Published for SISSA by Springer

Received: *July 25, 2023*
Accepted: *December 18, 2023*
Published: *December 21, 2023*

Singular spin structures and superstrings

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Abstract: There are two main problems in finding the higher genus superstring measure. The first one is that for $g \geq 5$ the super moduli space is not projected. Furthermore, the supermeasure is regular for $g \leq 11$, a bound related to the source of singularities due to the divisor in the moduli space of Riemann surfaces with even spin structure having holomorphic sections, such a divisor is called the θ-null divisor. A result of this paper is the characterization of such a divisor. This is done by first extending the Dirac propagator, that is the Szegö kernel, to the case of an arbitrary number of zero modes, that leads to a modification of the Fay trisecant identity, where the determinant of the Dirac propagators is replaced by the product of two determinants of the Dirac zero modes. By taking suitable limits of points on the Riemann surface, this *holomorphic Fay trisecant identity* leads to identities that include points dependent rank 3 quadrics in $\mathbb{P}^{g-1}$. Furthermore, integrating over the homological cycles gives relations for the Riemann period matrix which are satisfied in the presence of Dirac zero modes. Such identities characterize the θ-null divisor. Finally, we provide the geometrical interpretation of the above points dependent quadrics and show, via a new θ-identity, its relation with the Andreotti-Mayer quadric.

Keywords: Differential and Algebraic Geometry, Superstrings and Heterotic Strings

ArXiv ePrint: 2307.12666

https://doi.org/10.1007/JHEP12(2023)150

JHEP12(2023)150

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1 Introduction

Despite the progress in formulating superstring theories, unlike the Polyakov string, there are still many unsolved problems. In the case of the bosonic string, the Mumford form on the moduli space of Riemann surfaces is known in terms of theta functions and prime forms. However, its expression in terms of theta constants is known only up to genus 4 [1–5]. In particular, the genus 4 Polyakov volume form, conjectured by Belavin and Knizhnik and proved in [5], shows that the main obstacle for the extension to arbitrary genus is strictly related to the problem of characterizing the Jacobian locus, that is the Schottky problem [6–10]. In this respect, in [5] it has been shown that the higher genus Mumford form is expressed in terms of vector-valued modular forms associated to the Schottky locus, which in turn suggests that the string partition function is a multiresidue on the Siegel upper half-space. Such investigations strongly suggest that, as observed by Belavin and Knizhink, any conformally invariant string theory can be expressed in terms of algebraic objects on the moduli space of Riemann surfaces. This is in the spirit of the Serre GAGA principle. In this respect, it has been shown that one can introduce modular invariant regularization of string determinants [11] and formulate finite string theories, even in 4 dimension, in terms of products of Mumford forms of different weight [12].

Finding the superstring measure for arbitrary genus g is much harder than in the bosonic case [13–42]. In particular, its explicit form is known only in the case of genus 1 and 2 and for $g \geq 5$ the super moduli space is not projected [15]. Furthermore, it turns out that the supermeasure is regular for $g \leq 11$, with the bound related to possible singularities in the locus in the moduli space of supercurves having even θ-characteristics with nontrivial sections [16]. A related result has been first argued by Witten who suggested possible problems starting at genus 11 [14]. His argument is based on the observation that for an even θ-characteristic with 2 zero-modes there are, in 10 dimension, 20 in all, and since there are $2g-2$ odd moduli it follows that for $g \geq 11$ there are sufficient picture-changing operators to absorb the fermion zero-modes and this may lead to a singularity.[1] The critical point is just when the dimension of $H^0(\Sigma; Ber(\Sigma))$ is not $g|0$ and this happens when there are spin bundles L_δ such that

$$h^0(\Sigma_{\rm red}, L_\delta) \neq 0\,, \tag{1.1}$$

with $\Sigma_{\rm red}$ the reduced surface of the super Riemann surface. In this case the isomorphism $Ber_1 = \det \mathcal{V}$, breaks down [14]. The divisor in the moduli space of Riemann surfaces with an even spin structure $\mathcal{M}_{g,{\rm spin}+}$ corresponding to an even θ-characteristic with $h^0(\Sigma, L_\delta) \geq 2$ is called the θ-null divisor.

One of the main results of the present paper is the characterization, given in eq. (4.17), of such a divisor. We start by investigating the properties of singular spin bundles, that is the ones associated to θ-characteristics, both odd and even, such that $\theta[\delta](0)$ and its gradient vanish. In other words, we will consider the spin bundles L_δ with

$$n = h^0(L_\delta) > 1\,. \tag{1.2}$$

We start by showing that by the Riemann vanishing theorem for singular spin structures one gets properties and relations that characterize holomorphic sections of $H^0(L_\delta)$. Such relations correspond to a subset of a wider class of holomorphic sections, namely the ones associated to the singular θ-divisor.

As a first step we introduce the Dirac propagator, or Szegö kernel, $S_\delta(z-w)$, in the presence of an arbitrary number of Dirac zero modes n, extending the case $n=1$, reported, for example, in [17]. We then note that removing the term $z-w$ from the argument of the θ-function in the modified Szëgo kernel gives

$$\frac{\theta[\delta]\left(\sum_1^n (z_i - w_i)\right)}{\prod_{i,j} E(z_i, w_j)}\,, \tag{1.3}$$

which is a holomorphic section that, after adding the prime forms to get a single valued expression, reproduces one side of the Fay trisecant identity [43]. The novelty is that now this expression is a holomorphic one. Furthermore, it turns out that the determinant of the Szegö kernel, which is the other side of the trisecant identity, is now replaced by the product of two determinants of a basis of $H^0(L_\delta)$, that is of the Dirac zero modes. In this respect, we note that whereas the Fay trisecant formula can be obtained starting from the Cauchy determinantal formula on the complex plane $\mathbb{C}$ by replacing the factors $z_i - w_j$, $z_i - z_j$ and $w_i - w_j$ by the corresponding prime forms and then adding a theta-function to absorb the multivaluedness, such a formula, that we call *holomorphic Fay trisecant identity* has no counterpart on $\mathbb{C}$. We will see that such an identity also implies relations between different spin structures, that include some invariants.

By taking suitable limits of points on the Riemann surface, the holomorphic Fay trisecant formula leads to identities that include points dependent quadrics in $\mathbb{P}^{g-1}$ of rank 3. Furthermore, integration over the homological cycles leads to relations for the Riemann period matrix τ which are satisfied in the presence of Dirac zero modes. These relations characterize the Jacobian locus when τ admits singular spin bundles with $n>1$, and, in

[1] I thank E. Witten for useful comments on this point.

particular, must be satisfied by the elements of the θ-null divisor. Finally, we provide the geometrical interpretation of the rank 3 quadric and shows, via a new θ-identity, its relation with the Andreotti-Mayer quadric [44].

2 Dirac propagator for singular spin structures

Let us start by introducing basic facts on the Riemann θ-functions. Excellent references are [43, 45].

Let C be a compact Riemann surface of genus g and K_C its cotangent bundle. Let α_i, and β_i, $i = 1, \ldots, g$, be a symplectic basis of $H_1(C, \mathbb{Z})$ and ω_i, $i = 1, \ldots, g$, the basis of $H^0(K_C)$ such that $\oint_{\alpha_i} \omega_j = \delta_{ij}$. Denote by $\tau_{ij} := \oint_{\beta_i} \omega_j$ the Riemann period matrix.

Let $C_n = \mathrm{Sym}^n(C)$, $n \in \mathbb{N}_+$, be the space of effective divisors of degree n on C and $J_n(C)$ the principal homogeneous space of linear equivalence classes of divisors of degree n on C. Denote by

$$J(C) := \mathbb{C}^g / L_\tau \,, \qquad L_\tau := \mathbb{Z}^g + \tau \mathbb{Z}^g \,, \tag{2.1}$$

the Jacobian of C. Choose an arbitrary point $p_0 \in C$ and denote by $I : C \to J(C)$, $I(p) := (I_1(p), \ldots, I_g(p))$, $I_i(p) := \int_{p_0}^{p} \omega_i$, $p \in C$, the Abel-Jacobi map, which is an embedding of C into the Jacobian. Note that $J(C)$ is identified with $J_0(C)$: each point of $J_0(C)$ can be expressed as $D_2 - D_1$ with D_1 and D_2 effective divisors of the same degree, this corresponds to $I(D_1 - D_2) \in J(C)$, where $I(\sum_i n_i p_i) := \sum_i n_i I(p_i)$, $p_i \in C$, $n_i \in \mathbb{Z}$. Note that all the maps $C^g \to C_g \to J(C)$ are surjective.

Let us introduce the θ-function with characteristic $\delta \equiv \binom{\delta'}{\delta''}$, with $\delta', \delta'' \in \{0, 1/2\}^{2g}$

$$\theta[\delta](x, \tau) = \sum_{n \in \mathbb{Z}^g} e^{\pi i (n+\delta')\tau(n+\delta'') + 2\pi i (n+\delta')(x+\delta'')} \tag{2.2}$$

$$= e^{\pi i \delta' \tau \delta' + 2\pi i \delta'(x+\delta'')} \theta(x + \tau\delta' + \delta'', \tau) \,, \tag{2.3}$$

$x \in \mathbb{C}^g$, where $\theta(x, \tau) \equiv \theta \left[\begin{smallmatrix}0\\0\end{smallmatrix}\right] (x, \tau)$. The θ-function has the quasi-periodicity properties

$$\theta[\delta](x + n + \tau m, \tau) = e^{-\pi i m \tau m - 2\pi i m x + 2\pi i (\delta' n - \delta'' m)} \theta[\delta](x, \tau) \,, \tag{2.4}$$

$m, n \in \mathbb{Z}^g$. The parity of the θ-function, and of the θ-characteristic δ, is the same of the parity of the integer $4\delta' \cdot \delta''$. There are 2^{2g} different characteristics of definite parity, $2^{g-1}(2^g + 1)$ even and $2^{g-1}(2^g - 1)$ odd. By Abel Theorem each one of such characteristics determines the divisor class of a spin bundle $L_\delta \simeq K_C^{1/2}$, so that we can call them spin structures. In the following we will frequently denote the Abel-Jacobi map of points on C by the points themselves, the meaning being clear from the context.

Denote by Δ the vector of Riemann constants. The results of the present investigation are strictly related to the Riemann Vanishing Theorem: if z, p_i are arbitrary points of C, then

$$\theta\left(z - \sum_1^g p_i + \Delta, \tau\right) \,, \tag{2.5}$$

either vanishes identically or else it has g zeros at $z = p_1, \ldots, p_g$. This implies that if $n = h^0(L_\delta) > 0$, then for arbitrary points $x_i, y_i \in C$, $i = 1, \ldots, n$,

$$\Big(\theta[\delta](0), \theta[\delta](x_1 - y_1), \ldots, \theta[\delta]\Big(\sum_1^{n-1}(x_i - y_i)\Big)\Big) = (0, 0, \ldots, 0)\,. \tag{2.6}$$

Recall that the parity of n is the same of the one of δ. Denote by Θ the θ-divisor, that is the set of all e such that $\theta(e) = 0$, and by Θ_s the singular θ-divisor, that is the sublocus of Θ whose elements are zeros of θ of order greater than 1. By Riemann's Singularity Theorem it follows that the dimension of Θ_s for $g \geq 4$ is $g - 3$ in the hyperelliptic case and $g - 4$ if C is canonical. The curves admitting singular spin structures form a sublocus of codimension one in the moduli space of genus g canonical curves.

Let δ be a non-singular odd θ-characteristic, i.e. with $n = 1$. We consider the prime form[2] [43]

$$E(y, x) = \frac{\theta[\delta](y - x)}{h_\delta(x) h_\delta(y)}\,, \qquad \forall x, y \in C\,, \tag{2.7}$$

where $h_\delta(x)$ is the square root of the holomorphic 1 differential

$$h_\delta^2(x) = \sum_1^g \theta_i[\delta](0)\omega_i(x)\,, \tag{2.8}$$

and $\theta_i[\delta](0) \equiv \partial_{X_i}\theta[\delta](X)_{|X=0}$, $X \in \mathbb{C}^g$. $E(y, x)$ is a holomorphic section of a line bundle on $C \times C$, with the multi-valuedness properties

$$E(y + \alpha n + \beta m, x) = e^{-\pi i m\tau m - 2\pi i m I(x-y)} E(y, x)\,, \tag{2.9}$$

$m, n \in \mathbb{Z}^g$, and such that $E(y, x) = -E(y, x)$. In particular, it only vanishes if $y = x$, and if t is a local coordinate at $y \in C$ such that $h_\delta = dt$, then

$$E(y, x) = \frac{t(y) - t(x)}{\sqrt{dt(y)}\sqrt{dt(x)}}(1 + \mathcal{O}((t(y) - t(x))^2))\,. \tag{2.10}$$

For any even θ-characteristic with $n = 0$ one defines the Szegö kernel, that is the Dirac propagator [43]

$$S_\delta(z, w) = \frac{\theta[\delta](z - w)}{\theta[\delta](0)E(z, w)}\,. \tag{2.11}$$

The first result concerns the Szegö kernel in the case of arbitrary n.

Proposition 1. Let $p_1, \ldots, p_n, q_1, \ldots, q_n$, $n = h^0(L_\delta)$, be pairwise distinct points of C. The corresponding Szegö kernel (Dirac propagator) is

$$S_\delta(z, w) = \frac{\theta[\delta]\left(z - w + \sum_1^n(p_i - q_i)\right)}{E(z, w)\theta[\delta]\left(\sum_1^n(p_i - q_i)\right)} \prod_1^n \frac{E(z, p_i)E(w, q_i)}{E(z, q_i)E(w, p_i)}\,, \qquad \forall z, w \in C\,. \tag{2.12}$$

The proof is by inspection. One may check that $S_\delta(z, w)$ is single valued with respect to all points and is a meromorphic function with respect to the p_i's and q_i's. Furthermore, besides the pole at $z = w$, it has poles also for $z = p_1, \ldots, p_n$ and for $w = q_1, \ldots, q_n$.

[2] Fay denotes the so defined prime form by $E(x, y)$. Here we choose the notation $E(y, x)$.

3 Dirac zero modes and the holomorphic Fay trisecant identity

Note that by (2.6) it follows that the ratio

$$\frac{\theta[\delta]\left(\sum_1^n(z_i-w_i)\right)}{\prod_{i,j}E(z_i,w_j)}\,, \tag{3.1}$$

has no poles. More generally, if $e\in J(C)$ is a zero of θ of order n, that is $h^0(e\otimes\Delta)=n$, then

$$\frac{\theta\left(\sum_1^n(z_i-w_i)+e\right)}{\prod_{i,j}E(z_i,w_j)}\,, \tag{3.2}$$

has no poles. By a suitable insertion of prime forms, (3.1) becomes single-valued, preserving holomorphicity, namely

$$F_\delta(\{z_i\};\{w_i\})=\frac{\theta[\delta]\left(\sum_1^n(z_i-w_i)\right)\prod_{i<j}E(z_i,z_j)E(w_j,w_i)}{\prod_{i,j}E(z_i,w_j)}\,, \tag{3.3}$$

which is a holomorphic 1/2 differential with respect to all points $\{z_i\}$ and $\{w_i\}$ and with spin structure δ. Note that we used $E(w_j,w_i)$ instead of $E(w_i,w_j)$ to avoid factors such as $(-1)^{n(n-1)/2}$. An expression analogous to (3.3) follows by inserting the same prime forms in (3.2).

The above analysis suggests the following theorem that we will prove by using an adaptation of the proof by Fay of his Proposition 2.16 and Corollary 2.18 in [43]; see also his Lemma 1.1 in [46] and Theorem 3.1 in [45].

Theorem 2. Let $z_1,\dots,z_n,w_1,\dots,w_n$ be pairwise distinct points of C and $e\in J(C)$ a zero of θ of order n. If $\phi_1^+,\dots,\phi_n^+$ (resp. $\phi_1^-,\dots,\phi_n^-$) is a suitable normalized basis for $H^0(e\otimes\Delta)$ (resp. $H^0((-e)\otimes\Delta)$), then

$$\frac{\theta\left(\sum_1^n(z_i-w_i)+e\right)\prod_{i<j}E(z_i,z_j)E(w_j,w_i)}{\prod_{i,j}E(z_i,w_j)}=\det\phi_i^-(z_j)\det\phi_i^+(w_j)\,, \tag{3.4}$$

that, in the case of spin bundles L_δ with $n=h^0(L_\delta)>0$, reads

$$\frac{\theta[\delta]\left(\sum_1^n(z_i-w_i)\right)\prod_{i<j}E(z_i,z_j)E(w_j,w_i)}{\prod_{i,j}E(z_i,w_j)}=\det h_{\delta i}(z_j)\det h_{\delta i}(w_j)\,, \tag{3.5}$$

with $h_{\delta 1}(x),\dots,h_{\delta n}(x)$ a suitably normalized basis of $H^0(L_\delta)$.

Proof. If $Z=z_1+\ldots+z_n$ is a generic divisor of distinct points with $\det\phi_i^-(z_j)\neq 0$, then, since the zero set of $\det\phi_i^-(z_j)\neq 0$ is base independent, it follows that there is no section of $H^0((-e)\otimes\Delta)$ vanishing at Z, therefore $h^0(\Delta-e-Z)=0$. On the other hand, by Riemann-Roch theorem

$$h^0(\Delta-e-Z)-h^0(\Delta+e+Z)=\deg(\Delta-e-Z)-g+1\,, \tag{3.6}$$

so that $h^0(\Delta+e+Z)=n$. Then $h^0(\Delta+e+Z-W)=0$ for generic $W=w_1+\ldots+w_n$ and thus $\theta(Z-W+e)$ is not identically vanishing in w_1, say, for generic $w_2,\dots,w_g$ and generic Z. By Riemann's vanishing theorem $\theta(Z-W+e)$ has a divisor R (in w_1) of degree g, the unique R with

$$R \equiv \sum_j z_j - \sum_{j\neq 1} w_j + e + \Delta \in J_g \,. \tag{3.7}$$

The quotient between the left and the right-hand sides of (3.4) is thus a singlevalued (not identically 0) function in w_1 for generic fixed $w_2,\ldots,w_n$ and $z_1,\ldots,z_n$, provided $\det\phi_i^-(z_j)\neq 0$. Now for generic $w_2,\ldots,w_g$, the zeroes of $\det\phi_i^+(w_j)$ in w_1 are at

$$\sum_{j\neq 1} w_j + D_1 \equiv \Delta + e\,, \tag{3.8}$$

where D_1 is a divisor of degree $g-n$. By Riemann's theorem, R is therefore the divisor $\sum_j z_j + D_1$ and thus the ratio between the left- and the right-hand sides of (3.4) has no zeroes or poles in w_1 and is a non-zero constant in w_1, and likewise constant for all w_j's. Therefore, such a ratio is a non-zero constant $c_1(z_1,\ldots,z_n)$ for all w_j's and generic Z. Repeating the argument for the divisor $-e$ and with Z, W interchanged, this function of the z_j's is a constant $c_2(w_1,\ldots,w_n)$ for generic W with $\det\phi_i^+(w_j)\neq 0$. Thus $c_1=c_2=c$ is independent of Z and W but dependent on e and the bases that can be taken to be 1 by choosing them appropriately. Eq. (3.5) follows trivially by (3.4).

3.1 Determinants of Dirac zero modes and θ-relations

We now use the above results to derive relations for the Dirac zero modes. Let us first observe that in the case $n=1$, which is the non-singular odd spin structure, eq. (3.5) reduces to the definition of prime form. This suggests considering the generalization of (2.7) using (3.5). To this end we take the limits $w_i\to z_i$, $i=1,\ldots,n$, of $F_\delta(\{z_i\};\{w_i\})$ to get

$$\det{}^2 h_{\delta i}(z_j) = F_\delta(\{z_i\};\{z_i\})\,, \tag{3.9}$$

that is

$$\det{}^2 h_{\delta i}(z_j) = \frac{1}{n!}\sum_{i_1,\ldots,i_n}^{g} \theta_{i_1\ldots i_n}[\delta](0)\omega_{i_1}(z_1)\cdots\omega_{i_n}(z_n)\,, \tag{3.10}$$

showing that the right side has double zeros, so that it has a holomorphic square root

$$\det h_{\delta i}(z_j) = \left(\frac{1}{n!}\sum_{i_1,\ldots,i_n}^{g} \theta_{i_1\ldots i_n}[\delta](0)\omega_{i_1}(z_1)\cdots\omega_{i_n}(z_n)\right)^{1/2}. \tag{3.11}$$

Corollary 3. If $z_1,\ldots,z_n,w_1,\ldots,w_n$ are pairwise distinct points of C, with $n=h^0(L_\delta)>0$, then

$$F_\delta(\{z_i\};\{w_i\}) = (F_\delta(\{z_i\};\{z_i\})F_\delta(\{w_i\};\{w_i\}))^{1/2}\,, \tag{3.12}$$

and

$$\frac{\theta[\delta](\sum_1^n(z_i-w_i))\prod_{i<j}E(z_i,z_j)E(w_j,w_i)}{\prod_{i,j}E(z_i,w_j)}$$
$$= \frac{1}{n!}\left[\left(\sum_{i_1,\ldots,i_n}^{g}\theta_{i_1\ldots i_n}[\delta](0)\omega_{i_1}(z_{i_1})\cdots\omega_{i_n}(z_{i_n})\right)\left(\sum_{i_1,\ldots,i_n}^{g}\theta_{i_1\ldots i_n}[\delta](0)\omega_{i_1}(w_{i_1})\cdots\omega_{i_n}(w_{i_n})\right)\right]^{1/2}. \tag{3.13}$$

In the case $n > 1$, this implies

$$\sum_{i_1,\ldots,i_n}^{g} \theta_{i_1\ldots i_n}[\delta](0)\omega_{i_1}(z)\omega_{i_2}(z)\omega_{i_3}(z_3)\cdots\omega_{i_n}(z_n) = 0\,, \tag{3.14}$$

that follows by setting $z_2 = z_1 \equiv z$ in (3.11). Furthermore, we have the following factorization property

$$\frac{\theta\left(\sum_1^n(z_i - w_i) + e\right)}{\prod_1^n \theta[\nu_i](z_i - w_i)} = g_-(\{z_j\})g_+(\{w_i\})\,, \tag{3.15}$$

with g_- and g_+ some sections on C, where for any $e \in J(C)$ zero of θ of order n and with $\{\nu_1,\ldots,\nu_n\}$ any arbitrary set of odd non-singular θ-characteristics.

Eq. (3.12) is the trivial identity $(\det h_{\delta i}(z_j)\det h_{\delta i}(w_j))^2 = \det^2 h_{\delta i}(z_j)\det^2 h_{\delta i}(w_j)$ and (3.5). Eq. (3.13) then follows by (3.11) and (3.12). Eq. (3.15) follows by replacing the prime forms in (3.4) by their expression in (2.7) and then removing all the terms which are products of sections of $\{z_i\}$ times sections of $\{w_i\}$. The factorization property (3.15) is a phenomenon that does not appear on the Riemann sphere. In the context of string theory, this appears as a loop correction, reminiscent of the cluster property.

The prime form is defined in terms of an arbitrary odd non singular theta characteristic, in this respect the above investigation shows a natural generalization of the prime form, which has the same structure but now constructed in terms of $\theta[\delta](\sum_1^n(z_i - w_i))$, $n = h^0(L_\delta) > 1$ rather than $\theta[\nu](z - w)$, $h^0(L_\nu) = 1$

$$E_n(\{z_i\},\{w_i\}) := \frac{\theta[\delta]\left(\sum_1^n(z_i - w_i)\right)\prod_{i<j}E(z_i,z_j)E(w_j,w_i)}{\det h_{\delta i}(z_j)\det h_{\delta i}(w_j)}\,, \tag{3.16}$$

that, as expected, by (3.5) coincides with the product of prime forms

$$E_n(\{z_i\},\{w_i\}) = \prod_{i,j} E(z_i,w_j)\,. \tag{3.17}$$

We have the following relations between sections of different spin structures.

Remark 4. Let δ_1 and δ_2 be two θ-characteristics such that $h^0(L_{\delta_1}) = h^0(L_{\delta_2}) = n > 0$. For any set of points $z_1,\ldots,z_n,w_1,\ldots,w_n$ in C, we have the invariant ratio

$$\frac{\theta[\delta_1]\left(\sum_1^n(z_i - w_i)\right)}{\det h_{\delta_1 i}(z_j)\det h_{\delta_1 i}(w_j)} = \frac{\theta[\delta_2]\left(\sum_1^n(z_i - w_i)\right)}{\det h_{\delta_2 i}(z_j)\det h_{\delta_2 i}(w_j)}\,. \tag{3.18}$$

Furthermore, if α and δ are θ-characteristics such that $h^0(L_\alpha) = 0$ and $h^0(L_\delta) = n > 0$, then

$$\det h_{\delta i}(z_j)\det h_{\delta i}(w_j) = \frac{\theta[\alpha](0)\theta[\delta]\left(\sum_1^n(z_i - w_i)\right)}{\theta[\alpha]\left(\sum_1^n(z_i - w_i)\right)}\det S_\alpha(z_i - w_j)\,, \tag{3.19}$$

or, equivalently,

$$\det(h_\delta S_\alpha^{-1} h_\delta) = \frac{\theta[\alpha](0)\theta[\delta]\left(\sum_1^n(z_i - w_i)\right)}{\theta[\alpha]\left(\sum_1^n(z_i - w_i)\right)}\,. \tag{3.20}$$

Eq. (3.18) is an immediate consequence of (3.5), whereas Eq. (3.19) follows by (3.5) and the Fay trisecant identity [43], stating that if $e \in \mathbb{C}^g$, with $\theta(e) \neq 0$, then

$$\frac{\theta\left(\sum_1^m(z_i - w_i) - e\right)\theta^{m-1}(e)\prod_{i<j}E(z_i,z_j)E(w_j,w_i)}{\prod_{i,j}E(z_i,w_j)} = \det\left(\frac{\theta(z_i - w_j - e)}{E(z_i,w_j)}\right)\,, \tag{3.21}$$

for any set of points, $z_1,\ldots,z_m,w_1,\ldots,w_m$ of C.

4 Quadrics in $\mathbb{P}^{g-1}$ and characterization of the θ-null divisor

We now show that Theorem 2 leads to rank 3 quadrics in $\mathbb{P}^{g-1}$ that depend on points of C. In section 5 we will show its relation with the Andreotti-Mayer quadric [44]

$$\sum_{i,j}^{g} \theta_{ij}(e)\omega_i(z)\omega_j(z) = 0\,, \tag{4.1}$$

$e \in \Theta_s$.

We start by restricting to the case of spin bundles with $h^0(L_\delta) = 2$ and then will consider the case $h^0(L_\delta) > 2$. This leads to relations that characterize the θ-null divisor. The results will be generalized to the case of arbitrary $e \in \Theta_s$ in subsection 4.2.

4.1 Two Dirac zero modes

Let us first show that an immediate consequence of eq. (3.12) of Corollary 3 leads to a quadric that depends on points of C. We then will use such a derivation to provide an alternative derivation that involves a divisor analysis and that, in section 5, will lead to the rulings of the quadric.

The derivation of the quadric just follows by noticing that eq. (3.12) implies

$$F_\delta(z,p;z,p)F_\delta(z,q;z,q) = F_\delta^2(z,p;z,q)\,, \tag{4.2}$$

which is a quadric involving only three holomorphic 1 differentials with respect to z.

Corollary 5. If $h^0(L_\delta) = 2$, then there is the following points dependent rank three quadric in $\mathbb{P}^{g-1}$

$$\frac{1}{4}\sum_{i,j}^{g} \theta_{ij}[\delta](0)\omega_i(z)\omega_j(p)\sum_{k,l}^{g} \theta_{kl}[\delta](0)\omega_k(z)\omega_l(q) = \left(\sum_{1}^{g} \frac{\theta_i[\delta](p-q)\omega_i(z)}{E(p,q)}\right)^2\,, \tag{4.3}$$

with $p \neq q$.

Note that in the cases $p = z$ or $q = z$ Eq. (4.3) vanishes so that it reduces to

$$\sum_{i,j}^{g} \theta_{ij}[\delta](0)\omega_i(z)\omega_j(z) = 0\,. \tag{4.4}$$

This is immediate by eq. (4.2) recalling that by (3.5)

$$F_\delta(\{z_i\};\{w_i\}) = \det h_{\delta i}(z_j)\det h_{\delta i}(w_j)\,. \tag{4.5}$$

We now provide an alternative proof of the above theorem which is based on a divisor analysis. Let us first recall that for any $n = h^0(L_\delta)$ one may arbitrarily choose $n-1$ points for each one of the divisors corresponding to the holomorphic section of L_δ; once the choice is done the remaining $g-n$ points are then uniquely fixed. Let us first consider one of the two divisors associated to $h^0(L_\delta) = 2$. By (2.3) and the Riemann vanishing theorem it follows that if $h^0(L_\delta) = 2$, then

$$\tau\delta' + \delta'' = -p - p_2 - \ldots - p_{g-1} + \Delta\,, \tag{4.6}$$

where the first point is denoted p instead of p_1 to emphasize that, given an arbitrary $p \in C$, there are always $g-2$ points $p_2(p), \ldots, p_{g-1}(p) \in C$ such that (4.6) is satisfied. Adding the second divisor, we have

$$\tau\delta' + \delta'' = -p - p_2 - \ldots - p_{g-1} + \Delta = -q - q_2 - \ldots - q_{g-1} + \Delta\,, \tag{4.7}$$

with $p, q \in C$ arbitrary. In the case $n = 2$ Eq. (3.3) corresponds to the following holomorphic $1/2$ differential with respect to $z, p, w, q \in C$

$$F_\delta(z,p;w,q) = \frac{\theta[\delta](z-w+p-q)E(z,p)E(q,w)}{E(z,w)E(z,q)E(p,w)E(p,q)}\,, \qquad \forall p,q,w,z \in C\,. \tag{4.8}$$

Recall that $p_2, \ldots, p_{g-1}$ depend only on p, whereas $q_2, \ldots, q_{g-1}$ depend only on q. Since the derivative of the θ-function in (4.8) with respect to w, evaluated at $w = z$ and divided by $-E(p,q)$, equals $F_\delta(z,p;z,q)$, by Theorem 2 we have

$$g_\delta(z) := \det\begin{pmatrix} h_{\delta 1}(z) & h_{\delta 2}(z) \\ h_{\delta 1}(p) & h_{\delta 2}(p)\end{pmatrix} \det\begin{pmatrix} h_{\delta 1}(z) & h_{\delta 2}(z) \\ h_{\delta 1}(q) & h_{\delta 2}(q)\end{pmatrix} = \sum_1^g \frac{\theta_i[\delta](p-q)\omega_i(z)}{E(p,q)}\,. \tag{4.9}$$

Let us show that

$$\operatorname{div} g_\delta(z) = p + \sum_{i=2}^{g-1} p_i + q + \sum_{i=2}^{g-1} q_i\,. \tag{4.10}$$

We first note that (2.6) and (4.7) imply the following identity between divisors with respect to z

$$\operatorname{div}\theta[\delta](z-w+p-q) = \operatorname{div}\theta[0](z-w-q-p_2-\ldots-p_{g-1}+\Delta) = w+q+\sum_{i=2}^{g-1} p_i\,. \tag{4.11}$$

Therefore, the divisor of $g_\delta(z)$ includes $q + \sum_{i=2}^{g-1} p_i$. On the other hand, (4.9) shows that $g_\delta(z)E(p,q)$ is antisymmetric with respect to p and q, so that even p is a zero of $g_\delta(z)$. It follows that $g_\delta(z)$ is a holomorphic 1 differential with zeros at $z = q, p, p_2, \ldots, p_{g-1}$, so that, since it has fixed g zeros, it follows that this is the unique holomorphic 1 differential with such zeros, and since there are sections of $H^0(L_\delta)$ with divisors $p, p_2, \ldots, p_{g-1}$ and $q, q_2, \ldots, q_{g-1}$, it follows that $g_\delta(z)$ is their product.

We now introduce two holomorphic 1 bi-differentials. The first one is $1/2$ of the derivative of $-\sum_1^g \theta_i[\delta](p-q)\omega_i(z)$ in (4.9) with respect to q evaluated at $q = p$, that coincides with $F_\delta(z,p;z,p)$

$$f_{\delta 1}(z) := \det{}^2\begin{pmatrix} h_{\delta 1}(z) & h_{\delta 2}(z) \\ h_{\delta 1}(p) & h_{\delta 2}(p)\end{pmatrix} = \frac{1}{2}\sum_{i,j}^g \theta_{ij}[\delta](0)\omega_i(z)\omega_j(p)\,. \tag{4.12}$$

The other holomorphic 1 bi-differential is $F_\delta(z,q;z,q)$

$$f_{\delta 2}(z) := \det{}^2\begin{pmatrix} h_{\delta 1}(z) & h_{\delta 2}(z) \\ h_{\delta 1}(q) & h_{\delta 2}(q)\end{pmatrix} = \frac{1}{2}\sum_{i,j}^g \theta_{ij}[\delta](0)\omega_i(z)\omega_j(q)\,. \tag{4.13}$$

Next, note that since $p, p_2, \ldots, p_{g-1}$ are independent of q, it follows that the zeros of $f_{\delta 1}(z)$ include $p, p_2, \ldots, p_{g-1}$. On the other hand, $f_{\delta 1}(z)$ is even under the exchange of p with z,

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so that p is a double zero of z. Therefore $f_{\delta 1}(z)$ is a holomorphic 1 differential in z with fixed g zeros, so that it must be unique up to a multiplicative constant. This means that it is the square of the 1/2 differential with zeros at $p, p_2, \ldots, p_{g-1}$.

Next, changing the role of p and q, one sees that $f_{\delta 2}(z)$ has double zeros in $z = q, q_2 \ldots, q_{g-1}$, so that is the square of the other zero mode. Comparing the divisors we get the rank three quadric

$$f_{\delta 1}(z) f_{\delta 2}(z) = g_\delta^2(z)\,, \tag{4.14}$$

that is Eq. (4.3).

4.2 Generalization to arbitrary n and characterization of the θ-null divisor

The generalization of the points dependent quadric (4.3) to arbitrary n is again a consequence of Theorem 2.

Corollary 6. If z, w_j, z_k, $j, k = 2, \ldots, n$, are arbitrary but distinct points of C, then, for any $n = h^0(L_\delta) > 1$ there is the following rank three quadric in $\mathbb{P}^{g-1}$

$$\begin{aligned} &F_\delta(z, z_2, \ldots, z_n; z, z_2, \ldots, z_n) F_\delta(z, w_2, \ldots, w_n; z, w_2, \ldots, w_n) \\ &\quad = F_\delta^2(z, z_2, \ldots, z_n; z, w_2, \ldots, w_n)\,, \end{aligned} \tag{4.15}$$

that is

$$\begin{aligned} &\frac{1}{(n!)^2} \sum_{i_1,\ldots,i_n}^{g} \theta_{i_1\ldots i_n}[\delta](0)\omega_{i_1}(z)\omega_{i_2}(z_2)\cdots\omega_{i_n}(z_n) \sum_{j_1,\ldots,j_n}^{g} \theta_{j_1\ldots j_n}[\delta](0)\omega_{j_1}(z)\omega_{j_2}(w_2)\cdots\omega_{j_n}(w_n) \\ &\quad = \left(\sum_{k=1}^{g} \frac{\theta_k[\delta]\left(\sum_2^n (z_i - w_i)\right)\omega_k(z) \prod_{2\le i<j} E(z_i, z_j) E(w_j, w_i)}{\prod_{i=2,j=2}^{n} E(z_i, w_j)} \right)^2 . \end{aligned} \tag{4.16}$$

Furthermore, integrating (4.16) along the canonical basis of homological cycles of C, we get the following relations involving the Riemann period matrix

$$\begin{aligned} &\frac{1}{(n!)^2} \sum_{i_1,i_m,\ldots,i_n}^{g} \theta_{i_1\ldots i_n}[\delta](0)\omega_{i_1}(z)\tau_{i_m k_m}\cdots\tau_{i_n k_n} \sum_{j_1,j_p,\ldots,j_n}^{g} \theta_{j_1\ldots j_n}[\delta](0)\omega_{j_1}(z)\tau_{j_p l_p}\cdots\tau_{j_n l_n} \\ &\quad = \sum_{j,k}^{g} (\delta|k_m, \ldots, k_n; l_p, \ldots, l_n)_{j,k}\omega_j(z)\omega_k(z)\,, \end{aligned} \tag{4.17}$$

$n > 1$, where we introduced the tensor

$$\begin{aligned} &(\delta|k_m, \ldots, k_n; l_p, \ldots, l_n)_{j,k} \\ &\quad := \oint_{\gamma_m}\oint_{\gamma_p} \frac{\theta_j[\delta]\left(\sum_2^n (z_r - w_r)\right)\theta_k[\delta]\left(\sum_2^n (z_r - w_r)\right)\prod_{2\le r<s} E^2(z_r, z_s)E^2(w_s, w_r)}{\prod_{r=2,s=2}^{n} E^2(z_r, w_s)}\,, \end{aligned} \tag{4.18}$$

$$\gamma_m = \alpha_{i_1} \cup \ldots \cup \alpha_{i_{m-1}} \cup \beta_{k_m} \cup \ldots \cup \beta_{k_n}\,, \qquad \gamma_p = \alpha_{i_1} \cup \ldots \cup \alpha_{i_{p-1}} \cup \beta_{l_p} \cup \ldots \cup \beta_{l_n}\,,$$

$m, p = 1, \ldots, n$, and $j, k = 1, \ldots, g$.

We note that the relations (4.17) characterize the period matrices τ admitting a singular θ-characteristic, and then the θ-null divisor.

The generalization of the above results to the case of any $e \in \Theta_s$ is immediate. In particular, we have

Corollary 7. If $e \in J(C)$ is a zero of order n of θ and $z_1, \dots, z_n$, $w_1, \dots, w_n$, are pairwise distinct points of C, then

$$\det \phi_i^-(z_j) \det \phi_i^+(z_j) = \frac{1}{n!} \sum_{i_1,\dots,i_n}^{g} \theta_{i_1\dots i_n}(e)\omega_{i_1}(z_1)\cdots\omega_{i_n}(z_n)\,, \tag{4.19}$$

and

$$\begin{aligned} &\frac{\theta(\sum_1^n(z_i-w_i)+e)\theta(\sum_1^n(z_i-w_i)-e)\prod_{i<j}E^2(z_i,z_j)E^2(w_j,w_i)}{\prod_{i,j}E^2(z_i,w_j)} \\ &\quad = \frac{(-1)^n}{(n!)^2}\left(\sum_{i_1,\dots,i_n}^{g}\theta_{i_1\dots i_n}(e)\omega_{i_1}(z_{i_1})\cdots\omega_{i_n}(z_{i_n})\right)\left(\sum_{i_1,\dots,i_n}^{g}\theta_{i_1\dots i_n}(e)\omega_{i_1}(w_{i_1})\cdots\omega_{i_n}(w_{i_n})\right), \end{aligned} \tag{4.20}$$

that, setting $z_1 = w_1 \equiv z$, gives the points dependent quadric

$$\begin{aligned} &\frac{(-1)^n}{(n!)^2}\sum_{i_1,\dots,i_n}^{g}\theta_{i_1\dots i_n}(e)\omega_{i_1}(z)\omega_{i_2}(z_2)\cdots\omega_{i_n}(z_n)\sum_{j_1,\dots,j_n}^{g}\theta_{j_1\dots j_n}(e)\omega_{j_1}(z)\omega_{j_2}(w_2)\cdots\omega_{j_n}(w_n) \\ &\quad = \frac{\sum_{j,k}^{g}\theta_j\left(\sum_2^n(z_i-w_i)+e\right)\omega_j(z)\theta_k\left(\sum_2^n(z_i-w_i)-e\right)\omega_k(z)\prod_{2\le i<j}E^2(z_i,z_j)E^2(w_j,w_i)}{\prod_{i=2,j=2}^{n}E^2(z_i,w_j)}\,. \end{aligned} \tag{4.21}$$

Eq. (4.19) follows by taking the limits $w_i \to z_i$, $i = 1, \dots, g$, of (3.4). Eq. (4.20) follows by the identity

$$\begin{aligned} &(\det \phi_i^-(z_j)\det\phi_i^+(w_j))(\det\phi_i^-(w_j)\det\phi_i^+(z_j)) \\ &\quad = (\det\phi_i^-(z_j)\det\phi_i^+(z_j))(\det\phi_i^-(w_j)\det\phi_i^+(w_j))\,, \end{aligned} \tag{4.22}$$

and then expressing its left-hand side by the left-hand side of (3.4), noticing that interchanging z_i's and w_i's gives the factor $(-1)^n$, whereas its right-hand side is replaced by the right-hand side of (4.19).

We conclude this section noticing that integrating (4.21) along the homological cycles leads to a straightforward generalization of the relations (4.17).

5 Geometrical interpretation of the points dependent quadric

We now adapt to the case of singular points of order 2 the divisor analysis that led to the proof of (4.14), reported also in (4.3). The aim is to show the relation between (4.3) and the Andreotti-Mayer quadric (4.1). The generalization of (4.3) to the case $H^0(\pm e \otimes \Delta)$ corresponds to (4.20) with $n = 2$.

Set $X = (X_1, \dots, X_g) \in \mathbb{C}^g$, equivalently $[X] \in \mathbb{P}^{g-1}$. The canonical curve is then $[\omega(C)] := [\omega(X)] \in \mathbb{P}^{g-1}$ for all $X = \omega(z) = (\omega_1(z), \dots, \omega_g(z)) \in \mathbb{C}^g$, $z \in C$.

For a divisor $D = p_1 + \ldots + p_{g-1} \in C_{g-1}$ with $h^0(D) = 2$, the matrix $\omega_i(p_j)$ has rank $g-2$; set

$$\Sigma_D = \mathrm{span}\,\{\omega(p_1), \ldots, \omega(p_{g-1})\} \subset \mathbb{C}^g\,. \tag{5.1}$$

We have $\dim \Sigma_D = g-2$, so that $[\Sigma_D] \leftrightarrow \mathbb{P}^{g-3}$.

If $\theta(z)$ vanishes to second order at $z = e \in \mathbb{C}^g$, that is e is a singular point of order two on Θ, then $Q(X) := \sum_{i,j} \theta_{ij}(e) X_i X_j = 0$ is a quadric in $\mathbb{P}^{g-1}$ containing $[\omega(C)]$, that is eq. (4.1). If $e \equiv D - \Delta \in J_0(C)$, then

$$X \in \Sigma_D \implies Q(X) = 0\,, \tag{5.2}$$

that is

$$\sum_{i,j} \theta_{ij}(e)\omega_i(p_\alpha)\omega_j(p_\beta) = 0\,, \tag{5.3}$$

for all α and β in $[1, g-1]$, which is Theorem 3.6 of [10] in the case $n = 2$.

Σ_D depends on the $\mathbb{P}^1$-family of $D \in C_{g-1}$, all giving rise to the same $e \in J_0(C)$. For any $p, q \in C$ set

$$\begin{cases} \Delta + e \equiv D_p^+ \equiv D_q^+\,, \quad D_p^+ = p + \xi_p\,, \quad D_q^+ = q + \xi_q\,, \\ \Delta - e \equiv D_p^- \equiv D_q^-\,, \quad D_p^- = p + \eta_p\,, \quad D_q^- = q + \eta_q\,, \end{cases} \tag{5.4}$$

for divisors ξ_*, η_* of degree $g-2$ depending on $*$.

Setting

$$K^\pm_{p,q}(X) = \sum_i \theta_i(p - q \pm e) E(p,q)^{-1} X_i\,, \tag{5.5}$$

$$H_p(X) = \sum_{i,j} \theta_{ij}(e) X_i \omega_j(p)\,, \tag{5.6}$$

we have $K^\pm_{p,p} = H_p$, $K^-_{p,q} = K^+_{q,p}$ and

$$K^+_{p,q}(\omega(z)) = \det \begin{pmatrix} \phi_1^-(z) & \phi_2^-(z) \\ \phi_1^-(p) & \phi_2^-(p) \end{pmatrix} \det \begin{pmatrix} \phi_1^+(z) & \phi_2^+(z) \\ \phi_1^+(q) & \phi_2^+(q) \end{pmatrix} = K^-_{q,p}(\omega(z))\,. \tag{5.7}$$

$$H_p(\omega(z)) = \det \begin{pmatrix} \phi_1^-(z) & \phi_2^-(z) \\ \phi_1^-(p) & \phi_2^-(p) \end{pmatrix} \det \begin{pmatrix} \phi_1^+(z) & \phi_2^+(z) \\ \phi_1^+(p) & \phi_2^+(p) \end{pmatrix}\,. \tag{5.8}$$

These hyperplanes intersect $[\omega(C)] \in \mathbb{P}^{g-1}$ at

$$\begin{cases} \mathrm{div}\, H_p(\omega(z)) = 2p + \xi_p + \eta_p\,, & \mathrm{div}\, H_q(\omega(z)) = 2q + \xi_q + \eta_q\,, \\ \mathrm{div}\, K^+_{p,q}(\omega(z)) = p + q + \xi_p + \eta_p\,, & \mathrm{div}\, K^-_{p,q}(\omega(z)) = p + q + \xi_p + \eta_q\,, \end{cases} \tag{5.9}$$

and if $(Q) = \mathrm{div}\, Q(X) \subset \mathbb{P}^{g-1}$, the rulings of $Q = 0$ by the $\Sigma_* \leftrightarrow \mathbb{P}^{g-3}$'s are

$$\begin{cases} \Sigma_{D_p^+} \cup \Sigma_{D_p^-} \subset (H_p) \cap (Q)\,, & \Sigma_{D_q^+} \cup \Sigma_{D_q^-} \subset (H_q) \cap (Q)\,, \\ \Sigma_{D_p^-} \cup \Sigma_{D_q^+} \subset (K^+_{p,q}) \cap (Q)\,, & \Sigma_{D_p^+} \cup \Sigma_{D_q^-} \subset (K^-_{p,q}) \cap (Q)\,. \end{cases} \tag{5.10}$$

That $Q(X) = 0$ is a rank ≤ 4-quadric can be expressed by the following relation.

Theorem 8.

$$H_p(X)H_q(X) - K^+_{p,q}(X)K^-_{p,q}(X) = c_{p,q}Q(X)\,, \tag{5.11}$$

for all $X \in \mathbb{C}^g$, and constant

$$c_{p,q} = \frac{1}{2}\sum_{\alpha,\beta}\theta_{\alpha\beta}(e)\omega_\alpha(p)\omega_\beta(q) \neq 0\,, \tag{5.12}$$

for generic p and q.

Eq. (5.11) follows by the Fay trisecant identity for $n = 2$

$$\begin{aligned}\theta(x-p-e)\theta(y-q-e)E(x,q)E(p,y) + \theta(x-q-e)\theta(y-p-e)E(x,p)E(y,q)\\ = \theta(x+y-p-q-e)\theta(e)E(x,y)E(p,q)\,.\end{aligned} \tag{5.13}$$

In particular, differentiating this identity with respect to e_m and e_n, evaluated at the singular e, and then setting $x = p$, $y = q$, one gets

$$\begin{aligned}&-[\theta_m(p-q-e)\theta_n(p-q+e) + \theta_n(p-q-e)\theta_m(p-q+e)]E^{-2}(p,q)\\ &\quad= \sum_{\alpha,\beta}(\theta_{mn}(e)\theta_{\alpha\beta}(e) - \theta_{m\alpha}(e)\theta_{n\beta}(e) - \theta_{n\alpha}(e)\theta_{m\beta}(e))\omega_\alpha(p)\omega_\beta(q)\,.\end{aligned} \tag{5.14}$$

Interesting relations, similar to (5.11), follow for the tangent cone

$$\sum_{i_1,\ldots,i_n}\theta_{i_1\ldots i_n}(e)X_{i_1}\cdots X_{i_n}\,, \qquad n\geq 3\,, \tag{5.15}$$

for $X = (X_1,\ldots,X_g) \in \mathbb{C}^g$ at singular $e \in \Theta$.

Acknowledgments

It is a pleasure to thank John Fay and Roberto Volpato for important suggestions and remarks and Samuel Grushevsky and Edward Witten for key comments. I also gratefully acknowledge support from the Simons Center for Geometry and Physics, Stony Brook University at which the final part of this paper was performed.

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