Electroweak corrections to the Charm-Top-Quark Contribution to ϵ_K

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Abstract: We calculate the leading-logarithmic and next-to-leading-logarithmic electroweak corrections to the charm-top-quark contribution to the effective $|\Delta S| = 2$ Lagrangian, relevant for the parameter ϵ_K. We find that these corrections lead to a -0.5% shift in the corresponding Wilson coefficient. Moreover, our calculation removes an implicit ambiguity in the standard-model prediction of ϵ_K, by fixing the renormalization scheme of the electroweak input parameters.

Contents

1 Introduction

Indirect CP violation in the neutral kaon system, parameterized by ϵ_K, is one of the most sensitive precision probes of new physics. The parameter ϵ_K can be expressed to excellent approximation as [1]

$$\epsilon_K \equiv e^{i\phi_\epsilon} \sin\phi_\epsilon \frac{1}{2} \arg\left(\frac{-M_{12}}{\Gamma_{12}}\right). \tag{1.1}$$

Here, $\phi_\epsilon = \arctan(2\Delta M_K/\Delta\Gamma_K)$, with ΔM_K and $\Delta\Gamma_K$ the mass and lifetime difference of the weak eigenstates K_L and K_S. M_{12} and Γ_{12} are the Hermitian and anti-Hermitian parts of the Hamiltonian that determines the time evolution of the neutral kaon system. The short-distance contributions to ϵ_K are then contained in the matrix element $M_{12} = -\langle K^0|\mathcal{L}_{f=3}^{\Delta S=2}|\bar{K}^0\rangle/(2\Delta M_K)$, up to higher powers in the operator-product expansion.

Experimentally, $|\epsilon_K| = (2.228 \pm 0.0011) \times 10^{-3}$ [2], with an uncertainty at the permil level. From the theory side, recent progress indicates that we will be able to predict ϵ_K in the Standard Model (SM) with an uncertainty at the percent level in the not-so-far future. Currently, the combined perturbative uncertainty is of the order of 3%, while the non-perturbative uncertainty is of the order of 3.5% [3]. Interestingly, both these errors can in principle be reduced by perturbative calculations, the first by computing the three-loop QCD corrections to the top-quark contribution to ϵ_K, and the second by computing the two-loop conversion to the $\overline{\text{MS}}$ scheme of the hadronic matrix element. The non-local long-distance contributions to ϵ_K, estimated in refs. [1] and [4], can be improved in the future with lattice calculations (see ref. [5] for recent results).

With theory uncertainties approaching the percent level, also parametrically smaller corrections have been taken into consideration recently. The power corrections to the effective Lagrangian [6] have been revisited in an extended analysis [7], leading to a one-percent increase of the SM prediction of ϵ_K. On the perturbative side, the electroweak corrections to the top-quark contribution to ϵ_K have been calculated by some of us [8].

In this work, we complete the analysis of the leading perturbative electroweak and QED corrections to ϵ_K by considering the mixed charm-top contributions. The paper is organized as follows. The analytic results, including the details of our calculation, are presented in section 2. The numerical evaluation, as well as a discussion of the results, can be found in section 3, where we also give an updated SM prediction for ϵ_K. Appendix A contains the definition of evanescent operators used in our calculation.

2 Electroweak corrections in the charm-top sector

In this section we provide the details of the renormalization-group (RG) analysis. We will show the analytic results only for the electroweak and QED corrections; the QCD corrections to $\tilde{C}_{S2}^{ut}$ have already been presented in refs. [9, 10] and can be transcribed to our convention for the effective Lagrangian as explained in ref. [3]. In particular, the NNLL QCD results are not needed as an ingredient of our calculation. Of course, they are included in our final numerics.

2.1 The effective Lagrangians

As shown in ref. [3], it is advantageous to choose the effective Lagrangian describing the $|\Delta S| = 2$ transition in the three-flavor theory as

$$\mathcal{L}_{f=3}^{|\Delta S|=2} = -\frac{G_F^2 M_W^2}{4\pi^2} [\lambda_u^2 \tilde{C}_{S2}^{uu}(\mu) + \lambda_t^2 \tilde{C}_{S2}^{tt}(\mu) + \lambda_u \lambda_t \tilde{C}_{S2}^{ut}(\mu)] \tilde{Q}_{S2} + \text{h.c.} + \dots, \qquad (2.1)$$

because then the higher-order QCD corrections are small. Here, G_F denotes Fermi's constant, and M_W the W-boson mass. The parameters $\lambda_i \equiv V_{is}^* V_{id}$ comprise the Cabibbo-Kobayashi-Maskawa (CKM) matrix elements. The $|\Delta S| = 2$ transition is induced by the local operator

$$\tilde{Q}_{S2} = (\bar{s}_L \gamma_\mu d_L)(\bar{s}_L \gamma_\mu d_L), \qquad (2.2)$$

where s_L, d_L denote the left-handed strange- and down-quark fields, respectively. The ellipsis in eq. (2.1) denotes non-local contributions as well as the contribution of higher dimension operators [6, 7]. The combination of CKM matrix elements, λ_u, is real and the coefficient $\tilde{C}_{S2}^{uu}$ does not affect ϵ_K (it does contribute to the kaon mass difference ΔM_K). The coefficient $\tilde{C}_{S2}^{tt}$ depends on the top-quark mass and is independent of the charm-quark mass to excellent approximation. It is known including next-to-leading-logarithmic (NLL) QCD corrections [11], while the electroweak corrections have been presented in ref. [8]. The coefficient $\tilde{C}_{S2}^{ut}$, on the other hand, depends on both the charm and top masses and has been predicted including next-to-next-to-leading-logarithmic (NNLL) QCD corrections [3, 9, 10]. Here, we calculate the electroweak corrections to $\tilde{C}_{S2}^{ut}$.

The Lagrangian (2.1) is valid below the charm-quark scale. Its Wilson coefficients are obtained by matching from the effective four- and five flavor Lagrangians

$$\mathcal{L}_{f=4,5}^{\text{eff}} = -\frac{4G_F}{\sqrt{2}} \left[\sum_{q,q'=u,c} V_{qs}^* V_{q'd}(C_+ Q_+^{qq'} + C_- Q_-^{qq'}) - \lambda_t \sum_{i=3,6} C_i Q_i \right] +$$
$$- \frac{G_F^2 M_W^2}{4\pi^2} \lambda_t^2 C_{S2}^{tt} Q_{S2} - 8 G_F^2 \left(\lambda_u \lambda_t + \lambda_t^2 \right) \tilde{C}_7 \tilde{Q}_7 + \text{h.c.}, \qquad (2.3)$$

after the appropriate RG evolution, as described below. The first line in eq. (2.3) contains the $|\Delta S| = 1$ current-current operators, defined as

$$Q_{\pm}^{qq'} = \frac{1}{2}\left((\bar{s}_L^\alpha \gamma_\mu q_L^\alpha)(\bar{q}_L'^\beta \gamma^\mu d_L^\beta) \pm (\bar{s}_L^\alpha \gamma_\mu q_L^\beta)(\bar{q}_L'^\beta \gamma^\mu d_L^\alpha)\right). \qquad (2.4)$$

Here, α, β are SU(3) color indices. The QCD-penguin operators Q_i, $i = 3, \ldots, 6$, are defined, e.g., in ref. [10]. They are neglected in this work as they constitute a percent-level correction to our numerically small results (see section 3 below; cf. also ref. [12]). The $|\Delta S| = 1$ operators mix, via bilocal insertions, into the local $|\Delta S| = 2$ operator. For the contributions proportional to $\lambda_u \lambda_t$, the Glashow-Iliopoulos-Maiani mechanism ensures that the mixing starts at order m_c^2; it is therefore convenient to define a rescaled version of the Q_{S2} operator as

$$\tilde{Q}_7 = \frac{m_c^2}{g_s^2 \mu^{2\epsilon}}(\bar{s}_L \gamma_\mu d_L)(\bar{s}_L \gamma_\mu d_L). \qquad (2.5)$$

This operator is formally of dimension eight. The appearance of the strong coupling constant in the denominator takes account of the large logarithm in the LO result.

2.2 RG analysis of the charm-top contribution

To begin, we briefly discuss the structure of the RG-improved perturbation series. Recall that the leading QCD RG evolution of $\tilde{C}_{S2}^{ut}$ reproduces the large logarithm $\log(m_c^2/M_W^2)$ that appears in the (fixed-order) Inami-Lim function [13], and sums this logarithm to all orders. As these leading-order boxes involve no gluon exchange, it is conventional to rescale the $|\Delta S| = 2$ effective operator (2.2) with an inverse power of g_s^2 (see eq. (2.5)). In this way, the leading-logarithmic (LL) series has the standard form, with terms proportional to $(\alpha_s \log)^n$, where $n = 1, \ldots$. The terms in the NLL and NNLL series are then proportional to $\alpha_s(\alpha_s \log)^n$ and $\alpha_s^2(\alpha_s \log)^n$, respectively.

Here, we will sum the two series whose terms are proportional to $\alpha \alpha_s^n \log^{n+1}$ ("LL QED") and $\alpha(\alpha_s \log)^n$ ("NLL QED"). The former series receives contributions only from the one-loop QED running of the current-current operators (see below), while the latter requires the calculation of the one-loop electroweak initial conditions in the current-current sector, the mixed QED-QCD RG evolution of the current-current operators, and the QED corrections to the anomalous dimension tensor, encoding the mixing of current-current operators into the $|\Delta S| = 2$ sector. In contrast to the case of $\tilde{C}_{S2}^{tt}(\mu)$, the two-loop electroweak initial condition of the Wilson coefficient in the $|\Delta S| = 2$ sector is not needed, due to the presence of the large logarithm in the leading-order (LO) result. In addition to summing these series, our calculation fixes the renormalization scheme of the electroweak input parameters. That is achieved by normalizing [14] the initial conditions of the current-current Wilson coefficients to the Wilson coefficient for muon decay, in analogy to the procedure in ref. [8]. In this way, a large part of the radiative corrections is absorbed into the measured value of the muon decay rate [15], and G_F is the only requisite electroweak input parameter for our calculation.

The actual RG analysis involves the determination of the initial conditions of the Wilson coefficients at the electroweak scale, and the subsequent RG evolution down to the

hadronic scale where the local hadronic matrix element is evaluated. The steps of this analysis have been discussed extensively in the literature and no new conceptual questions arise in our analysis, so we can afford to be brief in our exposition.

Expanding all Wilson coefficients in the five- and four-flavor effective theories as

$$C_i(\mu) = C_i^{(0)}(\mu) + \frac{\alpha}{\alpha_s(\mu)} C_i^{(e)}(\mu) + \frac{\alpha}{4\pi} C_i^{(es)}(\mu) \,, \tag{2.6}$$

we find by performing an explicit matching calculation at the electroweak scale

$$C_\pm^{(0)}(\mu_W) = 1 \,, \quad C_\pm^{(e)}(\mu_W) = 0 \,, \quad C_\pm^{(es)}(\mu_W) = -\frac{22}{9} - \frac{4}{3} \log \frac{\mu_W^2}{M_Z^2} \,, \tag{2.7}$$

consistent with the results in the literature [16, 17]. The initial conditions for the Wilson coefficients in the $|\Delta S| = 2$ sector vanish at this order, i.e., we have $\tilde{C}_7^{(e)}(\mu_W) = \tilde{C}_7^{(es)}(\mu_W) = 0$.

In order to evolve the Wilson coefficients down to the hadronic scale, we need to solve the set of RG equations

$$\mu \frac{d}{d\mu} C_i(\mu) = C_j(\mu)\gamma_{ji} \,, \quad i,j = +, - \,, \tag{2.8}$$

and

$$\mu \frac{d}{d\mu} \tilde{C}_7(\mu) = \tilde{C}_7(\mu)\tilde{\gamma}_{77} + \sum_{k,l=+,-} C_k(\mu)C_l(\mu)\hat{\gamma}_{kl,7} \,. \tag{2.9}$$

Here, $\tilde{\gamma}_{77} = \tilde{\gamma}_{S2} + 2\gamma_m + 2\beta$ is given in terms of the anomalous dimension $\tilde{\gamma}_{S2}$ of the local operator $\tilde{Q}_{S2}$. The quark anomalous dimension and the beta function appear because of the explicit factors of m_c and g_s in the definition of $\tilde{Q}_7$. Further, γ_{ij} denotes the anomalous dimension matrix in the current-current sector, and $\hat{\gamma}_{kn,7}$ is the anomalous dimension tensor, describing the mixing of the dimension-six operators into $\tilde{Q}_7$. Defining $dg_s/d\log\mu = \beta$, with

$$\beta(g_s, e) = -\beta_0 \frac{g_s^3}{16\pi^2} - \beta_1 \frac{g_s^5}{(16\pi^2)^2} - \beta_{es} \frac{e^2 g_s^3}{(16\pi^2)^2} + \dots \,, \tag{2.10}$$

and $dm/d\log\mu = -m\gamma_m$, with

$$\gamma_m(g_s, e) = \gamma_m^{(0)} \frac{g_s^2}{16\pi^2} + \gamma_m^{(1)} \frac{g_s^4}{(16\pi^2)^2} + \gamma_m^{(e)} \frac{e^2}{16\pi^2} + \gamma_m^{(es)} \frac{e^2 g_s^2}{(16\pi^2)^2} + \dots \,, \tag{2.11}$$

we have [8, 18]

$$\tilde{\gamma}_{S2}^{(0)} = 4 \,, \quad \tilde{\gamma}_{S2}^{(e)} = \frac{4}{3} \,, \quad \tilde{\gamma}_{S2}^{(es)} = -\frac{148}{9} \,, \tag{2.12}$$

and

$$\gamma_m^{(0)} = 8 \,, \qquad \gamma_m^{(e)} = \frac{8}{3} \,, \qquad \gamma_m^{(es)} = \frac{32}{9} \,, \tag{2.13}$$

$$\beta_0 = 11 - \frac{2}{3}f \,, \qquad \beta_e = 0 \,, \qquad \beta_{es} = -\frac{8}{9}\left(f_u + \frac{f_d}{4}\right) \,, \tag{2.14}$$
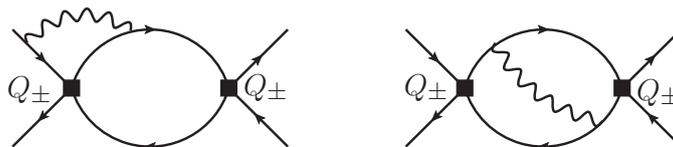
Figure 1. Sample Feynman diagrams with bilocal insertions of the current-current operators $Q^{qq'}_{\pm}$. Solid lines denote appropriate quark flavors, and wavy lines denote photons.

where f_u and f_d denote the number of up- and down-type quark flavours, and $f = f_u + f_d$. Moreover [19]

$$\gamma^{(0)} = \begin{pmatrix} 4 & 0 \\ 0 & -8 \end{pmatrix}, \quad \gamma^{(e)} = \begin{pmatrix} -\frac{8}{3} & 0 \\ 0 & -\frac{8}{3} \end{pmatrix}, \quad \gamma^{(es)} = \begin{pmatrix} \frac{107}{9} & -18 \\ -9 & \frac{38}{9} \end{pmatrix}, \quad (2.15)$$

and

$$\hat{\gamma}^{(0)}_{kl,7} = \begin{pmatrix} 3 & -1 \\ -1 & 1 \end{pmatrix}, \quad \hat{\gamma}^{(e)}_{kl,7} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \hat{\gamma}^{(es)}_{kl,7} = \begin{pmatrix} 43 & -\frac{43}{3} \\ -\frac{43}{3} & \frac{43}{3} \end{pmatrix}, \quad (2.16)$$

with an expansion defined in analogy to eq. (2.11). The result for $\hat{\gamma}^{(es)}_{kl,7}$ is new. It has been calculated in terms of the renormalization constants [20] for bilocal insertions of current-current operators (see figure 1 for sample Feynman diagrams). All diagrams have been calculated using self-written FORM [21] routines, implementing the two-loop recursion presented in refs. [22, 23]. The amplitudes were generated using qgraf [24]. We used the algorithm in ref. [25] to isolate the UV divergences.

Solving the inhomogeneous system of differential equations (2.8) and (2.9) is tedious. It is, however, straightforward to verify that eqs. (2.8) and (2.9) are equivalent to the homogeneous system of equations

$$\mu \frac{d}{d\mu} D(\mu) = D(\mu)\gamma, \quad (2.17)$$

with[1]

$$D(\mu) = \begin{pmatrix} C_+(\mu)^2 \\ C_+(\mu)C_-(\mu) \\ C_-(\mu)^2 \\ \tilde{C}_7 \end{pmatrix} \quad (2.18)$$

and

$$\gamma = \begin{pmatrix} 2\gamma_{++} & \gamma_{+-} & 0 & \hat{\gamma}_{++,7} \\ 2\gamma_{-+} & \gamma_{++} + \gamma_{--} & 2\gamma_{+-} & \hat{\gamma}_{+-,7} + \hat{\gamma}_{-+,7} \\ 0 & \gamma_{-+} & 2\gamma_{--} & \hat{\gamma}_{--,7} \\ 0 & 0 & 0 & \tilde{\gamma}_{77} \end{pmatrix}. \quad (2.19)$$

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[1]All penguin contributions to the QCD RG evolution are included in our final numerics, and have been evaluated using a straightforward generalization of these definitions. We checked explicitly that we reproduce the QCD results in the literature, up to NNLL.

This can be solved using standard methods (see, for instance, Reference [26]). The RG evolution can be conveniently written in terms of an evolution matrix, such that $D(\mu) = D(\mu_0)U(\mu_0, \mu, \alpha)$. We expand

$$U(\mu_0, \mu, \alpha) = U^{(0)}(\mu_0, \mu) + \frac{\alpha}{\alpha_s(\mu)}U^{(e)}(\mu_0, \mu) + \frac{\alpha}{4\pi}U^{(se)}(\mu_0, \mu) + \dots . \tag{2.20}$$

Here, μ_0 and μ denote the generic "high" and "low" scale of the RG evolution (i.e., $\mu_0 = \mu_t$ and $\mu = \mu_b$ in the five-flavor, and $\mu_0 = \mu_b$ and $\mu = \mu_c$ in the four-flavor theory). We find the following contributions to the Wilson coefficient at the low scale:

$$D^{(0)}(\mu) = D^{(0)}(\mu_0)U^{(0)}(\mu_0, \mu), \tag{2.21}$$

$$D^{(e)}(\mu) = D^{(0)}(\mu_0)U^{(e)}(\mu_0, \mu) + \eta^{-1}D^{(e)}(\mu_0)U^{(0)}(\mu_0, \mu), \tag{2.22}$$

$$D^{(se)}(\mu) = \eta D^{(1)}(\mu_0)U^{(e)}(\mu_0, \mu) + \eta^{-1}D^{(e)}(\mu_0)U^{(1)}(\mu_0, \mu) \\ + D^{(se)}(\mu_0)U^{(0)}(\mu_0, \mu) + D^{(0)}(\mu_0)U^{(se)}(\mu_0, \mu), \tag{2.23}$$

where we have introduced the ratio $\eta = \alpha_s(\mu_0)/\alpha_s(\mu)$. The explicit expression for the evolution matrix, in terms of the anomalous dimensions of the Wilson coefficients, can be found in ref. [27].

At the bottom threshold, $\mu_b \sim m_b$, the bottom quark is removed as a dynamical degree of freedom. Numerically, the impact of this threshold correction is small. In fact, since we neglect the contribution of penguin operators for the QED and electroweak corrections, the only effect is the decoupling of α_s from $f = 5$ to $f = 4$:

$$\alpha_s^{(5)} = \alpha_s^{(4)}\left(1 + \frac{2}{3}\frac{\alpha_s^{(4)}}{4\pi}\log\left(\mu_b^2/m_b(\mu_b)^2\right)\right), \tag{2.24}$$

leading to an additional logarithmic contribution to all Wilson coefficients. Requiring the equality of all Green's functions at the matching scale and writing $\delta C(\mu_b) = C_{f=5}(\mu_b) - C_{f=4}(\mu_b)$ we find, for the dimension-six Wilson coefficients,

$$\delta C_i^{(0)} = 0, \quad \delta C_i^{(e)} = 0, \quad \delta C_i^{(es)} = \frac{2}{3}C_i^{(e)}\log\left(\frac{\mu_b^2}{m_b(\mu_b)^2}\right), \tag{2.25}$$

and for the dimension-eight Wilson coefficient (taking into account the factor m_c^2/g_s^2 in the definition of the operator)

$$\delta\tilde{C}_7^{(0)} = 0, \quad \delta\tilde{C}_7^{(e)} = 0, \quad \delta\tilde{C}_7^{(es)} = \frac{4}{3}\tilde{C}_7^{(e)}\log\left(\frac{\mu_b^2}{m_b(\mu_b)^2}\right). \tag{2.26}$$

At the scale $\mu_c \sim m_c$ the charm quark is removed from the theory as a dynamical degree of freedom, and the effective Lagrangian is now given by eq. (2.1). Requiring the equality of the Green's functions in both the four-flavor and three-flavor theories at the charm-quark scale leads to the matching condition

$$\sum_{k,l=+,-} C_k(\mu_c)C_l(\mu_c)\langle Q_kQ_l\rangle(\mu_c) + \tilde{C}_7(\mu_c)\langle\tilde{Q}_7\rangle(\mu_c) = \frac{M_W^2}{32\pi^2}\tilde{C}_{S2}^{ut}(\mu_c)\langle\tilde{Q}_{S2}\rangle(\mu_c), \tag{2.27}$$

where the angle brackets denote the partonic $|\Delta S| = 2$ matrix elements. We parameterize these matrix elements in the following way:

$$\langle \tilde{Q}_7 \rangle = r_{S2} \langle \tilde{Q}_7 \rangle^{(0)} , \quad \langle \tilde{Q}_{S2} \rangle = r_{S2} \langle \tilde{Q}_{S2} \rangle^{(0)} , \quad \text{and} \quad \langle Q_i Q_j \rangle (\mu_c) = \frac{m_c^2(\mu_c)}{32\pi^2 M_W^2} r_{ij,S2}^{ut} \langle \tilde{Q}_{S2} \rangle^{(0)} . \tag{2.28}$$

Taking into account the explicit factor of m_c^2/g_s^2 in the definition of $\tilde{Q}_7$, we expand the Wilson coefficient $\tilde{C}_{S2}^{ut}$ as

$$\tilde{C}_{S2}^{ut} = \frac{4\pi}{\alpha_s^{(3)}} \tilde{C}_{S2}^{ut(0)} + \frac{4\pi\alpha}{(\alpha_s^{(3)})^2} \tilde{C}_{S2}^{ut(e)} + \frac{\alpha}{\alpha_s^{(3)}} \tilde{C}_{S2}^{ut(es)} + \dots , \tag{2.29}$$

and find the following contributions to the matching:

$$\tilde{C}_{S2}^{ut(0)}(\mu_c) = 2 \frac{m_c^2(\mu_c)}{M_W^2} \tilde{C}_7^{(0)}(\mu_c) , \tag{2.30}$$

$$\tilde{C}_{S2}^{ut(e)}(\mu_c) = 2 \frac{m_c^2(\mu_c)}{M_W^2} \tilde{C}_7^{(e)}(\mu_c) , \tag{2.31}$$

$$\tilde{C}_{S2}^{ut(es)}(\mu_c) = 2 \frac{m_c^2(\mu_c)}{M_W^2} \left[\tilde{C}_7^{(es)}(\mu_c) - \frac{4}{3} \tilde{C}_7^{(e)}(\mu_c) \log \frac{\mu_c^2}{m_c(\mu_c)^2} \right]$$
$$+ \frac{m_c^2(\mu_c)}{M_W^2} \left[C_i^{(0)}(\mu_c) C_j^{(e)}(\mu_c) + C_i^{(e)}(\mu_c) C_j^{(0)}(\mu_c) \right] r_{ij,S2}^{ut,(0)} . \tag{2.32}$$

Here, $m_c(\mu_c)$ denotes the running charm-quark mass, including the leading QED running. We see that for the electroweak corrections, the LO matching result is sufficient. According to ref. [3], it can be taken as $r_{ij,S2}^{ut} = 2 r_{ij,S2}^{cc} - r_{ij,S2}^{ct}$ in terms of the results in refs. [3, 9, 10]. We find

$$r_{ij,S2}^{ut,(0)} = \begin{pmatrix} \frac{9}{2} - 3\log \frac{\mu_c^2}{m_c^2(\mu_c)} & -\frac{3}{2} + \log \frac{\mu_c^2}{m_c^2(\mu_c)} \\ -\frac{3}{2} + \log \frac{\mu_c^2}{m_c^2(\mu_c)} & \frac{3}{2} - \log \frac{\mu_c^2}{m_c^2(\mu_c)} \end{pmatrix} . \tag{2.33}$$

Finally, the RG evolution in the effective three-flavor theory involves only the single physical operator $\tilde{Q}_{S2}$, with anomalous dimension given in eq. (2.12). As discussed in detail in ref. [8], the mixed two-loop anomalous dimension $\tilde{\gamma}_{S2}^{(es)}$ is renormalization-scheme independent, which prevents us from extending the definition of the scheme-independent correction factors η_{ut} to include electroweak corrections. Instead, we have to work with Wilson coefficients directly. In particular, our result for $\tilde{C}_{S2}^{ut}$ is not independent of the renormalization scheme.

Of course, this residual scheme dependence will cancel once we multiply $\tilde{C}_{S2}^{ut}$ by the hadronic matrix element of the local operator $\tilde{Q}_{S2}$, evaluated including the leading QED corrections. While this matrix element is a non-perturbative quantity, and the QED corrections are not (yet) available, it is easy to calculate the scheme-dependent part [8]. As a cross check of our calculation, we kept the definition of all contributing evanescent operators arbitrary (see appendix A) and verified that all scheme dependence completely cancels in the product of the Wilson coefficient and (the scheme-dependent part of) the
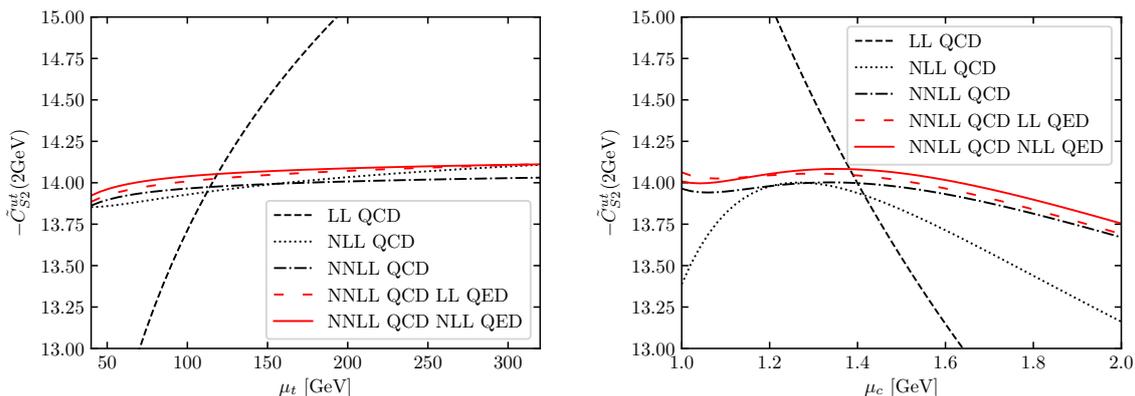
Figure 2. Residual dependence of the Wilson cofficient $\tilde{C}_{S2}^{ut}(2\,\text{GeV})$ on the electroweak (left panel) and charm-threshold (right panel) matching scales. The short-dashed, dotted, and dash-dotted lines show the LL, NLL, and NNLL QCD results, respectively. The long-dashed and solid lines show the results including also the LL and NLL electroweak corrections.

matrix element. In particular, we verified that the only left-over dependence of $\tilde{C}_{S2}^{ut}$ is on the parameter a_{11}; all other parameters cancel.[2] We will discuss the numerical size of the scheme-dependent term in the following section.

3 Discussion and conclusion

To obtain a numerical estimate of the size of the electroweak corrections, as well as an estimate of the remaining perturbative uncertainties, we evaluate the Wilson coefficient $\tilde{C}_{S2}^{ut}(2\,\text{GeV})$, including now all known QCD corrections, and varying the electroweak and charm-threshold matching scales in the intervals $40\,\text{GeV} \leq \mu_t \leq 320\,\text{GeV}$ and $1\,\text{GeV} \leq \mu_c \leq 2\,\text{GeV}$. (The dependence on the bottom-quark matching scale is negligible in comparison.) The resulting residual scale variation is displayed in figure 2.

To obtain a final value, we fix $\mu_t = m_t$ and take the average of the highest and lowest value of $\tilde{C}_{S2}^{ut}$ in the interval for the variation of μ_c, and half the difference between the highest and lowest values as the uncertainty. Retaining only the QCD corrections up to NNLL, we find $\tilde{C}_{S2}^{ut,\text{QCD}} = -13.84 \pm 0.17$. Including also the LL and NLL electroweak corrections gives $\tilde{C}_{S2}^{ut} = -13.92 \pm 0.16$. This amounts to a -0.5% shift, while the uncertainty is essentially unchanged.

What is the numerical impact of the unmatched scheme-dependent term on this result? First, recall that the only dependence is on the parameter a_{11} (see appendix A); all other scheme dependence fully cancels against the corresponding terms in the hadronic matrix

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[2]In this context we note that one of the statements made below eq. (C.12) in ref. [8] is not correct: even for scheme-independent $\tilde{\gamma}_{S2}^{(es)}$, the corresponding evolution matrix $U^{(se)}$ does depend on the renormalization scheme via its dependence on the two-loop QCD anomalous dimension, $\tilde{\gamma}_{S2}^{(1)}$. However, our analytic check shows that this dependence drops out completely in the product of Wilson coefficient and the (known) QCD part of the hadronic matrix element. The same is true in the top-quark sector.

element.[3] Part of the dependence on a_{11} of our result arises from the dependence on the two-loop QCD anomalous dimension; any such dependence is also canceled by the corresponding scheme dependence of the hadronic matrix element. The only residual dependence on a_{11} is that related to the leading QED corrections to the matrix element; numerically, it is tiny:

$$\tilde{C}_{S2}^{ut}(2\,\text{GeV}) = -14.075 + 0.001 a_{11}\,. \tag{3.1}$$

We obtained this number by setting all threshold matching scales to their respective quark masses.

While a consistent estimate of the full electroweak and QED corrections can be obtained only once a lattice calculation (or another systematic estimate) of the QED correction to the hadronic matrix element becomes available, we point out that this correction is not enhanced by a large logarithm and thus of order $\alpha/(4\pi) \sim 10^{-4}$, the same as the residual scheme dependence. It is expected to be numerically negligible compared to the -0.5% shift found above. It follows that for our numerics we can safely adopt the standard definition of evanescent operators with $a_{11} = 4$.

To summarize, given the uncancelled (but small) residual scheme dependence of our result, we propose a temporary prescription in analogy to the case of the top-quark contribution [8]: we rescale the NNLL QCD value of $\eta_{ut} = 0.402(5)$ [3] by a factor of 1.005, to take account of the electroweak corrections.[4] Including also the power correction presented in ref. [7], this leads to an updated SM prediction of

$$|\epsilon_K| = \left(2.170 \pm 0.065_{\text{pert.}} \pm 0.076_{\text{nonpert.}} \pm 0.153_{\text{param.}}\right) \times 10^{-3}\,. \tag{3.2}$$

Here, the quoted errors correspond to the residual perturbative, non-perturbative, and parametric uncertainties, respectively; see ref. [3] for details. We obtained this number by employing the phenomenological expression in ref. [18], including the long-distance corrections presented in refs. [4, 28].

All parametric inputs are taken from PDG [2]. In particular, as input for the top-quark mass we use the $\overline{\text{MS}}$ mass $m_t(m_t) = 162.92(67)\,\text{GeV}$, obtained by converting the pole mass $M_t = 172.5(7)\,\text{GeV}$ [2] to $\overline{\text{MS}}$ at three-loop accuracy using RUNDEC [28].

In summary, we calculated the leading and next-to-leading electroweak corrections to the charm-top contribution $\tilde{C}_{S2}^{ut}$ to the effective $|\Delta S| = 2$ effective Lagrangian, using RG-improved perturbation theory. We find a small negative shift of the Wilson coefficient, and a corresponding small positive shift of ϵ_K. A systematic estimate of the QED corrections to the hadronic matrix element would complete our analysis.

As consistency checks, we performed the calculation in generalized R_ξ gauge for gluons and photon and verified the gauge-parameter independence of our results. We analytically checked that our results are independent of all matching scales, and that the dependence

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[3]In practice, the hadronic matrix element is only converted to the $\overline{\text{MS}}$ scheme at NLO, such that part of the NNLO QCD scheme dependence is not included. However, in the conventional formalism of the η correction factors, the *perturbative* part of the result is scheme independent including NNLO, as far as QCD is concerned.

[4]Recall that η_{ut} is defined via $\tilde{C}_{S2}^{ut} = 2\eta_{ut}\mathscr{S}_{ut}(x_c, x_t)$, with the modified Inami-Lim function [3] being negative.

on the renormalization scheme is canceling by the corresponding scheme dependence of the hadronic matrix element.

The aim of this work is to provide a further step in the prediction of ϵ_K with residual theoretical uncertainty at the percent level. Further important directions of improvement are the calculation of the three-loop QCD corrections in the top-quark sector of the effective Lagrangian, and the NLO scheme conversion from RI/SMOM to $\overline{\text{MS}}$ for the hadronic matrix element of the local $|\Delta S| = 2$ operator.

Acknowledgments

A Definition of evanescent operators

In the context of dimensional regularization, evanescent operators arise in intermediate stages of the calculation because certain relations (such as Dirac algebra and Fierz transformations) are valid only in four space-time dimensions. In the dimension-six sector, we define them as

$$E_1^{qq'(1)} = (\bar{s}_L \gamma_{\mu_1\mu_2\mu_3} T^a q_L) \otimes (\bar{q}'_L \gamma^{\mu_1\mu_2\mu_3} T^a d_L) - (16 - A_{11}\epsilon - A_{12}\epsilon^2)Q_1^{qq'} , \tag{A.1}$$

$$E_2^{qq'(1)} = (\bar{s}_L \gamma_{\mu_1\mu_2\mu_3} q_L) \otimes (\bar{q}'_L \gamma^{\mu_1\mu_2\mu_3} d_L) - (16 - B_{11}\epsilon - B_{12}\epsilon^2)Q_2^{qq'} , \tag{A.2}$$

$$E_1^{qq'(2)} = (\bar{s}_L \gamma_{\mu_1\mu_2\mu_3\mu_4\mu_5} T^a q_L) \otimes (\bar{q}'_L \gamma^{\mu_1\mu_2\mu_3\mu_4\mu_5} T^a d_L) - \left(256 - A_{21}\epsilon - A_{22}\epsilon^2\right)Q_1^{qq'} , \tag{A.3}$$

$$E_2^{qq'(2)} = (\bar{s}_L \gamma_{\mu_1\mu_2\mu_3\mu_4\mu_5} q_L) \otimes (\bar{q}'_L \gamma^{\mu_1\mu_2\mu_3\mu_4\mu_5} d_L) - \left(256 - B_{21}\epsilon - B_{22}\epsilon^2\right)Q_2^{qq'} . \tag{A.4}$$

while the evanescent operators in the dimension-eight sector have been chosen as

$$\tilde{E}_F = \frac{m_c^2}{g^2\mu^{2\epsilon}}(\bar{s}_L^\alpha \gamma_\mu d_L^\beta) \otimes (\bar{s}_L^\beta \gamma^\mu d_L^\alpha) - \tilde{Q}_7 , \tag{A.5}$$

$$\tilde{E}_7^{(1)} = \frac{m_c^2}{g^2\mu^{2\epsilon}}(\bar{s}_L^\alpha \gamma_{\mu_1\mu_2\mu_3} d_L^\alpha) \otimes (\bar{s}_L^\beta \gamma^{\mu_1\mu_2\mu_3} d_L^\beta) - (16 - a_{11}\epsilon - a_{12}\epsilon^2)\tilde{Q}_7 , \tag{A.6}$$

$$\tilde{E}_8^{(1)} = \frac{m_c^2}{g^2\mu^{2\epsilon}}(\bar{s}_L^\alpha \gamma_{\mu_1\mu_2\mu_3} d_L^\beta) \otimes (\bar{s}_L^\beta \gamma^{\mu_1\mu_2\mu_3} d_L^\alpha) - (16 - b_{11}\epsilon - b_{12}\epsilon^2)(\tilde{Q}_7 + \tilde{E}_F) , \tag{A.7}$$

$$\tilde{E}_7^{(2)} = \frac{m_c^2}{g^2\mu^{2\epsilon}}(\bar{s}_L^\alpha \gamma_{\mu_1\mu_2\mu_3\mu_4\mu_5} d_L^\alpha) \otimes (\bar{s}_L^\beta \gamma^{\mu_1\mu_2\mu_3\mu_4\mu_5} d_L^\beta) - (256 - a_{21}\epsilon - a_{22}\epsilon^2)\tilde{Q}_7 , \tag{A.8}$$

$$\tilde{E}_8^{(2)} = \frac{m_c^2}{g^2\mu^{2\epsilon}}(\bar{s}_L^\alpha \gamma_{\mu_1\mu_2\mu_3\mu_4\mu_5} d_L^\beta) \otimes (\bar{s}_L^\beta \gamma^{\mu_1\mu_2\mu_3\mu_4\mu_5} d_L^\alpha) - (256 - b_{21}\epsilon - b_{22}\epsilon^2)(\tilde{Q}_7 + \tilde{E}_F) . \tag{A.9}$$

Note that, to facilitate an additional check on our calculation, we have kept the coefficients in front of the ϵ terms arbitrary. In the conventional definition of these operators, $A_{1i} =$

$B_{1i} = a_{1i} = b_{1i} = 4$ where $i = 1, 2$, $A_{21} = B_{21} = a_{21} = b_{21} = 224$, $a_{22} = 5712/25$, $B_{22} = 10032/25$, and $a_{22} = b_{22} = 108\,816/325$. The evanescent operators related to $\tilde{Q}_{S2}$ are defined with the same coefficients. We have checked explicitly that the terms quadratic in ϵ do not contribute to the two-loop anomalous dimensions. All results quoted in the main body of the paper correspond to the conventional definition of evanescent operators.

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