Off-shell higher-spin fields in AdS_4 and external currents

N.G. Misuna

*Tamm Department of Theoretical Physics, Lebedev Physical Institute,
Leninsky prospect 53, 119991, Moscow, Russia*

E-mail: misuna@lpi.ru

ABSTRACT: We construct an unfolded system for off-shell fields of arbitrary integer spin in $4d$ anti-de Sitter space. To this end we couple an on-shell system, encoding Fronsdal equations, to external Fronsdal currents for which we find an unfolded formulation. We present a reduction of the Fronsdal current system which brings it to the unfolded Fierz-Pauli system describing massive fields of arbitrary integer spin. Reformulating off-shell higher-spin system as the set of Schwinger–Dyson equations we compute propagators of higher-spin fields in the de Donder gauge directly from the unfolded equations. We discover operators that significantly simplify this computation, allowing a straightforward extraction of wave equations from an unfolded system.

KEYWORDS: Higher Spin Gravity, Higher Spin Symmetry

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1 Introduction

Higher-spin (HS) gravity represents a theory of interacting massless fields of all spins. Up to date the most complete formulation of higher-spin gravity is provided by Vasiliev theory [1, 2]. It represents a set of first-order generating equations that encode classical equations of motion of HS fields. This method of formulating a theory in the form of first-order constraints on exterior forms is referred to as an unfolded dynamics approach [3]. It provides a coordinate-independent and manifestly gauge-invariant description of the model under consideration. Both these features are substantial when dealing with HS theory, because it includes gravity (a massless spin-2 field) and possesses an infinite-dimensional HS gauge symmetry. But extracting physical quantities from Vasiliev equations represents a complicated (both technically and conceptually) problem. By now only cubic HS vertices

have been completely derived and analyzed [4–9], and also some partial results on quartic and quintic vertices are available [10–12].

One of the central difficulties of HS gravity is that its full nonlinear action is unknown. This prevents one from straightforward verification of various *AdS/CFT* conjectures relating HS gravity to boundary conformal vectorial models [13–18] and from systematic study of quantum features of HS gravity. A possible way to the action was proposed in [19], where it was shown that all nontrivial gauge-invariant functionals of the unfolded system are in one-to-one correspondence with cohomologies of a certain operator Q determined by unfolded equations. An unfolded system is said to lie off-shell if all unfolded equations only express descendant fields in terms of primaries, while primary fields remain unconstrained. Then Q -cohomologies of the off-shell unfolded system list all candidates for gauge-invariant actions of the theory in question. However, a direct calculation of Q -cohomologies of Vasiliev equations (and their putative off-shell completion) seems to be unfeasible. Even for the simplest theories Q -cohomology problem requires some efforts (see e.g. [20] where Q -cohomology analysis was carried out for Wess-Zumino model).

Other alternatives for an action principle of HS gravity were also put forward, see e.g. [21–28]. And in spite of the lack of the full canonical action, certain quantum calculations in HS gravity were successfully performed. These include e.g. evaluations of a 1-loop vacuum partition function of HS gravity in different geometries and comparing results with *AdS/CFT* predictions [29–38], computations of some AdS-amplitudes (including loop corrections) for HS fields [39–46], study of the chiral HS gravity which revealed various cancellations of UV divergences and one-loop finiteness [47–52] etc. In [53] a different approach to the problem of action and quantization of HS gravity was proposed. It was argued that the procedure of the off-shell completion of a given unfolded system amounts to switching on external sources for all primary fields. Then the resulting off-shell system can be reformulated as the set of Schwinger–Dyson functional equations of the quantized theory. This allows one to directly proceed to the systematic computation of the quantum generating functional and correlation functions. In [53] this was illustrated by constructing an off-shell system for free HS fields in $4d$ Minkowski space.

This paper represents the first step in the program of quantization of Vasiliev theory within the unfolded framework. Here we present an off-shell unfolded system for free massless bosonic HS fields in $4d$ anti-de Sitter space. We start with the unfolded on-shell HS equations, which represent a linear limit of the full Vasiliev equations, and consistently couple them to the external Fronsdal HS currents. For these currents we construct an unfolded system. It turns out that this HS current system allows a simple reduction to the unfolded system for Fierz-Pauli massive HS fields. Resulting description of the massive HS fields is non-gauge, providing an alternative to unfolding massive HS fields in [54–61] by composing them from the couple of massless ones with all necessary helicities. Using an off-shell completed HS system, we compute propagators for massless HS fields in the de Donder gauge. The procedure of computation gets substantially simplified due to the application of “conjugate operators” D^* we found, which allows one to directly extract wave equations for component fields from the unfolded system.

The paper is organized as follows. In section 2 we introduce and discuss necessary concepts and features of the unfolded dynamics approach and explain our method of constructing off-shell completion. In section 3 we consider an on-shell unfolded system for free HS fields in AdS_4 , the so-called Central On-Mass-Shell Theorem, which is a starting point for our analysis, and present a quick way to recover Fronsdal equations from it. Section 4 is devoted to the detailed analysis of an off-shell completion of a scalar field, including consideration of arbitrary mass-shell reductions and calculation of a bulk-to-bulk propagator, in order to illustrate our idea. In section 5 we construct an unfolded system describing Fronsdal HS currents and discuss its on-shell reduction, leading to the system for the Fierz-Pauli HS fields of arbitrary mass. Finally, in section 6 we couple unfolded Fronsdal currents to the Central On-Mass-Shell Theorem, thus formulating an off-shell unfolded system for HS fields in AdS_4 , and make use of it in order to calculate bulk-to-bulk propagators of Fronsdal HS fields in the de Donder gauge. Section 7 contains our conclusions. In appendix A notations and conventions used throughout the paper are collected. In the second appendix B we present a different off-shell extension for HS fields, which contains simpler equations for HS currents but leads to non-diagonal coupling of these currents to Fronsdal fields.

2 Essentials of the unfolded dynamics approach

Unfolded formulation [1–3, 19, 62] of the theory implies its representation as the set of equations of the form

$$dW^A(x) + G^A(W) = 0, \quad (2.1)$$

where d is the de Rham differential on the spacetime manifold M^d with local coordinates x ; $W^A(x)$ are unfolded fields representing spacetime exterior forms, with A denoting all their indices; $G^A(W)$ is built from exterior products of W (we will omit the wedge symbol throughout the paper). The nilpotency of the de Rham differential $d^2 \equiv 0$ imposes a consistency condition on G

$$G^B \frac{\delta G^A}{\delta W^B} \equiv 0, \quad (2.2)$$

which plays the crucial role in the unfolded analysis. An unfolded system (2.1) is manifestly invariant under a set of infinitesimal gauge transformations

$$\delta W^A = d\varepsilon^A(x) - \varepsilon^B \frac{\delta G^A}{\delta W^B} \quad (2.3)$$

(in checking the invariance one should make use of (2.2)). A gauge parameter $\varepsilon^A(x)$, representing a rank- $(n-1)$ form, is associated with a gauge transformation generated by rank- n unfolded field W^A . 0-forms do not give rise to gauge symmetries and are transformed only by gauge transformations of higher-rank fields due to the second term in (2.3).

The spacetime geometry in the unfolded approach is described by a generalized 1-form connection $\Omega = dx^a \Omega_a^A(x) T_A$ that takes values in the Lie algebra of spacetime symmetries with generators T_A . Maximally symmetric gravitational background arises via imposing zero-curvature condition on Ω

$$d\Omega + \frac{1}{2}[\Omega, \Omega] = 0 \quad (2.4)$$

(square brackets stand for the Lie-algebra commutator). Fixing some particular solution Ω_0 to this equation breaks a gauge symmetry

$$\delta\Omega = d\varepsilon(x) + [\Omega, \varepsilon] \quad (2.5)$$

to the leftover global symmetry ε_{glob} that leaves Ω_0 invariant and thus satisfies

$$d\varepsilon_{glob} + [\Omega_0, \varepsilon_{glob}] = 0. \quad (2.6)$$

In this paper we deal with $d = 4$ anti-de Sitter space AdS_4 , so we introduce a corresponding connection of its symmetry algebra $so(3, 2)$

$$\Omega^{AdS} = e^{\alpha\dot{\beta}} P_{\alpha\dot{\beta}} + \Omega^{\alpha\beta} M_{\alpha\beta} + \bar{\Omega}^{\dot{\alpha}\dot{\beta}} \bar{M}_{\dot{\alpha}\dot{\beta}}, \quad (2.7)$$

where $P_{\alpha\dot{\alpha}}$, $M_{\alpha\beta}$ and $\bar{M}_{\dot{\alpha}\dot{\beta}}$ represent generators of spacetime translations and (selfdual and anti-selfdual part of) rotations, $e^{\alpha\dot{\beta}}$ and $\Omega^{\alpha\beta}$ ($\bar{\Omega}^{\dot{\alpha}\dot{\beta}}$) are 1-forms of vierbein and Lorentz connection. Expansion of (2.4) in generators gives

$$de^{\alpha\dot{\beta}} + \Omega^\alpha{}_\gamma e^{\gamma\dot{\beta}} + \bar{\Omega}^{\dot{\beta}}{}_{\dot{\gamma}} e^{\alpha\dot{\gamma}} = 0, \quad (2.8)$$

$$d\Omega^{\alpha\beta} + \Omega^\alpha{}_\gamma \Omega^{\gamma\beta} = -\lambda^2 E^{\alpha\beta}, \quad (2.9)$$

$$d\bar{\Omega}^{\dot{\alpha}\dot{\beta}} + \bar{\Omega}^{\dot{\alpha}}{}_{\dot{\gamma}} \bar{\Omega}^{\dot{\gamma}\dot{\beta}} = -\lambda^2 \bar{E}^{\dot{\alpha}\dot{\beta}}. \quad (2.10)$$

Here $E^{\alpha\beta} = e^\alpha{}_\gamma e^{\beta\dot{\gamma}} \bar{E}^{\dot{\alpha}\dot{\beta}} = e_\gamma{}^{\dot{\alpha}} e^{\gamma\dot{\beta}} \bar{E}^{\dot{\alpha}\dot{\beta}}$ are basis 2-forms and λ denotes an inverse AdS-radius. If now one chooses some particular solution to (2.8)–(2.10), i.e. fixes some coordinate frame in AdS_4 , then solutions of (2.6) will provide an explicit realization of AdS global symmetries in these coordinates.

To illustrate the unfolded approach in more detail and explain our notations let us consider an unfolded formulation of a free massless scalar field. The scalar field is described by an unfolded module

$$C(Y|x) = \sum_{N=0}^{\infty} C_N(Y|x) = \sum_{N=0}^{\infty} C_{\alpha(N), \dot{\alpha}(N)}(x) y^{\alpha_1} \dots y^{\alpha_N} \bar{y}^{\dot{\alpha}_1} \dots \bar{y}^{\dot{\alpha}_N} \quad (2.11)$$

where $Y = (y^\alpha, \bar{y}^{\dot{\alpha}})$ is a pair of auxiliary commuting $sp(2)$ -spinors (see appendix A for notation details). Unfolded equations for C are

$$DC + ie^{\alpha\dot{\beta}} \partial_\alpha \bar{\partial}_{\dot{\beta}} C - ie^{\alpha\dot{\beta}} y_\alpha \bar{y}_{\dot{\beta}} C = 0, \quad (2.12)$$

where

$$D = \frac{1}{\lambda} \left(d + \Omega^{\alpha\beta} y_\alpha \partial_\beta + \bar{\Omega}^{\dot{\alpha}\dot{\beta}} \bar{y}_{\dot{\alpha}} \bar{\partial}_{\dot{\beta}} \right) \quad (2.13)$$

is a dimensionless 1-form of a Lorentz-covariant derivative. To study the content of (2.12) one expands D in vierbein basis as $D = e^{\alpha\dot{\beta}} D_{\alpha\dot{\beta}}$ which yields

$$D_{\alpha\dot{\beta}} C + i \partial_\alpha \bar{\partial}_{\dot{\beta}} C - i y_\alpha \bar{y}_{\dot{\beta}} C = 0. \quad (2.14)$$

Contracting this with $iy^\alpha \bar{y}^{\dot{\beta}}$ leads to

$$C_N(Y|x) = \frac{1}{(N!)^2} \left(iy^\alpha \bar{y}^{\dot{\beta}} D_{\alpha\dot{\beta}} \right)^N C(0|x), \quad (2.15)$$

while contracting with $-\frac{\lambda^2}{2}(D^{\alpha\dot{\beta}} - i\partial^\alpha \bar{\partial}^{\dot{\beta}} + iy^\alpha \bar{y}^{\dot{\beta}})$ produces

$$\square C_N + \lambda^2 (N^2 + 2N + 2) C_N = 0, \quad (2.16)$$

where $\square = -\frac{\lambda^2}{2} D^{\alpha\dot{\beta}} D_{\alpha\dot{\beta}}$ is the wave operator in AdS_4 . So for the primary field $\phi(x) = C_0(0|x)$ this gives

$$\square \phi + 2\lambda^2 \phi = 0. \quad (2.17)$$

Thus unfolded system (2.12) indeed describes a massless scalar field $\phi(x)$ in AdS_4 .¹

One sees from (2.15) that $C_N(Y|x)$ with $N > 0$ are descendant fields that form a tower of totally symmetrized traceless derivatives of the primary scalar $\phi(x)$. The system is on-shell because the primary field is subjected to the differential constraint (2.17). Global transformations (2.6) for AdS-connection (2.7) induce a massless scalar representation of AdS_4 -algebra $so(3,2)$ on the unfolded module (2.11) via the general formula (2.3).

The system (2.12) contains an infinite number of unfolded equations on infinite number of unfolded fields. This is a typical situation: while unfolded, a system with an infinite number of d.o.f. (like a relativistic field) generates an infinite number of descendants which parameterize all these d.o.f (compare with three unfolded equations (2.8)–(2.10) determining non-dynamical AdS background). If the initial system is on-shell, the basis of unfolded descendants is “incomplete”: some subset of possible descendants is absent — namely those that correspond to the l.h.s. of e.o.m. and all their differential consequences. In the scalar field example the basis of descendants (2.15) contains only traceless symmetrized derivatives, and there are no descendants containing d’Alembertians of ϕ , because those are fixed by the mass-shell constraint (2.17).²

Thus, the problem of constructing an off-shell extension for a given on-shell unfolded system amounts to the completion of the basis of unfolded descendants. This can be performed at the level of the unfolded equations by restoring all absent descendants and introducing them to the G^A of the on-shell system (2.1) in the way compatible with the consistency requirement (2.2). However, as was proposed in [53], another way can be followed, if the spectrum of the primary fields and their e.o.m. are known. This input either can be known from the very beginning or can be gained by carrying out the σ_- -analysis [19, 66] that allows for the systematic extraction of the dynamical content of the unfolded system.

To get the idea of [53], consider once again a free massless scalar field. Conventional on-shell formulation is provided by equation (2.17) and unfolding this results in (2.12).

¹As usual, in AdS by massless we mean a scalar field which is conformally coupled to the AdS-curvature, that fixes the mass-like term in (2.17).

²Also there are no descendants containing antisymmetrization of derivatives, but this is because due to AdS commutator $[D, D] \sim \Lambda$ such combinations reduce to the lower descendants from (2.15).

Now let us couple ϕ to an external source $J(x)$. Then (2.17) turns to

$$\square\phi + 2\lambda^2\phi = J. \quad (2.18)$$

If J takes some prescribed value, then (2.18) determines a corresponding backreaction of ϕ to this J . But if J is an a priori unknown function, with (2.18) being the only relation involving it, one can treat (2.18) as the definition of J . In this case the theory in question can be considered as lying off-shell: it describes two scalar fields, $\phi(x)$ and $J(x)$, with primary ϕ totally unconstrained and descendant J defined by (2.18). Unfolding this system is equivalent to unfolding off-shell scalar ϕ . So to construct an off-shell completion of the given on-shell theory one should couple it to the non-fixed external sources, which from the standpoint of the unfolded approach will play a role of the “prodigal sons”, i.e. previously absent descendants.

This method of constructing an off-shell completion has as important advantage as it paves the way to the quantization of the theory directly within the unfolded framework. In the conventional QFT, if the classical e.o.m. for fields $\{\varphi_k(x)\}$, arising from the action S , are

$$\frac{\delta S}{\delta \varphi_n}(\varphi_k) = 0, \quad (2.19)$$

then Schwinger–Dyson equations for the partition function $Z[J] = \int \mathcal{D}\varphi \exp \{iS[\varphi] - iJ_k\varphi^k\}$ of the quantum theory are

$$\frac{\delta S}{\delta \varphi_n} \left(i \frac{\delta}{\delta J_k} \right) Z = J_n Z. \quad (2.20)$$

(Formulation of Schwinger–Dyson equations for non-Lagrangian theories, involving the construction of the so-called Lagrange anchor, is considered in [63–65].) So having at one’s disposal an off-shell system or equivalently an on-shell system coupled to the external currents

$$\frac{\delta S}{\delta \varphi_n}(\varphi_k) = J_n, \quad (2.21)$$

by a substitution $\varphi_k \rightarrow i \frac{\delta}{\delta J_k}$ one arrives at Schwinger–Dyson equations (2.20) that determine the partition function Z and, hence, the whole quantum theory.

3 Unfolded on-shell HS fields and Fronsdal equations

Our aim is to provide an unfolded off-shell formulation for bosonic HS fields in AdS_4 . In conventional Lagrangian language, a massless integer spin- s field propagating in AdS_4 is described, as was found by Fronsdal [67], by a double-traceless rank- s Lorentz tensor $\varphi_{\underline{a}(s)}(x)$ with classical e.o.m.

$$\square\varphi_{\underline{a}(s)} - sD_{\underline{a}}D^{\underline{b}}\varphi_{\underline{b}\underline{a}(s-1)} + \frac{s(s-1)}{2}D_{\underline{a}}D_{\underline{a}}\varphi_{\underline{b}\underline{b}\underline{a}(s-2)} - \lambda^2(s^2-2s-2)\varphi_{\underline{a}(s)} - \lambda^2s(s-1)g_{\underline{a}\underline{a}}\varphi_{\underline{b}\underline{b}\underline{a}(s-2)} = 0, \quad (3.1)$$

and a gauge transformation law (for $s > 0$)

$$\delta\varphi_{\underline{a}(s)} = D_{\underline{a}}\epsilon_{\underline{a}(s-1)}. \quad (3.2)$$

In the unfolded language, the same theory is described by a 1-form $\omega(Y|x)$ (encoding Fronsdal gauge spin- s field and its first $(s-1)$ derivatives) and 0-form $C(Y|x)$ (encoding gauge-invariant HS curvatures and infinite towers of their descendants)

$$\omega(Y|x) = \sum_{n,m} \omega_{\underline{a}|\alpha(n),\dot{\beta}(m)} y^{\alpha_1} \dots y^{\alpha_n} \bar{y}^{\dot{\beta}_1} \dots \bar{y}^{\dot{\beta}_m} dx^{\underline{a}}, \quad C(Y|x) = \sum_{n,m} C_{\alpha(n),\dot{\beta}(m)} y^{\alpha_1} \dots y^{\alpha_n} \bar{y}^{\dot{\beta}_1} \dots \bar{y}^{\dot{\beta}_m}. \quad (3.3)$$

Powers of y and $\bar{y}$ in (3.3) for a concrete spin s are not independent. To fix them we introduce following important operators

$$N = y^\alpha \partial_\alpha, \quad \bar{N} = \bar{y}^{\dot{\alpha}} \bar{\partial}_{\dot{\alpha}}, \quad (3.4)$$

$$\varsigma = \frac{1}{2} (N + \bar{N}), \quad \tau = \frac{1}{2} (N - \bar{N}), \quad (3.5)$$

so that N and $\bar{N}$ count the number of y and $\bar{y}$, respectively, while ς and τ count their half-sum and half-difference. Then for a spin- s field one has

$$\varsigma \omega = (s-1)\omega, \quad |\tau|C = sC. \quad (3.6)$$

Relations (3.6) have a simple interpretation in terms of the Young diagrams of AdS-tensors that ω and C span: they mean that ω includes all one- and two-row Young diagrams with the upper row length equal to $(s-1)$, while C includes all two-row diagrams with the lower row length equal to s . A primary double-traceless Fronsdal spin- s field is identified with

$$\varphi_{\underline{a}(s)} = \omega_{\underline{a}|\alpha(s-1),\dot{\alpha}(s-1)} \left(e_{\underline{a}}^{\alpha\dot{\alpha}} \right)^{s-1}. \quad (3.7)$$

The non-gauge scalar field is degenerate from this point of view: it does not have ω , while its C spans all one-row diagrams as follows from (2.11).

In terms of ω and C , the Fronsdal theory is described by unfolded equations referred to as Central On-Mass-Shell Theorem [62]

$$D\omega + e^{\alpha\dot{\beta}} y_\alpha \bar{\partial}_{\dot{\beta}} \omega + e^{\alpha\dot{\beta}} \partial_\alpha \bar{y}_{\dot{\beta}} \omega = \frac{i}{4} \eta \bar{E}^{\dot{\alpha}\dot{\beta}} \bar{\partial}_{\dot{\alpha}} \bar{\partial}_{\dot{\beta}} C|_{N=0} + \frac{i}{4} \bar{\eta} E^{\alpha\beta} \partial_\alpha \partial_\beta C|_{\bar{N}=0}, \quad (3.8)$$

$$DC + ie^{\alpha\dot{\beta}} \partial_\alpha \bar{\partial}_{\dot{\beta}} C - ie^{\alpha\dot{\beta}} y_\alpha \bar{y}_{\dot{\beta}} C = 0, \quad (3.9)$$

where η is an arbitrary unimodular $\eta\bar{\eta} = 1$ phase parameter accounting for parity breaking. These equations arise in the linear limit of the full nonlinear Vasiliev equations [2] after solving for auxiliary generating variables.

Let us see how (3.8) encodes Fronsdal equation (3.1). To this end one expands ω in the vierbein basis as

$$\omega(Y|x) = e^{\alpha\dot{\alpha}}(x) \omega_{\alpha\dot{\alpha}}(Y|x) \quad (3.10)$$

and then decompose a 0-form $\omega_{\alpha\dot{\alpha}}(Y|x)$ as

$$\omega_{\alpha\dot{\alpha}}(Y|x) = \partial_\alpha \bar{\partial}_{\dot{\alpha}} \phi(Y|x) + y_\alpha \bar{y}_{\dot{\alpha}} \tilde{\phi}(Y|x) + \bar{y}_{\dot{\alpha}} \partial_\alpha \chi(Y|x) + y_\alpha \bar{\partial}_{\dot{\alpha}} \bar{\chi}(Y|x). \quad (3.11)$$

Because of the constraint (3.6), for constituent 0-forms one has

$$\varsigma \phi = s\phi, \quad \varsigma \tilde{\phi} = (s-2)\tilde{\phi}, \quad \varsigma \chi = (s-1)\chi, \quad \varsigma \bar{\chi} = (s-1)\bar{\chi}. \quad (3.12)$$

Contracting (3.11) with $y^\alpha \bar{y}^{\dot{\alpha}}$, $\partial^\alpha \bar{\partial}^{\dot{\alpha}}$, $y^\alpha \bar{\partial}^{\dot{\alpha}}$ or $\bar{y}^{\dot{\alpha}} \partial^\alpha$ one finds

$$\phi = \frac{1}{N\bar{N}} y^\alpha \bar{y}^{\dot{\alpha}} \omega_{\alpha\dot{\alpha}}, \quad \tilde{\phi} = \frac{1}{(N+2)(\bar{N}+2)} \partial^\alpha \bar{\partial}^{\dot{\alpha}} \omega_{\alpha\dot{\alpha}}, \quad (3.13)$$

$$\chi = -\frac{1}{N(\bar{N}+2)} y^\alpha \bar{\partial}^{\dot{\alpha}} \omega_{\alpha\dot{\alpha}}, \quad \bar{\chi} = -\frac{1}{(N+2)\bar{N}} \bar{y}^{\dot{\alpha}} \partial^\alpha \omega_{\alpha\dot{\alpha}}. \quad (3.14)$$

Plugging this back into (3.11) one obtains a following resolution of the identity for $\omega_{\alpha\dot{\alpha}}$:

$$\omega_{\alpha\dot{\alpha}} = \mathbb{P}_{\alpha\dot{\alpha}}^{\beta\dot{\beta}} \omega_{\beta\dot{\beta}}, \quad \mathbb{P}_{\alpha\dot{\alpha}}^{\beta\dot{\beta}} = \frac{1}{(N+1)(\bar{N}+1)} \left(\partial_\alpha \bar{\partial}_{\dot{\alpha}} y^\beta \bar{y}^{\dot{\beta}} + y_\alpha \bar{y}_{\dot{\alpha}} \partial^\beta \bar{\partial}^{\dot{\beta}} - y_\alpha \bar{\partial}_{\dot{\alpha}} \bar{y}^{\dot{\beta}} \partial^\beta - \bar{y}_{\dot{\alpha}} \partial_\alpha y^{\dot{\beta}} \bar{\partial}^{\dot{\beta}} \right). \quad (3.15)$$

Four terms comprising $\mathbb{P}_{\alpha\dot{\alpha}}^{\beta\dot{\beta}}$ form a complete set of orthogonal projectors. They correspond to the decomposition of $\omega_{\alpha\dot{\alpha}}(Y|x)$ in terms of the Young diagrams for Lorentz tensors: an additional, as compared to the Young diagrams of 1-form $\omega(Y|x)$, cell $\alpha\dot{\alpha}$ of $\omega_{\alpha\dot{\alpha}}$ can be added to the first row (first term in (3.15)), subtracted from the first row (second term), or added or subtracted from the second row (third/fourth term depending on the sign of τ).

Now one can construct first-order operators which, being formally (-2) -forms, directly extract wave equations for constituent 0-forms from unfolded equations on ω . For ϕ and $\tilde{\phi}$ they look as, respectively,

$$D_\phi^* = \frac{\lambda^2}{4\varsigma(\varsigma+1)} \left(y^\alpha \bar{y}^{\dot{\alpha}} D_{\alpha\dot{\alpha}} y^\beta \partial^\gamma + \frac{(1-\tau)}{(\varsigma+\tau)} \bar{y}^{\dot{\alpha}} \partial^\alpha D_{\alpha\dot{\alpha}} y^\beta y^\gamma + (\varsigma-\tau+1)(1+\tau) y^\beta \bar{y}^{\dot{\gamma}} \right) \cdot \frac{\partial^2}{\partial e^\beta_{\dot{\beta}} \partial e^{\gamma\dot{\gamma}}} + h.c. \quad (3.16)$$

$$D_{\tilde{\phi}}^* = \frac{\lambda^2}{4(\varsigma+1)(\varsigma+2)} \left(\partial^\alpha \bar{\partial}^{\dot{\alpha}} D_{\alpha\dot{\alpha}} y^\beta \partial^\gamma - \frac{(1+\tau)}{(\varsigma+\tau+2)} y^\alpha \bar{\partial}^{\dot{\alpha}} D_{\alpha\dot{\alpha}} \partial^\beta \partial^\gamma - (\varsigma-\tau+1)(1-\tau) \partial^\beta \partial^\gamma \right) \cdot \frac{\partial^2}{\partial e^\beta_{\dot{\beta}} \partial e^{\gamma\dot{\gamma}}} + h.c. \quad (3.17)$$

Here $h.c.$ operation amounts to the exchange of dotted and undotted variables and hence $\varsigma \rightarrow \varsigma$, $\tau \rightarrow -\tau$ as follows from (3.5). Acting with (3.16) and (3.17) on the first equation of the Central On-Mass-Shell Theorem (3.8) produces respectively

$$\begin{aligned} \square \phi + \frac{\lambda^2}{2s} \left(y^\beta \bar{y}^{\dot{\beta}} D_{\beta\dot{\beta}} \right) \left(\partial^\alpha \bar{\partial}^{\dot{\alpha}} D_{\alpha\dot{\alpha}} \right) \phi - \frac{\lambda^2}{2s} \left(y^\beta \bar{y}^{\dot{\beta}} D_{\beta\dot{\beta}} \right)^2 \tilde{\phi} - \lambda^2 \left(s^2 - 2s - 2 - \tau^2 \right) \phi = \\ = \frac{i\bar{\eta}\lambda^2(s-1)}{4(s+1)} \left(s+1 - \frac{(s-2)}{(2s-1)} \left(\bar{y}^{\dot{\beta}} \partial^\beta D_{\beta\dot{\beta}} \right) \right) C|_{\tau=s} + h.c. \end{aligned} \quad (3.18)$$

$$\square \tilde{\phi} - \frac{\lambda^2}{2s} \left(\partial^\alpha \bar{\partial}^{\dot{\alpha}} D_{\alpha\dot{\alpha}} \right) \left(y^\beta \bar{y}^{\dot{\beta}} D_{\beta\dot{\beta}} \right) \tilde{\phi} + \frac{\lambda^2}{2s} \left(\partial^\alpha \bar{\partial}^{\dot{\alpha}} D_{\alpha\dot{\alpha}} \right)^2 \phi - \lambda^2 \left(s^2 + 2s - 2 - \tau^2 \right) \tilde{\phi} = 0. \quad (3.19)$$

Considering $\tau = 0$ sector, where $\phi_{\alpha(s),\dot{\alpha}(s)}$ and $\tilde{\phi}_{\alpha(s-2),\dot{\alpha}(s-2)}$ are identified with the traceless and trace parts of spin- s Fronsdal field $\varphi_{\underline{a}(s)}$, one sees that equations (3.18) and (3.19)

reproduce traceless and trace parts of the Fronsdal equation (3.1). Thus, Central On-Mass-Shell Theorem (3.8)–(3.9) indeed provides an unfolded formulation for the Fronsdal theory.

The main goal of the paper is to construct an off-shell extension of the system (3.8)–(3.9). To this end, as we discussed in the previous section, one should consistently couple it to external HS currents. From (3.1) it follows that these HS currents represent gauge-invariant double-traceless fields $J_{\underline{a}(s)}^F(x)$ subjected to the generalized conservation law [67]

$$D^b J_{\underline{b}\underline{a}(s-1)}^F = \frac{(s-1)}{2} D_{\underline{a}} J_{\underline{b}\underline{a}(s-2)}^{Fb}. \quad (3.20)$$

So the first task is to unfold the system of $J_{\underline{a}(s)}^F(x)$ with “on-shell constraint” (3.20). But let us start with a simpler degenerate spin-0 case, which is helpful to illustrate the general technique.

4 Off-shell completion of the scalar field

An external current for a scalar field is another scalar field which is unconstrained. So an unfolded system for a scalar source should represent some deformation of (2.12). First, we introduce an unfolded module describing spin-0 source $J^{(0)}$ constrained by condition $(\square + 2\lambda^2)J^{(0)} = 0$,

$$J^{(0)}(Y|x) = \sum_{N=0}^{\infty} J_N^{(0)}(Y|x) = \sum_{N=0}^{\infty} J_{\alpha(N),\dot{\alpha}(N)}^{(0)}(x)(y^\alpha)^N(\bar{y}^{\dot{\alpha}})^N, \quad (4.1)$$

$$DJ^{(0)} + ie\partial\bar{\partial}J^{(0)} - iey\bar{y}J^{(0)} = 0. \quad (4.2)$$

(From now on we omit contracted spinor indices as explained in appendix A.) We add $J^{(0)}$ to the r.h.s. of (2.12) with some coefficients $k_N^{(0)}$ dependent on N

$$DC + ie\partial\bar{\partial}C - iey\bar{y}C = iey\bar{y}k_N^{(0)}J^{(0)}. \quad (4.3)$$

Values of $k_N^{(0)}$ are constrained by consistency condition (2.2), as we will see shortly. Now, in order to relax the constraint $(\square + 2\lambda^2)J^{(0)} = 0$ one can similarly introduce “source for source” $J^{(1)}$ to the r.h.s. of (4.2). Then, to relax $(\square + 2\lambda^2)J^{(1)} = 0$, one has to introduce $J^{(2)}$ and so on. At the end of the day, one arrives at an infinite sequence of sources $J(Y|x) = \sum_{n=0}^{\infty} J^{(n)}$ subjected to unfolded equations

$$DJ^{(n)} + ie\partial\bar{\partial}J^{(n)} - iey\bar{y}J^{(n)} = iey\bar{y}k_N^{(n+1)}J^{(n+1)}. \quad (4.4)$$

Consistency condition (2.2) requires

$$(N+2)k_N^{(n)} = Nk_{N-1}^{(n)}, \quad (4.5)$$

which is the only constraint on coefficients $k_N^{(n)}$. There is always certain (usually very large) freedom in the choice of coefficients in unfolded equations, because new descendants can be introduced with arbitrary multipliers, when first appearing in an unfolded system.

This freedom can be used to simplify the form of equations, but the choice may affect the further analysis. We will face such a situation below. Here we fix coefficients to be

$$k_N^{(n)} = -\frac{g\lambda^{-2}}{(N+1)(N+2)} \quad (4.6)$$

with g playing the role of the coupling constant. It is convenient to organize all $J^{(n)}$ into a single master-source $J(Y|p|x)$ as a formal expansion in some auxiliary parameter p

$$J(Y|p|x) = \sum_{n=0}^{\infty} \frac{p^n}{n!} J^{(n)}(Y|x) = \sum_{n,N=0}^{\infty} \frac{p^n}{n!} J_{\alpha(N),\dot{\alpha}(N)}^{(n)}(x) (y^\alpha)^N (\bar{y}^{\dot{\alpha}})^N. \quad (4.7)$$

Then an unfolded system describing off-shell scalar field takes the form

$$DC + ie\partial\bar{\partial}C - iey\bar{y}C = -iey\bar{y} \frac{g\lambda^{-2}}{(N+1)(N+2)} J^{(0)}, \quad (4.8)$$

$$DJ + ie\partial\bar{\partial}J - iey\bar{y}J = -iey\bar{y} \frac{g\lambda^{-2}}{(N+1)(N+2)} \frac{\partial}{\partial p} J. \quad (4.9)$$

Acting on (4.8) with an operator

$$D_C^* = -\frac{\lambda^2}{2} \left(D^{\alpha\dot{\beta}} - i\partial^\alpha \bar{\partial}^{\dot{\beta}} + iy^\alpha \bar{y}^{\dot{\beta}} \right) \frac{\partial}{\partial e^{\alpha\dot{\beta}}}, \quad (4.10)$$

which represents an analogue of (3.16) and (3.17) for the second equation of the Central On-Mass-Shell Theorem (3.9), extracts wave equations for components

$$\square C_N + \lambda^2 (N^2 + 2N + 2) C_N = \frac{ig}{2} y^\alpha \bar{y}^{\dot{\beta}} D_{\alpha\dot{\beta}} \frac{1}{(N+1)(N+2)} J_{N-1}^{(0)} + \frac{g(N+2)}{2(N+1)} J_N^{(0)}. \quad (4.11)$$

For the primary field $\phi(x) = C_0(0|x)$ this gives

$$\square\phi + 2\lambda^2\phi = gJ_0^{(0)}(x). \quad (4.12)$$

So (4.8)–(4.9) indeed describe a scalar field $\phi(x)$ coupled to an external current $J_0^{(0)}(x)$ with coupling constant g or, interpreted differently, they describe an unfolded off-shell scalar field, with $J(Y|p|x)$ encoding descendants which contain d'Alembertians of the primary scalar $\phi(x)$.

From the point of view of representation theory, the above construction goes as follows. One starts with $\text{SO}(3,2)$ -module $D(1,0) \oplus D(2,0)$, corresponding to an AdS_4 massless scalar field (two submodules here are due to two different boundary asymptotics of the AdS-scalar), and “glues” it to an external current module. This results in an indecomposable representation which contains the external current as an invariant subspace. The peculiarity of the scalar field is that its external current module itself represents an infinite chain of the same gluings: all $J^{(n)}(Y|x)$ in the expansion (4.7) are isomorphic as $\text{SO}(3,2)$ -modules by construction, and, as seen from (4.8)–(4.9), for any n_0 a subspace of unfolded fields

$$J^{(>n_0)} := \sum_{n=n_0+1}^{\infty} \frac{p^n}{n!} J^{(n)}(Y|x) \quad (4.13)$$

corresponds to an invariant subspace in the dual Verma module. So the off-shell scalar module we constructed represents an infinitely indecomposable sequence of glued $D(1,0) \oplus D(2,0)$ modules, such that each subsequent $D(1,0) \oplus D(2,0)$ is nested as an invariant subspace.

Now, having in hand this off-shell system, one can move in two different directions. First, one can perform various consistent reductions of the unfolded module, getting rid of some part of descendants. This leads to different on-shell theories with different equations of motion. As we will see, on this way it is possible to describe an on-shell scalar field with arbitrary mass. Second, one can reformulate (4.8)–(4.9) as an unfolded Schwinger–Dyson system and quantize the theory this way.

4.1 On-shell reduction: a scalar field of arbitrary mass

We want to find an unfolded system that describes a scalar field $\phi(x)$ subjected to e.o.m.

$$\left(\square + 2\lambda^2 - m^2\right)\phi = 0. \quad (4.14)$$

This should arise as some reduction of the off-shell system (4.8)–(4.9). Looking at (4.12) one sees that g_0 should be identified with m^2 , and $J_0^{(0)}(x)$ with $\phi(x)$. Then from (4.9) it follows that all $J^{(n)}(Y|x)$ and their equations have to represent exact copies of $C(Y|x)$. Thus the submodule $J(Y|p|x)$ actually disappears and one is left with an unfolded module $C(Y|x)$ subjected to

$$DC + ie\partial\bar{\partial}C - iey\bar{y}C + iey\bar{y}\frac{(m/\lambda)^2}{(N+1)(N+2)}C = 0. \quad (4.15)$$

This provides an unfolded formulation of the free on-shell scalar field with mass m . To see this one acts on (4.15) with an operator

$$D_{C,m}^* = -\frac{\lambda^2}{2}(D^{\alpha\dot{\beta}} - i\partial^\alpha\bar{\partial}^{\dot{\beta}} + iy^\alpha\bar{y}^{\dot{\beta}} - i\bar{y}^\alpha y^{\dot{\beta}}\frac{(m/\lambda)^2}{(N+1)(N+2)})\frac{\partial}{\partial e^{\alpha\dot{\beta}}}, \quad (4.16)$$

which is a generalization of (4.10) for nonzero mass. It recovers following wave equations from (4.15)

$$\square C_N + \lambda^2(N^2 + 2N + 2)C_N - m^2 C_N = 0, \quad (4.17)$$

so for a primary scalar $\phi(x) = C_0(0|x)$ one has (4.14).

Also, one can define an unfolded system that corresponds to the “off-shell scalar field with mass m ”. By this we mean an unfolded system which describes the coupling of the external current directly to (4.14). Although in terms of content it gives nothing new compared to (4.8)–(4.9), it is handy when dealing with the quantization problem considered below. To construct this system one notices that from the standpoint of formal consistency of (4.8)–(4.9) the only requirement for $g\lambda^{-2}\frac{\partial}{\partial p}$ is to commute with all other operators in equations that act on C and J . So if one shifts it by $(m/\lambda)^2$ the consistency will be preserved. The resulting equations are

$$DC + ie\partial\bar{\partial}C - iey\bar{y}C + iey\bar{y}\frac{(m/\lambda)^2}{(N+1)(N+2)}C = -iey\bar{y}\frac{g\lambda^{-2}}{(N+1)(N+2)}J^{(0)}, \quad (4.18)$$

$$DJ + ie\partial\bar{\partial}J - iey\bar{y}J + iey\bar{y}\frac{(m/\lambda)^2}{(N+1)(N+2)}J = -iey\bar{y}\frac{g\lambda^{-2}}{(N+1)(N+2)}\frac{\partial}{\partial p}J, \quad (4.19)$$

and application of (4.16) reveals a desired constraint

$$\left(\square + 2\lambda^2 - m^2\right) \phi = gJ_0^{(0)}. \quad (4.20)$$

One can also write down a more general off-shell system with varying mass parameter: formally, one can take $m = m(p)$ being an arbitrary function of p in (4.19). In this case the current module represents an infinite sequence of glued $\text{SO}(3, 2)$ -modules of scalar fields with arbitrary masses.

Finally, as was pointed out in [53], it is possible to impose higher-order equations of motion. Consider for simplicity $m = 0$ case (4.8)–(4.9). Then, if one restricts the upper limit of n -summation in (4.7) with some n_0 , resulting system will encode the following order- $(2n_0 + 4)$ equation of motion³

$$\left(\square + 2\lambda^2\right)^{n_0+2} \phi(x) = 0. \quad (4.21)$$

In terms of the representation theory this restriction is tantamount to the quotienting of the infinitely indecomposable off-shell scalar Verma module by its submodule dual to (4.13). Resulting representation is $(n_0 + 1)$ times indecomposable, with remaining $(n_0 + 1)$ nested submodules being dual to spaces of solutions to $(\square + 2\lambda^2)^k \phi = 0$, $k = [1, (n_0 + 1)]$. And the maximal on-shell reduction $J = 0$, leading to Klein–Gordon equation (2.17), implements a quotient by the maximal submodule (dual to the whole J), which results in an irreducible (after fixing boundary conditions) module $D(1, 0) \oplus D(2, 0)$.

4.2 Quantization: a scalar field propagator

Here we present a sample calculation which illustrates how one can calculate correlation functions from the off-shell unfolded system. To this end we treat (4.18)–(4.19) as Schwinger–Dyson equations and try to solve them. As we deal with a free theory with linear e.o.m. it is convenient to introduce a connected generating functional $W = i \log Z$. Then a transition from classical e.o.m. for field φ_n to Schwinger–Dyson equation for W is performed by a substitution $\varphi_k \rightarrow \frac{\delta W}{\delta J_k}$ and addition of J_n to the r.h.s. Thus if we manage to solve (4.18) for J we will be able to restore two-point function of a massive scalar field in AdS_4 .

Applying (4.16) to (4.18) leads to

$$\left(\square + \left(N^2 + 2N + 2\right) \lambda^2 - m^2\right) C = \frac{g}{2(N+1)} \left(\frac{iy^\alpha \bar{y}^{\dot{\alpha}}}{(N+2)} D_{\alpha\dot{\alpha}} + N + 2 \right) J. \quad (4.22)$$

This can be solved by means of a standard AdS_4 scalar propagator $G(x, x')$ that solves

$$\left(\square - \mu^2\right) G(x, x') = \delta(x, x'). \quad (4.23)$$

The propagator is expressed via the hypergeometric function as [68]

$$G_\Delta(x, x') = \lambda^2 A_\Delta \xi^\Delta F\left(\frac{\Delta}{2}, \frac{\Delta+1}{2}; \Delta - \frac{1}{2}; \xi^2\right), \quad (4.24)$$

³If one starts from (4.18)–(4.19), it is possible to get a general $(n_0 + 2)$ -degree polynomial in $\square$ with arbitrary coefficients on the l.h.s. of (4.21).

where ξ in the Poincaré coordinates is

$$\xi = \frac{2zz'}{z^2 + z'^2 + (x - x')^2}, \quad (4.25)$$

conformal weight is determined from

$$(\mu/\lambda)^2 = \Delta(\Delta - 3) \quad (4.26)$$

and A_Δ is a Δ -dependent normalization constant. Thus for (4.22) conformal weights are determined by

$$\Delta_{m,N}(\Delta_{m,N} - 3) = (m/\lambda)^2 - (N^2 + 2N + 2), \quad (4.27)$$

and a solution for (4.22) is

$$C(Y|x) = \int dx'^4 G_{\Delta_{m,N}}(x, x') \frac{g}{2(N+1)} \left(\frac{iy^\alpha \bar{y}^{\dot{\alpha}}}{(N+2)} D_{\alpha\dot{\alpha}} + N + 2 \right) J(Y|x'). \quad (4.28)$$

This encodes two-point functions for all components of the unfolded module $C(Y|x)$ in terms of the unfolded sources from $J(Y|x)$. To extract them explicitly in terms of the primary scalar field $\phi(x) = C(0|x)$ one has to express descendants from $C(Y|x)$ in terms of ϕ and descendants from $J(Y|x)$ in terms of the primary source $J_0^{(0)} = J(Y=0|p=0|x)$. Then by replacing $\phi \rightarrow \frac{\delta W}{\delta J_0^{(0)}}$ in (4.28) one can evaluate two-point functions. For instance, for the propagator of the primary ϕ one has

$$\langle \phi(x) \phi(x') \rangle = g G_{\Delta_{m,0}}(x, x'), \quad (4.29)$$

as, of course, it should be.

5 Unfolded system for Fronsdal current

Our goal is to apply the method of off-shell completion reasoned in section 3 and illustrated in section 4 to the Fronsdal field of arbitrary integer spin. To this end we should unfold Fronsdal current (3.20) and then couple it to the unfolded Fronsdal equations (3.8)–(3.9). This section is devoted to the first part of the problem.

Double-traceless Fronsdal current $J_{\underline{a}(s)}^F(x)$ can be decomposed into two traceless currents: $J_{\underline{a}(s)}(x)$ (traceless part of J^F) and $T_{\underline{a}(s-2)}(x)$ (trace of J^F). Then the content of the generalized conservation law (3.20) is that the divergence $D^b J_{\underline{b}\underline{a}(s-1)}$ is proportional to the first symmetrized derivative $D_{\underline{a}} T_{\underline{a}(s-2)}$. Apart from this, $J_{\underline{a}(s)}$ and $T_{\underline{a}(s-2)}$ are unconstrained. So we unfold the Fronsdal current in the following way: first, in subsection 5.1 we unfold traceless conserved $J_{\underline{a}(s)}$; based on this, in subsection 5.2 we find an unfolded formulation for unconstrained traceless $T_{\underline{a}(s-2)}$; finally, in subsection 5.3 we couple T to J in such a way as to impose (3.20), which completes the procedure. Also, as we show in subsection 5.1.1, the unfolded system for conserved J admits a reduction to an unfolded system that describes on-shell massive HS fields subjected to Fierz-Pauli conditions.

5.1 Conserved traceless HS current

The first task is to find an unfolded system for traceless spin- s current $J_{\underline{a}(s)}(x)$ subjected to the conservation condition

$$D^b J_{b\underline{a}(s-1)} = 0. \quad (5.1)$$

We solve the problem in two steps. First, we construct an unfolded system for $J_{\underline{a}(s)}$ that in addition to (5.1) also satisfies “masslessness” condition

$$\square J_{\underline{a}(s)} + 2\lambda^2 J_{\underline{a}(s)} = 0. \quad (5.2)$$

Then we remove (5.2) by introducing additional descendants, similar to what we did for the scalar field.

An appropriate unfolded module for (5.1), (5.2) is $J(Y|x)$ such that

$$\varsigma \geq s, \quad |\tau| \leq s. \quad (5.3)$$

In Lorentz tensor language this module corresponds to the space of all one- and two-row traceless Young diagrams with the upper (the only for one-row diagrams) row length at least s and lower row length at most s . This is because the primary source $J_{\underline{a}(s)}$ represents a Young diagram with one row of length s , and successive differentiation of $J_{\underline{a}(s)}$ will add new cells to the diagram, corresponding to the traceless-symmetrized derivatives. All contractions or antisymmetrizations of derivatives give nothing new due to (5.1), (5.2) and the commutator of AdS-covariant derivatives.

The most general Ansatz for an unfolded equation is

$$DJ + ie\partial\bar{\partial}a_{N,\bar{N}}J - ie\bar{y}\bar{y}b_{N,\bar{N}}J + e\bar{y}\bar{\partial}c_{N,\bar{N}}J + e\bar{y}\partial\bar{c}_{N,\bar{N}}J = 0. \quad (5.4)$$

Coefficients $a_{N,\bar{N}}$, $b_{N,\bar{N}}$, $c_{N,\bar{N}}$ and $\bar{c}_{N,\bar{N}}$ are (partially) fixed by the consistency requirement (2.2). Imposing it one arrives at the following recurrent system

$$(N+2)a_{N+1,\bar{N}-1}c_{N,\bar{N}} - Na_{N,\bar{N}}c_{N-1,\bar{N}-1} = 0, \quad (5.5)$$

$$Nb_{N-1,\bar{N}+1}\bar{c}_{N,\bar{N}} - (N+2)b_{N,\bar{N}}\bar{c}_{N+1,\bar{N}+1} = 0, \quad (5.6)$$

$$(N+2)a_{N+1,\bar{N}+1}b_{N,\bar{N}} - Na_{N,\bar{N}}b_{N-1,\bar{N}-1} + (N+2)c_{N,\bar{N}}\bar{c}_{N+1,\bar{N}-1} - N\bar{c}_{N,\bar{N}}c_{N-1,\bar{N}+1} = 2, \quad (5.7)$$

plus conjugate equations resulting from $N \leftrightarrow \bar{N}$, $a_{K,\bar{L}} \rightarrow a_{L,\bar{K}}$, $b_{K,\bar{L}} \rightarrow b_{L,\bar{K}}$ and $c_{K,\bar{L}} \leftrightarrow \bar{c}_{L,\bar{K}}$.

As mentioned previously, partial leftover freedom in fixing unfolded coefficients helps to simplify the system, but may affect further nonlinear deformations or couplings to other systems. In fact, such a simplification is even necessary to some extent, because it seems unfeasible to explicitly solve all consistency conditions in their full generality. The final goal of the current analysis is to couple an unfolded module $J(Y|x)$ to the unfolded Fronsdal equations (3.8)–(3.9) with $J_{\alpha(s),\dot{\alpha}(s)}(x)$ playing the role of the traceless part of the Fronsdal spin- s current. It turns out that not every possible solution of (5.5)–(5.7) leads to this result.

The proper choice is to put

$$a_{N,\bar{N}} = 1, \quad \varsigma > s, \quad (5.8)$$

$$a_{N,\bar{N}} = 0, \quad \varsigma = s, \quad (5.9)$$

where “boundary condition” (5.9) is dictated by self-consistency: nonzero $a_{N,\bar{N}}$ with $\varsigma = s$ would give rise to descendants with tensor rank $(s-1)$ which are absent in J by construction. A different solution, which has somewhat simpler-looking coefficients but does not allow the identification of $J_{\alpha(s),\dot{\alpha}(s)}$ with a traceless part of Fronsdal current, is given in appendix B.

Applying (5.8) to (5.5)–(5.6) and their conjugate allows one to solve for $b_{N,\bar{N}}$, $c_{N,\bar{N}}$ and $\bar{c}_{N,\bar{N}}$

$$b_{N,\bar{N}} = \frac{b_\varsigma}{(N+1)(N+2)(\bar{N}+1)(\bar{N}+2)}, \quad (5.10)$$

$$c_{N,\bar{N}} = \frac{c_\tau}{(N+1)(N+2)}, \quad (5.11)$$

$$\bar{c}_{N,\bar{N}} = \frac{\bar{c}_\tau}{(\bar{N}+1)(\bar{N}+2)}, \quad (5.12)$$

where b_ς and c_τ , $\bar{c}_\tau$ are so far arbitrary functions of ς and τ respectively. To determine them one adds to (5.7) its conjugate and substituting (5.10)–(5.11) arrives at

$$2\left((\varsigma+1)^4 - (\varsigma+1)^2 - \tau^4 + \tau^2\right) + (\varsigma+2)b_{\varsigma-1} - \varsigma b_\varsigma + (\tau-1)c_\tau \bar{c}_{\tau+1} - (\tau+1)\bar{c}_\tau c_{\tau-1} = 0. \quad (5.13)$$

From here one finds, separating ς and τ variables,

$$b_\varsigma = (\varsigma+1)^2(\varsigma+2)^2 + (\varsigma+1)(\varsigma+2)b, \quad (5.14)$$

$$\bar{c}_\tau c_{\tau-1} = \tau^2(\tau-1)^2 + \tau(\tau-1)c, \quad (5.15)$$

with b and c being arbitrary constants. To fix them one notices that restrictions (5.3) put following “boundary conditions” on b_ς , c_τ and $\bar{c}_\tau$

$$b_{\varsigma=s-1} = 0, \quad c_{\tau=s} = 0, \quad \bar{c}_{\tau=-s} = 0. \quad (5.16)$$

To satisfy them one chooses

$$b = c = -s(s+1). \quad (5.17)$$

Then an unfolded system for the traceless spin- s current $J_{\underline{a}(s)}$ subjected to (5.1) and (5.2) is

$$\begin{aligned} DJ + ie\partial\bar{\partial}J - ie\bar{y}\bar{y} \frac{(\varsigma+1)(\varsigma+2)(\varsigma-s+1)(\varsigma+s+2)}{(N+1)(N+2)(\bar{N}+1)(\bar{N}+2)} J + \\ + e\bar{y}\bar{\partial} \frac{\tau(\tau-s)}{(N+1)(N+2)} J + e\bar{y}\partial \frac{\tau(\tau+s)}{(\bar{N}+1)(\bar{N}+2)} J = 0. \end{aligned} \quad (5.18)$$

Now one has to relax masslessness condition (5.2). The idea is the same as for the scalar field: one introduces an infinite sequence of descendant modules $J^{(n)}(Y|x)$, successively

contributing to the r.h.s. of (5.2) for each other, and collects them into single $J(Y|p|x) = \sum_{n=0}^{\infty} \frac{p^n}{n!} J^{(n)}(Y|x)$. So one seeks a generalization of (5.18) of the form

$$\begin{aligned}
 & DJ + ie\partial\bar{\partial}J - iey\bar{y} \frac{(\varsigma+1)(\varsigma+2)(\varsigma-s+1)(\varsigma+s+2)}{(N+1)(N+2)(\bar{N}+1)(\bar{N}+2)} J + \\
 & + ey\bar{\partial} \frac{\tau(\tau-s)}{(N+1)(N+2)} J + e\bar{y}\partial \frac{\tau(\tau+s)}{(\bar{N}+1)(\bar{N}+2)} J + \\
 & + ie\partial\bar{\partial}\hat{a}_{N,\bar{N}} \frac{\partial}{\partial p} J + iey\bar{y}\hat{b}_{N,\bar{N}} \frac{\partial}{\partial p} J + ey\bar{\partial}\hat{c}_{N,\bar{N}} \frac{\partial}{\partial p} J + e\bar{y}\partial\hat{c}_{N,\bar{N}} \frac{\partial}{\partial p} J = 0, \tag{5.19}
 \end{aligned}$$

with some coefficients $\hat{a}_{N,\bar{N}}$, $\hat{b}_{N,\bar{N}}$, $\hat{c}_{N,\bar{N}}$ and $\hat{\bar{c}}_{N,\bar{N}}$. In principle, one could allow these coefficients to depend on p , but this is an unnecessary overgeneralization, as follows from the isomorphism of all $J^{(n)}$.

Once again, it is possible to fix a part of unknown coefficients from the very beginning. New descendants $J^{(n)}$ encode d'Alembertians of the primary source so, from the standpoint of Young diagrams, these descendants arise from removal of cells. So we keep only those terms that correspond to such removing and take

$$\hat{a}_{N,\bar{N}} = 0, \quad \hat{c}_{N,\bar{N}}|_{\tau < 0} = 0, \quad \hat{\bar{c}}_{N,\bar{N}}|_{\tau > 0} = 0. \tag{5.20}$$

Then formal consistency (2.2) requires

$$(N+2)\hat{c}_{N,\bar{N}} - N\hat{c}_{N-1,\bar{N}-1} = 0, \tag{5.21}$$

$$\begin{aligned}
 & \frac{(\varsigma+1)(\varsigma+2)(\varsigma-s+1)(\varsigma+s+2)}{(N+1)(\bar{N}+3)} \hat{\bar{c}}_{N,\bar{N}} + (N+2) \frac{\tau(\tau+s)}{(\bar{N}+3)} \hat{b}_{N,\bar{N}} - \\
 & - \frac{(\varsigma+1)(\varsigma+2)(\varsigma-s+1)(\varsigma+s+2)}{(N+1)(\bar{N}+1)} \hat{\bar{c}}_{N+1,\bar{N}+1} - N \frac{\tau(\tau+s)}{(\bar{N}+1)} \hat{b}_{N-1,\bar{N}+1} = 0. \tag{5.22}
 \end{aligned}$$

$$\begin{aligned}
 & N\hat{b}_{N-1,\bar{N}-1} + \frac{\tau(\tau-s)}{(\bar{N}+1)} \hat{\bar{c}}_{N+1,\bar{N}-1} + (N+2) \frac{(\tau+1)(\tau+s+1)}{\bar{N}(\bar{N}+1)} \hat{c}_{N,\bar{N}} - \\
 & - (N+2)\hat{b}_{N,\bar{N}} - \frac{(\tau-1)(\tau-s-1)}{(N+1)} \hat{\bar{c}}_{N,\bar{N}} - N \frac{\tau(\tau+s)}{(\bar{N}+1)(\bar{N}+2)} \hat{c}_{N-1,\bar{N}+1} = 0, \tag{5.23}
 \end{aligned}$$

$$(N+2)\hat{b}_{N,\bar{N}}\hat{\bar{c}}_{N+1,\bar{N}+1} - N\hat{b}_{N-1,\bar{N}+1}\hat{\bar{c}}_{N,\bar{N}} = 0, \tag{5.24}$$

$$(N+2)\hat{\bar{c}}_{N+1,\bar{N}-1}\hat{c}_{N,\bar{N}} - N\hat{c}_{N-1,\bar{N}+1}\hat{\bar{c}}_{N,\bar{N}} = 0, \tag{5.25}$$

plus conjugate equations. From (5.21) and conjugate one finds

$$\hat{c}_{N,\bar{N}}|_{\tau \geq 0} = \frac{\hat{c}_{\tau}}{(N+1)(N+2)}, \tag{5.26}$$

$$\hat{\bar{c}}_{N,\bar{N}}|_{\tau \leq 0} = \frac{\hat{\bar{c}}_{\tau}}{(\bar{N}+1)(\bar{N}+2)}, \tag{5.27}$$

with $\hat{c}_\tau$ being arbitrary function of τ and $\hat{\bar{c}}_\tau$ being its conjugate $\hat{\bar{c}}_\tau = \hat{c}_{-\tau}$ in order to ensure reality. After substituting (5.27) to (5.22) terms with $\hat{\bar{c}}_\tau$ cancel and for $\hat{b}_{N,\bar{N}}$ one obtains

$$\hat{b}_{N,\bar{N}} = \frac{\hat{b}_\varsigma}{(N+1)(N+2)(\bar{N}+1)(\bar{N}+2)} \quad (5.28)$$

with arbitrary ς -dependent $\hat{b}_\varsigma$. Now, the sum and the difference of (5.23) and its conjugate yield, respectively,

$$\begin{aligned} & (\tau+1) \left((\tau-1)(\tau-1-s)\hat{c}_\tau + \tau(\tau+s)\hat{c}_{\tau-1} \right) + (\varsigma+2)\hat{b}_{\varsigma-1} - \\ & - (\tau-1) \left((\tau+1)(\tau+1+s)\hat{c}_\tau + \tau(\tau-s)\hat{c}_{\tau+1} \right) - \varsigma\hat{b}_\varsigma = 0. \end{aligned} \quad (5.29)$$

$$\begin{aligned} & (\varsigma+1) \left((\tau+1)(\tau+1+s)\hat{c}_\tau + \tau(\tau-s)\hat{c}_{\tau+1} \right) + \tau\hat{b}_\varsigma - \\ & - (\varsigma+1) \left((\tau-1)(\tau-1-s)\hat{c}_\tau + \tau(\tau+s)\hat{c}_{\tau-1} \right) - \tau\hat{b}_{\varsigma-1} = 0, \end{aligned} \quad (5.30)$$

Separating ς and τ variables and making use of the reality condition $\hat{c}_\tau = \hat{\bar{c}}_{-\tau}$ one easily finds, taking into account (5.20),

$$\hat{b}_\varsigma = \hat{b} + \hat{c}(\varsigma+1)(\varsigma+2), \quad (5.31)$$

$$\hat{c}_{\tau \geq 0} = -\frac{\hat{b} + \hat{c}\tau(\tau+1)}{(\tau+1)(\tau+1+s)}, \quad (5.32)$$

$$\hat{\bar{c}}_{\tau \leq 0} = -\frac{\hat{b} + \hat{c}\tau(\tau-1)}{(\tau-1)(\tau-1-s)}, \quad (5.33)$$

with $\hat{b}$ and $\hat{c}$ being arbitrary constants. Restrictions on ς and τ values (5.3) require $\hat{\bar{c}}_{-s} = 0$ and $\hat{c}_s = 0$, which entail

$$\hat{b} = -\hat{c}s(s+1). \quad (5.34)$$

Then by rescaling p in (5.19) one can always set

$$\hat{c} = 1, \quad \hat{b} = -s(s+1). \quad (5.35)$$

Remaining equations (5.24), (5.25) and their conjugate now hold identically and give no further constraints.

So, now one is in a position to write down an unfolded system that describes a conserved traceless spin- s current. It is handy to introduce projectors Π^+ and Π^- to components with $\tau \geq 0$ and $\tau \leq 0$

$$\Pi^+ F_{N,\bar{N}}(Y) = \begin{cases} F_{N,\bar{N}}(Y), & N \geq \bar{N} \\ 0, & N < \bar{N} \end{cases}; \quad \Pi^- F_{N,\bar{N}}(Y) = \begin{cases} F_{N,\bar{N}}(Y), & N \leq \bar{N} \\ 0, & N > \bar{N} \end{cases}. \quad (5.36)$$

Then the system in question is realized by an unfolded module $J(Y|p|x)$ with $\varsigma \geq s$, $|\tau| \leq s$ subjected to unfolded equation

$$\begin{aligned}
 & DJ + ie\partial\bar{\partial}J - iey\bar{y} \frac{(\varsigma+1)(\varsigma+2)(\varsigma-s+1)(\varsigma+s+2)}{(N+1)(N+2)(\bar{N}+1)(\bar{N}+2)} J + \\
 & + ey\bar{\partial} \frac{\tau(\tau-s)}{(N+1)(N+2)} J + e\bar{y}\partial \frac{\tau(\tau+s)}{(\bar{N}+1)(\bar{N}+2)} J + iey\bar{y} \frac{(\varsigma-s+1)(\varsigma+s+2)}{(N+1)(N+2)(\bar{N}+1)(\bar{N}+2)} \frac{\partial}{\partial p} J - \\
 & - ey\bar{\partial} \frac{(\tau-s)}{(\tau+1)(N+1)(N+2)} \frac{\partial}{\partial p} \Pi^+ J - e\bar{y}\partial \frac{(\tau+s)}{(\tau-1)(\bar{N}+1)(\bar{N}+2)} \frac{\partial}{\partial p} \Pi^- J = 0. \quad (5.37)
 \end{aligned}$$

5.1.1 On-shell reduction: Fierz-Pauli massive HS fields

Interestingly, the unfolded formulation for conserved spin- s current we found also allows for a description of the on-shell massive HS fields. Concretely, equation (5.37) admits a simple reduction that sends it to the unfolded system, corresponding to the (AdS-generalization of) Fierz-Pauli equations for a massive spin- s field

$$(\square - m^2) \phi_{\underline{a}(s)} = 0, \quad (5.38)$$

$$D^b \phi_{\underline{ba}(s-1)} = 0, \quad (5.39)$$

$$\phi^b_{\underline{ba}(s-2)} = 0. \quad (5.40)$$

A general idea is the same as was used for the scalar field in subsection 4.1: from the standpoint of formal consistency of (5.37), the only requirement for $\frac{\partial}{\partial p}$ is to commute with all other operators in the equation. So one can reduce the unfolded module $J(Y|p|x) \rightarrow J(Y|x)$ replacing $\frac{\partial}{\partial p}$ in (5.37) with $(m/\lambda)^2$ and this will not ruin the consistency. Then one arrives at

$$\begin{aligned}
 & DJ + ie\partial\bar{\partial}J - iey\bar{y} \frac{(\varsigma-s+1)(\varsigma+s+2)((\varsigma+1)(\varsigma+2) - (m/\lambda)^2)}{(N+1)(N+2)(\bar{N}+1)(\bar{N}+2)} J + \\
 & + ey\bar{\partial} \frac{(\tau-s)}{(N+1)(N+2)} \left(\tau - \frac{(m/\lambda)^2}{(\tau+1)} \Pi^+ \right) J + e\bar{y}\partial \frac{(\tau+s)}{(\bar{N}+1)(\bar{N}+2)} \left(\tau - \frac{(m/\lambda)^2}{(\tau-1)} \Pi^- \right) J = 0. \quad (5.41)
 \end{aligned}$$

Acting with a “conjugate operator”

$$\begin{aligned}
 D_{FP}^* = & -\frac{\lambda^2}{2} \left(D^{\alpha\dot{\beta}} - i\partial^\alpha \bar{\partial}^{\dot{\beta}} + iy^\alpha \bar{y}^{\dot{\beta}} \frac{(\varsigma-s+1)(\varsigma+s+2)((\varsigma+1)(\varsigma+2) - (m/\lambda)^2)}{(N+1)(N+2)(\bar{N}+1)(\bar{N}+2)} - \right. \\
 & \left. - y^\alpha \bar{\partial}^{\dot{\beta}} \frac{(\tau-s)}{(N+1)(N+2)} \left(\tau - \frac{(m/\lambda)^2}{(\tau+1)} \Pi^+ \right) J - \bar{y}^{\dot{\beta}} \partial^\alpha \frac{(\tau+s)}{(\bar{N}+1)(\bar{N}+2)} \left(\tau - \frac{(m/\lambda)^2}{(\tau-1)} \Pi^- \right) \right) \frac{\partial}{\partial e^{\alpha\dot{\beta}}} \quad (5.42)
 \end{aligned}$$

on (5.41), one reveals the following wave equation for J

$$\square J - m^2 J + \lambda^2(\varsigma(\varsigma+1) + \tau^2 - s(s+1) + 2)J = 0. \quad (5.43)$$

For the primary field $J_{\alpha(s),\dot{\alpha}(s)}(x)$ this gives

$$\left(\square - m^2 + 2\lambda^2\right) J_{\alpha(s),\dot{\alpha}(s)} = 0. \quad (5.44)$$

Tracelessness and conservation condition (5.1) were built-in from the very beginning. Thus (5.41) indeed provides an unfolded formulation for Fierz-Pauli massive HS fields. (5.41) also allows for an immediate off-shell completion: one just has to restore p -dependence of the module $J(Y|p|x)$ and add $\frac{\partial}{\partial p}$ to each $(m/\lambda)^2$ in (5.41). Resulting system for “off-shell HS field with mass m ” will be equivalent to (5.37) up to field redefinitions (and will coincide literally in $m = 0$ case), like it was for “off-shell scalar field with mass m ” in subsection 4.1.

5.2 Trace of the Fronsdal current

The next task is to build an unfolded system for the trace of spin- s Fronsdal current. This represents a rank- $(s-2)$ symmetric traceless tensor $T_{\underline{a}(s-2)}(x)$ whose first symmetrized derivative is proportional to the divergence of J according to (3.20). So we are going to unfold an unconstrained symmetric traceless rank- $(s-2)$ tensor and then add T -dependent terms to (5.37).

The distinction between T -module and J -module studied before is that now a primary field possesses unconstrained divergences, which generate unfolded submodules corresponding to lower-rank tensors. In order to account for them we introduce another expansion parameter q in addition to p

$$T(Y|p, q|x) = \sum_{n=0}^{\infty} \sum_{m=0}^{s-2} \frac{p^n}{n!} \frac{q^m}{m!} T^{(n,m)}(Y|x). \quad (5.45)$$

Submodules $T^{(n,m)}$, corresponding to higher q -powers m , contain order- m divergences of the submodules $T^{(n,0)}$, so in each $T^{(n,m)}$ a primary field is a tensor of rank $(s-m-2)$. Thus instead of (5.3) now one has for $T(Y|p, q|x)$

$$\varsigma \geq s-2-\nu, \quad |\tau| \leq s-2-\nu, \quad (5.46)$$

where a q -power counting operator ν is defined as

$$\nu = q \frac{\partial}{\partial q}. \quad (5.47)$$

Then an appropriate Ansatz for the unfolded system in question is

$$\begin{aligned} & DT + ie\partial\bar{\partial}T - ie\bar{y}\bar{y} \frac{(\varsigma-s+\nu+3)(\varsigma+s-\nu)}{(N+1)(N+2)(\bar{N}+1)(\bar{N}+2)} \left((\varsigma+1)(\varsigma+2) - \frac{\partial}{\partial p} - \mu_\nu \right) T + \\ & + e\bar{y}\bar{\partial} \frac{\tau(\tau+s-\nu-2)}{(\bar{N}+1)(\bar{N}+2)} T + e\bar{y}\bar{\partial} \frac{\tau(\tau-s+\nu+2)}{(N+1)(N+2)} T - \\ & - e\bar{y}\bar{\partial} \frac{(\tau-s+\nu+2)}{(\tau+1)(N+1)(N+2)} \left(\frac{\partial}{\partial p} + \mu_\nu \right) \Pi^+ T - e\bar{y}\bar{\partial} \frac{(\tau+s-\nu-2)}{(\tau-1)(\bar{N}+1)(\bar{N}+2)} \left(\frac{\partial}{\partial p} + \mu_\nu \right) \Pi^- T + \\ & + ie\bar{y}\bar{y} \tilde{b}_{N,\bar{N}}^\nu \frac{\partial}{\partial q} T - e\bar{y}\bar{\partial} \tilde{c}_{N,\bar{N}}^\nu \Pi^+ \frac{\partial}{\partial q} T - e\bar{y}\bar{\partial} \tilde{c}_{N,\bar{N}}^\nu \Pi^- \frac{\partial}{\partial q} T = 0. \end{aligned} \quad (5.48)$$

This Ansatz represents a generalization of (5.37) (for a $(s-2)$ -rank tensor) which now includes the terms encoding divergences (the last line) and accounts for reducing of the tensor rank of divergences (so all s have been replaced with $(s-\nu-2)$ as compared to (5.37)). We also allowed for mass-like terms (reproducing those from subsection 5.1.1) with ν -dependent parameter μ_ν . Motivation for introducing them is that divergences of massless AdS-field of high spin have conformal dimensions different from dimensions of massless fields of lower spins (e.g. the dimension of a massless spin-1 divergence $D_n A^n(x)$ does not coincide with that of massless scalar ϕ) which fact should reveal itself as emergence of effective “masses” for the lower-rank descendant modules. To put it differently, in AdS space the wave operator does not commute with covariant derivatives, therefore descendants obey different wave equations than the primary.

Now one has to solve for the consistency conditions in order to fix coefficients $\tilde{b}_{N,\bar{N}}^\nu$, $\tilde{c}_{N,\bar{N}}^\nu$ and μ_ν . These comprise following equations (and their conjugate)

$$(N+2)\tilde{c}_{N,\bar{N}}^\nu \Pi^+ - N\tilde{c}_{N-1,\bar{N}-1}^\nu \Pi^+ = 0, \quad (5.49)$$

$$\begin{aligned} & \frac{(\bar{N}+2)\tau(\tau-s+\nu+2)}{(N+3)}\tilde{b}_{N,\bar{N}}^\nu - \bar{N}\frac{\tau(\tau-s+\nu+3)}{(N+1)}\tilde{b}_{N+1,\bar{N}-1}^\nu + \\ & + \frac{(\varsigma-s+\nu+4)(\varsigma+s-\nu-1)}{(N+1)(\bar{N}+1)}((\varsigma+1)(\varsigma+2)-\mu_{\nu-1})\tilde{c}_{N+1,\bar{N}+1}^\nu \Pi^+ - \\ & - \frac{(\varsigma-s+\nu+3)(\varsigma+s-\nu)}{(N+3)(\bar{N}+1)}((\varsigma+1)(\varsigma+2)-\mu_\nu)\tilde{c}_{N,\bar{N}}^\nu \Pi^+ + \\ & + \frac{\bar{N}(\tau-s+3+\nu)\mu_{\nu-1}}{(\tau+1)(N+1)}\tilde{b}_{N+1,\bar{N}-1}^\nu \Pi^+ - \frac{(\bar{N}+2)(\tau-s+2+\nu)\mu_\nu}{(\tau+1)(N+3)}\tilde{b}_{N,\bar{N}}^\nu \Pi^+ = 0, \end{aligned} \quad (5.50)$$

$$\begin{aligned} & \frac{(\varsigma-s+3+\nu)(\varsigma+s-\nu)}{(N+3)(\bar{N}+1)}\tilde{c}_{N,\bar{N}}^\nu \Pi^+ - \frac{(\varsigma-s+4+\nu)(\varsigma+s-\nu-1)}{(N+1)(\bar{N}+1)}\tilde{c}_{N+1,\bar{N}+1}^\nu \Pi^+ + \\ & + \bar{N}\frac{(\tau-s+3+\nu)}{(\tau+1)(N+1)}\tilde{b}_{N+1,\bar{N}-1}^\nu \Pi^+ - (\bar{N}+2)\frac{(\tau-s+2+\nu)}{(\tau+1)(N+3)}\tilde{b}_{N,\bar{N}}^\nu \Pi^+ = 0, \end{aligned} \quad (5.51)$$

$$\begin{aligned} & \bar{N}\tilde{b}_{N-1,\bar{N}-1}^\nu - (\bar{N}+2)\tilde{b}_{N,\bar{N}}^\nu + \frac{(\tau+1)(\tau-1+s-\nu)}{(\bar{N}+1)}\tilde{c}_{N,\bar{N}}^\nu \Pi^+ - \frac{\tau(\tau+s-\nu-3)}{(\bar{N}+1)}\tilde{c}_{N-1,\bar{N}+1}^\nu \Pi^+ + \\ & + \bar{N}\frac{\tau(\tau-s+\nu+3)}{(N+1)(N+2)}\tilde{c}_{N+1,\bar{N}-1}^\nu \Pi^- - (\bar{N}+2)\frac{(\tau-1)(\tau+1-s+\nu)}{N(N+1)}\tilde{c}_{N,\bar{N}}^\nu \Pi^- = 0, \end{aligned} \quad (5.52)$$

$$\bar{N}\tilde{b}_{N+1,\bar{N}-1}^\nu \tilde{c}_{N,\bar{N}}^{\nu+1} \Pi^+ - (\bar{N}+2)\tilde{b}_{N,\bar{N}}^{\nu+1} \tilde{c}_{N+1,\bar{N}+1}^\nu \Pi^+ = 0. \quad (5.53)$$

From (5.49) it follows that

$$\tilde{c}_{N,\bar{N}}^\nu = \frac{\tilde{c}_\tau^\nu}{(N+1)(N+2)}. \quad (5.54)$$

Considering (5.50) with $\tau < 0$ excludes all terms with Π^+ and allows to solve for negative- τ $\tilde{b}_{N,\bar{N}}^\nu$. Combining this with positive- τ answer resulting from conjugate equation one can write

$$\tilde{b}_{N,\bar{N}}^\nu = \frac{\tilde{b}_\varsigma^\nu}{(|\tau| + s - \nu - 2)(N+1)(N+2)(\bar{N}+1)(\bar{N}+2)}. \quad (5.55)$$

Plugging this into (5.51) leads to

$$\tilde{b}_{N,\bar{N}}^\nu = \frac{\tilde{b}^\nu}{(|\tau| + s - \nu - 2)(N+1)(N+2)(\bar{N}+1)(\bar{N}+2)}, \quad (5.56)$$

$$\tilde{c}_{N,\bar{N}}^\nu = \frac{\tilde{b}^\nu}{(\tau+1)(\tau+s-\nu-2)(\tau+s-\nu-1)(N+1)(N+2)}, \quad (5.57)$$

and by conjugacy

$$\tilde{\bar{c}}_{N,\bar{N}}^\nu = \frac{\tilde{b}^\nu}{(|\tau|+1)(|\tau|+s-\nu-2)(|\tau|+s-\nu-1)(\bar{N}+1)(\bar{N}+2)}. \quad (5.58)$$

$\tilde{b}^\nu$ in (5.56)–(5.58) is an arbitrary function of ν . One can always set it to one

$$\tilde{b}^\nu = 1 \quad (5.59)$$

by rescaling the q variable. Finally, returning to (5.50) one now determines μ_ν to be

$$\mu_\nu = \mu - \nu(2s - \nu - 3) \quad (5.60)$$

with arbitrary constant μ . The rest of equations (5.52), (5.53) and conjugate then hold identically. Thus, unfolded system (5.48) with coefficients (5.56)–(5.60) describes unrestricted symmetric traceless rank- $(s-2)$ tensor field.

5.3 Double-traceless Fronsdal current

Now to build an unfolded system for Fronsdal current one adds T -dependent terms to (5.37) in such a way that (5.1) for the primary of the unfolded J -module gets deformed to (3.20) with $J_{\alpha(s),\dot{\alpha}(s)}$ and $T_{\alpha(s-2),\dot{\alpha}(s-2)}$ being traceless and trace parts of $J_{\underline{a}(s)}^F$. According to (3.20) T must couple as a divergence of J , so one has the following Ansatz

$$\begin{aligned} & DJ + ie\partial\bar{\partial}J - iey\bar{y} \frac{(\varsigma+1)(\varsigma+2)(\varsigma-s+1)(\varsigma+s+2)}{(N+1)(N+2)(\bar{N}+1)(\bar{N}+2)} J + \\ & + e\bar{y}\partial \frac{\tau(\tau+s)}{(\bar{N}+1)(\bar{N}+2)} J + ey\bar{\partial} \frac{\tau(\tau-s)}{(N+1)(N+2)} J + iey\bar{y} \frac{(\varsigma-s+1)(\varsigma+s+2)}{(N+1)(N+2)(\bar{N}+1)(\bar{N}+2)} \frac{\partial}{\partial p} J - \\ & - ey\bar{\partial} \frac{(\tau-s)}{(\tau+1)(N+1)(N+2)} \frac{\partial}{\partial p} \Pi^+ J - e\bar{y}\partial \frac{(\tau+s)}{(\tau-1)(\bar{N}+1)(\bar{N}+2)} \frac{\partial}{\partial p} \Pi^- J + \\ & + iey\bar{y} \frac{\check{b}_\tau}{(N+1)(N+2)(\bar{N}+1)(\bar{N}+2)} T - ey\bar{\partial} \frac{\check{c}_\tau}{(N+1)(N+2)} \Pi^+ T - \\ & - e\bar{y}\partial \frac{\check{\bar{c}}_\tau}{(\bar{N}+1)(\bar{N}+2)} \Pi^- T = 0. \end{aligned} \quad (5.61)$$

Consistency requires following relations (and their conjugate) to be fulfilled

$$\frac{\check{b}_{\tau+1}}{(\tau+1)(\tau+s-1)}\Pi^+ - \check{c}_\tau \Pi^+ = 0, \quad (5.62)$$

$$(\varsigma-s+1)(\varsigma+s+2)\check{c}_\tau \Pi^+ - \frac{(\tau-s)}{(\tau+1)}\check{b}_\tau \Pi^+ + \frac{(\tau-s+2)}{(\tau+1)}\check{b}_{\tau+1}\Pi^+ - (\varsigma-s+3)(\varsigma+s)\check{c}_\tau \Pi^+ = 0, \quad (5.63)$$

$$2\check{b}_\tau + N(\tau+1)(\tau+1+s)\check{c}_\tau \Pi^+ - (N+2)(\tau-1)(\tau-1-s)\check{c}_\tau \Pi^- - (N+2)\tau(\tau+s-2)\check{c}_{\tau-1}\Pi^+ + N\tau(\tau-s+2)\check{c}_{\tau+1}\Pi^- = 0, \quad (5.64)$$

$$(\varsigma+1)(\varsigma+2)(\varsigma-s+1)(\varsigma+s+2)\check{c}_\tau \Pi^+ - \frac{(\tau-s+2)}{(\tau+1)}(\mu-(s-2)(s-1))\check{b}_{\tau+1}\Pi^+ - \tau(\tau-s)\check{b}_\tau + \tau(\tau-s+2)\check{b}_{\tau+1} - (\varsigma-s+3)(\varsigma+s)((\varsigma+1)(\varsigma+2)+\mu-(s-2)(s-1))\check{c}_\tau \Pi^+ = 0. \quad (5.65)$$

(5.62) determines $\check{c}_\tau$ in terms of $\check{b}_\tau$. Substituting this into (5.63) one obtains an answer for $\check{b}_\tau \Pi^+$. Combining this with a conjugate expression one can write

$$\check{b}_\tau = \frac{\check{b}}{(|\tau|+s-1)(|\tau|+s)}. \quad (5.66)$$

Then from (5.62) one has

$$\check{c}_\tau = \frac{\check{b}}{(\tau+1)(\tau+s-1)(\tau+s+1)(\tau+s)}, \quad (5.67)$$

and by conjugacy

$$\check{\bar{c}}_\tau = \frac{\check{b}}{(-\tau+1)(-\tau+s-1)(-\tau+s+1)(-\tau+s)}. \quad (5.68)$$

One can always set an arbitrary constant $\check{b}$ to one $\check{b} = 1$ by overall rescaling of T -module. Now (5.64) holds identically, while (5.65) fixes the value of constant μ to be

$$\mu = s(s+1). \quad (5.69)$$

Finally, one can formulate an unfolded system for the double-traceless spin- s Fronsdal current obeying (3.20). This comprises two modules $J(Y|p|x)$ and $T(Y|p,q|x)$ constrained by (5.3) and (5.46) respectively and obeying unfolded equations

$$\begin{aligned} & DJ + ie\partial\bar{\partial}J + iey\bar{y}\frac{1}{(N+1)(N+2)(\bar{N}+1)(\bar{N}+2)}\left(-(\varsigma+1)(\varsigma+2)(\varsigma-s+1)(\varsigma+s+2)J + \right. \\ & \left. (\varsigma-s+1)(\varsigma+s+2)\frac{\partial}{\partial p}J + \frac{1}{(|\tau|+s)(|\tau|+s-1)}T\right) + \\ & + ey\bar{\partial}\frac{1}{(N+1)(N+2)}\left(\tau(\tau-s)J - \frac{(\tau-s)}{(\tau+1)}\Pi^+ \frac{\partial}{\partial p}J - \frac{\Pi^+}{(\tau+1)(\tau+s+1)(\tau+s)(\tau+s-1)}T\right) + \\ & + ey\bar{\partial}\frac{1}{(\bar{N}+1)(\bar{N}+2)}\left(\tau(\tau+s)J - \frac{(\tau+s)}{(\tau-1)}\Pi^- \frac{\partial}{\partial p}J - \frac{\Pi^-}{(\tau-1)(\tau-s-1)(\tau-s)(\tau-s+1)}T\right) = 0. \end{aligned} \quad (5.70)$$

$$\begin{aligned}
 & DT + ie\partial\bar{\partial}T + iey\bar{y}\frac{1}{(N+1)(N+2)(\bar{N}+1)(\bar{N}+2)}\left(-(\varsigma+1)(\varsigma+2)(\varsigma-s+\nu+3)(\varsigma+s-\nu)+\right. \\
 & \left. +(\varsigma-s+\nu+3)(\varsigma+s-\nu)\left(\frac{\partial}{\partial p}+(\nu+2)(2s-\nu-1)\right)+\frac{1}{(s-\nu+|\tau|-2)}\frac{\partial}{\partial q}\right)T+ \\
 & +ey\bar{\partial}\frac{1}{(N+1)(N+2)}\left(\tau(\tau-s+\nu+2)-\frac{(\tau-s+\nu+2)(\nu+2)(2s-\nu-1)}{(\tau+1)}\Pi^+-\right. \\
 & \left.-\frac{(\tau-s+\nu+2)}{(\tau+1)}\frac{\partial}{\partial p}\Pi^+-\frac{1}{(\tau+1)(\tau+s-\nu-2)(\tau+s-\nu-1)}\Pi^+\frac{\partial}{\partial q}\right)T+ \\
 & +e\bar{y}\partial\frac{1}{(\bar{N}+1)(\bar{N}+2)}\left(\tau(\tau+s-\nu-2)-\frac{(\tau+s-\nu-2)(\nu+2)(2s-\nu-1)}{(\tau-1)}\Pi^--\right. \\
 & \left.-\frac{(\tau+s-\nu-2)}{(\tau-1)}\frac{\partial}{\partial p}\Pi^--\frac{1}{(\tau-1)(\tau-s+\nu+2)(\tau-s+\nu+1)}\Pi^-\frac{\partial}{\partial q}\right)T=0. \tag{5.71}
 \end{aligned}$$

6 Off-shell HS fields unfolded

We proceed to the final part of the main problem: in this section we couple unfolded Fronsdal currents (5.70)–(5.71) to the Central On-Mass-Shell Theorem (3.8)–(3.9) that encodes Fronsdal equations. This way in subsection 6.1 we arrive at unfolded system for the off-shell HS fields in AdS_4 . Then in subsection 6.2 we use this system to compute propagators of Fronsdal fields in the de Donder gauge.

6.1 Coupling of HS currents to the Fronsdal system

In order to write down an Ansatz for coupling currents J and T to unfolded spin- s Fronsdal equations (3.8)–(3.9), one compares restrictions on ς - and τ -dependence for J and T in (5.3), (5.46) with that for ω and C in (3.6). Then one can write

$$\begin{aligned}
 D\omega + ey\bar{\partial}\omega + e\bar{y}\partial\omega &= \frac{i}{4}\eta\bar{E}\bar{\partial}\bar{\partial}C|_{N=0} + \frac{i}{4}\bar{\eta}E\partial\partial C|_{\bar{N}=0} + \\
 &+ \frac{i}{4}\bar{E}\bar{\partial}\bar{\partial}f_{N,\bar{N}}J + \frac{i}{4}E\partial\partial\bar{f}_{N,\bar{N}}J + iEyyg_{N,\bar{N}}T + i\bar{E}\bar{y}\bar{y}\bar{g}_{N,\bar{N}}T, \tag{6.1}
 \end{aligned}$$

$$\begin{aligned}
 DC + ie\partial\bar{\partial}C - iey\bar{y}C &= -ie\partial\bar{\partial}h_{N,\bar{N}}J - iey\bar{y}\frac{k_{N,\bar{N}}}{(N+1)(N+2)(\bar{N}+1)(\bar{N}+2)}J - \\
 &- ey\bar{\partial}\frac{\ell_{N,\bar{N}}}{(N+1)(N+2)}J - e\bar{y}\partial\frac{\bar{\ell}_{N,\bar{N}}}{(\bar{N}+1)(\bar{N}+2)}J. \tag{6.2}
 \end{aligned}$$

Here one does not include p -dependent components of $J(Y|p|x)$ and p - or q -dependent components of $T(Y|p,q|x)$ to the r.h.s. of the Ansatz, because those are identified with descendants containing d'Alembertians or divergences of the primary sources, so they have too high order in derivatives. Similar reasoning about the power of derivatives excludes terms like $E\partial\partial T$ from (6.1). Such an identification, however, is not absolute and, generally speaking, depends on the freedom in the choice of coefficients before higher-order descendants in unfolded equations, mentioned above. An example of a different non-diagonal coupling, including p -dependent components of J , is presented in appendix B.

We start with the study of consistency condition for (6.2). As follows from (3.6), for components of C , τ takes only two values: either s or $-s$. We consider $\tau = s$ case, then restoring $\tau = -s$ sector from reality. With τ fixed, N and $\bar{N}$ are not independent, so $h_{N,\bar{N}}$, $k_{N,\bar{N}}$, $\ell_{N,\bar{N}}$ and $\bar{\ell}_{N,\bar{N}}$ in (6.2) depend, in fact, only on one variable. It is convenient to consider them as functions of ς . In $\tau = s$ sector, $\bar{\ell}_\varsigma$ vanishes, because there is no J with $\tau = s + 1$, while only such J could contribute under the action of $e\bar{y}\partial$. Considering terms with $\frac{\partial}{\partial p}$ in the consistency conditions, one sees that h_ς has to vanish as well, because through (5.70) it generates contributions that have no counterparts to cancel with, like

$$-\frac{1}{2}Ey\partial\frac{h_{\varsigma+1}}{(\varsigma+s+1)}\frac{\partial}{\partial p}J \quad (6.3)$$

and others. Then one is left with a following list of consistency conditions

$$\ell_\varsigma - \ell_{\varsigma-1} = 0, \quad (6.4)$$

$$(\varsigma-s+1)(\varsigma+s+2)\ell_{\varsigma+1} - \frac{1}{s}k_\varsigma = 0, \quad (6.5)$$

$$\frac{k_\varsigma(s-1)}{(\varsigma+s+1)(\varsigma+s+2)(\varsigma-s+2)} - \frac{\ell_{\varsigma+1}(\varsigma+2)(\varsigma+1)(\varsigma-s+1)}{(\varsigma+s+1)(\varsigma-s+2)} + (\varsigma-s+1)\ell_\varsigma = 0, \quad (6.6)$$

$$\frac{2s^2\ell_\varsigma}{(\varsigma+s)} + \frac{k_{\varsigma-1}}{(\varsigma+s)} - \frac{k_\varsigma}{(\varsigma+s+2)} = 0, \quad (6.7)$$

$$\frac{k_{\varsigma-1}}{(\varsigma-s)} - \frac{k_\varsigma}{(\varsigma-s+2)} - \frac{2s^2\ell_\varsigma}{(\varsigma-s+2)} = 0. \quad (6.8)$$

From (6.4) it follows that

$$\ell_\varsigma = \ell \quad (6.9)$$

is constant. Then (6.5) yields

$$k_\varsigma = s(\varsigma-s+1)(\varsigma+s+2)\ell \quad (6.10)$$

and rest of equations (6.6)–(6.8) hold identically.

Now one has to determine coupling coefficients f and g in (6.1). First, one notes that due to (3.6) f and g are functions of τ only. Then, analyzing consistency relations for (6.1) that involve $\frac{\partial}{\partial p}$, one sees that all four terms with $\frac{\partial}{\partial p}$ are different and do not cancel, so the only way out is to put coefficients before them to zero, which amounts to

$$f_{\tau<0} = 0, \quad \bar{f}_{\tau>0} = 0, \quad g_{\tau<0} = 0, \quad \bar{g}_{\tau>0} = 0. \quad (6.11)$$

Then consistency conditions read

$$\frac{N\tau(\tau+s-2)}{(\bar{N}+1)(\bar{N}+2)}g_{\tau-1} + \frac{\bar{N}\tau(\tau-s+2)}{(N+1)(N+2)}\bar{g}_{\tau+1} - (N+3)g_\tau - (\bar{N}+3)\bar{g}_\tau = 0, \quad (6.12)$$

$$\frac{f_\tau}{(N+1)(N+2)(\bar{N}+1)(|\tau|+s-1)} + Ng_\tau = 0, \quad (6.13)$$

$$\frac{\tau(\tau+s)}{(\bar{N}+1)}f_{\tau-1} - (\bar{N}-1)f_\tau - \frac{\bar{\eta}\ell}{(N+1)}|_{\bar{N}=1} = 0, \quad (6.14)$$

plus conjugate ones. From (6.12) one finds, taking into account (3.6) and (6.11),

$$g_\tau = g \frac{(s + \tau - 2)!^2 (s - \tau - 2)! (s - \tau - 1)! \tau!}{(s + \tau + 1)!} \Pi^+, \quad (6.15)$$

$$\bar{g}_\tau = g \frac{(s + |\tau| - 2)!^2 (s - |\tau| - 2)! (s - |\tau| - 1)! |\tau|!}{(s + |\tau| + 1)!} \Pi^- \quad (6.16)$$

with g being arbitrary coupling constant. Now (6.13) yields

$$f_\tau = -g(s - \tau - 2)! (s - \tau)! (s + \tau)! \tau! \Pi^+. \quad (6.17)$$

Finally, (6.14) expresses ℓ in terms of the coupling constant g and the phase factor η as

$$\ell = -2g\eta(2s - 1)!s!. \quad (6.18)$$

So a consistent coupling of J and T to ω and C is

$$\begin{aligned} D\omega + ey\bar{\partial}\omega + e\partial\bar{y}\omega = & \left(\frac{i}{4}\eta\bar{E}\bar{\partial}\bar{\partial}C|_{N=0} + \frac{g}{4}\bar{E}\bar{\partial}\bar{\partial}(s - \tau - 2)!(s - \tau)!(s + \tau)! \tau! \Pi^+ J - \right. \\ & \left. - \frac{g}{4}Eyy \frac{(s + \tau - 2)!^2 (s - \tau - 2)! (s - \tau - 1)! \tau!}{(s + \tau + 1)!} \Pi^+ T + h.c. \right) \end{aligned} \quad (6.19)$$

$$DC + ie\partial\bar{\partial}C - ie\bar{y}\bar{y}C = -g \left(iey\bar{y} \frac{\eta(2s)!s!}{(\varsigma + s + 1)(\varsigma - s + 2)} \Pi^+ J + ey\bar{\partial} \frac{2\eta(2s - 1)!s!}{(\varsigma + s)(\varsigma + s + 1)} \Pi^+ J + h.c. \right) \quad (6.20)$$

This completes unfolding of the off-shell HS fields. Equations (6.19)–(6.20) and (5.70)–(5.71) provide an unfolded formulation of a massless spin- s field coupled to the external Fronsdal current. Interpreted differently, they form an off-shell completion of the Central On-Mass-Shell Theorem (3.8)–(3.9): the only primary field is $\omega_{s-1,s-1}(Y|x)$, which encodes unconstrained double-traceless spin- s Fronsdal field and generates Fronsdal gauge symmetry through (2.3), while other components of ω as well as the whole modules C , J and T form a complete basis of its descendants. A reverse on-shell reduction of (6.19)–(6.20) to the unfolded Fronsdal equations (3.8)–(3.9) is trivially achieved by sending the coupling constant to zero $g = 0$.

Translating into the language of representation theory, we did the following: we took $SO(3, 2)$ -module $D(s + 1, s)$ (accompanied by an appropriate gauge module in the 1-form sector), which corresponds to an AdS_4 massless spin- s field, and “glued” it to the Fronsdal current module corresponding to J and T , in order to get an indecomposable off-shell spin- s Verma module. This resulting module has more complicated structure in comparison with the scalar off-shell module constructed in section 4 and does not allow for the same straightforward and simple analysis. However, the most distinctive features remain the same: the representation is infinitely indecomposable; the module represents an infinite sequence of nested submodules; quotient by some submodule leads to an on-shell system with (in general higher-derivative) e.o.m.; reduction to the initial on-shell Fronsdal system amounts to the quotient by the maximal submodule dual to entire J and T .

Now we will demonstrate how one can use the off-shell system we built in order to quantize the theory. Namely, we will evaluate HS propagators in the de Donder gauge from (6.19), similarly to what we did in the scalar field example in subsection 4.2.

6.2 Quantization: massless HS propagators in the de Donder gauge

Having in hand unfolded equations for off-shell Fronsdal fields and conjugate operators extracting wave equations therefrom, one can compute correlation functions.

To this end one first fixes the gauge. Making use of the unfolded gauge transformation (2.3), which for (6.19) takes the form

$$\delta\omega = D\varepsilon + (ey\bar{\partial})\varepsilon + (e\bar{y}\partial)\varepsilon \quad (6.21)$$

and in terms of constituent 0-forms (3.11) reads as

$$\delta\phi = \frac{1}{N\bar{N}}(y^\alpha\bar{y}^{\dot{\alpha}}D_{\alpha\dot{\alpha}})\varepsilon, \quad \delta\tilde{\phi} = \frac{1}{(N+2)(\bar{N}+2)}(\partial^\alpha\bar{\partial}^{\dot{\alpha}}D_{\alpha\dot{\alpha}})\varepsilon, \quad (6.22)$$

one can impose generalized de Donder gauge

$$\left(\partial^\alpha\bar{\partial}^{\dot{\alpha}}D_{\alpha\dot{\alpha}}\right)\phi - \left(y^\alpha\bar{y}^{\dot{\alpha}}D_{\alpha\dot{\alpha}}\right)\tilde{\phi} = 0. \quad (6.23)$$

Then, applying conjugate operators (3.16), (3.17) to (6.19) and accounting for (6.23), one has in the primary $\tau = 0$ sector

$$\left(\square - \lambda^2(s^2 - 2s - 2)\right)\phi_{\alpha(s),\dot{\alpha}(s)} = \frac{g}{2}(s-1)!^2 s! J_{\alpha(s),\dot{\alpha}(s)}, \quad (6.24)$$

$$\left(\square - \lambda^2(s^2 + 2s - 2)\right)\tilde{\phi}_{\alpha(s-2),\dot{\alpha}(s-2)} = \frac{g}{2s}(s-2)!^3 T_{\alpha(s-2),\dot{\alpha}(s-2)}. \quad (6.25)$$

Replacing $\phi_{\alpha(s),\dot{\alpha}(s)} \rightarrow \frac{\delta W}{\delta J^{\alpha(s),\dot{\alpha}(s)}}$, $\tilde{\phi}_{\alpha(s-2),\dot{\alpha}(s-2)} \rightarrow \frac{\delta W}{\delta T^{\alpha(s-2),\dot{\alpha}(s-2)}}$, one restores propagators for traceless and trace parts of the Fronsdal spin- s field in the de Donder gauge

$$\langle\phi_{\alpha(s),\dot{\alpha}(s)}(x)\phi_{\beta(s),\dot{\beta}(s)}(x')\rangle = \frac{g}{2}(s-1)!^2 s! (\epsilon_{\alpha\beta})^s (\epsilon_{\dot{\alpha}\dot{\beta}})^s G_{\Delta_s}(x, x'), \quad (6.26)$$

$$\langle\tilde{\phi}_{\alpha(s-2),\dot{\alpha}(s-2)}(x)\tilde{\phi}_{\beta(s-2),\dot{\beta}(s-2)}(x')\rangle = \frac{g}{2s}(s-2)!^3 (\epsilon_{\alpha\beta})^{s-2} (\epsilon_{\dot{\alpha}\dot{\beta}})^{s-2} G_{\tilde{\Delta}_s}(x, x'), \quad (6.27)$$

where $G_\Delta(x, x')$ is the scalar Green function (4.24) with conformal weights determined from

$$\Delta_s(\Delta_s - 3) - s = s^2 - 2s - 2, \quad \tilde{\Delta}_s(\tilde{\Delta}_s - 3) - (s-2) = s^2 + 2s - 2. \quad (6.28)$$

7 Conclusion

In the paper we have constructed an off-shell completion for unfolded on-shell system of free massless bosonic fields in AdS_4 . This has been done by coupling them to the external HS currents, which from the standpoint of unfolded approach represent some special set of descendants of primary Fronsdal fields. We have constructed an appropriate unfolded system, which describes these HS currents. The process of construction goes as follows: first, one determines an appropriate unfolded module, which should include all possible descendants (excluding those vanishing due to differential constraints) of a primary field; then one writes down a suitable Ansatz for unfolded equations; finally, one fixes coefficients in the Ansatz, partially by solving for consistency constraints and

partially based on convenience reasons (this reflects a large freedom in choice of basis in the descendant space). It turns out that this basis choice may seriously affect a further analysis: although any option leads to a consistent unfolded system of HS currents, thus solving the problem, not any such system can be diagonally coupled to unfolded Fronsdal equations. By this we mean that in general some linear combination of unfolded currents appears in the r.h.s. of the Fronsdal equation, and only in certain cases, achieved by a particular smart choice of coefficients, there will be a single unfolded current, that can be therefore unambiguously identified with the Fronsdal source.

The possibility of such identification is of particular importance if one is about to study the quantum aspects of the off-shell unfolded system. As we have demonstrated, classical unfolded off-shell system can be reformulated as the set of Schwinger–Dyson equations that determine a partition function of the underlying quantum theory. To this end one should treat primary unfolded fields and primary unfolded currents as conjugated variables upon action on a partition function. This way we have managed to restore AdS_4 -propagators for Fronsdal fields in the de Donder gauge. Although we have not presented here a rigorous comprehensive prescription for quantization of a general unfolded field theory, which should be a topic for a separate thorough study, our analysis demonstrates at least the principal possibility of extracting quantum answers from an off-shell unfolded system.

This paves the way for exploring quantum features of nonlinear higher-spin gravity. Since the on-shell HS system we started with arises from a linearization of Vasiliev equations, its off-shell extension we have constructed should represent a linear limit of a would-be off-shell completion of Vasiliev theory. Constructing this off-shell completed Vasiliev theory together with formulating general quantization prescription for unfolded systems will allow one to start systematic investigation of quantum HS gravity and, among other, hopefully will help to proceed in the most urgent problem of spacetime- and spin-locality of HS interactions.

From the point of view of representation theory, the off-shell system we have constructed corresponds to an indecomposable representation of $SO(3,2)$, which leads to the initial on-shell system after quotienting by a maximal submodule, which is the external current module. This off-shell representation is in fact infinitely indecomposable, being an infinite sequence of successively nested submodules. It would be interesting to explore the structure of the off-shell module in more detail, in order to clarify the representation-theory picture of our construction.

As a byproduct of our analysis, we have also discovered a simple way of extracting wave equations for component fields from an unfolded system by means of “conjugate operators” D^* (3.16), (3.17), (4.16), (5.42) and have found an interesting reduction of the unfolded system for HS currents, which turns it to the unfolded Fierz-Pauli system (5.41) for massive HS fields.

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A Notations and conventions

Here we list conventions used in calculations and collect notations introduced in the paper.

We deal with $4d$ anti-de Sitter space AdS_4 with a negative cosmological constant Λ and an inverse radius λ such that $\Lambda = -3\lambda^2$. Tensor indices referring to this space are denoted by underlined lowercase Latin letters $\underline{a}, \underline{b}, \underline{c}, \dots = \{0, 1, 2, 3\}$ and are transforming by global AdS symmetry group $SO(3, 2)$. $g_{\underline{ab}}$ is AdS metric, $D_{\underline{a}}$ is AdS-covariant derivative and $\square = D_{\underline{a}} D^{\underline{a}}$ is covariant d'Alembertian.

To work with unfolded modules we introduce a flat fiber space which is linked to the base AdS_4 -manifold through a vierbein field $e_{\underline{a}}^{\alpha\dot{\beta}}(x)$, where α and $\dot{\beta}$, taking two values $\{1, 2\}$, correspond to two spinor representations of the fiber Lorentz algebra $so(3, 1) \approx sp(2, \mathbb{C})$. Raising and lowering of spinor indices are carried out by means of antisymmetric spinor metric

$$\epsilon_{\alpha\beta} = \epsilon_{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \epsilon^{\alpha\beta} = \epsilon^{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad (\text{A.1})$$

according to conventions

$$v_{\alpha} = \epsilon_{\beta\alpha} v^{\beta}, \quad v^{\alpha} = \epsilon^{\alpha\beta} v_{\beta}, \quad \bar{v}_{\dot{\alpha}} = \epsilon_{\dot{\beta}\dot{\alpha}} \bar{v}^{\dot{\beta}}, \quad \bar{v}^{\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}} \bar{v}_{\dot{\beta}}. \quad (\text{A.2})$$

For vierbein we fix the following normalization

$$e_{\underline{a}}^{\alpha\dot{\alpha}} e_{\underline{b}}^{\beta\dot{\beta}} \epsilon_{\alpha\beta} \epsilon_{\dot{\alpha}\dot{\beta}} = -\frac{1}{2} g_{\underline{ab}}, \quad e_{\underline{a}}^{\alpha\dot{\alpha}} e_{\underline{b}}^{\beta\dot{\beta}} g^{\underline{ab}} = -\frac{1}{2} \epsilon^{\alpha\beta} \epsilon^{\dot{\alpha}\dot{\beta}}. \quad (\text{A.3})$$

We define a dimensionless covariant derivative $D_{\alpha\dot{\beta}}$ as

$$D^{\alpha\dot{\beta}} = -\frac{2}{\lambda} e_{\underline{a}}^{\alpha\dot{\beta}} D^{\underline{a}} \implies D = dx^{\underline{a}} e_{\underline{a}}^{\alpha\dot{\beta}} D_{\alpha\dot{\beta}}, \quad \square = -\frac{\lambda^2}{2} D_{\alpha\dot{\beta}} D^{\alpha\dot{\beta}}. \quad (\text{A.4})$$

We make use of condensed notations for higher-rank symmetric tensors and multispinors, so

$$T_{\underline{a}(n)} = T_{\underline{a_1 a_2 \dots a_n}}, \quad T_{\alpha(n), \dot{\beta}(m)} = T_{\alpha_1 \alpha_2 \dots \alpha_n, \dot{\beta}_1 \dot{\beta}_2 \dots \dot{\beta}_m}. \quad (\text{A.5})$$

To deal with unfolded fields we introduce a pair of auxiliary $sp(2, \mathbb{C})$ -spinors $Y = \{y^{\alpha}, \bar{y}^{\dot{\alpha}}\}$ which are commutative variables, so

$$y^{\alpha} y^{\beta} \epsilon_{\alpha\beta} = 0, \quad \bar{y}^{\dot{\alpha}} \bar{y}^{\dot{\beta}} \epsilon_{\dot{\alpha}\dot{\beta}} = 0, \quad (\text{A.6})$$

and corresponding derivatives

$$\partial_{\alpha} y^{\beta} = \delta_{\alpha}^{\beta}, \quad \bar{\partial}_{\dot{\alpha}} \bar{y}^{\dot{\beta}} = \delta_{\dot{\alpha}}^{\dot{\beta}}. \quad (\text{A.7})$$

From (2.13) and (2.8)–(2.10) one finds for an AdS-derivatives commutator acting on some unfolded module $F(Y|x)$

$$[D_{\alpha\dot{\alpha}}, D_{\beta\dot{\beta}}] F = -\epsilon_{\alpha\beta} (\bar{y}_{\dot{\alpha}} \bar{\partial}_{\dot{\beta}} + \bar{y}_{\dot{\beta}} \bar{\partial}_{\dot{\alpha}}) F - \epsilon_{\dot{\alpha}\dot{\beta}} (y_{\alpha} \partial_{\beta} + y_{\beta} \partial_{\alpha}) F. \quad (\text{A.8})$$

Y -counting operators N and $\bar{N}$ and their linear combinations ς and τ are

$$N = y^\alpha \partial_\alpha, \quad \bar{N} = \bar{y}^{\dot{\alpha}} \bar{\partial}_{\dot{\alpha}}, \quad \varsigma = \frac{1}{2} (N + \bar{N}), \quad \tau = \frac{1}{2} (N - \bar{N}). \quad (\text{A.9})$$

Π^+ and Π^- are projectors on non-negative and non-positive τ components of the unfolded module

$$\Pi^+ F_{N,\bar{N}}(Y) = \begin{cases} F_{N,\bar{N}}(Y), & \tau \geq 0 \\ 0, & \tau < 0 \end{cases}; \quad \Pi^- F_{N,\bar{N}}(Y) = \begin{cases} F_{N,\bar{N}}(Y), & \tau \leq 0 \\ 0, & \tau > 0 \end{cases}. \quad (\text{A.10})$$

To simplify notations we omit contracted indices between vierbein 1-form $e^{\alpha\dot{\beta}} = dx^{\underline{a}} e_{\underline{a}}^{\alpha\dot{\beta}}$ and $\{y, \bar{y}, \partial, \bar{\partial}\}$ in unfolded equations, so that

$$ey\bar{y} = e^{\alpha\dot{\beta}} y_\alpha \bar{y}_{\dot{\beta}}, \quad e\partial\bar{\partial} = e^{\alpha\dot{\beta}} \partial_\alpha \bar{\partial}_{\dot{\beta}}, \quad ey\bar{\partial} = e^{\alpha\dot{\beta}} y_\alpha \bar{\partial}_{\dot{\beta}}, \quad e\bar{y}\partial = e^{\alpha\dot{\beta}} \bar{y}_{\dot{\beta}} \partial_\alpha, \quad (\text{A.11})$$

and for basis 2-forms

$$E^{\alpha\beta} = e^\alpha_{\dot{\gamma}} e^{\beta\dot{\gamma}}, \quad \bar{E}^{\dot{\alpha}\dot{\beta}} = e_{\dot{\gamma}}^{\dot{\alpha}} e^{\gamma\dot{\beta}}, \quad (\text{A.12})$$

the same “top-down” rules for contractions with $\{y, \bar{y}, \partial, \bar{\partial}\}$ hold.

B Non-diagonal spin-s off-shell extension

One can study (5.5)–(5.7) without fixing $a_{N,\bar{N}} = 1$ as in (5.8). Then after some tedious algebra one arrives at following consistency relations

$$a_{N+1,\bar{N}+1} b_{N,\bar{N}} = \frac{(\varsigma+1)(\varsigma+2)(\varsigma-s+1)(\varsigma+s+2)}{(N+1)(N+2)(\bar{N}+1)(\bar{N}+2)}, \quad (\text{B.1})$$

$$c_{N,\bar{N}+1} \bar{c}_{N+1,\bar{N}} = \frac{(\tau-\frac{1}{2})(\tau+\frac{1}{2})(\tau-s-\frac{1}{2})(\tau+s+\frac{1}{2})}{(N+1)(N+2)(\bar{N}+1)(\bar{N}+2)}, \quad (\text{B.2})$$

and finds that $a_{N,\bar{N}}$, $b_{N,\bar{N}}$, $c_{N,\bar{N}}$ and $\bar{c}_{N,\bar{N}}$ can be presented as

$$a_{N,\bar{N}} = a_\varsigma \alpha_N \bar{\alpha}_{\bar{N}}, \quad b_{N,\bar{N}} = b_\varsigma \beta_N \bar{\beta}_{\bar{N}}, \quad \bar{c}_{N,\bar{N}} = \bar{c}_\tau \alpha_N \bar{\beta}_{\bar{N}}, \quad c_{N,\bar{N}} = c_\tau \bar{\alpha}_{\bar{N}} \beta_N, \quad (\text{B.3})$$

with the following constraints

$$(N+2)\alpha_{N+1}\beta_N = N\alpha_N\beta_{N-1}, \quad (\bar{N}+2)\bar{\alpha}_{\bar{N}+1}\bar{\beta}_{\bar{N}} = \bar{N}\bar{\alpha}_{\bar{N}}\bar{\beta}_{\bar{N}-1}, \quad (\text{B.4})$$

$$a_{\varsigma+1} b_\varsigma = (\varsigma+1)(\varsigma+2)(\varsigma-s+1)(\varsigma+s+2), \quad (\text{B.5})$$

$$c_{\tau-\frac{1}{2}} \bar{c}_{\tau+\frac{1}{2}} = (\tau-\frac{1}{2})(\tau+\frac{1}{2})(\tau-s-\frac{1}{2})(\tau+s+\frac{1}{2}). \quad (\text{B.6})$$

One can choose a partial solution to this system which provides a more simpler-looking form (with more symmetric coefficients) for the equations on J than (5.18). Then instead of (5.37) one arrives at

$$\begin{aligned} & DJ + ie\partial\bar{\partial} \frac{(\varsigma+s+1)}{(N+1)(\bar{N}+1)} \left(\varsigma + \frac{\partial}{\partial p} + 1 \right) J - ie y\bar{y} \frac{(\varsigma-s+1)}{(N+1)(\bar{N}+1)} \left(\varsigma - \frac{\partial}{\partial p} + 1 \right) J + \\ & + e\bar{y}\partial \frac{(\tau+s)}{(N+1)(\bar{N}+1)} \left(\tau + \frac{\partial}{\partial p} \right) J + ey\bar{\partial} \frac{(\tau-s)}{(N+1)(\bar{N}+1)} \left(\tau - \frac{\partial}{\partial p} \right) J = 0. \end{aligned} \quad (\text{B.7})$$

However, coupling of this current to the Central On-Mass-Shell-Theorem takes the form

$$\left(D + ey\bar{\partial} + e\partial\bar{y}\right)\omega = \frac{i\bar{\eta}}{4}E\partial\partial(C - f(2s)!J)|_{\bar{N}=0} + \frac{ig}{4}E\partial\partial\frac{\left(s - 2\tau - \frac{\partial}{\partial p}\right)}{(s+\tau+1)(s-\tau+1)}J + h.c., \quad (\text{B.8})$$

$$\begin{aligned} \left(D + ie\partial\bar{\partial} - ie\bar{y}\bar{\partial}\right)C &= ifey\bar{y}\frac{(\varsigma+s)!}{(\varsigma-s+2)!}\left(s - \frac{\partial}{\partial p} - 1\right)\Pi^+J + \\ &+ ife\partial\bar{\partial}\frac{(\varsigma+s-1)!}{(\varsigma-s+1)!}\left(s - \frac{\partial}{\partial p} - 1\right)\Pi^+J + fey\bar{\partial}\frac{(\varsigma+s-1)!}{(\varsigma-s+2)!}\left(s - \frac{\partial}{\partial p} - 1\right)\Pi^+J + h.c. \end{aligned} \quad (\text{B.9})$$

with arbitrary unrelated coupling constants g , f and $\bar{f}$. From here one sees that $J_{\alpha(s),\dot{\alpha}(s)}(x)|_{p=0}$ cannot be unambiguously identified with the primary Fronsdal current because J linear in p also contributes to the r.h.s. of Fronsdal equations. Moreover, now C cannot be strictly identified with on-shell d.o.f. of Fronsdal field because J arises at the same places in ω -equations as C does. Thus, although this system also provides an off-shell unfolded formulation for spin- s field, an interpretation of different descendants is obscure, that, in particular, obstructs the procedure of quantization. So one concludes that the basis of descendants fixed by this choice of coefficients $a_{N,\bar{N}}$, $b_{N,\bar{N}}$, $c_{N,\bar{N}}$ and $\bar{c}_{N,\bar{N}}$ is “non-diagonal”.

The reason behind this is just the excessive symmetry of coefficients in (B.7), which requires $\frac{\partial}{\partial p}$ to be presented in all terms. Because of this, already the very first equation for $J_{\alpha(s),\dot{\alpha}(s)}(x)|_{p=0}$ involves component of J linear in p , that obstructs local expression of p -dependent components in terms of the p -independent ones. From the standpoint of representation theory, it means that the module set by (B.7) is not of a lowest-weight type. Thus the primary Fronsdal current turns to be smeared over two $J_{\alpha(s),\dot{\alpha}(s)}$, p -independent and p -linear, and this is indeed what one sees from (B.8). Of course, since both “non-diagonal” (B.7) and “diagonal” (5.37) describe the same dynamical system (spin- s conserved current), they must be related by some field redefinition, but this would be severely non-local in terms of Y and p .

Thus, among the large set of formally consistent unfolded systems for conserved spin- s current, only a special class corresponds to the lowest-weight modules, which can be diagonally coupled to the Fronsdal system. And the choice (5.8), (5.20) fixes a particular representative from this class.

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