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New duality-invariant models for nonlinear supersymmetric electrodynamics

Sergei M. Kuzenko and Jake C. Stirling

Department of Physics M013, The University of Western Australia,
35 Stirling Highway, Perth W.A. 6009, Australia

E-mail: sergei.kuzenko@uwa.edu.au, jake.stirling@research.uwa.edu.au

Abstract: We propose a new family of $\mathsf{U}(1)$ duality-invariant models for nonlinear $\mathcal{N}=1$ supersymmetric electrodynamics coupled to supergravity. It includes the Cribiori-Farakos-Tournoy-van Proeyen supergravity-matter theory for spontaneously broken local supersymmetry with a novel Fayet-Iliopoulos term without gauged R-symmetry. We present superconformal duality-invariant models, as well as new $\mathsf{U}(1)$ duality-invariant models for spontaneously broken local supersymmetry.

Keywords: Supergravity Models, Superspaces, Supersymmetric Effective Theories, Supersymmetry and Duality

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Contents

1 Introduction

The general theory of duality invariance for nonlinear models with $\mathcal{N} = 1$ and $\mathcal{N} = 2$ vector supermultiplets was developed in 2000 [1, 2] and soon after extended to the locally supersymmetric case [3–5]. This formalism is a generalisation of the classic results on the structure of self-dual models for nonlinear electrodynamics in four dimensions [6–10] (see [2, 11] for a review) in conjunction with the self-duality properties [12, 13] of the $\mathcal{N} = 1$ supersymmetric Born-Infeld action [14–16] and its generalisations.

At the turn of the millennium, the main motivation to study supersymmetric self-dual systems was the existence of deep yet (still) mysterious connections between nonlinear self-duality and supersymmetry. These are:

- In the case of partial spontaneous supersymmetry breakdown $\mathcal{N} = 2 \rightarrow \mathcal{N} = 1$, the Maxwell-Goldstone multiplet [15, 16] (coinciding with the $\mathcal{N} = 1$ supersymmetric Born-Infeld action [14]) and the tensor Goldstone multiplet [16, 17] were shown in [1, 2] to be invariant under $\mathsf{U}(1)$ duality rotations.[1]

- Extending the earlier incomplete proposal of [19], it was suggested in [2] that the Maxwell-Goldstone multiplet for partial $\mathcal{N} = 4 \rightarrow \mathcal{N} = 2$ supersymmetry breakdown (proposed to be the $\mathcal{N} = 2$ supersymmetric Born-Infeld action) is a unique $\mathcal{N} = 2$ vector multiplet theory with the following properties: (i) it possesses $\mathsf{U}(1)$ duality invariance; and (ii) it is invariant under a nonlinearly realised central charge bosonic symmetry. Within the perturbative approach to constructing the $\mathcal{N} = 2$ supersymmetric Born-Infeld action elaborated in [2], the uniqueness of the action was demonstrated to order W^{10} in powers of the chiral superfield strength W. A year later, a powerful formalism of nonlinear realisations for the partial $\mathcal{N} = 4 \rightarrow \mathcal{N} = 2$

[1] The Maxwell-Goldstone multiplet for partial $\mathcal{N} = 2 \rightarrow \mathcal{N} = 1$ supersymmetry breaking has also been extended [18] to the following maximally supersymmetric backgrounds: (i) $\mathbb{R} \times S^3$; (ii) $\mathrm{AdS}_3 \times \mathbb{R}$; and (iii) a supersymmetric plane wave. This theory possesses $\mathsf{U}(1)$ duality invariance.

supersymmetry breaking was developed [20] which supported the uniqueness of the $\mathcal{N}=2$ supersymmetric Born-Infeld action and reproduced [21] the perturbative results of [2]. Further progress towards the construction of the $\mathcal{N}=2$ supersymmetric Born-Infeld action has been achieved in [22–24].

- The $\mathcal{N}=4$ super Yang-Mills (SYM) theory is believed to be self-dual [25, 26] (see also [27]). This conjecture was put forward in the late 1970s as a duality between the conventional and soliton sectors of the theory. In 1998 it was suggested [28] that self-duality might be realised in terms of a low-energy effective action of the theory on its Coulomb branch.[2] Here the gauge group $\mathsf{SU}(N)$ is spontaneously broken to $\mathsf{SU}(N-1)\times\mathsf{U}(1)$ and the dynamics is described by a nonlinear $\mathcal{N}=2$ *superconformal* action for the $\mathcal{N}=2$ vector multiplet associated with the $\mathsf{U}(1)$ factor of the unbroken group. Two different realisations of self-duality for the $\mathcal{N}=4$ SYM effective action in $\mathcal{N}=2$ superspace were proposed: (i) self-duality under Legendre transformation [28]; and (ii) self-duality under $\mathsf{U}(1)$ duality rotations [1].[3] Both proposals have not yet been derived from first principles, although each of them is consistent with the one-loop [33–37] and two-loop [38] calculations.

- For a large class of *nonlinear* $\mathsf{U}(1)$ duality-invariant models for $\mathcal{N}=1$ supersymmetric electrodynamics [1], it was demonstrated [4] that the component fermionic action, which is obtained by switching the bosonic fields off, is equivalent (modulo a nonlinear field redefinition) to the Akulov-Volkov action for the Goldstino [39, 40].[4]

A new motivation to revisit the general structure of $\mathsf{U}(1)$ duality-invariant models for the $\mathcal{N}=1$ vector multiplet emerged five years ago when Cribiori et al. [42] discovered a novel Fayet-Iliopoulos term in supergravity without gauged R-symmetry. In order to explain this motivation, it is pertinent to give a summary of the formalism introduced in [1] and its generalisation advocated in [43].

Let $S[W,\bar{W}]$ be the action describing the dynamics of a single vector supermultiplet in Minkowski superspace. It is assumed that the action is a functional of the gauge-invariant chiral spinor field strength $W_\alpha=-\frac{1}{4}\bar{D}^2D_\alpha V$ and its conjugate $\bar{W}_{\dot\alpha}=-\frac{1}{4}D^2\bar{D}_{\dot\alpha}V$, where $V=\bar{V}$ is a gauge prepotential [44]. In order for this theory to possess $\mathsf{U}(1)$ duality invariance, the action must be a solution of the so-called self-duality equation [1]

$$\mathrm{Im}\int \mathrm{d}^4x\mathrm{d}^2\theta\,\Big\{W^\alpha W_\alpha+M^\alpha M_\alpha\Big\}=0\,,\qquad M_\alpha:=-2\mathrm{i}\,\frac{\delta}{\delta W^\alpha}\,S[W,\bar{W}]\,. \tag{1.1}$$

Since the equation (1.1) is nonlinear, its solutions are difficult to generate. Inspired by the bosonic approach due to Ivanov and Zupnik [45–48], new formulations were developed for $\mathcal{N}=1$ and $\mathcal{N}=2$ supersymmetric duality-invariant theories coupled to supergravity

[2]This was inspired in part by the Seiberg-Witten theory [29, 30] and also by the AdS/CFT correspondence [31].

[3]It was also conjectured by Schwarz [32] that the world-volume action of a probe D3-brane in an $\mathrm{AdS}_5\times\mathrm{S}^5$ background of type IIB superstring theory, with one unit of flux, can be reinterpreted as the exact (or highly) effective action for $\mathsf{U}(2)$ $\mathcal{N}=4$ super Yang-Mills theory on the Coulomb branch.

[4]An explanation of this result was given in [41].

ten years ago [49]. The method makes use of an auxiliary unconstrained chiral superfield (a spinor in the $\mathcal{N}=1$ case and a scalar for $\mathcal{N}=2$) and is characterised by the fundamental property that $\mathsf{U}(1)$ duality invariance is equivalent to the manifest $\mathsf{U}(1)$ invariance of the self-interaction. In the $\mathcal{N}=1$ rigid supersymmetric case, analogous results were independently obtained in [43].

It is assumed in (1.1) that W_α is an unrestricted chiral spinor superfield, and the action $S[W,\bar W]$ is "analytically" continued from the original functional, which depends on W_α satisfying the Bianchi identity $D^\alpha W_\alpha = \bar D_{\dot\alpha}\bar W^{\dot\alpha}$, to a functional of the unrestricted chiral spinor W_α. Such a continuation is obviously not unique, and additional conditions are required to fix it. For instance, consider a supersymmetric theory of the form

$$S[W,\bar W] = \frac{1}{4}\int \mathrm{d}^4x\mathrm{d}^2\theta\, W^2 + \text{c.c.} + \frac{1}{4}\int \mathrm{d}^4x\mathrm{d}^2\theta\mathrm{d}^2\bar\theta\, W^2\,\bar W^2\Omega\left(u,\bar u, D^\alpha W_\alpha\right)\,, \tag{1.2}$$

where $W^2 = W^\alpha W_\alpha$ and $u = \frac{1}{8}D^2W^2$. A possible way to extend this functional to the case when the Bianchi identity $D^\alpha W_\alpha = \bar D_{\dot\alpha}\bar W^{\dot\alpha}$ is no longer required, consists in replacing $\Omega\left(u,\bar u,D^\alpha W_\alpha\right) \to \Omega\left(u,\bar u,\gamma D^\alpha W_\alpha + \bar\gamma \bar D_{\dot\alpha}\bar W^{\dot\alpha}\right)$, for a complex parameter γ such that $\gamma+\bar\gamma = 1$.

The ambiguity with analytic continuation is naturally resolved for the family of nonlinear models studied in [1]:

$$S[W,\bar W] = \frac{1}{4}\int \mathrm{d}^4x\mathrm{d}^2\theta\, W^2 + \text{c.c.} + \frac{1}{4}\int \mathrm{d}^4x\mathrm{d}^2\theta\mathrm{d}^2\bar\theta\, W^2\,\bar W^2\Lambda\left(u,\bar u\right)\,. \tag{1.3}$$

Here the self-duality equation (1.1) implies that

$$\mathrm{Im}\Big\{\Gamma - \bar u\Gamma^2\Big\} = 0\,, \qquad \Gamma := \partial_u(u\Lambda)\,. \tag{1.4}$$

This equation coincides in form with that arising in $\mathsf{U}(1)$ duality-invariant nonlinear electrodynamics $L(F_{ab})$ (see [2] for the technical details),

$$\mathrm{Im}\left\{\frac{\partial(\omega\,\Lambda)}{\partial\omega} - \bar\omega\,\left(\frac{\partial(\omega\,\Lambda)}{\partial\omega}\right)^2\right\} = 0\,, \tag{1.5}$$

provided the Lagrangian is expressed in terms of invariants of the electromagnetic field

$$L(F_{ab}) = -\frac{1}{2}\Big(\omega+\bar\omega\Big) + \omega\,\bar\omega\;\Lambda(\omega,\bar\omega)\,, \tag{1.6a}$$

where we have introduced

$$\omega = \alpha + \mathrm{i}\,\beta\,, \qquad \alpha = \frac{1}{4}\,F^{ab}F_{ab}\,, \qquad \beta = \frac{1}{4}\,F^{ab}\tilde F_{ab}\,. \tag{1.6b}$$

Therefore, every $\mathsf{U}(1)$ duality-invariant model for nonlinear electrodynamics (1.6) possesses the $\mathcal{N}=1$ supersymmetric extension (1.3) which is also $\mathsf{U}(1)$ duality invariant.

Instead of worrying about a procedure for analytic continuation to start with, one can follow a different path proposed by Ivanov, Lechtenfeld and Zupnik [43]. Their starting point is the assumption that some procedure of analytic continuation has been chosen, and for the unconstrained chiral spinor W_α the action reads

$$S[W,\bar{W}] = \frac{1}{4}\int \mathrm{d}^4x\mathrm{d}^2\theta\, W^2 + \text{c.c.} + \frac{1}{4}\int \mathrm{d}^4x\mathrm{d}^2\theta\mathrm{d}^2\bar{\theta}\, W^2\,\bar{W}^2\Lambda\left(u,\bar{u},p,\bar{p}\right)\,, \tag{1.7}$$

where $p := D^\alpha W_\alpha$. For such a model to be $\mathsf{U}(1)$ duality-invariant, the action must satisfy the equation (1.1), and the latter implies that

$$\mathrm{Im}\Big\{\Gamma - \bar{u}\Gamma^2 + 2u\bar{u}(\partial_p\Lambda)^2\Big\} = 0\,. \tag{1.8}$$

To the best of our knowledge, no solution of the self-duality equation (1.8) has so far been found with $\partial_p\Lambda \neq 0$. In the present paper, we propose a family of such solutions.

Functionals of the type (1.2) naturally appear as low-energy effective actions in quantum supersymmetric gauge theories, see, e.g., [50–52]. However, the combination $D^\alpha W_\alpha$ is nothing but the free equation of motion of the $\mathcal{N}=1$ vector multiplet. It is known that those contributions to the effective action, which contain factors of the classical equations of motion, are ambiguous. That is why only action functionals of the form (1.3) were studied in refs. [1, 2]. On the other hand, there may exist microscopic models that involve $D^\alpha W_\alpha$ in the superfield Lagrangian, and then one is forced to deal with models (1.2). This is exactly the case with the model of [42] proposed to describe a novel Fayet-Iliopoulos term in supergravity without gauged R-symmetry. Restricted to a flat superspace background, the corresponding vector multiplet action is

$$S[W,\bar{W}] = \frac{1}{4}\int \mathrm{d}^4x\mathrm{d}^2\theta\, W^2 + \text{c.c.} + \frac{\zeta}{4}\int \mathrm{d}^4x\mathrm{d}^2\theta\mathrm{d}^2\bar{\theta}\,\frac{W^2\bar{W}^2}{u\bar{u}}D^\alpha W_\alpha\,, \tag{1.9}$$

with ζ a coupling constant. For this action to be well defined, $u = \frac{1}{8}D^2W^2$ should be nowhere vanishing. This requirement is consistent with the equations of motion, since the auxiliary field of the vector multiplet develops a non-zero VEV on-shell.

The model (1.9) was demonstrated in [42] to be self-dual under a superfield Legendre transformation. In this paper, we will prove that (1.9) possesses $\mathsf{U}(1)$ duality invariance. It is a general property of $\mathsf{U}(1)$ duality-invariant theories that every solution of the self-duality equation (1.1) is self-dual under the Legendre transformation [1, 2]. Therefore, our analysis in this paper implies the self-duality property established in [42].

Before turning to the technical part of this paper, it is necessary to point out several recent developments. Although the idea to combine $\mathsf{U}(1)$ duality invariance with $\mathcal{N}=2$ superconformal symmetry was put forward in 2000 [1], the first duality-invariant and (super)conformal theories have only recently been derived in closed form for $\mathcal{N}<2$. Bandos, Lechner, Sorokin and Townsend [53] constructed the so-called ModMax theory, which is a unique nonlinear duality-invariant and conformal extension of Maxwell's equations (see [54] for a related analysis). Its $\mathcal{N}=1$ supersymmetric extension was given in [55, 56]. Ref. [56] also derived the $\mathcal{N}=2$ superconformal $\mathsf{U}(1)$ duality-invariant model proposed to describe the low-energy effective action for $\mathcal{N}=4$ super-Yang-Mills theory, thus completing the program initiated in [1]. Duality-invariant (super)conformal higher-spin models were constructed for $\mathcal{N}\leq 2$ in [57] on arbitrary conformally flat backgrounds. A supersymmetric

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nonlinear σ-model analogue of the ModMax theory was constructed in [58] building on the concept of self-dual supersymmetric nonlinear σ-models [59]. Supersymmetric duality-invariant theories have found numerous applications in the framework of $T\bar{T}$ deformations, see [60–62] and references therein. A remarkable relation has been established between helicity conservation for the tree-level scattering amplitudes and the electric-magnetic duality [63].

This paper is organised as follows. In section 2 we briefly review the $\mathcal{N}=1$ results of [1, 3, 43] and then present our new family of $\mathsf{U}(1)$ duality-invariant models for nonlinear $\mathcal{N}=1$ supersymmetric electrodynamics coupled to supergravity. In section 3 we provide a brief review of the $\mathcal{N}=1$ auxiliary superfield formulation of [49] and then recast the novel features of [43] (as compared with [49]) in a locally supersymmetric framework. Section 4 is devoted to superconformal $\mathsf{U}(1)$ duality-invariant models. Section 5 discusses the obtained results and introduces new $\mathsf{U}(1)$ duality-invariant models for spontaneously broken local supersymmetry.

Our two-component spinor notation and conventions follow [64], and are similar to those adopted in [65]. We make use of the Grimm-Wess-Zumino superspace geometry [66] as described in [64, 67].

2 Duality-invariant supersymmetric models

We consider a dynamical system describing an Abelian $\mathcal{N}=1$ vector multiplet in curved superspace and denote by $S[W,\bar{W}]$ the corresponding action functional. The action is assumed to depend on the chiral spinor field strength W_α and its conjugate $\bar{W}_{\dot\alpha}$ which are constructed in terms of a real unconstrained gauge prepotential V [68] as

$$W_\alpha = -\frac{1}{4}(\bar{\mathcal{D}}^2 - 4R)\mathcal{D}_\alpha V\,, \qquad \bar{\mathcal{D}}_{\dot\beta} W_\alpha = 0\,. \tag{2.1}$$

Here $\mathcal{D}_\alpha$ and $\bar{\mathcal{D}}_{\dot\alpha}$ are the spinor covariant derivatives in curved superspace, and R is the chiral scalar torsion tensor.[5] The prepotential is defined modulo gauge transformations

$$\delta_\xi V = \xi + \bar{\xi}\,, \qquad \bar{\mathcal{D}}_{\dot\alpha}\xi = 0\,, \tag{2.2}$$

such that $\delta_\xi W_\alpha = 0$. The gauge-invariant field strengths W_α and $\bar{W}_{\dot\alpha}$ obey the Bianchi identity

$$\mathcal{D}^\alpha W_\alpha = \bar{\mathcal{D}}_{\dot\alpha}\bar{W}^{\dot\alpha}\,, \tag{2.3}$$

and thus W_α is a reduced chiral superfield. We assume that $S[W,\bar{W}]$ does not involve the combination $\mathcal{D}^\alpha W_\alpha$ as an independent variable, and therefore it can unambiguously be defined as a functional of a *general* chiral superfield W_α and its conjugate $\bar{W}_{\dot\alpha}$. Then, introducing a covariantly chiral spinor superfield M_α,

$$\mathrm{i}\, M_\alpha := 2\,\frac{\delta}{\delta W^\alpha}\, S[W,\bar{W}]\,, \qquad \bar{\mathcal{D}}_{\dot\beta} M_\alpha = 0\,, \tag{2.4}$$

[5] Our normalisation of the torsion tensors of the Grimm-Wess-Zumino geometry follows [64, 67].

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the equation of motion for V is

$$\mathcal{D}^\alpha M_\alpha = \bar{\mathcal{D}}_{\dot\alpha}\bar{M}^{\dot\alpha}\,. \tag{2.5}$$

The variational derivative $\delta S/\delta W^\alpha$ in (2.4) is defined by

$$\delta S = \int \mathrm{d}^4x\,\mathrm{d}^2\theta\,\mathcal{E}\,\delta W^\alpha \frac{\delta S}{\delta W^\alpha} \;+\; \text{c.c.}\,, \tag{2.6}$$

where $\mathcal{E}$ denotes the chiral integration measure, and W^α is assumed to be an unrestricted covariantly chiral spinor.

Since the Bianchi identity (2.3) and the equation of motion (2.5) have the same functional form, one may consider $\mathsf{U}(1)$ duality rotations

$$\delta W_\alpha = \lambda M_\alpha\,, \qquad \delta M_\alpha = -\lambda W_\alpha\,, \tag{2.7}$$

with $\lambda \in \mathbb{R}$ a constant parameter. The condition for duality invariance is the so-called self-duality equation

$$\mathrm{Im}\int \mathrm{d}^4x\mathrm{d}^2\theta\,\mathcal{E}\Big\{W^\alpha W_\alpha + M^\alpha M_\alpha\Big\} = 0\,, \tag{2.8}$$

in which W_α is chosen to be a general chiral spinor superfield.

In what follows, we shall treat W_α and $\bar{W}_{\dot\alpha}$ as unconstrained chiral superfields which are not subjected to the Bianchi identity (2.3). Now let us introduce the following general model for nonlinear $\mathcal{N}=1$ supersymmetric electrodynamics

$$S[W,\bar{W}] = \frac{1}{4}\int \mathrm{d}^4x\mathrm{d}^2\theta\,\mathcal{E}\,W^2 + \text{c.c.} + \frac{1}{4}\int \mathrm{d}^4x\mathrm{d}^2\theta\mathrm{d}^2\bar\theta\,E\,W^2\,\bar{W}^2\,\Lambda\left(u,\bar{u},p,\bar{p}\right)\,, \tag{2.9}$$

where the complex variables u and p are defined by

$$u := \frac{1}{8}(\mathcal{D}^2 - 4\bar{R})W^2\,, \qquad p := \mathcal{D}^\alpha W_\alpha\,. \tag{2.10}$$

For this model the self-duality equation (2.8) amounts to

$$\mathrm{Im}\int \mathrm{d}^4x\mathrm{d}^2\theta\mathrm{d}^2\bar\theta\,E\,W^2\,\bar{W}^2\Big\{\Gamma - \bar{u}\Gamma^2 + 2u\bar{u}(\partial_p\Lambda)^2\Big\} = 0\,, \qquad \Gamma := \partial_u(u\Lambda)\,. \tag{2.11}$$

In this equation the covariantly chiral spinor W_α has to be completely arbitrary, and therefore we conclude that it suffices for the equation (1.8) to hold.

A super-Weyl invariant equivalent of the model (2.9) is given by

$$\begin{aligned} S[W,\bar{W};\Upsilon] = {}& \frac{1}{4}\int \mathrm{d}^4x\mathrm{d}^2\theta\,\mathcal{E}\,W^2 + \text{c.c.} \\ &+ \frac{1}{4}\int \mathrm{d}^4x\mathrm{d}^2\theta\mathrm{d}^2\bar\theta\,E\,\frac{W^2\,\bar{W}^2}{\Upsilon^2}\Lambda\left(\frac{u}{\Upsilon^2},\frac{\bar{u}}{\Upsilon^2},\frac{p}{\Upsilon},\frac{\bar{p}}{\Upsilon}\right)\,, \end{aligned} \tag{2.12}$$

where Υ is a nowhere vanishing real scalar with the super-Weyl transformation

$$\delta_\sigma\Upsilon = (\sigma+\bar\sigma)\Upsilon\,, \qquad \bar{\mathcal{D}}_{\dot\beta}\sigma = 0\,, \tag{2.13}$$

with σ being the super-Weyl parameter.[6] The transformation law (2.13) implies that (2.12) is super-Weyl invariant. One may readily check that if $\Lambda(u, \bar{u}, p, \bar{p})$ is a solution of the equation (1.8), then

$$\widetilde{\Lambda}(u, \bar{u}, p, \bar{p}; \Upsilon) := \frac{1}{\Upsilon^2} \Lambda\left(\frac{u}{\Upsilon^2}, \frac{\bar{u}}{\Upsilon^2}, \frac{p}{\Upsilon}, \frac{\bar{p}}{\Upsilon}\right) \tag{2.14}$$

is also a solution of the same equation, and thus the model (2.12) is $\mathsf{U}(1)$ duality-invariant.

The family of $\mathsf{U}(1)$ duality-invariant theories analysed in [4, 49] is given by

$$\begin{aligned} S[W, \bar{W}; \Upsilon] = & \frac{1}{4} \int \mathrm{d}^4x \mathrm{d}^2\theta\, \mathcal{E}\, W^2 + \text{c.c.} \\ & + \frac{1}{4} \int \mathrm{d}^4x \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta}\, E\, \frac{W^2 \bar{W}^2}{\Upsilon^2} \Lambda\left(\frac{u}{\Upsilon^2}, \frac{\bar{u}}{\Upsilon^2}\right). \end{aligned} \tag{2.15}$$

In this case the self-duality equation (1.8) turns into (1.4).

We now present several simple solutions to the self-duality equation (1.8). Consider a polynomial interaction homogeneous in p and $\bar{p}$,

$$\Lambda^{(n)}(u, \bar{u}, p, \bar{p}) = \frac{1}{u\bar{u}} \sum_{k=0}^{n} a_k p^k \bar{p}^{n-k}, \qquad a_{n-k} = \bar{a}_k = a_k. \tag{2.16}$$

The self-duality equation (1.8) amounts to the following conditions on the coefficients a_k:

$$k^2 a_k^2 - (n-k+1)^2 a_{n-k+1}^2 = 0, \qquad k = 1, \ldots, n, \tag{2.17a}$$

$$kl a_k a_l - (n-k+1)(n-l+1) a_{n-k+1} a_{n-l+1} = 0, \qquad l \neq k. \tag{2.17b}$$

We end up with

$$a_j = \frac{n!}{j!(n-j)!} a_n, \qquad j = \begin{cases} 1, \ldots, \dfrac{n}{2} & \text{for } n \text{ even} \\ 1, \ldots, \dfrac{n+1}{2} & \text{for } n \text{ odd} \end{cases}. \tag{2.18}$$

As a result, the interaction (2.16) can be written in the form

$$\Lambda^{(n)}(u, \bar{u}, p, \bar{p}) = \frac{\zeta}{u\bar{u}} \left(\frac{p+\bar{p}}{2}\right)^n, \tag{2.19a}$$

with ζ a coupling constant. This solution of the self-duality equation (1.8) has an obvious polynomial generalisation

$$\Lambda^{[n]}(u, \bar{u}, p, \bar{p}) = \frac{1}{u\bar{u}} \sum_{k=0}^{n} \zeta_k \left(\frac{p+\bar{p}}{2}\right)^k. \tag{2.19b}$$

In order for the couplings (2.19) to be well-defined, the descendant u should be nowhere vanishing. In order for this condition to be consistent with the equations of motion, in general we should require the coefficient ζ_1 to be non-zero. We will come back to a discussion of this issue in section 5.[7]

[6] See, e.g., [67] for a review of super-Weyl transformations within the Grimm-Wess-Zumino geometry.

[7] At this stage we are only interested in generating solutions of the self-duality equation (1.8) without worrying about consistency issues.

Of special interest is the $n = 1$ case in (2.19a), since it corresponds to the model introduced in [42] and describes a Fayet-Iliopoulos term in supergravity without gauged R-symmetry, eq. (1.9).[8]

The duality-invariant model constructed, eq. (2.19), can be generalised as follows

$$\Lambda(u, \bar{u}, p, \bar{p}) = \frac{1}{u\bar{u}} \mathfrak{D}\left(\frac{p+\bar{p}}{2}\right), \tag{2.20}$$

where $\mathfrak{D}(x)$ is a real function of a real argument. It is easy to see that (2.20) is a solution to the self-duality equation (1.8). This is a special case of more general solutions

$$\Lambda(u, \bar{u}, p, \bar{p}) = \Lambda(u, \bar{u}) + \frac{1}{u\bar{u}} \mathfrak{D}\left(\frac{p+\bar{p}}{2}\right), \tag{2.21}$$

where $\Lambda(u, \bar{u})$ is an arbitrary solution of the self-duality equation (1.4).

More general solutions of the self-duality equation (1.8) may be obtained. Let $\Lambda(u, \bar{u}; \gamma)$ be a solution of (1.4), which depends on a real duality-invariant parameter γ. Then the following self-interaction

$$\Lambda(u, \bar{u}, p, \bar{p}) := \Lambda\left(u, \bar{u}; \mathfrak{D}\left(\frac{p+\bar{p}}{2}\right)\right) \tag{2.22}$$

is a solution of the self-duality equation (1.8), for any real function $\mathfrak{D}(x)$ of a real variable. A converse statement also holds. If $\Lambda(u, \bar{u}, p, \bar{p}) := \boldsymbol{\Lambda}(u, \bar{u}; \frac{1}{2}p + \frac{1}{2}\bar{p})$ is a solution of (1.8), then $\Lambda(u, \bar{u}; \gamma) := \boldsymbol{\Lambda}(u, \bar{u}, \gamma)$ is a solution of the self-duality equation (1.4), with γ being the duality-invariant parameter.

It should be emphasised that the parameter γ in $\Lambda(u, \bar{u}; \gamma)$ above is duality invariant. However, the combination $(p + \bar{p})$ in (2.22) is not duality invariant, even on-shell in the nonlinear case, as follows from the infinitesimal duality transformation (2.7). The duality transformation acts on all components of the vector multiplet, and not only on the Maxwell tensor.

As an example, let us consider the supersymmetric ModMax theory [55, 56]

$$S_{\text{sModMax}} = \frac{1}{4} \int \mathrm{d}^4x \mathrm{d}^2\theta\, \mathcal{E}\, W^2 \cosh\gamma + \text{c.c.} + \frac{1}{4} \int \mathrm{d}^4x \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta}\, E\, \frac{W^2\, \bar{W}^2}{\sqrt{u\bar{u}}} \sinh\gamma\,, \tag{2.23}$$

where γ is the duality-invariant parameter. With this form of the action, the deformation prescription (2.22), that is $\gamma \to \mathfrak{D}\left(\frac{p+\bar{p}}{2}\right)$ is not applicable. However, (2.23) can be rewritten in the alternative form

$$S_{\text{sModMax}} = \frac{1}{4} \int \mathrm{d}^4x \mathrm{d}^2\theta\, \mathcal{E} W^2 + \text{c.c.} + \frac{1}{4} \int \mathrm{d}^4x \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta}\, E W^2\, \bar{W}^2 \Lambda_{\text{SC}}\left(u, \bar{u}; \gamma\right), \tag{2.24a}$$

where we have introduced

$$\Lambda_{\text{SC}}(u, \bar{u}; \gamma) = \frac{1}{2}(1 - \cosh\gamma)\left(\frac{1}{u} + \frac{1}{\bar{u}}\right) + \frac{\sinh\gamma}{\sqrt{u\bar{u}}}\,. \tag{2.24b}$$

Since $\Lambda_{\text{SC}}(u, \bar{u}; \gamma)$ is a solution of (1.4), the deformation prescription (2.22) is applicable and leads to a solution of the self-duality equation (1.8).

[8]Models with more general Fayet-Iliopoulos terms described in [69] do not appear to be duality invariant.

3 Auxiliary superfield formalism

In this section we describe an alternative formulation for the self-dual models of the $\mathcal{N}=1$ vector multiplet described in the previous section. In particular, we extend the results obtained in [43] to the case of curved superspace.

We start with a brief summary of the construction given in [49]. Consider an auxiliary action of the form

$$S[W,\bar{W},\eta,\bar{\eta}] = \int \mathrm{d}^4x\mathrm{d}^2\theta\,\mathcal{E}\Big\{\eta W - \frac{1}{2}\eta^2 - \frac{1}{4}W^2\Big\} + \text{c.c.} + \mathfrak{S}_{\text{int}}[\eta,\bar{\eta}]\,. \tag{3.1}$$

Here the spinor superfield η_α is constrained to be covariantly chiral, $\bar{\mathcal{D}}_{\dot\beta}\eta_\alpha = 0$, but otherwise it is completely arbitrary. By definition, the second term on the right, $\mathfrak{S}_{\text{int}}[\eta,\bar{\eta}]$, contains cubic, quartic and higher powers of η_α and its conjugate.

The above model is equivalent to a theory with action

$$S[W,\bar{W}] = \frac{1}{4}\int \mathrm{d}^4x\mathrm{d}^2\theta\,\mathcal{E}\,W^2 + \text{c.c.} + S_{\text{int}}[W,\bar{W}]\,, \tag{3.2}$$

describing the dynamics of the vector multiplet. Indeed, under reasonable assumptions the equation of motion for η^α

$$W_\alpha = \eta_\alpha - \frac{\delta}{\delta\eta^\alpha}\mathfrak{S}_{\text{int}}[\eta,\bar{\eta}] \tag{3.3}$$

allows one to express η_α as a functional of W_α and its conjugate, $\eta_\alpha = \Psi_\alpha[W,\bar{W}]$. Plugging this functional and its conjugate into (3.1) leads to a vector-multiplet model of the form (3.2). If $S[W,\bar{W}]$ is a solution of the self-duality equation (2.8), then the self-interaction $\mathfrak{S}_{\text{int}}[\eta,\bar{\eta}]$ in (3.1) proves to be invariant under rigid $\mathsf{U}(1)$ phase transformations of η_α and its conjugate,

$$\mathfrak{S}_{\text{int}}[\mathrm{e}^{-\mathrm{i}\lambda}\eta,\mathrm{e}^{\mathrm{i}\lambda}\bar{\eta}] = \mathfrak{S}_{\text{int}}[\eta,\bar{\eta}]\,, \qquad \lambda\in\mathbb{R}\,. \tag{3.4}$$

The duality rotation (2.7) acts on the chiral spinor η_α as

$$\delta\eta_\alpha = -\mathrm{i}\lambda\eta_\alpha\,, \tag{3.5}$$

see [49] for the technical details.

We now restrict our attention to a subclass of the models (3.1) of the form:

$$\begin{aligned} S[W,\bar{W},\eta,\bar{\eta}] = &\int \mathrm{d}^4x\mathrm{d}^2\theta\,\mathcal{E}\Big\{\eta W - \frac{1}{2}\eta^2 - \frac{1}{4}W^2\Big\} + \text{c.c.} \\ &+\frac{1}{4}\int \mathrm{d}^4x\mathrm{d}^2\theta\mathrm{d}^2\bar{\theta}\,E\,\eta^2\bar{\eta}^2\mathfrak{F}(v,\bar{v},q,\bar{q})\,, \end{aligned} \tag{3.6a}$$

in which

$$v := \frac{1}{8}(\mathcal{D}^2 - 4\bar{R})\eta^2\,, \qquad q := \mathcal{D}^\alpha\eta_\alpha \tag{3.6b}$$

and $\mathfrak{F}(v,\bar{v},q,\bar{q})$ is a real function. Such models in a flat background were analysed in [43].

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We aim to integrate out from (3.6a) the auxiliary spinor variables η_α and $\bar{\eta}_{\dot{\alpha}}$ in order to bring the action to the form (2.9). The equation of motion for η^α is

$$\begin{aligned} W_\alpha = \eta_\alpha \bigg\{ 1 + \frac{1}{8}(\bar{\mathcal{D}}^2 - 4R)\Big[\bar{\eta}^2\Big(\mathfrak{F} + \frac{1}{8}(\mathcal{D}^2 - 4\bar{R})(\eta^2\,\partial_v\mathfrak{F})\Big)\Big]\bigg\} \\ -\frac{1}{16}(\bar{\mathcal{D}}^2 - 4R)\left[\bar{\eta}^2\mathcal{D}_\alpha(\eta^2\partial_q\mathfrak{F})\right] . \end{aligned} \tag{3.7}$$

Its immediate implications are

$$\eta W = \eta^2 \bigg\{ 1 + \frac{1}{8}(\bar{\mathcal{D}}^2 - 4R)\Big[\bar{\eta}^2\Big(\partial_v(v\mathfrak{F}) + \frac{1}{2}q\partial_q\mathfrak{F}\Big)\Big]\bigg\} , \tag{3.8a}$$

$$\begin{aligned} W^2 = \eta^2\bigg\{ \Big[1 + \frac{1}{8}(\bar{\mathcal{D}}^2 - 4R)\left\{\bar{\eta}^2\partial_v(v\mathfrak{F})\right\}\Big]^2 + \frac{1}{8}(\bar{\mathcal{D}}^2 - 4R)\left(\bar{\eta}^2 q\partial_q\mathfrak{F}\right) \\ +\frac{1}{64}(\bar{\mathcal{D}}^2 - 4R)\left(\bar{\eta}^2\partial_v(v\mathfrak{F})\right)(\bar{\mathcal{D}}^2 - 4R)\left(\bar{\eta}^2 q\partial_q\mathfrak{F}\right) \\ +\frac{1}{128}(\bar{\mathcal{D}}^2 - 4R)\left(\bar{\eta}^2(\mathcal{D}^\alpha\eta^\beta)\partial_q\mathfrak{F}\right)(\bar{\mathcal{D}}^2 - 4R)\left(\bar{\eta}^2(\mathcal{D}_\alpha\eta_\beta)\partial_q\mathfrak{F}\right)\bigg\} , \end{aligned} \tag{3.8b}$$

$$W^2\bar{W}^2 = \eta^2\bar{\eta}^2\mathfrak{H}\bar{\mathfrak{H}} , \tag{3.8c}$$

where we have introduced

$$\mathfrak{H} := [1 + \partial_v(v\bar{v}\mathfrak{F})]^2 + \bar{v}q\partial_q\mathfrak{F}\,[1 + \partial_v(v\bar{v}\mathfrak{F})] - 2v\bar{v}^2(\partial_q\mathfrak{F})^2 . \tag{3.9}$$

It should be noted that in deriving (3.8b) we have made use of the identity

$$\eta^2(\mathcal{D}_\alpha\eta^\beta)(\mathcal{D}^\alpha\eta_\beta) = 4v\eta^2 . \tag{3.10}$$

Eq. (3.8b) and (3.7) imply that

$$u \approx v\mathfrak{H} , \tag{3.11}$$

$$p \approx q\,[1 + \partial_v(v\bar{v}\mathfrak{F})] - 4v\bar{v}\partial_q\mathfrak{F} \tag{3.12}$$

respectively. The symbol $\approx$ is used to indicate that the result holds modulo terms proportional to η_α and $\bar{\eta}_{\dot{\alpha}}$ (or, equivalently, to W_α and $\bar{W}_{\dot{\alpha}}$). The equations (3.11) and (3.12) are the "effective" relations used to relate the auxiliary variables $(v, \bar{v}, q, \bar{q})$ to the original multiplet variables $(u, \bar{u}, p, \bar{p})$.

The identities (3.8) may be used to derive the following integral relations

$$\begin{aligned} \int \mathrm{d}^4x\mathrm{d}^2\theta\,\mathcal{E}\,\eta W = \int \mathrm{d}^4x\mathrm{d}^2\theta\,\mathcal{E}\,\eta^2 \\ -\frac{1}{2}\int \mathrm{d}^4x\mathrm{d}^2\theta\mathrm{d}^2\bar{\theta}\,E\,\eta^2\bar{\eta}^2\Big(\partial_v(v\mathfrak{F}) + \frac{1}{2}q\partial_q\mathfrak{F}\Big) , \end{aligned} \tag{3.13a}$$

$$\begin{aligned} \int \mathrm{d}^4x\mathrm{d}^2\theta\,\mathcal{E}\,W^2 = \int \mathrm{d}^4x\mathrm{d}^2\theta\,\mathcal{E}\,\eta^2 \\ -\int \mathrm{d}^4x\mathrm{d}^2\theta\mathrm{d}^2\bar{\theta}\,E\,\eta^2\bar{\eta}^2\Big\{\partial_v(v\mathfrak{F}) + \frac{1}{2}\bar{v}[\partial_v(v\mathfrak{F})]^2 \\ +\frac{1}{2}q\partial_q\mathfrak{F}(1 + \partial_v(v\bar{v}\mathfrak{F})) - v\bar{v}(\partial_q\mathfrak{F})^2\Big\} . \end{aligned} \tag{3.13b}$$

These relations along with (3.8c) allow us to rewrite the action (3.6a) in terms of the vector multiplet,

$$S[W, \bar{W}] = \frac{1}{4}\int \mathrm{d}^4x\mathrm{d}^2\theta\,\mathcal{E}\,W^2 + \text{c.c.} + \frac{1}{4}\int \mathrm{d}^4x\mathrm{d}^2\theta\mathrm{d}^2\bar{\theta}\,E\,W^2\bar{W}^2\Lambda(u,\bar{u},p,\bar{p})\,, \tag{3.14}$$

where we have introduced

$$\Lambda(u,\bar{u},p,\bar{p}) := \frac{\mathfrak{F}+\mathfrak{G}+\bar{\mathfrak{G}}}{\mathfrak{H}\bar{\mathfrak{H}}}\,, \tag{3.15}$$

and

$$\mathfrak{G} := \bar{v}[\partial_v(v\mathfrak{F})]^2 + q\partial_q\mathfrak{F}\partial_v(v\bar{v}\mathfrak{F}) - 2v\bar{v}(\partial_q\mathfrak{F})^2\,. \tag{3.16}$$

The super-Weyl invariant version of the model (3.6) is given by

$$\begin{aligned} S[W,\bar{W},\eta,\bar{\eta};\Upsilon] = &\int \mathrm{d}^4x\mathrm{d}^2\theta\,\mathcal{E}\Big\{\eta W - \frac{1}{2}\eta^2 - \frac{1}{4}W^2\Big\} + \text{c.c.} \\ &+\frac{1}{4}\int \mathrm{d}^4x\mathrm{d}^2\theta\mathrm{d}^2\bar{\theta}\,E\,\frac{\eta^2\,\bar{\eta}^2}{\Upsilon^2}\mathfrak{F}\left(\frac{v}{\Upsilon^2},\frac{\bar{v}}{\Upsilon^2},\frac{q}{\Upsilon},\frac{\bar{q}}{\Upsilon}\right)\,, \end{aligned} \tag{3.17}$$

in which the auxiliary variable η_α transforms as

$$\delta_\sigma\eta_\alpha = \frac{3}{2}\sigma\eta_\alpha\,, \tag{3.18}$$

in conjunction with the transformation of Υ, eq. (2.13).

The condition of $\mathsf{U}(1)$ duality invariance (2.8) in the auxiliary variable formalism is equivalent to (3.4). For the model (3.6a) this means manifest $\mathsf{U}(1)$ invariance of the auxiliary interaction function $\mathfrak{F}$ and thus

$$\mathfrak{F}(v,\bar{v},q,\bar{q}) = f\left(\frac{v}{q^2},\frac{\bar{v}}{\bar{q}^2},q\bar{q}\right)\,. \tag{3.19}$$

To demonstrate how the construction discussed above works, we provide here a simple example. Consider the following $\mathsf{U}(1)$-invariant auxiliary interaction

$$\mathfrak{F}^{(0)}(v,\bar{v},q,\bar{q}) = \frac{\kappa}{v\bar{v}}\,, \qquad \kappa\in\mathbb{R}\,. \tag{3.20}$$

The effective relations (3.11) and (3.12) are trivial,

$$u\approx v\,, \qquad p\approx q\,. \tag{3.21}$$

The self-interaction defined by (3.15) takes the form

$$\Lambda^{(0)}(u,\bar{u},p,\bar{p}) = \frac{\kappa}{u\bar{u}}\,, \tag{3.22}$$

which coincides with the $n=0$ case of (2.19) when $\kappa=\zeta$. The corresponding action

$$S[W,\bar{W}] = \frac{1}{2}\int \mathrm{d}^4x\mathrm{d}^2\theta\,\mathcal{E}\,W^2 + \frac{\kappa}{4}\int \mathrm{d}^4x\mathrm{d}^2\theta\mathrm{d}^2\bar{\theta}\,E\,\frac{W^2\,\bar{W}^2}{u\bar{u}} \tag{3.23}$$

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contains a cosmological term,

$$\kappa \int \mathrm{d}^4 x\, e\,, \qquad e = \det({e_m}^a)\,, \tag{3.24}$$

at the component level, where ${e_m}^a(x)$ is the vielbein.

Model (3.23) is a new solution to the self-duality equation (1.4) with $\Lambda(u,\bar u) = \kappa(u\bar u)^{-1}$, where κ is the coupling constant. It was not derived in the early 2000s since refs. [1, 2] studied only those models for supersymmetric nonlinear electrodynamics which possess a weak field expansion.

4 Superconformal duality-invariant models

As an example of applying the formalism described in the previous section, we now turn to deriving a new superconformal duality-invariant model. The duality-invariant model defined by eqs. (3.17) and (3.19) is superconformal if the action is independent of Υ. The most general duality-invariant and superconformal model is described by

$$\mathfrak{F}_{\rm SC}(v,\bar v,q,\bar q) = \frac{1}{q\bar q}\varphi\Big(\frac{v}{q^2},\frac{\bar v}{\bar q^2}\Big)\,, \tag{4.1}$$

for some function $\varphi(z,\bar z)$. Choosing $\varphi(z,\bar z) = \kappa/\sqrt{z\bar z}$, with $\kappa\in\mathbb{R}$, gives the model [56]

$$\mathfrak{F}_{\rm SC}(v,\bar v) = \frac{\kappa}{\sqrt{v\bar v}}\,, \tag{4.2}$$

which is the only member of the family (4.1) without dependence on q and $\bar q$. Eliminating the auxiliary chiral η_α and antichiral $\bar\eta_{\dot\alpha}$ variables leads to the supersymmetric ModMax theory [55, 56].

Here we will study a different duality-invariant and superconformal model defined by $\varphi(z,\bar z) = \kappa/(z\bar z)$, with a real coupling constant κ, which leads to

$$\mathfrak{F}_{\rm SC}(v,\bar v,q,\bar q) = \kappa\frac{q\bar q}{v\bar v}\,. \tag{4.3}$$

In this case the effective relations (3.11) and (3.12) become

$$u \approx v + \kappa q\bar q - 2\kappa^2\bar q^2\,, \qquad \bar u \approx \bar v + \kappa q\bar q - 2\kappa^2 q^2\,, \tag{4.4}$$

$$p \approx q - 4\kappa\bar q\,, \qquad \bar p \approx \bar q - 4\kappa q\,. \tag{4.5}$$

Using the effective relations (4.5), we can express the auxiliary variables q and $\bar q$ in terms of the multiplet variables p and $\bar p$,

$$q \approx \frac{p+4\kappa\bar p}{1-(4\kappa)^2}\,, \qquad \bar q \approx \frac{\bar p + 4\kappa p}{1-(4\kappa)^2}\,. \tag{4.6}$$

Substituting these expressions (4.6) into (4.4) allows one to express the remaining auxiliary variables v and $\bar v$ purely in terms of the multiplet variables as

$$\begin{aligned} v &\approx u + \frac{4\kappa^2 p^2(8\kappa^2-1) - 2\kappa^2\bar p^2 - \kappa p\bar p}{(1-(4\kappa)^2)^2}\,,\\ \bar v &\approx \bar u + \frac{4\kappa^2 \bar p^2(8\kappa^2-1) - 2\kappa^2 p^2 - \kappa p\bar p}{(1-(4\kappa)^2)^2}\,. \end{aligned} \tag{4.7}$$

With the aid of these relations (4.6) and (4.7), we can read off the self-interaction (3.15) as a function of the multiplet variables,

$$\Lambda(u,\bar u,p) = \left(\frac{\kappa}{1-4\kappa}\right)\frac{p^2}{u\bar u}\,. \tag{4.8}$$

It should be noted that for the purposes of our analysis, we have treated p and $\bar p$ as independent, and only at the end of the calculation is the Bianchi identity (2.3) imposed.

The model derived above (4.8) corresponds to the $n=2$ case of the family of duality-invariant solutions in (2.19a) with $p=\bar p$ and $\zeta=\kappa/(1-4\kappa)$. The outcome of our analysis is the superconformal duality-invariant model,

$$S[W,\bar W] = \frac{1}{4}\int \mathrm{d}^4x\mathrm{d}^2\theta\,\mathcal{E}\,W^2 + \text{c.c.} + \frac{\zeta}{4}\int \mathrm{d}^4x\mathrm{d}^2\theta\mathrm{d}^2\bar\theta\,E\,W^2\bar W^2\frac{(\mathcal{D}W)^2}{u\bar u}\,. \tag{4.9}$$

It is a member of the family of the superconformal vector multiplet models [70]

$$\begin{aligned} S[W,\bar W] =& \frac{1}{4}\int \mathrm{d}^4x\mathrm{d}^2\theta\,\mathcal{E}\,W^2 + \text{c.c.} \\ &+\frac{1}{4}\int \mathrm{d}^4x\mathrm{d}^2\theta\mathrm{d}^2\bar\theta\,E\,\frac{W^2\bar W^2}{(\mathcal{D}W)^2}\,\Lambda\left(\frac{u}{(\mathcal{D}W)^2},\frac{\bar u}{(\bar{\mathcal{D}}\bar W)^2}\right)\,, \end{aligned} \tag{4.10}$$

where $\Lambda(\omega,\bar\omega)$ is a real function of one complex variable. If $\Lambda(\omega,\bar\omega)$ is a solution of the self-duality equation (1.5), then replacing $\mathcal{D}W \to \frac{1}{2}(p+\bar p)$ in the action (4.10) leads to a superconformal duality-invariant theory.

5 Discussion

In this paper we have derived for the first time the family of solutions of the self-duality equation (1.8) with $\partial_p\Lambda \neq 0$. They have the general form

$$\Lambda(u,\bar u,p,\bar p) := \boldsymbol{\Lambda}\Big(u,\bar u;\frac{1}{2}p+\frac{1}{2}\bar p\Big) \tag{5.1a}$$

where $\boldsymbol{\Lambda}(u,\bar u;\gamma)$ is a solution of the self-duality equation (1.4) depending on a duality-invariant parameter γ, the latter may be a superfield such as the supergravity compensator. Less interesting solutions of (1.8) are of the form

$$\widetilde\Lambda(u,\bar u,p,\bar p) := \boldsymbol{\Lambda}\Big(u,\bar u;\gamma+\frac{\mathrm{i}}{2}p-\frac{\mathrm{i}}{2}\bar p\Big)\,. \tag{5.1b}$$

Unlike (5.1a), $\widetilde\Lambda(u,\bar u,p,\bar p)$ reduces to $\boldsymbol{\Lambda}(u,\bar u;\gamma)$ for $p=\bar p$. It remains an open problem to derive solutions of the self-duality equation (1.8) with $\partial_p\Lambda \neq \pm\partial_{\bar p}\Lambda$.

In our opinion, the most interesting solutions in the family (5.1a) are

$$\Lambda(u,\bar u,p,\bar p) := \Lambda(u,\bar u) + \frac{\xi}{16}\frac{p+\bar p}{u\bar u}\,, \tag{5.2}$$

where $\Lambda(u,\bar u)$ is a solution of (1.4) and ξ a constant parameter. For $\Lambda(u,\bar u)\neq 0$, such solutions generate new duality-invariant models for spontaneously broken local supersymmetry described by actions of the form

$$S = S_{\rm SG} + \frac{1}{4}\int {\rm d}^4x{\rm d}^2\theta\,\mathcal{E}\,W^2 + {\rm c.c.} + \frac{1}{4}\int {\rm d}^4x{\rm d}^2\theta{\rm d}^2\bar\theta\,E\,\frac{W^2\,\bar W^2}{\Upsilon^2}\Lambda\left(\frac{u}{\Upsilon^2},\frac{\bar u}{\Upsilon^2}\right) \\ +\frac{\xi}{8}\int {\rm d}^4x{\rm d}^2\theta{\rm d}^2\bar\theta\,E\,\Upsilon\frac{W^2\,\bar W^2}{u\bar u}\mathcal{D}^\alpha W_\alpha\,, \tag{5.3}$$

where $S_{\rm SG}$ is the action for off-shell supergravity coupled to matter multiplets, and Υ the corresponding compensator. Within the old minimal formulation for $\mathcal{N}=1$ supergravity (see, e.g., [71]), Υ is given by

$$\Upsilon = \bar S_0 {\rm e}^{-\frac{1}{3}K(\phi,\bar\phi)}S_0\,, \tag{5.4}$$

where S_0 is the chiral compensator, and $K(\phi,\bar\phi)$ is the Kähler potential for a Kähler-Hodge manifold in which the matter chiral superfields ϕ take their values. The matter-coupled supergravity action is

$$S_{\rm SG} = -3\int {\rm d}^4x{\rm d}^2\theta{\rm d}^2\bar\theta\,E\,\bar S_0\,{\rm e}^{-\frac{1}{3}K(\phi,\bar\phi)}S_0 + \left\{\int {\rm d}^4x{\rm d}^2\theta\,\mathcal{E}\,S_0^3 W(\phi) + {\rm c.c.}\right\}. \tag{5.5}$$

The fact that Υ should have the form (5.4) to preserve the Kähler invariance, was first pointed out in [72, 73], see also [70].

The supergravity-matter system (5.3) with $\Lambda(u,\bar u)=0$ was proposed in [42] (in conjunction with the clarifying comments given in [72, 73]). In the $\Lambda(u,\bar u)\neq 0$ case, this $\mathsf{U}(1)$ duality-invariant theory is new, to the best of our knowledge.

In the main body of this paper, we concentrated on generating solutions to the self-duality equation (1.8) without worrying about consistency of such duality-invariant models on the mass shell. The main technical issue here is related to the coupling (2.20). In order for such a coupling to be well-defined, the descendant u is required to be nowhere vanishing, and this requirement should be consistent with the equations of motion. To discuss this issue, it suffices to consider a flat superspace background. We introduce the component fields of the vector multiplet following [2]

$$W_\alpha| = \psi_\alpha\,, \qquad -\frac{1}{2}D^\alpha W_\alpha| = D\,, \qquad D_{(\alpha}W_{\beta)}| = 2{\rm i}F_{\alpha\beta} = {\rm i}(\sigma^{bc})_{\alpha\beta}F_{bc}\,, \tag{5.6}$$

where the bar-projection $U|$ of a superfield U means, as usual, switching off the superspace Grassmann variables, see e.g. [64, 74]. It holds that

$$u| \equiv \frac{1}{8}D^2W^2| = \boldsymbol{u} + \text{fermionic terms}\,, \qquad \boldsymbol{u} = F^{\alpha\beta}F_{\alpha\beta} - \frac{1}{2}D^2 = \omega - \frac{1}{2}D^2\,, \tag{5.7}$$

where ω is given by (1.6b). For the coupling (2.20), the vector multiplet action in the Minkowski background is

$$S[W,\bar W] = \frac{1}{4}\int {\rm d}^4x{\rm d}^2\theta\,W^2 + {\rm c.c.} + \frac{1}{4}\int {\rm d}^4x{\rm d}^2\theta{\rm d}^2\bar\theta\,\frac{W^2\,\bar W^2}{u\bar u}\mathfrak{D}(D^\alpha W_\alpha) \\ = \int {\rm d}^4x\left\{-\frac{1}{4}F^{ab}F_{ab} + \frac{1}{2}D^2 + \mathfrak{D}(-2D)\right\} + \text{fermionic terms}\,. \tag{5.8}$$

All the non-analytic contributions are concentrated in the fermionic sector. Since the equation of motion for D is

$$D = 2\mathfrak{D}'(-2D) + \text{fermionic terms}\,, \tag{5.9}$$

there are two ways for $\boldsymbol{u}$ to be nowhere vanishing. The first option is that (5.9) has a non-zero solution $D_0 \neq 0$, which is the case for $\mathfrak{D}(y) = \frac{\zeta}{8}y^2 + \frac{\xi}{2}y$. The second option is realised when the solution to (5.9) is $D = 0$ modulo fermionic contributions, and then $F^{\alpha\beta}F_{\alpha\beta}$ should be restricted to be nowhere vanishing, as in the ModMax theory [53].

The bosonic sector of the supergravity-matter system (5.3) with $\xi = 0$ (coupled to the dilaton-axion multiplet) was computed in [4], to which the reader is referred for the details. Switching off the supergravity multiplet in (5.3), including the compensator, the bosonic action takes the form

$$S_{\text{boson}} = \int \mathrm{d}^4x\,\mathcal{L}\,, \qquad \mathcal{L} = -\frac{1}{4}F^{ab}F_{ab} + \frac{1}{2}D^2 + \boldsymbol{u}\bar{\boldsymbol{u}}\Lambda(\boldsymbol{u},\bar{\boldsymbol{u}}) - \xi D\,, \tag{5.10}$$

and the equation of motion for D becomes [41]

$$D\Big[1 - \bar{\boldsymbol{u}}\Gamma(\boldsymbol{u},\bar{\boldsymbol{u}}) - \boldsymbol{u}\bar{\Gamma}(\boldsymbol{u},\bar{\boldsymbol{u}})\Big] = \xi\,. \tag{5.11}$$

Generically, the auxiliary field develops a non-vanishing expectation value, $\langle D\rangle \neq 0$, which must satisfy an algebraic nonlinear equation that follows from (5.11) by setting $F^{\alpha\beta}F_{\alpha\beta} = 0$. As a result, the supersymmetry becomes spontaneously broken. For example, the $\mathcal{N} = 1$ supersymmetric Born-Infeld action [14–16] is described by

$$\Lambda_{\text{SBI}}(u,\bar{u};g) = \frac{g^2}{1 + \frac{1}{2}g^2(u+\bar{u}) + \sqrt{1 + g^2(u+\bar{u}) + \frac{1}{4}g^4(u-\bar{u})^2}}\,. \tag{5.12}$$

The corresponding bosonic Lagrangian density, eq. (5.10), can be written in the form

$$\mathcal{L}_{\text{SBI}} = \frac{1}{g^2}\left\{1 - \sqrt{1 + g^2(\boldsymbol{u}+\bar{\boldsymbol{u}}) + \frac{1}{4}g^4(\boldsymbol{u}-\bar{\boldsymbol{u}})^2}\right\} - \xi\,D\,. \tag{5.13}$$

In this case the equation (5.11) is solved as [41]

$$D = \frac{\xi}{\sqrt{1+g^2\xi^2}}\sqrt{1 + g^2(\omega+\bar{\omega}) + \frac{1}{4}g^4(\omega-\bar{\omega})^2}\,. \tag{5.14}$$

In general, the component reduction of the supergravity-matter system (5.3) may be obtained by applying the technique developed in [55] using the earlier construction of [75]. This will be discussed elsewhere.

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