

Truncated cluster algebras and Feynman integrals with algebraic letters

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ABSTRACT: We propose that the symbol alphabet for classes of planar, dual-conformal-invariant Feynman integrals can be obtained as truncated cluster algebras purely from their kinematics, which correspond to boundaries of (compactifications of) $G_+(4, n)/T$ for the n -particle massless kinematics. For one-, two-, three-mass-easy hexagon kinematics with $n = 7, 8, 9$, we find finite cluster algebras D_4 , D_5 and D_6 respectively, in accordance with previous result on alphabets of these integrals. As the main example, we consider hexagon kinematics with two massive corners on opposite sides and find a truncated affine D_4 cluster algebra whose polytopal realization is a co-dimension 4 boundary of that of $G_+(4, 8)/T$ with 39 facets; the normal vectors for 38 of them correspond to g-vectors and the remaining one gives a limit ray, which yields an alphabet of 38 rational letters and 5 algebraic ones with the unique four-mass-box square root. We construct the space of integrable symbols with this alphabet and physical first-entry conditions, whose dimension can be reduced using conditions from a truncated version of cluster adjacency. Already at weight 4, by imposing last-entry conditions inspired by the $n = 8$ double-pentagon integral, we are able to uniquely determine an integrable symbol that gives the algebraic part of the most generic double-pentagon integral. Finally, we locate in the space the $n = 8$ double-pentagon ladder integrals up to four loops using differential equations derived from Wilson-loop $d \log$ forms, and we find a remarkable pattern about the appearance of algebraic letters.

KEYWORDS: Scattering Amplitudes, String theory and cosmic strings

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1 Introduction and review

Recent years have witnessed enormous progress in computing and understanding analytic structures of scattering amplitudes in QFT. Not only do these developments greatly push the frontier of perturbative calculations relevant for high energy experiments, but they also offer deep insights into the theory itself and exhibit surprising connections with mathematics. An outstanding example is the $\mathcal{N} = 4$ super-Yang-Mills theory (SYM), where one can perform calculations that were unimaginable before and discover rich mathematical structures underlying them. For example, positive Grassmannian [1] and the amplituhedron [2] have provided a new geometric formulation for its planar integrand to all loop orders.

A related direction which we focus on in this paper concerns the deep connection between singularities of loop amplitudes in planar SYM and cluster algebras related to positive Grassmannians [3]. It was first discovered in [1] that Grassmannian cluster algebra [4] naturally appear from the quiver dual to plabic graphs that compute loop integrand of the theory. Remarkably, it has been realized in [1] that cluster algebras of Grassmannian $G(4, n)$ are directly relevant for branch cuts of loop amplitudes with n particles. More precisely, the \mathcal{A} coordinates of $G(4, n)$ cluster algebras are related to symbol [5, 6] letters of amplitudes: the 9 letters of six-particle amplitudes and 42 letters of seven-particle ones are nicely explained by A_3 and E_6 cluster algebras, respectively;¹ they have been exploited for bootstrap program to impressively high orders [7–18]. Perhaps even more surprisingly, cluster algebras seem to dictate how different singularities of amplitudes are related to each other, known as “cluster adjacency” [14, 19, 20], which are closely related to the so-called

¹The rank of the cluster algebra is given by the dimension of the kinematic space parameterized by momentum twistors, which is $3(n - 5)$ for $G_+(4, n)/T$ due to dual conformal symmetry.

extended Steinmann relations [12, 13, 16, 18]. For $n = 6, 7$, all known amplitudes exhibit a remarkable pattern that only \mathcal{A} coordinates that belong to the same cluster can appear adjacent in the symbol.

Beyond $n = 6, 7$, Grassmannian cluster algebras for $G(4, n)$ for $n \geq 8$ become infinite, thus it is an important question how to truncate it to give a finite symbol alphabet. Moreover, as already seen for one-loop N^2 MHV, amplitudes with $n \geq 8$ generally involve letters that cannot be expressed as rational functions of Plücker coordinates of the kinematics $G(4, n)/T$; more non-trivial algebraic (= non-rational throughout the paper) letters appear in new computations based on \bar{Q} equations [21] for two-loop NMHV amplitudes for $n = 8$ and $n = 9$ [22, 23]. This means that in addition to the truncation, new ingredients are needed in the context of Grassmannian cluster algebras to explain these and more algebraic letters. A solution to both problems has been proposed using tropical positive Grassmannian [4] and related tools for $n = 8$ [24–29] and very recently for $n = 9$ [30, 31].² Another method for explaining the alphabet has been proposed using Yangian invariants or the associated collections of plabic graphs [39–42].

On the other hand, $\mathcal{N} = 4$ SYM has been proven to be an extremely fruitful laboratory for the study of Feynman integrals (cf. [43–48] and references therein). Remarkably, the connection to cluster algebras seems to extend to individual Feynman integrals as well, e.g. the same A_3 and E_6 control $n = 6, 7$ multi-loop integrals in $\mathcal{N} = 4$ SYM [19, 45]. Very recently, cluster algebra structures have been discovered for Feynman integrals beyond those in planar $\mathcal{N} = 4$ SYM [49]: there is strong evidence for a C_2 cluster algebra and adjacency for four-point Feynman integrals with an off-shell leg, and various cluster-algebra alphabets have been found for one-loop integrals, and general five-particle alphabet which play an important role in recent two-loop computations [50–52]. Apart from connection to cluster algebras, knowledge of alphabet (and further information) can be used for bootstrapping Feynman integrals [46, 53] (see also [54]). In [55], we identified cluster algebras and certain adjacency conditions for a class of finite, dual conformal invariant (DCI) [56, 57] Feynman integrals to high loops, based on recently-proposed Wilson-loop $d \log$ representation [58] (see [47] for a closely-related Feynman-parameter representation). For ladder integrals with possible “chiral pentagons” on one or both ends (without any square roots), we find a sequence of cluster algebras D_2, D_3, \dots, D_6 for their alphabets, depending on n and the kinematic configurations.

Note that some integrals share the same (or almost the same) alphabet, such as A_3 or E_6 , as the amplitudes for $n = 6, 7$ since the kinematics is just that of the n massless particles; other integrals depend on less kinematic variables, e.g. double-penta-ladder integrals for $n = 7$ (with two legs on a corner) depend on 4 out of 6 variables and the alphabet turns out to be D_4 . What is non-trivial about results in [55] is that for such a class of Feynman integrals we always find a cluster algebra, which is a sub-algebra of that of $G(4, n)$ (as opposed to a random subset), which is already interesting for $n = 7$ but more so for $n = 8, 9$

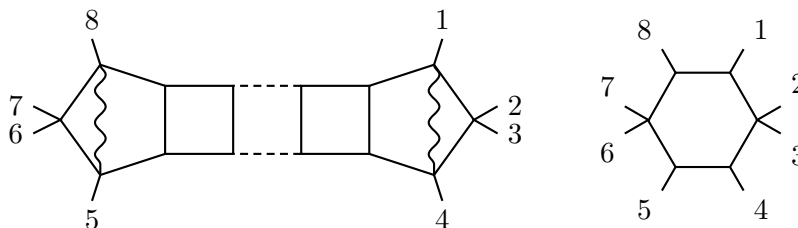
²In this paper, we consider polytopal realization of $G_+(4, n)/T$ as explicitly computed in [32] using Minkowski sum based on stringy integrals [33], which is dual to tropical positive Grassmannian. We will not consider tropical Grassmannian explicitly, though the latter can be recovered from our polytope easily. Also see [34–38] for recent studies in a different context.

etc.³ Another intriguing observation is that the alphabet and cluster algebra structure of these DCI Feynman integrals seem to be independent of details such as numerators or loop orders, but controlled by the kinematics only. It is natural to ask if one can predict the alphabet and possible adjacency conditions for these DCI integrals, as well as those with algebraic letters, from cluster algebra considerations. In this paper, we take a first step in making prediction for the alphabet of DCI Feynman integrals from cluster algebras associated with their kinematics, which correspond to boundaries $G_+(4, n)/T$. We propose that for certain kinematics which can be parametrized by a positroid cell of $G_+(4, n)$, the candidate alphabet for Feynman integrals is given by a cluster algebra obtained from an initial quiver which is the dual of the plabic graph; we are done if the resulting cluster algebra is finite (these include the type- D cases in [55]), but generically it becomes infinite just as $G(4, n)$ cluster algebra for $n \geq 8$, and we need truncation. The procedure is equivalent to that in [25–27] (see also [32]): we construct a polytopal realization for this boundary of $G_+(4, n)/T$ by taking Minkowski sum of Newton polytopes of (non-vanishing) Plücker coordinates, and the facets of the polytope (dual to the rays of tropical positive Grassmannian) teaches us how to truncate the cluster algebra and possibly include algebraic letters. We will loosely refer to the alphabet that comes from this procedure as a *truncated cluster algebra* associated with the kinematics.⁴ We expect that the truncation using Minkowski sum or tropicalization of all non-vanishing Plücker coordinates commutes with taking boundaries in $G_+(4, n)/T$, thus alternatively we can just take the truncated cluster algebra of the latter and go to the corresponding boundary. However, the computation for $G_+(4, n)/T$ becomes extremely complicated beyond $n = 8$, thus for low-dimensional boundaries it makes no sense to do the full computation and then go down. Our proposal makes it more practical to predict symbol alphabet of higher-point Feynman integrals, especially those with more massive corners (whose kinematics depend on less variables). Moreover, boundaries of $G_+(4, n)/T$ and corresponding truncated cluster algebras deserve investigations on their own(cf. [59]); a systematic study of these boundaries is beyond the scope of this paper, and we will illustrate our proposal with a few examples which can be applied to classes of Feynman integrals we are interested in. In particular we find a co-dimension 4 boundary of $G_+(4, 8)/T$ whose cluster algebra is an affine D_4 type. The Minkowski sum gives a polytope with 39 facets, and we obtain 38 rational letters plus 5 algebraic ones.

Another motivation for our study comes from interests in Feynman integrals (and scattering amplitudes) with algebraic letters, which poses certain challenges for multi-loop computations. Using direct integration (either as $d \log$ forms [58] or in Feynman parametrization form [47]), it is straightforward to evaluate such DCI Feynman integrals to high loops for cases without any square roots. The presence of algebraic letters makes computation

³At one or two loops, we usually only see a subset of the full alphabet, but at high enough loops, the alphabet becomes stable and exactly correspond to e.g. those type- D cluster algebras.

⁴It is important to note that compactification introduced by Minkowski sum/tropicalization always give truncations, even for cases with finite cluster algebra; e.g. for $G_+(4, 7)/T$, there are various different compactifications which all give an alphabet with 42 letters, but the polytopes/tropical fans are different! In this paper we stick to the analog of Speyer-Williams fan by using Minkowski sum using all (non-vanishing) Plücker coordinates.



difficult and structures obscured due to the need of rationalization and cancellation of spurious square roots [60, 61]; for example, the symbol of most general double-pentagon integrals contains 16 square roots of four-mass-box type, and for each of them there are 5 (multiplicative independent) algebraic letters. The technical difficulty involved is almost identical to that in computing two-loop NMHV amplitudes from \bar{Q} equations [22, 23], and extensions to higher loops become more and more difficult. It is an interesting and difficult problem in computing (the symbol) of these integrals and amplitudes at higher loops, and understanding structures of the result involving algebraic letters.

Among these integrals we consider, arguably the simplest all-loop series involving non-trivial algebraic letters is the class of double-penta-ladder integrals $\Omega_L(1, 4, 5, 8)$ [47, 58] (as shown in the figure above); the kinematics involved can be drawn as a hexagon with two massive corners on the opposite sides, which corresponds to the co-dimension 4 boundary of $G_+(4, 8)/T$. As we will explain, the L -loop integral can be written as two-fold integral of $(L - 1)$ -loop on (alternatively a pair of nice first-order differential operators reduce the former to the latter). However, unlike the seven-point counterpart (or those higher-point cases without square roots), performing such integrations become tricky due to the presence of square roots, which also prevents us from seeing underlying structures concerning algebraic letters. Now equipped with the alphabet from truncated cluster algebra (and physical discontinuity conditions), we can construct the space of all possible multiple polylogarithm functions (MPL) at symbol level which can be further reduced by adjacency conditions, and we conjecture that the space includes all DCI integrals with this kinematics. This “bootstrap” strategy can be viewed as an extension of results in [46] to include algebraic letters.⁵ Already at weight 4, we find that simply by imposing last entry conditions implied by $d \log$ form or differential equations, the part containing square root is uniquely determined! Moreover, this weight 4 function with algebraic letters turns out to be the “seed” for (the algebraic part of) the most general $n = 12$ double-pentagon integrals: the latter can be obtained by the sum of 16 functions with relabeling; this suggests that the $n = 12$ case contains 16 such truncated cluster algebras.

Moving to higher weights, we can easily determine $\Omega_L(1, 4, 5, 8)$ to four loops (as a strong support for the alphabet and adjacency conditions), by imposing differential equations and boundary conditions obtained from $d \log$ forms, which circumvent the need of rationalization at all. Furthermore, we discover some nice pattern about the algebraic let-

⁵This is in spirit a bit different from bootstrapping amplitudes/form factors since in principle we have Wilson-loop $d \log$ forms/differential equations which determine the answer; in some sense all we need to do is to locate the solution.

ters, which confirms a conjecture we had for these integrals: at least through four loops, non-trivial algebraic letters only appear on the third entry of the symbol, with the accompanying first two entries being that of the four-mass box! Thus for the algebraic part, the highly non-trivial procedure of performing $d \log$ integrals/solving differential equations amounts to simply “translating” the first three entries, and “attaching” rational letters in subsequent entries. Similar observations have been made for $k + \ell = 3$ level of $n = 8$ amplitudes [22], and we hope our results can provide a starting point for future investigation into similar structures of multi-loop integrals and amplitudes with algebraic letters.

The rest of the paper is organized as follows. We first give a quick review of cluster algebras and polytopes from certain stringy integrals, which will be used in our study of truncated cluster algebras. In section 2, we describe our procedure for predicting alphabet of Feynman integrals: after presenting warm-up examples for finite cases D_4, D_5 and D_6 , we give the truncated affine D_4 as the alphabet for “two-mass-opposite” hexagon kinematics. It consists of 38 rational letters associated with g -vectors, and 5 algebraic ones associated with the limit ray (containing the unique four-mass-box square root). In section 3, we move to constructing the cluster function space at symbol level and determine $\Omega_L(1, 4, 5, 8)$ inside the space. With first entry conditions and constraints from a truncated version of cluster adjacency, we obtain the reduced space up to weight 6, and already at weight 4 one can determine a unique function responsible for the algebraic part of most generic double-pentagon integrals. We then discuss constraints for $\Omega_L(1, 4, 5, 8)$ including last entries, differential equations etc. Finally, we determine $\Omega_L(1, 4, 5, 8)$ up to four loops and discuss the pattern concerning the algebraic letters.

1.1 Review of cluster algebras and polytopes from stringy integrals

Let us first give a lightning review of cluster algebras [62–65], where we only give necessary ingredients needed in this paper. Cluster algebras are commutative algebras with a particular set of generators \mathcal{A}_i , known as the cluster \mathcal{A} -coordinates; they are grouped into *clusters* which are subsets of rank n . From an initial cluster, one can construct all the \mathcal{A} -coordinates by *mutations* acting on \mathcal{A} 's (the so-called frozen coordinates or coefficients can also be included, which do not mutate). Cluster variables in a cluster are related by arrows, which forms a quiver Q (without 2-cycles, i.e. arrows $* \rightarrow \cdot \rightarrow *$). Then we associate Q with an skew-symmetric exchange matrix $B(Q) = (b_{ij})$ by $b_{ij} = -b_{ji} = l$ whenever there are l arrows from vertex i to vertex j . Suppose we mutate the vertex k , then the exchange matrix of the mutated quiver reads

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k \\ b_{ij} + \text{sgn}(b_{ik}) [b_{ik} b_{kj}]_+ & \text{otherwise,} \end{cases}$$

where $[x]_+ := \max(x, 0)$, and the cluster variable x_k on vertex k is mutated to x'_k given by

$$x'_k x_k = \prod_{i=1}^n x_i^{[b_{ik}]_+} + \prod_{i=1}^n x_i^{[-b_{ik}]_+}.$$

In general, mutations generate infinite number of cluster variables. As classified in [63], only cluster algebras whose quiver can be mutated from a Dynkin diagram of type

A, B, C, D, E, F, G have finite number of cluster variables, known as the cluster algebra of finite type. The number of cluster variables (dimension) N for finite types read:

$$\begin{aligned} N(A_n) &= n(n+3)/2, & N(B_n) &= N(C_n) = n(n+1), & N(D_n) &= n^2, \\ N(E_6) &= 42, & N(E_7) &= 70, & N(E_8) &= 128, & N(F_4) &= 28, & N(G_2) &= 8. \end{aligned}$$

According to [65], one can further assign a coefficient to a vertex, where the coefficient should be a monomial of some given free variables. Then the mutation rule for cluster variable x_k and the coefficient y_k on vertex k reads

$$y'_j = \begin{cases} y_k^{-1} & \text{if } j = k, \\ y_j y_k^{[b_{kj}]_+} (y_k \oplus 1)^{-b_{kj}} & \text{if } j \neq k, \end{cases} \quad (1.1)$$

and

$$x'_k x_k = \frac{y_k}{y_k \oplus 1} \prod_{i=1}^n x_i^{[b_{ik}]_+} + \frac{1}{y_k \oplus 1} \prod_{i=1}^n x_i^{[-b_{ik}]_+}, \quad (1.2)$$

where the addition \oplus for monomials of free variables $\{u_i\}$ is defined by

$$\prod_j u_j^{a_j} \oplus \prod_j u_j^{b_j} = \prod_j u_j^{\min(a_j, b_j)}.$$

If the coefficients of a cluster are exactly the chosen free variables, then we call that this cluster has principal coefficients. Therefore, starting from a cluster $\{x_i\}_{i=1, \dots, n}$ with exchange matrix $B = (b_{ij})$ and principal coefficients $\{y_i\}_{i=1, \dots, n}$, the cluster variable on vertex k after a series of mutations of vertices \mathbf{v} is a rational function of initial cluster variables and coefficients

$$X_{\mathbf{v},k} = X_{\mathbf{v},k}(x_1, \dots, x_n; y_1, \dots, y_n).$$

Furthermore, if one defines a \mathbb{Z}^n -grading on $\mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}; y_1, \dots, y_n]$ by $\deg(x_i) = \mathbf{e}_i$ (1 in the i -th component and 0 in the rest) and $\deg(y_j) = -\sum_i b_{ij} \mathbf{e}_i$, then $X_{\mathbf{v},k}$ is homogeneous, and its degree $g_{\mathbf{v},k} = (g_{\mathbf{v},k}^1, \dots, g_{\mathbf{v},k}^n) \in \mathbb{Z}^n$ is called its g -vector. Another useful polynomial related to $X_{\mathbf{v},k}$ is the F -polynomial

$$F_{\mathbf{v},k}(y_1, \dots, y_n) := X_{\mathbf{v},k}(1, \dots, 1; y_1, \dots, y_n). \quad (1.3)$$

Once known the F -polynomial and g -vector, we can recover the whole $X_{\mathbf{v},k}$ by

$$X_{\mathbf{v},k}(x_1, \dots, x_n; y_1, \dots, y_n) = x_1^{g_{\mathbf{v},k}^1} \dots x_n^{g_{\mathbf{v},k}^n} F_{\mathbf{v},k} \left(y_1 \prod_i x_i^{b_{i1}}, \dots, y_n \prod_i x_i^{b_{in}} \right). \quad (1.4)$$

There is even a conjecture [65] to read g -vector from F -polynomial alone: if $F_{\mathbf{v},k} \neq 1$, then

$$y_1^{g_{\mathbf{v},k}^1} \dots y_n^{g_{\mathbf{v},k}^n} = \frac{F_{\mathbf{v},k}|_{\text{Trop}}(y_1^{-1}, \dots, y_n^{-1})}{F_{\mathbf{v},k}|_{\text{Trop}}(\prod_i y_i^{b_{i1}}, \dots, \prod_i y_i^{b_{in}})}, \quad (1.5)$$

where $F_{\mathbf{v},k}|_{\text{Trop}}$ means that $+$ is replaced by \oplus in the F -polynomial.

Associated with a finite-type cluster algebra (or more generally a truncated cluster algebra), there is a natural space of polylogarithm functions, whose alphabet is given cluster variables (equivalently they can be chosen as $N-n$ F -polynomials and the n principal coefficients). A cluster function $I^{(w)}$ [66, 67] of transcendental weight w is defined such that its differential has the form

$$dI^{(w)} = \sum_i I_i^{(w-1)} d \log X_i \tag{1.6}$$

where $I_i^{(w-1)}$ are cluster functions of transcendental weight $w-1$ and X_i are cluster \mathcal{A} -coordinates (or F -polynomials). We see that the alphabet of a cluster function is by definition the corresponding cluster algebra.

Already for finite-type cluster algebras, it is natural to consider the so-called *cluster string integrals* which are “stringy canonical forms” [33] associated with cluster polytopes (they also give natural “cluster configuration spaces” [59] which will not be discussed here). For a finite-type (denoted as Φ_n) cluster algebra with principal coefficients $\mathbf{f} = (f_1 \cdots f_n)$ and F -polynomials $F_I(\mathbf{f})$ for $I = n+1 \cdots N$,⁶ we define:

$$\mathcal{I}_{\Phi_n}(\{s\}) = (\alpha')^n \int_{\mathbb{R}_{>0}^n} \prod_{i=1}^n d \log f_i \prod_{I=1}^N F_I(\mathbf{f})^{\alpha' s_I}. \tag{1.7}$$

As $\alpha' \rightarrow 0$, leading order of the integral computes the canonical function of cluster polytope of type Φ_n , which is nicely given by the Minkowski sum of the Newton polytopes of the F -polynomials. Vertices of this polytope correspond to clusters: whenever two vertices are connected by an edge, one can mutate from one to the other in the cluster algebra. Furthermore, $N(\Phi_n)$ facets of the polytopes correspond to cluster variables $X_{\mathbf{v},k}(x_1, \dots, x_n; f_1, \dots, f_n)$, where we can compute outward normal vectors of these facets [33, 68] in terms of the exponents of $\mathbf{f} = \{f_1, \dots, f_n\}$, i.e. $\{s_1, \dots, s_n\}$. Very nicely, these vectors are nothing but the corresponding g -vectors. Note that these g -vectors for cluster variables in any cluster give a cone: the cones for different clusters are non-overlapping, and the union of all cones (known as the cluster fan) covers the full space in any finite type. As mentioned, Grassmannian cluster algebras for $G(4,6)$ and $G(4,7)$ are A_3 and E_6 respectively, and starting at $n=8$ they become infinite. A natural way for truncating an infinite cluster algebra to be finite has been proposed in [32] using a similar *Grassmannian string integrals*. With a positive parametrization of $G_+(k,n)/T$, we can write the integral where the positive polynomials are instead given by all (or a reasonable subset of) the Plücker coordinates of $G_+(k,n)$. The leading order as $\alpha' \rightarrow 0$ is given by the Minkowski sum of Newton polytopes of these polynomials, and one obtains a polytope for the compactification of $G_+(k,n)/T$.⁷ For infinite type, e.g. $G(4,n)$ with $n \geq 8$ (or $G(3,n)$ with $n \geq 9$), by taking the normal vectors for facets of the polytope, we truncate

⁶From here we will denote principal coefficients using $f_I \equiv F_I$ for $I = 1, \dots, n$, which in our subsequent studies can be chosen to be *face variables* of a plabic graph.

⁷For $k=2$, this is the well-known Deligne-Mumford compactification [69–71] for the moduli space $\mathcal{M}_{0,n}^+$, which gives the $(n-3)$ -dimensional associahedron. For $(k,n) = (3,6), (3,7), (3,8)$ we have polytopes that are related to D_4, E_6, E_8 cluster algebras, respectively.

the infinite cluster algebra by identifying a finite set of g -vectors, and there will also be some normal vectors which are not g -vectors (called exceptional rays). We remark that the truncation is not unique since it depends on the choice of Plücker coordinates, and it is equivalent to tropical Grassmannian method since the normal vectors are exactly the rays when choosing the same set of Plücker coordinates for tropicalization [24–26]. For $G(4, 8)$, if we choose the polynomials to be all Plücker coordinates, the Minkowski sum gives a polytope with 360 facets, where 356 normal vectors are g -vectors of $G(4, 8)$ cluster variables, and the remaining 4 are exceptional rays; if we only keep those of the form $\langle ii+1jj+1 \rangle$ and $\langle i-1ii+1j \rangle$ (which respect parity), we get a 274-facet polytope, where 272 are g -vectors and the other 2 are exceptional ones. Moreover, as we will see shortly, at least for $G(4, 8)$ case, an exceptional ray turns out to be a *limit ray* which naturally give algebraic letters associated with a square root, in addition to those rational ones corresponding to g -vectors.

We mention an important subtlety of our algorithm: while the polytope from Minkowski sum must be dual to the tropical fan from tropicalization, the identification of alphabets from these two methods requires some care. Note that instead of computing cluster variables from mutations of Grassmannian cluster algebras, we find it more convenient to compute F polynomials of the cluster variables, where we choose principal coefficients as *face variables* of the corresponding $G_+(4, n)$ or its positroids. As we have mentioned, naively the F -polynomials only capture partial information of cluster variables, the latter can be recovered as in (1.4) once we obtain the corresponding g -vectors. While we do not have a mathematically rigorous proof, we find that the resulting alphabets from these two methods are multiplicatively equivalent; more precisely, the alphabet of F polynomials and face variables (which are frozen) amount to DCI combinations of A coordinates obtained from tropicalization method. For example, we have applied our method to $G_+(4, 8)/T$ (top cell), where we find that the 356 normal vectors, which are g -vectors, precisely correspond to 347 F polynomials and remaining 9 trivial ones with $F = 1$ (since there are 9 face variables). Altogether the 356 polynomials of face variables are DCI combinations of the $356 + 8$ A coordinates (including 8 frozen Plücker coordinates), which span the same space under multiplications.

2 Truncated cluster algebras for Feynman integrals

In this section, we propose an algorithm which predicts symbol alphabet for classes of DCI Feynman integrals with same kinematics. Here the kinematics simply mean the m dual points which the class of Feynman integrals universally depend on, without referring to actual propagator structure or possible numerators. We will refer to such a kinematical configuration as an m -gon with certain massless and massive corners, where for each massless (massive) corner, we put one (two) massless legs, with n legs in total for $n \geq m$;⁸ when $n = m$, all dual points are null separated, which is the kinematics of n massless legs. For example, all off-shell four-point integrals relevant for four-point CFT correlators share the kinematics of an $n = 8$ square ($m = 4$, with all four corners massive), and in

⁸We can trivially add more than two legs at a massive corner, which gives higher-point Feynman integrals with the same kinematics, thus the n here is the minimal number of legs.

particular all-loop box ladder integrals belong to this class. The $n = 7, 8$ pentagon-box ladder proposed in [44] belong to $n = 7, 8$ pentagon ($m = 5$) with two or three massive corners, respectively. It is fun to count the dimension of such kinematics with DCI: for each dual point we have 4 degree of freedoms, but when two points are null separated the degree of freedom is reduced by one, and DCI means subtracting 15 in the end. For two- or three-mass pentagon, the dimension is $4 \times 5 - 3(2) - 15 = 2(3)$ as expected; for four-mass square it is trickier: the kinematics is so special that one of the 15 redundancies no longer exists, thus we have $4 \times 4 - 14 = 2$ dimensions as expected.⁹

For $m = n = 6, 7$, the symbol alphabet of the amplitude and all DCI integrals computed so far is dictated by the kinematics, which is given by A_3 and E_6 respectively. What we propose here is a natural extension to more general kinematics with $m < n$, where we identify it as certain boundaries of $G_+(4, n)/T$. This first gives an equivalent way of counting: from $G_+(4, n)/T$ which has dimension $3(n - 5)$, generically for each massive corner we go down in dimension by 2. It is generally unclear how to identify which boundary of $G_+(4, n)/T$ corresponds to a given kinematics, and to find a truncated cluster algebra for its symbol alphabet. In this paper we focus on special cases where the boundary can be identified with a positroid cell of $G_+(4, n)$ (mod torus action) [47], which can be labelled by plabic graphs.

The algorithm we propose consists of the following steps, which crucially depends on the fact that the kinematics is associated with a positroid cell.

- By imposing conditions on Plücker coordinates of n momentum twistors according to the kinematics, we identify a positroid cell Γ of $G_+(4, n)$ represented by a plabic graph G_Γ , which gives a positive parametrization \mathbf{Z}_Γ of the kinematics (after modding out torus action). More precisely, $\mathbf{Z}_\Gamma(\{f\})$ depends on internal face variables f_i for $i = 1, 2, \dots, d$ where d is the dimension of Γ/T (we set all but one external face variables to 1).
- We define the cluster algebra \mathcal{A}_Γ by applying mutations from the initial quiver diagram, which is the dual of the plabic graph. We use the face variables as principal coefficients which parametrize the positive part of the cluster variety, and we are interested in the F -polynomials. We obtain a finite alphabet if the cluster algebra is a finite type.
- We consider all non-vanishing Plücker coordinates (or a subset of them) of \mathbf{Z}_Γ , which are positive polynomials of f 's (a subset of F -polynomials); we take the Minkowski sum of their Newton polytopes, which gives a polytope denoted as \mathcal{P}_Γ . We conjecture that \mathcal{P}_Γ is always a boundary of the polytopal realization of $G_+(4, n)/T$ (which is dual to tropical $G_+(4, n)$).
- At least a subset of normal vectors for facets of the polytope \mathcal{P}_Γ should coincide with certain g -vectors of \mathcal{A}_Γ , and the rational alphabet consists of these F -polynomials which are associated with these facets (as well as f_1, \dots, f_d). For those exceptional normal vectors that do not correspond to g -vectors, we conjecture that they are associated with non-rational letters etc. which need to be treated differently.

⁹We thank Nima Arkani-Hamed for first explaining this to us.

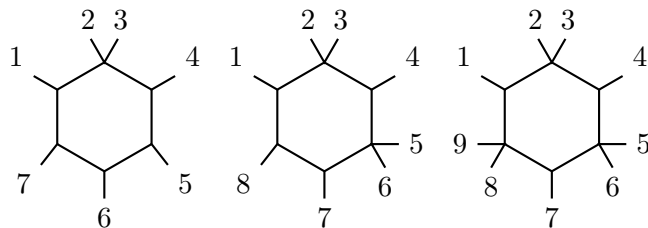
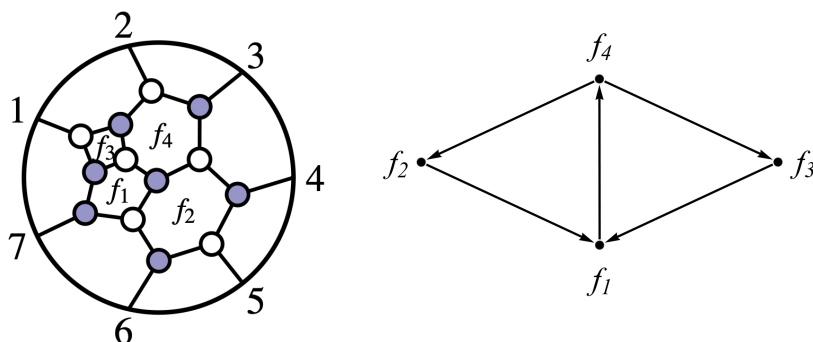


Figure 1. One-, two-, three-mass-easy hexagon kinematics with $n = 7, 8, 9$ legs.

2.1 Warm-up examples: truncated D_4 , D_5 and D_6 cluster algebras

Let us begin with warm-up examples for one-, two- and three-mass easy hexagon kinematics with $n = 7, 8, 9$. We will not give details of the computation for these finite-type cases, and simply list the positroid cells given in [47] (with decorated permutations and plabic graphs), positive parametrizations of the kinematics, the polytopes from Minkowski sum and the resulting cluster algebras.

Let's first consider one-mass kinematics with dual points $(x_1, x_2, x_4, x_5, x_6, x_7)$, which should correspond to a co-dimension 2 positroid. As explained in [47], the latter can be specified by $\langle n123 \rangle = \langle 2345 \rangle = 0$, which gives a decorated permutation $\sigma = \{6, 5, 7, 8, 9, 11, 10\}$, and we find plabic graph



For modding out torus action, we fix all but one external face variables to be unity, and the resulting \mathbf{Z} matrix which positively parametrize the kinematics reads

$$\begin{pmatrix} f_3 f_4 (1 + f_3) f_4 & 1 + f_4 + f_3 f_4 & 1 & 0 & 0 & 0 \\ 0 & f_1 f_2 f_4 & f_2 (1 + f_1 + f_1 f_4) & 1 + f_2 + f_1 f_2 & 1 & 0 & 0 \\ 0 & 0 & f_2 & 1 + f_2 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix}.$$

We have drawn the dual quiver diagram of the plabic graph, where we ignored all external facets, on the right. It is easy to see that this is a quiver for the D_4 cluster algebra, and as mentioned above, the face variables correspond to principal coefficients assigned to each node. By applying mutation rules we find 16 cluster variables, which can

be identified with f_1, f_2, f_3, f_4 and 12 F -polynomials of f 's:

$$\{1 + f_1, 1 + f_2, 1 + f_3, 1 + f_4, 1 + f_2 + f_1 f_2, 1 + f_3 + f_1 f_3, 1 + f_1 + f_1 f_4, 1 + f_4 + f_2 f_4, 1 + f_4 + f_3 f_4, 1 + f_2 + f_3 + f_2 f_3 + f_1 f_2 f_3, 1 + f_4 + f_2 f_4 + f_3 f_4 + f_2 f_3 f_4 + f_1 f_2 f_3 f_4\} \quad (2.1)$$

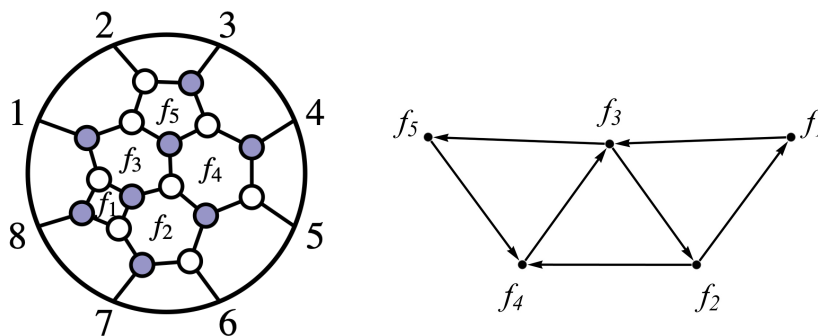
Since this is a finite type we have a finite alphabet: we conjecture that any DCI integral with this one-mass hexagon kinematics has a symbol alphabet of the 16 letters.

On the other hand, by computing all non-vanishing Plücker coordinates of the \mathbf{Z} matrix, we find 15 positive polynomials, which actually already contain 15 of the above alphabet, except for $1 + f_4 + f_2 f_4 + f_3 f_4 + f_2 f_3 f_4$. Now, we compute the Minkowski sum of Newton polytopes of these 15 polynomials, remarkably we find a polytope with 16 facets whose f -vector is

$$\mathbf{f} = (1, 49, 99, 66, 16, 1)$$

which is almost a D_4 polytope (which has $\mathbf{f} = (1, 50, 100, 66, 16, 1)$). Moreover, the (outward) normal vectors of all these 16 facets are nothing but the g -vectors of the 16 letters, which allow us to identify each letter with a facet of the polytope. Note that both codimension 1 and 2 boundaries of this polytope agree with those of D_4 polytope, but it misses one edge and one vertex (and becomes slightly non-simple). We can of course include the last F -polynomial for Minkowski sum/tropicalization, which will then give exactly the D_4 polytope.

Next we consider two-mass-easy case with dual points $(x_1, x_2, x_4, x_5, x_7, x_8)$: the (codimension 4) positroid is given by the two conditions above and $\langle 3456 \rangle = \langle 5678 \rangle = 0$, and we have decorated permutation $\sigma = \{7, 5, 6, 9, 8, 10, 12, 11\}$ and plabic graph



We obtain the following \mathbf{Z} matrix as a positive parametrization (after modding out torus action):

$$\begin{pmatrix} f_5 & f_5 & 1+f_5 & 1 & 0 & 0 & 0 & 0 \\ 0 & f_1 f_2 f_3 f_4 f_5 & f_2 f_4 (1+f_1+f_1 f_3+f_1 f_3 f_5) & 1+f_2+f_1 f_2+f_2 f_4+f_1 f_2 f_4+f_1 f_2 f_3 f_4 & 1+f_2+f_1 f_2 & 1 & 0 & 0 \\ 0 & 0 & f_2 f_4 & 1+f_2+f_2 f_4 & 1+f_2 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}.$$

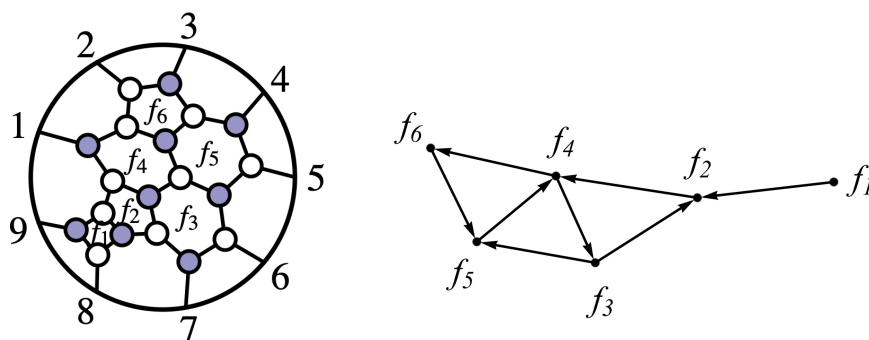
The dual quiver diagram (on the right) is also one for the D_5 cluster algebra; by applying mutation rules we find 25 letters, including f_1, \dots, f_5 and 20 F -polynomials of f 's. We will not explicitly write this D_5 alphabet: it suffices to say that it consists of 23 positive polynomials from all non-vanishing Plücker coordinates of the \mathbf{Z} matrix above, and two missing letters, which are $1 + f_3$ and $1 + f_2 + f_5 + f_2f_5 + f_2f_4f_5$.

To obtain a truncated cluster algebra, we take the Minkowski sum of Newton polytopes of the 23 polynomials, and we obtain a polytope with 25 facets whose f -vector is

$$\mathbf{f} = (1, 178, 449, 408, 160, 25, 1),$$

which is a truncated D_5 polytope. The normal vectors of these facets turn out to be exactly the g -vectors of the 25 letters we find. We see that again it differs from D_5 polytope starting from co-dimension 3 boundaries, and by including the two missing F -polynomials we of course recover the D_5 polytope.

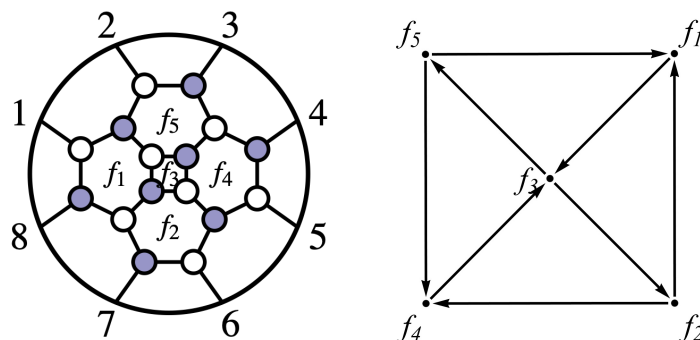
As our last warm-up example, we consider three-mass-easy kinematics with dual points $(x_1, x_2, x_4, x_5, x_7, x_8)$: the (co-dimension 6) positroid is given by the four conditions above and additionally $\langle 6789 \rangle = \langle 8912 \rangle = 0$, thus we have the decorated permutation $\sigma = \{7, 5, 6, 10, 8, 9, 13, 11, 12\}$, and the plabic graph



and after modding out torus action, the \mathbf{Z} matrix reads

$$\begin{pmatrix} f_6 & f_6 & 1+f_6 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & f_1f_2f_3f_4f_5f_6 & f_3f_5(1+f_1+f_1f_2+f_1f_2f_4+f_1f_2f_4f_6) & * & 1+f_1+f_3+f_1f_3+f_1f_2f_3 & 1+f_1 & 0 & -1 & 0 \\ 0 & 0 & f_3f_5 & 1+f_3+f_3f_5 & 1+f_3 & 1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \end{pmatrix}$$

where $*$ = $1+f_1+f_3+f_1f_3+f_3f_5+f_1f_2f_3+f_1f_3f_5+f_1f_2f_3f_5+f_1f_2f_3f_4f_5$. As we draw on the right, the quiver diagram is one for the D_6 cluster algebra, and the resulting alphabet consists of f_1, \dots, f_6 and 30 F -polynomials. They can be identified as the 33 positive polynomials from non-vanishing Plücker coordinates of the above \mathbf{Z} matrix, except for the following 3: $1 + f_4, 1 + f_3 + f_6 + f_3f_6 + f_3f_5f_6$, and $f_1f_3f_2^2 + f_1f_3f_5f_2^2 + f_1f_3f_4f_5f_2^2 + f_1f_2 + 2f_1f_3f_2 + f_3f_2 + 2f_1f_3f_5f_2 + f_3f_5f_2 + f_1f_3f_4f_5f_2 + f_1 + f_1f_3 + f_3 + f_1f_3f_5 + f_3f_5 + 1$. By taking the Minkowski sum of Newton polytopes of these 33 polynomials, we find a



truncated D_6 polytope with 36 facets and f -vector

$$\mathbf{f} = (1, 657, 1986, 2292, 1257, 330, 36, 1).$$

The normal vectors agree with all the g -vectors of the 36 letters, but it differs from the D_6 polytope starting at co-dimension 3 boundaries.

As checked to at least three loops in [55], these alphabets apply to all DCI integrals we computed with such kinematics, including double-penta-ladder integrals for $n = 6, 7, 8$ with various possible numerators. It is remarkable that their symbol alphabets seem to be determined by truncated cluster algebras naturally associated with the kinematics.

In the next subsection, we move to a more non-trivial case, where the cluster algebra from the dual quiver is an infinite type (affine D_4). The Minkowski sum becomes crucial for this case since it provides a natural truncation that gives a finite (rational) alphabet, as well as limit ray(s) that gives non-rational letters.

2.2 Truncated affine D_4 cluster algebras

The main example we are interested in is the hexagon with two massive corners on opposite sides, where we have dual points $(x_1, x_2, x_4, x_5, x_6, x_8)$. The (co-dimension 4) positroid can be obtained by $\langle 8123 \rangle = \langle 2345 \rangle = \langle 4567 \rangle = \langle 6781 \rangle = 0$, thus the decorated permutation reads $\sigma = \{6, 5, 8, 7, 10, 9, 12, 11\}$. We have a rather symmetric plabic graph, and the dual quiver diagram can be identified with one for affine D_4 type. This is an infinite-type cluster algebra (though it is mutation finite), and we must rely on Minkowski sum to obtain a finite alphabet. After modding out the torus action, we have the \mathbf{Z} matrix:

$$\begin{pmatrix} f_1 f_2 f_3 f_4 f_5 & f_1 f_2 f_3 f_4 f_5 & f_2 f_4 (-1 + f_1 f_3 f_5) & -1 - f_2 - f_2 f_4 & -1 - f_2 & -1 & 0 & 0 \\ 0 & f_3 f_4 f_5 & f_4 (1 + f_3 + f_3 f_5) & 1 + f_4 + f_3 f_4 & 1 & 0 & 0 & 0 \\ 0 & 0 & f_2 f_4 & 1 + f_2 + f_2 f_4 & 1 + f_2 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

From all non-vanishing Plücker coordinates, we find exactly 25 positive polynomials, which we record as W_i for $i = 1, \dots, 25$ (anticipating that they will be part of the full

alphabet). The first 10 of them are linear in f 's: which we write as

$$W_i = f_i, \quad W_{5+i} = 1 + f_i, \quad \text{for } i = 1, \dots, 5; \quad (2.2)$$

The next 8 letters are degree-2 polynomials of the form $w_{i,j} := 1 + f_j + f_i f_j$:

$$\begin{aligned} W_{11} &= w_{1,2}, & W_{12} &= w_{3,1}, & W_{13} &= w_{2,3}, & W_{14} &= w_{4,2}, \\ W_{15} &= w_{3,4}, & W_{16} &= w_{1,5}, & W_{17} &= w_{5,3}, & W_{18} &= w_{4,5}. \end{aligned} \quad (2.3)$$

Finally, the last 7 letters involve polynomials of degree 3, 4 or 5; introducing $w_{i,j,k} := 1 + f_i + f_j + f_i f_j + f_i f_j f_k$; we have

$$\begin{aligned} W_{19} &= w_{1,4,3}, W_{20} = 1 + f_3(f_2 + w_{2,5}), W_{21} = 1 + f_2 w_{1,4,3}, W_{22} = 1 + f_3 w_{2,5,1}, \\ W_{23} &= 1 + f_5 w_{1,4,3}, W_{24} = 1 + f_3 w_{2,5,4}, W_{25} = 1 + f_3(w_{2,5,1} + w_{3,1} f_2 f_4 f_5). \end{aligned} \quad (2.4)$$

By taking the Minkowski sum of their Newton polytopes, we obtain a 5-dimensional polytope with f -vector

$$\mathbf{f} = (1, 280, 739, 694, 272, 39, 1),$$

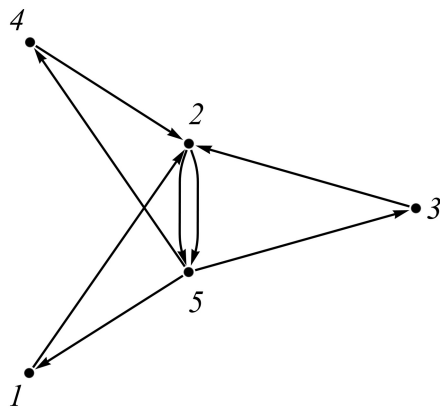
and it is easy to compute the normal vectors of these 39 facets. By comparing these 39 vectors with g -vectors of the affine D_4 cluster algebra above, we see that 38 of them correspond to g -vectors, and for completeness we record them here. For W_1, \dots, W_5 , their g -vectors are $g_i = \mathbf{e}_i$ for $i = 1, \dots, 5$, and the remaining 33 g_i for $i = 6, 7, \dots, 38$ read:

$$\begin{aligned} &(-1, 0, 1, 0, 0), (1, -1, 0, 1, 0), (0, 1, -1, 0, 1), (0, 0, 1, -1, 0), (1, 0, 0, 1, -1), (0, -1, 0, 1, 0), (-1, 0, 0, 0, 0), \\ &(0, 0, -1, 0, 1), (1, -1, 0, 0, 0), (0, 0, 0, -1, 0), (0, 0, 0, 1, -1), (0, 1, -1, 0, 0), (1, 0, 0, 0, -1), (-1, 0, 1, -1, 0), \\ &(1, 0, -1, 1, 0), (0, -1, 0, 0, 0), (0, 0, -1, 1, 0), (0, 0, 0, 0, -1), (1, 0, -1, 0, 0), (0, 0, -1, 0, 0), (0, -1, 1, 0, 0), \\ &(1, -1, 0, 2, -1), (0, 0, 1, 0, -1), (2, -1, 0, 1, -1), (0, -1, 1, 1, -1), (0, -1, 0, 1, -1), (1, -1, -1, 1, 0), \\ &(1, -1, 1, 0, -1), (1, -1, 0, 0, -1), (0, -1, 1, 0, -1), (1, -2, 0, 1, -1), (1, 0, -1, 1, -1), (1, -1, 0, 1, -2) \end{aligned}$$

These 38 facets then give F -polynomials including the above 25 polynomials, and we find additionally 13 polynomials. We denote these letters as W_{26}, \dots, W_{38} . Note that some of the remaining ones are relabelling of what we have seen in the first 25 letters. For example, $W_{26} = 1 + f_2(f_1 + w_{1,4})$, $W_{27} = w_{2,5,1}$, $W_{28} = 1 + f_5(f_1 + w_{1,5})$, etc. All 38 rational letters are recorded in the supplementary material. Note that these letters can also be obtained by simply parametrizing the 356 rational letters of $G_+(4, 8)/T$ using our \mathbf{Z} matrix. It is interesting to see that if we start with the smaller (rational) alphabet with 272 letters for $G_+(4, 8)/T$, we obtain only 33 letters with $\{W_{30}, W_{33}, W_{35}, W_{36}, W_{38}\}$ missing, and we will come back to this smaller alphabet later. It is, however, not clear to us how to directly obtain the 33 letters (plus algebraic ones) by Minkowski sum; e.g. if we use parity-invariant subset of non-vanishing minors of our \mathbf{Z} matrix, we obtain a polytope with only 18 facets and all of them correspond to rational letters, which is insufficient.

There is a remaining normal vector, $g_\infty = (1, -1, 0, 1, -1)$. After extensive search, it turns out not to be any g -vector of the infinite cluster algebra. As shown in [25], after

infinite sequences of mutations on a quiver with doubled arrow, the directions of g -vectors on two ends of the doubled arrow will asymptote to the so-called *limit ray*. Difference between the two g -vectors on the end of doubled arrow will stay invariant in the infinite mutations, giving the limit rays they asymptote to, which is exactly our g_∞ ! For instance, after mutation series $\{5, 1, 4\}$ from the initial cluster, the quiver turns out to be



with $g(2) = g_2$, $g(5) = g_{10}$. It is straightforward to check that $g_\infty = g_{10} - g_2$, and the difference stays invariant in the infinite mutations.

As can be computed from the algorithm in [25–27], g_∞ is associated with exactly the square root for the unique four-mass-box (x_2, x_4, x_6, x_8) for this kinematics, which is defined as

$$\Delta := \sqrt{(1 - u_3 - v_3)^2 - 4u_3v_3}, \quad u_3 = \frac{\langle 1234 \rangle \langle 5678 \rangle}{\langle 1256 \rangle \langle 3478 \rangle}, \quad v_3 = \frac{\langle 1278 \rangle \langle 3456 \rangle}{\langle 1256 \rangle \langle 3478 \rangle}, \quad (2.5)$$

where the two cross-ratios can be expressed using the letters as $u_3 = 1/W_{25}$, $v_3 = W_1W_2W_3^2W_4W_5/W_{25}$. It is convenient to introduce the two roots $\alpha_\pm = \frac{1}{2}(1 + u_3 - v_3 \pm \Delta)$ (such that $\alpha_+ - \alpha_- = \Delta$), which appear in the (second entry of) symbol of the famous four-mass box

$$\mathcal{S}[F(x_2, x_4, x_6, x_8)] = -\frac{1}{2}(v_3 \otimes L_1 + u_3 \otimes L_2),$$

where the two simplest *algebraic letters* are denoted as $L_1 = \frac{\alpha_+}{\alpha_-}$ and $L_2 = \frac{1-\alpha_-}{1-\alpha_+}$. In addition, we find infinite sequences of mutations which produce these and other algebraic letters, similar to what was done in [25, 26]. The upshot is that we find a space of 5 multiplicative independent algebraic letters: L_1, L_2 and

$$L_3 = \frac{W_{17}^{-1} - \alpha_-}{W_{17}^{-1} - \alpha_+}, \quad L_4 = \frac{W_{13}/W_{25} - \alpha_-}{W_{13}/W_{25} - \alpha_+}, \quad L_5 = \frac{(1 - W_1W_2W_3)^{-1} - \alpha_+}{(1 - W_1W_2W_3)^{-1} - \alpha_-}. \quad (2.6)$$

It is remarkable that this is precisely the 5-dimensional space of algebraic letters found for double-pentagon integral $\Omega_2(1, 4, 5, 8)$ [60]!

3 The cluster function space and double-penta ladders to four loops

3.1 First entries, cluster adjacency and algebraic letters

Having obtained the alphabet with 38 rational letters and 5 non-rational ones, it is natural to construct the space of cluster functions, and we will content ourselves with first building all integrable symbols. There are two important constraints we can impose: first, we are interested in symbols whose first entries consist of only physical discontinuities, which can be chosen to be 5 independent space-time cross-ratios. Moreover, we will impose cluster adjacency conditions, i.e. only letters that appear in the same cluster (of the truncated cluster algebras) can be adjacent to each other in the symbol.

As discussed in [55], the 5 independent cross-ratios which can appear on the first entry are u_3, v_3 defined above, as well as the following three:

$$u_1 = \frac{\langle 1245 \rangle \langle 5681 \rangle}{\langle 1256 \rangle \langle 4581 \rangle} = \frac{1}{W_8}, \quad u_2 = \frac{\langle 3481 \rangle \langle 4578 \rangle}{\langle 3478 \rangle \langle 4581 \rangle} = \frac{W_{13} W_{17}}{W_8 W_{25}}, \quad u_4 = \frac{\langle 1234 \rangle \langle 4581 \rangle}{\langle 1245 \rangle \langle 3481 \rangle} = \frac{W_8}{W_{17}}$$

With the alphabet and first entries, we are ready to build functions or integrable symbols, starting from the $\log(u_1), \log(u_2), \log(u_3), \log(v_3), \log(u_4)$ at weight 1. Our construction is recursive: at each weight w , we consider all integrable symbols of weight $w - 1$ tensored with any of the 38 + 5 letters, and impose integrability conditions on the final two entries. We start from the ansatz $\sum_{i,j} c_{i,j} S_i^{(w-1)} d \log l_j$ where $S_i^{(w-1)}$ denotes weight- $(w-1)$ integrable symbols, and l_j the letters, i.e. $W_1, \dots, W_{38}, L_1, \dots, L_5$. The integrability condition reads

$$\sum_{i,j,m} c_{i,j} S_{i;m}^{(w-2)} d \log l_m \wedge d \log l_j = 0, \quad (3.1)$$

where $S_{i;m}^{(w-2)}$ denote coefficients of $d \log l_m$ in $S_i^{(w-1)}$, which are linear combinations of weight- $(w-2)$ integrable symbols. Therefore, all we need is to find all linear relations among $\binom{43}{2}$ $d \log$ 2-forms (some of them vanish identically, e.g. $d \log f_i \wedge d \log(1 + f_i) = 0$), and all such relations are recorded in the supplementary material. In this way, we can easily construct the space to relatively high weight: it turns out that there are 5, 24, 113, 530 such integrable symbols at weight $w = 1, 2, 3, 4$.

Now we turn to possible cluster adjacency conditions to reduce the space, which forbid letters that cannot be in the same cluster to appear next to each other in the symbol. More precisely, we will use the truncated cluster algebra and its polytope for imposing these conditions: if two letters have facets that intersect in the polytope, then clearly they belong to the same cluster, otherwise we claim that they are a *forbidden pair* in the truncated cluster algebra. We do not know if there exists a cluster in the infinite affine D_4 cluster algebra which includes a forbidden pair, but for our purpose we will use this “truncated” version of cluster adjacency and forbid such a pair to appear next to each other in the symbol.¹⁰

¹⁰This may sounds too strong, but in fact what we have done is we first “bootstrapped” the integrals $\Omega_L(1, 4, 5, 8)$ up to $L = 4$ without using such adjacency conditions; the result does respect these conditions, which means that they can indeed be imposed, and in the following we present the improved bootstrap in this reduced space.

We apply this version of cluster adjacency to the rational letters W_1, \dots, W_{38} (it is not clear to us how to extend it to include the remaining 5 L 's which are assigned to the same facet). Since W_1, \dots, W_5 are not F -polynomials, we do not consider them in the study of adjacency conditions; in other words, we list the facets for W_i with $i = 6, 7, \dots, 38$, and find all pairs that do not intersect in the polytopes. In this way, we find 350 forbidden pairs out of $\frac{33 \times 34}{2}$, which we record in the supplementary material (in practice, this can be trivially done by using e.g. **polymake**). By applying these adjacency conditions to the construction, and we find that the space is reduced significantly (more and more so for higher weights). For $w = 1, 2, 3, 4$, the dimension of the space is reduced to 5, 23, 93, 340. Moreover, we have computed the reduced space for $w = 5, 6$, and find 1141, 3585 such integrable symbols respectively. The physical meaning of such adjacency is unclear as usual, but we conjecture that this reduced space contains DCI Feynman integrals with such “two-mass opposite” kinematics, and we can use it to bootstrap such integrals at least up to three loops.

Before proceeding, we remark that similarly one can bootstrap for these warm-up cases such as D_4 functions for one-mass hexagon kinematics with $n = 7$. Note that this D_4 alphabet can be obtained as a boundary of our truncated cluster algebra e.g. by sending $u_3 \rightarrow 0$. It is straightforward to construct the space of integrable symbols for D_4 functions, with first entries given by u_1, u_2, u_4, v_3 . The dimensions of the space at weight 1, 2, 3, 4 are 4, 16, 63, 246; we can impose adjacency conditions which forbid 30 pairs out of $\frac{12 \times 13}{2}$ pairs of F -polynomials, and these conditions reduce the dimensions to 4, 15, 50, 155 up to weight 4. Nicely, any integral with one-mass hexagon kinematics up to weight 4 that we know of can be found in the space.

What can we say about non-rational letters? Although we do not know how to impose conditions such as cluster adjacency on them, it turns out that they are still constrained at least at low weights. The first observation is that there is only one weight 2 function involving them: the four-mass box whose symbol we record above, where we have L_1 and L_2 in the second entry. Similarly we find only 11 weight 3 functions with algebraic letters. Among them, the first five have the form

$$\mathcal{S}(F(x_2, x_4, x_6, x_8)) \otimes L_i + \text{rational part} \tag{3.2}$$

with $i = 1 \dots 5$. While other 6 functions are linear combinations of

$$\{\mathcal{S}(F(x_2, x_4, x_6, x_8)) \otimes W_j, \quad \mathcal{S}(\text{dilog. with } W_j) \otimes L_i\} \tag{3.3}$$

Note that under a “parity” $\Delta \rightarrow -\Delta$, those symbols in (3.2) stay invariant, while those in (3.3) picks up a minus sign. For any “parity-even” amplitude or integral, what we need are those even functions in (3.2), or those odd ones in (3.3) dressed with a prefactor that is a odd function in Δ , such as $1/\Delta$.¹¹

For higher weights, the number of functions involving algebraic letters grows rapidly. However, we are mostly interested in a particular class of functions starting at weight 4. In the next subsection, we will locate $\Omega_L(1, 4, 5, 8)$ up to $L = 4$, and for now let's see how

¹¹This is also true for the “odd” four-mass box at weight 2, which can be normalize with a prefactor $1/\Delta$ to make it “even”, e.g. when appearing in one-loop amplitudes.

much it takes to determine the part that contains algebraic letters at $L = 2$. We will show shortly how to determine the *last entries* of $\Omega_L(1, 4, 5, 8)$ for $L \geq 2$ from the Wilson-loop $d \log$ form or differential equations: starting $L = 2$, the symbol of $\Omega_L(1, 4, 5, 8)$ contains exactly 5 last entries, which we denote as $\{z_i\}_{i=1 \dots 5}$:

$$z_1 = -W_3, z_2 = -\frac{W_2 W_3 W_5 W_{12} W_{15}}{W_{13} W_{17}}, z_3 = \frac{W_2 W_3}{W_{13}}, z_4 = \frac{W_3 W_5}{W_{17}}, z_5 = \frac{W_2 W_3^2 W_5 W_{12}}{W_{13} W_{17}} \quad (3.4)$$

Therefore it is natural to see what symbols with algebraic letters and only these 5 last entries can we find in the space. Surprisingly after imposing last-entry conditions on weight 4, only *one* independent weight-4 functions containing algebraic letters L_i is left, and we record this *integrable* symbol up to the part involving purely rational letters W_j 's (the rational part depends on our basis of weight-4 functions):

$$\mathcal{S}_{2,4,6,8} := \mathcal{S}(F_{2,4,6,8}) \otimes \left(\frac{L_2 L_5}{L_1 L_3} \otimes z_1 + \frac{L_2 L_5}{L_1 L_4} \otimes z_2 + \frac{L_5}{L_1^2 L_3 L_4} \otimes z_3 + \frac{L_5}{L_1} \otimes z_4 + \frac{L_1^2 L_3 L_4}{L_2 L_5^2} \otimes z_5 \right) + \text{rational} \quad (3.5)$$

where we have denoted the four-mass box as $F_{2,4,6,8} := F(x_2, x_4, x_6, x_8)$. We see that by restricting to the five last entries, exactly the first five weight-3 functions described above contribute, which can be viewed as generating the first derivatives ∂_{z_i} for $i = 1, 2, \dots, 5$ of $\mathcal{S}_{2,4,6,8}$. In fact, the first two weight-3 functions (involving L_1 and L_2) can be chosen to be the two weight-3 functions that appear when solving differential equation for double-box integral [44], and it is nice to see that we just have three additional weight-3 functions involving L_3, L_4, L_5 , when solving weight-4 double-pentagon. We record the symbol of $\Omega_2(1, 4, 5, 8)$ in the supplementary material, and it is easy to see that the algebraic part is given by (3.5).

Having obtained a function that captures the algebraic part of $\Omega_2(1, 4, 5, 8)$, we remark that from it we can easily obtain the algebraic part of the most general double-pentagon integral; we denote it as $\Omega_2(i, j, k, l)$ with the first fully general case for $n = 12$ and e.g. $(i, j, k, l) = (1, 4, 7, 10)$ [60]. $\Omega_2(i, j, k, l)$ contains $2^4 = 16$ four-mass-box square roots, labelled by (x_a, x_b, x_c, x_d) with $(a, b, c, d) = (i + \sigma_i, j + \sigma_j, k + \sigma_k, l + \sigma_l)$ with $\sigma = 0, 1$ [60]. For each (a, b, c, d) , all we need to do is simply relabel the momentum twistors of $\Omega_2(1, 4, 5, 8)$ by $\{1 \rightarrow i, 4 \rightarrow j, 5 \rightarrow k, 8 \rightarrow l\}$ and $\{2 \rightarrow i \pm 1, 3 \rightarrow j \pm 1, 6 \rightarrow k \pm 1, 7 \rightarrow l \pm 1\}$ where the choice ± 1 depends on σ 's, e.g. for $\sigma_i = 1$ ($a = i + 1$), $2 \rightarrow i + 1$. By summing over 16 such relabelled symbol (with alternating signs), we obtain an integrable symbol that contains the algebraic part of $\Omega_2(i, j, k, l)$:

$$\mathcal{S}(\Omega_2(i, j, k, l)) = \sum_{\{\sigma\}} (-)^{\sum \sigma} \mathcal{S}_{i+\sigma_i, j+\sigma_j, k+\sigma_k, l+\sigma_l} + \mathcal{S}(R) \quad (3.6)$$

where the sum is over $2^4 = 16$ choices of σ 's with a minus sign when $\sigma_i + \sigma_j + \sigma_k + \sigma_l$ is odd; R denotes a weight-4 function with only rational letters. It is remarkable that, up to this R function, the most generic double-pentagon integral can be obtained using 16 weight-4 integrable symbols found in our space.

3.2 Double-penta-ladders: last entries, differential equations etc.

Now we move to the computation of double-penta ladder integrals, $\Omega_L(1, 4, 5, 8)$, which can be defined directly from Wilson loop $d \log$ representation: we can rewrite an L -loop ladder

as a two-fold integral over a $(L - 1)$ -loop integral as

$$\Omega_L(1, 4, 5, 8) = \int d \log \langle 148Y \rangle d \log \frac{\langle 1X4Y \rangle}{t} \times \frac{Y}{X} \times \text{Diagram} \quad (3.7)$$

where $X = Z_1 - tZ_3$, $Y = Z_3 - sZ_5$ with t and s integrated on $\mathbb{R}_{\geq 0}^2$. We can rescale t and s to make DCI property manifest, and we arrive at the recursion

$$\begin{aligned} \Omega_{L+\frac{1}{2}}(u_1, u_2, u_4, u_3, v_3) &= \int_0^\infty d \log \frac{t+1}{t} \Omega_L \left(\frac{u_1(t+u_4)}{t+u_1u_4}, u_2, \frac{u_4(t+1)}{t+u_4}, \frac{u_3(t+1)}{t+u_1u_4}, \frac{tv_3}{t+u_1u_4} \right), \quad (3.8) \\ \Omega_{L+1}(u_1, u_2, u_4, u_3, v_3) &= \int_0^\infty d \log(s+1) \Omega_{L+\frac{1}{2}} \left(u_1, \frac{u_2(s+1)}{u_2s+1}, \frac{s+u_4}{s+1}, \frac{u_3(1+s/u_4)}{1+su_2}, \frac{v_3}{1+su_2} \right), \end{aligned}$$

Note that at the limit $u_3 \rightarrow 0$, $\Omega_L(1, 4, 5, 8)$ and the recursions degenerate to the $\Omega_L(1, 4, 5, 7)$ case. The source of the recursion is the one-loop 8-pt chiral hexagon whose result is well known [58] (e.g. in box expansion including $F_{2,4,6,8}$):

$$\Omega_1(1, 4, 5, 8) = \frac{3}{2} \times \text{Diagram} \quad (3.9)$$

Note that at two loops, after some tedious calculation based on rationalization, the symbol (and even function [47]) of $\mathcal{S}(\Omega_2)$ can be computed from the recursion (3.9). Its alphabet consists of 5 algebraic letters $\{L_1, L_2, L_3, L_4, L_5\}$ and 21 rational letters which are $\{W_1, \dots, W_{25}\}$ with $\{W_6, W_9, W_{22}, W_{24}\}$ absent. As mentioned, the last entries of the answer are the five z -variables (3.4), which are related to the cross ratios $\{u_1, u_2, u_3, u_4, v_3\}$ by

$$u_1 = \frac{1}{1-z_1}, \quad u_2 = \frac{1}{1-z_2}, \quad u_4 = 1-z_4, \quad u_3 = \frac{(1-z_3)(1-z_4)}{(1-z_1)(1-z_2)}, \quad v_3 = -\frac{(z_1z_2-z_5)(z_3z_4-z_5)}{(1-z_1)(1-z_2)z_5}.$$

These z -variables make many properties of the ladder integrals $\Omega_L(1, 4, 5, 8)$ manifest, and we will use them extensively in the following discussions. For instance, the integrals have two axial symmetries, which are given by

$$z_1 \leftrightarrow z_2 \quad \text{and} \quad z_3 \leftrightarrow z_4.$$

The deformations are also simplified in terms of z -variables to

$$\Omega_{L+\frac{1}{2}}(z_1, \dots, z_5) = \int_0^\infty d \log \frac{t+1}{t} \Omega_L \left(\frac{tz_1}{t-z_4+1}, z_2, z_3, \frac{tz_4}{t-z_4+1}, \frac{tz_5}{t-z_4+1} \right) \quad (3.10)$$

and

$$\Omega_{L+1}(z_1, \dots, z_5) = \int_0^\infty d \log(s+1) \Omega_{L+\frac{1}{2}}\left(z_1, \frac{z_2}{s+1}, z_3, \frac{z_4}{s+1}, \frac{z_5}{s+1}\right). \quad (3.11)$$

Following the same algorithm in determining last entries of all-loop penta-box integrals [58], it is straightforward to see that last entries of $\Omega_L(1, 4, 5, 8)$ remain unchanged for $L \geq 2$. Recall that with constants a and b , last entries of the integral $\int_0^\infty F(t) \otimes (t+b) d \log(t+a)$ are a or $(b-a)$ and those of $\int_0^\infty F(t) \otimes b d \log(t+a)$ are a or b . After the first-step integration eq. (3.10) with $d \log((t+1)/t)$, the five original last entries give six last entries $\{z_1, z_2, z_3, z_4, 1-z_4, z_5\}$, where the new one $1-z_4$ is from the integration

$$\int_0^\infty F(t) \otimes (t-z_4+1) d \log(t).$$

However, deformed $1-z_4$ only contributes z_4 as last entry in the second-step integration eq. (3.11) as well, since after the deformation it only contributes terms like

$$\int_0^\infty F(s) \otimes \frac{s+1-z_4}{s+1} d \log(s+1).$$

Therefore by induction, we have proven that last entries of the integral $\Omega_L(1, 4, 5, 8)$ are always $\{z_1, \dots, z_5\}$ for arbitrary L .

Using z variables, we also find remarkably simple first-order differential equations:

$$\Omega_{L+\frac{1}{2}} = (z_2 \partial_{z_2} + z_4 \partial_{z_4} + z_5 \partial_{z_5}) \Omega_{L+1} \quad (3.12)$$

and

$$\Omega_L = (z_4 - 1)(z_1 \partial_{z_1} + z_4 \partial_{z_4} + z_5 \partial_{z_5}) \Omega_{L+\frac{1}{2}}. \quad (3.13)$$

For example, consider the deformation of $L + 1/2 \rightarrow L + 1$:

$$\begin{aligned} \Omega_{L+\frac{1}{2}}(z_1, \dots, z_5) &= \int_0^\infty d \log(s+1) \Omega_L\left(z_1, \frac{z_2}{s+1}, z_3, \frac{z_4}{s+1}, \frac{z_5}{s+1}\right) \\ &= \int_0^{z_5} d \log t \Omega_L\left(z_1, \frac{z_2}{z_5} t, z_3, \frac{z_4}{z_5} t, t\right), \end{aligned}$$

its derivative with respect to z_5 is

$$z_5 \partial_{z_5} \Omega_{L+\frac{1}{2}} = \Omega_L(z_1, \dots, z_5) - \frac{1}{z_5} \int_0^{z_5} d \log t (z_2 t \partial_2 + z_4 t \partial_4) \Omega_L\left(z_1, \frac{z_2}{z_5} t, z_3, \frac{z_4}{z_5} t, t\right),$$

where ∂_2 and ∂_4 denotes partial derivative acting on the second and fourth argument respectively, and then (3.12) is given by the following identity

$$\frac{1}{z_5} (z_2 t \partial_2 + z_4 t \partial_4) \Omega_L\left(z_1, \frac{z_2}{z_5} t, z_3, \frac{z_4}{z_5} t, t\right) = (z_2 \partial_{z_2} + z_4 \partial_{z_4}) \Omega_L\left(z_1, \frac{z_2}{z_5} t, z_3, \frac{z_4}{z_5} t, t\right).$$

(3.13) can be found in a similar way from the deformation of $L \rightarrow L + 1/2$. With the DE (3.12) and the symmetry, it is also easy to see that the last entries of Ω_L for $L \geq 2$ can only be $\{z_1, \dots, z_5\}$ since $\Omega_{L+\frac{1}{2}}$ is pure for $L \geq 1$.

Finally from the recursion, we can easily impose certain boundary conditions. Since the boundary value of $d \log$ form $d \log \frac{t+1}{t}$ diverges at $t = 0$ in eq. (3.10), deformed function $\Omega_L(1, 4, 5, 8)$ should vanish when $t \rightarrow 0$, which gives the constraint:

$$\lim_{t \rightarrow 0} \Omega_L(tz_1, z_2, z_3, tz_4, tz_5) = 0. \tag{3.14}$$

We expect that in the space, differential equations (3.12), (3.13), together with boundary conditions, should determine the symbol of $\Omega_L(1, 4, 5, 8)$ recursively. This will be confirmed up to weight 8 in the next subsection.

Finally, as we have mentioned, setting $u_3 \rightarrow 0$, i.e. $z_3 \rightarrow 1$, $\Omega_L(1, 4, 5, 8)$ degenerates to 7-point ladder integrals $\Omega_L(1, 4, 5, 7)$, which have been computed in [55, 58] up to $L = 4$ easily. We use this colinear limit as a cross check for our result.

3.3 Locating the integrals and the pattern for algebraic letters

We have constructed the space with given first entries and adjacency conditions in the section 3.1, in this subsection we using the conditions above to bootstrap Ω_L up to $L = 4$. As mentioned above, DE and boundary conditions are sufficient for the task, but computationally it is easier if we first impose last-entry conditions and symmetry of the integral.

To impose DE explicitly, we use the derivative formula of a symbol:

$$\partial_a(F \otimes w) = F \frac{\partial}{\partial a} \log w.$$

In practice, the derivative in DE (3.12) takes the ansatz $\sum_i F_i \otimes z_i$ into

$$(z_2 \partial_{z_2} + z_4 \partial_{z_4} + z_5 \partial_{z_5}) \sum_i F_i \otimes z_i = F_2 + F_4 + F_5.$$

For the other DE (3.13), one may need to calculate the derivative of letters by

$$\frac{\partial W_i}{\partial z_j} = \sum_k \frac{\partial W_i}{\partial f_k} \frac{\partial f_k}{\partial z_j},$$

but it is more convenient to first require that the last entries of $F_2 + F_4 + F_5$ are $\{z_1, z_2, z_3, z_4, 1 - z_4, z_5\}$ which is proven for $\Omega_{L+1/2}$ in the last subsection, and then the derivative $(z_4 - 1)(z_1 \partial_{z_1} + z_4 \partial_{z_4} + z_5 \partial_{z_5})$ is trivial. To impose the symmetry, we rewrite two symmetries $z_1 \leftrightarrow z_2$ and $z_3 \leftrightarrow z_4$ in terms of alphabet. For example, the transformation $z_3 \leftrightarrow z_4$ is simply

$$\left\{ \begin{aligned} &W_2 \leftrightarrow W_5, W_7 \leftrightarrow W_{10}, W_{11} \leftrightarrow W_{16}, W_{13} \leftrightarrow W_{17}, W_{14} \leftrightarrow W_{18}, W_{21} \leftrightarrow W_{23}, \\ &W_{26} \leftrightarrow W_{28}, W_{32} \leftrightarrow W_{37}, W_{36} \leftrightarrow W_{38}, L_3 \rightarrow \frac{1}{L_1 L_4}, L_4 \rightarrow \frac{1}{L_1 L_3}, L_5 \rightarrow \frac{L_5}{L_1 L_3 L_4} \end{aligned} \right\}.$$

The other symmetry $z_1 \leftrightarrow z_2$ behaves more complicated because under the transformation $z_1 \leftrightarrow z_2$, rational letters $\{W_{30}, W_{33}, W_{35}, W_{36}, W_{38}\}$ will produce new factors, thus for bootstrapping Ω_L , we only need the other 33 rational letters. It is remarkable that this is exactly the smaller rational alphabet obtained from the parity-invariant $G_+(4, 8)/T$ as

conditions	# free parameters
weight-6 function space	3585
last entry	257
symmetry $z_3 \leftrightarrow z_4$	146
symmetry $z_1 \leftrightarrow z_2$	56
DE	3
boundary conditions	0

Table 1. Number of free parameters left after using constrains on the left column for bootstrapping Ω_3 .

mentioned above! It is intriguing that this smaller alphabet is exactly the one that respects the symmetry; for our purpose, it is sufficient to use only these 33 rational letters (plus 5 algebraic ones).

Before going to higher loops, it is already interesting to re-derive Ω_2 from the bootstrapping strategy. As mentioned in subsection 3.1, after imposing the last entry condition, there is only one algebraic function left. By imposing DE (with the weight 2 function being one loop hexagon (3.9)) and boundary conditions, we arrive at the unique symbol of Ω_2 , which is recorded in the supplementary material.

We continue to determine Ω_3 in this way: the number of free parameters of the ansatz during the bootstrap of Ω_3 is given in the table 1. Note that the letters in the symbol after imposing the last entries is dramatically reduced, only $29 + 5$ letters left. These 29 rational letters behave well under the transformation $z_1 \leftrightarrow z_2$. Then imposing the derivative $(z_2\partial_{z_2} + z_4\partial_{z_4} + z_5\partial_{z_5})$, we get an ansatz of $\Omega_{2+1/2}$ whose last entries are proven to be $\{z_1, z_2, z_3, z_4, z_5, 1 - z_4\}$, but here in the ansatz naively we have 9 extra last entries. It is convenient to just eliminate these “spurious” last entries, and then apply the second DE and boundary conditions which allow us to immediately determine the symbol of Ω_3 .

Next, we want to determine Ω_4 . Since the function space (even after using adjacency) is too large at weight 8, we find it useful to directly impose last-entry conditions when constructing the space for weight 7 and above (we also use the alphabet with 33 rational letter which respect the symmetry $z_1 \leftrightarrow z_2$). After obtaining the reduced space, it becomes straightforward to apply DE and boundary conditions, which uniquely determine the symbol of Ω_4 (it takes a few hours on a laptop).

The symbol of Ω_3 has about 8×10^4 terms which is recorded in the supplementary material, while Ω_4 has more than 10^6 terms which is too lengthy to be recorded. Although the complexity of the result grows fast with the number of loops, we find some hidden simplicity at least for the algebraic part. Just like Ω_2 , the part of Ω_3 and Ω_4 containing algebraic letters take a strikingly simple form:

$$\sum_{i=1}^5 \mathcal{S}(F(2, 4, 6, 8)) \otimes L_i \otimes \mathcal{S}(F_i)$$

where F_i are weight 3 (5) MPL functions with rational letters only, for Ω_3 (Ω_4) respectively. This means that in addition to L_1, L_2 in the second entry as part of $\mathcal{S}(F(2, 4, 6, 8))$, the

5 algebraic letters *only* appear on the third entries but not any subsequent ones. This phenomenon was observed at the special $R^{1,1}$ kinematics [72] as well; in $R^{1,1}$ Ω_L is a rational A_2 function $\Omega_L(v, w)$, and only non-trivial algebraic letters, L_3, L_4, L_5 become the “mixing letter” $v - w$. We have proven that in $R^{1,1}$ kinematics, the part of $\Omega_L(v, w)$ L with mixing letter reads [72]:

$$\mathcal{S}(F(2, 4, 6, 8)) \otimes (v - w) \otimes \mathcal{S}\left(\log^{2L-3}\left(\frac{v}{w}\right)\right).$$

This indicate that for all L , algebraic letters L_i for $i = 3, 4, 5$ can only appear at the third entries (with symbol of $F(2, 4, 6, 8)$ in the first two), but still does not exclude the possibility that simpler algebraic letters, L_1, L_2 , may appear in subsequent entries. Here we confirm that at least through four loops, no algebraic letters appear beyond the third entry. Recall that $\Omega_L(1, 4, 5, 8)$ is given by a two-fold integral of $\Omega_{L-1}(1, 4, 5, 8)$, and the pattern we observe means that the first three entries of the algebraic part remain unchanged! It is an interesting problem to prove this by carefully analysing rationalization and possible cancellation of spurious square roots in our integration routine. Note that the same phenomenon is expected to hold for three-loop MHV amplitudes, which follow from similar pattern of two-loop NMHV ones via \bar{Q} equations, as observed in [73] in $R^{1,1}$ kinematics.

We also note that the rational alphabet of Ω_3 and Ω_4 does not contain 38 or even 33 rational letters but only the first 25 $\{W_1, \dots, W_{25}\}$, which can already be found in all non-zero Plücker coordinates. We believe that other integrals sharing the same kinematics as $\Omega_L(1, 4, 5, 8)$ may contain some of the remaining letters, and we leave it to future work for finding and studying such integrals.

4 Conclusion and discussions

In this paper, we have conjectured that symbol alphabets for certain classes of DCI Feynman integrals can be determined by truncated cluster algebras purely from their kinematics, which are boundaries of $G_+(4, n)/T$. The main example we study is the two-mass-opposite hexagon kinematics, and our method produces an alphabet of 38 rational letters and 5 algebraic ones (as a truncated affine D_4 cluster algebra). We construct the space of integrable symbols after imposing physical first-entry conditions and a truncated version of cluster adjacency, which we believe to be universal. When restricting to $\Omega_L(1, 4, 5, 8)$, we derive differential equations and last-entry conditions from our $d \log$ recursion, which allow us to locate its symbol in the space up to weight 8. We also find a remarkable pattern up to four loops for the appearance of algebraic letters, which begs for some explanations. Since the rationalization is very similar to those required for computing multi-loop amplitudes using \bar{Q} equations, it is tempting to look for similar pattern for higher-loop $n = 8$ amplitudes.

We have only focused on cases where the kinematics can be naturally given in terms of positroid cells of $G_+(4, n)$. If the kinematics for a class of Feynman integrals cannot be labelled by positroids, our method does not directly apply since we do not know the quiver to begin with. For example, currently we do not know any positroid cell for two-mass-hard hexagon kinematics. In [49], the alphabet of the latter was conjectured to be a subset of octagon alphabet that are annihilated by first-order differential operators encoding the

kinematics. This seems to be a general method when we know the alphabet of $G_+(4, n)/T$, and it would be interesting to study the relation of such subsets to our truncated cluster algebras. For example, if we apply the differential operator to our two-mass-opposite case, we have 33 of the 38 rational letters and 5 algebraic letters. An important difference is that these subsets generally do not correspond to boundaries of $G_+(4, n)/T$ (while we expect our truncated cluster algebras do). We have looked at higher-dimensional cases, e.g. for one-mass heptagon kinematics with $n = 8$, our method gives a co-dimension 2 boundary of $G_+(4, 8)/T$ which has 100+1 facets, where we have 100 g -vectors and 1 limit ray (the subset from differential operators of [49] is smaller). Since the computation for $G_+(4, n)/T$ cluster algebra becomes very difficult beyond $n = 8$ (there are recent results for $n = 9$ using a subset of all Plücker coordinates [30]), it is crucial to develop both methods for studying higher-point DCI integrals. It is also an interesting mathematical problem to systematically classify the boundaries of $G_+(4, n)/T$ (see [28]) and study their relevance for Feynman integrals.

Both for amplitudes and integrals in $\mathcal{N} = 4$ SYM, the possibility of studying symbol alphabets pure from kinematics sounds like magic: despite more and more data supporting such conjectures, we do not have a good understanding of the mechanism. Compared to scattering amplitudes, there might be a better chance to systematically understand why alphabets of certain DCI Feynman integrals are related to such truncated cluster algebra, especially via canonical differential equations [74, 75]. It would be also highly desirable to connect alphabets for these integrals to certain $4k$ -dimensional plabic graph of $G_+(k, n)$ as have been studied for amplitudes, which essentially amount to maps from such cells in $G_+(k, n)$ to (boundaries of) $G_+(4, n)/T$. A pressing question is to see if and how more complicated algebraic letters including those containing higher-order roots appear in truncated cluster algebras for corresponding integrals (one could even speculate something “elliptic” might appear for the “alphabet” of the kinematics for two-loop $n = 10$ double-box integral). Last but not least, cluster algebra structures have been observed for Feynman integrals that are not DCI including those with IR divergence; their alphabet can sometimes be obtained from that in DCI case, e.g. the pentagon alphabet with one-massive leg can be obtained from sending a dual point to infinity in the two-mass-hard hexagon kinematics [49]. It would be extremely interesting to find possible truncated cluster algebras for these more general integrals, which should again be directly related to their canonical differential equations.

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