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The large-N limit of 4d superconformal indices for general BPS charges

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Abstract: We study the superconformal index of $\mathcal{N}=1$ quiver theories at large-N for general values of electric charges and angular momenta, using both the Bethe Ansatz formulation and the more recent elliptic extension method. We are particularly interested in the case of unequal angular momenta, $J_1 \neq J_2$, which has only been partially considered in the literature. We revisit the previous computation with the Bethe Ansatz formulation with generic angular momenta and extend it to encompass a large class of competing exponential terms. In the process, we also provide a simplified derivation of the original result. We consider the newly-developed elliptic extension method as well; we apply it to the $J_1 \neq J_2$ case, finding a good match with the Bethe Ansatz results. We also investigate the relation between the two different approaches, finding in particular that for every saddle of the elliptic action there are corresponding terms in the Bethe Ansatz formula that match it at large-N.

Keywords: AdS-CFT Correspondence, Black Holes in String Theory

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Contents

1 Introduction

The AdS/CFT duality offers the ideal framework to investigate the microscopical origin of the entropy carried by black holes that are asymptotically-AdS. The microstates of a black hole in the bulk correspond holographically to an ensemble of states of the dual CFT on the boundary; the black hole entropy can be determined by counting these states, and the result should be compared with the semiclassical prediction given by the Bekenstein-Hawing formula [1–4]. The first successful microstate counting of this type was obtained for a class of static dyonic BPS black holes in $\mathrm{AdS}_4 \times S^7$ [5–7], and has been followed by an extensive literature.[1]

Recently, the case of Kerr-Newman BPS black holes in AdS_5 has attracted a lot of interest. There are known BPS black hole solutions for type IIB supergravity in $\mathrm{AdS}_5 \times S^5$ which depend on two angular momenta $J_{1,2}$ and three electric charges $Q_{1,2,3}$, with a non-linear constraint among them [9–13]. In the weak-gravity / large-N limit it should be possible to reproduce the entropy of these black holes by counting the 1/16 BPS states of

[1] A comprehensive list of references can be found in the review [8].

the dual boundary theory, that is $\mathcal{N}=4$ Super Yang-Mills on $S^3 \times \mathbb{R}$. These states are counted, with a $(-1)^F$ sign, by the superconformal index [14, 15]. However, early attempts to compute the superconformal index [15–18] did not reproduce the expected $\mathcal{O}(N^2)$ growth, leading to the belief that large cancellations between fermions and bosons caused by the $(-1)^F$ sign made this approach non-viable. Recently, a solution to this puzzle has been found: when the fugacities associated to electric charges and angular momenta are extended to complex values the cancellations between fermions and bosons states are obstructed [19, 20]. At leading order the resulting expression for the superconformal index does indeed match the *entropy function* of $\mathrm{AdS}_5 \times S^5$ black holes [21], both in the large-N limit [20, 22–26] and in the limit of large conserved charges (i.e. the Cardy limit) [19, 23, 27–34]. The entropy function is the Legendre transform of the black hole entropy; it can also be written as the complexified on-shell action of the Euclidean black hole geometry [35, 36]. These results for the computation of the superconformal index have also been extended to the case of more general $\mathcal{N}=1$ gauge theories [24, 29, 32, 33, 37–43].

There are primarily two distinct methods that have been used to compute the superconformal index of $\mathcal{N}=1$ quiver theories at large-N.[2] The first one makes use of the *Bethe Ansatz formula* [44, 45], which recasts the standard integral representation of the index [14, 15, 46] as a sum over the solutions of a set of transcendental equations, the Bethe Ansatz equations (BAE). The Bethe Ansatz formula simplifies considerably in the particular case of equal angular momenta; for this reason the computations of [20, 23, 26] for $\mathcal{N}=4$ Super Yang-Mills and [37, 38] for more generic quiver theories were restricted to $J_1 = J_2$. The $J_1 \neq J_2$ case was finally addressed in [24]: a particular contribution to the Bethe Ansatz formula for the index has been shown to reproduce the entropy of Kerr-Newman BPS black holes with arbitrary charges. However, a notable limitation of [24] is that we only computed a single exponentially growing term out of the many competing ones that contribute to the Bethe Ansatz formula.

The other approach to the large-N computation of the superconformal index is the *elliptic extension* method [22, 39, 47]. It consists of a saddle point analysis of the matrix integral representation of the index, with the peculiarity that the integrand is not extended analytically outside its contour of integration; instead, it is extended to a doubly periodic function. The action of the matrix integral is thus well-defined on a torus, and a large class of saddle point solutions can be found by taking advantage of its periodicity properties. This method was pioneered in [22] for $\mathcal{N}=4$ Super Yang-Mills and later generalized to other quiver gauge theories in [39]; furthermore, a reformulation of this approach that is exact even at finite values of N has been developed in [47]. So far this type of saddle point analysis has been employed only for the case of equal angular momenta; the reason behind this technical restriction is that the modulus of the torus is taken to be equal the chemical potential of the angular momentum $J \equiv J_1 = J_2$.

The primary motivation behind the work presented in this paper is to better understand the large-N behavior of the superconformal index for general values of BPS charges,

[2]There is also the approach of [25], which considered a truncated matrix model for the index and showed that higher order corrections are numerically small.

especially in the case of unequal angular momenta, $J_1 \neq J_2$. We consider both approaches, the elliptic extension method and the Bethe Ansatz formalism, in order to provide an estimate of the large-N limit of the index. We also investigate the relation between the two methods, focusing in particular on what the saddles of the elliptic action correspond to in the Bethe Ansatz formalism.

First, we extend the saddle point analysis of [22, 39] to the $J_1 \neq J_2$ case. We achieve this by employing the same trick as [45]: we can assume without loss of generality that the angular chemical potentials are integer multiples of the same quantity, that is $\tau = a\omega$ and $\sigma = b\omega$, so that we can take advantage of the properties of the elliptic gamma functions [48] to rewrite the action as a function that is well-defined on a torus of modulus $ab\omega$. We find that the class of solutions to the saddle point equations described in [39] can be easily generalized to the $\tau \neq \sigma$ case, and we compute their effective action.

We then consider the Bethe Ansatz approach to the large-N computation of the index, proceeding as following.

- We revisit the computation of [24], which focused only on a single contribution to the Bethe Ansatz formula, and extend it to encompass a large class of competing exponential terms, finding a good match with the effective action of the elliptic saddles. Our large-N estimate of the superconformal index is thus verified in both formalisms.

- We provide a simplified derivation of the same large-N result of [24]. The most laborious step in the computation of [24] is proving that a particular simplification does not affect the large-N leading order of the index. We show how this step can be avoided altogether, provided that N and ab are coprime.

- We study the relation between the saddles of the elliptic extension method and the solutions to the Bethe Ansatz equations (BAE), with the intent to shed light on the connection between the two different approaches. We find that in the $J_1 = J_2$ case every elliptic saddle corresponds exactly to a BAE solution; however this is no longer true when $J_1 \neq J_2$, since the elliptic action and the Bethe Ansatz equations have different periodicities. Nonetheless, we show that matching elliptic saddles with holonomy configurations that contribute to the Bethe Ansatz formula is always possible, as long as the role of the auxiliary integer variables m_i present in the Bethe Ansatz formalism is taken into consideration. This matching is not always exact: sometimes the two differ by $\mathcal{O}(1/N)$ corrections, which can be shown to produce a negligible effect at leading order.

This paper is organized as follows. In section 2 we introduce the integral representation of the superconformal index of $\mathcal{N} = 1$ quiver theories and we define the elliptic extension of the integrand. In section 3 we describe the saddles of the elliptic action and compute their effective action. In section 4 we switch to the Bethe Ansatz formalism: in subsection 4.2 we study the relation between solutions of the Bethe Ansatz equations and saddles of the elliptic action, while in subsection 4.3 we evaluate the large-N limit of the contributions to the Bethe Ansatz formula that correspond to holonomy distributions that match the saddles;

lastly, in subsection 4.4 we elaborate on the relation between our results and the ones of [24]. In section 5 we provide a summary of our results and discuss some open questions.

2 The superconformal index of quiver theories

We are interested in computing the large-N limit of the superconformal index of a broad class of four dimensional $\mathcal{N}=1$ quiver gauge theories. We will focus on theories whose gauge group can be written as the direct sum of SU(N) subgroups, and with matter fields that transform in either the adjoint or the bifundamental representation. The exact field content of these theories can be summarized in the quiver diagram, a directed graph with $|G|$ nodes and n_χ arrows (oriented edges) between them, according to the following rules:

- Each node of the quiver denotes a SU(N) subgroup of the gauge group G.
- An arrow between two distinct nodes denotes a chiral multiplet in the bifundamental representation of the two SU(N) groups associated to the respective nodes.
- An arrow that has both ends attached to the same node denotes a chiral multiplet in the adjoint representation of the respective SU(N) subgroup.

In this section we briefly review some of the generalities regarding the superconformal index; we introduce the integral representation of the index and discuss how to extend its integrand to a function that is well-defined on a torus.

The superconformal index of a $\mathcal{N}=1$ field theory on $\mathbb{R}\times S^3$ is defined by the following trace [14, 15]:

$$\mathcal{I}(p,q,v_a)=\mathrm{tr}\,(-1)^F\,e^{-\beta\{\mathcal{Q},\mathcal{Q}^\dagger\}}\,p^{J_1+\frac{R}{2}}\,q^{J_2+\frac{R}{2}}\prod_{a=1}^{\mathrm{rk}(G_F)}(v_a)^{q_a}\,, \tag{2.1}$$

where $\mathcal{Q}$ is a complex supercharge, R is the generator of the R-symmetry, $J_{1,2}$ are the angular momenta relative to S^3, and lastly the q_a are the Cartan generators of the flavor symmetry group G_F. Only the BPS states of the theory that are annihilated by $\mathcal{Q}$ and $\mathcal{Q}^\dagger$ give a non-vanishing contribution to the trace; in particular this means that the index does not depend on the value of β.

The complex numbers p, q and v_a are the fugacities associated to the angular momenta and the flavor charges respectively. It is useful to define the chemical potentials τ, σ, ξ_a by

$$p=e^{2\pi i\,\tau},\qquad q=e^{2\pi i\,\sigma},\qquad v_a=e^{2\pi i\,\xi_a}\,. \tag{2.2}$$

There is a more convenient parametrization for the flavor chemical potentials ξ_a. If $I=1,\dots,n_\chi$ is an index that runs over all the chiral multiplets of the theory, we can set $\xi_I\equiv\omega_I(\xi)$, where ω_I is the weight of the representation of G_F under which the I-th chiral multiplet transforms. The new chemical potentials ξ_I are not linearly independent, they satisfy the following constraint for every superpotential term W in the Lagrangian:

$$\sum_{I\in W}\xi_I=0\,. \tag{2.3}$$

This is a consequence of the invariance of the theory under the flavor group G_F.

It is convenient to define yet another set of chemical potentials as

$$\Delta_I = \xi_I + r_I \, \frac{\tau+\sigma}{2}\,, \tag{2.4}$$

where r_I is the R-charge of the I-th chiral multiplet. Notably, the superconformal index as a function of τ, σ, Δ_I is invariant under integer shifts of its arguments.[3] For each superpotential term W in the Lagrangian we have that

$$\sum_{I\in W} \Delta_I = \tau + \sigma + n_W\,, \tag{2.5}$$

where $n_W \in \mathbb{Z}$ can be chosen arbitrarily, considering that the Δ_I are only defined up to integers. This constraint follows from (2.3) and the fact that each superpotential term must have R-charge 2:

$$\sum_{I\in W} r_I = 2\,. \tag{2.6}$$

The superconformal index has an integral representation [14, 15, 46], which for a quiver theory with SU(N) groups and matter in the adjoint and bifundamental representations takes the following form:

$$\mathcal{I} = \kappa \int [D\underline{u}] \prod_{\alpha=1}^{|G|} \prod_{i\neq j=1}^{N} \Gamma_e\Big(u_{ij}^{\alpha} + \tau + \sigma\,;\tau,\sigma\Big) \cdot \prod_{\alpha,\beta=1}^{|G|} \prod_{I_{\alpha\beta}} \prod_{i,j=1}^{N} \Gamma_e\Big(u_{ij}^{\alpha\beta} + \Delta_I\,;\tau,\sigma\Big)\,. \tag{2.7}$$

This formula is only valid when the chemical potentials are inside the domain

$$\operatorname{Im}(\tau+\sigma) > \operatorname{Im}\Delta_I > 0\,, \qquad \operatorname{Im}\tau > 0\,, \qquad \operatorname{Im}\sigma > 0\,. \tag{2.8}$$

We will only consider values of the chemical potentials within this domain throughout this paper.

The integration variables in formula (2.7) are the *holonomies* u_i^α, which parametrize the Cartan subalgebra of the gauge group; the index α runs over the $|G|$ nodes of the quiver, while the index i runs from 1 to N. For brevity we use the notation $u_{ij}^\alpha \equiv u_i^\alpha - u_j^\alpha$ and $u_{ij}^{\alpha\beta} \equiv u_i^\alpha - u_j^\beta$. Each holonomy is integrated over the interval $[\,0,1)$; the integration measure is given by

$$[D\underline{u}\,] = \frac{1}{(N!)^{|G|}} \prod_{\alpha=1}^{|G|} \delta\Big(\sum_{j=1}^{N} u_j^\alpha\Big) \prod_{i=1}^{N} du_i^\alpha\,. \tag{2.9}$$

The delta functions ensure that the matrix $\operatorname{diag}(u_1^\alpha,\dots,u_N^\alpha)$ belongs in the Cartan subalgebra of SU(N) for every value of α.

As usual $I = 1,\dots,n_\chi$ is an index that runs over the chiral multiplets of the theory. We used the symbol $\prod_{I_{\alpha\beta}}$ to denote the product over a subset of the possible values of I, specifically the subset that includes only the multiplets that in the quiver diagram are represented by an arrow that goes from the node α to the node β.

[3] In general the same is not true when the index is written as a function of τ, σ and ξ_a. The reason is that the index is not a single-valued function of the fugacities p, q and v_a, unless all the R-charges of the theory are even.

JHEP12(2022)013

The special function Γ_e that appears in the integrand of (2.7) is the elliptic gamma function [48], it is defined by

$$\Gamma_e\Big(z\,;\tau,\sigma\Big) = \prod_{j,k=0}^{\infty} \frac{1-e^{2\pi i\,\big((j+1)\tau+(k+1)\sigma-z\big)}}{1-e^{2\pi i\,\big(j\tau+k\sigma+z\big)}}\,. \tag{2.10}$$

This infinite product is convergent for $\tau,\,\sigma\in\mathbb{H}$, where $\mathbb{H}=\{z\in\mathbb{C}\mid \operatorname{Im} z>0\}$.

Lastly, the prefactor κ that appears in front of (2.7) is given by

$$\kappa = \left[\prod_{k=1}^{\infty}\Big(1-e^{2\pi ik\tau}\Big)\Big(1-e^{2\pi ik\sigma}\Big)\right]^{|G|(N-1)} \prod_{\alpha=1}^{|G|}\prod_{I_{\alpha\alpha}}\Gamma_e\Big(\Delta_I\,;\tau,\sigma\Big)^{-1}\,. \tag{2.11}$$

Since $\log\kappa=\mathcal{O}(N)$, in the large-$N$ limit this term gives a subleading contribution and can be neglected.

In the integral representation of the index, the contour of integration for the holonomies u_i^α lies exclusively on the real axis. The integrand of (2.7) can be extended analytically to the rest of the complex plane, since it is a product of elliptic gamma functions, which are meromorphic. However it is possible to consider different extensions to the complex plane; one of the key ideas behind the saddle-point approach of [22, 39] for the large-N limit of the index is to forgo the analytic extension of the integrand in favor of a doubly periodic one. Focusing exclusively on the $\tau=\sigma$ case, the authors of [22, 39] rewrote the integral representation of the index in terms of the function $Q_{c,d}(z\,;\tau)$, which is a doubly periodic function in z with periodicities $1\,,\ \tau$ that matches the elliptic gamma function on the real axis as following:

$$Q_{c,d}(x\,;\tau) = \Gamma_e(x+(c+1)\tau+d\,;\tau,\tau)^{-1}\,,\qquad \forall\,x\in\mathbb{R}\,. \tag{2.12}$$

For all $c,d\in\mathbb{R}$ the $Q_{c,d}$ function is defined by [22]

$$Q_{c,d}\,(z\,;\tau) = e^{\pi i\tau\left(\frac{c^3}{3}-\frac{c}{6}\right)}\,\frac{Q(z+c\tau+d\,;\tau)}{P(z+c\tau+d\,;\tau)^c}\,, \tag{2.13}$$

where the functions P and Q are defined by (A.17) and (A.19) respectively [49–51]. There is an ambiguity in the definition of the phase of P, Q which will play an important role in the discussion of section 3.1.

One of the goals of this paper is to extend the computation of [22, 39] to the case of unequal angular momenta. We can assume without loss of generality that the angular chemical potential τ, σ are such that τ/σ is a rational number. Indeed, the set $\{(\tau,\sigma)\in\mathbb{H}^2\mid\tau/\sigma\in\mathbb{Q}\}+\mathbb{Z}^2$ is dense in $\mathbb{H}^2$; considering that the index as a function of τ, σ, Δ_I is invariant under integer shifts and it is continuous, the value of the index for generic angular chemical potentials can be inferred from the $\tau/\sigma\in\mathbb{Q}$ case [45]. It is then useful to define $\omega\in\mathbb{H}$ and integers a, b such that

$$\tau = a\omega\,,\qquad \sigma = b\omega\,,\qquad \gcd(a,b)=1\,. \tag{2.14}$$

JHEP12(2022)013

We can now take advantage of the following gamma function identity [48]:

$$\Gamma_e(z\,;a\omega,b\omega)=\prod_{r=0}^{a-1}\prod_{s=0}^{b-1}\Gamma_e(z+(as+br)\omega\,;ab\omega,ab\omega)\,, \tag{2.15}$$

which follows from (A.3) and it allows us to write an analogue of (2.12) valid for $\tau\neq\sigma$:

$$\Gamma_e(x+(c+1)ab\omega+d\,;a\omega,b\omega)^{-1}=\prod_{r=0}^{a-1}\prod_{s=0}^{b-1}Q_{\frac{r}{a}+\frac{s}{b}+c\,,\,d}(x\,;ab\omega)\,,\qquad\forall\,x\in\mathbb{R}\,. \tag{2.16}$$

We can use this relation to rewrite the integral representation of the index (2.7) in terms of a new integrand which is doubly periodic but not meromorphic:

$$\begin{aligned}\mathcal{I}=\kappa\int[D\underline{u}]\prod_{r=0}^{a-1}\prod_{s=0}^{b-1}\Bigg[\prod_{\alpha=1}^{|G|}\prod_{i\neq j=1}^{N}Q_{\frac{r+1}{a}+\frac{s+1}{b}-1\,,\,0}\Big(u_{ij}^{\alpha}\,;ab\omega\Big)\cdot\\ \prod_{\alpha,\beta=1}^{|G|}\prod_{I_{\alpha\beta}}\prod_{i,j=1}^{N}Q_{\frac{r}{a}+\frac{s}{b}+(\Delta_I)_2-1\,,\,(\Delta_I)_1}\Big(u_{ij}^{\alpha\beta}\,;ab\omega\Big)\Bigg]^{-1},\end{aligned} \tag{2.17}$$

where $(\Delta_I)_{1,2}$ are defined by

$$\Delta_I\equiv(\Delta_I)_1+ab\omega(\Delta_I)_2\,,\qquad(\Delta_I)_{1,2}\in\mathbb{R}\,. \tag{2.18}$$

This integral representation will be the starting point of the saddle point analysis of section 3.

3 Large-N saddle points and the effective action

In this section we compute the large-N limit of quiver theories for general angular momenta by following the same saddle-point approach as [22, 39]. First, we write the matrix model (2.17) as

$$\mathcal{I}=\int[D\underline{u}]\exp\Big(-S(\underline{u})\Big)\,, \tag{3.1}$$

where the action $S(\underline{u})$ takes the following form:

$$S(\underline{u})=S_0+\sum_{\alpha=1}^{|G|}\sum_{i\neq j=1}^{N}V(u_{ij}^{\alpha},\tau+\sigma)+\sum_{\alpha,\beta=1}^{|G|}\sum_{I_{\alpha\beta}}\sum_{i,j=1}^{N}V(u_{ij}^{\alpha\beta},\Delta_I)\,. \tag{3.2}$$

Here S_0 is a constant that does not depend on the holonomies $\underline{u}$ and whose value is subleading at large-N, while the function V is defined as following:

$$V(z,\Delta)=\sum_{r=0}^{a-1}\sum_{s=0}^{b-1}\log Q_{\frac{r}{a}+\frac{s}{b}+(\Delta)_2-1\,,\,(\Delta)_1}(z\,;ab\omega) \tag{3.3}$$

Since $Q_{c,d}(z;ab\omega)$ is doubly-periodic in the variable z with periodicities 1 and $ab\omega$, so is the function V.

The saddle point equations are obtained by varying the quantity

$$S(\underline{u},\bar{\underline{u}})-\sum_{\alpha=1}^{|G|}\sum_{i=1}^{N}\left(\lambda^\alpha u_i^\alpha+\widetilde{\lambda}^\alpha\overline{u_i^\alpha}\right) \tag{3.4}$$

with respect to the holonomies $\{u_i^\alpha\}$ and their complex conjugates $\{\overline{u_i^\alpha}\}$. The quantities λ^α and $\widetilde{\lambda}^\alpha$ are Lagrange multipliers required to enforce the SU(N) constraint. We have denoted the action (3.2) as $S(\underline{u},\bar{\underline{u}})$ to stress the fact that it is not meromorphic and thus $\partial_{\overline{u_i^\alpha}}S\neq 0$. Varying with respect to u_i^α leads to the following equation:

$$\sum_{j=1}^{N}\left(\partial V(u_{ij}^\alpha,\tau+\sigma)-\partial V(u_{ji}^\alpha,\tau+\sigma)+\sum_{\beta=1}^{|G|}\sum_{I_{\alpha\beta}}\partial V(u_{ij}^{\alpha\beta},\Delta_I)-\sum_{\gamma=1}^{|G|}\sum_{I_{\gamma\alpha}}\partial V(u_{ji}^{\gamma\alpha},\Delta_I)\right)=\lambda^\alpha. \tag{3.5}$$

Here ∂V is a shorthand for $\partial_z V(z,\bar{z},\Delta)$. A similar equation with $\bar{\partial}V$ and $\widetilde{\lambda}^\alpha$ replacing ∂V and λ^α is obtained when we vary with respect to $\overline{u_i^\alpha}$.

When $a=b=1$ equation (3.5) and its analogue for $\bar{\partial}V$ match the saddle point equations discussed in [39]. A large class of solutions for the $a=b=1$ case has been found in [22, 39] using only the periodicity properties of V. When $ab\neq 1$ the expression for V becomes more complicated, but it still remains a doubly periodic function and thus the solutions known for the $a=b=1$ case can be easily generalized; we will now briefly review them.

Because of the periodicities of V the solutions to equation (3.5) live in the torus $E_T\equiv\mathbb{C}/(\mathbb{Z}+T\,\mathbb{Z})$, where $T\equiv ab\omega$. The solutions that we consider are such that $u_i^\alpha=u_i^\beta\equiv u_i$ for all α,β; the advantage of this ansatz is that equation (3.5) can now be solved simply by searching for configurations $\{u_i\}_{i=1}^N$ such that the sum

$$\sum_{j=1}^{N}\partial V(u_j-u_i,\Delta) \tag{3.6}$$

does not depend on the value of the index i. This can be achieved by taking $\{u_i\}_{i=1}^N=\mathcal{U}+\bar{u}$, where $\mathcal{U}$ is a finite subgroup of the torus E_T and $\bar{u}$ is some constant ($\bar{u}$ vanishes when we take the difference u_j-u_i). Indeed, for any $u_i\in\mathcal{U}$ we have that $\{u-u_i\}_{u\in\mathcal{U}}$ and $\mathcal{U}$ are the same set, and thus the following sum does not actually depend on the value of u_i:

$$\sum_{u\in\mathcal{U}}\partial V(u-u_i,\Delta)=\sum_{u\in\mathcal{U}}\partial V(u,\Delta)\,. \tag{3.7}$$

Thus equation (3.5) is solved by the choice $\{u_i\}_{i=1}^N=\mathcal{U}+\bar{u}$, and the same is true for the analogue equation for $\bar{\partial}V$. In particular this means that these solutions to the saddle point equations can be classified by homomorphisms of abelian groups of order N into the torus E_T [39].

Any abelian group G of order N is isomorphic to a product of cyclic groups:

$$G\cong(\mathbb{Z}\,/\,N_1\,\mathbb{Z})\times\ldots\times(\mathbb{Z}\,/\,N_\ell\,\mathbb{Z})\,, \tag{3.8}$$

JHEP12(2022)013

where $N_1 \dots N_\ell = N$. Furthermore, we can assume without loss of generality that each N_i is a divisor of N_{i-1},[4] which we write compactly as $N_i \mid N_{i-1}$. The most general homomorphism of the cyclic group of order N into the torus can be written as

$$i \mapsto \frac{i}{N}(m\,T + n)\,, \qquad i \in \mathbb{Z}\,/\,N\,\mathbb{Z}\,, \tag{3.9}$$

for some $m, n \in \mathbb{Z}$. Hence, the most general saddle point configuration that corresponds to a finite group homomorphism in the torus takes the following form:

$$u^{\alpha}_{i_1 \dots i_\ell} = \frac{i_1}{N_1}(m_1\,T + n_1) + \dots + \frac{i_\ell}{N_\ell}(m_\ell\,T + n_\ell) + \bar{u}\,, \tag{3.10}$$

where $N_1 \dots N_\ell = N$ and $N_i \mid N_{i-1}$. The value for the constant $\bar{u}$ is chosen so that the SU(N) constraint is satisfied:

$$\sum_{i=1}^{N} u_i^{\alpha} = 0\,. \tag{3.11}$$

Since (3.2) only depends on differences between holonomies $\bar{u}$ ultimately cancels out in all the relevant equations. From now on we will omit $\bar{u}$ completely.

We note that different choices of integers $\{N_i, m_i, n_i\}_{i=1}^{\ell}$ may lead to equivalent solutions, that is solutions that match under the periodicities of the torus E_T or permutations of the index i of u_i^α. As an example, (3.10) is invariant under $u_i^\alpha \mapsto -u_i^\alpha$, or equivalently $\{m_i, n_i\}_{i=1}^{\ell} \mapsto \{-m_i, -n_i\}_{i=1}^{\ell}$. For this reason we can assume without loss of generality that $m_1 \geq 0$.

3.1 Contour deformation

It is not sufficient for the saddles (3.10) to be stationary points of the action (3.2) for them to contribute to the integral representation of the index (2.17); it is necessary for the contour of integration to pass through the saddle point as well. There is a problem: the integrand of (2.17) is not meromorphic, and thus it is not possible to use the Cauchy theorem to change contour. An alternative procedure for the deformation of the contour has been used in [22, 39] for the analysis of the $\tau = \sigma$ case; in this section we will show that it can be adapted to the $\tau \neq \sigma$ case as well.

In both integral representations of the superconformal index, (2.7) and (2.17), each holonomy variable u_i^α is integrated over the interval $[\,0, 1)$. In the case of (2.7) the integrand is meromorphic and we are free to deform the contour as long as we don't cross any poles; however the saddles (3.10) are only stationary points of the integral representation with a doubly periodic integrand (2.17). The key insight of [22, 39] is that the integrands of (2.7) and (2.17) are equal when evaluated on any given saddle, as long as the phase of the $Q_{c,d}$ function is chosen appropriately. The idea is to deform the contour of integration of the meromorphic integrand to one that passes thought the saddle point, and then show

[4]This is due to the fact that $(\mathbb{Z}\,/\,m\,\mathbb{Z}) \times (\mathbb{Z}\,/\,n\,\mathbb{Z}) \cong (\mathbb{Z}\,/\,nm\,\mathbb{Z})$ if $\gcd(m, n) = 1$, hence there are multiple factorizations $N = N_1 \times \dots \times N_\ell$ that, up to isomorphisms, define the same abelian group G, and it is always possible to find one that satisfies $N_i \mid N_{i-1}\ \forall\, i$ [39].

that the meromorphic integrand can be substituted with the doubly periodic one up to subleading corrections.

In order to show that the argument of [22, 39] can be adapted the $\tau \neq \sigma$ case we only need to check that the integrand of (2.7), which is a product of elliptic gamma functions Γ_e, and the integrand of (2.17), which depends on the $Q_{c,d}$ function and the choice of its phase, match when the holonomies u_i^α take (3.10) as their value.

When z is real the functions $Q_{c,d}(z;\tau)$ and $\Gamma_e(z+(c+1)\tau+d\,;\tau,\tau)^{-1}$ match exactly; otherwise their relation is given by the following formula [22], obtained by substituting (A.19) in (2.13):

$$Q_{c,d}(z;\tau) = e^{2\pi i\,\alpha_Q(z+c\tau+d)}\, e^{-2\pi i\tau A_c(z_2)}\, \frac{P(z+c\tau+d\,;\tau)^{z_2}}{\Gamma_e(z+(c+1)\tau+d\,;\tau,\tau)}\,. \tag{3.12}$$

Here $z_{1,2}\in\mathbb{R}$ are defined by $z \equiv z_1+\tau\, z_2$. The phase of $Q_{c,d}$ depends on the particular choice for the real-valued function α_Q. Apart from the constraint $\alpha_Q(x)=0\ \forall\, x\in\mathbb{R}$, α_Q can be chosen arbitrarily in the fundamental domain $0\le z_{1,2}<1$; its value on the rest of the complex plain is then fixed by the requirement that $Q_{c,d}(z;\tau)$ must be doubly periodic in z with periods 1, τ.

The rest of this subsection will be dedicated to showing that the integrands of (2.7) and (2.17) are equal in absolute value when evaluated on any given saddle. It is then possible to choose α_Q appropriately so that the integrands match in phase as well, and thus the contour deformation argument of [22, 39] can also be applied to the $\tau\neq\sigma$ case.

The function A_c that appears in (3.12) denotes the following cubic polynomial:

$$A_c(x) = \tfrac{1}{6}x^3+\tfrac{1}{2}cx^2+\tfrac{1}{2}c^2x-\tfrac{1}{12}x\,. \tag{3.13}$$

We can show that the total contribution of A_c to the integrand of (2.17) vanishes when evaluated at the saddle points, that is

$$\sum_{r,s}\sum_{i,j}\left[|G|\,A_{\frac{r+1}{a}+\frac{s+1}{b}-1}\big((u_{ij})_2\big)+\sum_I A_{\frac{r}{a}+\frac{s}{b}+(\Delta_I)_2-1}\big((u_{ij})_2\big)\right]\Bigg|_{u_i\text{ as in (3.10)}}=0\,. \tag{3.14}$$

First, we note that the odd powers of x in $A_c(x)$ vanish when we sum over i,j since $(u_{ij})_2$ changes sign when i and j are exchanged. This leaves only the quadratic term in x, which is proportional to c; when we sum over r,s and all the multiplet contributions the c-terms vanish:

$$\begin{aligned}\sum_{r=0}^{a-1}\sum_{s=0}^{b-1}&\left[|G|\left(\frac{r+1}{a}+\frac{s+1}{b}-1\right)+\sum_I\left(\frac{r}{a}+\frac{s}{b}+(\Delta_I)_2-1\right)\right]\\ &=\frac{a+b}{2}\left[|G|+\sum_I\left(\frac{2ab(\Delta_I)_2}{a+b}-1\right)\right]=0\,.\end{aligned} \tag{3.15}$$

The term in the square bracket in the second line can be shown to be vanishing by imposing the $\mathrm{U}(1)_\mathrm{R}$ - gauge^2 anomaly cancellation condition, which for the quiver theories that we are considering can be written as following:[5]

$$|G| + \sum_I (\widetilde{r}_I - 1) = 0\,. \tag{3.16}$$

This relation is valid for *any* R-symmetry. Then (3.15) follows from (3.16) if we consider the R-symmetry obtained by assigning the following charges to each chiral multiplet:

$$\widetilde{r}_I \equiv \frac{2ab(\Delta_I)_2}{a+b}\,. \tag{3.17}$$

Because of relation (2.5) this choice of R-charges does indeed satisfy

$$\sum_{I\in W} \widetilde{r}_I = 2 \tag{3.18}$$

for every superpotential term W in the Lagrangian.

The contribution of $\log|P|$ to the integrand is vanishing as well:

$$\sum_{i,j=1}^{N} (u_{ij})_2 \, \log \Big| P(u_{ij} + (c+1) + d\,;\tau) \Big| = 0\,. \tag{3.19}$$

This relation can be derived from the double Fourier expansion of $\log|P|$ (A.18) and the fact that sums of the following type vanish:

$$\sum_{i_k,j_k=1}^{N_k} (i_k - j_k)\, e^{2\pi i \left(\frac{i_k - j_k}{N_k}\,(m_k n - n_k m)\right)} = 0\,. \tag{3.20}$$

Since the contribution of the A_c and $\log|P|$ terms is overall zero, from (3.12) we see that the integrands of (2.7) and (2.17) are equal in absolute value on the saddles, which is what we needed to show.

We conclude this subsection by mentioning that in [47] a more rigorous framework for this type of saddle point analysis has been presented, based on Atiyah-Bott-Beligne-Vergne equivariant integration formula [52–56]. The method of [47] is also applicable at finite N, and it provides more solid evidence for the fact that the (3.10) saddles do indeed contribute to index.

3.2 Continuum limit

In the large-N limit the saddles (3.10) become uniform continuous distributions. We can make the substitutions

$$u_i^\alpha \longmapsto u^\alpha(x)\,, \qquad \sum_{i=1}^{N} \longmapsto N \int_0^1 dx \tag{3.21}$$

[5]Condition (3.16) is equivalent to the statement that $\operatorname{tr} \widetilde{R} = \mathcal{O}(1)$ at large-N, for any R-symmetry $\widetilde{R}$. For more details we refer to appendix B of [39].

and replace the discrete action (3.2) with a large-N effective action $S_{\text{eff}}[u]$, which is a functional of the distribution $u^\alpha(x)$ and is given by

$$S_{\text{eff}}[u] = N^2 \int_0^1 dx \int_0^1 dy \left[\sum_{\alpha=1}^{|G|} V\left(u^\alpha(x) - u^\alpha(y)\,, \tau+\sigma\right) + \sum_{\alpha,\beta=1}^{|G|} \sum_{I_{\alpha\beta}} V\left(u^\alpha(x) - u^\beta(y), \Delta_I\right) \right]. \tag{3.22}$$

The stationary points of this action can be found by extremising the functional

$$S_{\text{eff}}[u] - \sum_{\alpha=1}^{|G|} \int_0^1 dx \Big(\lambda^\alpha u^\alpha(x) + \widetilde{\lambda}^\alpha \overline{u^\alpha(x)} \Big)\,, \tag{3.23}$$

and correspond to the continuum limit of the discrete saddles (3.10). The superconformal index at large-N can then be written as a sum over these stationary points:

$$\mathcal{I} \sim \sum_{u \in \{\text{saddles}\}} \exp\left(- S_{\text{eff}}[u] \right). \tag{3.24}$$

In order to take the continuum limit of the saddles (3.10) we need to distinguish between a few cases. Each saddle depends on a particular factorization of N, that is $N \equiv N_1 \dots N_\ell$ with $N_i \mid N_{i-1}\ \forall\, i$:

$$u^\alpha_{i_1 \dots i_\ell} = \frac{i_1}{N_1}(m_1 T + n_1) + \dots + \frac{i_\ell}{N_\ell}(m_\ell T + n_\ell)\,. \tag{3.25}$$

Hence, the $N \to \infty$ limit can be realized in multiple ways.

Let us consider the case of saddles with $\ell = 1$ first. After the substitution $i_1/N_1 \mapsto x$ they become

$$u^\alpha(x) = x\,(m T + n)\,. \tag{3.26}$$

We omitted the subscript on m_1 and n_1 as it is no longer needed. The effective action for these saddles can be written as

$$\begin{aligned} S_{\text{eff}}(m,n) &= N^2 \int_0^1 dx\, dy \left[|G|\, V\Big((x-y)(mT+n), \tau+\sigma\Big) + \sum_I V\Big((x-y)(mT+n), \Delta_I\Big) \right] \\ &= N^2 \int_0^1 dx \left[|G|\, V\Big(x(mT+n), \tau+\sigma\Big) + \sum_I V\Big(x(mT+n), \Delta_I\Big) \right]. \end{aligned} \tag{3.27}$$

The second equality follows from the fact that $mT+n$ is a period of V. Another consequence of the periodicity of V is that $S_{\text{eff}}(m,n) = S_{\text{eff}}(m/h, n/h)$, where $h \equiv \gcd(m,n)$. This is expected, considering that the (m,n) saddle describes a distribution of holonomies equivalent to the one of the $(m/h, n/h)$ saddle. Thus for the $\ell = 1$ "string-like" saddles we can assume that $\gcd(m,n) = 1$ without loss of generality. We postpone the computation of $S_{\text{eff}}(m,n)$ to section 3.3.

We consider the $\ell = 2$ saddles now. Let us first assume that $N_2 \sim \mathcal{O}(1)$ at large-N. We can make the substitution $i_1/N_1 \mapsto x$ and write the $\ell = 2$ saddles in the continuum limit as

$$u^\alpha_{i_2}(x) = x\,(m_1 T + n_1) + \frac{i_2}{N_2}(m_2 T + n_2)\,. \tag{3.28}$$

If we want to write the saddle without the extra index i_2 we can change variables to $x_{\rm new} \equiv x/N_2 + i_2/N_2$ so that

$$u^\alpha(x) = \{N_2 x\}\,(m_1\,T + n_1) + \frac{\lfloor N_2\,x \rfloor}{N_2}\,(m_2\,T + n_2)\,, \tag{3.29}$$

where $\{N_2\,x\} \equiv N_2\,x - \lfloor N_2\,x \rfloor$. It is straightforward to see that (3.29) extremises the effective action (3.22) for any value of N_2. For convenience we will use representation (3.28) and keep the index i_2; the effective action is then given by

$$\begin{aligned} S_{\rm eff}(m_1,n_1;m_2,n_2,N_2) = \frac{N^2}{N_2}\sum_{i_2=1}^{N_2}\int_0^1 dx\,\Bigg[|G|\,V\left(x(m_1T+n_1) + \frac{i_2}{N_2}(m_2\,T+n_2),\tau+\sigma\right) \\ +\sum_I V\left(x(m_1T+n_1) + \frac{i_2}{N_2}(m_2\,T+n_2),\Delta_I\right)\Bigg]. \end{aligned} \tag{3.30}$$

Again, without loss of generality we can assume that $\gcd(m_1,n_1) = \gcd(m_2,n_2) = 1$. We postpone the computation of (3.30) to section 3.4.

If we take $N_2 \to \infty$ in (3.28) we obtain the "surface" saddles:

$$u^\alpha(x,y) = x\,(m_1\,T + n_1) + y\,(m_2\,T + n_2)\,. \tag{3.31}$$

The effective action for these saddles is the same as (3.30), provided that the following substitutions are made:

$$\frac{i_2}{N_2} \longmapsto y\,, \qquad \frac{1}{N_2}\sum_{i_2=1}^{N} \longmapsto \int_0^1 dy\ . \tag{3.32}$$

Because of the periodicity of the potential V, as long as (m_1,n_1) and (m_2,n_2) are linearly independent the effective action of surface saddles does not depend on any of these integers:

$$\int_0^1 dx\int_0^1 dy\,V\Big(x(m_1T+n_1)+y(m_2T+n_2),\Delta\Big) = \int_0^1 dx\int_0^1 dy\,V\Big(x\,T+y\,,\Delta\Big)\,. \tag{3.33}$$

On the other hand if (m_1,n_1) and (m_2,n_2) are linearly dependent the saddle (3.31) is just equivalent to one of the "string-like" saddles (3.26).

The saddles with $\ell \geq 3$ in the continuum limit are always equivalent to one of the already discussed cases, (3.26), (3.28) or (3.31). To see why, let us fist assume that $m_1 \neq 0$. We can rewrite the $\ell = 2$ saddle (3.28) by shifting $x \mapsto x - (i_2/N_2)(m_2/m_1)$, obtaining the following equivalent expression:

$$u^\alpha_{i_2}(x) = x\,(m_1\,T + n_1) + \frac{i_2}{N_2}\,\frac{m_1 n_2 - m_2 n_1}{m_1}\,. \tag{3.34}$$

Similarly, a generic saddle with $\ell = 3$ and $m_1 \neq 0$ after the $i_1/N_1 \mapsto x$ substitution and analogue shifts can be written as

$$u^\alpha_{i_2,i_3}(x) = x\,(m_1\,T + n_1) + \frac{i_2}{N_2}\,\frac{m_1 n_2 - m_2 n_1}{m_1} + \frac{i_3}{N_3}\,\frac{m_1 n_3 - m_3 n_1}{m_1}\,. \tag{3.35}$$

Considering that $N_3 \mid N_2$ and $\gcd(m_1, n_1) = 1$, it is always possible to find $(\widetilde{m}_2, \widetilde{n}_2)$ such that the $\ell = 2$ saddle with $m_1, n_1, \widetilde{m}_2, \widetilde{n}_2, N_2$ is equivalent to (3.35). The $m_1 = 0$ case is similar: the saddle

$$u^{\alpha}_{i_2,i_3}(x) = x + \frac{i_2}{N_2}(m_2 T + n_2) + \frac{i_3}{N_3}(m_3 T + n_3) \tag{3.36}$$

can be rewritten as

$$u^{\alpha}_{i_2,i_3}(x) = x + \frac{i_2}{N_2} m_2 T + \frac{i_3}{N_3} m_3 T \tag{3.37}$$

by shifting $x \mapsto x - (i_2/N_2)n_2 - (i_3/N_3)n_3$; it is then always possible to find $\widetilde{N}_2$ such that the saddle is equivalent to

$$u^{\alpha}_{\widetilde{i}_2}(x) = x + \frac{\widetilde{i}_2}{\widetilde{N}_2} T\,. \tag{3.38}$$

In conclusion, there is no need to consider saddles with $\ell \geq 3$ in the continuum limit. This argument does not hold at finite N however; we will discuss the saddles (3.10) at finite N in more detail in section 4.2.

3.3 String-like saddles

In this section we focus of the saddles $u^{\alpha}(x) = x(mT + n)$ and compute their effective action $S_{\text{eff}}(m, n)$. Without loss of generality we can assume that $\gcd(m, n) = 1$ and $m \geq 0$. Given (3.3) and (3.27), the effective action of these saddles takes the following form:

$$\begin{aligned} S_{\text{eff}}(m,n) = N^2 \int_0^1 dx \sum_{r=0}^{a-1}\sum_{s=0}^{b-1} \Bigg[& |G| \log Q_{\frac{r+1}{a}+\frac{s+1}{b}-1\,,\,0}\Big(x(mab\omega + n)\,; ab\omega\Big) \\ & + \sum_I \log Q_{\frac{r}{a}+\frac{s}{b}+(\Delta_I)_2-1\,,\,(\Delta_I)_1}\Big(x(mab\omega+n)\,; ab\omega\Big)\Bigg]\,. \end{aligned} \tag{3.39}$$

When $m \neq 0$ the integral can be computed using formula (A.22), which we can write as

$$\int_0^1 dx \log Q_{c,d}\Big(x(m\tau + n)\,;\tau\Big) = -\frac{\pi i}{6} c\tau + \frac{\pi i}{3}\frac{B_3([m(c\tau+d)]'_{m\tau+n})}{m(m\tau+n)^2} + \Big(\text{purely imaginary}\Big)\,, \tag{3.40}$$

where the function $[\,\cdot\,]'_\tau$ is defined as follows:

$$[x + y\tau]'_\tau = \begin{cases} x - \lfloor x \rfloor + y\tau & \text{for } x \in \mathbb{R} \setminus \mathbb{Z},\ y \in \mathbb{R} \\ \text{either } \ y\tau \ \text{ or } \ y\tau + 1 & \text{for } x \in \mathbb{Z},\ y \in \mathbb{R} \end{cases}\,. \tag{3.41}$$

There is an ambiguity in the definition of $[z]'_\tau$ when $z \in \mathbb{Z} + \tau\,\mathbb{R}$; however, because of property (A.13) of the Bernoulli polynomials and the fact that $B_n(0) = B_n(1)$, one can see that equation (3.40) is unaffected by this ambiguity. The purely imaginary terms left out from equation (3.40) do not actually contribute to the large-N leading order of the effective action, considering that S_{eff} is defined up to multiples of $2\pi i$.

Using (3.40) we find the contribution of a single multiplet to the effective action:

$$
\begin{aligned}
&N^2 \sum_{r=0}^{a-1}\sum_{s=0}^{b-1}\int_0^1 dx\, \log Q_{\frac{r}{a}+\frac{s}{b}+(\Delta)_2-1\,,\,(\Delta)_1}\Big(x(mab\omega+n)\,;ab\omega\Big) \qquad (3.42)\\
&= -\frac{\pi i}{6}\, abN^2\left(ab\omega(\Delta)_2-\frac{\tau+\sigma}{2}\right)+\pi i N^2\sum_{r=0}^{a-1}\sum_{s=0}^{b-1}\frac{B_3([m\Delta+m\omega(as+br-ab)]'_{mab\omega+n})}{3m(mab\omega+n)^2},
\end{aligned}
$$

where $\Delta\equiv\tau+\sigma$ for vector multiplets and $\Delta\equiv\Delta_I$ for the I-th chiral multiplet. When we sum over all multiplet contributions the first term in the second line of (3.42) gives an overall null contribution because of anomaly cancellation relations; it is indeed the same (3.15) term that we discussed in section 3.1, up to a proportionality constant. The effective action for (m,n) saddles with $m\neq 0$ can thus be written as

$$
S_{\text{eff}}(m,n)=\pi i N^2\Bigg(|G|\,\Psi_{m,n}(\tau+\sigma)+\sum_I \Psi_{m,n}(\Delta_I)\Bigg)\,, \qquad (3.43)
$$

where $\Psi_{m,n}(\Delta)$ denotes the following quantity:

$$
\Psi_{m,n}(\Delta)=\sum_{r=0}^{a-1}\sum_{s=0}^{b-1}\frac{B_3([m\Delta+m\omega(as+br-ab)]'_{mab\omega+n})}{3m(mab\omega+n)^2}\,. \qquad (3.44)
$$

As a simple check, we notice that (3.44) is invariant under $(m,n)\mapsto(-m,-n)$, as expected. Using that $[-z]'_\tau=1-[z]'_\tau$ and property (A.11) of the Bernoulli polynomials, we can see that under $(m,n)\mapsto(-m,-n)$ the numerator of the summand in (3.44) changes sign; since the denominator changes sign as well, (3.44) is indeed invariant.

When $a=b=1$ we have $\tau=\sigma=\omega$ and the effective action (3.43) matches the analogous result obtained in [39]. It is also in accord with the results [20, 23, 26, 37, 38] derived from the Bethe Ansatz formula. As for the $a\neq b$ case, the effective action of the $(m,n)=(1,0)$ saddle matches perfectly the contribution to the index computed in [24] using the Bethe Ansatz formalism; we will discuss in more detail the relation between the saddle point and the Bethe Ansatz approaches in section 4.2.

As already noted in [24], expression (3.44) can be simplified significantly when $(m,n)=(1,0)$. Using the translation property of the Bernoulli polynomials (A.13) it is possible to write $\Psi_{1,0}(\Delta)$ as

$$
\begin{aligned}
\Psi_{1,0}(\Delta)&=\frac{1}{3(ab\omega)^2}\sum_{r=0}^{a-1}\sum_{s=0}^{b-1}B_3\Big([\Delta]'_\omega+\omega(as+br-ab)\Big) \qquad (3.45)\\
&=\frac{1}{3}\sum_{r=0}^{a-1}\sum_{s=0}^{b-1}\sum_{k=0}^{3}\binom{3}{k}(ab\omega)^{k-2}\left(\frac{r}{a}+\frac{s}{b}+\frac{a+b}{2ab}-1\right)^k B_{3-k}\left([\Delta]'_\omega-\frac{\tau+\sigma}{2}\right).
\end{aligned}
$$

The sum over r and s can now be easily computed by means of a simple trick; we consider the following power series

$$\sum_{k=0}^{\infty}\sum_{r=0}^{a-1}\sum_{s=0}^{b-1}\left(\frac{r}{a}+\frac{s}{b}+\frac{a+b}{2ab}-1\right)^k\frac{(2t)^k}{k!}=\sum_{r=0}^{a-1}\sum_{s=0}^{b-1}e^{2t\left(\frac{r}{a}+\frac{s}{b}+\frac{a+b}{2ab}-1\right)}=\frac{\sinh^2 t}{\sinh\frac{t}{a}\sinh\frac{t}{b}}$$
$$=ab+\frac{t^2}{6}\left(2ab-\frac{a}{b}-\frac{b}{a}\right)+O(t^4)\,, \tag{3.46}$$

which gives us immediately the relations that we need:

$$\sum_{r=0}^{a-1}\sum_{s=0}^{b-1}\left(\frac{r}{a}+\frac{s}{b}+\frac{a+b}{2ab}-1\right)^k=\begin{cases}ab & \text{for }\; k=0\\ 0 & \text{for }\; k=1\\ \frac{1}{12}\left(2ab-\frac{a}{b}-\frac{b}{a}\right) & \text{for }\; k=2\\ 0 & \text{for }\; k=3\end{cases} \tag{3.47}$$

Substituting (3.47) in (3.45) we get

$$\Psi_{1,0}(\Delta)=\frac{1}{3\tau\sigma}B_3\left([\Delta]'_\omega-\frac{\tau+\sigma}{2}\right)+\frac{1}{12}\left(2ab-\frac{a}{b}-\frac{b}{a}\right)B_1\left([\Delta]'_\omega-\frac{\tau+\sigma}{2}\right). \tag{3.48}$$

When we sum over all the multiplets, the total contribution to the effective action $S_{\text{eff}}(1,0)$ coming from the B_1 terms is purely imaginary and at leading N^2 order can be neglected. Indeed the ω-dependent part of the B_1 term gives a total contribution proportional to the term in the second line of (3.15). Therefore, we can equivalently define the function $\Psi_{1,0}(\Delta)$ as

$$\Psi_{1,0}(\Delta)\equiv\frac{1}{3\tau\sigma}B_3\left([\Delta]'_\omega-\frac{\tau+\sigma}{2}\right). \tag{3.49}$$

The disappearance of the term proportional to $2ab$ is not surprising considering that the index is ultimately a continuous function of $\tau=a\omega$ and $\sigma=b\omega$.

The $(m,n)=(0,1)$ saddle. So far we have assumed $m\neq 0$; let us now discuss the $m=0$ case. The requirement $\gcd(m,n)=1$ only leaves $n=\pm 1$ as possible choices, and they are equivalent; hence, there is only one saddle with $m\neq 0$. As we will now show, the effective action of this saddle is zero at the leading N^2 order, which is coherent with the results obtained in [20, 22, 39] for the $\tau=\sigma$ case.

For the $(m,n)=(0,1)$ saddle the $\{u_i^\alpha\}$ are all real and thus the doubly periodic function $Q_{c,d}$ simply coincides with the analytic elliptic gamma, and thus the action $S(\underline{u})$ given by (3.2) and (3.3) is just minus the logarithm of the integrand of (2.7). We find it easier in this case to work with the elliptic gamma functions directly rather than the $Q_{c,d}$.

First, let us look at the contribution to the effective action of the $(m,n)=(0,1)$ saddle coming from a chiral multiplet. Using the property (A.4) and the definition (A.1) of the elliptic gamma function we can write it as

$$-\sum_{i,j=1}^{N}\log\Gamma_e\Big(\Delta_I+\frac{i-j}{N}\,;a\omega,b\omega\Big)=-N\log\Gamma_e\Big(N\Delta_I\,;Na\omega,Nb\omega\Big) \tag{3.50}$$
$$=N\sum_{j,k=0}^{\infty}\left[\log\left(1-e^{2\pi i\,N\left(ja\omega+kb\omega+\Delta_I\right)}\right)-\log\left(1-e^{2\pi i\,N\left((j+1)a\omega+(k+1)b\omega-\Delta_I\right)}\right)\right].$$

If either $\left((j+1)a\omega+(k+1)b\omega-\Delta_I\right)$ or $\left(ja\omega+kb\omega+\Delta_I\right)$ had a negative imaginary part the respective logarithm term would be $\mathcal{O}(N)$ and we would get nonzero contributions at the N^2 order. However in the domain (2.8) the imaginary part of these terms is always positive and at large-N all the logarithms are exponentially suppressed. Hence, the chiral multiplet contribution is null at the N^2 order.

The contribution to the effective action coming from the vector multiplets is subleading as well. We can write it as

$$-|G|\sum_{i\neq j=1}^{N}\log\Gamma_e\Big(a\omega+b\omega+\frac{i-j}{N};a\omega,b\omega\Big)=-N|G|\sum_{\ell=1}^{N-1}\log\Gamma_e\Big(a\omega+b\omega+\frac{\ell}{N};a\omega,b\omega\Big)\,. \tag{3.51}$$

This term is of order $\mathcal{O}(N\log N)$. Indeed, if we substitute the definition (A.1) of the elliptic gamma function in the following product

$$\begin{aligned}&\prod_{\ell=1}^{N-1}\Gamma_e\Big(a\omega+b\omega+\frac{i}{N};a\omega,b\omega\Big) \\ &\quad=\prod_{\ell=1}^{N-1}\left[\Big(1-e^{-2\pi i\frac{\ell}{N}}\Big)\Big(1-e^{2\pi i\left(a\omega-\frac{\ell}{N}\right)}\Big)\Big(1-e^{2\pi i\left(b\omega-\frac{\ell}{N}\right)}\Big)\prod_{j,k=1}^{\infty}\frac{1-e^{2\pi i\left(a\omega+b\omega-\frac{\ell}{N}\right)}}{1-e^{2\pi i\left(a\omega+b\omega+\frac{\ell}{N}\right)}}\right],\end{aligned} \tag{3.52}$$

then we can use a slight modification of identity (A.5),

$$\prod_{\ell=1}^{N-1}\Big(1-e^{-2\pi i\frac{\ell}{N}}\,z\Big)=\frac{1-z^N}{1-z}=1+z+\ldots+z^{N-1}\,, \tag{3.53}$$

to conclude that

$$\prod_{\ell=1}^{N-1}\Gamma_e\Big(a\omega+b\omega+\frac{i}{N};a\omega,b\omega\Big)=N\ \frac{1-e^{2\pi iNa\omega}}{1-e^{2\pi ia\omega}}\ \frac{1-e^{2\pi iNb\omega}}{1-e^{2\pi ib\omega}}=\mathcal{O}(N)\,, \tag{3.54}$$

and thus (3.51) does not contribute to the leading N^2 order either.

3.4 General saddles

In section 3.3 we considered the particular case of the $u^\alpha(x)=x(mT+n)$ saddles; we will now evaluate the effective action of the other saddles discussed in section 3.2. Other than the surface saddles (3.31), in the continuum limit the only type of saddles that we still need to account for are the "two-factor" saddles (3.28), whose effective action $S_{\text{eff}}(m_1,n_1;m_2,n_2,N_2)$ is given by (3.30). We start from the two-factor saddles and postpone the discussion about surface saddles at the end of this section.

We will assume that $m_1\neq 0$; without loss of generality we can take $m_1>0$ and $\gcd(m_1,n_1)=\gcd(m_2,n_2)=1$. The contribution to the effective action coming from a single multiplet is given by the following expression:

$$\frac{N^2}{N_2}\sum_{i_2=1}^{N_2}\int_0^1 dx\sum_{r=0}^{a-1}\sum_{s=0}^{b-1}\log Q_{\frac{r}{a}+\frac{s}{b}+(\Delta)_2-1,\,(\Delta)_1}\Big(x(m_1ab\omega+n_1)+\frac{i_2}{N_2}(m_2ab\omega+n_2);ab\omega\Big)\,, \tag{3.55}$$

where as usual Δ is equal to $\tau+\sigma$ for vector multiplets and to Δ_I for the I-th chiral multiplet. In order to compute (3.55) we first generalize formula (3.40) to include the sum over the new index i_2. Using (A.22) and ignoring purely imaginary terms we find that

$$\begin{aligned}
&\frac{1}{N_2}\sum_{i_2=1}^{N_2}\int_0^1 dx\log Q_{c,d}\Big(x(m_1\tau+n_1)+\frac{i_2}{N_2}(m_2\tau+n_2)\,;\tau\Big)\\
&\quad=-\frac{\pi i}{6}\,c\tau+\frac{\pi i}{3}\,\frac{1}{N_2}\sum_{i_2=1}^{N_2}\frac{B_3\left(\{m_1d-n_1c+\frac{i_2}{N_2}(m_1n_2-m_2n_1)\}+c(m_1\tau+n_1)\right)}{m_1(m_1\tau+n_1)^2}\\
&\quad=-\frac{\pi i}{6}\,c\tau+\frac{\pi i}{3}\,\frac{B_3([m(c\tau+d)]'_{m\tau+n})}{m(m\tau+n)^2}\,.
\end{aligned}\tag{3.56}$$

In the last equality we used formula (A.15) to simplify the sum of Bernoulli polynomials and we defined the integers m and n as following:

$$(m,n)\equiv\frac{N_2}{\gcd(N_2,m_1n_2-m_2n_1)}\cdot(m_1,n_1)\,.\tag{3.57}$$

Given the similarity between the last line of (3.55) and the right-hand side of (3.40), the rest of the computation is identical to the one in section 3.3.

In conclusion the effective action for the (3.28) saddles can also be expressed in terms of the $\Psi_{m,n}(\Delta)$ function (3.44) as

$$S_{\text{eff}}(m_1,n_1;m_2,n_2,N_2)=\pi iN^2\Big(|G|\,\Psi_{m,n}(\tau+\sigma)+\sum_I\Psi_{m,n}(\Delta_I)\Big).\tag{3.58}$$

The difference between this expression and (3.43) lies in the definition of the integers m,n: for the latter they could be any pair of coprime integers, $\gcd(m,n)=1$, while in the case of the former they are given by (3.57) and $\gcd(m,n)=N_2/\gcd(N_2,m_1n_2-m_2n_1)$. If we set $N_2=1$ the two-factor saddles (3.28) become simple string-like saddles (3.26); in this case the integers m,n in (3.57) simply match m_1,n_1, and expressions (3.43) and (3.58) are in agreement. Furthermore, in the particular case of $a=b=1$ the effective action (3.58) matches the one computed in [39].

An explanation for the similarity between (3.58) and (3.43) can be found by recasting the saddles (3.28) in a new form. Starting from expression (3.34), we can make the following manipulations:

$$\begin{aligned}
u^\alpha_{i_2}(x)&=x\,(m_1\,T+n_1)+\frac{i_2}{N_2}\,\frac{m_1n_2-m_2n_1}{m_1}\\
&=\{m_1x\}\left(T+\frac{n_1}{m_1}\right)+\lfloor m_1x\rfloor\frac{n_1}{m_1}+\frac{i_2}{N_2}\,\frac{m_1n_2-m_2n_1}{m_1}\quad\mod T\,.
\end{aligned}\tag{3.59}$$

If we set $x_{\text{new}}\equiv\{m_1x\}$ and $j\equiv n_1\lfloor m_1x\rfloor(m/m_1)+i_2(m_1n_2-m_2n_1)/\gcd(N_2,m_1n_2-m_2n_1)$ mod m, we can thus rewrite the two-factor saddle as

$$u^\alpha_j(x)=\frac{j}{m}+x\left(T+\frac{n}{m}\right)\,,\tag{3.60}$$

where m,n are the same as in (3.57).

JHEP12(2022)013

From result (3.58) we find the following estimate for the large-N limit of the superconformal index:

$$\log\mathcal{I} \gtrsim \max_{\substack{m,n\,\in\mathbb{Z}\\ m\neq 0}}\Big[-\pi i N^2\Big(|G|\,\Psi_{m,n}(\tau+\sigma)+\sum_I \Psi_{m,n}(\Delta_I)\Big)\Big]+o(N^2)\,, \tag{3.61}$$

where the maximum is taken with respect to the real part. In regions of the parameter space where there is no maximum all the competing exponentially growing contributions to the index should be summed. In this case information about the phase of each term would be necessary to accurately compute the index, and that would require an analysis of the $o(N^2)$ terms. Hence, estimate (3.61) does not apply in these regions. The same can be said for the codimension-one surfaces where multiple contributions have the same real component (i.e. Stokes lines).

In this paper we will not try to determine which contribution maximizes (3.61) in each region of the parameter space. The large-N phase structure of the index has been studied in the case of equal angular momenta in [20, 22, 23, 38, 39].

Surface saddles. The last type of saddles that we still need to account for are the surface saddles (3.31). Assuming that $(m_1,n_1)\neq(m_2,n_2)$, the following relation follows from formula (A.22):

$$\int_0^1 dx\,dy\,\log Q_{c,d}\Big(x(m_1\tau+n_1)+y(m_2\tau+n_2)\,;\tau\Big)=\pi i\tau\left(\frac{c^3}{3}-\frac{c}{6}\right)+\Big(\text{purely imaginary}\Big). \tag{3.62}$$

As expected there is no dependence on the specific value of the integers m_1,n_1,m_2,n_2. This formula can also be found by taking the $N_2\to\infty$ limit of (3.56). In particular this means that surface saddles correspond to the $m,n\sim\mathcal{O}(N)$ terms in estimate (3.61).

Using relation (3.62) we can compute contribution to the effective action of the surface saddle coming from a single multiplet:

$$\begin{aligned}&\frac{\pi i}{3}\,N^2\,ab\omega\sum_{r=0}^{a-1}\sum_{s=0}^{b-1}\left[\left(\frac{r}{a}+\frac{s}{b}+(\Delta)_2-1\right)^3-\frac{1}{2}\left(\frac{r}{a}+\frac{s}{b}+(\Delta)_2-1\right)\right]\\ &\quad=\frac{\pi i}{3}\,N^2\,a^2b^2\omega\Big((\Delta)_2-\frac{a+b}{2ab}\Big)^3-\frac{\pi i}{12}\,N^2\,(a^2+b^2)\,\omega\Big((\Delta)_2-\frac{a+b}{2ab}\Big).\end{aligned} \tag{3.63}$$

The sums over r,s in the first line are calculated quickly with the help of relations (3.47). When we sum over all multiplet contributions the second term in the second line of (3.63) sums to zero: it is proportional to (3.15). If we define a set of trial R-charges $\widehat{\Delta}_{\text{trial}\,,I}$ as

$$\widehat{\Delta}_{\text{trial}\,,I}=\frac{2ab\omega(\Delta_I)_2}{\tau+\sigma}\,, \tag{3.64}$$

then the effective action of surface saddles can be expressed in terms of the cubic 't Hooft anomaly for this trial R-symmetry:

$$S_{\text{eff}}=\frac{\pi i}{24}\,\frac{(\tau+\sigma)^3}{\tau\sigma}\,\operatorname{tr}R^3(\widehat{\Delta}_{\text{trial}})\,, \tag{3.65}$$

where the trace is taken over the fermions of the theory. When $\tau=\sigma$ this result matches the one of [39].

4 The large-N limit with the Bethe Ansatz formula

In this section we will consider a different approach to the computation of the superconformal index at large-N. Our starting point will not be the matrix model (2.7), but rather the Bethe Ansatz formula [44, 45]. A contribution to the Bethe Ansatz formula that reproduces the entropy of black holes with unequal angular momenta was found in [24]; in this section we will revisit the computation of [24] and also expand it to include more contributions. The results we will find reaffirm estimate (3.61), thus providing a double check for the saddle-point analysis of section 3.

This section is organized as follows. We begin with a brief review of the Bethe Ansatz formula in subsection 4.1. Then in subsection 4.2 we study the relation between the holonomy distributions that contribute to the Bethe Ansatz formula and the saddles (3.10) found in [22, 39]. If the reader is not interested in the technical details of subsection 4.2 it is possible to skip directly to subsection 4.3, in which we evaluate the large-N limit of the index with the Bethe Ansatz formula. Lastly, in subsection 4.4 we elaborate on the relation between our results and the ones of [24].

4.1 The Bethe Ansatz formula

As always we assume that the angular chemical potentials τ and σ are integer multiples of the same quantity $\omega \in \mathbb{H}$, that is $\tau = a\omega$, $\sigma = b\omega$. The Bethe Anstatz formula recasts the integral representation (2.7) of the index as following:

$$\mathcal{I} = \frac{\kappa}{(N!)^{|G|}} \sum_{\hat{u} \in \text{BAE}} \sum_{\{m_i^\alpha\}=1}^{ab} \mathcal{Z}(\hat{u} - m\omega\,; \Delta, \tau, \sigma)\, H^{-1}(\hat{u}\,; \Delta, \omega)\,. \tag{4.1}$$

The function $\mathcal{Z}(u\,; \Delta, \tau, \sigma)$ that appears in (4.1) denotes the integrand of matrix model (2.7), or more accurately its analytic continuation to the complex plane with respect to the holonomies $\{u_i^\alpha\}$, and it is given by

$$\mathcal{Z}(u\,; \Delta, \tau, \sigma) = \prod_{\alpha=1}^{|G|} \prod_{i \neq j=1}^{N} \Gamma_e\Big(u_{ij}^\alpha + \tau + \sigma\,; \tau, \sigma\Big) \cdot \prod_{\alpha,\beta=1}^{|G|} \prod_{I_{\alpha\beta}} \prod_{i,j=1}^{N} \Gamma_e\Big(u_{ij}^{\alpha\beta} + \Delta_I\,; \tau, \sigma\Big)\,. \tag{4.2}$$

The first out of the two sums in formula (4.1) runs over the set of inequivalent solutions to the following transcendental equations:

$$1 = Q_i^\alpha(u\,; \Delta, \omega) \equiv e^{2\pi i \lambda^\alpha} \prod_{j=1}^{N} \frac{\prod_{\beta=1}^{|G|} \prod_{I_{\alpha\beta}} \exp\Big(2\pi i u_i^\alpha \left(\frac{1}{2} - \frac{1}{\omega}\Delta_I\right)\Big)\, \theta_0(-u_{ij}^{\alpha\beta} + \Delta_I\,; \omega)}{\prod_{\gamma=1}^{|G|} \prod_{I_{\gamma\alpha}} \exp\Big(-2\pi i u_i^\alpha \left(\frac{1}{2} - \frac{1}{\omega}\Delta_I\right)\Big)\, \theta_0(u_{ij}^{\alpha\gamma} + \Delta_I\,; \omega)}\,, \tag{4.3}$$

where λ^α is a Lagrange multiplier and the function θ_0 is defined in (A.6). Equations (4.3) are called Bethe Ansatz equations (BAE). A notable property of the Bethe Ansatz operators Q_i^α is that they are doubly periodic with periods 1 and ω in each holonomy u_j^β. Thus solutions to the BAE are equivalent if they match under the following identifications:

$$u_i^\alpha \sim u_i^\alpha + 1 \sim u_i^\alpha + \omega \tag{4.4}$$

Additionally solutions that differ by a Weyl group transformation are also considered equivalent. In our case Weyl group transformation consist in permutations of the N holonomies associated to each SU(N) subgroup of the gauge group.

We will focus our attention on the class of solutions to the BAE found in [57], often referred to as Hong-Liu solutions. Given any choice of three integers $\{p,\, q,\, r\}$ such that $p\cdot q = N$ and $0 \leq r < q$,[6] the following configuration of complex holonomies solves the Bethe Ansatz equations:

$$u^{\alpha}_{jk} = \frac{j}{p} + \frac{k}{q}\left(\omega + \frac{r}{p}\right) + \bar{u}\,, \tag{4.5}$$

where $j = 0, \dots, p-1$ and $k = 0, \dots, q-1$ constitute a new parametrization of the index $i = 1, \dots, N$, while $\bar{u}$ is a constant needed to satisfy the SU(N) constraint

$$\sum_{i=1}^{N} u^{\alpha}_{i} = 0\,. \tag{4.6}$$

We point out that the Hong-Liu solutions (4.5) are such that $u^{\alpha}_{j_1k_1} \neq u^{\alpha}_{j_2k_2} \mod 1, \omega$ whenever $(j_1, k_1) \neq (j_2, k_2)$, or in other words they are not invariant under nontrivial Weyl group transformations. As argued in [45], BAE solutions that do not fit this requirement give an overall null contribution to the superconformal index when plugged in the Bethe Anstatz formula (4.1).

Other than the discrete class of solutions (4.5) there is evidence in favor of the existence of other solutions to the BAE, either isolated or belonging to continuous families of solutions [23, 58, 59]. In this paper we will not account for the contribution of these "non-standard" solutions, we will instead focus on the standard Hong-Liu solutions exclusively.

The other sum that appears in formula (4.1) is a sum over a collection of integers $\{m^{\alpha}_{i}\}$. When $i \neq N$ the possible values that m^{α}_{i} can take range from 1 to ab; on the other hand m^{α}_{N} is fixed by the SU(N) constraint:

$$m^{\alpha}_{N} = -\sum_{i=1}^{N-1} m^{\alpha}_{i}\,. \tag{4.7}$$

However in the large-N limit we can ignore this constraint and set m^{α}_{N} to whatever is most convenient: the leading order of $\log \mathcal{Z}(u\,; \Delta, \tau, \sigma)$ is unaffected by a change in value of a single holonomy u^{α}_{i}, and thus changing m^{α}_{N} from (4.7) to something else entirely does not impact the computation of the index [24].

Lastly, the quantity $H(u\,; \Delta, \omega)$ that appears in formula (4.1) is a Jacobian given by

$$H(u\,;\Delta,\omega) = \det\left[\frac{1}{2\pi i}\,\frac{\partial\big(\log Q^{1}_{1}, \dots, \log Q^{1}_{N}, \dots, \log Q^{|G|}_{1}, \dots, \log Q^{|G|}_{N}\big)}{\partial\big(u^{1}_{1}, \dots, u^{1}_{N-1}, \lambda^{1}, \dots, u^{|G|}_{1}, \dots, u^{|G|}_{N-1}, \lambda^{|G|}\big)}\right]. \tag{4.8}$$

In this expression the holonomies $\{u^{\alpha}_{N} \,|\, \alpha = 1, \dots, |G|\}$ are not considered independent variables, they are instead treated like functions of the other holonomies, $u^{\alpha}_{N} \equiv -\sum_{i=1}^{N-1} u^{\alpha}_{i}$. The Lagrange multipliers λ^{α} on the other hand are regarded as independent variables.

[6] Taking into account identifications (4.4), we could substitute r with $r + nq$ in (4.5) for any $n \in \mathbb{Z}$ and the solution would be the same up to a redefinition of the index j, that is $j_{\text{new}} \equiv j + n \mod p$. For this reason the range of r can be limited to $0 \leq r < q$.

4.2 BAE solutions and saddle points of the elliptic action

For a direct comparison of the saddle point analysis with the Bethe Ansatz formula it is important to understand the relation between the saddles found in [22, 39] with the configurations that arise from the discrete solutions to the Bethe Ansatz equations; this will be the goal of this section. The bulk of the computation of the large-N limit of the index will be in section 4.3, and it is possible for the reader to skip ahead.

In the first half of this section we will show that the saddles given by (3.10) can always be written in a form similar to the Hong-Liu solutions (4.5), namely it is possible to find integers p, q and r and a new set of indices $j = 0, \dots, p-1$ and $k = 0, \dots, q-1$ such that[7]

$$u^{\alpha}_{i_1 \dots i_\ell} \equiv \frac{i_1}{N_1}(m_1 T + n_1) + \dots + \frac{i_\ell}{N_\ell}(m_\ell T + n_\ell) = \frac{j}{p} + \frac{k}{q}\left(T + \frac{r}{p}\right) \mod 1, T\,. \qquad (4.9)$$

This expression generalizes relation (3.60), which is valid only in the continuum limit, to the case of finite N. The main difference between the right-hand side of (4.9) and the BAE solutions (4.5) is that the saddles of the doubly periodic action have $T \equiv ab\omega$ as their period, while the solutions to the Bethe Ansatz equations have periodicity ω. We will address this discrepancy in the second half of this section, where we will discuss the role played the vector of integers m that appears in the Bethe Ansatz formula (4.1).

We can ignore without loss of generality saddle point configurations that repeat values, or in other words saddles such that $u^{\alpha}_{i_1 \dots i_\ell} = u^{\alpha}_{j_1 \dots j_\ell} \mod 1, T$ for some $(i_1, \dots i_\ell) \neq (j_1, \dots, j_\ell)$. Since the saddles given by (3.10) can be thought as homomorphisms of finite abelian groups into the torus, repetitions occur only if the kernel is nontrivial. If the kernel contains n elements, then the image group in the torus is the same as the image group of a $\mathrm{SU}(N/n)$ saddle point configuration with no repetitions. Therefore (4.9) holds for these saddles as long as we take $p \cdot q = N/n$, assuming (4.9) is true for saddles that don't repeat values. Furthermore, we note that solutions to the Bethe Ansatz equations that repeat values give an overall null contribution to the index because they are not invariant under nontrivial Weyl group transformations. For these reasons we will only consider configurations without repetitions from now on.

For $\ell = 1$ the relation (4.9) has already been proven in [20]. The idea is to take $p = \gcd(m_1, N_1)$, $q = N_1/p$ and defining the new indices $k = 0, \dots, q-1$, $\hat{\jmath} = 0, \dots, p-1$ so that $i_1 = sk + q\hat{\jmath} \mod N_1$, where s is a positive integer such that $s m_1/p \mod q = 1$; such an integer must exist since m_1/p and q are coprime. Furthermore, s cannot have factors in common with q, and thus the set $\{sk + q\hat{\jmath} \,|\, k = 0, \dots, q-1,\, \hat{\jmath} = 0, \dots, p-1\}$ covers all residue classes modulo N_1 once. The saddle can then be written as

$$\frac{i_1}{N_1}(m_1 T + n_1) = (sk + q\hat{\jmath})\left(\frac{m_1/p}{q} T + \frac{n_1}{N_1}\right) = \frac{n_1 \hat{\jmath}}{p} + \frac{k}{q}\left(T + \frac{n_1 s}{p}\right) \mod 1, T\,, \quad (4.10)$$

which matches the right-hand side of (4.9) for $r \equiv n_1 s \mod q$, $j \equiv n_1 \hat{\jmath} \mod p$.

[7] The vice versa however does not hold: we will later provide an explicit example of a choice of integers p, q and r such that the right-hand side of (4.9) does not correspond to any of the saddles given by (3.10).

We can now prove (4.9) in the general case using induction. Let us assume that there are positive integers p_1, q_1 and r_1 such that $p_1 q_1 = N_1 \dots N_{\ell-1} = N/N_\ell$ and

$$\frac{i_1}{N_1}(m_1 T + n_1) + \dots + \frac{i_{\ell-1}}{N_{\ell-1}}(m_{\ell-1} T + n_{\ell-1}) = \frac{j_1}{p_1} + \frac{k_1}{q_1}\left(T + \frac{r_1}{p_1}\right) \mod 1,\, T\,. \tag{4.11}$$

The left-hand side of this identity is missing the following piece:

$$\frac{i_\ell}{N_\ell}(m_\ell T + n_\ell) \equiv \frac{j_2}{p_2} + \frac{k_2}{q_2}\left(T + \frac{r_2}{p_2}\right) \mod 1,\, T\,, \tag{4.12}$$

where the integers p_2, q_2 and r_2 are determined as in the $\ell = 1$ case. In this case p_2, q_2 satisfy $p_2 q_2 = N_\ell$; furthermore the condition $N_\ell \mid N_{\ell-1}$ implies that $p_2 q_2 \mid p_1 q_1$. The left-hand side of (4.9) can thus be written as

$$\frac{j_1}{p_1} + \frac{k_1}{q_1}\left(T + \frac{r_1}{p_1}\right) + \frac{j_2}{p_2} + \frac{k_2}{q_2}\left(T + \frac{r_2}{p_2}\right) \mod 1,\, T\,. \tag{4.13}$$

We will now show that this expression doesn't repeat values only when p_1, p_2, q_1, q_2 satisfy $\gcd(p_1, p_2) = \gcd(q_1, q_2) = 1$.

The necessity of the condition $\gcd(p_1, p_2) = 1$ can be inferred just from the $j_1/p_1 + j_2/p_2$ portion of (4.13), considering that the set

$$\left\{ \frac{j_1 p_2 + j_2 p_1}{\gcd(p_1, p_2)} \,\middle|\, j_1 = 0, \dots, p_1 - 1,\; j_2 = 0, \dots, p_2 - 1 \right\} \tag{4.14}$$

covers every residue class modulo $p_1 p_2/\gcd(p_1, p_2)$ exactly $\gcd(p_1, p_2)$ times. Therefore if p_1 and p_2 were not coprime $(j_1/p_1 + j_2/p_2 \mod 1)$ would repeat values, and so would (4.13) for fixed k_1 and k_2.

It is easy to see that requiring $\gcd(q_1, q_2) = 1$ in addition to $\gcd(p_1, p_2) = 1$ is sufficient to ensure that (4.13) doesn't repeat values. Indeed if q_1 and q_2 were coprime $(k_1/q_1 + k_2/q_2)T$ modulo T wouldn't repeat and therefore all possible combinations of j_1, j_2, k_1, k_2 would give rise to unique values for expression (4.13). On the other hand it is a little trickier to show that the condition $\gcd(q_1, q_2) = 1$ is necessary, as we need to take in account the terms proportional to r_1 and r_2 as well. Since $j_1/p_1 + j_2/p_2$ covers all multiples of $1/p_1 p_2$ modulo 1 once, if (4.13) doesn't have repetitions modulo 1, T then the same expression without $j_1/p_1 + j_2/p_2$ won't have repetitions modulo $1/p_1 p_2\,,\, T$. Each possible value of $(k_1/q_1 + k_2/q_2)T$ modulo T is repeated $\gcd(q_1, q_2)$ times, which means that either $\gcd(q_1, q_2) = 1$ or the following expression doesn't have repetitions:

$$\frac{k_1 r_1}{p_1 q_1} + \frac{k_2 r_2}{p_2 q_2} \mod \frac{1}{p_1 p_2} = \frac{1}{p_1 q_1}\left(k_1 r_1 + k_2 r_2 \frac{p_1 q_1}{p_2 q_2} \mod \frac{q_1}{p_2}\right). \tag{4.15}$$

Considering that both $p_1 q_1/p_2 q_2$ and q_1/p_2 are integers[8] and that the pair (k_1, k_2) can take a total of $q_1 q_2$ distinct values, the term in the parenthesis will take the same values multiple times unless $p_1 = q_1 = 1$, which cannot be possible as it would imply $N_1 = \dots = N_{\ell-1} = 1$. Therefore we must have $\gcd(q_1, q_2) = 1$.

[8]The condition $\gcd(p_1, p_2) = 1$ together with $p_2 q_2 \mid p_1 q_1$ implies that $p_2 \mid q_1$.

Let us now show that (4.13) can be written in the same form as the right-hand side of (4.9), assuming that $\gcd(p_1, p_2) = \gcd(q_1, q_2) = 1$. First we define $k \equiv k_1 q_2 + k_2 q_1 \mod q_1 q_2$; since q_1 and q_2 are coprime k is an index that runs from 0 to $q_1 q_2 - 1$ once. Let us ignore for the moment terms that are integer multiples of $1/p_1 p_2$; we can write the rest as

$$\frac{k_1}{q_1}\left(T + \frac{r_1}{p_1}\right) + \frac{k_2}{q_2}\left(T + \frac{r_2}{p_2}\right) = \frac{1}{q_1 q_2}\left(kT + \frac{k_1 r_1 p_2 q_2}{p_1 p_2}\right) \mod \frac{1}{p_1 p_2}\,, T\,. \tag{4.16}$$

The term proportional to r_2 is a multiple of $1/p_1 p_2$, considering that $p_2 q_2 \mid p_1 q_1$ and $\gcd(q_1, q_2) = 1$ imply that $q_2 \mid p_1$. Since q_1 and q_2 don't have factors in common it is possible to find an integer n such that $r_1 p_2 + n q_1 = 0 \mod q_2$, which is going to help us rewrite (4.16) solely in terms of k:

$$\begin{aligned} k_1 r_1 p_2 q_2 &= k_1 q_2 (r_1 p_2 + n q_1) && \mod q_1 q_2 \\ &= (k - k_2 q_1)(r_1 p_2 + n q_1) && \mod q_1 q_2 \\ &= k(r_1 p_2 + n q_1) && \mod q_1 q_2\,. \end{aligned} \tag{4.17}$$

Defining $p \equiv p_1 p_2$, $q \equiv q_1 q_2$ and $r \equiv r_1 p_2 + n q_1 \mod q$, equation (4.16) becomes

$$\begin{aligned} \frac{k_1}{q_1}\left(T + \frac{r_1}{p_1}\right) + \frac{k_2}{q_2}\left(T + \frac{r_2}{p_2}\right) &= \frac{k}{q}\left(T + \frac{r}{p}\right) \mod \frac{1}{p}\,, T \\ &\equiv \frac{k}{q}\left(T + \frac{r}{p}\right) + \frac{n_k}{p} \mod 1\,, T \end{aligned} \tag{4.18}$$

for some k-dependent integer n_k. At last we can define $j \equiv j_1 p_2 + j_2 p_1 + n_k \mod p$, so that

$$\frac{j_1}{p_1} + \frac{k_1}{q_1}\left(T + \frac{r_1}{p_1}\right) + \frac{j_2}{p_2} + \frac{k_2}{q_2}\left(T + \frac{r_2}{p_2}\right) = \frac{j}{p} + \frac{k}{q}\left(T + \frac{r}{p}\right) \mod 1,\, T\,, \tag{4.19}$$

which concludes the proof of (4.9).

Vice versa, we can show that there exist some choices of integers p, q and r such that the set of points

$$\left\{ \frac{j}{p} + \frac{k}{q}\left(T + \frac{r}{p}\right) \,\middle|\, j = 0, \ldots, p-1,\; k = 0, \ldots, q-1 \right\} \tag{4.20}$$

does not match any of the saddles given by (3.10), modulo 1, T. One way to see this is to look at the greatest common divisor of p, q and r obtained by the procedure above.

First, in the case of saddles with $\ell = 1$ the steps outlined in (4.10) lead to values of p, q and r that don't have factors in common:

$$\gcd(p, q, r) = \gcd(m_1, N_1, q, n_1 s) = 1\,, \tag{4.21}$$

where we used that $\gcd(m_1, n_1) = 1 = \gcd(q, s)$. For more general saddles on the other hand we find that

$$\gcd(p, q, r) = N_2 \ldots N_\ell\,. \tag{4.22}$$

We can show this by means of induction, by writing $\gcd(p,q,r)$ in terms of $\gcd(p_1,q_1,r_1)$. Considering that $kr = k_1 r_1 p_2 q_2 \mod q$ for all possible values of k, we must have that $\gcd(q,r) = \gcd(q, r_1 p_2 q_2)$. The greatest common divisor of p, q, r is thus given by

$$\gcd(p,q,r) = \gcd(p,q,r_1 p_2 q_2) = p_2 q_2 \gcd\left(\frac{p_1}{q_2}, \frac{q_1}{p_2}, r_1\right) = p_2 q_2 \gcd(p_1, q_1, r_1)\,. \tag{4.23}$$

In the last step we used that $\gcd(p_1,p_2) = \gcd(q_1,q_2) = 1$. Since $p_2 q_2 = N_\ell$, by induction we find formula (4.22).

Let us consider for example the case $p = 4$, $q = 4$, $r = 2$. We want to try to find a saddle point configuration that matches the set (4.23) for these values of p, q, r. Using formula (4.22) we find that such a saddle would have $N_2 \ldots N_\ell = \gcd(p,q,r) = 2$, which implies $\ell = 2$ and $N_2 = 2$. Consequently, there are only two possible values that $\gcd(m_2, N_2)$ can take, either 1 or 2, and neither of them works: the former would lead to $q_1 = q_2 = 2$, while the latter would lead to $p_1 = p_2 = 2$, and in both cases $\gcd(p_1,p_2) = \gcd(q_1,q_2) = 1$ is not satisfied. Hence there is no saddle that reproduces the configuration with $p = 4$, $q = 4$, $r = 2$.

The different periodicities. The most jarring difference between the saddles (3.10) of the doubly-periodic action and the Hong-Liu solutions (4.5) to the Bethe Ansatz equations lies in the different value for T, the modulus of the torus, which is $ab\omega$ for the former and just ω for the latter.

In the particular case of equal angular momenta we have $\tau = \sigma \equiv \omega$, which implies $a = b = 1$ and thus this discrepancy between the known saddles and the standard BAE solutions disappears.

When $ab \neq 1$ the solution to the problem comes from a key element that we haven't taken in consideration yet: the presence of the vector of integers m in the Bethe Ansatz formula (4.1). Each of its entries m_i^α takes values that range from 1 to ab, and its shifts the corresponding holonomy inside the argument of the integrand $\mathcal{Z}$ as $\hat{u}_i^\alpha - m_i^\alpha \omega$, where $\hat{u}$ is the BAE solution that we are considering. Rather than trying to match the saddles (3.10) with the Hong-Liu solutions directly, it is more sensible to compare them with configurations of the type $\hat{u} - m\omega$. That is, given any choice of integers $\{p,q,r\}$, we search for a BAE solution $\hat{u}$ and a choice of vector m such that

$$\frac{j}{p} + \frac{k}{q}\left(ab\omega + \frac{r}{p}\right) = \hat{u}_i^\alpha - m_i^\alpha \omega + \text{constant} \mod 1,\, ab\omega\,. \tag{4.24}$$

The constant term ultimately vanishes because the integrand (4.2) only depends on differences between holonomies.

We point out that there is a large number of valid $\hat{u} - m\omega$ configurations other than the ones that satisfy (4.24). We won't try to account for all possible $(\hat{u}, m)$ combinations, especially considering that the number of possible values that the vector m can take is $(ab)^{|G|(N-1)}$, which grows exponentially with N.

In order to find a $(\hat{u}, m)$ combination that satisfies (4.24) for a given choice of $\{p,q,r\}$, we need to search for integers $\widetilde{p}$, $\widetilde{q}$ and $\widetilde{r}$ that satisfy $\widetilde{p}\,\widetilde{q} = pq$ and a new set of indices

$\widetilde{\jmath} = 0, \dots, \widetilde{p}-1$ and $\widetilde{k} = 0, \dots, \widetilde{q}-1$ such that

$$\frac{j}{p} + \frac{k}{q}\left(ab\omega + \frac{r}{p}\right) = \frac{\widetilde{\jmath}}{\widetilde{p}} + \frac{\widetilde{k}}{\widetilde{q}}\left(\omega + \frac{\widetilde{r}}{\widetilde{p}}\right) \mod 1,\, \omega\,. \tag{4.25}$$

Unfortunately this isn't always possible; to see why, let us set $h \equiv \gcd(q, ab)$ and define a new parametrization of the index k in terms of new indices $k' = 0, \dots, q/h-1$ and $k'' = 0, \dots, h-1$ such that $k \equiv k' + (q/h)\, k''$. The left-hand side of (4.25) then becomes

$$\frac{j}{p} + \frac{k}{q}\left(ab\omega + \frac{r}{p}\right) = \frac{j}{p} + \frac{k'}{q}\left(ab\omega + \frac{r}{p}\right) + \frac{k'' r}{hp} \mod 1,\, \omega\,. \tag{4.26}$$

If r and h are not coprime then the right hand side of (4.26) manifestly repeats values when k'' varies while j and k' are fixed. This means that unless $\gcd(q, r, ab) = 1$ it is not possible to match the right-hand side of (4.25), since the latter never repeats values modulo 1, ω as $\widetilde{\jmath}$ and $\widetilde{k}$ vary.

Even if it is not possible to find a $(\hat{u}, m)$ combination that satisfies (4.24) when q and r are such that $\gcd(q, r, ab) \neq 1$, it is always possible to find $\hat{u}$ and m that approximate the left-hand side of (4.24) well enough in the large-N limit; we will discuss this in more detail in appendix B.

Let us consider the case $\gcd(q, r, ab) = 1$ and show that it is indeed possible to obtain (4.25) starting from (4.26). Since ab/h and q/h are coprime $k'(ab/h) \mod q/h$ takes all the values from 0 to $q/h - 1$ once; therefore if we set $\widetilde{q} \equiv q/h$ and $\widetilde{k} \equiv k'(ab/h) \mod \widetilde{q}$ we can match the ω-dependent portion of (4.25) and (4.26) as follows:

$$\frac{k'}{q}\, ab\omega = \frac{1}{q/h}\left(k'\,\frac{ab}{h}\right)\omega = \frac{\widetilde{k}}{\widetilde{q}}\,\omega \mod \omega\,. \tag{4.27}$$

Since k' also appears in the ω-independent term proportional to r, we need to rewrite this term as well in terms of the new index $\widetilde{k}$; to do so, we will ignore for the moment the role of integer multiples of $1/hp$ so that we can write

$$\frac{k' r}{pq} = \frac{k'(r + nq/h)}{pq} \mod \frac{1}{hp} = \left(\frac{1}{q/h}\, k'\, \frac{ab}{h}\right)\frac{1}{hp}\left(\frac{r + nq/h}{ab/h}\right) \mod \frac{1}{hp}\,, \tag{4.28}$$

where n is an arbitrary integer that we have introduced. Once again we make use of the fact that ab/h and q/h are coprime: $r + nq/h$ for $n \in \mathbb{Z}$ covers all the residue classes modulo ab/h, which means that we can always choose n such that $\widetilde{r} \equiv (r + nq/h)/(ab/h)$ is an integer. Setting $\widetilde{p} \equiv hp$ and noticing that the other term inside parentheses is equal to $\widetilde{k}/\widetilde{q} \mod 1$, we find that

$$\frac{k' r}{pq} = \frac{\widetilde{k}\,\widetilde{r}}{\widetilde{p}\,\widetilde{q}} \mod \frac{1}{\widetilde{p}} \equiv \frac{\widetilde{k}\,\widetilde{r}}{\widetilde{p}\,\widetilde{q}} + \frac{n_k}{\widetilde{p}} \mod 1 \tag{4.29}$$

for some k-dependent integer n_k. We notice that the way $\widetilde{p}$ and $\widetilde{q}$ have been defined is such that $\widetilde{p}\,\widetilde{q}$ is equal to pq, as it should be. At last we make use of the fact that we are assuming that $\gcd(q, r, ab) = 1$, which is equivalent to the statement that h and r are coprime, and thus $\widetilde{\jmath} \equiv k'' r + hj + n_k \mod \widetilde{p}$ is a proper definition for an index that runs from 0 to

$\widetilde{p}-1$ a single time; using (4.29) and the definition of the new index $\widetilde{\jmath}$ we can match the ω-independent portion of (4.25) and (4.26):

$$\frac{j}{p}+\frac{k'r}{pq}+\frac{k''r}{hp}=\frac{\widetilde{\jmath}}{\widetilde{p}}+\frac{\widetilde{k}\,\widetilde{r}}{\widetilde{p}\,\widetilde{q}}\quad \mod 1\,. \tag{4.30}$$

This concludes the proof of the existence of integers $\{\widetilde{p},\widetilde{q},\widetilde{r}\}$ for which the rewrite (4.25) is possible, under the assumption that $\gcd(q,r,ab)=1$.

4.3 Evaluation of the index

In this section we will evaluate the contribution to the Bethe Ansatz formula (4.1) coming from distributions of holonomies of the following type:

$$(u-m\omega)^{\alpha}_{jk}=\frac{j}{p}+\frac{k}{q}\left(ab\omega+\frac{\widehat{r}}{p}\right)+\text{const}\,. \tag{4.31}$$

As usual $p\cdot q=N$ and $0\leq\widehat{r}<q$; we have added the hat on $\widehat{r}$ in order to avoid confusion with the index r that appears in definition (3.44). For simplicity when we take the large-N limit we will keep p and $\widehat{r}$ fixed and send $q\to\infty$.

In section 4.2 we have shown that configurations like (4.31) are possible only when $\gcd(ab,q,\widehat{r})=1$. Throughout this section we will assume that q satisfies this condition; in appendix B we will show how this restriction can be removed.

The quantity that we need to compute is the following:

$$\lim_{q\to\infty}\log\Big(\kappa\,(N!)^{-|G|}\,\mathcal{Z}(u-m\omega\,;\Delta,\tau,\sigma)\,H^{-1}(u\,;\Delta,\omega)\Big)\Big|_{(4.31)}\,. \tag{4.32}$$

The prefactor κ is given by (2.11); as already mentioned in section 2, it is subleading at large-N, specifically $\log\kappa=\mathcal{O}(N)$. The factor $(N!)^{-|G|}$ is subleading as well, it is $\mathcal{O}(N\log N)$ by Stirling formula. We are left with the Jacobian H, given by (4.8), and the integrand $\mathcal{Z}$, given by (4.2).

Let us start from the Jacobian: we can show that generically it gives a subleading contribution and can be neglected, using an argument similar to the one given in [20, 38]. The Jacobian H is the determinant of the matrix whose elements are the partials derivatives of the Bethe Ansatz operators Q_i^{α} defined in (4.3); let us examine these partial derivatives. First, the derivatives of Q_i^{α} with respect to the Lagrange multipliers are simply

$$\frac{\partial\log Q_i^{\alpha}}{\partial\lambda^{\beta}}=2\pi i\,\delta_{\alpha\beta}\,. \tag{4.33}$$

We can ignore these terms as they are just $\mathcal{O}(1)$. On the other hand, the derivatives of with respect to the holonomies are given by

$$\begin{aligned}\frac{\partial\log Q_i^{\alpha}}{\partial u_j^{\beta}}=&-\delta_{\alpha\beta}\,(\delta_{ij}-\delta_{iN})\sum_{k=1}^{N}\sum_{\gamma=1}^{|G|}\Bigg[\sum_{I_{\alpha\gamma}}F(-u_{ik}^{\alpha\gamma}+\Delta_I)+\sum_{I_{\gamma\alpha}}F(u_{ik}^{\alpha\gamma}+\Delta_I)\Bigg]\\&+\sum_{I_{\alpha\beta}}\Big[F(-u_{ij}^{\alpha\beta}+\Delta_I)-F(-u_{iN}^{\alpha\beta}+\Delta_I)\Big]+\sum_{I_{\beta\alpha}}\Big[F(u_{ij}^{\alpha\beta}+\Delta_I)-F(u_{iN}^{\alpha\beta}+\Delta_I)\Big]\,,\end{aligned} \tag{4.34}$$

where F is the following function:

$$F(u) \equiv \frac{2\pi i}{\omega}\, u - \pi i + \frac{\partial_u \theta_0(u;\omega)}{\theta_0(u;\omega)}\,. \tag{4.35}$$

This function becomes singular only at the zeros of the θ_0, that is when $u \in \mathbb{Z} + \omega\,\mathbb{Z}$. If the chemical potentials Δ_I are such that the distribution of points $u_{ij}^{\alpha\beta} + \Delta_I$ doesn't accumulate around any of these poles in the limit $N \to \infty$, then $F(u_{ij}^{\alpha\beta} + \Delta_I) \sim \mathcal{O}(1)$ and

$$\frac{\partial \log Q_i^\alpha}{\partial u_j^\beta} = \delta_{\alpha\beta}\left(\delta_{ij} - \delta_{iN}\right)\cdot \mathcal{O}(N) \;+\; \mathcal{O}(1)\,. \tag{4.36}$$

In our case u_i^α is given by (4.31), and in the $q \to \infty$ limit the poles of F are provided that

$$\Delta_I \notin \frac{\gcd(ab, \widehat{r})}{pab}\,\mathbb{Z} + (pab\omega + \widehat{r})\,\mathbb{R}\,. \tag{4.37}$$

As long as this condition is satisfied for all chemical potentials only the diagonal elements and the $i = N$, $\alpha = \beta$ elements are of order $\mathcal{O}(N)$, while all the others are just $\mathcal{O}(1)$. In particular this means that the determinant (4.8) grows like $N^{(|G|N)}$, and thus $\log H = \mathcal{O}(N \log N)$.

Since the Jacobian is subleading as long as the chemical potentials satisfy (4.37), the large-N leading order of (4.32) is determined solely by the integrand $\mathcal{Z}$. The computation boils down to the evaluation of the following quantity:

$$\Phi_{p,q,\widehat{r}}(\Delta) \equiv \sum_{j_1,j_2=0}^{p-1}\;\sum_{k_1\neq k_2=0}^{q-1} \log\Gamma_e\left(\Delta + \frac{j_1 - j_2}{p} + \frac{k_1 - k_2}{q}\left(ab\omega + \frac{\widehat{r}}{p}\right); a\omega, b\omega\right). \tag{4.38}$$

In terms of this function we can write $\log \mathcal{Z}$ as[9]

$$\log \mathcal{Z}(u - m\omega\,; \Delta, \tau, \sigma)\Big|_{(4.31)} = |G|\,\Phi_{p,q,\widehat{r}}(\tau+\sigma) + \sum_I \Phi_{p,q,\widehat{r}}(\Delta_I) + \mathcal{O}(N)\,. \tag{4.39}$$

A quick comparison with (3.58) tells us that in the large-N limit we should expect $\Phi_{p,q,\widehat{r}}(\Delta)$ to be related to the function $\Psi_{m,n}(\Delta)$ defined in (3.44). More specifically, because of (3.60) we expect the leading order of $\Phi_{p,q,\widehat{r}}(\Delta)$ to be equal to $-\pi i N^2\, \Psi_{p,\widehat{r}}(\Delta)$.

Using formula (A.4) we can take care of the sum over j_1, j_2:

$$\Phi_{p,q,\widehat{r}}(\Delta) = p \sum_{k_1\neq k_2=0}^{q-1} \log\Gamma_e\left(p\Delta + \frac{k_1 - k_2}{q}\,(pab\omega + \widehat{r})\,; pa\omega, pb\omega\right). \tag{4.40}$$

In order to compute the $q \to \infty$ limit of this sum we can take advantage of the following result found in [20]:

$$\sum_{i\neq j=1}^{N} \log\Gamma_e\left(\Delta + \frac{i-j}{N}\,\omega\,; \omega, \omega\right) = -\pi i N^2 \frac{B_3([\Delta - \omega]'_\omega)}{3\omega^2} + o(N^2)\,. \tag{4.41}$$

[9] If p is fixed as $N \to \infty$ then the sum of all the terms that have $k_1 = k_2$ is subleading: $q|G|\sum_{j_1\neq j_2=0}^{p-1} \log\Gamma_e\big(a\omega + b\omega + \frac{j_1-j_2}{p}\,; a\omega, b\omega\big) + q\sum_I \sum_{j_1,j_2=0}^{p-1} \log\Gamma_e\big(\Delta + \frac{j_1-j_2}{p}\,; a\omega, b\omega\big) = \mathcal{O}(N)$.

Here the subleading terms are of order $\mathcal{O}(N)$ when $\Delta \notin \mathbb{Z} + \mathbb{R} \cdot \omega$, and $\mathcal{O}(N \log N)$ for the $\Delta = 0$ case. We can recast (4.40) in a form similar to (4.41) by making use of the following identity:

$$\log \Gamma_e\Big(z\,; pa\omega, pb\omega\Big) = \sum_{r=0}^{a-1}\sum_{s=0}^{b-1} \log \Gamma_e\Big(z + p\omega(as+br)\,; pab\omega + \widehat{r}, pab\omega + \widehat{r}\Big)\,, \tag{4.42}$$

which follows from (A.3) and the invariance of the elliptic gamma under integer shifts of any of its arguments. Denoting $pab\omega + \widehat{r}$ as $\widetilde{\omega}$ for convenience, (4.40) becomes

$$\Phi_{p,q,\widehat{r}}(\Delta) = p \sum_{r=0}^{a-1}\sum_{s=0}^{b-1} \sum_{k_1 \neq k_2 = 0}^{q-1} \log \Gamma_e \left(p\Delta + p\omega(as+br) + \frac{k_1 - k_2}{q}\,\widetilde{\omega}\,; \widetilde{\omega}, \widetilde{\omega} \right). \tag{4.43}$$

Applying formula (4.41), we get at last the following expression for the large-N leading order of $\Phi_{p,q,\widehat{r}}(\Delta)$:

$$\begin{aligned} \Phi_{p,q,\widehat{r}}(\Delta) &= -\frac{\pi i N^2}{3p(pab\omega + \widehat{r})^2} \sum_{r=0}^{a-1}\sum_{s=0}^{b-1} B_3\Big(\big[p\Delta + p\omega(as+br-ab)\big]'_{pab\omega+\widehat{r}}\Big) + o(N^2) \\ &= -\pi i N^2\, \Psi_{p,\widehat{r}}(\Delta) + o(N^2)\,, \end{aligned} \tag{4.44}$$

where as usual the function $\Psi_{m,n}(\Delta)$ is defined by (3.44).

This result is consistent with the ones we obtained in sections 3.3 and 3.4 with the elliptic extension approach. In particular, we find that the large-N estimate for the index (3.61) is verified by the Bethe Ansatz formalism as well.

As already mentioned in section 3.3, the $p = 1$, $\widehat{r} = 0$ case matches the contribution to the Bethe Ansatz formula computed in [24], which reproduces the entropy function of AdS_5 black holes. In the next subsection we will elaborate more on how result (4.44) and the one of [24] compare.

4.4 Relation with previous work and competing exponential terms

Let us discuss the relation between the computation of subsection 4.3 and the one of [24], which also estimated the large-N leading order of the index with $\tau \neq \sigma$ using the Bethe Ansatz formula.

In [24] we focused exclusively on a single contribution to the index, the one coming from the following distribution of holonomies:

$$(u - m\omega)_i^\alpha \;=\; \frac{i}{N}\,\omega + (i \bmod ab)\,\omega + \text{const}\,. \tag{4.45}$$

Up to $\mathcal{O}(1/N)$ terms, this configuration matches the right-hand side of (4.31) with $p = 1$, $\widehat{r} = 0$. For clarity, let us assume that N is a multiple of ab; we can then reparametrize the index i in terms of new indices $i' = 0, \dots, N/ab - 1$ and $i'' = 0, \dots, ab - 1$ such that $i \equiv i'ab + i''$, and rewrite (4.45) as following:

$$\frac{i}{N}\,\omega + (i \bmod ab)\,\omega = \frac{i' + i''(N/ab)}{N}\, ab\omega + \frac{i''}{N}\,\omega \equiv \frac{\widetilde{\imath}}{N}\, ab\omega + \mathcal{O}\left(\frac{1}{N}\right). \tag{4.46}$$

In the second step we defined a new index $\widetilde{\imath} = 0, \ldots, N-1$ as $\widetilde{\imath} \equiv i' + i''(N/ab)$. As argued in appendix A of [24], these $\mathcal{O}(1/N)$ terms can be neglected in the large-N limit, if we are only interested in the leading order. Accordingly, the contribution to the index coming from the distribution of holonomies (4.45) computed in [24] does indeed match the result that we have obtained for the $p = 1$, $\widehat{r} = 0$ case.

Proving that the $\mathcal{O}(1/N)$ terms in (4.46) do not affect the large-N leading order is possibly the most laborious step in the large-N computation of [24]. In this section we have shown that it is possible to avoid this step completely by choosing a different set up for $u - m\omega$, i. e. (4.31), at least as long as the assumption $\gcd(ab, q, \widehat{r}\,) = 1$ is valid. Since the superconformal index is a continuous function of $\tau = a\omega$ and $\sigma = b\omega$, it is natural to expect that the $\gcd(ab, q, \widehat{r}\,) = 1$ condition doesn't actually play a role in the large-N behavior of the index; rather, this condition should be a byproduct of focusing strictly on holonomy distributions that can be written as (4.31).[10]

We want to stress the fact that the distributions of holonomies (4.45) and (4.31) (with $p = 1$, $\widehat{r} = 0$) are distinct from one another, even if in the large-N limit they differ just by $\mathcal{O}(1/N)$ terms. This raises a problem: the contributions to the index coming from these two distributions are exponentially growing terms whose logarithms match at leading N^2 order, and it is easy to see that there are many other similar contributions;[11] all these competing exponential terms must be summed together since there is no guarantee that one of them clearly dominates over the others.

Let us first estimate how many competing exponential terms there are. Any possible choice of $u - m\omega$ that matches $\frac{i}{N}\, ab\omega$ up to $\mathcal{O}(1/N)$ terms must have $u_i \equiv \frac{i}{N}\, \omega$, since this is the only Hong-Liu solution (4.5) whose holonomies are strictly proportional to ω. There are then $(ab)^{|G|(N-1)}$ possible choices for the vector m, at most. If we were to assume that all the competing exponential terms interfere constructively, we would get at most a $|G|(N-1)\log(ab)$ correction to our previous estimate for the large-N limit, which is subleading and thus negligible. In other words the leading N^2 order does not receive corrections from the multiplicity of the competing exponentials. However it would be possible, albeit very unlikely, for all these terms to interfere destructively in such a way that they cancel completely. In order to determine whether this is the case, we would need to calculate the exact phase of all the competing contributions, which is unfeasible. Given that in the saddle point analysis of section 3 this problem does not occur at all, we are lead to believe that such a cancellation does not happen and the leading N^2 order is unaffected.

[10] In appendix B we will verify that this intuition is indeed correct: for any possible choice of integers p, q, $\widehat{r}$ there are contributions to the Bethe Anstatz formula that in the large-N limit give the same result as (4.44), even when the condition $\gcd(ab, q, \widehat{r}\,) = 1$ is not satisfied; the price to pay is that we will have to deal with the $\mathcal{O}(1/N)$ terms once again.

[11] For example, as already argued in [24] changing the value of a single holonomy does not impact the large N leading order, and it is always possible to change the value of a single holonomy by changing the value of one of the entries of the vector of integers m. Hence, for any contribution to the index that we have computed there are always many possible competing exponential terms.

JHEP12(2022)013

5 Summary and discussion

In this paper we have estimated the large-N limit of the superconformal index of $\mathcal{N}=1$ quiver theories with adjoint and bifundamental matter for general values of BPS charges, using both the elliptic extension approach of [22, 39, 47] and the Bethe Ansatz formula [44, 45]. We have found a good accord between the two methods, resulting in the following estimate for the index:

$$\log \mathcal{I}\left(\{\Delta_I\};\tau,\sigma\right) \gtrsim \max_{\substack{m,n\,\in\,\mathbb{Z}\\ m\neq 0}}\Big[-\pi i N^2\Big(|G|\,\Psi_{m,n}(\tau+\sigma)+\sum_I \Psi_{m,n}(\Delta_I)\Big)\Big]+o(N^2)\,, \quad (5.1)$$

where the function $\Psi_{m,n}(\Delta)$ is defined by

$$\Psi_{m,n}(\Delta)\equiv\sum_{r=0}^{a-1}\sum_{s=0}^{b-1}\frac{B_3([m\Delta+m\omega(as+br-ab)]'_{mab\omega+n})}{3m(mab\omega+n)^2} \qquad (\tau\equiv a\omega\,,\ \sigma\equiv b\omega) \quad (5.2)$$

and the parentheses $[\,\cdot\,]'_T$ are such that $[\,x+y\,T\,]'_T=x-\lfloor x\rfloor+y\,T$ for real x, y.

Our results extend the saddle point analysis of [22, 39] to the case of unequal angular momenta ($\tau\neq\sigma$). They also extend the computation of [24] to include multiple competing exponentially-growing contributions to the Bethe Ansatz formula; the single contribution computed in [24] corresponds to the $m=1$, $n=0$ term in (5.1).

In section 4.2 we have shown that the saddles of the elliptic action found in [22, 39] can always be written in the form

$$\frac{j}{p}+\frac{k}{q}\left(T+\frac{r}{p}\right)+\text{const.} \quad (5.3)$$

for some integers p, q, r, with $T\equiv ab\omega$. This is the same form that the standard Hong-Liu solutions [57] to the Bethe Ansatz equations (BAE) take, with the only difference being that the latter are defined on a torus with a modulus $T\equiv\omega$. When $a=b=1$ this means that each saddle has a matching BAE solution, and thus a corresponding term in the Bethe Ansatz equations; however for general a,b the different values of T cause a mismatch between saddles and BAE solutions. In this paper we have shown how the two different pictures can be reconciled: we have to consider that each contribution to the Bethe Ansatz formula is labeled not only by the BAE solution u but also by the choice of value for the auxiliary integer parameters $\{m_i\}$ that shift the BAE solution as $u_i\mapsto u_i-m_i\,\omega\,.$ We have found that for each saddle of the elliptic action there is a $(u,\{m_i\})$ combination that matches it, either exactly or up to $\mathcal{O}(1/N)$ corrections that are negligible at large-N.

There are still some open questions concerning the matching between the two approaches. Most notably, the number of $(u,\{m_i\})$ combinations that label each contribution to the Bethe Ansatz formula is exponentially bigger than the number of known saddles of the elliptic action. In this paper we have computed only the contribution of the $(u,\{m_i\})$ combinations that match a saddle, but there are many other contributions that are unaccounted for. It is not feasible to try to evaluate all of them, given their exponentially large number: the integers $\{m_i\}$ can take $(ab)^{|G|(N-1)}$ different values. Furthermore, the formulas that we have used in section 4.3 would not apply in general. Nonetheless, trying to understand what role do all these terms play remains an interesting question. The simplest possible answer would be that only the $(u, \{m_i\})$ combinations that match one of the elliptic saddles up to negligible corrections give a contribution that at large-N dominates in some region of the space of parameters; further work is however still needed to test the correctness of such a conjecture.

In this paper we have not analyzed which contribution maximizes (5.1) in each region of the parameter space. A detailed study of the phase structure of the index at large-N for general values of BPS charges is a possible direction for future research. In our analysis we focus exclusively on the $\mathcal{O}(N^2)$ leading order; an interesting generalization would be to compute some lower order corrections.

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A The elliptic gamma and related functions

The elliptic gamma function. The elliptic gamma function [48] is defined by the following infinite product:

$$\Gamma_e\Big(z\,;\tau,\sigma\Big) = \prod_{j,k=0}^{\infty} \frac{1 - e^{2\pi i\,\big((j+1)\tau+(k+1)\sigma-z\big)}}{1 - e^{2\pi i\,\big(j\tau+k\sigma+z\big)}}\,, \tag{A.1}$$

which is convergent as long as $\operatorname{Im}\tau > 0$ and $\operatorname{Im}\sigma > 0$. It is meromorphic in z, with poles in $z \in \mathbb{Z} + \tau\,\mathbb{Z}_{\leq 0} + \sigma\,\mathbb{Z}_{\leq 0}$ and zeros in $z \in \mathbb{Z} + \tau\,\mathbb{Z}_{\geq 1} + \sigma\,\mathbb{Z}_{\geq 1}$. It is manifestly invariant under integer shifts in z, τ, σ and symmetric under the exchange of τ and σ.

The elliptic gamma satisfies the following inversion relation:

$$\Gamma_e\Big(z\,;\tau,\sigma\Big) = \Gamma_e\Big(\tau+\sigma-z\,;\tau,\sigma\Big)^{-1}\,, \tag{A.2}$$

and the following product formulas:

$$\prod_{k=0}^{n-1}\Gamma_e\Big(z+\frac{k}{n}\tau\,;\tau,\sigma\Big) = \Gamma_e\Big(z\,;\frac{\tau}{n},\sigma\Big)\,, \tag{A.3}$$

$$\prod_{j=0}^{m-1}\Gamma_e\Big(z+\frac{j}{m}\,;\tau,\sigma\Big) = \Gamma_e\Big(mz\,;m\tau,m\sigma\Big)\,. \tag{A.4}$$

Relations (A.2) and (A.3) follow directly from definition (A.1); as for relation (A.4), it is a consequence of the following polynomial identity:

$$\prod_{j=0}^{m-1}\Big(1 - e^{2\pi i(j/m)}z\Big) = 1 - z^m\,. \tag{A.5}$$

The θ_0 function. The q-theta function θ_0 is defined as follows:

$$\theta_0(z\,;\tau) = \prod_{k=0}^{\infty} \left(1 - e^{2\pi i(z+k\tau)}\right)\left(1 - e^{2\pi i(-z+(k+1)\tau)}\right) . \tag{A.6}$$

It is analytic in z and its zeros are in $z \in \mathbb{Z} + \tau\,\mathbb{Z}$. It is related to the elliptic gamma function by the following shift identity:

$$\Gamma_e(z+\tau\,;\tau,\sigma) = \theta_0(z\,;\sigma)\,\Gamma_e(z\,;\tau,\sigma)\,. \tag{A.7}$$

On the other hand the shift identity for the θ_0 itself is the following:

$$\theta_0(z+\tau\,;\sigma) = -\,e^{-2\pi i\,z}\,\theta_0(z\,;\sigma)\,. \tag{A.8}$$

Bernoulli polynomials. The Bernoulli polynomials are defined by the following generating function:

$$\frac{te^{xt}}{e^t-1} = \sum_{n=0}^{\infty} B_n(x)\,\frac{t^n}{n!}\,. \tag{A.9}$$

They satisfy the following relations:

$$B_n(x+1) = B_n(x) + nx^{n-1}\,, \tag{A.10}$$

$$B_n(1-x) = (-1)^n B_n(x)\,. \tag{A.11}$$

As a consequence of (A.11) the Bernoulli polynomials are either even or odd when expressed in the variable $2x-1$; in particular the first few polynomials can be written as

$$\begin{aligned} B_1(x) &= \frac{1}{2}\,(2x-1) \\ B_2(x) &= \frac{1}{4}\,(2x-1)^2 - \frac{1}{12} \\ B_3(x) &= \frac{1}{8}\,(2x-1)^3 - \frac{1}{8}\,(2x-1)\,. \end{aligned} \tag{A.12}$$

Some useful identities include the translation property,

$$B_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} B_k(x)\,y^{n-k}\,, \tag{A.13}$$

and the multiplication formula,

$$\sum_{j=0}^{m-1} B_n\left(z + \frac{j}{m}\right) = m^{1-n}\,B_n\big(mz\big)\,. \tag{A.14}$$

The multiplication formula can also be written as follows:

$$\sum_{j=0}^{m-1} B_n\left(z + \left\{x + \frac{j}{m}\right\}\right) = m^{1-n}\,B_n\big(mz + \{mx\}\big)\,, \tag{A.15}$$

where the brackets $\{\cdot\}$ denote the fractional part, which is defined by $x \equiv \lfloor x \rfloor + \{x\}$. Relation (A.15) follows directly from (A.14) if we consider that

$$m \cdot \min_j \left\{x + \frac{j}{m}\right\} = \{mx\}\,. \tag{A.16}$$

The P, Q functions. Given z, $\tau \in \mathbb{C}$, $\mathrm{Im}\,\tau > 0$, throughout the rest of this appendix we will denote with z_1 and z_2 the real numbers such that $z \equiv z_1 + \tau\, z_2$.

The function P is defined by [49]

$$P(z\,;\tau) = e^{2\pi i\,\alpha_P(z)}\, e^{\pi i \tau B_2(z_2)}\, \theta_0(z\,;\tau)\,, \tag{A.17}$$

where α_P can be any real-valued function that satisfies the following two constraints: α_P should vanish on the real axis, and it must be chosen so that $P(z\,;\tau)$ is invariant under translations by the lattice $\mathbb{Z} + \tau\,\mathbb{Z}$ in z. The second requirement can always be fulfilled because $|P|$ is manifestly invariant under integer shifts, and it is also invariant under shifts in τ, once (A.8) is taken into consideration. This can also be seen from the second Kronecker limit formula [49]:

$$\log|P(z;\tau)| = -\lim_{s\to 1} \frac{(\mathrm{Im}\tau)^s}{2\pi} \sum_{\substack{m,n\in\mathbb{Z}\\(m,n)\neq(0,0)}} \frac{e^{2\pi i(nz_2 - mz_1)}}{|m\tau+n|^{2s}}\,. \tag{A.18}$$

Similarly, the function Q is defined by [50, 51]

$$Q(z\,;\tau) = e^{2\pi i\,\alpha_Q(z)}\, e^{2\pi i\left(\frac{1}{3}B_3(z_2) - \frac{1}{2}z_2 B_2(z_2)\right)}\, \frac{P(z\,;\tau)^{z_2}}{\Gamma_e(z+\tau\,;\tau,\tau)}\,, \tag{A.19}$$

where α_Q is a real-valued function chosen according to the same criteria as α_P. Hence Q is a doubly periodic function in z as well, with periods 1 and τ. Its double Fourier expansion is given by [22]

$$\log Q(z\,;\tau) = -\frac{1}{4\pi^2} \sum_{\substack{m,n\in\mathbb{Z}\\ m\neq 0}} \frac{e^{2\pi i(nz_2 - mz_1)}}{m(m\tau+n)^2} + \frac{2\pi i\tau}{3} B_3(\{z_2\}) + \pi i \phi(z)\,, \tag{A.20}$$

where ϕ is a real-valued doubly periodic function related to α_Q.

Let m, n be integers such that $\gcd(m,n) = 1$ and $m \neq 0$; from (A.18) and (A.20) it is possible to derive the following integral formulas [22, 39]:

$$\begin{aligned} \int_0^1 dx\, \log P\Big(x(m\tau+n) + c\tau + d\,;\tau\Big) &= -\,\pi i\, \frac{B_2(\{md - nc\})}{m(m\tau+n)} + \pi i\, \varphi_P(m,n)\,, \\ \int_0^1 dx\, \log Q\Big(x(m\tau+n) + c\tau + d\,;\tau\Big) &= \frac{\pi i}{3}\, \frac{B_3(\{md-nc\})}{m(m\tau+n)^2} + \pi i\, \varphi_Q(m,n)\,. \end{aligned} \tag{A.21}$$

Here ϕ_P and ϕ_Q are real functions whose precise value depends on the particular choice of the phases α_P and α_Q. We point out that $B_n(\{\cdot\})$ is a continuous function on the real axis because identity (A.10) implies that $B_n(0) = B_n(1)$.

From (A.21) and definition (2.13) we can find a similar formula for the $Q_{c,d}$ function (2.13); making use of the identity (A.13) for the Bernoulli polynomials we can write it as

$$\begin{aligned} &\int_0^1 dx\, \log Q_{c,d}\Big(x(m\tau+n) + z\,;\tau\Big) \\ &\qquad = -\frac{\pi i}{6}\, c\tau + \frac{\pi i}{3}\, \frac{B_3(\{m(d+z_1) - n(c+z_2)\} + c(m\tau+n))}{m(m\tau+n)^2} \\ &\qquad\quad - \frac{\pi i}{m}\, c^2 B_1(\{m(d+z_1) - n(c+z_2)\}) - \frac{\pi i n}{3m}\, c^3 - \pi i\big(\phi_Q(m,n) - c\,\phi_P(m,n)\big)\,. \end{aligned} \tag{A.22}$$

All the terms in the last line of this equation are purely imaginary.

B Subleading terms in the Bethe Ansatz formula

In section 4.3 we assumed for simplicity that the integers q and $\widehat{r}$ satisfy $\gcd(ab, q, \widehat{r}\,) = 1$; we will now show how the large-N computation of the superconformal index with the Bethe Ansatz formula can be done without this assumption.

When $\gcd(ab, q, \widehat{r}\,) \neq 1$ the problem is that it is not possible to find a BAE solution u and a valid choice for the vector of integers m such that $u - m\omega$ satisfies (4.31). The workaround is to search for u and m that approximate the right-hand side of (4.31) instead; to be precise we want to find a choice of integers $\{\widetilde{p}, \widetilde{q}, \widetilde{r}\}$, indices $\widetilde{\jmath} = 0, \ldots, \widetilde{p} - 1$ and $\widetilde{k} = 0, \ldots, \widetilde{q} - 1$, and vector of integers $m_{\widetilde{\jmath}\widetilde{k}}$, such that $\widetilde{p} \cdot \widetilde{q} = N$ and

$$\frac{\widetilde{\jmath}}{\widetilde{p}} + \frac{\widetilde{k}}{\widetilde{q}}\left(\omega + \frac{\widetilde{r}}{\widetilde{p}}\right) - m_{\widetilde{\jmath}\widetilde{k}}\,\omega = \frac{j}{p} + \frac{k}{q}\left(ab\omega + \frac{\widehat{r}}{p}\right) + \mathcal{O}\left(\frac{1}{q}\right) \qquad \text{mod } 1\,. \tag{B.1}$$

Then, we will show that these extra $\mathcal{O}(1/q)$ terms can be neglected without affecting the leading order of the integrand $\mathcal{Z}$:

$$\begin{aligned}
&\sum_{\widetilde{\jmath}_1,\widetilde{\jmath}_2=0}^{\widetilde{p}-1}\ \sum_{\widetilde{k}_1\neq\widetilde{k}_2=0}^{\widetilde{q}-1} \log\Gamma_e\left(\Delta + \frac{\widetilde{\jmath}_1 - \widetilde{\jmath}_2}{\widetilde{p}} + \frac{\widetilde{k}_1 - \widetilde{k}_2}{\widetilde{q}}\left(\omega + \frac{\widetilde{r}}{\widetilde{p}}\right) - \left(m_{\widetilde{\jmath}_1\widetilde{k}_1} - m_{\widetilde{\jmath}_2\widetilde{k}_2}\right)\omega\,; a\omega, b\omega\right)\\
&\quad = \sum_{j_1,j_2=0}^{p-1}\ \sum_{k_1\neq k_2=0}^{q-1} \log\Gamma_e\left(\Delta + \frac{j_1 - j_2}{p} + \frac{k_1 - k_2}{q}\left(ab\omega + \frac{\widehat{r}}{p}\right); a\omega, b\omega\right) + o(q^2)\,.
\end{aligned} \tag{B.2}$$

The quantity in the second line of (B.2) is the same as (4.38). Hence, the rest of the computation is identical to the one in section 4.3.

Let us set $\widehat{h} \equiv \gcd(ab, q, \widehat{r}\,)$; if we reparametrize the index k in terms of new indices $k' = 0, \ldots, q/\widehat{h} - 1$ and $k'' = 0, \ldots, \widehat{h} - 1$ such that $k \equiv k' + (q/\widehat{h})k''$, we can write the following:

$$\begin{aligned}
\frac{j}{p} + \frac{k}{q}\left(ab\omega + \frac{\widehat{r}}{p}\right) &= \frac{j}{p} + \frac{k}{q/\widehat{h}}\left(\frac{ab}{\widehat{h}}\,\omega + \frac{\widehat{r}/\widehat{h}}{p}\right) = \\
&= \frac{j}{p} + \frac{k'}{q/\widehat{h}}\left(\frac{ab}{\widehat{h}}\,\omega + \frac{\widehat{r}/\widehat{h}}{p}\right) + \frac{k''(\widehat{r}/\widehat{h})}{p} \qquad \text{mod } 1,\ \omega\\
&= \frac{j'}{p} + \frac{k'}{q/\widehat{h}}\left(\frac{ab}{\widehat{h}}\,\omega + \frac{\widehat{r}/\widehat{h}}{p}\right) \qquad \text{mod } 1,\ \omega\,.
\end{aligned} \tag{B.3}$$

In the last step we defined a new index j' as $j' \equiv j + k''(\widehat{r}/\widehat{h}) \mod p$. The dependence on the index k'' has dropped completely modulo $1, \omega$; considering that BAE solutions cannot repeat values modulo $1, \omega$, we will have to reintroduce the dependence on k'' as a part of the $\mathcal{O}(1/q)$ term.

As a consequence of the definition of $\widehat{h}$, we have that $\gcd(ab/\widehat{h}, q/\widehat{h}, \widehat{r}/\widehat{h}) = 1$. Therefore if we set $h \equiv \gcd(q/\widehat{h}, ab)$, $\widetilde{p} \equiv hp$ and $\widetilde{q} \equiv N/\widetilde{p}$, we can find indices $\widetilde{\jmath} = 0, \ldots, \widetilde{p} - 1$, $\widetilde{k} = 0, \ldots, \widetilde{q}/\widehat{h} - 1$ and an integer $\widetilde{r}$ such that

$$\frac{j'}{p} + \frac{k'}{q/\widehat{h}}\left(\frac{ab}{\widehat{h}}\,\omega + \frac{\widehat{r}/\widehat{h}}{p}\right) = \frac{\widetilde{\jmath}}{\widetilde{p}} + \frac{\widetilde{k}}{\widetilde{q}/\widehat{h}}\left(\omega + \frac{\widetilde{r}}{\widetilde{p}}\right) \qquad \text{mod } 1,\ \omega\,. \tag{B.4}$$

This relation can be obtained by following the same steps used to prove (4.25), with $ab/\widehat{h}$, $q/\widehat{h}$ and $\widehat{r}/\widehat{h}$ taking the place of ab, q and r respectively.

We can now chose the value for the vector of integers m so that the following identity holds:

$$\frac{j}{p}+\frac{k}{q}\left(ab\omega+\frac{\widehat{r}}{p}\right) \equiv \frac{\widetilde{j}}{\widetilde{p}}+\frac{\widehat{k}}{\widetilde{q}/\widehat{h}}\left(\omega+\frac{\widetilde{r}}{\widetilde{p}}\right)+m_{\widetilde{j}\widehat{k}}\,\omega \mod 1\,. \tag{B.5}$$

Lastly, we define the index $\widetilde{k}=0,\dots,\widetilde{q}-1$ as $\widetilde{k}\equiv k''+\widehat{k}\,\widehat{h}$, which gives us

$$\frac{\widetilde{j}}{\widetilde{p}}+\frac{\widetilde{k}}{\widetilde{q}}\left(\omega+\frac{\widetilde{r}}{\widetilde{p}}\right) \equiv \frac{\widetilde{j}}{\widetilde{p}}+\frac{\widehat{k}}{\widetilde{q}/\widehat{h}}\left(\omega+\frac{\widetilde{r}}{\widetilde{p}}\right)+\frac{k''}{\widetilde{q}}\left(\omega+\frac{\widetilde{r}}{\widetilde{p}}\right). \tag{B.6}$$

Considering that $k''/\widetilde{q}=\mathcal{O}(1/q)$, if we combine relations (B.5) and (B.6) together we finally obtain (B.1). The only thing left to do is to verify that the simplification (B.2) works at leading order.

Let us set $Z\equiv\Delta+(\widetilde{j}_1-\widetilde{j}_2)/\widetilde{p}$. We need to verify that the following is true:

$$\begin{aligned}&\sum_{\widehat{k}_1\neq\widehat{k}_2=0}^{\widetilde{q}/\widehat{h}-1}\log\Gamma_e\left(Z+\frac{\widehat{k}_1-\widehat{k}_2}{\widetilde{q}/\widehat{h}}\left(\omega+\frac{\widetilde{r}}{\widetilde{p}}\right)+\frac{k''_1-k''_2}{\widetilde{q}}\left(\omega+\frac{\widetilde{r}}{\widetilde{p}}\right)-(m_{\widetilde{j}_1\widehat{k}_1}-m_{\widetilde{j}_2\widehat{k}_2})\,\omega\,;a\omega,b\omega\right)\\&\quad=\sum_{\widehat{k}_1\neq\widehat{k}_2=0}^{\widetilde{q}/\widehat{h}-1}\log\Gamma_e\left(Z+\frac{\widehat{k}_1-\widehat{k}_2}{\widetilde{q}/\widehat{h}}\left(\omega+\frac{\widetilde{r}}{\widetilde{p}}\right)-(m_{\widetilde{j}_1\widehat{k}_1}-m_{\widetilde{j}_2\widehat{k}_2})\,\omega\,;a\omega,b\omega\right)+o(q^2)\,.\end{aligned} \tag{B.7}$$

We are ignoring the sums over $\widetilde{j}_1,\widetilde{j}_2=0,\dots,\widetilde{p}-1$ and $k''_1,k''_2=0,\dots,\widehat{h}-1$ because $\widetilde{p},\,\widehat{h}\sim\mathcal{O}(1)$; if (B.7) holds, then (B.2) would immediately follow.

A relation similar to (B.7), albeit simpler, has already been proven in [24], and we can use it as a staring point. Let us define the following function:

$$f(z;\tau)=\sum_{\gamma\neq\delta=1}^{\widetilde{N}}\log\Gamma_e\Big(z+\frac{\gamma-\delta}{\widetilde{N}}\,\tau\,;n\tau,n\tau\Big)\,, \tag{B.8}$$

where n is any positive integer. As long as $z+t\,\tau$ does not cross a zero or a pole of Γ_e for any $t\in(-1,0)\cup(0,1)$, this function has been shown to satisfy the following bound:

$$\Big|f(z+C\tau/\widetilde{N}\,;\tau)-f(z;\tau)\Big|\leq\mathcal{O}(\widetilde{N}\log\widetilde{N})\,, \tag{B.9}$$

for any $C\in(-1,1)$. There are a few details about the proof of (B.9) that will be useful; let us review them briefly. The first step in the proof is to use the mean value theorem to write

$$\Big|f(z+C\tau/\widetilde{N}\,;\tau)-f(z;\tau)\Big|\leq\frac{|\tau|}{\widetilde{N}}\left(\Big|\partial_z f(z+\bar{c}_1\tau/\widetilde{N}\,;\tau)\Big|+\Big|\partial_z f(z+\bar{c}_2\tau/\widetilde{N}\,;\tau)\Big|\right) \tag{B.10}$$

for some $\bar{c}_1,\bar{c}_2\in\mathbb{R}$, with $|\bar{c}_{1,2}|<|C|$.[12] Then the authors of [24] have shown that for any $\bar{c}\in(-1,1)$ the following bound holds:

$$\frac{1}{\widetilde{N}}\Big|\partial_z f(z+\bar{c}\,\tau/\widetilde{N}\,;\tau)\Big|\leq\frac{1}{\widetilde{N}}\sum_{\gamma\neq\delta=1}^{\widetilde{N}}\left|\frac{\partial_z\Gamma_e\Big(z+\frac{\gamma-\delta+\bar{c}}{\widetilde{N}}\,\tau\,;n\tau,n\tau\Big)}{\Gamma_e\Big(z+\frac{\gamma-\delta+\bar{c}}{\widetilde{N}}\,\tau\,;n\tau,n\tau\Big)}\right|\leq\mathcal{O}(\widetilde{N}\log\widetilde{N})\,. \tag{B.11}$$

Relation (B.9) then follows from (B.10) and (B.11).

[12] The mean value theorem is applied to the real and imaginary part separately, which is the reason for the need of two constants, $\bar{c}_1$ and $\bar{c}_2$.

JHEP12(2022)013

We can't use formula (B.9) to prove (B.7) directly; we need to generalize (B.9) a bit first. Given any two subsets S, S' of the set $\{1, \ldots, \widetilde{N}\}$, we consider the following function:

$$f_{S,S'}(z;\tau) = \sum_{\substack{\gamma \in S,\, \delta \in S' \\ \gamma \neq \delta}}^{\widetilde{N}} \log \Gamma_e\Big(z + \frac{\gamma - \delta}{\widetilde{N}}\tau\,; n\tau, n\tau\Big)\,. \tag{B.12}$$

Then a similar relation to (B.9) holds for $f_{S,S'}$ as well:

$$\Big| f_{S,S'}(z + C\tau/\widetilde{N}\,;\tau) - f_{S,S'}(z;\tau) \Big| \leq \mathcal{O}(\widetilde{N}\log \widetilde{N})\,. \tag{B.13}$$

Indeed, the following trivial inequality:

$$\sum_{\substack{\gamma \in S,\, \delta \in S' \\ \gamma \neq \delta}}^{\widetilde{N}} \left| \frac{\partial_z \Gamma_e\Big(z + \frac{\gamma - \delta + \bar{c}}{\widetilde{N}}\tau\,; n\tau, n\tau\Big)}{\Gamma_e\Big(z + \frac{\gamma - \delta + \bar{c}}{\widetilde{N}}\tau\,; n\tau, n\tau\Big)} \right| \leq \sum_{\gamma \neq \delta = 1}^{\widetilde{N}} \left| \frac{\partial_z \Gamma_e\Big(z + \frac{\gamma - \delta + \bar{c}}{\widetilde{N}}\tau\,; n\tau, n\tau\Big)}{\Gamma_e\Big(z + \frac{\gamma - \delta + \bar{c}}{\widetilde{N}}\tau\,; n\tau, n\tau\Big)} \right| \tag{B.14}$$

together with (B.11) and an analogue of (B.10) imply (B.13).

We can use relation (B.13) to show that

$$\begin{aligned} &\sum_{\gamma \neq \delta = 1}^{\widetilde{N}} \log \Gamma_e\Big(z + \frac{\gamma - \delta}{\widetilde{N}}\tau - (m_\delta - m_\gamma)\,\omega + C\tau/\widetilde{N}\,; n\tau, n\tau\Big) \\ &\qquad = \sum_{\gamma \neq \delta = 1}^{\widetilde{N}} \log \Gamma_e\Big(z + \frac{\gamma - \delta}{\widetilde{N}}\tau - (m_\delta - m_\gamma)\,\omega\,; n\tau, n\tau\Big) + \mathcal{O}(\widetilde{N}\log \widetilde{N})\,, \end{aligned} \tag{B.15}$$

where $\{m_\delta\}_{\delta=1}^{\widetilde{N}}$ is a vector of integers between 1 and ab (with $ab \sim \mathcal{O}(\widetilde{N}^0)$), $\omega \in \mathbb{C}$ and z is such that $z - (m_\delta - m_\gamma)\,\omega + t\,\tau$ does not cross a zero or a pole of Γ_e for any $t \in (-1, 0) \cup (0, 1)$ and any possible value of $(m_\delta - m_\gamma)$. Indeed, we can write

$$\sum_{\gamma \neq \delta = 1}^{\widetilde{N}} \log \Gamma_e\Big(z + \frac{\gamma - \delta}{\widetilde{N}}\tau - (m_\delta - m_\gamma)\,\omega\,; n\tau, n\tau\Big) \equiv \sum_{i_1, i_2 = 1}^{ab} f_{S(i_1), S(i_2)}\big(z - (i_1 - i_2)\,\omega\,; \tau\big)\,, \tag{B.16}$$

where $S(i) \equiv \{\delta \,|\, m_\delta = i\}$. Then (B.15) follows directly from (B.13).

At last, let us show the validity of simplification (B.7). In order to apply (B.15) we need to first change the moduli of the Γ_e from $(a\omega, b\omega)$ to $(n(\omega + \widetilde{r}\,/\,\widetilde{p}),\, n(\omega + \widetilde{r}\,/\,\widetilde{p}))$, where n is some positive integer. We can use the following identity:

$$\log \Gamma_e\Big(u\,; a\omega, b\omega\Big) = \sum_{\ell_1 = 0}^{b\widetilde{p}-1} \sum_{\ell_2 = 0}^{a\widetilde{p}-1} \log \Gamma_e\Big(u + (\ell_1 a + \ell_2 b)\omega\,; \widetilde{p}ab\omega + abr,\, \widetilde{p}ab\omega + abr\Big)\,, \tag{B.17}$$

which follows from (A.3) and the invariance of Γ_e under integers shifts. Then, for any given value of ℓ_1, ℓ_2 we can use (B.15) with $z \equiv Z + (\ell_1 a + \ell_2 b)\,\omega$, $\tau \equiv \omega + \widetilde{r}\,/\,\widetilde{p}$ and $n \equiv \widetilde{p}\,ab$. It is easy to verify that z satisfies the required condition necessary for avoiding zeros and poles, considering that the possible values for Δ are $\Delta \equiv \tau + \sigma$ and $\Delta \equiv \Delta_I$, with Δ_I satisfying condition (4.37). This concludes the proof of (B.7).

References

[1] J.D. Bekenstein, *Black holes and the second law*, *Lett. Nuovo Cim.* **4** (1972) 737 [INSPIRE].

[2] J.D. Bekenstein, *Black holes and entropy*, *Phys. Rev. D* **7** (1973) 2333 [INSPIRE].

[3] J.M. Bardeen, B. Carter and S.W. Hawking, *The four laws of black hole mechanics*, *Commun. Math. Phys.* **31** (1973) 161 [INSPIRE].

[4] S.W. Hawking, *Particle Creation by Black Holes*, *Commun. Math. Phys.* **43** (1975) 199 [*Erratum ibid.* **46** (1976) 206] [INSPIRE].

[5] F. Benini, K. Hristov and A. Zaffaroni, *Black hole microstates in AdS*$_4$ *from supersymmetric localization*, *JHEP* **05** (2016) 054 [arXiv:1511.04085] [INSPIRE].

[6] F. Benini and A. Zaffaroni, *Supersymmetric partition functions on Riemann surfaces*, *Proc. Symp. Pure Math.* **96** (2017) 13 [arXiv:1605.06120] [INSPIRE].

[7] F. Benini, K. Hristov and A. Zaffaroni, *Exact microstate counting for dyonic black holes in AdS*$_4$, *Phys. Lett. B* **771** (2017) 462 [arXiv:1608.07294] [INSPIRE].

[8] A. Zaffaroni, *AdS black holes, holography and localization*, *Living Rev. Rel.* **23** (2020) 2 [arXiv:1902.07176] [INSPIRE].

[9] J.B. Gutowski and H.S. Reall, *Supersymmetric AdS*$_5$ *black holes*, *JHEP* **02** (2004) 006 [hep-th/0401042] [INSPIRE].

[10] J.B. Gutowski and H.S. Reall, *General supersymmetric AdS*$_5$ *black holes*, *JHEP* **04** (2004) 048 [hep-th/0401129] [INSPIRE].

[11] Z.W. Chong, M. Cvetič, H. Lü and C.N. Pope, *Five-dimensional gauged supergravity black holes with independent rotation parameters*, *Phys. Rev. D* **72** (2005) 041901 [hep-th/0505112] [INSPIRE].

[12] Z.W. Chong, M. Cvetič, H. Lü and C.N. Pope, *General non-extremal rotating black holes in minimal five-dimensional gauged supergravity*, *Phys. Rev. Lett.* **95** (2005) 161301 [hep-th/0506029] [INSPIRE].

[13] H.K. Kunduri, J. Lucietti and H.S. Reall, *Supersymmetric multi-charge AdS*$_5$ *black holes*, *JHEP* **04** (2006) 036 [hep-th/0601156] [INSPIRE].

[14] C. Romelsberger, *Counting chiral primaries in* $N = 1$, $d = 4$ *superconformal field theories*, *Nucl. Phys. B* **747** (2006) 329 [hep-th/0510060] [INSPIRE].

[15] J. Kinney, J.M. Maldacena, S. Minwalla and S. Raju, *An index for 4 dimensional super conformal theories*, *Commun. Math. Phys.* **275** (2007) 209 [hep-th/0510251] [INSPIRE].

[16] Y. Nakayama, *Index for orbifold quiver gauge theories*, *Phys. Lett. B* **636** (2006) 132 [hep-th/0512280] [INSPIRE].

[17] A. Gadde, L. Rastelli, S.S. Razamat and W. Yan, *On the Superconformal Index of* $N = 1$ *IR Fixed Points: A Holographic Check*, *JHEP* **03** (2011) 041 [arXiv:1011.5278] [INSPIRE].

[18] R. Eager, J. Schmude and Y. Tachikawa, *Superconformal Indices, Sasaki-Einstein Manifolds, and Cyclic Homologies*, *Adv. Theor. Math. Phys.* **18** (2014) 129 [arXiv:1207.0573] [INSPIRE].

[19] S. Choi, J. Kim, S. Kim and J. Nahmgoong, *Large AdS black holes from QFT*, arXiv:1810.12067 [INSPIRE].

[20] F. Benini and E. Milan, *Black Holes in 4D $\mathcal{N} = 4$ Super-Yang-Mills Field Theory*, *Phys. Rev. X* **10** (2020) 021037 [arXiv:1812.09613] [INSPIRE].

[21] S.M. Hosseini, K. Hristov and A. Zaffaroni, *An extremization principle for the entropy of rotating BPS black holes in AdS_5*, *JHEP* **07** (2017) 106 [arXiv:1705.05383] [INSPIRE].

[22] A. Cabo-Bizet and S. Murthy, *Supersymmetric phases of 4d $\mathcal{N} = 4$ SYM at large N*, *JHEP* **09** (2020) 184 [arXiv:1909.09597] [INSPIRE].

[23] A. Arabi Ardehali, J. Hong and J.T. Liu, *Asymptotic growth of the 4d $\mathcal{N} = 4$ index and partially deconfined phases*, *JHEP* **07** (2020) 073 [arXiv:1912.04169] [INSPIRE].

[24] F. Benini, E. Colombo, S. Soltani, A. Zaffaroni and Z. Zhang, *Superconformal indices at large N and the entropy of $AdS_5 \times SE_5$ black holes*, *Class. Quant. Grav.* **37** (2020) 215021 [arXiv:2005.12308] [INSPIRE].

[25] C. Copetti, A. Grassi, Z. Komargodski and L. Tizzano, *Delayed deconfinement and the Hawking-Page transition*, *JHEP* **04** (2022) 132 [arXiv:2008.04950] [INSPIRE].

[26] O. Aharony, F. Benini, O. Mamroud and E. Milan, *A gravity interpretation for the Bethe Ansatz expansion of the $\mathcal{N} = 4$ SYM index*, *Phys. Rev. D* **104** (2021) 086026 [arXiv:2104.13932] [INSPIRE].

[27] M. Honda, *Quantum Black Hole Entropy from 4d Supersymmetric Cardy formula*, *Phys. Rev. D* **100** (2019) 026008 [arXiv:1901.08091] [INSPIRE].

[28] A. Arabi Ardehali, *Cardy-like asymptotics of the 4d $\mathcal{N} = 4$ index and AdS_5 blackholes*, *JHEP* **06** (2019) 134 [arXiv:1902.06619] [INSPIRE].

[29] A. González Lezcano, J. Hong, J.T. Liu and L.A. Pando Zayas, *Sub-leading Structures in Superconformal Indices: Subdominant Saddles and Logarithmic Contributions*, *JHEP* **01** (2021) 001 [arXiv:2007.12604] [INSPIRE].

[30] K. Goldstein, V. Jejjala, Y. Lei, S. van Leuven and W. Li, *Residues, modularity, and the Cardy limit of the 4d $\mathcal{N} = 4$ superconformal index*, *JHEP* **04** (2021) 216 [arXiv:2011.06605] [INSPIRE].

[31] A. Amariti, M. Fazzi and A. Segati, *The SCI of $\mathcal{N} = 4 \mathrm{USp}(2N_c)$ and $\mathrm{SO}(N_c)$ SYM as a matrix integral*, *JHEP* **06** (2021) 132 [arXiv:2012.15208] [INSPIRE].

[32] D. Cassani and Z. Komargodski, *EFT and the SUSY Index on the 2nd Sheet*, *SciPost Phys.* **11** (2021) 004 [arXiv:2104.01464] [INSPIRE].

[33] A. Arabi Ardehali and S. Murthy, *The 4d superconformal index near roots of unity and 3d Chern-Simons theory*, *JHEP* **10** (2021) 207 [arXiv:2104.02051] [INSPIRE].

[34] V. Jejjala, Y. Lei, S. van Leuven and W. Li, $\mathrm{SL}(3, Z)$ *Modularity and New Cardy limits of the $\mathcal{N} = 4$ superconformal index*, *JHEP* **11** (2021) 047 [arXiv:2104.07030] [INSPIRE].

[35] A. Cabo-Bizet, D. Cassani, D. Martelli and S. Murthy, *Microscopic origin of the Bekenstein-Hawking entropy of supersymmetric AdS_5 black holes*, *JHEP* **10** (2019) 062 [arXiv:1810.11442] [INSPIRE].

[36] D. Cassani and L. Papini, *The BPS limit of rotating AdS black hole thermodynamics*, *JHEP* **09** (2019) 079 [arXiv:1906.10148] [INSPIRE].

[37] A. González Lezcano and L.A. Pando Zayas, *Microstate counting via Bethe Ansätze in the 4d $\mathcal{N} = 1$ superconformal index*, *JHEP* **03** (2020) 088 [arXiv:1907.12841] [INSPIRE].

[38] A. Lanir, A. Nedelin and O. Sela, *Black hole entropy function for toric theories via Bethe Ansatz*, *JHEP* **04** (2020) 091 [arXiv:1908.01737] [INSPIRE].

[39] A. Cabo-Bizet, D. Cassani, D. Martelli and S. Murthy, *The large-N limit of the 4d $\mathcal{N} = 1$ superconformal index*, *JHEP* **11** (2020) 150 [arXiv:2005.10654] [INSPIRE].

[40] J. Kim, S. Kim and J. Song, *A 4d $\mathcal{N} = 1$ Cardy Formula*, *JHEP* **01** (2021) 025 [arXiv:1904.03455] [INSPIRE].

[41] A. Cabo-Bizet, D. Cassani, D. Martelli and S. Murthy, *The asymptotic growth of states of the 4d $\mathcal{N} = 1$ superconformal index*, *JHEP* **08** (2019) 120 [arXiv:1904.05865] [INSPIRE].

[42] A. Amariti, I. Garozzo and G. Lo Monaco, *Entropy function from toric geometry*, *Nucl. Phys. B* **973** (2021) 115571 [arXiv:1904.10009] [INSPIRE].

[43] A. Amariti, M. Fazzi and A. Segati, *Expanding on the Cardy-like limit of the SCI of 4d $\mathcal{N} = 1$ ABCD SCFTs*, *JHEP* **07** (2021) 141 [arXiv:2103.15853] [INSPIRE].

[44] C. Closset, H. Kim and B. Willett, *$\mathcal{N} = 1$ supersymmetric indices and the four-dimensional A-model*, *JHEP* **08** (2017) 090 [arXiv:1707.05774] [INSPIRE].

[45] F. Benini and E. Milan, *A Bethe Ansatz type formula for the superconformal index*, *Commun. Math. Phys.* **376** (2020) 1413 [arXiv:1811.04107] [INSPIRE].

[46] F.A. Dolan and H. Osborn, *Applications of the Superconformal Index for Protected Operators and q-Hypergeometric Identities to $N = 1$ Dual Theories*, *Nucl. Phys. B* **818** (2009) 137 [arXiv:0801.4947] [INSPIRE].

[47] A. Cabo-Bizet, *From multi-gravitons to Black holes: The role of complex saddles*, arXiv:2012.04815 [INSPIRE].

[48] G. Felder and A. Varchenko, *The elliptic gamma function and* $\mathrm{SL}(3, Z) \ltimes Z^3$, *Adv. Math.* **156** (2000) 44 [math/9907061].

[49] A. Weil, *Elliptic functions according to eisenstein and kronecker*, *Ergeb. Math. Grenzgeb. A* **8** (1976) [DOI].

[50] W. Duke, *On a formula of bloch*, *Funct. Approximatio Comment. Math.* **37** (2007) 109.

[51] V. Paşol and W. Zudilin, *A study of elliptic gamma function and allies*, *Res. Math. Sci.* **5** (2018) [arXiv:1801.00210] [INSPIRE].

[52] J.J. Duistermaat and G.J. Heckman, *On the Variation in the cohomology of the symplectic form of the reduced phase space*, *Invent. Math.* **69** (1982) 259.

[53] J. Duistermaat and G. Heckman, *Addendum to "on the variation in the cohomology of the symplectic form of the reduced phase space"*, *Invent. Math.* **72** (1983) 153.

[54] E. Witten, *Supersymmetry and Morse theory*, *J. Diff. Geom.* **17** (1982) 661 [INSPIRE].

[55] N. Berline and M. Vergne, *Zeros d'un champ de vecteurs et classes caracteristiques equivariantes*, *Duke Math. J.* **50** (1983) 539.

[56] M.F. Atiyah and R. Bott, *The moment map and equivariant cohomology*, *Topology* **23** (1984) 1 [INSPIRE].

JHEP12(2022)013

[57] J. Hong and J.T. Liu, *The topologically twisted index of $\mathcal{N} = 4$ super-Yang-Mills on $T^2 \times S^2$ and the elliptic genus*, *JHEP* **07** (2018) 018 [arXiv:1804.04592] [INSPIRE].

[58] A.G. Lezcano, J. Hong, J.T. Liu and L.A. Pando Zayas, *The Bethe-Ansatz approach to the $\mathcal{N} = 4$ superconformal index at finite rank*, *JHEP* **06** (2021) 126 [arXiv:2101.12233] [INSPIRE].

[59] F. Benini and G. Rizi, *Superconformal index of low-rank gauge theories via the Bethe Ansatz*, *JHEP* **05** (2021) 061 [arXiv:2102.03638] [INSPIRE].

JHEP12(2022)013