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Non-invertible duality defect and non-commutative fusion algebra

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Abstract: We study non-invertible duality symmetries by gauging a diagonal subgroup of a non-anomalous U(1)×U(1) global symmetry. In particular, we employ the half-space gauging to $c = 2$ bosonic torus conformal field theory (CFT) in two dimensions and pure U(1)×U(1) gauge theory in four dimensions. In $c = 2$ bosonic torus CFT, we show that the non-invertible symmetry obtained from the diagonal gauging becomes emergent on an *irrational* CFT point. We also calculate the fusion rules concerning the duality defect. We find out that the fusion algebra is *non-commutative*. We also obtain a similar result in pure U(1)×U(1) gauge theory in four dimensions.

Keywords: Global Symmetries, Duality in Gauge Field Theories, Scale and Conformal Symmetries

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Contents

1 Introduction and summary

Background: global symmetry has always been a pivotal concept in the analysis of quantum field theories (QFTs). One of the most prominent and successful applications of global symmetries is the 't Hooft anomaly matching [1], which aids in our comprehension of the strongly coupled systems. Toward giving further insights into the non-perturbative dynamics of QFTs, the notion of global symmetry has been generalized in [2]. There, it has been revealed that a global symmetry is associated with the existence of the *topological* defect, and the symmetry transformation can be realized as a boundary condition on the topological defect. Although various types of generalized global symmetries have been concerned so far, the non-invertible symmetry has gained significant attention above all.[1] Unlike the ordinary symmetries, the non-invertible symmetry has no inverse operation, hence the resulting fusion algebra forms the fusion category rather than a group [29–33]. In recent years, numerous non-invertible symmetries have been discovered, offering new predictions into the dynamics of QFTs, e.g., constraints on renormalization group flows and realistic QFTs, across diverse dimensions [34–94]. (See [95, 96] for the comprehensive review on the non-invertible symmetry.)

Half-space gauging: in these developments, the half-space gauging plays a crucial role in systematically constructing the non-invertible duality defects [45, 46]. To consider half-space gauging, we first split the spacetime manifold X into the left and right regions separated by the co-dimension one interface as depicted in figure 1. We perform gauging a non-anomalous

[1] Historically, non-invertible symmetries have been considered in the context of rational conformal field theories [3–28].

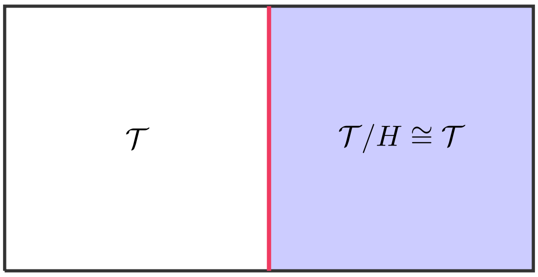

Figure 1. Pictorical representation of the half-space gauging.

discrete global symmetry H of a theory $\mathcal{T}$ only in half of the spacetime and impose the Dirichlet boundary condition on the H gauge field at the interface. Then, for some special cases, the theory becomes invariant under gauging H: $\mathcal{T}/H \cong \mathcal{T}$, and the interface becomes (topological) non-invertible defect $\mathcal{N}$. Here, let us briefly summarize the non-invertible symmetries constructed from the half-space gauging in $c = 1$ compact boson model [45, section 4.1];

$$\frac{R^2}{4\pi} \int_{X_2} d\phi \wedge \star d\phi \,, \tag{1.1}$$

where X_2 is a two-dimensional orientable manifold, and ϕ is the compact boson with the periodicity 2π. In this case, we gauge the discrete shift symmetry $\mathbb{Z}_N \subset \mathrm{U}(1)^{\text{shift}}$ only in the right region. By using T-duality, we can achieve $\mathcal{T}/\mathbb{Z}_N \cong \mathcal{T}$ only if we tune the compact radius such that $R = \sqrt{N}$, which describes the rational conformal field theory (RCFT). Notably, the non-invertible duality defect $\mathcal{N}$ can be expressed by the following action;

$$\mathcal{N} \;:\; \mathrm{i}\frac{N}{2\pi} \int_{x=0} \phi_{\mathrm{L}} d\phi_{\mathrm{R}} \,, \tag{1.2}$$

where ϕ_{L} and ϕ_{R} are the compact boson fields that live in the left and right regions, respectively. The fusion algebra concerning the non-invertible duality defect $\mathcal{N}$ and the $\mathbb{Z}_N$ shift symmetry generator η are given by the following Tambara-Yamagami category [97];

$$\begin{aligned} \mathcal{N} \times \mathcal{N} &= \mathcal{C} \,, \\ \eta \times \mathcal{N} = \mathcal{N} \times \eta &= \mathcal{N} \,, \\ \eta^N &= 1 \,, \end{aligned} \tag{1.3}$$

where $\mathcal{C} = 1 + \eta + \eta^2 + \cdots \eta^{N-1}$ is the projection operator of the $\mathbb{Z}_N$ shift symmetry up to normalization. In summary, in $c = 1$ compact boson CFT, the non-invertible symmetry obtained by the half-space gauging becomes emergent at $R = \sqrt{N}$, namely RCFT point, and the fusion algebra is given by the Tambara-Yamagami category (1.3).

Motivations: the most natural and simplest generalization of the above $c = 1$ compact boson CFT is the $c = 2$ bosonic torus CFT;[2]

$$\frac{1}{4\pi} \int_{X_2} G_{IJ}\, d\phi^I \wedge \star\, d\phi^J \quad , \qquad I, J = 1\,, 2\,, \tag{1.4}$$

[2]For simplicity, we do not include the topological term in the action (1.4) in this paper.

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where ϕ^I is the compact boson with periodicity 2π. Then, inspired by the above example of the $c = 1$ compact boson theory, the following two questions naturally arise;

- Where do the non-invertible symmetries obtained from the half-space gauging become emergent on the conformal manifold? In particular, are these non-invertible symmetries found at rational or irrational CFT points?
- What is the fusion algebra associated with the non-invertible symmetry defect?

The main aim of this paper is to address these questions. As we will see later, the landscape of non-invertible symmetries from the half-space gauging in the $c = 2$ bosonic torus CFT is richer than the $c = 1$ case. We also apply the half-space gauging to the pure $\mathrm{U}(1)\times\mathrm{U}(1)$ gauge theory in four dimensions;

$$\frac{1}{4\pi}\int_{M_4} \mathcal{G}_{IJ}\, dA^I \wedge \star\, dA^J \ , \qquad I, J = 1\,, 2\,, \tag{1.5}$$

and investigate the non-invertible structures of this theory. In the remainder of the Introduction, we present a concise summary of our work.

Summary: the $c = 2$ bosonic torus CFT has the zero-form shift-symmetry $\mathrm{U}(1)_1^{\text{shift}} \times \mathrm{U}(1)_2^{\text{shift}}$. Its charged operator is the vertex operator $e^{\mathrm{i}\vec{n}\cdot\vec{\phi}}$ which is characterized by two integers: $\vec{n} \in \mathbb{Z}\times\mathbb{Z}$. Under $\mathrm{U}(1)_1^{\text{shift}}\times\mathrm{U}(1)_2^{\text{shift}}$, the vertex operator is transformed in the following way;

$$\mathrm{U}(1)_1^{\text{shift}} \times \mathrm{U}(1)_2^{\text{shift}} \ : \ e^{\mathrm{i}\vec{n}\cdot\vec{\phi}} \mapsto e^{\mathrm{i}\vec{\theta}\cdot\vec{n}}\, e^{\mathrm{i}\vec{n}\cdot\vec{\phi}}\,, \qquad \theta^1, \theta^2 \in [0, 2\pi)\,. \tag{1.6}$$

As a discrete subgroup of the shift symmetry to be gauged, we choose the diagonal subgroup $(\mathbb{Z}_{2N'}^{[0]})_{\text{diag}}$, whose generator is specified by $(\, e^{\mathrm{i}\frac{\pi}{N'}}, e^{\mathrm{i}\frac{\pi}{N'}}\,)$. As a result of the gauging $(\mathbb{Z}_{2N'}^{[0]})_{\text{diag}}$, the charge lattice of the original theory $\mathbb{Z}\times\mathbb{Z}$ is reduced to its sublattice $\Lambda_{2N'}$ defined by;

$$\begin{aligned} \Lambda_{2N'} &\equiv \{\vec{n} \in \mathbb{Z}\times\mathbb{Z} \mid n_1 + n_2 = 0 \mod 2N'\} \\ &= \mathrm{Span}(\vec{\ell}_1, \vec{\ell}_2)\,, \end{aligned} \tag{1.7}$$

where the charge lattice $\Lambda_{2N'}$ is spanned by the two orthogonal vectors $\vec{\ell}_1$ and $\vec{\ell}_2$ (see the upper right lattice in figure 2.);

$$\vec{\ell}_1 = \begin{pmatrix} N' \\ N' \end{pmatrix}\,, \qquad \vec{\ell}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}\,. \tag{1.8}$$

For later convenience, we put these two basis vectors $\vec{\ell}_1$ and $\vec{\ell}_2$ into the matrix K defined by;[3]

$$K \equiv (\vec{\ell}_1, \vec{\ell}_2) = \begin{pmatrix} N' & -1 \\ N' & 1 \end{pmatrix}\,, \tag{1.9}$$

which clearly carries an information of the charge lattice $\Lambda_{2N'}$. In order for the theory to be invariant under the diagonal gauging $(\mathbb{Z}_{2N'}^{[0]})_{\text{diag}}$, we must perform the rotation and rescaling on the charge lattice $\Lambda_{2N'}$ and bring it back to the original one $\mathbb{Z}\times\mathbb{Z}$. The charge lattice

[3]Throughout this paper, we choose the charge matrix K such that $\det K = +2N'$.

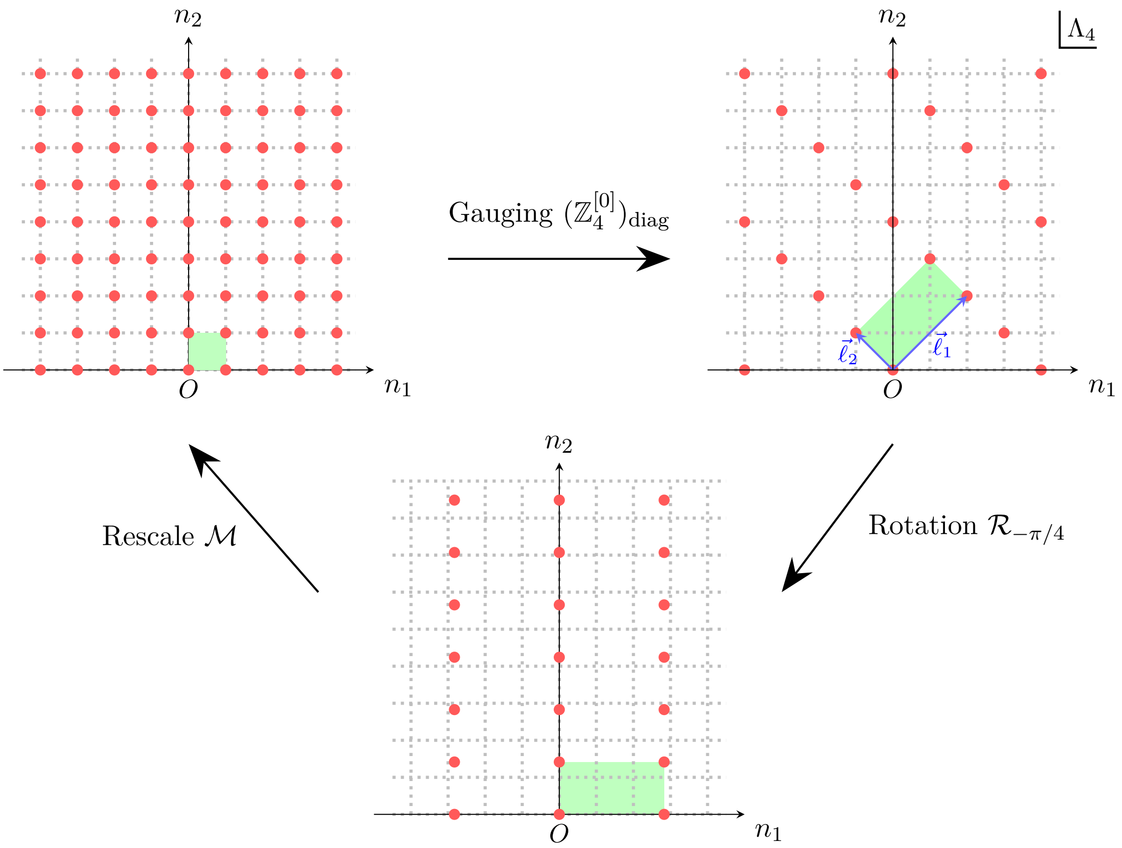

Figure 2. The transition of the charged lattice corresponding to the gauging $(\mathbb{Z}_4^{[0]})_{\text{diag}}$. The horizontal/vertical axis denotes the $\mathrm{U}(1)_1^{\text{shift}}/\,\mathrm{U}(1)_2^{\text{shift}}$ charge of the vertex operator $e^{\mathrm{i}\vec{n}\cdot\vec{\phi}}$, respectively. The red points in each diagram mean the properly quantized charges and the green realm shows the unit cell. In order to bring the charge lattice after gauging $(\mathbb{Z}_4^{[0]})_{\text{diag}}$ back to the original one $\mathbb{Z}\times\mathbb{Z}$, we need to perform rotation and rescaling as depicted in the figure.

transition under these operations is depicted in figure 2. Note that the rotation is a peculiar operation to the $c = 2$ bosonic torus CFT. Moreover, by using the T-duality, we can perfectly restore the original theory $\mathcal{T}/(\mathbb{Z}_{2N'}^{[0]})_{\text{diag}} \cong \mathcal{T}$, if the kinetic matrix G_{IJ} is tuned to satisfy the following self-duality condition;

$$G = K^{\mathrm{T}} G^{-1} K \,. \tag{1.10}$$

The solution G^*_{IJ} to the self-duality condition (1.10) is given by

$$G^* = \sqrt{-\frac{N}{D}} \begin{pmatrix} +2K_{11} & K_{12}+K_{21} \\ K_{12}+K_{21} & 2K_{22} \end{pmatrix} , \qquad D \equiv (K_{12}+K_{21})^2 - 4K_{11}K_{22}\,, \tag{1.11}$$

and we can show that the solution G^*_{IJ} corresponds to the complex multiplication (CM) point[4] [98], which is a wider class of the $c = 2$ bosonic torus RCFTs. We also prove that the

[4]Here, we call the CM point when either the complex structure modulus τ *or* the complexified Kähler modulus ρ (see (3.18) and (3.21) for their definitions) belongs to an imaginary quadratic field. If these two moduli are elements of the *same* imaginary quadratic field, the CM point is enhanced to the RCFT one.

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CM point can be promoted to the RCFT if and only if the charge matrix K is symmetric;

$$K \ : \ \text{symmetric} \quad \Longleftrightarrow \quad \text{RCFT}\,. \tag{1.12}$$

Hence, from the explicit formula (1.9), we conclude that the non-invertible symmetry constructed from the half-space gauging associated with the diagonal gauging becomes emergent on the *irrational* CFT.

Furthermore, we also show that the non-invertible symmetry defect $\mathcal{D}$ associated with the gauging $(\mathbb{Z}^{[0]}_{2N'})_{\text{diag}}$ can be put into the following Lagrangian form;

$$\mathcal{D} \ : \ \frac{\mathrm{i}}{2\pi}\int_{x=0} K_{IJ}\phi^I_{\mathrm{L}}\, d\phi^J_{\mathrm{R}}\,, \tag{1.13}$$

and derive the fusion algebra. We find that the resulting fusion algebra is *infinitely* generated and *non-commutative*. We also discover the closed fusion subalgebra. To see this, we must put the various global symmetry generators on the duality defect $\mathcal{D}$, and define the *dressed* duality defect $\widehat{\mathcal{D}}_{s_1,s_2}$ $(s_1, s_2 = 0, 1, \cdots 2N'-1)$. (See section 3.3 for the definition.) Thereby, the projection operator should also be replaced by the dressed one $\widehat{\mathcal{C}}_{s_1,s_2}$. As a result of this dressing, the fusion algebra concerning $\widehat{\mathcal{D}}_{s_1,s_2}$, $\widehat{\mathcal{C}}_{s_1,s_2}$, and the $(\mathbb{Z}^{[0]}_{2N'})_{\text{diag}}$ shift symmetry generator $\eta_{\vec{p}}$ can be summarized as follows;

Non-commutative fusion subalgebra at the irrational CFT point:

$$\begin{aligned}
&\widehat{\mathcal{D}}_{s_1,s_2}\times\widehat{\mathcal{D}}_{s_3,s_4} = \widehat{\mathcal{C}}_{s_2+s_3,s_1+s_4}\,,\\
&\widehat{\mathcal{D}}_{s_1,s_2}\times\eta_{\vec{p}} = \eta_{\vec{p}}\times\widehat{\mathcal{D}}_{s_1,s_2} = \widehat{\mathcal{D}}_{s_1,s_2}\,,\\
&\widehat{\mathcal{D}}_{s_1,s_2}\times\widehat{\mathcal{C}}_{s_3,s_4} = 2N'\,\widehat{\mathcal{D}}_{s_1+s_3,s_2+s_4}\,,\\
&\widehat{\mathcal{C}}_{s_1,s_2}\times\widehat{\mathcal{D}}_{s_3,s_4} = 2N'\,\widehat{\mathcal{D}}_{s_2+s_3,s_1+s_4}\,,\\
&\widehat{\mathcal{C}}_{s_1,s_2}\times\widehat{\mathcal{C}}_{s_3,s_4} = 2N'\,\widehat{\mathcal{C}}_{s_1+s_3,s_2+s_4}\,,\\
&\eta_{\vec{p}}^{2N'} = 1\,.
\end{aligned} \tag{1.14}$$

We also consider the half-space gauging with respect to the product group $\mathbb{Z}_{N_1}\times\mathbb{Z}_{N_2}$, instead of the diagonal one. In this case, the non-invertible symmetry arises on the RCFT point, and the fusion algebra is given by the standard Tambara-Yamagami category (1.3) since $c = 2$ bosonic torus CFT is reduced to the two sets of the $c = 1$ compact boson CFTs. We emphasize that this is perfectly consistent with the promoting condition (1.12) since the charge matrix K is diagonal. Our main results described above are summarized in table 1.

Finally, we apply the half-space gauging and explore the non-invertible symmetries in the pure U(1)×U(1) gauge theory in four dimensions. This theory has the $\mathrm{U}(1)^{\text{ele}}_1\times\mathrm{U}(1)^{\text{ele}}_2$ electric one-form symmetry, whose charged object is the Wilson loop. In a similar manner to the two dimensions, we construct the non-invertible symmetries from gauging the diagonal subgroup $(\mathbb{Z}^{[1]}_N)_{\text{diag}}\subset\mathrm{U}(1)^{\text{ele}}_1\times\mathrm{U}(1)^{\text{ele}}_2$. By utilizing the electric-magnetic duality transformation, we find out the special gauge couplings where the non-invertible symmetries appear. As in the two dimensions, we construct the duality defect associated with the diagonal gauging and calculate the fusion rules concerning the duality defect. The resulting fusion algebra is again *infinitely* generated and *non-commutative*. We also find out the

Gauging group	Charge matrix K	Emergent point	Fusion algebra
Diagonal group $(\mathbb{Z}_{2N'}^{[0]})_{\text{diag}}$	$K^{\text{T}} \neq K$	Irrational CFT	Non-commutative (1.14)
Product group $\mathbb{Z}_{N_1}^{[0]} \times \mathbb{Z}_{N_2}^{[0]}$	$K^{\text{T}} = K$	RCFT	Tambara-Yamagami (1.3)

Table 1. Summary of the main results in $c = 2$ bosonic torus CFT.

closed fusion subalgebra which is mostly similar to (1.14). It remains an open question how we interpret the obtained non-commutative fusion algebra in the framework of the higher category [33].

The rest of the paper is organized as follows. In section 2, we describe our method to construct non-invertible symmetries from the half-space gauging in arbitrary even dimensions. In particular, we give a detailed explanation of each step from the diagonal gauging to the rotation and the rescaling of the charge lattice, to the duality. In section 3, we discuss the non-invertible symmetries in $c = 2$ bosonic torus CFT. In section 3.1, we derive the self-duality condition (1.10), and show that the solution corresponds to the CM point. Also, we derive the condition (1.12) for promoting the CM point to the RCFT. In section 3.2, we construct the duality defect action (1.13), and describe various aspects of the duality defect e.g., the boundary condition on the defect, topological property, and the orientation reversion. In section 3.3, we elucidate the precise definition of the dressed duality defect $\widehat{\mathcal{D}}_{s_1,s_2}$ and discuss the fusion algebra. In section 4, we consider the pure $\mathrm{U}(1){\times}\mathrm{U}(1)$ gauge theory in four dimensions and explain that the non-invertible symmetries from the half-space gauging from the diagonal gauging can be constructed in a very similar manner to the two-dimensions. Furthermore, we show that the resulting fusion algebra is also non-commutative, and end with giving an open question on our fusion algebra. In section 5, we briefly summarize this paper and discuss the future directions. In appendix (A), we derive some selected fusion rules, skipped in the main text.

2 Non-invertible symmetry from half-space gauging $(\mathbb{Z}_{2N'}^{[q]})_{\text{diag}}$

In this section, we describe the general method to construct the non-invertible symmetry of the theory $\mathcal{T}_g$ whose collective couplings are symbolically denoted by g. We assume that the global symmetry contains a non-anomalous $q = \frac{d-2}{2}$ form symmetry $\mathrm{U}(1)_1^{[q]} \times \mathrm{U}(1)_2^{[q]}$ whose q dimensional charged object is denoted by $V_{\vec{n}}^{[q]}$. The charged operator $V_{\vec{n}}$ transforms as;

$$\mathrm{U}(1)_1^{[q]} \times \mathrm{U}(1)_2^{[q]}: \quad V_{\vec{n}}^{[q]} \mapsto e^{\mathrm{i}\vec{\theta}\cdot\vec{n}}\, V_{\vec{n}}^{[q]}\,, \tag{2.1}$$

where $\vec{\theta} \equiv (\theta^1, \theta^2)$ are rotational angles. Importantly, in order for the 2π rotation to be trivial, the $\mathrm{U}(1)_1^{[q]}{\times}\mathrm{U}(1)_2^{[q]}$ charge $\vec{n}$ must be quantized to be integers;

$$\vec{n} \in \mathbb{Z} \times \mathbb{Z}\,. \tag{2.2}$$

	Theory	$\mathrm{U}(1)_1^{[q]}\times\mathrm{U}(1)_2^{[q]}$	Charged ops. $V_{\vec{n}}^{[q]}$	Duality
$d=2$ $(q=0)$	$c=2$ bosonic torus CFT (section 3)	$\mathrm{U}(1)_1^{\text{shift}}\times\mathrm{U}(1)_2^{\text{shift}}$	$e^{\mathrm{i}\vec{n}\cdot\vec{\phi}}$	T-duality
$d=4$ $(q=1)$	$\mathrm{U}(1)\times\mathrm{U}(1)$ gauge theory (section 4)	$\mathrm{U}(1)_1^{\text{ele}}\times\mathrm{U}(1)_2^{\text{ele}}$	$e^{\mathrm{i}\vec{n}\cdot\int_\gamma\vec{A}}$	Electric-Magnetic duality

Table 2. Examples treated in this paper and the corresponding notations with section 2. Detailed explanations for the two-dimensional and four-dimensional examples are deferred to section 3 and (4), respectively.

We refer to the set of properly quantized charges of $V_{\vec{n}}^{[q]}$ as the *charge lattice*. In this sense, the original theory $\mathcal{T}_g$ has the charge lattice $\mathbb{Z}\times\mathbb{Z}$. (See table 2 for referring examples treated in this paper.)

Our construction of the non-invertible symmetry in the theory $\mathcal{T}_g$ can be schematically summarized as the following diagram;

$$\mathcal{T}_g \xrightarrow{\text{Gauging } (\mathbb{Z}_{2N'}^{[q]})_{\text{diag}}} \mathcal{T}_g/(\mathbb{Z}_{2N'}^{[q]})_{\text{diag}} \xrightarrow{\mathcal{R}_{-\theta} \text{ and } \mathcal{M}} \mathcal{T}'_{g'} \xrightarrow{\text{Duality}} \widehat{\mathcal{T}'}_{\widehat{g}'} \cong \mathcal{T}_g\,. \tag{2.3}$$

In particular, the transition of the charge lattice at each step is shown below;

$$\mathbb{Z}\times\mathbb{Z} \xrightarrow{\text{Gauging } (\mathbb{Z}_{2N'}^{[q]})_{\text{diag}}} \Lambda_{2N'} \xrightarrow{\mathcal{R}_{-\theta} \text{ and } \mathcal{M}} \mathcal{M}\mathcal{R}_{-\theta}\,\Lambda_{2N'} \cong \mathbb{Z}\times\mathbb{Z} \xrightarrow{\text{Duality}} \widehat{\mathbb{Z}\times\mathbb{Z}} \cong \mathbb{Z}\times\mathbb{Z}\,. \tag{2.4}$$

In the rest of this section, we provide a detailed explanation of each step in (2.3) and (2.4).

2.1 Gauging $(\mathbb{Z}_{2N'}^{[q]})_{\text{diag}}$

The story begins with gauging the diagonal discrete subgroup $(\mathbb{Z}_{2N'}^{[q]})_{\text{diag}} \subset \mathrm{U}(1)_1^{[q]}\times\mathrm{U}(1)_2^{[q]}$. Here, the diagonal group $(\mathbb{Z}_{2N'}^{[q]})_{\text{diag}}$ is generated by $(\,e^{\mathrm{i}\frac{\pi}{N'}}, e^{\mathrm{i}\frac{\pi}{N'}}\,)$. From (2.1), the charged operator $V_{\vec{n}}^{[q]}$ is transformed under $(\mathbb{Z}_{2N'}^{[q]})_{\text{diag}}$ as follows;

$$(\mathbb{Z}_{2N'}^{[q]})_{\text{diag}}:\ V_{\vec{n}}^{[q]} \mapsto e^{\mathrm{i}\frac{\pi}{N'}\vec{p}\cdot\vec{n}}\,V_{\vec{n}}^{[q]}\,,\qquad \vec{p}=(1,1)^{\mathrm{T}}\,. \tag{2.5}$$

Therefore, as a consequence of the diagonal gauging, the charge lattice of the original theory $\mathbb{Z}\times\mathbb{Z}$ is projected out to its sublattice $\Lambda_{2N'}$ defined by;

$$\Lambda_{2N'} \equiv \{\vec{n}\in\mathbb{Z}\times\mathbb{Z}\mid n_1+n_2=0 \mod 2N'\}\,, \tag{2.6}$$

which is depicted in figure 3. The new charge lattice $\Lambda_{2N'}$ is spanned by the two orthogonal vectors $\vec{\ell}_1$ and $\vec{\ell}_2$;

$$\vec{\ell}_1=\begin{pmatrix}N'\\N'\end{pmatrix}\,,\qquad \vec{\ell}_2=\begin{pmatrix}-1\\1\end{pmatrix}\,, \tag{2.7}$$

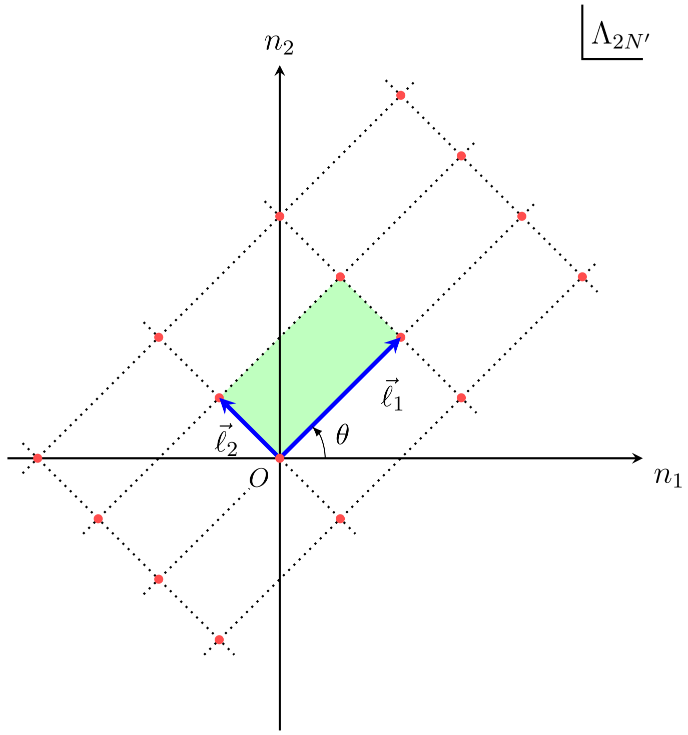

Figure 3. Picture of the charge lattice $\Lambda_{2N'}$. The elements of the charge lattice $\Lambda_{2N'}$ are depicted by red points, with the origin denoted by O. The unit cell is shown by the green realm, and its orthogonal vectors $\vec{\ell}_1$ and $\vec{\ell}_2$ are represented by two blue arrows. Also, for later convenience, we introduce the angle θ between the first axis and the vector $\vec{\ell}_1$.

and we define the charge matrix K as follows;

$$K \equiv (\vec{\ell}_1, \vec{\ell}_2) = \begin{pmatrix} N' & -1 \\ N' & 1 \end{pmatrix} . \tag{2.8}$$

2.2 Rotation and rescaling

As a result of the diagonal gauging, the charge lattice $\Lambda_{2N'}$ is clearly different from the original one $\mathbb{Z} \times \mathbb{Z}$. Therefore, in order to construct a symmetry, the charge lattice $\Lambda_{2N'}$ must be restored to the original one $\mathbb{Z} \times \mathbb{Z}$. To achieve this, two operations are needed. Firstly, the charge lattice $\Lambda_{2N'}$ must be rotated by an angle of $-\theta = -\pi/4$. Secondly, the rotated charge lattice should be rescaled to accomplish a grid scale of one. We denote these operations as $\mathcal{R}_{-\theta}$ and $\mathcal{M}$, respectively. Under these operations, indeed, the basis vectors $\vec{\ell}_1$ and $\vec{\ell}_2$ are transformed to $(1,0)^{\mathrm{T}}$ and $(0,1)^{\mathrm{T}}$, respectively;

$$\begin{aligned} \mathcal{M}\mathcal{R}_{-\theta} \ &: \ \vec{\ell}_1 \xrightarrow{\mathcal{R}_{-\theta}} R_{-\theta}\, \vec{\ell}_1 = (\ell_1, 0)^{\mathrm{T}} \ \xrightarrow{\mathcal{M}} \ (M^{-1} R_{-\theta})\, \vec{\ell}_1 = (1,0)^{\mathrm{T}} , \\ \mathcal{M}\mathcal{R}_{-\theta} \ &: \ \vec{\ell}_2 \xrightarrow{\mathcal{R}_{-\theta}} R_{-\theta}\, \vec{\ell}_2 = (0, \ell_2)^{\mathrm{T}} \ \xrightarrow{\mathcal{M}} \ (M^{-1} R_{-\theta})\, \vec{\ell}_2 = (0,1)^{\mathrm{T}} , \end{aligned} \tag{2.9}$$

where $\ell_1 \equiv |\vec{\ell}_1|$, $\ell_2 \equiv |\vec{\ell}_2|$, and the matrices $R_{-\theta}$ and M are defined as follows;

$$R_{-\theta} \equiv \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} , \qquad M \equiv \begin{pmatrix} \ell_1 & 0 \\ 0 & \ell_2 \end{pmatrix} . \tag{2.10}$$

Therefore, we have succeeded in bringing the charge lattice $\Lambda_{2N'}$ to the original one $\mathbb{Z}\times\mathbb{Z}$;

$$\mathcal{M}\mathcal{R}_{-\theta}\,\Lambda_{2N'} \cong \mathbb{Z}\times\mathbb{Z}\,. \tag{2.11}$$

We refer the reader to consult the figure 2, where the above operations from gauging to rotation to rescaling are illustrated in the case of $c=2$ bosonic torus CFT. Finally, we notice that the charge matrix K defined by (2.8) can be written in terms of the rotation and rescaling matrices;

$$K = R_\theta\, M\,. \tag{2.12}$$

2.3 Duality

While the charge lattice indeed comes back to the original one with the above steps, the theory has not yet been restored to the original one $\mathcal{T}_g$. This is because the coupling constants g' of the theory $\mathcal{T}'_{g'}$ typically differ from the original ones g. We, however, can make use of the duality transformation, which maps the theory $\mathcal{T}'_{g'}$ to the dual one $\widehat{\mathcal{T}'}_{\widehat{g}'}\cong\mathcal{T}_g$, and connects the value of couplings g' to the dual one $\widehat{g}\,'$. Hence, by tuning the original couplings g such that they satisfy the following self-duality condition;

$$\widehat{g}\,' = g\,, \tag{2.13}$$

then the dual theory $\widehat{\mathcal{T}'}_{\widehat{g}'}$ becomes equivalent to the original one $\mathcal{T}_g$.[5] Dualities, of course, depend on theories, e.g., T-duality for the $c=2$ bosonic torus CFT and electric-magnetic duality for the pure U(1)×U(1) gauge theory. Hence, we relegate the details of self-duality conditions to the subsequent sections.

Following the all above steps, we can conclude that the theory $\mathcal{T}_g$ is invariant under the diagonal gauging: $\mathcal{T}_g/(\mathbb{Z}^{[q]}_{2N'})_{\rm diag}\cong\mathcal{T}_g$. Then, we anticipate some (generalized) symmetry associated with the diagonal gauging, which will later be identified with the *non-invertible* one.

3 Example in two dimensions: $c=2$ bosonic torus CFT

In this section, we explore the non-invertible symmetries in the $c=2$ bosonic torus CFT following the method described in section 2. This theory can be described by the following action;

$$S[\phi^1,\phi^2] = \frac{1}{4\pi}\int_{X_2} G_{IJ}\, d\phi^I\wedge\star\, d\phi^J\ , \qquad I,J=1\,,2\,, \tag{3.1}$$

where X_2 is the orientable two-dimensional manifold,[6] and ϕ^I is the compact boson with the periodicity 2π;

$$\phi^I\sim\phi^I+2\pi\,. \tag{3.2}$$

[5] Here, we assume that the charge lattice $\widehat{\mathbb{Z}\times\mathbb{Z}}$ of the dual theory $\widehat{\mathcal{T}'}_{\widehat{g}'}$ is equivalent to the original one $\mathbb{Z}\times\mathbb{Z}$. Indeed, all examples treated in this paper satisfy this property.

[6] Throughout this paper, we only concern the Euclidean spacetime.

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Also, G_{IJ} is the real kinetic matrix, which must satisfy the following stability condition;

$$G_{11} > 0\,, \quad G_{22} > 0\,, \quad \det G > 0\,. \tag{3.3}$$

In this paper, we consider a particular element of the T-duality group $\mathrm{O}(2,2,\mathbb{Z})$, which maps the kinetic matrix to its inverse;

$$\text{T-duality} \;:\; G \mapsto G^{-1}\,. \tag{3.4}$$

The global symmetry of the $c = 2$ bosonic torus CFT contains the shift symmetry $\mathrm{U}(1)_1^{\text{shift}} \times \mathrm{U}(1)_2^{\text{shift}}$, which acts on the vertex operator $V_{\vec{n}}^{[0]} \equiv e^{\mathrm{i}\vec{n}\cdot\vec{\phi}}$ as follows;

$$\mathrm{U}(1)_1^{\text{shift}} \times \mathrm{U}(1)_2^{\text{shift}} \;:\; V_{\vec{n}}^{[0]} \mapsto e^{\mathrm{i}\vec{\theta}\cdot\vec{n}}\, V_{\vec{n}}^{[0]}\,. \tag{3.5}$$

As a first step toward realizing our program (2.3), we perform gauging the diagonal subgroup $(\mathbb{Z}_{2N'}^{[0]})_{\text{diag}} \subset \mathrm{U}(1)_1^{\text{shift}} \times \mathrm{U}(1)_2^{\text{shift}}$. Then, the original charge lattice $\mathbb{Z}\times\mathbb{Z}$ is reduced to the sublattice defined by (2.6);

$$\mathbb{Z}\times\mathbb{Z} \longrightarrow \Lambda_{2N'}\,. \tag{3.6}$$

As explained in section 2, we can bring the charge lattice $\Lambda_{2N'}$ to the original one by performing the rotation and rescaling successively. All we need is to perform the duality transformation, hence we proceed to give details on it below.

3.1 Self-duality condition

In this subsection, we first derive the kinetic matrix after a series of operations (diagonal gauging, rotation, and rescaling), which typically differs from the original one. Next, we show that the T-duality manifestly makes the diagonal gauging $(\mathbb{Z}_{2N'}^{[0]})_{\text{diag}}$ a symmetry when the kinetic matrix G_{IJ} satisfies the self-duality condition;

$$G = K^{\mathrm{T}} G^{-1} K\,. \tag{3.7}$$

First of all, under the diagonal gauging and the rotation $\mathcal{R}_{-\theta}$, the compact boson $\vec{\phi}$ is transformed as follows;

$$\vec{\phi} \mapsto \vec{\phi}' = R_{-\theta}\,\vec{\phi}\,, \tag{3.8}$$

where $R_{-\theta}$ is defined by (2.10), and the periodicity condition for $\vec{\phi}'$ reads;

$$\phi'^1 \sim \phi'^1 + \frac{2\pi}{\ell_1}\,, \qquad \phi'^2 \sim \phi'^2 + \frac{2\pi}{\ell_2}\,. \tag{3.9}$$

We should note that the periodicity of $\vec{\phi}'$ is not 2π, and this corresponds to the fact that the grid scale of the charged lattice $\Lambda_{2N'}$ is not one. (See also the flow from the upper left lattice to the upper right one to the bottom middle one in figure 2.) Therefore, to restore the original theory, we must make the periodicity of the compact boson 2π. This can be done by the following rescaling transformation;

$$\vec{\phi}' \mapsto \vec{\phi}'' = M\,\vec{\phi}'\,, \tag{3.10}$$

where the matrix M is the rescaling matrix defined in (2.10). We should notice that the periodicity of $\vec{\phi}''$ is 2π, and this restoration can be seen in the flow from the bottom middle lattice to the upper left one in figure 2. In summary, the diagonal gauging $(\mathbb{Z}_{2N'}^{[0]})_{\text{diag}}$, rotation and rescaling transform the compact boson field $\vec{\phi}$ as follows;

$$\begin{aligned}\vec{\phi} \mapsto \vec{\phi}'' &= M R_{-\theta}\, \vec{\phi} \\ &= K^{\mathrm{T}} \vec{\phi}\,,\end{aligned} \tag{3.11}$$

where we used (2.12). Since the periodicities of both compact bosons $\vec{\phi}$ and $\vec{\phi}''$ are 2π, the above map (3.11) can be rephrased by the transformation law for the kinetic matrix G_{IJ};

$$G \,\mapsto\, K^{-1}\, G\, (K^{\mathrm{T}})^{-1}\,. \tag{3.12}$$

As stressed in earlier times, the kinetic matrix after the series of operations, takes the different values from the original one. However, by making full use of the T-duality, we can put the theory back to the original one. The T-duality transformation (3.4), indeed, maps the deformed kinetic matrix $K^{-1}G(K^{\mathrm{T}})^{-1}$ to its inverse;

$$\text{T-duality:}\;\; K^{-1}G(K^{\mathrm{T}})^{-1} \mapsto K^{\mathrm{T}}G^{-1}K\,. \tag{3.13}$$

Therefore, if we choose the kinetic matrix G_{IJ} such that it satisfies the following self-duality condition;

$$G = K^{\mathrm{T}}G^{-1}K\,, \tag{3.14}$$

the diagonal gauging $(\mathbb{Z}_{2N'}^{[0]})_{\text{diag}}$ becomes a true symmetry. In the following, we denote the solution to (3.14) by G^*. By noticing $\det G = 2N'$, we easily obtain the solution to the self-duality condition;

$$G^* = \sqrt{-\frac{2N'}{D}} \begin{pmatrix} 2K_{11} & K_{12}+K_{21} \\ K_{12}+K_{21} & 2K_{22} \end{pmatrix}\,, \tag{3.15}$$

where D is defined by

$$D \equiv (K_{12}+K_{21})^2 - 4K_{11}K_{22}\,. \tag{3.16}$$

We should note that the kinetic matrix G_{IJ} must be real in physical theory, hence D takes the negative value;

$$D < 0\,. \tag{3.17}$$

Interestingly, the self-dual solution (3.15) is the same one known as *complex multiplication (CM) point* which is a more generic point than a RCFT in the $c = 2$ bosonic torus CFT[7] [98]. To see this, we put the kinetic matrix into the complex structure modulus τ defined by;

$$\tau \equiv \frac{G_{12}}{G_{22}} + \mathrm{i}\frac{\sqrt{\det G}}{G_{22}}\,, \tag{3.18}$$

[7] We thank Justin Kaidi for plentiful discussions on this point.

then the complex structure modulus at the self-dual point τ^* satisfies the following quadratic equation;

$$K_{22}\,(\tau^*)^2 - (K_{12}+K_{21})\tau^* + K_{11} = 0\,. \tag{3.19}$$

Importantly, the discriminat of the above quadratic equation (3.19) is precisely same as D defined in (3.16), and its negative property $D<0$ ensures that τ^* belongs to the imaginary quadratic number field $\mathbb{Q}(\sqrt{D})$ [98];

$$\tau^* \in \mathbb{Q}(\sqrt{D})\,. \tag{3.20}$$

Since it is known that the elliptic curves with the modular parameter τ^* satisfying (3.20) have complex multiplication properties, such modulus is called the CM point.

Then, the following natural question arises; when can the CM point be lifted up to the RCFT? We can show that only if the charge matrix K is symmetric, namely $K^{\mathrm{T}} = K$, this promoting occurs. The proof is as follows. In order for this lifting to be achieved, it is sufficient to show that the complexified Kähler modulus ρ also needs to belong to the *same* imaginary quadratic number field $\mathbb{Q}(\sqrt{D})$ [98]. Now, the complexified Kähler modulus ρ is given by the pure imaginary number due to the absence of the B-field;

$$\rho \equiv \mathrm{i}\sqrt{\det G}\,, \tag{3.21}$$

and at the self-duality point, the modulus ρ becomes $\rho^* = \mathrm{i}\sqrt{2N'}$. Hence, when there exist the integers α, β and γ such that;

$$\alpha(\rho^*)^2 + \beta\rho^* + \gamma = 0 \qquad \text{and} \qquad \beta^2 - 4\alpha\gamma = D\,, \tag{3.22}$$

the CM points can get promoted to RCFT ones. We first notice that $\beta = 0$ due to the pure imaginary property of ρ^*, then resulting in $\gamma = 2N'\alpha$. Next, we can rewrite the discriminant D given in (3.16) as follows;

$$D = (K_{12} - K_{21})^2 - 8N'\,, \tag{3.23}$$

where the formula $\det K = 2N'$ is used. Therefore, to realize $\rho^* \in \mathbb{Q}(\sqrt{D})$, there must exist some integer α such that

$$\alpha^2 = 1 - \frac{(K_{12}-K_{21})^2}{8N'}\,. \tag{3.24}$$

Since α^2 must take its value in positive integers, K_{12} must be equal to K_{21}, which completes the proof. Note that we cannot find the symmetric charge matrix K in the case of the diagonal gauging,[8] as clearly seen from (2.8). Hence, we arrive at the following conclusion;

> *A (generalized) symmetry associated with the diagonal gauging $(\mathbb{Z}_{2N'}^{[0]})_{diag}$ becomes emergent at the irrational CFT point.*

[8] One may notice we can make the charge matrix K be symmetric if we exchange the two basis vectors $\vec{\ell}_1$ and $\vec{\ell}_2$. In that case, however, we can never obtain the integer α because of $\det K = -2N'$.

Emergent $\mathbb{Z}_2$ symmetry at the self-dual point. Finally, we close this subsection by mentioning the non-trivial emergent $\mathbb{Z}_2$ symmetry at the irrational CFT point. We should note that the self-duality condition (3.14) is invariant under the replacement of the charge matrix K with its transposed one K^{T};

$$K^{\mathrm{T}} G^{-1} K = G \qquad \Longleftrightarrow \qquad K G^{-1} K^{\mathrm{T}} = G\,. \tag{3.25}$$

This implies some emergent $\mathbb{Z}_2$ symmetry at the self-dual point, and we find out that the mapping $K \mapsto K^{\mathrm{T}}$ can be realized by the transformation of the compact boson field;

$$\phi^I \mapsto \phi'^I = \mathsf{S}_{IJ}\, \phi^J\,, \qquad \mathsf{S} = \begin{pmatrix} 1 & 0 \\ 1-N' & -1 \end{pmatrix}. \tag{3.26}$$

Indeed, the matrix S satisfies the following properties;

$$\mathsf{S}^2 = 1_{2\times 2}\,, \qquad \mathsf{S}^{\mathrm{T}}\, G^*\, \mathsf{S} \;\; = G^*\,, \qquad \mathsf{S}^{\mathrm{T}}\, K\, \mathsf{S} = K^{\mathrm{T}}\,, \tag{3.27}$$

and we can easily check that the theory at the self-dual point is invariant under the $\mathbb{Z}_2$ transformation (3.26). This emergent $\mathbb{Z}_2$ symmetry plays a crucial role in discussing the fusion algebra, and we denote the topological defect associated with this emergent $\mathbb{Z}_2$ symmetry by $\mathcal{S}$.

3.2 Duality defect

In this subsection, following the spirit of [45, 46], we derive the duality defect associated with the diagonal gauging $(\mathbb{Z}_{2N'}^{[0]})_{\mathrm{diag}}$. First of all, we divide the ambient spacetime into the left and right regions separated by the co-dimension one defect residing at $x = 0$. Then, we propose that the duality defect $\mathcal{D}$ can be expressed by the following Lagrangian;

$$\mathcal{D} \;:\; \frac{\mathrm{i}}{2\pi} \int_{x=0} K_{IJ}\, \phi_{\mathrm{L}}^I\, d\phi_{\mathrm{R}}^J\,, \tag{3.28}$$

where ϕ_{L} and ϕ_{R} are the compact boson fields which are located in the bulks $x < 0$ and $x > 0$, respectively. We should note that the duality defect (3.28) is gauge-invariant since the charge matrix K is an integer matrix. In the following, we explicitly show that only when the bulk kinetic matrix is tuned to be self-dual one G^*, the duality defect $\mathcal{D}$ correctly reflects the sequence of operations in (2.3) by seeing the boundary conditions of the left and right fields. Finally, we give some comments on the topological property of the duality defect and its orientation-reversion.

The combined system of the bulk theory and the duality defect can be described in the following action (see figure 4.);

$$\frac{1}{4\pi} \int_{\mathrm{L}} G_{IJ}^*\, d\phi_{\mathrm{L}}^I \wedge \star\, d\phi_{\mathrm{L}}^J + \frac{1}{4\pi} \int_{\mathrm{R}} G_{IJ}^*\, d\phi_{\mathrm{R}}^I \wedge \star\, d\phi_{\mathrm{R}}^J + \frac{\mathrm{i}}{2\pi} \int_{x=0} K_{IJ}\, \phi_{\mathrm{L}}^I\, d\phi_{\mathrm{R}}^J\,, \tag{3.29}$$

then the variations of the left and right compact boson fields give rise to the following boundary conditions at $x = 0$;

$$x = 0 \;:\; \mathrm{i}\, G_{IJ}^* \star d\phi_{\mathrm{L}}^J = K_{IJ}\, d\phi_{\mathrm{R}}^J\,, \tag{3.30}$$

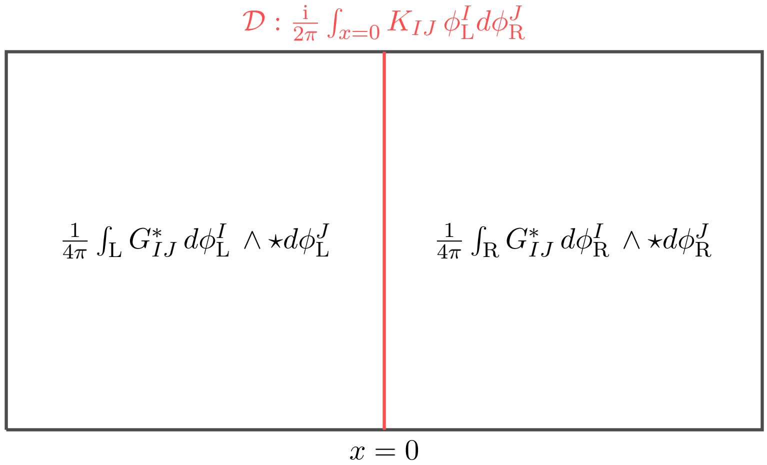

Figure 4. Pictorical representation of the duality defect $\mathcal{D}$.

$$x=0\ :\ \mathrm{i}\,G^*_{IJ}\star d\phi_{\mathrm{R}}^J = K_{JI}\,d\phi_{\mathrm{L}}^J\,. \tag{3.31}$$

We can readily check the equivalence of these two conditions by using the self-duality condition (3.14). Concretely speaking, we can obtain the latter boundary condition (3.31) from the former one (3.30) by acting the Hodge dual operation $\star$ on both hand sides in (3.30), and using the self-duality condition (3.14), and vice versa. Furthermore, we should note that by rewriting the matrix K in terms of the rotation matrix and the rescaling one, the boundary condition becomes;

$$x=0\ :\ \mathrm{i}\,G^*_{IJ}\,d\phi_{\mathrm{R}}^J = \star\left((MR_{-\theta})_{IJ}\,d\phi_{\mathrm{L}}^J\right)\,. \tag{3.32}$$

This can be interpreted as first performing a $(\mathbb{Z}_{2N'}^{[0]})_{\mathrm{diag}}$ gauging to rotate the compact boson by angle $-\theta$ to rescale by the matrix M, and finally performing the T-duality transformation. This observation corroborates that our construction (2.3) can be realized by insertion of the duality defect $\mathcal{D}$ defined by (3.28) into the spacetime.

In addition, we insist that the duality defect $\mathcal{D}$ becomes topological when $G=G^*$. Although this is clear from the viewpoint of the half-space gauging [45, 46], we provide another proof in the spirit of [53, 99]. To show this topological property, it is enough to show that the energy-momentum tensors must satisfy the following matching condition;

$$x=0\ :\ n_\mu(T_{\mathrm{L}}^{\mu\nu}-T_{\mathrm{R}}^{\mu\nu})=0\,. \tag{3.33}$$

Here, $T_{\mathrm{L}}^{\mu\nu}$ and $T_{\mathrm{R}}^{\mu\nu}$ are the energy-momentum tensors in the left and right regions, and given by

$$\begin{aligned} T_{\mathrm{L}}^{\mu\nu} &= \frac{1}{4\pi}G_{IJ}\,\partial_\alpha\phi_{\mathrm{L}}^I\partial_\beta\phi_{\mathrm{L}}^J\left(\frac{1}{2}\delta^{\alpha\beta}\delta^{\mu\nu}-\delta^{\mu\alpha}\delta^{\nu\beta}\right)\,,\\ T_{\mathrm{R}}^{\mu\nu} &= \frac{1}{4\pi}G_{IJ}\,\partial_\alpha\phi_{\mathrm{R}}^I\partial_\beta\phi_{\mathrm{R}}^J\left(\frac{1}{2}\delta^{\alpha\beta}\delta^{\mu\nu}-\delta^{\mu\alpha}\delta^{\nu\beta}\right)\,, \end{aligned} \tag{3.34}$$

respectively, and n^μ is the normal vector to the duality defect $\mathcal{D}$. We can easily prove that the matching condition (3.33) can be achieved by using (3.14) and (3.30). As a result of this reasoning, we can conclude that the duality defect $\mathcal{D}$ is topological.

Finally, we comment on the orientation-reversing of the duality defect $\mathcal{D}$. Its orientation reversal $\overline{\mathcal{D}}(M)$ is defined by [47, 52];

$$\overline{\mathcal{D}}(M) = \mathcal{D}(\overline{M})\,, \tag{3.35}$$

where M is the support manifold of the duality defect, namely $x = 0$, and $\overline{M}$ denotes the orientation reversal of M. The defect action of $\overline{\mathcal{D}}$ can be obtained by swapping ϕ_L with ϕ_R, and flipping the overall sign stemming from the orientation-reversion of M;

$$\overline{\mathcal{D}}(M) \ : \ \frac{\mathrm{i}}{2\pi}\int_{x=0} K_{JI}\,\phi^I_{\mathrm{L}}\,d\phi^J_{\mathrm{R}}\,. \tag{3.36}$$

Note that we can also obtain $\overline{\mathcal{D}}$ only by replacing the charge matrix K with its transposed one in the duality defect $\mathcal{D}$. We can realize this replacement by utilizing the emergent $\mathbb{Z}_2$ symmetry discussed in section 3.1, and write the orientation-reversed duality defect $\overline{\mathcal{D}}$ in terms of $\mathcal{D}$ and $\mathcal{S}$;

$$\overline{\mathcal{D}} = \mathcal{S}\times\mathcal{D}\times\mathcal{S}\,. \tag{3.37}$$

We should notice that the orientation-reversed duality defect $\overline{\mathcal{D}}$ can be interpreted as the duality defect obtained by gauging $\mathbb{Z}^{[0]}_{2N'}$ symmetry generated by $\eta_{\mathsf{S}\vec{p}}$;

$$\eta_{\mathsf{S}\vec{p}} = \mathcal{S}\times\eta_{\vec{p}}\times\mathcal{S}\,. \tag{3.38}$$

This is because, after gauging this $\mathbb{Z}^{[0]}_{2N'}$ symmetry, the charge matrix is given by the transposed matrix K^{T}.

3.3 Non-commutative fusion algebra

In this subsection, we describe various fusion rules involving the duality defect $\mathcal{D}$ introduced in section 3.2. Since the derivations of the fusion algebra require somewhat technical calculations, we just digest our results here. If the readers have some interest in the detailed calculations, we refer to reading appendix (A), where some skipped derivations are demonstrated.

First of all, we consider the fusion rules between the duality defect $\mathcal{D}$ and the $(\mathbb{Z}^{[0]}_{2N'})_{\mathrm{diag}}$ shift symmetry generator $\eta_{\vec{p}}(\Sigma)$ defined by;

$$\eta_{\vec{p}}(\Sigma) \equiv \exp\left[-\frac{p^I}{2N'}\int_\Sigma G_{IJ}\star d\phi^J\right]\,, \qquad \vec{p}\equiv(1,1)^{\mathrm{T}}\,, \tag{3.39}$$

where Σ is the parallel line to the duality defect $\mathcal{D}$. Interestingly, unlike the ordinary fusion rules of the duality defect, $\eta_{\vec{p}}\times\mathcal{D}$ and $\mathcal{D}\times\eta_{\vec{p}}$ do not give rise to the same results in general. If we bring the $(\mathbb{Z}^{[0]}_{2N'})_{\mathrm{diag}}$ shift symmetry generator $\eta_{\vec{p}}$ closer to the duality defect $\mathcal{D}$ from the left side, the fusion rule $\eta_{\vec{p}}\times\mathcal{D}$ reads;

$$\eta_{\vec{p}}\times\mathcal{D} \ : \ \frac{\mathrm{i}}{2\pi}\int_{x=0} K_{IJ}\,\phi^I_{\mathrm{L}}\,d\phi^J_{\mathrm{R}} - \frac{\mathrm{i}p^I K_{IJ}}{2N'}\int_{x=0} d\phi^J_{\mathrm{R}}\,. \tag{3.40}$$

Here, we should recall $p^I K_{IJ} = 0 \mod 2N'$, hence the last term in (3.40) becomes trivial and can be dropped. This implies that the symmetry generator $\eta_{\vec{p}}$ is absorbed into the duality defect $\mathcal{D}$, and the fusion rule $\eta_{\vec{p}} \times \mathcal{D}$ becomes as follows;

$$\eta_{\vec{p}} \times \mathcal{D} = \mathcal{D}\,. \tag{3.41}$$

From this, it turns out that the duality defect $\mathcal{D}$ is non-invertible.[9] On the other hand, if we put the $(\mathbb{Z}^{[0]}_{2N'})_{\text{diag}}$ shift symmetry generator $\eta_{\vec{p}}$ to the duality defect $\mathcal{D}$ from the right, the *non-trivial* $\mathbb{Z}^{[0]}_{2N'}$ winding symmetry generator becomes emergent on the left side of the duality defect;

$$\mathcal{D} \times \eta_{\vec{p}} \ : \ \frac{\mathrm{i}}{2\pi}\int_{x=0} K_{IJ}\,\phi^I_{\mathrm{L}}\, d\phi^J_{\mathrm{R}} - \frac{\mathrm{i}p^J K_{IJ}}{2N'}\int_{x=0} d\phi^I_{\mathrm{L}}\,, \tag{3.42}$$

since $p^J K_{IJ} \neq 0 \mod 2N'$ in general. We denote this emergent $\mathbb{Z}^{[0]}_{2N'}$ winding symmetry generator by $\widetilde{\eta}_{K\vec{p}}$, then the fusion rule $\mathcal{D} \times \eta_{\vec{p}}$ becomes as follows;

$$\mathcal{D} \times \eta_{\vec{p}} = \widetilde{\eta}_{K\vec{p}} \times \mathcal{D}\,. \tag{3.43}$$

By comparing the above results (3.41) and (3.43), we can conclude that the obtained fusion rules are non-commutative, namely $\eta_{\vec{p}} \times \mathcal{D} \neq \mathcal{D} \times \eta_{\vec{p}}$. What happens if we close the $\mathbb{Z}^{[0]}_{2N'}$ winding symmetry generator $\widetilde{\eta}_{K\vec{p}}$ to the right side of the duality defect $\mathcal{D}$? The fusion rule $\mathcal{D} \times \widetilde{\eta}_{K\vec{p}}$ can be calculated as follows;

$$\mathcal{D} \times \widetilde{\eta}_{K\vec{p}} \ : \ \frac{\mathrm{i}}{2\pi}\int_{x=0} K_{IJ}\,\phi^I_{\mathrm{L}}\, d\phi^J_{\mathrm{R}} + \frac{(\vec{p}^{\,\mathrm{T}} K^{\mathrm{T}} K^{-1})^I}{2N'}\int_{x=0} G_{IJ} \star d\phi^I_{\mathrm{L}}\,. \tag{3.44}$$

From (3.39), the last term in (3.44) is none other than the shift symmetry generator $\eta_{MK\vec{p}}$;

$$\eta_{MK\vec{p}}(\Sigma) \equiv \exp\left[-\frac{(\vec{p}^{\,\mathrm{T}} K^{\mathrm{T}} M^{\mathrm{T}})^I}{2N'}\int_{\Sigma} G_{IJ} \star d\phi^J\right]\,, \qquad M \equiv (K^{\mathrm{T}})^{-1}\,, \tag{3.45}$$

and the fusion rule $\mathcal{D} \times \widetilde{\eta}_{K\vec{p}}$ becomes as follows;

$$\mathcal{D} \times \widetilde{\eta}_{K\vec{p}} = \eta_{MK\vec{p}} \times \mathcal{D}\,. \tag{3.46}$$

We should notice that $\eta_{MK\vec{p}}$ is the symmetry generator associated with $\mathbb{Z}^{[0]}_{(2N')^2}$ shift symmetry. If we moreover put this the shift symmetry generator $\eta_{MK\vec{p}}$ to $\mathcal{D}$ from the right, $\mathbb{Z}^{[0]}_{(2N')^2}$ winding symmetry generator $\widetilde{\eta}_{KMK\vec{p}}$ appears in the left side;

$$\mathcal{D} \times \eta_{MK\vec{p}} = \widetilde{\eta}_{KMK\vec{p}} \times \mathcal{D}\,. \tag{3.47}$$

We can straightforwardly keep going the above discussions, and eventually obtain the following fusion rules:

$$\mathcal{D} \times \eta_{(MK)^i\vec{p}} = \widetilde{\eta}_{K(MK)^i\vec{p}} \times \mathcal{D}\,, \quad \mathcal{D} \times \widetilde{\eta}_{K(MK)^i\vec{p}} = \eta_{(MK)^{i+1}\vec{p}} \times \mathcal{D}\,, \quad i = 0,1,2,\cdots\,, \tag{3.48}$$

[9]This can be readily checked as follows. Suppose that the duality defect $\mathcal{D}$ is invertible, i.e., $\mathcal{D} \times \mathcal{D}^{-1} = 1$, this contradicts with the fusion rule derived in (3.41);

$$\mathcal{D} \times \mathcal{D}^{-1} = \eta_{\vec{p}} \times \mathcal{D} \times \mathcal{D}^{-1} = \eta_{\vec{p}} \neq 1\,.$$

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where $\eta_{(MK)^i\,\vec{p}}$ and $\widetilde{\eta}_{K(MK)^i\,\vec{p}}$ are $\mathbb{Z}^{[0]}_{(2N')^i}$ shift and winding symmetry generators, respectively. This implies that the fusion algebra concerning the non-invertible duality defect constructed from the diagonal gauging is *infinitely* generated. Notably, this is consistent with the general property of the irrational CFT where the number of topological defect lines is expected to be infinite.

Interestingly, we find out the closed fusion subalgbera of the infinitely generated fusion algebra described above. To see this, we first redefine the duality defect $\mathcal{D}$ by dressing the $\mathbb{Z}^{[0]}_{2N'}$ winding symmetry generator;

$$\mathcal{D}_s(\Sigma)\ :\ \int_\Sigma \left(\frac{\mathrm{i}}{2\pi}K_{IJ}\,\phi^I_{\mathrm{L}}\,d\phi^J_{\mathrm{R}} - \frac{\mathrm{i}s\,p^J K_{IJ}}{2N'}\,d\phi^I_{\mathrm{L}}\right)\,, \tag{3.49}$$

which is labelled by the $\mathbb{Z}_{2N'}$ element $s = 0, 1, \cdots 2N'-1$. The fusion rules between the dressed duality defect $\mathcal{D}_s$ and the $(\mathbb{Z}^{[0]}_{2N'})_{\mathrm{diag}}$ shift symmetry generator $\eta_{\vec{p}}$ becomes;

$$\eta_{\vec{p}} \times \mathcal{D}_s = \mathcal{D}_s\,, \tag{3.50}$$

$$\mathcal{D}_s \times \eta_{\vec{p}} = \mathcal{D}_{s+1}\,. \tag{3.51}$$

However, as it is, the fusion algebra is not closed, which can be seen from the direct calculation of $\mathcal{D}_{s_1} \times \mathcal{D}_{s_2}$;

$$\mathcal{D}_{s_1} \times \mathcal{D}_{s_2}\ :\ \frac{\mathrm{i}}{2\pi}\int_{x=0}(K_{IJ}\phi^I_{\mathrm{L}} - K_{JI}\phi^I_{\mathrm{R}})d\phi^J_{\mathrm{M}} - \frac{\mathrm{i}s_2\,p^J K_{IJ}}{2N'}\int_{x=0} d\phi^I_{\mathrm{M}}\,. \tag{3.52}$$

To our best effort, we cannot write the above result in a closed form by using the known topological defects. Hence, further modification for the duality defect is needed to close the fusion algebra. After some trial and error, we find out that the following combination works well for the closure of the fusion algebra;[10]

$$\widehat{\mathcal{D}}_{s_1,s_2} \equiv (\eta_{\mathsf{S}\vec{p}})^{s_1} \times \mathcal{D}_{s_2} \times \mathcal{S}\,,\quad s_1, s_2 = 0, \cdots, 2N'-1\,, \tag{3.53}$$

and we arrive at the following conclusion;

Non-commutative fusion subalgebra at the irrational CFT point. *Non-invertible symmetry constructed from the half-space gauging of $(\mathbb{Z}^{[0]}_{2N'})_{diag}$ becomes emergent at the irrational CFT point. The fusion algebra is non-commutative and infinitely generated. We find out the closed fusion subalgebra;*

$$\begin{aligned}
\widehat{\mathcal{D}}_{s_1,s_2} \times \widehat{\mathcal{D}}_{s_3,s_4} &= \widehat{\mathcal{C}}_{s_2+s_3,s_1+s_4}\,,\\
\widehat{\mathcal{D}}_{s_1,s_2} \times \eta_{\vec{p}} = \eta_{\vec{p}} \times \widehat{\mathcal{D}}_{s_1,s_2} &= \widehat{\mathcal{D}}_{s_1,s_2}\,,\\
\eta_{\vec{p}} \times \widehat{\mathcal{C}}_{s_1,s_2} = \widehat{\mathcal{C}}_{s_1,s_2} \times \eta_{\vec{p}} &= \widehat{\mathcal{C}}_{s_1,s_2}\,,
\end{aligned}$$

[10] We note that orientation-reversed dressed duality defect $\overline{\widehat{\mathcal{D}}}_{s_1,s_2}(M) \equiv \widehat{\mathcal{D}}_{s_1,s_2}(\overline{M})$ can be expressed in terms of the dressed duality defect $\widehat{\mathcal{D}}_{s_1,s_2}$ as follows;

$$\begin{aligned}
\overline{\widehat{\mathcal{D}}}_{s_1,s_2} &= \eta^{-s_2}_{\mathsf{S}\vec{p}} \times \mathsf{S} \times \overline{\mathcal{D}} \times \eta^{s_1}_{\mathsf{S}\vec{p}}\\
&= \eta^{-s_2}_{\mathsf{S}\vec{p}} \times \mathcal{D} \times \mathsf{S} \times \eta^{s_1}_{\mathsf{S}\vec{p}}\\
&= \widehat{\mathcal{D}}_{-s_2,-s_1}\,.
\end{aligned}$$

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$$
\begin{aligned}
&\widehat{\mathcal{D}}_{s_1,s_2}\times\widehat{\mathcal{C}}_{s_3,s_4}=2N'\,\widehat{\mathcal{D}}_{s_1+s_3,s_2+s_4}\,,\\
&\widehat{\mathcal{C}}_{s_1,s_2}\times\widehat{\mathcal{D}}_{s_3,s_4}=2N'\,\widehat{\mathcal{D}}_{s_2+s_3,s_1+s_4}\,,\\
&\widehat{\mathcal{C}}_{s_1,s_2}\times\widehat{\mathcal{C}}_{s_3,s_4}=2N'\,\widehat{\mathcal{C}}_{s_1+s_3,s_2+s_4}\,,\\
&\eta_{\vec{p}}^{2N'}=1\,,
\end{aligned}
\tag{3.54}
$$

where $\widehat{\mathcal{C}}_{s_1,s_2}$ *is the projection operator associated with* $(\mathbb{Z}_{2N'}^{[0]})_{diag}$ *shift symmetry, dressed with the* $\mathbb{Z}_{2N'}^{[0]}$ *winding symmetry generator;*

$$
\begin{aligned}
\widehat{\mathcal{C}}_{s_1,s_2}(\Sigma)\equiv&\exp\left[-\frac{\mathrm{i}s_1p^JK_{IJ}}{2N'}\int_\Sigma d\phi_{\mathrm{L}}^I\right]\\
&\times\int\mathcal{D}\varphi\exp\left[-\frac{\mathrm{i}}{2\pi}\int_\Sigma K_{IJ}\left(\phi_{\mathrm{L}}^I-\phi_{\mathrm{R}}^I-\frac{2\pi s_2}{2N'}\,\mathsf{S}^{IK}p_K\right)d\varphi^J\right].
\end{aligned}
\tag{3.55}
$$

We can easily check that the above fusion algebra satisfies the associativity condition. For instance, the fusion rule $(\widehat{\mathcal{D}}_{s_1,s_2}\times\widehat{\mathcal{D}}_{s_3,s_4})\times\widehat{\mathcal{D}}_{s_5,s_6}$ becomes as follows;

$$
\begin{aligned}
(\widehat{\mathcal{D}}_{s_1,s_2}\times\widehat{\mathcal{D}}_{s_3,s_4})\times\widehat{\mathcal{D}}_{s_5,s_6}&=\widehat{\mathcal{C}}_{s_2+s_3,s_1+s_4}\times\widehat{\mathcal{D}}_{s_5,s_6}\\
&=2N'\widehat{\mathcal{D}}_{s_1+s_4+s_5,s_2+s_3+s_6}\,.
\end{aligned}
\tag{3.56}
$$

On the other hand, the fusion rule $\widehat{\mathcal{D}}_{s_1,s_2}\times(\widehat{\mathcal{D}}_{s_3,s_4}\times\widehat{\mathcal{D}}_{s_5,s_6})$ can be evaluated as follows;

$$
\begin{aligned}
\widehat{\mathcal{D}}_{s_1,s_2}\times(\widehat{\mathcal{D}}_{s_3,s_4}\times\widehat{\mathcal{D}}_{s_5,s_6})&=\widehat{\mathcal{D}}_{s_1,s_2}\times\widehat{\mathcal{C}}_{s_4+s_5,s_3+s_6}\\
&=2N'\widehat{\mathcal{D}}_{s_1+s_4+s_5,s_2+s_3+s_6}\,.
\end{aligned}
\tag{3.57}
$$

From 3.56 and 3.57, the associativity condition does hold;

$$
(\widehat{\mathcal{D}}_{s_1,s_2}\times\widehat{\mathcal{D}}_{s_3,s_4})\times\widehat{\mathcal{D}}_{s_5,s_6}=\widehat{\mathcal{D}}_{s_1,s_2}\times(\widehat{\mathcal{D}}_{s_3,s_4}\times\widehat{\mathcal{D}}_{s_5,s_6})\,.
\tag{3.58}
$$

For other defects, we can show the associativity in a similar manner to the above.

Here, it is instructive to compare the above fusion algebra 3.54 with the one obtained by gauging the product subgroup $\mathbb{Z}_{N_1}\times\mathbb{Z}_{N_2}\subset\mathrm{U}(1)_1\times\mathrm{U}(1)_2$. In this case, the charge matrix is given by the following diagonal form;

$$
K=\begin{pmatrix}N_1&0\\0&N_2\end{pmatrix},
\tag{3.59}
$$

then the self-dual kinetic matrix is also diagonal;

$$
G=\begin{pmatrix}N_1&0\\0&N_2\end{pmatrix}.
\tag{3.60}
$$

The above result 3.60 shows that we can split the $c=2$ bosonic torus CFT into the double $c=1$ compact boson CFTs, whose radiuses are given by $\sqrt{N_1}$ and $\sqrt{N_2}$. By recalling the facts about non-invertible symmetries in $c=1$ compact boson CFT [45, 46], we can convince that the non-invertible symmetry from the gauging $\mathbb{Z}_{N_1}\times\mathbb{Z}_{N_2}$ becomes emergent at a RCFT point, and the resulting fusion algebra is given by the Tambara-Yamagami category (1.3). This observation is consistent with the fact that we can undress totally of various global symmetry generators from the dressed duality defect $\widehat{\mathcal{D}}_{s_1,s_2}$, and the resulting fusion algebra 3.54 is reduced to the Tambara-Yamagami category.

4 Example in four dimensions: pure U(1)×U(1) gauge theory

In this section, we provide a four-dimensional example that has a non-invertible symmetry obtained from the diagonal gauging. In particular, we consider the pure U(1)×U(1) gauge theory;

$$S[A^1, A^2] = \frac{1}{4\pi} \int_{M_4} \mathcal{G}_{IJ}\, dA^I \wedge \star dA^J \ , \qquad I, J = 1\,, 2\,, \tag{4.1}$$

where A^1 and A^2 are U(1) gauge fields. Note that this model is the higher-dimensional analog of the $c = 2$ bosonic torus CFT, and has the same symmetry structures with that. (See table 2.) Therefore, we can parallelly discuss the non-invertible symmetry in this model based on the method accomplished in section 3.

This theory has the electric one-form symmetry $\mathrm{U}(1)^{\mathrm{ele}}_1 \times \mathrm{U}(1)^{\mathrm{ele}}_2$ [2] and the electric-magnetic duality: $\mathcal{G} \mapsto \mathcal{G}^{-1}$. First, we gauge the diagonal subgroup $(\mathbb{Z}^{[1]}_{2N'})_{\mathrm{diag}} \subset \mathrm{U}(1)^{\mathrm{ele}}_1 \times \mathrm{U}(1)^{\mathrm{ele}}_2$, generated by;

$$\eta_{\vec{p}}(\Sigma_2) = \exp\left[-\frac{p^I}{2N'} \int_{\Sigma_2} \mathcal{G}_{IJ} \star dA^J \right] \ , \quad \vec{p} \equiv (1, 1)\,, \tag{4.2}$$

where Σ_2 is a two-dimensional closed manifold. When the kinetic matrix $\mathcal{G}_{IJ}$ satisfies the self-duality condition $K^{\mathrm{T}} \mathcal{G}^{-1} K = \mathcal{G}$, namely its solution is given by

$$\mathcal{G}^* = \sqrt{-\frac{2N'}{D}} \begin{pmatrix} 2K_{11} & K_{12} + K_{21} \\ K_{12} + K_{21} & 2K_{22} \end{pmatrix} \ , \quad D = (K_{12} + K_{21})^2 - 4K_{11}K_{22}\,, \tag{4.3}$$

the diagonal gauging $(\mathbb{Z}^{[1]}_{2N'})_{\mathrm{diag}}$ becomes a non-invertible symmetry. Furthermore, the (topological) duality defect $\mathcal{D}$ and its orientation reversal $\overline{\mathcal{D}}$ is obtained by the same procedure as discussed in section 2, and its defect action can be written as the following;

$$\mathcal{D} \ : \ \frac{\mathrm{i}}{2\pi} \int_{M_3} K_{IJ}\, A^I_{\mathrm{L}}\, dA^J_{\mathrm{R}}\,, \tag{4.4}$$

$$\overline{\mathcal{D}} \ : \ -\frac{\mathrm{i}}{2\pi} \int_{M_3} K_{JI}\, A^I_{\mathrm{L}}\, dA^J_{\mathrm{R}}\,, \tag{4.5}$$

where M_3 is a three dimensional manifold, and A_{L} and A_{R} are gauge fields living in left and right regions, respectively. As with the two dimensions (3.37), we can rewrite $\overline{\mathcal{D}}$ in a following way;

$$\overline{\mathcal{D}} = \mathcal{S} \times \mathcal{D} \times \mathcal{S} \times \mathcal{U}\,, \tag{4.6}$$

where $\mathcal{S}$ and $\mathcal{U}$ are $\mathbb{Z}_2$ symmetry defects which act on the gauge field A^I as follows;

$$\mathcal{S} \ : \ A^I \mapsto \mathsf{S}_{IJ}\, A^J\,, \qquad \mathcal{U} \ : \ A^I \mapsto -A^I\,. \tag{4.7}$$

Here, the matrix S is defined in (3.26). The resulting fusion algebra is again infinitely generated and non-commutative, and in a similar manner to the two dimensions, we can find the closed subalgebra by defining the dressed duality defect $\widehat{\mathcal{D}}_{\boldsymbol{t}_1, \boldsymbol{t}_2}$ as follows;

$$\widehat{\mathcal{D}}_{\boldsymbol{t}_1, \boldsymbol{t}_2}(M_3) \equiv \eta_{\mathsf{S}\vec{p}}(\boldsymbol{t}_1) \times \mathcal{D}_{\boldsymbol{t}_2}(M_3) \times \mathcal{S}\,, \tag{4.8}$$

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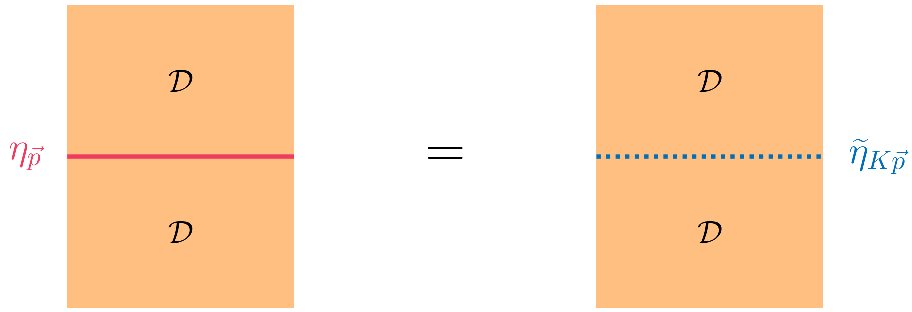

Figure 5. The result of the fusion rule $\mathcal{D} \times \eta_{\vec{p}}$. Here, we imagine that the duality defect $\mathcal{D}$ (orange plane) is sitting on this paper. In the left (right) diagram, the electric (magnetic) symmetry defect $\eta_{\vec{p}}$ ($\widetilde{\eta}_{K\vec{p}} \equiv \exp(\frac{\mathrm{i}\, p^J K_{IJ}}{2N'} \int dA^I)$) is living in the front (back) side of the duality defect. Both symmetry defects serve like the 1-morphisms, mapping from the below duality defect $\mathcal{D}$ to the above one $\mathcal{D}$.

where $\boldsymbol{t}_1$ and $\boldsymbol{t}_2$ denote the homology cycles belonging to $H_2(M_3, \mathbb{Z}_{2N'})$, and

$$\eta_{\mathsf{S}\vec{p}}(\boldsymbol{t}_1) = \exp\left[-\frac{p^K \mathsf{S}_{IK}}{2N'} \int_{\boldsymbol{t}_1} \mathcal{G}_{IJ} * dA^J\right], \tag{4.9}$$

$$\mathcal{D}_{\boldsymbol{t}_2}(M_3) \equiv \exp\left[-\frac{\mathrm{i}}{2\pi} \int_{M_3} K_{IJ}\, A^I_{\mathrm{L}}\, dA^J_{\mathrm{R}} - \frac{\mathrm{i}\, p^J K_{IJ}}{2N'} \int_{\boldsymbol{t}_2} dA^I_{\mathrm{L}}\right]. \tag{4.10}$$

We also have the orientation-reversed duality defect $\overline{\widehat{\mathcal{D}}}_{\boldsymbol{t}_1,\boldsymbol{t}_2}(M) \equiv \widehat{\mathcal{D}}_{\boldsymbol{t}_1,\boldsymbol{t}_2}(\overline{M})$ as follows;

$$\overline{\widehat{\mathcal{D}}}_{\boldsymbol{t}_1,\boldsymbol{t}_2} = \widehat{\mathcal{D}}_{-\boldsymbol{t}_2,\boldsymbol{t}_1} \times \mathcal{U} = \mathcal{U} \times \widehat{\mathcal{D}}_{\boldsymbol{t}_2,-\boldsymbol{t}_1}. \tag{4.11}$$

Accordingly, we must also dress the condensation defect [32, 47, 100–104] as follows;

$$\begin{aligned} \widehat{\mathcal{C}}_{\boldsymbol{t}_1,\boldsymbol{t}_2}(M_3) \equiv \exp&\left[\frac{-\mathrm{i}p^J K_{IJ}}{2N'} \int_{\boldsymbol{t}_1} dA^I_{\mathrm{L}} - \frac{p^K \mathsf{S}_{JK}}{2N'} \int_{\boldsymbol{t}_2} \mathcal{G}_{JM} * dA^M_{\mathrm{L}}\right] \\ &\times \int \mathcal{D}a \exp\left[-\frac{\mathrm{i}}{2\pi} \int_{M_3} K_{IJ}(A^I_{\mathrm{L}} - A^I_{\mathrm{R}}) da^J\right]. \end{aligned} \tag{4.12}$$

Then, we can obtain a *non-commutative* fusion subalgebra concerning $\widehat{\mathcal{D}}_{\boldsymbol{t}_1,\boldsymbol{t}_2}$, $\widehat{\mathcal{C}}_{\boldsymbol{t}_1,\boldsymbol{t}_2}$ and the $(\mathbb{Z}^{[1]}_{2N'})_{\mathrm{diag}}$ symmetry generator $\eta_{\vec{p}}$ as follows;[11]

Non-commutative fusion subalgebra in pure U(1)×U(1) gauge theory.

$$\begin{aligned} \widehat{\mathcal{D}}_{\boldsymbol{t}_1,\boldsymbol{t}_2} \times \overline{\widehat{\mathcal{D}}}_{\boldsymbol{t}_3,\boldsymbol{t}_4} &= \widehat{\mathcal{C}}_{-\boldsymbol{t}_2+\boldsymbol{t}_4,\boldsymbol{t}_1-\boldsymbol{t}_3}, \\ \overline{\widehat{\mathcal{D}}}_{\boldsymbol{t}_1,\boldsymbol{t}_2} \times \widehat{\mathcal{D}}_{\boldsymbol{t}_3,\boldsymbol{t}_4} &= \widehat{\mathcal{C}}_{-\boldsymbol{t}_1+\boldsymbol{t}_3,-\boldsymbol{t}_2+\boldsymbol{t}_4}, \\ \overline{\widehat{\mathcal{D}}}_{\boldsymbol{t}_1,\boldsymbol{t}_2} &= \widehat{\mathcal{D}}_{-\boldsymbol{t}_2,\boldsymbol{t}_1} \times \mathcal{U} = \mathcal{U} \times \widehat{\mathcal{D}}_{\boldsymbol{t}_2,-\boldsymbol{t}_1}, \\ \widehat{\mathcal{D}}_{\boldsymbol{t}_1,\boldsymbol{t}_2} \times \eta_{\vec{p}} &= \eta_{\vec{p}} \times \widehat{\mathcal{D}}_{\boldsymbol{t}_1,\boldsymbol{t}_2} = \widehat{\mathcal{D}}_{\boldsymbol{t}_1,\boldsymbol{t}_2}, \\ \widehat{\mathcal{D}}_{\boldsymbol{t}_1,\boldsymbol{t}_2} \times \widehat{\mathcal{C}}_{\boldsymbol{t}_3,\boldsymbol{t}_4} &= \mathcal{Z}\, \widehat{\mathcal{D}}_{\boldsymbol{t}_1+\boldsymbol{t}_3,\boldsymbol{t}_2+\boldsymbol{t}_4}, \\ \widehat{\mathcal{C}}_{\boldsymbol{t}_1,\boldsymbol{t}_2} \times \widehat{\mathcal{D}}_{\boldsymbol{t}_3,\boldsymbol{t}_4} &= \mathcal{Z}\, \widehat{\mathcal{D}}_{\boldsymbol{t}_2+\boldsymbol{t}_3,-\boldsymbol{t}_1+\boldsymbol{t}_4}, \end{aligned}$$

[11]We have checked that the obtained fusion subalgebra satisfies the associativity condition.

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$$
\begin{aligned}
&\widehat{\mathcal{C}}_{\boldsymbol{t}_1,\boldsymbol{t}_2} \times \widehat{\mathcal{C}}_{\boldsymbol{t}_3,\boldsymbol{t}_4} = \mathcal{Z}\,\widehat{\mathcal{C}}_{\boldsymbol{t}_1+\boldsymbol{t}_3,\boldsymbol{t}_2+\boldsymbol{t}_4}\,,\\
&\eta_{\vec{p}}^{2N'} = 1\,,
\end{aligned}
\tag{4.13}
$$

where $\mathcal{Z}$ is the decoupled topological field theory defined by;

$$
\mathcal{Z}(M_3) \equiv \int \mathcal{D}a\mathcal{D}b\exp\left[-\frac{\mathrm{i}}{2\pi}\int_{M_3} K_{IJ}\, a^I db^J\right]\,. \tag{4.14}
$$

Here, a and b are dynamical U(1) gauge fields living in M_3, decoupled from the bulk theory.

Finally, we give some comments on the fusion rule between the duality defect $\mathcal{D}$ and the $(\mathbb{Z}^{[1]}_{2N'})_{\mathrm{diag}}$ symmetry generator $\eta_{\vec{p}}$. The fusion rule $\mathcal{D}\times\eta_{\vec{p}}$ can be evaluated as follows;

$$
\begin{aligned}
\mathcal{D}(M_3)\times\eta_{\vec{p}}(\Sigma_2) &= \exp\left(-\frac{p^I}{2N'}\int_{\Sigma_2}\mathcal{G}_{IJ}\star dA^J_{\mathrm{R}} - \frac{\mathrm{i}}{2\pi}\int_{M_3} K_{IJ}\,A^I_{\mathrm{L}}\,dA^J_{\mathrm{R}}\right)\\
&= \exp\left(-\frac{\mathrm{i}\,p^J K_{IJ}}{2N'}\int_{\Sigma_2} dA^I_{\mathrm{L}} - \frac{\mathrm{i}}{2\pi}\int_{M_3} K_{IJ}\,A^I_{\mathrm{L}}\,dA^J_{\mathrm{R}}\right)\,.
\end{aligned}
\tag{4.15}
$$

From the right-hand side, the electric and magnetic one-form symmetry defects look like the 1-morphisms which map the duality defect $\mathcal{D}$ to the same one in the context of the higher category [33, section 2]. Our “1-morphism” is, however, different from the standard 1-morphism in the higher category. This is because the different 1-morphisms, namely electric and magnetic one-form symmetry defects, appear on the left and right sides of the duality defect $\mathcal{D}$. It may be an interesting open question to explore the mathematical structures of our “1-morphism”.

5 Conclusion and outlook

In this paper, we explored the non-invertible symmetries by using the half-space gauging associated with the diagonal sub-group $(\mathbb{Z}^{[q]}_{2N'})_{\mathrm{diag}}$. In the $c = 2$ bosonic torus CFT, we showed that the diagonal gauging produces the non-invertible symmetry on the irrational CFT point, and derived the fusion algebra. In order for the algebra to close, we need to dress the duality defect with various global symmetry generators, and the resulting fusion algebra is non-commutative. Also, we apply the half-space gauging to the pure U(1)×U(1) gauge theory in four dimensions and discuss the fusion algebra in a very similar manner to the $c = 2$ bosonic torus CFT. The consequent fusion algebra in four dimensions is also non-commutative.

We conclude this paper by mentioning some future directions;

- In this paper, we mainly focused on the diagonal gauging, yet we can consider other gaugings. For instance, we can consider the gauging condition $p^1 n_1 + p^2 n_2 = 0$ mod N, which is more general compared with the diagonal gauging. Also, shift and winding symmetries in two dimensions (correspondingly, electric and magnetic one-form symmetries in four dimensions) have a mixed ’t Hooft anomaly. Hence, they cannot be gauged simultaneously and we cannot naively implement the half-space gauging associated to them. Even in that case, however, if we choose the nice discrete subgroup, we are free from any 't Hooft anomalies and can proceed with the half-space gauging[12] [87]. The half-space gauging via these other gaugings may result in new non-invertible symmetries which are not captured in this paper.

- In more general, we can add the topological terms by turning on the B-field and theta angle in two and four dimensions, respectively. In this paper, we only consider the case where the charge lattice after gauging is a rectangular type. However, if we include such topological terms in the actions, the charge lattice is not limited to the rectangular one. This is because the B-field or theta angle makes the axis of the charge lattice tilted. It is interesting to investigate this generalization.

Addressing these future directions would help with completing the landscape of non-invertible symmetries in the $c = 2$ bosonic torus CFT and the pure U(1)×U(1) gauge theory, and we leave them to intriguing avenues for future works.

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A Derivations of the selected fusion algebras in section 3.3

In this appendix, we provide concrete calculations, mainly focusing on the fusion algebras, which are omitted in section 3.3. Our methodology to derive the fusion rules closely follows the work [52, section 6].

- $\eta_{\vec{p}} \times \mathcal{D} = \mathcal{D}$ (3.41)

  In order to derive the fusion rule $\eta_{\vec{p}} \times \mathcal{D}$, it is sufficient to consider the left bulk and the defect actions, which are given by

$$\frac{1}{4\pi}\int_{x<0} G^*_{IJ}\, d\phi^I_{\mathrm{L}} \wedge \star d\phi^J_{\mathrm{L}} + \frac{\mathrm{i}}{2\pi}\int_{x=0} K_{IJ}\, \phi^I_{\mathrm{L}}\, d\phi^J_{\mathrm{R}} + \frac{p^I}{2N'}\int_{x=0} G^*_{IJ} \star d\phi^J_{\mathrm{L}} \,. \tag{A.1}$$

  By changing the path integral variable ϕ^J_{L};

$$\phi^J_{\mathrm{L}} = \phi'^J_{\mathrm{L}} - \frac{2\pi p^J}{2N'}\theta(-x)\,, \tag{A.2}$$

  where $\theta(x)$ is a step function defined as $\theta(x) = 0$ in $x \leq 0$ and $\theta(x) = 1$ in $x > 0$, the above combined action A.1 becomes;

$$\frac{1}{4\pi}\int_{x<0} G^*_{IJ}\, d\phi'^I_{\mathrm{L}} \wedge \star d\phi'^J_{\mathrm{L}} + \frac{\mathrm{i}}{2\pi}\int_{x=0} K_{IJ}\, \phi'^I_{\mathrm{L}}\, d\phi^J_{\mathrm{R}} - \frac{\mathrm{i}p^I K_{IJ}}{2N'}\int_{x=0} d\phi^J_{\mathrm{R}} \,. \tag{A.3}$$

[12] We thank Kantaro Ohmori for pointing out this possibility.

Note that $p^I K_{IJ} = 0 \mod 2N'$, therefore the last term can be dropped from the action. Then, we get the following fusion rule;

$$\eta_{\vec{p}} \times \mathcal{D} = \mathcal{D}\,. \tag{A.4}$$

- $\mathcal{D} \times \eta_{\vec{p}}$ (3.42)

  As stressed in the main text, the fusing the $(\mathbb{Z}_{2N'}^{[0]})_{\text{diag}}$ shift symmetry defect $\eta_{\vec{p}}$ to the duality defect $\mathcal{D}$ from the right shows a different behavior from the case of $\eta_{\vec{p}} \times \mathcal{D}$. To see this, we only have to consider the right bulk and the defect actions;

$$\frac{1}{4\pi}\int_{0<x} G^*_{IJ}\, d\phi^I_{\mathrm{R}} \wedge \star\, d\phi^J_{\mathrm{R}} + \frac{\mathrm{i}}{2\pi}\int_{x=0} K_{IJ}\, \phi^I_{\mathrm{L}}\, d\phi^J_{\mathrm{R}} + \frac{p^I}{2N'}\int_{x=0} G_{IJ} \star d\phi^J_{\mathrm{R}}\,. \tag{A.5}$$

  We perform the following field redefinition:

$$\phi^J_{\mathrm{R}} = \phi'^J_{\mathrm{R}} + \frac{2\pi p^J}{2N'}\theta(x)\,. \tag{A.6}$$

  Then, the composite action A.5 can be calculated as follows;

$$\frac{1}{4\pi}\int_{0<x} G^*_{IJ}\, d\phi'^I_{\mathrm{R}} \wedge \star\, d\phi'^J_{\mathrm{R}} + \frac{\mathrm{i}}{2\pi}\int_{x=0} K_{IJ}\, \phi^I_{\mathrm{L}}\, d\phi'^J_{\mathrm{R}} - \frac{\mathrm{i}p^J K_{IJ}}{2N'}\int_{x=0} d\phi^I_{\mathrm{L}}\,, \tag{A.7}$$

  which shows that the non-trivial $\mathbb{Z}_{2N'}$ winding symmetry generator appears on the left side of the duality defect $\mathcal{D}$ due to $p^J K_{IJ} \neq 0 \mod 2N'$.

The following fusion rules are related to our main result 3.54.

- $\widehat{\mathcal{D}}_{s_1,s_2} \times \widehat{\mathcal{D}}_{s_3,s_4} = \widehat{\mathcal{C}}_{s_2+s_3,s_1+s_4}$

  By using the associative property of symmetry generators, the fusion rule $\widehat{\mathcal{D}}_{s_1,s_2} \times \widehat{\mathcal{D}}_{s_3,s_4}$ can be reduced to as follows;

$$\begin{aligned}\widehat{\mathcal{D}}_{s_1,s_2} \times \widehat{\mathcal{D}}_{s_3,s_4} &= (\eta_{\mathsf{S}\vec{p}})^{s_1} \times (\mathcal{D}_{s_2} \times \mathcal{S}) \times (\eta_{\mathsf{S}\vec{p}})^{s_3} \times (\mathcal{D} \times \mathcal{S}) \times (\eta_{\mathsf{S}\vec{p}})^{s_4} \\ &= (\eta_{\mathsf{S}\vec{p}})^{s_1} \times \mathcal{D}_{s_2+s_3} \times (\mathcal{S} \times \mathcal{D} \times \mathcal{S}) \times (\eta_{\mathsf{S}\vec{p}})^{s_4} \\ &= (\eta_{\mathsf{S}\vec{p}})^{s_1} \times \mathcal{D}_{s_2+s_3} \times \overline{\mathcal{D}} \times (\eta_{\mathsf{S}\vec{p}})^{s_4}\,.\end{aligned} \tag{A.8}$$

  In the first line, we used the definition of the dressed duality defect $\widehat{\mathcal{D}}_{s_1,s_2}$ and (3.38). We also made use of (3.38) and (3.37) in the second and final lines, respectively. As easily can be checked, the fused defect action reads

$$\mathcal{D}_{s_2+s_3} \times \overline{\mathcal{D}} \;:\; \frac{\mathrm{i}(s_2+s_3)p^J K_{IJ}}{2N'}\int_{x=0} d\phi^I_{\mathrm{L}} + \frac{\mathrm{i}}{2\pi}\int_{x=0} K_{IJ}\,(\phi^I_{\mathrm{L}} - \phi^I_{\mathrm{R}})\, d\varphi^J\,. \tag{A.9}$$

  The first term is nothing but the $\mathbb{Z}_{2N'}$ winding symmetry generator, which originates from the dressed duality defects $\widehat{\mathcal{D}}_{s_2}$ and $\widehat{\mathcal{D}}_{s_3}$. The second term is the projection operator associated with the $\mathbb{Z}_{2N'}$ shift symmetry. To see this, we decompose the charge matrix K into the Smith normal form;

$$\begin{pmatrix} N' & -1 \\ N' & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2N' & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -N' & 1 \end{pmatrix}. \tag{A.10}$$

This shows that the second term can be split into the $\mathbb{Z}_{2N'}$ BF theory and the trivial one, and we can write it as a sum of $\mathbb{Z}_{2N'}$ generators (See [53, appendix E] for the derivation). Also, as we attached the $\mathbb{Z}_{2N'}$ element to the duality defect, we define the dressed projection operator $\widehat{\mathcal{C}}_{s_1,s_2}$ $(s_1, s_2 = 0, 1, \cdots 2N' - 1)$ as follows;

$$\begin{aligned}\widehat{\mathcal{C}}_{s_1,s_2}(\Sigma) \equiv \exp&\left[-\frac{\mathrm{i}s_1 p^J K_{IJ}}{2N'}\int_\Sigma d\phi^I_{\mathrm{L}}\right] \\ &\times \int \mathcal{D}\varphi \exp\left[-\frac{\mathrm{i}}{2\pi}\int_\Sigma K_{IJ}\left(\phi^I_{\mathrm{L}} - \phi^I_{\mathrm{R}} - \frac{2\pi s_2}{2N'}\mathsf{S}^{IK}p_K\right) d\varphi^J\right] .\end{aligned} \tag{A.11}$$

By using this dressed projection operator, the fusion rule $\mathcal{D}_{s_2+s_3} \times \overline{\mathcal{D}}$ can be written as

$$\mathcal{D}_{s_2+s_3} \times \overline{\mathcal{D}} = \widehat{\mathcal{C}}_{s_2+s_3,0} \,. \tag{A.12}$$

Then, we can evaluate the fusion rule $\widehat{\mathcal{D}}_{s_1,s_2} \times \widehat{\mathcal{D}}_{s_3,s_4}$ as follows;

$$\widehat{\mathcal{D}}_{s_1,s_2} \times \widehat{\mathcal{D}}_{s_3,s_4} = (\eta_{\mathsf{S}\vec{p}})^{s_1} \times \widehat{\mathcal{C}}_{s_2+s_3,0} \times (\eta_{\mathsf{S}\vec{p}})^{s_4} = \widehat{\mathcal{C}}_{s_2+s_3,s_1+s_4} \,. \tag{A.13}$$

In the last equality, we used the following fusion rule:

$$\widehat{\mathcal{C}}_{s_1,s_2} \times (\eta_{\mathsf{S}\vec{p}})^{s_3} = (\eta_{\mathsf{S}\vec{p}})^{s_3} \times \widehat{\mathcal{C}}_{s_1,s_2} = \widehat{\mathcal{C}}_{s_1,s_2+s_3} \,, \tag{A.14}$$

which can be understood as follows. After redefining the integral variables as follows:

$$\phi^J_{\mathrm{R}} = \phi'^J_{\mathrm{R}} + \frac{2\pi}{2N'}\mathsf{S}^{JK}p^K\theta(x) \,, \tag{A.15}$$

it turns out that the fusing $(\eta_{\mathsf{S}\vec{p}})^{s_3}$ to $\widehat{\mathcal{C}}_{s_1,s_2}$ from the right bulk results in the shift of ϕ^I_{R} on Σ by $\frac{2\pi s_3}{2N'}\mathsf{S}^{IK}p_K$. This is equivalent to the replacement the label s_2 in the dressed projection operator $\widehat{\mathcal{C}}_{s_1,s_2}$ as $s_2 \to s_2 + s_3$, hence the fusion rule $\widehat{\mathcal{C}}_{s_1,s_2} \times (\eta_{\mathsf{S}\vec{p}})^{s_3} = \widehat{\mathcal{C}}_{s_1,s_2+s_3}$ can be deduced. In a very similar manner, we can also derive the fusion rule $(\eta_{\mathsf{S}\vec{p}})^{s_3} \times \widehat{\mathcal{C}}_{s_1,s_2} = \widehat{\mathcal{C}}_{s_1,s_2+s_3}$.[13]

- $\eta_{\vec{p}} \times \widehat{\mathcal{D}}_{s_1,s_2} = \widehat{\mathcal{D}}_{s_1,s_2}$ and $\widehat{\mathcal{D}}_{s_1,s_2} \times \eta_{\vec{p}} = \widehat{\mathcal{D}}_{s_1,s_2}$

  Here, we derive the fusion rules $\eta_{\vec{p}} \times \widehat{\mathcal{D}}_{s_1,s_2}$ and $\widehat{\mathcal{D}}_{s_1,s_2} \times \eta_{\vec{p}}$ by using the associativity. The fusion $\eta_{\vec{p}} \times \widehat{\mathcal{D}}_{s_1,s_2}$ can be easily evaluated as follows;

$$\begin{aligned}\eta_{\vec{p}} \times \widehat{\mathcal{D}}_{s_1,s_2} &= \eta_{\vec{p}} \times (\eta_{\mathsf{S}\vec{p}})^{s_1} \times \mathcal{D}_{s_2} \times \mathcal{S} \\ &= (\eta_{\mathsf{S}\vec{p}})^{s_1} \times \eta_{\vec{p}} \times \mathcal{D}_{s_2} \times \mathcal{S} \\ &= (\eta_{\mathsf{S}\vec{p}})^{s_1} \times \mathcal{D}_{s_2} \times \mathcal{S} \\ &= \widehat{\mathcal{D}}_{s_1,s_2} \,,\end{aligned} \tag{A.16}$$

[13] Here, we should note that shift and winding symmetry generators commute with each other.

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where in the third line, we used the fusion rule 3.50. Likewise, we can derive the fusion rule $\widehat{\mathcal{D}}_{s_1,s_2} \times \eta_{\vec{p}}$ as follows;

$$\begin{aligned}\widehat{\mathcal{D}}_{s_1,s_2} \times \eta_{\vec{p}} &= (\eta_{\mathsf{S}\vec{p}})^{s_1} \times \mathcal{D} \times \eta_{\vec{p}}^{s_2} \times \mathcal{S} \times \eta_{\vec{p}} \\ &= (\eta_{\mathsf{S}\vec{p}})^{s_1} \times \mathcal{D} \times \eta_{\vec{p}}^{s_2} \times \eta_{\mathsf{S}\vec{p}} \times \mathcal{S} \\ &= (\eta_{\mathsf{S}\vec{p}})^{s_1} \times \mathcal{D} \times \eta_{\mathsf{S}\vec{p}} \times \eta_{\vec{p}}^{s_2} \times \mathcal{S} \\ &= (\eta_{\mathsf{S}\vec{p}})^{s_1} \times \mathcal{D} \times \eta_{\vec{p}}^{s_2} \times \mathcal{S} \\ &= \widehat{\mathcal{D}}_{s_1,s_2} \,. \end{aligned} \tag{A.17}$$

In the fourth line, we used the following formula;

$$\mathcal{D} \times \eta_{\mathsf{S}\vec{p}} = \mathcal{D} \,, \tag{A.18}$$

which can be derived by a similar way to (3.40) and (3.42).

- $\widehat{\mathcal{D}}_{s_1,s_2} \times \widehat{\mathcal{C}}_{s_3,s_4}$

We can rewrite $\widehat{\mathcal{D}}_{s_1,s_2} \times \widehat{\mathcal{C}}_{s_3,s_4}$ as follows;

$$\widehat{\mathcal{D}}_{s_1,s_2} \times \widehat{\mathcal{C}}_{s_3,s_4} = (\eta_{\mathsf{S}\vec{p}})^{s_1} \times \mathcal{D} \times \mathcal{S} \times (\eta_{\mathsf{S}\vec{p}})^{s_2} \times \widehat{\mathcal{C}}_{s_3,s_4} \tag{A.19}$$

$$= (\eta_{\mathsf{S}\vec{p}})^{s_1} \times \mathcal{S} \times \overline{\mathcal{D}} \times \widehat{\mathcal{C}}_{s_3,0} \times (\eta_{\mathsf{S}\vec{p}})^{s_2+s_4} \,. \tag{A.20}$$

In the last line, we used the following relation;

$$(\eta_{\mathsf{S}\vec{p}})^{s_2} \times \widehat{\mathcal{C}}_{s_3,s_4} = \widehat{\mathcal{C}}_{s_3,s_2+s_4} = \widehat{\mathcal{C}}_{s_3,0} \times (\eta_{\mathsf{S}\vec{p}})^{s_2+s_4} \,. \tag{A.21}$$

Therefore, in order to calculate the fusion rule $\widehat{\mathcal{D}}_{s_1,s_2} \times \widehat{\mathcal{C}}_{s_3,s_4}$, we firstly need to derive the fusion rule $\overline{\mathcal{D}} \times \widehat{\mathcal{C}}_{s_3,0}$. The defect action is given by

$$\overline{\mathcal{D}} \times \widehat{\mathcal{C}}_{s_3,0} : \frac{\mathrm{i} s_3 p^J}{2N'} \int_{x=0} K_{IJ} d\phi_{\mathrm{M}}^I + \frac{\mathrm{i}}{2\pi} \int_{x=0} K_{IJ} (\phi_{\mathrm{M}}^I - \phi_{\mathrm{R}}^I) d\varphi^J + \frac{\mathrm{i}}{2\pi} \int_{x=0} K_{JI} \, \phi_{\mathrm{L}}^I d\phi_{\mathrm{M}}^J \,. \tag{A.22}$$

By changing the path integral variables as follows:

$$\phi_{\mathrm{M}}^{\prime I} = \phi_{\mathrm{M}}^I - \phi_{\mathrm{R}}^I \,, \qquad \varphi^{\prime I} = \varphi^I - \phi_{\mathrm{L}}^I \,, \tag{A.23}$$

and the defect action becomes;

$$\frac{\mathrm{i}}{2\pi} \int_{x=0} K_{IJ} \, \phi_{\mathrm{M}}^{\prime I} d\varphi^{\prime J} + \frac{\mathrm{i} s_3 p^J}{2N'} \int_{x=0} K_{IJ} \, d\phi_{\mathrm{M}}^{\prime I} + \frac{\mathrm{i}}{2\pi} \int_{x=0} K_{JI} \, \phi_{\mathrm{L}}^I d\phi_{\mathrm{R}}^J + \frac{\mathrm{i} s_3 p^J}{2N'} \int_{x=0} K_{IJ} d\phi_{\mathrm{R}}^I \,. \tag{A.24}$$

The first two terms represent the decoupled TQFT $\mathcal{Z}$, and this can be written as the sum of $\mathbb{Z}_{2N'}$ symmetry generators. Hence, we have $\mathcal{Z} = 2N'$. Also, the last two terms are the defect actions of $\eta_{\vec{p}}^{s_3} \times \overline{\mathcal{D}}$ since the winding symmetry generator is changed to the shift one across the duality defect. Combining the above results, the fusion rule $\widehat{\mathcal{D}}_{s_1,s_2} \times \widehat{\mathcal{C}}_{s_3,s_4}$ can be derived as follows;

$$\begin{aligned}\widehat{\mathcal{D}}_{s_1,s_2} \times \widehat{\mathcal{C}}_{s_3,s_4} &= 2N'(\eta_{\mathsf{S}\vec{p}})^{s_1} \times \mathcal{S} \times \eta_{\vec{p}}^{s_3} \times \overline{\mathcal{D}} \times (\eta_{\mathsf{S}\vec{p}})^{s_2+s_4} \\ &= 2N'(\eta_{\mathsf{S}\vec{p}})^{s_1+s_3} \times \mathcal{D} \times \mathcal{S} \times (\eta_{\mathsf{S}\vec{p}})^{s_2+s_4} \\ &= 2N'(\eta_{\mathsf{S}\vec{p}})^{s_1+s_3} \times \mathcal{D} \times \eta_{\vec{p}}^{s_2+s_4} \times \mathcal{S} \\ &= 2N'\widehat{\mathcal{D}}_{s_1+s_3,s_2+s_4} \,.\end{aligned} \tag{A.25}$$

We can also calculate the fusion rule $\widehat{\mathcal{C}}_{s_1,s_2} \times \widehat{\mathcal{D}}_{s_3,s_4}$ in a similar manner to the above derivation.

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