $f_j(x) = \mathrm{vol}_{n-2}(K \cap (S + xu_j))$, $j = 1, 2, 3$, one gets, for $0 \le t \le 1$ and $0 \le s \le 1$,

$$\begin{aligned} f_3(r_3(s))^{1/(n-2)} &\ge (1-t) f_1(r_1(s))^{1/(n-2)} + t f_2(r_2(s))^{1/(n-2)} \\ &\ge f_1(r_1(s))^{(1-t)/(n-2)} f_2(r_2(s))^{t/(n-2)}. \end{aligned} \tag{5.3}$$

The first inequality in (5.3) follows from the Brunn–Minkowski inequality and the second inequality from the arithmetic-geometric mean inequality. The inequalities (5.3) are intermediate in a sequence of inequalities whose beginning and end give the inequality

$$\frac{r(u_3)}{\|u_1 + u_2\|_2} \ge (r(u_1)^{-1} + r(u_2)^{-1})^{-1}.$$

Thus, the equality (5.2) implies equality in the two inequalities of (5.3), for every $s \in (0, 1)$ and for certain values of t in $(0, 1)$.

By the equality case in Brunn–Minkowski we conclude from the first equality in (5.3), that, for every s, $K \cap (S + r_j(s)u_j)$, $j = 1, 2$, are homothetic. The second equality in (5.3) then shows that they have equal $(n-2)$-dimensional volumes, hence they are congruent. Further, another inequality in the above-mentioned chain of inequalities in [7, 8] is

$$\frac{1}{w_1 + w_2}(w_1 r(u_1))^{w_1/(w_1+w_2)}(w_2 r(u_2))^{w_2/(w_1+w_2)} \ge (r(u_1)^{-1} + r(u_2)^{-1})^{-1}, \tag{5.4}$$

where $w_j = r_j(s)^{-1}$, $j = 1, 2$. Inequality (5.4) is another case of the arithmetic-geometric inequality. Since equality in (5.2) implies equality in (5.4) for all s, we get $r_1(s) r(u_1) = r_2(s) r(u_2)$, or

$$\frac{r_1(s)}{r_2(s)} = \frac{r(u_2)}{r(u_1)} = \text{Const.}$$

We conclude that $K \cap S_{u_1}$ and $K \cap S_{u_2}$ are "one-dimensional dilations" of one another, which means that there exists a cylinder $\tilde{K} = M \times L$, where M is an $(n-1)$-dimensional convex body and L is a line orthogonal to the affine span of M, such that $K \cap S_{u_1} = \tilde{K} \cap S_{u_1}$ and $K \cap S_{u_2} = \tilde{K} \cap S_{u_2}$.

The above discussion shows also that, for all s,

$$f_3(r_3(s)) = f_2(r_2(s)) = f_1(r_1(s)),$$

so the same argument as above shows that $K \cap S_{u_3} = \tilde{K} \cap S_{u_3}$. By iteration of the process it now follows that for all $v \in [v_1, v_2]$ we have

$$K \cap v^{\perp} = \tilde{K} \cap v^{\perp}. \tag{5.5}$$

As $K = B_{\infty}^n$, the only central wedges cut from K, which are cylindrical wedges (in the sense of (5.5)), are cut from cylinders of the form $K_i = \{x \in \mathbb{R}^n; |x(j)| \leq 1, j = 1, \ldots, n, j \neq i\}$, $i = 1, \ldots, n$. Therefore (see the proof of (a) above) it follows that $[v_1, v_2] \subset G_i$ or $[v_1, v_2] \subset -G_i$ for one of the facets G_i of $B(F)$. □

Remarks. (i) The proof of (b) provides a general criterion for non-strict-convexity: the existence of a line segment on the boundary of the intersection body of a symmetric convex body K in $\mathbb{R}^n$ $(n \geq 3)$ is always equivalent to the existence of a central double wedge cut from K, which is cylindrical in the sense of (5.5).

(ii) The equality case in Busemann's theorem is investigated in [4] and [5]. The fact that such equality implies that $K \cap S_{u_1}$ and $K \cap S_{u_2}$ are what we call here "one-dimensional dilations" of each other, is proved there. For completeness we chose to include the above straightforward proof.

(iii) The formula (4.3) shows that, for $n \geq 4$, $\partial B(F)$ is twice differentiable at any point $v \in \partial B(F)$ with no zero coordinates. Does $\partial B(F)$ have positive Gauss curvature at such a point if this point is not in $\bigcup_{i=1}^n \pm G_i$?

Proposition 5.2. *Let $B(E)$ be the polar projection body of B_1^n $(n \geq 3)$. The sets*

$$H_i = \frac{(n-1)!}{2^{n-1}} \operatorname{conv}(\{e_i + e_j, e_i - e_j; j \neq i, j = 1, \ldots, n\}),$$

as well as the sets $-H_i$ $(i = 1, \ldots, n)$, are facets of $B(E)$.

Proof. As in the proof of Proposition 5.1, for every $v \in \mathbb{R}^n$ with $v(1) = 1$, $v \in (2^{n-1}/(n-1)!)H_1$ if and only if $P_v(e_1) \in \operatorname{conv}(\{P_v(e_2), \ldots, P_v(e_n)\})$. Thus, if $v \in H_1$ we have (using the notation $B_1^{n-1} = P_{e_1}(B_1^n)$)

$$\begin{aligned} \|v\|_E &= \|v\|_2 \operatorname{vol}_{n-1} P_v(B_1^n) = \|v\|_2 \operatorname{vol}_{n-1} P_v(B_1^{n-1}) \\ &= \|v\|_2 \frac{2^{n-1}}{(n-1)!} \left\langle \frac{v}{\|v\|_2}, e_1 \right\rangle = 1. \end{aligned} \tag{5.6}$$

Hence H_1 is contained in $\{x; x(1) = (n-1)!/2^{n-1}\} \cap B(E)$ which is thus a facet of $B(E)$. From (5.6) and the remark preceding it, it is also clear that if $v(1) = 1$ but $v \notin H_1$, then $\|v\|_E > 1$. We get

$$H_1 = \left\{x; x(1) = \frac{(n-1)!}{2^{n-1}}\right\} \cap B(E). \qquad \square$$

Proof of Theorem 1.2. In what follows $B(F)$ denotes the intersection body of B_∞^{n+1} and $B(E)$ is the projection body of B_1^{n+1}. The equality (2.4) means that, for $v \in \mathbb{R}^{n+1}$, $\mathrm{volprod}(P_v(B_1^{n+1}))$ is minimal if and only if there exist $\alpha > 0$ such that

$$\alpha B(E) \subset B(F) \quad \text{and} \quad v \in \alpha\partial B(E) \cap \partial B(F). \tag{5.7}$$

Denote $J_i = \mathrm{conv}(\{e_i + e_j, e_i - e_j;\ j = 1, \dots, n+1,\ j \neq i\})$. The facets G_i and H_i of $B(F)$ and $B(E)$, respectively, satisfy $G_i = 2^n J_i$, $H_i = (n!/2^n)J_i$. For $n = 2, \dots, 8$ we have, by Theorem 1.1, $\alpha = 4^n/n!$.

So for these dimensions $\alpha B(E)$ and $B(F)$ have joint facets $\pm\alpha H_i = \pm G_i$. For every v in such a joint facet, $v^\perp \cap B_\infty^{n+1}$ is a parallelotope of dimension n (see the proof of Proposition 5.1). Since $B(E)$ is a polytope, the strict convexity of $B(F)$, proved in Proposition 5.1, implies that a vector v which is not in any $\pm G_i$ can satisfy (5.7) only if v is an extreme point of $\alpha B(E)$.

The examination of the extreme points of $B(E)$ which has been done in Sections 3 and 4, shows that up to permutations and sign-changes of the coordinates, the only directions of extreme points of $B(E)$ for which $\mathrm{volprod}(v^\perp \cap B_\infty^{n+1})$ is minimal are: $(1, 1, 0, \dots, 0)$, $(1, 1, 1, 1)$ (for $n = 3$), and $(1, 1, 1, 1, 0, \dots, 0)$.

The vector $(1, 1, 0, \dots, 0)$ and its associates through permutations and sign-changes, are the directions of the extreme points of the facets $\pm G_i$.

If $v_0 = (1, 1, 1, 1)$, then (see the remarks which include (4.5) and (4.6)) $v_0^\perp \cap B_\infty^4$ is affinely isomorphic to the cross-polytope B_1^3.

If $n > 4$ and $v = (1, 1, 1, 1, 0, \dots, 0)$, then $v^\perp \cap B_\infty^{n+1} = (v_0^\perp \cap B_\infty^4) \times B_\infty^{n-3}$, which is affinely isomorphic (see the proof of Lemma 3.2) to the unit ball of $\ell_1^3 \oplus_\infty \ell_\infty^{n-3}$. $\square$

Remark. We conjecture that for any $n \geq 4$ the only directions v of extreme points of $B(E)$ for which $\mathrm{volprod}(v^\perp \cap B_\infty^{n+1})$ is minimal, are, up to permutations and sign changes, $(1, 1, 0, \dots, 0)$ and $(1, 1, 1, 1, 0, \dots, 0)$. If this conjecture is true, then the above proof would show that Theorem 1.2 is true for *every* n.

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