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Superfield approach to interacting $N = 2$ massive and massless supermultiplets in 3d flat space

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ABSTRACT: Massive arbitrary spin supermultiplets and massless (scalar and spin one-half) supermultiplets of the $N = 2$ Poincaré superalgebra in three-dimensional flat space are considered. Both the integer spin and half-integer spin supermultiplets are studied. For such massive and massless supermultiplets, a formulation in terms of light-cone gauge unconstrained superfields defined in a momentum superspace is developed. For the supermultiplets under consideration a superspace first derivative representation for all cubic interaction vertices is obtained. A superspace representation for dynamical generators of the $N = 2$ Poincaré superalgebra is also found.

KEYWORDS: Field Theories in Lower Dimensions, Extended Supersymmetry, Superspaces, Higher Spin Gravity

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Contents

1	Introduction	1
2	Light-cone gauge formulation of free massive and massless $N = 2$ supermultiplets	2
3	Superfield formulation of free massive and massless $N = 2$ supermultiplets	7
4	Kinematical symmetries of n-point dynamical generators of $N = 2$ Poincaré superalgebra	11
5	$N = 2$ Poincaré superalgebra restrictions for cubic vertex and light-cone gauge dynamical principle	13
6	Superspace first-derivative representation for cubic vertices	17
7	V_κ-vertices for superfields with non-critical masses	24
7.1	V_κ -vertices for three massive superfields	24
7.2	V_κ -vertices for two massive superfields and one massless superfield	25
7.3	V_κ -vertices for two massless superfields and one massive superfield	27
8	V_κ-vertices for superfields with critical masses	29
8.1	V_κ -vertices for three massive superfields	29
8.2	V_κ -vertices for two massive superfields and one massless superfield	31
9	Superspace first-derivative cubic vertices for massless fields	32
10	Conclusions	35
A	Notation and conventions	36
B	Properties of superfields $\Theta_{m,\lambda,u}^*$	38
C	Derivation of superspace representation for cubic vertex $p_{[3]}^-$ and densities $j_{[3]}^{-1}$, $q_{[3]}^{-R,L}$	39

1 Introduction

In view of simplicity and aesthetic features of field theories in three dimensions these theories have attracted a considerable interest during long period of time. Lagrangian formulation of free arbitrary spin massive bosonic and fermionic irreducible fields propagating in the flat $R^{2,1}$ space was obtained in ref. [1].¹ In ref. [12], by using the light-cone gauge formalism, we begun a systematic study of interacting massive arbitrary spin and massless (scalar and spin one-half) fields in three dimensions.² We studied both the fermionic and bosonic fields. We introduced a classification for cubic interactions and, by using a first-derivative representation for cubic interactions, obtained the explicit expressions for all cubic interaction vertices. In this paper, we study interacting arbitrary spin massive supermultiplets and massless (scalar and spin one-half) supermultiplets of the $N = 2$ Poincaré superalgebra in the $R^{2,1}$. Both the integer spin and half-integer spin supermultiplets are studied. For such supermultiplets, our aim in this paper, is to provide a first-derivative superspace representation for all cubic interactions. To this end we use a light-cone momentum superspace and unconstrained light-cone gauge superfields defined in such superspace. We note that, it is the light-cone gauge unconstrained superfields we introduce in this paper that allow us to construct a simple superspace representation for all cubic interaction vertices of the supermultiplets under consideration. For the reader convenience, we note that, in the past, the light-cone momentum superspace has been used successfully in various interesting studies of supergravity and superstring theories. We mention the use of the momentum superspace in 10d extended supergravity in ref. [13] and superstring field theories in ref. [14]. Application of the light-cone momentum superspace for the study of light-cone gauge 11d supergravity may be found in ref. [15].

In conclusion of the Introduction, let us briefly mention our main two long term motivations for our study of supersymmetric theories of massive and massless fields in the $R^{2,1}$ space. First, we believe that our results in this paper may be helpful in the search of yet unknown models of supersymmetric higher-spin massive fields in the $R^{2,1}$ space and their cousins in higher dimensions. Second, one expects that a spectrum of states of the superstring in the AdS_3 space is realized by arbitrary spin massive fields and low spin massless fields. We think then that our superspace representation for cubic vertices of light-cone gauge fields in the $R^{2,1}$ space might be a good starting point to understand cubic interactions of light-cone gauge fields in the AdS_3 space and hence might find applications in various studies of superstring in the AdS_3 space.³ Further discussion of our long term motivations for our research in this paper can be found in Conclusions.

This paper is organized in the following way.

¹Discussion of various interesting aspects of massless and massive higher-spin dynamics in three dimensions may be found in the (incomplete) list of refs. [2–11].

²In $3d$, all massless unitary irreps of the Poincaré algebra can be described by the spin-0 and spin- $\frac{1}{2}$ fields. For this reason, for the case of massless fields, we deal only with spin-0 and spin- $\frac{1}{2}$ fields.

³In the framework of light-cone gauge approach, superstring in the AdS_3 space was studied in ref. [16]. See also ref. [17] for studying 3-point correlator functions of AdS superstring. Interesting discussion of the $N = 2$ superstring model in the $R^{2,1}$ space may be found in ref. [18].

In section 2, we start with the presentation of a light-cone frame of the $N = 2$ Poincaré superalgebra. After that, by using light-cone gauge components fields, we review integer spin and half-integer spin massive and massless supermultiplets of the $N = 2$ Poincaré superalgebra.

Section 3 is devoted to a superfield description of the $N = 2$ Poincaré superalgebra supermultiplets. We introduce a momentum superspace and define the corresponding light-cone gauge unconstrained superfields. In terms of our superfields, we describe then a realization of the $N = 2$ Poincaré superalgebra on a space of the massive and massless supermultiplets under consideration.

In section 4 we discuss n -point interaction vertices. We present constraints on n -point interaction vertices which are obtained by using kinematical symmetries of the $N = 2$ Poincaré superalgebra.

Section 5 is devoted to cubic vertices. We consider constraints on cubic vertices which are obtained by using dynamical and kinematical symmetries of the $N = 2$ Poincaré superalgebra. After that, we formulate a light-cone gauge dynamical principle and discuss the complete system of equations which are required to determine the cubic interaction vertices uniquely.

In section 6, we present a superspace form for all our cubic vertices which describe interactions of arbitrary spin massive and massless supermultiplets of the $N = 2$ Poincaré superalgebra. We introduce a notion of critical and non-critical masses of fields entering the cubic vertices.

In section 7, we present solution for so called V -vertices entering superspace form of cubic vertices for the case of the non-critical masses, while, in section 8, we present solution for V -vertices entering superspace form of cubic vertices for the case of the critical masses.

Section 9 is devoted to the study of a superspace representation of cubic vertices for the case when all three $N = 2$ supermultiplets entering the cubic vertices are massless.

In section 10, we present our conclusions.

Our notation and conventions are presented in appendix A. Properties of our unconstrained superfields are discussed in appendix B. In appendix C, we outline the derivation of the superspace representation of our cubic vertices.

2 Light-cone gauge formulation of free massive and massless $N = 2$ supermultiplets

We use method suggested in ref. [19]. According to this method, the finding of a new light-cone gauge dynamical system amounts to the finding of solution for (anti)commutators of a symmetry algebra. For $N = 2$ supersymmetric theories of massive and massless fields in the flat space $R^{2,1}$, basic symmetries are associated with the $N = 2$ Poincaré superalgebra. In this section, we start therefore with the presentation of (anti)commutators of the $N = 2$ Poincaré superalgebra in light-cone frame. After that, we review $N = 2$ supermultiplets in light-cone frame and realization of Poincaré algebra on space of massive and massless component fields. Realization of the $N = 2$ Poincaré superalgebra on space of supermultiplets will be discussed in the next section by using unconstrained light-cone gauge superfields.

Light-cone frame for $N = 2$ Poincaré superalgebra in $3d$. For the flat space $R^{2,1}$, the $N = 2$ Poincaré superalgebra consists of the three translation generators P^μ , the three generators of the $so(2, 1)$ Lorentz algebra $J^{\mu\nu}$, the $U(1)$ R-symmetry generator denoted as U , and four supercharges. For the Poincaré algebra generators, we assume the following commutators:

$$[P^\mu, J^{\nu\rho}] = \eta^{\mu\nu} P^\rho - \eta^{\mu\rho} P^\nu, \quad [J^{\mu\nu}, J^{\rho\sigma}] = \eta^{\nu\rho} J^{\mu\sigma} + 3 \text{ terms}, \quad (2.1)$$

where $\eta^{\mu\nu}$ stands for the mostly positive Minkowski metric. Explicit form of the remaining (anti)commutators of the $N = 2$ Poincaré superalgebra will soon be given below.

In place of the Lorentz frame coordinates x^μ , $\mu = 0, 1, 2$, we use the light-cone basis coordinates $x^\pm$, x^1 , where the coordinates $x^\pm$ are defined as

$$x^\pm \equiv \frac{1}{\sqrt{2}}(x^2 \pm x^0), \quad (2.2)$$

and the coordinate x^+ is taken to be an evolution parameter. In the light-cone frame, the $so(2, 1)$ Lorentz algebra vector X^μ is decomposed as X^+, X^-, X . A scalar product of the $so(2, 1)$ Lorentz algebra vectors X^μ and Y^μ takes the form

$$\eta_{\mu\nu} X^\mu Y^\nu = X^+ Y^- + X^- Y^+ + X^1 Y^1. \quad (2.3)$$

Relation (2.3) tells us that, in the light-cone frame, the metric $\eta_{\mu\nu}$ has the following non-vanishing elements: $\eta_{+-} = \eta_{-+} = 1$, $\eta_{11} = 1$. This implies the rules $X^+ = X_-$, $X^- = X_+$, $X^1 = X_1$.

In the light-cone frame, commutators of the Poincaré algebra can simply be obtained from (2.1) by using the flat metric $\eta^{\mu\nu}$ which has the following non-vanishing elements $\eta^{+-} = \eta^{-+} = 1$, $\eta^{11} = 1$. Using notation $Q^{\pm R}$, $Q^{\pm L}$ for the four supercharges, we then note that, in the light-cone frame, the remaining (anti)commutators of the $N = 2$ Poincaré superalgebra takes the form,⁴

$$[J^{+-}, Q^{\pm R}] = \pm \frac{1}{2} Q^{\pm R}, \quad [J^{+-}, Q^{\pm L}] = \pm \frac{1}{2} Q^{\pm L}, \quad (2.4)$$

$$[Q^{-R}, J^{+1}] = -\frac{1}{\sqrt{2}} Q^{+R}, \quad [Q^{-L}, J^{+1}] = -\frac{1}{\sqrt{2}} Q^{+L}, \quad (2.5)$$

$$[Q^{+R}, J^{-1}] = \frac{1}{\sqrt{2}} Q^{-R}, \quad [Q^{+L}, J^{-1}] = \frac{1}{\sqrt{2}} Q^{-L}, \quad (2.6)$$

$$[U, Q^{\pm R}] = Q^{\pm R}, \quad [U, Q^{\pm L}] = -Q^{\pm L}, \quad (2.7)$$

$$\{Q^{+R}, Q^{+L}\} = P^+, \quad \{Q^{-R}, Q^{-L}\} = -P^-, \quad (2.8)$$

$$\{Q^{+R}, Q^{-L}\} = \frac{1}{\sqrt{2}} P^1, \quad \{Q^{+L}, Q^{-R}\} = \frac{1}{\sqrt{2}} P^1, \quad (2.9)$$

⁴We use the $N = 2$ Poincaré superalgebra without a central charge. Dimensional reduction of the $N = 1$ Poincaré superalgebra from $4d$ to $3d$ leads to the appearance of a central charge. Therefore we do not expect that our vertices in this paper can be obtained via dimensional reduction from the light-cone gauge vertices of the $N = 1$ massless supermultiplets in $4d$ obtained in ref. [20]. Study of the light-cone gauge vertices for the N -extended massless supermultiplets in $4d$ may be found in refs. [21, 22]. For the discussion of Lorentz covariant vertices for various massless supermultiplets in $4d$, see, e.g., refs. [23, 24] (and references therein).

where, we recall, that the U appearing in (2.7) is a generator of $U(1)$ symmetry (R -symmetry). For the generators, we assume the following hermitian conjugation rules:

$$\begin{aligned} P^{\pm\dagger} &= P^{\pm}, & P^{1\dagger} &= P^1, & J^{+-\dagger} &= -J^{+-}, & J^{\pm 1\dagger} &= -J^{\pm 1}, & U^\dagger &= U, \\ Q^{+R\dagger} &= Q^{+L}, & Q^{-R\dagger} &= Q^{-L}. \end{aligned} \quad (2.10)$$

Generators of the $N = 2$ Poincaré superalgebra can be separated into the following two groups:

$$P^+, \quad P^1, \quad J^{+1}, \quad J^{+-}, \quad U, \quad Q^{+R}, \quad Q^{+L}, \quad \text{kinematical generators}; \quad (2.11)$$

$$P^-, \quad J^{-1}, \quad Q^{-R}, \quad Q^{-L}, \quad \text{dynamical generators}. \quad (2.12)$$

In a field realization, with the exception of J^{+-} , the kinematical generators (2.11) are quadratic in fields,⁵ while, the dynamical generators (2.12) involve quadratic and higher order terms in fields.

To build a field theoretical realization of the $N = 2$ Poincaré superalgebra generators on massive and massless fields, we use a light-cone gauge formulation of the fields. Let us first explain our notation we use for component fields entering the $N = 2$ supermultiplets. Component fields we use in this paper are denoted as $\phi_{m,\lambda,u}(x)$ and $\psi_{m,\lambda,u}(x)$, where x stands for the space time-coordinates x^+ , x^- , x^1 , while the labels m , λ , and u denote the respective mass parameter, spin, and $U(1)$ charge of the fields. We recall that, for fields in three dimensions, the mass parameter m is allowed to be negative or positive. Sometimes, in the literature, fields with $m > 0$ and $m < 0$ are referred to as the respective self-dual and anti-self-dual massive fields. Obviously, the self-dual and anti-self-dual massive fields are associated with different irreps of the Poincaré algebra $iso(2, 1)$. For bosonic fields, the spin parameter λ takes integer values, $\lambda = s$, $s \in \mathbb{N}_0$, while for fermionic fields, the λ takes half-integer values, $\lambda = s + \frac{1}{2}$, $s \in \mathbb{N}_0$. The $U(1)$ charge takes values $u \in \mathbb{Z}$. We now proceed with the description of massive and massless supermultiplets of the $N = 2$ Poincaré superalgebra.

$N = 2$ massive spin- s supermultiplet. $N = 2$ massive spin- s supermultiplet which we denote as $\{m, s, u\}_{N=2}$ is described by the following set of massive fields:

$$\begin{aligned} \{m, s, u\}_{N=2} &= \phi_{m,s,u-1}^{\oplus}, \quad \phi_{m,s+\frac{1}{2},u}^{\oplus}, \quad \phi_{m,s-\frac{1}{2},u}^{\ominus}, \quad \phi_{m,s,u+1}^{\ominus}, \\ m &\neq 0, \quad s \in \mathbb{N}_0, \quad u \in \mathbb{Z}, \end{aligned} \quad (2.13)$$

where $\phi_{m,s,u-1}^{\oplus}$, $\phi_{m,s,u+1}^{\ominus}$ are bosonic fields, while $\phi_{m,s+\frac{1}{2},u}^{\oplus}$, $\phi_{m,s-\frac{1}{2},u}^{\ominus}$ are fermionic fields.⁶ For the fields in (2.13), we assume the following hermitian conjugation rules

$$\phi_{m,s,u}^{\oplus\dagger} = \phi_{m,s,-u}^{\ominus}, \quad \phi_{m,s+\frac{1}{2},u}^{\oplus\dagger} = \phi_{m,s+\frac{1}{2},-u}^{\oplus}, \quad \phi_{m,s-\frac{1}{2},u}^{\ominus\dagger} = \phi_{m,s-\frac{1}{2},-u}^{\ominus}. \quad (2.14)$$

From (2.14), we see that, for $u = 0$, the fermionic fields are real-valued and the supermultiplet (2.13) is self-adjoint. For $u \neq 0$, all fields in (2.13) are complex-valued.

⁵For interacting fields, the J^{+-} takes the form $J^{+-} = G_0 + ix^+ P^-$, where a generator G_0 is quadratic in fields, while the dynamical generator P^- (Hamiltonian) involves quadratic and higher order terms in fields.

⁶For $s = 0$, the field $\phi_{m,s-\frac{1}{2},u}^{\ominus}$ in (2.13) should read as $\phi_{-m,\frac{1}{2},u}$.

$N = 2$ massive spin- $(s + \frac{1}{2})$ supermultiplet. $N = 2$ massive spin- $(s + \frac{1}{2})$ supermultiplet denoted as $\{m, s + \frac{1}{2}, u\}_{N=2}$ is described by the following set of massive fields:

$$\{m, s + \frac{1}{2}, u\}_{N=2} = \psi_{m, s + \frac{1}{2}, u-1}^{\oplus}, \quad \psi_{m, s+1, u}^{\oplus}, \quad \psi_{m, s, u}^{\ominus}, \quad \psi_{m, s + \frac{1}{2}, u+1}^{\ominus},$$

$$m \neq 0, \quad s \in \mathbb{N}_0, \quad u \in \mathbb{Z}, \quad (2.15)$$

where $\psi_{m, s+1, u}^{\oplus}$, $\psi_{m, s, u}^{\ominus}$ are bosonic fields, while $\psi_{m, s + \frac{1}{2}, u-1}^{\oplus}$, $\psi_{m, s + \frac{1}{2}, u+1}^{\ominus}$ are fermionic fields. For the fields in (2.15), we use the following hermitian conjugation rules:

$$\psi_{m, s + \frac{1}{2}, u}^{\oplus \dagger} = \psi_{m, s + \frac{1}{2}, -u}^{\ominus}, \quad \psi_{m, s+1, u}^{\oplus \dagger} = \psi_{m, s+1, -u}^{\oplus}, \quad \psi_{m, s, u}^{\ominus \dagger} = \psi_{m, s, -u}^{\ominus}. \quad (2.16)$$

Relations (2.16) tell us that, for $u = 0$, the bosonic fields are real-valued and the supermultiplet (2.15) is self-adjoint. For $u \neq 0$, all fields in (2.15) are complex-valued.

$N = 2$ massless spin-0 supermultiplet. Fields of $N = 2$ massless spin-0 supermultiplets are introduced by analogy with the ones for the massive supermultiplets in (2.13),

$$\{0, 0, u\}_{N=2} = \phi_{0, 0, u-1}^{\oplus}, \quad \phi_{0, \frac{1}{2}, u}^{\oplus}, \quad \phi_{0, \frac{1}{2}, u}^{\ominus}, \quad \phi_{0, 0, u+1}^{\ominus}, \quad u \in \mathbb{Z}, \quad (2.17)$$

where $\phi_{0, 0, u-1}^{\oplus}$, $\phi_{0, 0, u+1}^{\ominus}$ are bosonic fields, while $\phi_{0, \frac{1}{2}, u}^{\oplus}$, $\phi_{0, \frac{1}{2}, u}^{\ominus}$ are fermionic fields. For the fields in (2.17), we assume the following hermitian conjugation rules

$$\phi_{0, 0, u}^{\oplus \dagger} = \phi_{0, 0, -u}^{\ominus}, \quad \phi_{0, \frac{1}{2}, u}^{\oplus \dagger} = \phi_{0, \frac{1}{2}, -u}^{\oplus}, \quad \phi_{0, \frac{1}{2}, u}^{\ominus \dagger} = \phi_{0, \frac{1}{2}, -u}^{\ominus}. \quad (2.18)$$

Fields in (2.17) are decomposed into two invariant subspaces. These invariant spaces will be described in section 3.

$N = 2$ massless spin- $\frac{1}{2}$ supermultiplet. Fields of $N = 2$ spin- $\frac{1}{2}$ massless supermultiplets are introduced by analogy with the ones for the massive supermultiplets in (2.15),

$$\left\{0, \frac{1}{2}, u\right\}_{N=2} = \psi_{0, \frac{1}{2}, u-1}^{\oplus}, \quad \psi_{0, 0, u}^{\oplus}, \quad \psi_{0, 0, u}^{\ominus}, \quad \psi_{0, \frac{1}{2}, u+1}^{\ominus}, \quad u \in \mathbb{Z}, \quad (2.19)$$

where $\psi_{0, 0, u}^{\oplus}$, $\psi_{0, 0, u}^{\ominus}$ are bosonic fields, while $\psi_{0, \frac{1}{2}, u-1}^{\oplus}$, $\psi_{0, \frac{1}{2}, u+1}^{\ominus}$ are fermionic fields. For the fields in (2.19), we use the following hermitian conjugation rules

$$\psi_{0, \frac{1}{2}, u}^{\oplus \dagger} = \psi_{0, \frac{1}{2}, -u}^{\ominus}, \quad \psi_{0, 0, u}^{\oplus \dagger} = \psi_{0, 0, -u}^{\oplus}, \quad \psi_{0, 0, u}^{\ominus \dagger} = \psi_{0, 0, -u}^{\ominus}. \quad (2.20)$$

Fields in (2.19) are decomposed into two invariant subspaces. These invariant spaces and some equivalencies between the massless spin-0 and the spin- $\frac{1}{2}$ supermultiplets are described in section 3.

In what follows, in place of the fields defined in x -space, we prefer to use fields defined in a momentum space. Namely, using the shortcut notation $\chi_{m, \lambda, u}(x)$ for all fields above discussed,

$$\chi_{m, \lambda, u} = \phi_{m, \lambda, u}^{\oplus, \ominus}, \quad \psi_{m, \lambda, u}^{\oplus, \ominus}, \quad (2.21)$$

we introduce fields $\chi_{m,\lambda,u}(x^+, p, \beta)$ by using the Fourier transform with respect to the coordinates x^-, x^1 ,

$$\chi_{m,\lambda,u}(x) = \int \frac{d^2 p}{2\pi} e^{i(\beta x^- + p x^1)} \chi_{m,\lambda,u}(x^+, p, \beta), \quad d^2 p \equiv dp d\beta. \quad (2.22)$$

In what follows, in place of the field $\chi_{m,\lambda,u}(x^+, p, \beta)$, we use the shortcut $\chi_{m,\lambda,u}(p)$. Note also that, throughout this paper, the field $\chi_{m,\lambda,u}^\dagger(p)$ is defined as $\chi_{m,\lambda,u}^\dagger(p) \equiv (\chi_{m,\lambda,u}(p))^\dagger$.

Realization of the Poincaré algebra in terms of the fields. We now recall a field theoretical realization of the Poincaré algebra and U(1) symmetry on the space of the component fields above discussed. As is known, the realizations of the Poincaré algebra (2.1) and U(1) symmetry in terms of differential operators acting on the component fields $\chi_{m,\lambda,u}(p)$ (2.22) is given as follows.

Realizations on space of fields $\chi_{m,\lambda,u}(p)$:

$$P^1 = p, \quad P^+ = \beta, \quad P^- = p^-, \quad p^- \equiv -\frac{p^2 + m^2}{2\beta}, \quad (2.23)$$

$$J^{+1} = ix^+ P^1 + \partial_p \beta, \quad J^{+-} = ix^+ P^- + \partial_\beta \beta - \frac{1}{2} e_\lambda, \quad (2.24)$$

$$J^{-1} = -\partial_\beta p + \partial_p p^- + \frac{im\lambda}{\beta} + \frac{p}{2\beta} e_\lambda, \quad (2.25)$$

$$U = u, \quad (2.26)$$

where partial derivatives ∂_β , ∂_p , and symbol e_λ are defined as

$$\partial_\beta \equiv \partial/\partial\beta, \quad \partial_p \equiv \partial/\partial p, \quad (2.27)$$

$$e_\lambda = 0 \quad \text{for integer } \lambda, \quad e_\lambda = 1 \quad \text{for half-integer } \lambda. \quad (2.28)$$

Making use of the relations (2.23)–(2.28) we can build a field representation for the symmetry algebras. Namely, to quadratic order in the component fields, a field-theoretical realization for generators of the Poincaré algebra (2.1) and the $u(1)$ algebra is given by

$$G_{[2]} = 2 \int d^2 p \beta^{e_{\lambda+\frac{1}{2}}} \chi_{m,\lambda,u}^\dagger(p) G_{\text{diff}} \chi_{m,\lambda,u}(p), \quad (2.29)$$

where G_{diff} are given in (2.23)–(2.28), while $G_{[2]}$ stands for the field-theoretical realization for generators of the Poincaré algebra and the $u(1)$ algebra. To avoid unnecessary complicated expressions in terms of the component fields, the field-theoretical realization of the supercharges $Q^{\pm R,L}$ will be given in section 3, by using superfield formulation of the $N = 2$ supermultiplets.

The fields $\chi_{m,\lambda,u}$ satisfy the Poisson-Dirac equal-time (anti)commutation relations

$$[\chi_{m,\lambda,u}(p), \chi_{m,\lambda',u'}^\dagger(p')]_\pm = \frac{1}{2} \beta^{-e_{\lambda+\frac{1}{2}}} \delta^2(p-p') \delta_{\lambda\lambda'} \delta_{uu'}, \quad (2.30)$$

where the notation $[a, b]_\pm$ is used for a graded commutator, $[a, b]_\pm = (-)^{\varepsilon_a \varepsilon_b + 1} [b, a]_\pm$. Making use of the relations (2.29), (2.30), it is easy to check the standard equal-time commutators between the components fields and the Grassmann even generators

$$[\chi_{m,\lambda,u}, G_{[2]}] = G_{\text{diff}} \chi_{m,\lambda,u}. \quad (2.31)$$

Free light-cone gauge action for the component field $\chi_{m,\lambda,u}(p)$ takes the well known form

$$S_{[2]} = \int dx^+ d^2p \beta^{-e_\lambda} \chi_{m,\lambda,u}^\dagger(p) (2i\beta\partial^- - p^2 - m^2) \chi_{m,\lambda,u}(p), \quad (2.32)$$

where $\partial^- \equiv \partial/\partial x^+$, while e_λ is given in (2.28).

3 Superfield formulation of free massive and massless $N = 2$ supermultiplets

To develop a light-cone gauge superfield formulation of the $N = 2$ supermultiplets under consideration we introduce two Grassmann-odd momenta denoted by p_θ and p_η . Our momentum superspace is parametrized then by the light-cone time x^+ , the momenta p , β and the two Grassmann momenta p_θ , p_η ,

$$x^+, p, \beta, p_\theta, p_\eta. \quad (3.1)$$

We now introduce various unconstrained superfields on the superspace (3.1) in turn.

Superfield for $N = 2$ massive spin- s supermultiplet. For $N = 2$ massive spin- s supermultiplet (2.13) we introduce Grassmann even superfield $\Phi_{m,s,u}$ defined as

$$\Phi_{m,s,u}(p, p_\theta, p_\eta) = \beta\phi(p) + p_\theta\phi_\theta(p) + p_\eta\phi_\eta(p) + p_\theta p_\eta\phi_{\theta\eta}(p), \quad (3.2)$$

where component fields entering the Grassmann momenta expansion of the superfield in (3.2) are expressed in terms of the component fields of the supermultiplets (2.13) as

$$\phi = \phi_{m,s,u-1}^\oplus, \quad \phi_{\theta\eta} = \phi_{m,s,u+1}^\ominus, \quad (3.3)$$

$$\phi_\theta = \frac{1}{\sqrt{2}} \left(i\phi_{m,s+\frac{1}{2},u}^\oplus + \phi_{m,s-\frac{1}{2},u}^\ominus \right), \quad \phi_\eta = \frac{1}{\sqrt{2}} \left(i\phi_{m,s+\frac{1}{2},u}^\oplus - \phi_{m,s-\frac{1}{2},u}^\ominus \right). \quad (3.4)$$

An argument p of superfields and fields stands for the momenta p and β entering the momentum superspace (3.1). In terms of the component fields in (3.1), the hermitian conjugation rules in (2.14) take the form

$$\phi_{\theta\eta}^\dagger(p) = \phi(-p)|_{u \rightarrow -u}, \quad \phi_\theta^\dagger(p) = -\phi_\eta(-p)|_{u \rightarrow -u}. \quad (3.5)$$

We recall that, for all fields, we use the convention $\chi^\dagger(p) \equiv (\chi(p))^\dagger$.

Superfield for $N = 2$ massive spin- $(s + \frac{1}{2})$ supermultiplet. For $N = 2$ massive spin- $(s + \frac{1}{2})$ supermultiplet (2.15) we introduce Grassmann odd superfield $\Psi_{m,s+\frac{1}{2},u}$ defined as

$$\Psi_{m,s+\frac{1}{2},u}(p, p_\theta, p_\eta) = \beta\psi(p) + \beta p_\theta\psi_\theta(p) + \beta p_\eta\psi_\eta(p) + p_\theta p_\eta\psi_{\theta\eta}(p), \quad (3.6)$$

where component fields entering the Grassmann momenta expansion of the superfield in (3.6) are expressed in terms of the component fields of the supermultiplets defined in (2.15) as

$$\psi = \psi_{m,s+\frac{1}{2},u-1}^\oplus, \quad \psi_{\theta\eta} = \psi_{m,s+\frac{1}{2},u+1}^\ominus, \quad (3.7)$$

$$\psi_\theta = \frac{1}{\sqrt{2}} \left(i\psi_{m,s+1,u}^\oplus + \psi_{m,s,u}^\ominus \right), \quad \psi_\eta = \frac{1}{\sqrt{2}} \left(i\psi_{m,s+1,u}^\oplus - \psi_{m,s,u}^\ominus \right). \quad (3.8)$$

In terms of the component fields (3.6), the hermitian conjugation rules (2.16) take the form

$$\psi_{\theta\eta}^\dagger(p) = \psi(-p)|_{u \rightarrow -u}, \quad \psi_\theta^\dagger(p) = -\psi_\eta(-p)|_{u \rightarrow -u}. \quad (3.9)$$

Superfield for $N = 2$ massless spin-0 supermultiplet. For $N = 2$ massless spin-0 supermultiplet (2.17), we introduce Grassmann even superfield denoted as $\Phi_{0,0,u}$. This superfield takes the Grassmann momenta expansion identical to the one in (3.2), where fields entering the Grassmann momenta expansion of the superfield $\Phi_{0,0,u}$ are expressed in terms of component fields of the supermultiplets defined in (2.17) as

$$\phi = \phi_{0,0,u-1}^{\oplus}, \quad \phi_{\theta\eta} = \phi_{0,0,u+1}^{\ominus}, \quad (3.10)$$

$$\phi_{\theta} = \frac{1}{\sqrt{2}} \left(i\phi_{0,\frac{1}{2},u}^{\oplus} + \phi_{0,\frac{1}{2},u}^{\ominus} \right), \quad \phi_{\eta} = \frac{1}{\sqrt{2}} \left(i\phi_{0,\frac{1}{2},u}^{\oplus} - \phi_{0,\frac{1}{2},u}^{\ominus} \right). \quad (3.11)$$

In terms of the component fields entering the superfield $\Phi_{0,0,u}$, hermitian conjugation rules in (2.18) take the same form as in (3.5). Note that the massless fields ϕ , ϕ_{θ} and ϕ_{η} , $\phi_{\theta\eta}$ in (3.10), (3.11) constitute two invariant subspaces under transformations of the $N = 2$ Poincaré superalgebra.

Superfield for $N = 2$ massless spin- $\frac{1}{2}$ supermultiplet. For $N = 2$ massless spin- $\frac{1}{2}$ supermultiplet (2.19), we introduce Grassmann odd superfield denoted as $\Psi_{0,\frac{1}{2},u}$. This superfield takes the Grassmann momenta expansion identical to the one in (3.6), where fields entering the Grassmann momenta expansion of the superfield $\Psi_{0,\frac{1}{2},u}$ are expressed in terms of the component fields of the supermultiplets defined in (2.19) as

$$\psi = \psi_{0,\frac{1}{2},u-1}^{\oplus}, \quad \psi_{\theta\eta} = \psi_{0,\frac{1}{2},u+1}^{\ominus}, \quad (3.12)$$

$$\psi_{\theta} = \frac{1}{\sqrt{2}} \left(i\psi_{0,0,u}^{\oplus} + \psi_{0,0,u}^{\ominus} \right), \quad \psi_{\eta} = \frac{1}{\sqrt{2}} \left(i\psi_{0,0,u}^{\oplus} - \psi_{0,0,u}^{\ominus} \right). \quad (3.13)$$

In terms of the component fields entering the superfield $\Psi_{0,\frac{1}{2},u}$, hermitian conjugation rules in (2.20) take the same form as in (3.9). The massless fields ψ , ψ_{θ} and ψ_{η} , $\psi_{\theta\eta}$ in (3.12), (3.13) constitute two invariant subspaces under transformations of the $N = 2$ Poincaré superalgebra. We note also some equivalencies between the $N = 2$ massless spin-0 and spin- $\frac{1}{2}$ supermultiplets. Namely, under action of the $N = 2$ Poincaré superalgebra generators, the fields ϕ , $i\phi_{\theta}$ entering the superfield $\Phi_{0,0,u}$ transform in the same way as the fields $i\psi_{\eta}$, $\psi_{\theta\eta}$ entering the superfield $\Psi_{0,\frac{1}{2},u-1}$, while, the fields $i\phi_{\eta}$, $\phi_{\theta\eta}$ entering the superfield $\Phi_{0,0,u}$ transform in the same way as the fields ψ , $i\psi_{\theta}$ entering the superfield $\Psi_{0,\frac{1}{2},u+1}$. This implies the following equivalence

$$\sum_{u \in \mathbb{Z}} \oplus \{0, 0, u\}_{N=2} \sim \sum_{u \in \mathbb{Z}} \oplus \left\{ 0, \frac{1}{2}, u \right\}_{N=2} \quad \text{for } N = 2 \text{ massless supermultiplets.} \quad (3.14)$$

In order to treat the above discussed superfields Φ and Ψ on an equal footing and to simplify our analysis we collect our superfields into new superfields denoted as $\Theta_{m,\lambda,u}$,

$$\Theta_{m,\lambda,u}(p, p_{\theta}, p_{\eta}), \quad \lambda = 0, \frac{1}{2}, 1, \dots, \infty, \quad m \neq 0; \quad (3.15)$$

$$\Theta_{0,\lambda,u}(p, p_{\theta}, p_{\eta}), \quad \lambda = 0, \frac{1}{2}, \quad m = 0, \quad (3.16)$$

where, depending on the values of λ and m , the new superfields Θ are identified with the Φ and Ψ in (3.2), (3.6) in the following way:

$$\begin{aligned}\Theta_{m,s,u} &\equiv \Phi_{m,s,u}, & \Theta_{m,s+\frac{1}{2},u} &\equiv \Psi_{m,s+\frac{1}{2},u}, & m \neq 0, & s = 0, 1, 2, \dots, \infty; \\ \Theta_{0,0,u} &\equiv \Phi_{0,0,u}, & \Theta_{0,\frac{1}{2},u} &\equiv \Psi_{0,\frac{1}{2},u}, & m = 0.\end{aligned}\quad (3.17)$$

We note that, for the integer λ , the new superfield $\Theta_{m,\lambda,u}$ is Grassmann even, while, for the half-integer λ , the new superfield $\Theta_{m,\lambda,u}$ is Grassmann odd. Therefore, making use of the notation $\text{GP}(\Theta_\lambda)$ for the Grassmann parity of the new superfields $\Theta_{m,\lambda,u}$ and recalling the definition of e_λ (2.28), we note the obvious relation,

$$\text{GP}(\Theta_{m,\lambda,u}) = e_\lambda. \quad (3.18)$$

$N = 2$ Poincaré superalgebra realization on superfield $\Theta_{m,\lambda,u}$. We find the realization of the $N = 2$ Poincaré superalgebra in terms of differential operators acting on the superfield $\Theta_{m,\lambda,u}(p, p_\theta)$:

$$P^1 = p, \quad P^+ = \beta, \quad P^- = p^-, \quad p^- = -\frac{p^2 + m^2}{2\beta}, \quad (3.19)$$

$$J^{+1} = ix^+ P^1 + \partial_p \beta, \quad J^{+-} = ix^+ P^- + \partial_\beta \beta + M^{+-}, \quad (3.20)$$

$$J^{-1} = -\partial_\beta p + \partial_p p^- + \frac{im}{\beta} \left(\lambda + \frac{1}{2} p_\theta \partial_{p_\eta} + \frac{1}{2} p_\eta \partial_{p_\theta} \right) - \frac{p}{\beta} M^{+-}, \quad (3.21)$$

$$U = 1 - u - p_\theta \partial_{p_\theta} - p_\eta \partial_{p_\eta}, \quad (3.22)$$

$$Q^{+R} = (-)^{e_\lambda} \beta \partial_{p_\theta}, \quad (3.23)$$

$$Q^{+L} = (-)^{e_\lambda} p_\theta, \quad (3.24)$$

$$Q^{-R} = \frac{(-)^{e_\lambda}}{\sqrt{2}} (p \partial_{p_\theta} + im \partial_{p_\eta}), \quad (3.25)$$

$$Q^{-L} = \frac{(-)^{e_\lambda}}{\sqrt{2} \beta} (p p_\theta - im p_\eta), \quad (3.26)$$

where operator M^{+-} is defined as

$$M^{+-} \equiv \frac{1}{2} p_\theta \partial_{p_\theta} + \frac{1}{2} p_\eta \partial_{p_\eta} - 1 - \frac{1}{2} e_\lambda, \quad (3.27)$$

while the definition of ∂_β , ∂_p and e_λ may be found in (2.27) and (2.28). Quantities ∂_{p_θ} , ∂_{p_η} stand for the left derivatives of the respective Grassmann momenta p_θ and p_η .

To discuss a light-cone gauge action, we find it convenient to use, in addition to the superfields $\Phi_{m,\lambda,u}$ and $\Psi_{m,\lambda,u}$, superfields which we denote as $\Phi_{m,\lambda,u}^*$ and $\Psi_{m,\lambda,u}^*$ respectively. The superfields $\Phi_{m,\lambda,u}^*$, $\Psi_{m,\lambda,u}^*$ are built in terms of the hermitian conjugated component fields appearing in (3.2), (3.6). Namely, for $m \neq 0$, the Grassmann even superfield $\Phi_{m,\lambda,u}^*$ and the Grassmann odd superfield $\Psi_{m,\lambda,u}^*$ are defined by the relations

$$\Phi_{m,s,u}^*(p, p_\theta, p_\eta) = \phi_{\theta\eta}^\dagger(p) - \frac{p_\theta}{\beta} \phi_\eta^\dagger(p) + \frac{p_\eta}{\beta} \phi_\theta^\dagger(p) + \frac{p_\theta p_\eta}{\beta} \phi^\dagger(p), \quad (3.28)$$

$$\Psi_{m,s+\frac{1}{2},u}^*(p, p_\theta, p_\eta) = \psi_{\theta\eta}^\dagger(p) + p_\theta \psi_\eta^\dagger(p) - p_\eta \psi_\theta^\dagger(p) + \frac{p_\theta p_\eta}{\beta} \psi^\dagger(p). \quad (3.29)$$

For the massless case, we introduce $\Phi_{0,0,u}^*$ and $\Psi_{0,\frac{1}{2},u}^*$ which take the same Grassmann momenta expansion as in (3.28) and (3.29) respectively. As before, in order to treat the superfields Φ^* , Ψ^* on an equal footing, we introduce superfields $\Theta_{m,\lambda,u}^*$,

$$\Theta_{m,\lambda,u}^*(p, p_\theta, p_\eta), \quad \lambda = 0, \frac{1}{2}, 1, \dots, \infty; \quad m \neq 0 \quad (3.30)$$

$$\Theta_{0,\lambda,u}^*(p, p_\theta, p_\eta), \quad \lambda = 0, \frac{1}{2}, \quad m = 0, \quad (3.31)$$

where, depending on the values of m and λ , the superfields Θ^* are identified with Φ^* , Ψ^* in the following way:

$$\Theta_{m,s,u}^* \equiv \Phi_{m,s,u}^*, \quad \Theta_{m,s+\frac{1}{2},u}^* \equiv \Psi_{m,s+\frac{1}{2},u}^*, \quad (3.32)$$

$$\Theta_{0,0,u}^* \equiv \Phi_{0,0,u}^*, \quad \Theta_{0,\frac{1}{2},u}^* \equiv \Psi_{0,\frac{1}{2},u}^*. \quad (3.33)$$

Grassmann parity of the superfield $\Theta_{m,\lambda,u}^*$ is given by the relation

$$\text{GP}(\Theta_{m,\lambda,u}^*) = e_\lambda, \quad (3.34)$$

which tells us, for the integer λ , the $\Theta_{m,\lambda,u}^*$ is Grassmann even, while, for the half-integer λ , the $\Theta_{m,\lambda,u}^*$ is Grassmann odd.

Relations (3.19)–(3.26) allow us to build a superfield representation for the $N = 2$ Poincaré superalgebra. Namely, to quadratic order in the superfields, a field-theoretical superspace representation for the generators of the $N = 2$ Poincaré superalgebra is given by

$$G_{[2]} = \sum_{2\lambda \in \mathbb{N}_0, u \in \mathbb{Z}} G_{[2], m, \lambda, u} \quad G_{[2], m, \lambda, u} = \int d^2p dp_\eta dp_\theta \beta^{e_{\lambda+\frac{1}{2}}} \Theta_{m,\lambda,u}^* G_{\text{diff}} \Theta_{m,\lambda,u}, \quad (3.35)$$

where G_{diff} are given in (3.19)–(3.26). A realization of G_{diff} on space of the superfield $\Theta_{m,\lambda,u}^*$ and some other properties of the superfield $\Theta_{m,\lambda,u}^*$ are discussed in appendix B.

The superfields Θ , Θ^* satisfy the Poisson-Dirac equal-time commutation relations

$$[\Theta_{m,\lambda,u}(p, p_\theta, p_\eta), \Theta_{m,\lambda',u'}^*(p', p'_\theta, p'_\eta)]_\pm = \frac{1}{2} \beta^{-e_{\lambda+\frac{1}{2}}} \delta^2(p - p') \delta(p_\theta - p'_\theta) \delta(p_\eta - p'_\eta) \delta_{\lambda\lambda'} \delta_{uu'}, \quad (3.36)$$

where, we recall, $[a, b]_\pm$ is a graded commutator, $[a, b]_\pm = (-)^{\varepsilon_a \varepsilon_b + 1} [b, a]_\pm$. Making use of the relations (3.35), (3.36), it is easy to verify the standard equal-time (anti)commutator between the generators and the superfields

$$[\Theta_{m,\lambda,u} G_{[2]}]_\pm = G_{\text{diff}} \Theta_{m,\lambda,u}, \quad (3.37)$$

where expressions for the G_{diff} may be found in (3.19)–(3.26).

Finally we note that, in terms of the superfields above discussed, the light-cone gauge action takes the form

$$S = \frac{1}{2} \sum_{2\lambda \in \mathbb{N}_0, u \in \mathbb{Z}} \int dx^+ d^2p dp_\eta dp_\theta \beta^{-e_\lambda} \Theta_{m,\lambda,u}^* (2i\beta \partial^- - p^2 - m^2) \Theta_{m,\lambda,u} + \int dx^+ P_{\text{int}}^-, \quad (3.38)$$

where $\partial^- \equiv \partial/\partial x^+$ and P_{int}^- stands for light-cone gauge Hamiltonian which describes interacting fields.

4 Kinematical symmetries of n -point dynamical generators of $N = 2$ Poincaré superalgebra

For interacting fields, the dynamical generators of the $N = 2$ Poincaré superalgebra receive corrections having higher powers of the superfields. Namely, for the dynamical generators, one has the following expansion in the superfields

$$G^{\text{dyn}} = \sum_{n=2}^{\infty} G_{[n]}^{\text{dyn}}, \quad (4.1)$$

where $G_{[n]}^{\text{dyn}}$ is a functional having n powers of the superfields Θ^* . We now describe restrictions imposed on the dynamical generators $G_{[n]}^{\text{dyn}}$, $n \geq 3$, by the kinematical symmetries of the $N = 2$ Poincaré superalgebra. We discuss the kinematical symmetry restrictions in turn.

Kinematical P^1 , P^+ , Q^{+L} symmetries. Making use of the (anti)commutators between the dynamical generators (2.12) and the kinematical generators P^1 , P^+ , Q^{+L} , we find that the dynamical generators $G_{[n]}^{\text{dyn}}$ with $n \geq 3$ can be cast into the form:

$$P_{[n]}^- = \int d\Gamma_{[n]} \langle \Theta_{[n]}^* | p_{[n]}^- \rangle, \quad (4.2)$$

$$Q_{[n]}^{-R} = \int d\Gamma_{[n]} \langle \Theta_{[n]}^* | q_{[n]}^{-R} \rangle, \quad (4.3)$$

$$Q_{[n]}^{-L} = \int d\Gamma_{[n]} \langle \Theta_{[n]}^* | q_{[n]}^{-L} \rangle, \quad (4.4)$$

$$J_{[n]}^{-1} = \int d\Gamma_{[n]} \langle \Theta_{[n]}^* | j_{[n]}^{-1} \rangle + \langle \mathbf{X}_{[n]}^1 \Theta_{[n]}^* | p_{[n]}^- \rangle \\ + \frac{1}{\sqrt{2}} \langle \mathbf{X}_{[n]} \theta \Theta_{[n]}^* | q_{[n]}^{-L} \rangle - \frac{1}{\sqrt{2}n} \langle \mathcal{P}_{[n]} \theta \Theta_{[n]}^* | q_{[n]}^{-R} \rangle, \quad (4.5)$$

where we are using the following notation

$$d\Gamma_{[n]} = d\Gamma_{[n]}^p d\Gamma_{[n]}^{p\eta p\theta}, \quad (4.6)$$

$$d\Gamma_{[n]}^p = \delta^2 \left(\sum_{a=1}^n p_a \right) \prod_{a=1}^n \frac{d^2 p_a}{2\pi}, \quad d^2 p_a = dp_a d\beta_a, \quad (4.7)$$

$$d\Gamma_{[n]}^{p\eta p\theta} \equiv dp_{\eta_1} dp_{\theta_1} \dots dp_{\eta_n} dp_{\theta_n} \delta \left(\sum_{a=1}^n p_{\theta_a} \right), \quad (4.8)$$

$$\mathbf{X}_{[n]}^1 = -\frac{1}{n} \sum_{a=1}^n \partial_{p_a}, \quad \mathbf{X}_{[n]} \theta = \frac{1}{n} \sum_{a=1}^n \partial_{p_{\theta_a}}, \quad \mathcal{P}_{[n]} \theta = \sum_{a=1}^n \frac{p_{\theta_a}}{\beta_a}. \quad (4.9)$$

In (4.2)–(4.5), the expressions $\langle \Theta_{[n]}^* | p_{[n]}^- \rangle$, $\langle \Theta_{[n]}^* | q_{[n]}^{-R,L} \rangle$, and $\langle \Theta_{[n]}^* | j_{[n]}^{-1} \rangle$ are shortcuts which are defined as

$$\langle \Theta_{[n]}^* | p_{[n]}^- \rangle \equiv \sum_{\sigma_1 \dots \sigma_n} \Theta_{\sigma_1 \dots \sigma_n}^* p_{\sigma_1 \dots \sigma_n}^-, \quad (4.10)$$

$$\langle \Theta_{[n]}^* | q_{[n]}^{-R,L} \rangle \equiv \sum_{\sigma_1 \dots \sigma_n} \Theta_{\sigma_1 \dots \sigma_n}^* q_{\sigma_1 \dots \sigma_n}^{-R,L}, \quad (4.11)$$

$$\langle \Theta_{[n]}^* | j_{[n]}^{-1} \rangle \equiv \sum_{\sigma_1 \dots \sigma_n} \Theta_{\sigma_1 \dots \sigma_n}^* j_{\sigma_1 \dots \sigma_n}^{-1}, \quad (4.12)$$

$$\Theta_{\sigma_1 \dots \sigma_n}^* \equiv \Theta_{\sigma_1}^*(p_1, p_{\theta_1}, p_{\eta_1}) \dots \Theta_{\sigma_n}^*(p_n, p_{\theta_n}, p_{\eta_n}), \quad (4.13)$$

where a label σ_a is a shortcut for mass parameters, spins, and U(1) charges

$$\sigma_a = m_a, \lambda_a, u_a, \quad (4.14)$$

and indices $a, b = 1, \dots, n$ are used to label superfields entering n -point vertices. Throughout this paper the p_a as argument of delta functions (4.7) and superfields (4.13) stands for the momenta p_a and β_a . Sometimes, the quantities $p_{\sigma_1 \dots \sigma_n}^-$, $q_{\sigma_1 \dots \sigma_n}^{-R,L}$, and $j_{\sigma_1 \dots \sigma_n}^{-1}$ (4.2)–(4.5), will shortly be denoted as $g_{[n]}$,

$$g_{[n]} = p_{\sigma_1 \dots \sigma_n}^-, \quad q_{\sigma_1 \dots \sigma_n}^{-R}, \quad q_{\sigma_1 \dots \sigma_n}^{-L}, \quad j_{\sigma_1 \dots \sigma_n}^{-1}. \quad (4.15)$$

The quantities $g_{[n]}$ (4.15) will be referred to as n -point densities. Often the density $p_{[n]}^-$ will be referred to as an n -point interaction vertex, while, for $n = 3$, the density $p_{[3]}^-$ will be referred to as cubic interaction vertex.

In general, the n -point densities $g_{[n]}$ (4.15) depend on the momenta p_a , β_a , Grassmann momenta p_{θ_a} , p_{η_a} , masses m_a , spins λ_a , and U(1) charges u_a , $a = 1, 2, \dots, n$,

$$g_{[n]} = g_{[n]}(p_a, p_{\theta_a}, p_{\eta_a}, \beta_a). \quad (4.16)$$

Also note that, in (4.5), the differential operators $\mathbf{X}_{[n]}^1$, $\mathbf{X}_{[n]\theta}$ act only on the arguments of the superfields. For example, the expression $\langle \mathbf{X}^1 \Theta_{[n]}^* | g_{[n]} \rangle$ should read as

$$\langle \mathbf{X}^1 \Theta_{[n]}^* | g_{[n]} \rangle = \sum_{\sigma_1, \dots, \sigma_n} (\mathbf{X}^1 \Theta_{\sigma_1 \dots \sigma_n}^*) g_{\sigma_1 \dots \sigma_n}. \quad (4.17)$$

Kinematical J^{+-} -symmetry. Making use of the commutators between the dynamical generators P^- , $Q^{-R,L}$, J^{-1} and the kinematical generator J^{+-} , we find the equations for the densities given by:

$$\sum_{a=1}^n \left(\beta_a \partial_{\beta_a} + \frac{1}{2} N_{p_{\theta_a}} + \frac{1}{2} N_{p_{\eta_a}} + \frac{1}{2} e_{\lambda_a} \right) g_{[n]} = \left(n - \frac{1}{2} \right) g_{[n]}, \quad \text{for } g_{[n]} = p_{[n]}^-, j_{[n]}^{-1}, \quad (4.18)$$

$$\sum_{a=1}^n \left(\beta_a \partial_{\beta_a} + \frac{1}{2} N_{p_{\theta_a}} + \frac{1}{2} N_{p_{\eta_a}} + \frac{1}{2} e_{\lambda_a} \right) g_{[n]} = n g_{[n]}, \quad \text{for } g_{[n]} = q_{[n]}^{-R,L}, \quad (4.19)$$

$$N_{p_{\theta_a}} \equiv p_{\theta_a} \partial_{p_{\theta_a}}, \quad N_{p_{\eta_a}} \equiv p_{\eta_a} \partial_{p_{\eta_a}}. \quad (4.20)$$

Kinematical J^{+1} , Q^{+R} -symmetries. Making use of the (anti)commutators between the dynamical generators P^- , $Q^{-R,L}$, J^{-1} and the kinematical generators J^{+1} , Q^{+R} , we verify that the dependence of the densities $g_{[n]}$ (4.15) on the momenta p_a and the Grassmann momenta p_{θ_a} is realized through new momentum variables $\mathbb{P}_{ab}$ and $\mathbb{P}_{\theta ab}$ defined by the relations

$$\mathbb{P}_{ab} \equiv p_a \beta_b - p_b \beta_a, \quad \mathbb{P}_{\theta ab} \equiv p_{\theta_a} \beta_b - p_{\theta_b} \beta_a. \quad (4.21)$$

Thus the densities $g_{[n]}$ (4.16) turn out to be functions of the new momenta $\mathbb{P}_{ab}$ and $\mathbb{P}_{\theta ab}$ in place of the generic momenta p_a , p_{θ_a} ,

$$g_{[n]} = g_{[n]}(\mathbb{P}_{ab}, \mathbb{P}_{\theta ab}, p_{\eta_a}, \beta_a). \quad (4.22)$$

Kinematical U(1)-symmetry. Making use of the commutators between the dynamical generators P^- , $Q^{-R,L}$, J^{-1} and the kinematical generator U , we find the equations for the densities given by:

$$\sum_{a=1}^n (N_{p_{\theta a}} + N_{p_{\eta a}} + u_a) p_{[n]}^- = (n-1) g_{[n]}^-, \quad \text{for } g_{[n]} = p_{[n]}^-, j_{[n]}^{-1}, \quad (4.23)$$

$$\sum_{a=1}^n (N_{p_{\theta a}} + N_{p_{\eta a}} + u_a) q_{[n]}^{-R} = (n-2) q_{[n]}^{-R}, \quad \sum_{a=1}^n (N_{p_{\theta a}} + N_{p_{\eta a}} + u_a) q_{[n]}^{-L} = n q_{[n]}^{-L}, \quad (4.24)$$

where we use the notation given in (4.20).

Let us now summarize the kinematical symmetry restrictions on the n -point densities.

- i) The kinematical P^1 , P^+ , and Q^{+L} symmetries lead to delta-functions in expressions for the dynamical generators P^- , $Q^{-R,L}$, J^{-1} (4.2)–(4.5). These symmetries imply the conservation laws for the momenta p_a , β_a and the Grassmann momenta $p_{\theta a}$.
- ii) The kinematical J^{+-} symmetry leads to the differential equations given in (4.18), (4.19).
- iii) The kinematical J^{+1} and Q^{+R} symmetries imply that the n -point densities $p_{[n]}^-$, $q_{[n]}^{-R,L}$, $j_{[n]}^{-1}$ turn out to be functions that depend on the new momenta $\mathbb{P}_{ab}$ and $\mathbb{P}_{\theta ab}$ (4.19) in place of the generic momenta p_a and $p_{\theta a}$ respectively. Conservation laws for the momenta p_a , β_a , $p_{\theta a}$ imply that there are $n-2$ independent momenta $\mathbb{P}_{ab}$ and $n-2$ independent Grassmann momenta $\mathbb{P}_{\theta ab}$ (4.19). For example, if $n=3$, then there is one independent momentum $\mathbb{P}$ and one independent Grassmann momentum $\mathbb{P}_{\theta}$ (see below).

5 $N=2$ Poincaré superalgebra restrictions for cubic vertex and light-cone gauge dynamical principle

In this section, firstly, we represent kinematical J^{+-} symmetry equations (4.18), (4.19) in terms of the momenta $\mathbb{P}_{ab}$ and $\mathbb{P}_{ab\theta}$. Secondly, we find restrictions imposed by dynamical symmetries. Finally, we discuss light-cone gauge dynamical principle and write down the complete system equations which allows us to determine the cubic vertices unambiguously.

Kinematical symmetries of the cubic densities. Making use of the momentum conservation laws

$$p_1 + p_2 + p_3 = 0, \quad \beta_1 + \beta_2 + \beta_3 = 0, \quad p_{\theta 1} + p_{\theta 2} + p_{\theta 3} = 0, \quad (5.1)$$

it is easy to check that $\mathbb{P}_{12}$, $\mathbb{P}_{23}$, $\mathbb{P}_{31}$ and Grassmann momenta $\mathbb{P}_{\theta 12}$, $\mathbb{P}_{\theta 23}$, $\mathbb{P}_{\theta 31}$ can be expressed in terms of momenta $\mathbb{P}$, $\mathbb{P}_{\theta}$,

$$\mathbb{P}_{12} = \mathbb{P}_{23} = \mathbb{P}_{31} = \mathbb{P}, \quad \mathbb{P}_{\theta 12} = \mathbb{P}_{\theta 23} = \mathbb{P}_{\theta 31} = \mathbb{P}_{\theta}, \quad (5.2)$$

where the momenta $\mathbb{P}$ and $\mathbb{P}_{\theta}$ are defined by the relations

$$\mathbb{P} \equiv \frac{1}{3} \sum_{a=1,2,3} \check{\beta}_a p_a, \quad \mathbb{P}_{\theta} \equiv \frac{1}{3} \sum_{a=1,2,3} \check{\beta}_a p_{\theta a}, \quad \check{\beta}_a \equiv \beta_{a+1} - \beta_{a+2}, \quad \beta_a \equiv \beta_{a+3}. \quad (5.3)$$

The use of the momenta (5.3) is convenient for us because these momenta turn out to be manifestly invariant upon cyclic permutations of the external line indices 1, 2, 3. Making use of the simplified notation for the cubic densities,

$$p_{[3]}^- = p_{\sigma_1\sigma_2\sigma_3}^-, \quad q_{[3]}^{-R,L} = q_{\sigma_1\sigma_2\sigma_3}^{-R,L}, \quad j_{[3]}^{-1} = j_{\sigma_1\sigma_2\sigma_3}^{-1}, \quad (5.4)$$

we note that the cubic densities $p_{[3]}^-$, $q_{[3]}^{-R,L}$, and $j_{[3]}^{-1}$ are functions of the momenta β_a , $\mathbb{P}$, the Grassmann momenta $\mathbb{P}_\theta$, p_{η_a} , and $\sigma_a = m_a, \lambda_a, u_a$, $a = 1, 2, 3$,

$$\begin{aligned} p_{[3]}^- &= p_{\sigma_1\sigma_2\sigma_3}^-(\mathbb{P}, \mathbb{P}_\theta, p_{\eta_a}, \beta_a), & q_{[3]}^{-R,L} &= q_{\sigma_1\sigma_2\sigma_3}^{-R,L}(\mathbb{P}, \mathbb{P}_\theta, p_{\eta_a}, \beta_a), \\ j_{[3]}^{-1} &= j_{\sigma_1\sigma_2\sigma_3}^{-1}(\mathbb{P}, \mathbb{P}_\theta, p_{\eta_a}, \beta_a). \end{aligned} \quad (5.5)$$

Thus we see that the dependence of the cubic densities on the momenta p_a and p_{θ_a} is realized through the respective momenta $\mathbb{P}$ and $\mathbb{P}_\theta$. In view of this, the study of cubic densities is considerably simplified. Let us now represent equations given in (4.18), (4.19) and (4.23), (4.24) in terms of the cubic densities given in (5.5).

$\mathbf{J}^{+-}$ -symmetry equations. Making use of the representation for cubic densities in (5.5), we verify that, for $n = 3$, equations (4.18), (4.19) can be represented as

$$\left(\mathbf{J}^{+-} - \frac{5}{2}\right)p_{[3]}^- = 0, \quad (\mathbf{J}^{+-} - 3)q_{[3]}^{-R,L} = 0, \quad \left(\mathbf{J}^{+-} - \frac{5}{2}\right)j_{[3]}^{-1} = 0, \quad (5.6)$$

where $\mathbf{J}^{+-}$ is defined by the relations

$$\mathbf{J}^{+-} \equiv N_{\mathbb{P}} + \frac{3}{2}N_{\mathbb{P}_\theta} + \sum_{a=1,2,3} \left(\beta_a \partial_{\beta_a} + \frac{1}{2}N_{p_{\eta_a}} + \frac{1}{2}e_{\lambda_a} \right), \quad (5.7)$$

$$N_{\mathbb{P}} \equiv \mathbb{P} \partial_{\mathbb{P}}, \quad N_{\mathbb{P}_\theta} \equiv \mathbb{P}_\theta \partial_{\mathbb{P}_\theta}, \quad N_{p_{\eta_a}} \equiv p_{\eta_a} \partial_{p_{\eta_a}}. \quad (5.8)$$

$\mathbf{U}(1)$ -symmetry equations. Making use of the representation for the cubic densities in (5.5), we verify that, for $n = 3$, equations (4.23), (4.24) can be represented as

$$\left(N_{\mathbb{P}_\theta} - 2 + \sum_{a=1,2,3} (N_{p_{\eta_a}} + u_a) \right) g_{[3]}^- = 0, \quad \text{for } g_{[3]} = p_{[3]}^-, \quad j_{[3]}^{-1}, \quad (5.9)$$

$$\left(N_{\mathbb{P}_\theta} - 1 + \sum_{a=1,2,3} (N_{p_{\eta_a}} + u_a) \right) q_{[3]}^{-R} = 0, \quad \left(N_{\mathbb{P}_\theta} - 3 + \sum_{a=1,2,3} (N_{p_{\eta_a}} + u_a) \right) q_{[3]}^{-L} = 0, \quad (5.10)$$

where we use the notation as in (5.8).

We now proceed with studying the restrictions imposed by the dynamical symmetries.

Dynamical symmetries restrictions on cubic densities. Restrictions on the cubic densities obtained from (anti)commutators between the dynamical generators will be referred to as dynamical symmetries restrictions in this paper. This is to say that now we discuss restrictions obtained from the following (anti)commutators:

$$[P^-, J^{-1}] = 0, \quad [P^-, Q^{-R,L}] = 0, \quad (5.11)$$

$$[Q^{-R,L}, J^{-1}] = 0, \quad \{Q^{-R}, Q^{-L}\} = -P^-. \quad (5.12)$$

Considering the commutators in (5.11), we find in the cubic approximation the following relations:

$$[P_{[2]}^-, J_{[3]}^{-1}] + [P_{[3]}^-, J_{[2]}^{-1}] = 0, \quad [P_{[2]}^-, Q_{[3]}^{-R,L}] + [P_{[3]}^-, Q_{[2]}^{-R,L}] = 0. \quad (5.13)$$

Making use of the commutators (5.13), we find the representation for the densities $q_{[3]}^{-R,L}$ and $j_{[3]}^{-1}$ in terms of the cubic vertex $p_{[3]}^-$,

$$\mathbf{P}^- q_{[3]}^{-R} = -\frac{\varepsilon}{\sqrt{2}} \left(\mathbb{P} \partial_{\mathbb{P}_\theta} + \sum_{a=1,2,3} i m_a \partial_{p_{\eta a}} \right) p_{[3]}^-, \quad (5.14)$$

$$\mathbf{P}^- q_{[3]}^{-L} = \frac{\varepsilon}{\sqrt{2}} \left(\frac{\mathbb{P} \mathbb{P}_\theta}{\beta} + \sum_{a=1,2,3} \frac{i m_a}{\beta_a} p_{\eta a} \right) p_{[3]}^-, \quad (5.15)$$

$$\mathbf{P}^- j_{[3]}^{-1} = -\mathbf{J}^{-1} p_{[3]}^-, \quad (5.16)$$

where in (5.14)–(5.16) we use the notation

$$\mathbf{P}^- \equiv \frac{\mathbb{P}^2}{2\beta} - \sum_{a=1,2,3} \frac{m_a^2}{2\beta_a}, \quad (5.17)$$

$$\mathbf{J}^{-1} \equiv -\frac{\mathbb{P}}{\beta} \mathbb{N}_\beta^{\eta E} + \sum_{a=1,2,3} \frac{\check{\beta}_a}{6\beta_a} m_a^2 \partial_{\mathbb{P}} - \mathcal{M} + \sum_{a=1,2,3} i m_a \left(\frac{\check{\beta}_a}{6\beta} \mathbb{P}_\theta \partial_{p_{\eta a}} - \frac{\check{\beta}_a}{6\beta_a} p_{\eta a} \partial_{\mathbb{P}_\theta} \right), \quad (5.18)$$

$$\mathbb{N}_\beta^{\eta E} \equiv \mathbb{N}_\beta + \sum_{a=1,2,3} \frac{1}{6} \check{\beta}_a (N_{p_{\eta a}} + e_{\lambda_a}), \quad \mathbb{N}_\beta \equiv \frac{1}{3} \sum_{a=1,2,3} \check{\beta}_a \beta_a \partial_{\beta_a}, \quad (5.19)$$

$$\mathcal{M} \equiv \sum_{a=1,2,3} \frac{i m_a \lambda_a}{\beta_a}, \quad (5.20)$$

$$\beta \equiv \beta_1 \beta_2 \beta_3, \quad \check{\beta}_a \equiv \beta_{a+1} - \beta_{a+2}, \quad (5.21)$$

$$\varepsilon \equiv (-)^{\mathbf{E}_\lambda}, \quad \mathbf{E}_\lambda \equiv e_{\lambda_1} + e_{\lambda_2} + e_{\lambda_3}, \quad (5.22)$$

while the symbol e_λ entering (5.22) is defined in (2.28).

Making use of the solution for $q_{[3]}^{-R,L}$ and $j_{[3]}^{-1}$ in (5.14)–(5.16), one can easily check that the kinematical symmetry equations for $q_{[3]}^{-R,L}$ and $j_{[3]}^{-1}$ in (5.6)–(5.10) are automatically satisfied provided the cubic vertex $p_{[3]}^-$ satisfies the kinematical symmetry equations given in (5.6), (5.9). Moreover, by using solutions for $q_{[3]}^{-R,L}$ and $j_{[3]}^{-1}$ in (5.14)–(5.16), one can check that, in the cubic approximation, the (anti)commutators in (5.12) are also automatically satisfied. Thus, in the cubic approximation, the kinematical symmetry equations for $p_{[3]}^-$ in (5.6), (5.9) and equations (5.14)–(5.16) exhaust all restrictions imposed by the (anti)commutators of the $N = 2$ Poincaré superalgebra.

Light-cone gauge dynamical principle. Equations in (5.6), (5.9) and (5.14)–(5.16) do not fix the cubic densities unambiguously. In order to fix the cubic densities unambiguously we should impose some additional restrictions on the cubic densities. Such additional restrictions will be referred to as light-cone gauge dynamical principle. We formulated the light-cone gauge dynamical principle as follows:

- i) All cubic densities $p_{[3]}^-$, $q_{[3]}^{-R,L}$, $j_{[3]}^{-1}$ should be polynomial in the momentum $\mathbb{P}$ and β -analytic;⁷
- ii) The cubic vertex $p_{[3]}^-$ should satisfy the following restriction

$$p_{[3]}^- \neq \mathbf{P}^- W, \quad W \text{ is polynomial in } \mathbb{P} \text{ and } \beta\text{-analytic}, \quad (5.23)$$

where $\mathbf{P}^-$ is defined in (5.17). We note that the restriction in (5.23) is motivated by the freedom of field redefinitions. Namely, upon field redefinitions, the cubic vertex $p_{[3]}^-$ is changed by terms proportional to $\mathbf{P}^-$ (see, e.g., appendix B in ref. [47]). Therefore ignoring restriction (5.23) leads to cubic vertex $p_{[3]}^-$ which can be removed by field redefinitions. However we are interested in the cubic vertex $p_{[3]}^-$ that cannot be removed by field redefinitions. For this reason we impose the restriction (5.23). For the reader convenience we note that the restriction in i) is the light-cone counterpart of locality condition used in Lorentz covariant and gauge invariant formulations.

Complete system of equations for cubic vertex. To summarize the discussion in this section, we note that, for the cubic vertex given by

$$p_{[3]}^- = p_{[3]}^-(\mathbb{P}, \mathbb{P}_\theta, p_{\eta_a}, \beta_a), \quad (5.24)$$

the complete system of equations which remains to be analysed is given by

$N = 2$ Poincaré superalgebra kinematical and dynamical restrictions:

$$\text{Equation for } p_{[3]}^- \text{ in (5.6),} \quad \text{kinematical } J^{+-} \text{ symmetry;} \quad (5.25)$$

$$\text{Equation for } p_{[3]}^- \text{ in (5.9),} \quad \text{kinematical } U(1) \text{ symmetry;} \quad (5.26)$$

$$\text{Equation (5.16),} \quad \text{dynamical } P^-, J^{-1} \text{ symmetries;} \quad (5.27)$$

$$\text{Equations (5.14) and (5.15),} \quad \text{dynamical } P^-, Q^{-R,L} \text{ symmetries;} \quad (5.28)$$

Light-cone gauge dynamical principle:

$$p_{[3]}^-, q_{[3]}^{-R,L}, j_{[3]}^{-1} \quad \text{should be polynomial in } \mathbb{P} \text{ and } \beta\text{-analytic;} \quad (5.29)$$

$$p_{[3]}^- \neq \mathbf{P}^- W, \quad W \text{ should be polynomial in } \mathbb{P} \text{ and } \beta\text{-analytic.} \quad (5.30)$$

It is the equations (5.25)–(5.30) that constitute our basic complete system of equations. If some vertex $p_{[3]}^-|_{\text{fix}}$ satisfies our system of equations, then the vertex $p_{[3]}^-$ obtained from $p_{[3]}^-|_{\text{fix}}$ via the field redefinition $p_{[3]}^- = p_{[3]}^-|_{\text{fix}} + \mathbf{P}^- f$ also satisfies our system of equations. This implies that in order to fix the vertex uniquely we should chose some representative of the vertex by using field redefinitions. Only after choosing a representative of the vertex our system of equations allows us to find all possible solutions for the cubic vertex $p_{[3]}^-$ and the densities $q_{[3]}^{-R,L}$, $j_{[3]}^{-1}$ uniquely. Below, we use a representative of cubic vertex which we refer to as first-derivative representation of the vertex.

⁷If function $f = f(\beta_1, \beta_2, \beta_3)$ can be presented as $f = g/h$ where g, h are polynomials in the momenta $\beta_1, \beta_2, \beta_3$, then we say the function f is β -analytic. Otherwise, we say that the function f is β -nonanalytic. Let us consider Taylor series expansion for the vertex $p_{[3]}^- = \sum_{n=0}^K f_n \mathbb{P}^n$. If all f_n are β -analytic, then we say the vertex $p_{[3]}^-$ as β -analytic. Otherwise, we say that the vertex $p_{[3]}^-$ is β -nonanalytic.

6 Superspace first-derivative representation for cubic vertices

Superspace first-derivative representation for cubic vertex. In general, the vertex $p_{[3]}^-$ is a degree- K polynomial in the momentum $\mathbb{P}$, where $K < \infty$. However, as noted in ref. [12], in the framework of light-cone gauge approach in $3d$, by making use of field redefinitions, any such vertex $p_{[3]}^-$ can be cast into a degree-1 polynomial in the momentum $\mathbb{P}$. Throughout this paper, the representation for the vertex $p_{[3]}^-$ in terms of degree-1 polynomial in the momentum $\mathbb{P}$ is referred to as first-derivative representation of vertex. Thus our first-derivative vertices are polynomials of degree-1 in the momentum $\mathbb{P}$. In this section, we present our superspace first-derivative representation for all cubic vertices $p_{[3]}^-$ and the corresponding densities $q_{[3]}^{-R,L}$, $j_{[3]}^{-1}$. In this section and sections 7, 8, we consider cubic vertices which involve at least one massive superfield. Cubic vertices for the case of three massless superfields are considered separately in section 9. Derivation of the results below presented may be found in appendix C. We now present our results.

We find that equations (5.14)–(5.16) lead to three types of cubic vertices which we refer as type- k cubic vertices, where $k = 1, 2, 3$. We note that the type- k cubic vertex is realized as a homogeneous degree- k polynomial in the Grassmann momenta $\mathbb{P}_\theta$ and $p_{\eta_1}, p_{\eta_2}, p_{\eta_3}$,

$$\left(N_{\mathbb{P}_\theta} + \sum_{a=1,2,3} N_{p_{\eta a}} \right) p_{[3]}^- = k p_{[3]}^-, \quad \text{for type-}k \text{ vertex}, \quad (6.1)$$

where we use the notation as in (5.8).⁸

To clarify further our classification of vertices we use requirement that the Hamiltonian $P_{[3]}^-$ (4.2) should be Grassmann even. To this end we note the following relations:

$$\text{GP}(d\Gamma_{[3]}^{p_\eta p_\theta}) = 1, \quad \text{GP}(\Theta_{[3]}^*) = \mathbf{E}_\lambda, \quad \text{GP}(p_{[3]}^-) = \mathbf{E}_\lambda + 1. \quad (6.2)$$

The 1st relation in (6.2) tells us that the integration measure $d\Gamma_{[3]}^{p_\theta p_\eta}$ (4.6) is Grassmann odd, while the 2nd relation for the Grassmann parity of the product of three superfields $\Theta_{[3]}^*$ (4.13) is obtained by using (3.34). It is the 3rd relation in (6.2) that is obtained by requiring $\text{GP}(P_{[3]}^-) = 0$. Note however that the Grassmann parity of the type- k cubic vertex is equal to $\text{GP}(p_{[3]}^-) = k$. Equating this relation to the 3rd relation (6.2), we find the restriction

$$\mathbf{E}_\lambda + 1 = k \pmod{2}. \quad (6.3)$$

Using (6.3), we can clarify further our classification of vertices. Namely, from (3.32)–(3.34), we note the relations $\text{GP}(\Phi^*) = 0$, $\text{GP}(\Psi^*) = 1$. Using these Grassmann parities of the superfields Φ^* , Ψ^* , we conclude that, for $k = 1, 3$, relation (6.3) is satisfied by the product $\Phi^* \Phi^* \Phi^*$ and $\Psi^* \Psi^* \Phi^*$ while, for $k = 2$, relation (6.3) is satisfied by the product $\Phi^* \Phi^* \Psi^*$ and $\Psi^* \Psi^* \Psi^*$. Omitting the asterisk of the superfields, we then conclude that we deal with

⁸The fact that the solutions for the vertex $p_{[3]}^-$ can be classified as homogeneous polynomials in the $\mathbb{P}_\theta$ and $p_{\eta_1}, p_{\eta_2}, p_{\eta_3}$ is obvious in view of the $U(1)$ symmetry restrictions in (5.9), (5.10).

the following vertices:

$$\text{type-1 } \Phi\Phi\Phi \text{ and } \Psi\Psi\Phi \text{ vertices;} \quad (6.4)$$

$$\text{type-3 } \Phi\Phi\Phi \text{ and } \Psi\Psi\Phi \text{ vertices;} \quad (6.5)$$

$$\text{type-2 } \Phi\Phi\Psi \text{ and } \Psi\Psi\Psi \text{ vertices.} \quad (6.6)$$

As the type-1 and type-3 cubic vertices turn out to be related by hermitian conjugation, we first discuss these vertices and after that we discuss the type-2 cubic vertex.

Type-1 $\Phi\Phi\Phi$ and $\Psi\Psi\Phi$ vertices. Superspace first-derivative representation for the type-1 cubic vertices $\Phi\Phi\Phi$ and $\Psi\Psi\Phi$ takes the form

$$\begin{aligned} p_{[3]}^- = & \frac{1}{2} \left(1 + \frac{i\mathbb{P}}{\kappa} \right) \left(\kappa\mathbb{P}_\theta - \beta \sum_{a=1,2,3} \frac{m_a}{\beta_a} p_{\eta a} \right) V \\ & - \frac{1}{2} \left(1 - \frac{i\mathbb{P}}{\kappa} \right) \left(\kappa\mathbb{P}_\theta + \beta \sum_{a=1,2,3} \frac{m_a}{\beta_a} p_{\eta a} \right) \bar{V}, \end{aligned} \quad (6.7)$$

where a quantity κ is defined as

$$\kappa^2 \equiv -\beta \sum_{a=1,2,3} \frac{m_a^2}{\beta_a}, \quad \beta \equiv \beta_1 \beta_2 \beta_3, \quad (6.8)$$

while V and $\bar{V}$ are vertices that depend only on the β -momenta $\beta_1, \beta_2, \beta_3$ and the labels $\sigma_1, \sigma_2, \sigma_3$ defined in (4.14). The vertices V and $\bar{V}$ satisfy the following decoupled differential equations:

$$\frac{\kappa}{\beta} \mathbb{N}_\beta^E V + i\mathcal{M}V = 0, \quad \mathbf{J}_\beta V = 0, \quad (6.9)$$

$$\frac{\kappa}{\beta} \mathbb{N}_\beta^E \bar{V} - i\mathcal{M}\bar{V} = 0, \quad \mathbf{J}_\beta \bar{V} = 0, \quad (6.10)$$

where we use the notation

$$\mathbb{N}_\beta^E \equiv \mathbb{N}_\beta + \frac{1}{6} \sum_{a=1,2,3} \check{\beta}_a e_{\lambda_a}, \quad \mathbf{J}_\beta \equiv \sum_{a=1,2,3} (\beta_a \partial_{\beta_a} + \frac{1}{2} e_{\lambda_a}), \quad (6.11)$$

while the quantities $\mathbb{N}_\beta, \mathcal{M}, \beta$ are defined in (5.19)–(5.21).

Thus, for the type-1 cubic vertex, equations (5.14)–(5.16) amount to superspace representation in (6.7) and first-order differential equations in (6.9), (6.10). As we discuss below, the equations (6.9), (6.10) allow us to fix vertices $V, \bar{V}$ uniquely up to coupling constants. Also we note that equations (6.9), (6.10) on an equal footing describe the type-1 $\Phi\Phi\Phi$ and the type-1 $\Psi\Psi\Phi$ vertices. Note however that, for the superfield $\Phi_{m,\lambda,u}^*$, $\lambda = s$, we use $e_\lambda = 0$, while, for the superfield $\Psi_{m,\lambda,u}^*$, $\lambda = s + \frac{1}{2}$, we use $e_\lambda = 1$. For this reason, the particular forms of equations in (6.9), (6.10) for $\Phi\Phi\Phi$ and type-1 $\Psi\Psi\Phi$ vertices are different. For example, to get equations (6.9), (6.10) for the type-1 $\Phi\Phi\Phi$ vertex, we use the values $\lambda_1 = s_1, \lambda_2 = s_2, \lambda_3 = s_3$ to fix the $\mathcal{M}$ (5.20) and the values $e_{\lambda_1} = 0, e_{\lambda_2} = 0, e_{\lambda_3} = 0$ to fix the operators in (6.11), while, for the type-1 $\Psi\Psi\Phi$ vertex, we use the values $\lambda_1 = s_1 + \frac{1}{2}, \lambda_2 = s_2 + \frac{1}{2}, \lambda_3 = s_3$ to fix the $\mathcal{M}$ (5.20) and the values $e_{\lambda_1} = 1, e_{\lambda_2} = 1, e_{\lambda_3} = 0$ to fix the operators in (6.11).

Type-3 $\Phi\Phi\Phi$ and $\Psi\Psi\Phi$ vertices. Superspace first-derivative representation for the type-3 $\Phi\Phi\Phi$ and $\Psi\Psi\Phi$ cubic vertices takes the form

$$p_{[3]}^- = \frac{1}{2\kappa} \left(1 + \frac{i\mathbb{P}}{\kappa} \right) \left(\kappa\mathbb{P}_\theta - \beta \sum_{a=1,2,3} \frac{m_a}{\beta_a} p_{\eta a} \right) \sum_{b=1,2,3} m_b p_{\eta b+1} p_{\eta b+2} V' \\ + \frac{1}{2\kappa} \left(1 - \frac{i\mathbb{P}}{\kappa} \right) \left(\kappa\mathbb{P}_\theta + \beta \sum_{a=1,2,3} \frac{m_a}{\beta_a} p_{\eta a} \right) \sum_{b=1,2,3} m_b p_{\eta b+1} p_{\eta b+2} \bar{V}'. \quad (6.12)$$

The vertices V' and $\bar{V}'$ depend only on the momenta $\beta_1, \beta_2, \beta_3$ and the labels $\sigma_1, \sigma_2, \sigma_3$ defined in (4.14). These two vertices satisfy the following decoupled equations:

$$\frac{\kappa}{\beta} \mathbb{N}_\beta^E V' + i\mathcal{M}V' = 0, \quad \mathbf{J}_\beta V' = 0, \quad (6.13)$$

$$\frac{\kappa}{\beta} \mathbb{N}_\beta^E \bar{V}' - i\mathcal{M}\bar{V}' = 0, \quad \mathbf{J}_\beta \bar{V}' = 0, \quad (6.14)$$

where we use the notation as in (6.11), while the quantities $\mathbb{N}_\beta, \mathcal{M}, \beta$ are defined in (5.19)–(5.21). We see that though the expressions for the type-1 and type-3 cubic vertices $p_{[3]}^-$ are different, the vertices $V, \bar{V}$ and their primed cousins satisfy the same equations (see (6.9), (6.10) and (6.13), (6.14)). Also we see that equations (6.13), (6.14) on an equal footing describe the type-3 $\Phi\Phi\Phi$ and the type-3 $\Psi\Psi\Phi$ vertices.

Type-2 $\Phi\Phi\Psi$ and $\Psi\Psi\Psi$ vertices. Superspace first-derivative representation for the type-2 cubic vertices $\Phi\Phi\Psi$ and $\Psi\Psi\Psi$ takes the form

$$p_{[3]}^- = \frac{1}{2} \left(1 + \frac{i\mathbb{P}}{\kappa} \right) \left(\kappa\mathbb{P}_\theta - \beta \sum_{a=1,2,3} \frac{m_a}{\beta_a} p_{\eta a} \right) \sum_{b=1,2,3} p_{\eta b} V_b \\ - \frac{1}{2} \left(1 - \frac{i\mathbb{P}}{\kappa} \right) \left(\kappa\mathbb{P}_\theta + \beta \sum_{a=1,2,3} \frac{m_a}{\beta_a} p_{\eta a} \right) \sum_{b=1,2,3} p_{\eta b} \bar{V}_b, \quad (6.15)$$

where V_a and $\bar{V}_a$, $a = 1, 2, 3$, are vertices that depend only on the momenta $\beta_1, \beta_2, \beta_3$ and the labels $\sigma_1, \sigma_2, \sigma_3$ (4.14). Equations for the three vertices V_a and their barred cousins are given by

$$\frac{\kappa}{\beta} \left(\mathbb{N}_\beta^E + \frac{\check{\beta}_a}{6} \right) V_a + i\mathcal{M}V_a + \frac{m_a}{6\kappa\beta_a} \sum_{b=1,2,3} \check{\beta}_b m_b V_b = 0, \quad a = 1, 2, 3; \quad (6.16)$$

$$\left(\mathbf{J}_\beta + \frac{1}{2} \right) V_a = 0, \quad a = 1, 2, 3; \quad \sum_{a=1,2,3} m_a V_a = 0, \quad (6.17)$$

$$\frac{\kappa}{\beta} \left(\mathbb{N}_\beta^E + \frac{\check{\beta}_a}{6} \right) \bar{V}_a - i\mathcal{M}\bar{V}_a + \frac{m_a}{6\kappa\beta_a} \sum_{b=1,2,3} \check{\beta}_b m_b \bar{V}_b = 0, \quad a = 1, 2, 3; \quad (6.18)$$

$$\left(\mathbf{J}_\beta + \frac{1}{2} \right) \bar{V}_a = 0, \quad a = 1, 2, 3; \quad \sum_{a=1,2,3} m_a \bar{V}_a = 0, \quad (6.19)$$

where we use the notation as in (6.11) and (5.19)–(5.21). We see that equations for the vertices V_a in (6.16), (6.17) are decoupled from the ones for the vertices $\bar{V}_a$ in (6.18), (6.19).

Note that equations (6.16)–(6.19) on an equal footing describe both the type-2 $\Phi\Phi\Psi$ and type-2 $\Psi\Psi\Psi$ vertices. Particular form of these equations corresponding to the type-2 $\Phi\Phi\Psi$ and type-2 $\Psi\Psi\Psi$ vertices is obtained by plugging the suitable values of λ_a and ϵ_{λ_a} into (6.16)–(6.19).

Superspace representation for densities $j_{[3]}^{-1}$, $q_{[3]}^{-R,L}$. We now discuss a superspace representation for the densities $j_{[3]}^{-1}$, $q_{[3]}^{-R,L}$. For all cubic vertices, expression for the density $j_{[3]}^{-1}$ takes the form

$$j_{[3]}^{-1} = 2i\mathbb{N}_\beta^{\eta E} V_{\mathbb{P}}, \quad (6.20)$$

where the operator $\mathbb{N}_\beta^{\eta E}$ is given in (5.19), while a vertex $V_{\mathbb{P}}$ is expressible in terms of the V -vertices. We now present explicit form of the vertex $V_{\mathbb{P}}$, and the supercharge densities $q_{[3]}^{-R,L}$ for all type- k vertices. Note that the quantity β , ϵ appearing below are defined in (5.21), (5.22).

$V_{\mathbb{P}}$ and $q_{[3]}^{-R,L}$ for type-1 $\Phi\Phi\Phi$ and $\Psi\Psi\Phi$ vertices:

$$V_{\mathbb{P}} = \frac{1}{2\kappa} \left(\kappa \mathbb{P}_\theta - \beta \sum_{a=1,2,3} \frac{m_a}{\beta_a} p_{\eta_a} \right) V + \frac{1}{2\kappa} \left(\kappa \mathbb{P}_\theta + \beta \sum_{a=1,2,3} \frac{m_a}{\beta_a} p_{\eta_a} \right) \bar{V}, \quad (6.21)$$

$$q_{[3]}^{-R} = -\frac{i\epsilon\beta}{\sqrt{2}} (V + \bar{V}), \quad (6.22)$$

$$q_{[3]}^{-L} = -\frac{i\epsilon\beta}{\sqrt{2}\kappa} \mathbb{P}_\theta \sum_{a=1,2,3} \frac{m_a}{\beta_a} p_{\eta_a} (V - \bar{V}). \quad (6.23)$$

$V_{\mathbb{P}}$ and $q_{[3]}^{-R,L}$ for type-3 $\Phi\Phi\Phi$ and $\Psi\Psi\Phi$ vertices:

$$\begin{aligned} V_{\mathbb{P}} = & \frac{1}{2\kappa^2} \left(\kappa \mathbb{P}_\theta - \beta \sum_{a=1,2,3} \frac{m_a}{\beta_a} p_{\eta_a} \right) \sum_{b=1,2,3} m_b p_{\eta_{b+1}} p_{\eta_{b+2}} V' \\ & - \frac{1}{2\kappa^2} \left(\kappa \mathbb{P}_\theta + \beta \sum_{b=1,2,3} \frac{m_b}{\beta_b} p_{\eta_b} \right) \sum_{b=1,2,3} m_b p_{\eta_{b+1}} p_{\eta_{b+2}} \bar{V}', \end{aligned} \quad (6.24)$$

$$q_{[3]}^{-R} = -\frac{i\epsilon\beta}{\sqrt{2}\kappa} \sum_{a=1,2,3} m_a p_{\eta_{a+1}} p_{\eta_{a+2}} (V' - \bar{V}'), \quad (6.25)$$

$$q_{[3]}^{-L} = -\frac{i\epsilon\beta}{\sqrt{2}\kappa^2} \mathbb{P}_\theta \sum_{a=1,2,3} \frac{m_a}{\beta_a} p_{\eta_a} \sum_{b=1,2,3} m_b p_{\eta_{b+1}} p_{\eta_{b+2}} (V' + \bar{V}'). \quad (6.26)$$

$V_{\mathbb{P}}$ and $q_{[3]}^{-R,L}$ for type-2 $\Phi\Phi\Psi$ and $\Psi\Psi\Psi$ vertices:

$$\begin{aligned} V_{\mathbb{P}} = & \frac{1}{2\kappa} \left(\kappa \mathbb{P}_\theta - \beta \sum_{a=1,2,3} \frac{m_a}{\beta_a} p_{\eta_a} \right) \sum_{b=1,2,3} p_{\eta_b} V_b \\ & + \frac{1}{2\kappa} \left(\kappa \mathbb{P}_\theta + \beta \sum_{a=1,2,3} \frac{m_a}{\beta_a} p_{\eta_a} \right) \sum_{b=1,2,3} p_{\eta_b} \bar{V}_b, \end{aligned} \quad (6.27)$$

$$q_{[3]}^{-R} = -\frac{i\epsilon\beta}{\sqrt{2}} \sum_{b=1,2,3} p_{\eta_b} (V_b + \bar{V}_b), \quad (6.28)$$

$$q_{[3]}^{-L} = -\frac{i\epsilon\beta}{\sqrt{2}\kappa} \mathbb{P}_\theta \sum_{a=1,2,3} \frac{m_a}{\beta_a} p_{\eta_a} \sum_{b=1,2,3} p_{\eta_b} (V_b - \bar{V}_b). \quad (6.29)$$

For critical and non-critical masses, a structure of solutions of the equations for the V -vertices is different. Therefore, before discussing solutions of the equations for the V -vertices, we provide the definition of critical and non-critical masses we use in this paper.

Critical and non-critical mass values. Consider a cubic vertex for three superfields which have masses m_1, m_2, m_3 . Let us introduce quantities $D, \mathbf{P}_{em}$ defined as

$$D \equiv m_1^4 + m_2^4 + m_3^4 - 2m_1^2 m_2^2 - 2m_2^2 m_3^2 - 2m_3^2 m_1^2, \quad (6.30)$$

$$\mathbf{P}_{em} \equiv \sum_{a=1,2,3} \epsilon_a m_a, \quad (6.31)$$

$$\epsilon_1^2 = 1, \quad \epsilon_2^2 = 1, \quad \epsilon_3^2 = 1, \quad (6.32)$$

$$D = (m_1 + m_2 + m_3)(m_1 + m_2 - m_3)(m_1 - m_2 + m_3)(m_1 - m_2 - m_3), \quad (6.33)$$

where, relation (6.33) provides us an alternative representation for the D defined in (6.30). If, for masses m_1, m_2, m_3 , we meet the inequality $D \neq 0$, then we refer to such masses as non-critical masses, while, if, for masses m_1, m_2, m_3 , we meet the equality $D = 0$, then such masses are referred to as critical masses. Note that the relation $\mathbf{P}_{em} = 0$ implies the relation $D = 0$, while the relation $D = 0$ implies that there exist $\epsilon_1, \epsilon_2, \epsilon_3$ (6.32) such that the relation $\mathbf{P}_{em} = 0$ holds true. Thus we have the relations

$$D \neq 0, \quad \text{for non-critical masses;} \quad (6.34)$$

$$D = 0, \quad \mathbf{P}_{em} = 0, \quad \text{for critical masses.} \quad (6.35)$$

General structure of solutions to V -vertices. We find that, for the non-critical masses (6.34), general structure of β -analytical solutions to vertices V, V', V_a , and their barred cousins takes the form

For non-critical masses:

$$V = CV_\kappa, \quad \bar{V} = CV_{-\kappa}, \quad \text{for type-1 } \Phi\Phi\Phi \text{ and } \Psi\Psi\Phi \text{ vertices;} \quad (6.36)$$

$$V' = C'V_\kappa, \quad \bar{V}' = C'V_{-\kappa}, \quad \text{for type-3 } \Phi\Phi\Phi \text{ and } \Psi\Psi\Phi \text{ vertices;} \quad (6.37)$$

$$V_a = C^{(1)}V_{a,\kappa}^{(1)} + C^{(2)}V_{a,\kappa}^{(2)}, \quad \bar{V}_a = C^{(1)}V_{a,-\kappa}^{(1)} + C^{(2)}V_{a,-\kappa}^{(2)}, \quad a = 1, 2, 3, \quad (6.38)$$

for type-2 $\Phi\Phi\Psi$ and $\Psi\Psi\Psi$ vertices;

where $C, C', C^{(1)}, C^{(2)}$ are coupling constants, while $V_\kappa, V_{a,\kappa}^{(1),(2)}$ are some fixed functions of $\beta_1, \beta_2, \beta_3$. Explicit expressions for these functions are given in sections 7.1–7.3. For the hermitian Hamiltonian $P_{[3]}^-$, the coupling constants entering vertices in (6.36)–(6.38) should satisfy the following rules:

$$C^* = I_u C', \quad (6.39)$$

$$C^{(1)*} = I_u C^{(2)}, \quad (6.40)$$

where I_u is a sign-inversion operator of the $U(1)$ charges (see (A.1)), while the asterisk stands for the complex conjugation.⁹ Let us make comments on the solution in (6.36)–(6.38).

⁹The coupling constants depend, among other things, on the $U(1)$ charges. Showing explicit dependence of the coupling constants on the $U(1)$ charges, $C = C_{u_1, u_2, u_3}, C' = C'_{u_1, u_2, u_3}$ we can represent, for example the relation in (6.39), as $C_{u_1, u_2, u_3}^* = C'_{-u_1, -u_2, -u_3}$.

- i) From (6.9), we see that the vertex V satisfies the two first-order differential equations with respect to β -momenta $\beta_1, \beta_2, \beta_3$. Note however that, in view of the conservation law $\beta_1 + \beta_2 + \beta_3 = 0$, the vertex V depends only on two β -momenta. This implies that, up to one coupling constant C the vertex V is determined uniquely. The same holds true for the vertex $\bar{V}$.
- ii) Comparing (6.13), (6.14) and (6.9), (6.10), we see that the vertices V' and $\bar{V}'$ satisfy the same equations as the respective vertices V and $\bar{V}$. For this reason, by module of the coupling constants, the solution for the vertices V' and $\bar{V}'$ (6.37) takes the same form as the one for the respective vertices V and $\bar{V}$ (6.36).
- iii) From (6.16), (6.17), we see that the three vertices V_a satisfy the six first-order differential equations with respect to β -momenta $\beta_1, \beta_2, \beta_3$ and one algebraic constraint. Note however that, in view of the conservation law $\beta_1 + \beta_2 + \beta_3 = 0$, the vertices V_a depend only on two β -momenta. This implies that, the three vertices V_a are determined uniquely up to two coupling constants denoted as $C^{(1)}$ and $C^{(2)}$ in (6.38). The same holds true for the vertices $\bar{V}_a$.
- iv) Equations for $\bar{V}$ (6.10) are obtained from the ones for V (6.9) by the replacement $\kappa \rightarrow -\kappa$. Therefore, if the solution for the vertex V is described by the function V_κ , then the solution for the vertex $\bar{V}$ is given by the $V_{-\kappa}$. The same holds true also for the vertices $V', \bar{V}'$ and $V_a, \bar{V}_a$.
- v) From (6.36), we see that the vertices V and $\bar{V}$ enter in the game with the one and the same coupling constant C . It is the requirement of the β analyticity that leads to the one and same coupling constant for the vertices V and $\bar{V}$. Namely, ignoring the β -analyticity, we find that the general solution for the vertex $\bar{V}$ takes the form $\bar{V} = \bar{C}V_{-\kappa}$. If $\bar{C} \neq C$, then the vertex $p_{[3]}^-$ is β -non-analytic, while, if $\bar{C} = C$, then the vertex $p_{[3]}^-$ becomes even function of κ , $p_{[3]}^-(\kappa) = p_{[3]}^-(-\kappa)$. Explicit expressions for the V_κ given in the next sections tell us that, in general, the V_κ is a rational function of κ . This implies that, for $\bar{C} = C$, the vertex $p_{[3]}^-$ depends on κ^2 and hence is β -analytic. The same holds true also for the vertices $V', \bar{V}'$ and $V_a, \bar{V}_a$ in (6.37), (6.38).

We now discuss vertices for the critical masses. We find that, for the critical masses (6.35), general structure of β -analytical solution to the vertices V, V', V_a , and their barred cousins takes the form

For critical masses:

$$V = CV_\epsilon, \quad \bar{V} = \bar{C}V_{-\epsilon}, \quad \text{for type-1 } \Phi\Phi\Phi \text{ and } \Psi\Psi\Phi \text{ vertices;} \quad (6.41)$$

$$V' = C'V_\epsilon, \quad \bar{V}' = \bar{C}'V_{-\epsilon}, \quad \text{for type-3 } \Phi\Phi\Phi \text{ and } \Psi\Psi\Phi \text{ vertices;} \quad (6.42)$$

$$V_a = C^{(1)}V_{a,\epsilon}^{(1)} + C^{(2)}V_{a,\epsilon}^{(2)}, \quad \bar{V}_a = \bar{C}^{(1)}V_{a,-\epsilon}^{(1)} + \bar{C}^{(2)}V_{a,-\epsilon}^{(2)}, \quad \text{for type-2 } \Phi\Phi\Psi \text{ and } \Psi\Psi\Psi \text{ vertices;} \quad (6.43)$$

where quantities $C, C', C^{(1)}, C^{(2)}$ and their barred cousins $\bar{C}, \bar{C}', \bar{C}^{(1)}, \bar{C}^{(2)}$ are coupling constants, while $V_\epsilon, V_{a,\epsilon}^{(1),(2)}$ are some fixed functions of $\beta_1, \beta_2, \beta_3$. Explicit expressions

for these functions are given in sections 8.1 and 8.2. For the hermitian Hamiltonian $P_{[3]}^-$, the coupling constants entering vertices in (6.41)–(6.43), unless otherwise specified, should satisfy the following rules:

$$C^* = I_u C', \quad \bar{C}^* = I_u \bar{C}', \quad (6.44)$$

$$C^{(1)*} = I_u C^{(2)}, \quad \bar{C}^{(1)*} = I_u \bar{C}^{(2)}. \quad (6.45)$$

Let us make comments on the expressions in (6.41)–(6.43).

i) For the critical masses, in view of the relation

$$-\beta \sum_{a=1,2,3} \frac{m_a^2}{\beta_a} = \mathbb{P}_{\epsilon m}, \quad \mathbb{P}_{\epsilon m} \equiv \frac{1}{3} \sum_{a=1,2,3} \check{\beta}_a \epsilon_a m_a, \quad \check{\beta}_a \equiv \beta_{a+1} - \beta_{a+2}, \quad (6.46)$$

the κ defined in (6.8) can be chosen as

$$\kappa = \mathbb{P}_{\epsilon m} \quad \text{for critical masses.} \quad (6.47)$$

Taking into account (6.46), (6.47), we see that equations for $\bar{V}$ (6.10) are obtained from the ones for V (6.9) by using the replacement $\epsilon_1, \epsilon_2, \epsilon_3 \rightarrow -\epsilon_1, -\epsilon_2, -\epsilon_3$. Therefore, if a solution for the vertex V is described by the function V_ϵ , then a solution for the vertex $\bar{V}$ is given by the $V_{-\epsilon}$. The same holds true also for the vertices V' , $\bar{V}'$ and V_a , $\bar{V}_a$.

ii) The κ given in (6.47) is β -analytic. Solutions for the V -vertices and their barred cousins turn also to be β -analytic. Therefore, in contrast to the relations in (6.36)–(6.38), there is no need to tune the coupling constants C , C' , $C^{(1)}$, $C^{(2)}$ and their barred cousins in (6.41)–(6.43).

As we said, for hermitian Hamiltonian $P_{[3]}^-$, the type-1 and type-3 vertices are related by hermitian conjugation rules. These rules are given explicitly by the relations (B.11), (B.12) in the appendix B. For the type-2 vertices, hermicity of the Hamiltonian leads to the rules given in (B.13)–(B.15). Hermitian conjugation rules in (B.11)–(B.15) are given in terms of the vertices. In terms of the coupling constants these rules are given in (6.39), (6.40) and (6.44), (6.45).

U(1) symmetry restrictions for type- k vertices. For the type- k vertices, by using the U(1) symmetry restriction (5.9), we can obtain constraints on the labels u_1 , u_2 , u_3 of the cubic vertex $p_{[3]}^- = p_{\sigma_1 \sigma_2 \sigma_3}^-$, where σ_a is defined in (4.14). Namely, using (5.9), we find that the u_1 , u_2 , u_3 of the cubic vertex $p_{\sigma_1 \sigma_2 \sigma_3}^-$ should satisfy the restrictions:

$$\mathbf{U} = 1 \quad \text{for type-1 vertices;} \quad \mathbf{U} = -1 \quad \text{for type-3 vertices;} \quad (6.48)$$

$$\mathbf{U} = 0 \quad \text{for type-2 vertices;} \quad (6.49)$$

$$\mathbf{U} \equiv u_1 + u_2 + u_3. \quad (6.50)$$

In other words, the type- k cubic vertices are non-trivial only for the values of $\mathbf{U}$ given in (6.48), (6.49). The vertex $p_{[3]}^- = p_{\sigma_1 \sigma_2 \sigma_3}^-$ and the product of three superfields $\Theta_{\sigma_1}^* \Theta_{\sigma_2}^* \Theta_{\sigma_3}^*$ entering the Hamiltonian (4.2) have the same labels σ_a , (4.10), (4.14). Therefore restrictions (6.48) are valid not only for the u -labels of cubic vertex but also for the u -labels of the product of three superfields $\Theta_{\sigma_1}^* \Theta_{\sigma_2}^* \Theta_{\sigma_3}^*$.

7 V_κ -vertices for superfields with non-critical masses

In this section, we deal with the non-critical masses defined by relations (6.34), where D is given in (6.30). Taking into account that some masses may be equal to zero, we will consider the following cases in turn:

$$m_1 \neq 0, \quad m_2 \neq 0, \quad m_3 \neq 0, \quad D \neq 0; \quad (7.1)$$

$$m_1 \neq 0, \quad m_2 \neq 0, \quad m_3 = 0, \quad |m_1| \neq |m_2|, \quad (7.2)$$

$$m_1 = 0, \quad m_2 = 0, \quad m_3 \neq 0. \quad (7.3)$$

Note that, for $m_3 = 0$, the inequality $D \neq 0$ amounts to the inequality $|m_1| \neq |m_2|$ in (7.2), while, for $m_1 = 0, m_2 = 0$, the inequality $D \neq 0$ amounts to the inequality $m_3 \neq 0$ in (7.3).

General expressions for the vertices we deal in this section are given in (6.36)–(6.38). From these expressions, we see that all that is required is to provide explicit expressions for the functions V_κ and $V_{a,\kappa}^{(1),(2)}$. Such expressions for the V_κ and $V_{a,\kappa}^{(1),(2)}$ are given below in sections 7.1–7.3.

7.1 V_κ -vertices for three massive superfields

$\Phi\Phi\Phi$ vertex. Three arbitrary spin massive superfields Φ . We consider a cubic vertex for the following three massive superfields:

$$\begin{aligned} \Phi_{m_1, s_1, u_1} - \Phi_{m_2, s_2, u_2} - \Phi_{m_3, s_3, u_3}, \quad s_1, s_2, s_3 \in \mathbb{N}_0, \\ m_1 \neq 0, \quad m_2 \neq 0, \quad m_3 \neq 0, \quad D \neq 0. \end{aligned} \quad (7.4)$$

Solution for the V_κ entering the vertices V and V' (6.36), (6.37) takes the form

$$V_\kappa = L_{\kappa,1}^{s_1} L_{\kappa,2}^{s_2} L_{\kappa,3}^{s_3}, \quad (7.5)$$

$$L_{\kappa,a} \equiv \frac{\kappa}{\beta_a} + \frac{\check{\beta}_a}{2\beta_a} m_a + \frac{m_{a+1}^2 - m_{a+2}^2}{2m_a}. \quad (7.6)$$

We note that the helpful relations for $L_{\kappa,a}$ (7.6) may be found in appendix A in ref. [12].

$\Psi\Psi\Phi$ vertex. Two arbitrary spin massive superfields Ψ and one arbitrary spin massive superfield Φ . We consider a cubic vertex for the following three massive superfields:

$$\begin{aligned} \Psi_{m_1, s_1 + \frac{1}{2}, u_1} - \Psi_{m_2, s_2 + \frac{1}{2}, u_2} - \Phi_{m_3, s_3, u_3}, \quad s_1, s_2, s_3 \in \mathbb{N}_0, \\ m_1 \neq 0, \quad m_2 \neq 0, \quad m_3 \neq 0, \quad D \neq 0. \end{aligned} \quad (7.7)$$

Solution for the V_κ entering the vertices V and V' (6.36), (6.37) takes the form

$$V_\kappa = K_\kappa L_{\kappa,1}^{s_1} L_{\kappa,2}^{s_2} L_{\kappa,3}^{s_3}, \quad K_\kappa \equiv \frac{1}{\beta_1 \beta_2} (\kappa + m_1 \beta_2 - m_2 \beta_1), \quad (7.8)$$

where $L_{\kappa,a}$ are given in (7.6).

$\Phi\Phi\Psi$ vertex. Two arbitrary spin massive superfields Φ and one arbitrary spin massive superfield Ψ . We consider a cubic vertex for the following three massive superfields:

$$\begin{aligned} \Phi_{m_1, s_1, u_1} - \Phi_{m_2, s_2, u_2} - \Psi_{m_3, s_3 + \frac{1}{2}, u_3}, \quad s_1, s_2, s_3 \in \mathbb{N}_0, \\ m_1 \neq 0, \quad m_2 \neq 0, \quad m_3 \neq 0, \quad D \neq 0. \end{aligned} \quad (7.9)$$

Solution for the functions $V_{a, \kappa}^{(1), (2)}$ entering the vertices V_a (6.38) takes the form

$$V_{a, \kappa}^{(1), (2)} = \mathcal{V}_{a, \kappa}^{(1), (2)} L_{\kappa, 1}^{s_1} L_{\kappa, 2}^{s_2} L_{\kappa, 3}^{s_3}, \quad a = 1, 2, 3, \quad (7.10)$$

where $L_{\kappa, a}$ are given in (7.6) and we use the notation

$$\mathcal{V}_{1, \kappa}^{(1)} = \frac{1}{\kappa} \left(L_{\kappa, 3} - \frac{m_1^2 + m_2^2 - m_3^2}{2m_3} \right), \quad (7.11)$$

$$\mathcal{V}_{2, \kappa}^{(1)} = \frac{m_1 m_2}{\kappa m_3}, \quad (7.12)$$

$$\mathcal{V}_{3, \kappa}^{(1)} = \frac{m_1}{\kappa m_3} \left(-L_{\kappa, 3} + \frac{m_1^2 - m_2^2 - m_3^2}{2m_3} \right), \quad (7.13)$$

$$\mathcal{V}_{1, \kappa}^{(2)} = -\frac{m_1 m_2}{\kappa m_3}, \quad (7.14)$$

$$\mathcal{V}_{2, \kappa}^{(2)} = \frac{1}{\kappa} \left(L_{\kappa, 3} + \frac{m_1^2 + m_2^2 - m_3^2}{2m_3} \right), \quad (7.15)$$

$$\mathcal{V}_{3, \kappa}^{(2)} = \frac{m_2}{\kappa m_3} \left(-L_{\kappa, 3} + \frac{m_1^2 - m_2^2 + m_3^2}{2m_3} \right). \quad (7.16)$$

$\Psi\Psi\Psi$ vertex. Three arbitrary spin massive superfields Ψ . We consider a cubic vertex for the following three massive superfields:

$$\begin{aligned} \Psi_{m_1, s_1 + \frac{1}{2}, u_1} - \Psi_{m_2, s_2 + \frac{1}{2}, u_2} - \Psi_{m_3, s_3 + \frac{1}{2}, u_3}, \quad s_1, s_2, s_3 \in \mathbb{N}_0, \\ m_1 \neq 0, \quad m_2 \neq 0, \quad m_3 \neq 0, \quad D \neq 0. \end{aligned} \quad (7.17)$$

Solution for the $V_{\kappa, a}^{(1), (2)}$ entering the vertices V_a (6.38) takes the form

$$V_{a, \kappa}^{(1), (2)} = K_\kappa \mathcal{V}_{a, \kappa}^{(1), (2)} L_{\kappa, 1}^{s_1} L_{\kappa, 2}^{s_2} L_{\kappa, 3}^{s_3}, \quad K_\kappa \equiv \frac{1}{\beta_1 \beta_2} (\kappa + m_1 \beta_2 - m_2 \beta_1), \quad (7.18)$$

where $L_{\kappa, a}$ are given in (7.6), while $\mathcal{V}_{a, \kappa}^{(1), (2)}$ take the same form as in (7.11)–(7.16)

7.2 V_κ -vertices for two massive superfields and one massless superfield

$\Phi\Phi\Phi$ vertex. Two arbitrary spin massive superfields Φ with masses $|m_1| \neq |m_2|$ and one massless superfield Φ . We consider a cubic vertex for the following three superfields:

$$\begin{aligned} \Phi_{m_1, s_1, u_1} - \Phi_{m_2, s_2, u_2} - \Phi_{0, 0, u_3}, \quad s_1, s_2 \in \mathbb{N}_0, \\ m_1 \neq 0, \quad m_2 \neq 0, \quad m_3 = 0, \quad |m_1| \neq |m_2|. \end{aligned} \quad (7.19)$$

Solution for the V_κ entering the vertices V and V' (6.36), (6.37) takes the form

$$V_\kappa = L_{\kappa,1}^{s_1} L_{\kappa,2}^{s_2}, \quad \kappa^2 = -\beta_2 \beta_3 m_1^2 - \beta_1 \beta_3 m_2^2, \quad (7.20)$$

$$L_{\kappa,1} \equiv \frac{\kappa}{\beta_1} + \frac{\check{\beta}_1}{2\beta_1} m_1 + \frac{m_2^2}{2m_1}, \quad L_{\kappa,2} \equiv \frac{\kappa}{\beta_2} + \frac{\check{\beta}_2}{2\beta_2} m_2 - \frac{m_1^2}{2m_2}. \quad (7.21)$$

$\Psi\Psi\Phi$ vertex. Two arbitrary spin massive superfields Ψ with masses $|m_1| \neq |m_2|$ and one massless superfield Φ . We consider a cubic vertex for the following three superfields:

$$\begin{aligned} & \Psi_{m_1, s_1 + \frac{1}{2}, u_3} - \Psi_{m_2, s_2 + \frac{1}{2}, u_2} - \Phi_{0,0,u_3}, \quad s_1, s_2 \in \mathbb{N}_0, \\ & m_1 \neq 0, \quad m_2 \neq 0, \quad m_3 = 0, \quad |m_1| \neq |m_2|. \end{aligned} \quad (7.22)$$

Solution for the V_κ entering the vertices V and V' (6.36), (6.37) takes the form

$$V_\kappa = K_\kappa L_{\kappa,1}^{s_1} L_{\kappa,2}^{s_2}, \quad K_\kappa \equiv \frac{1}{\beta_1 \beta_2} (\kappa + m_1 \beta_2 - m_2 \beta_1), \quad (7.23)$$

where $L_{\kappa,1}$, $L_{\kappa,2}$, and κ take the same form as in (7.20), (7.21).

$\Psi\Psi\Phi$ -vertex. One massless superfield Ψ , one arbitrary spin massive superfield Ψ and one arbitrary spin massive superfield Φ with masses $|m_2| \neq |m_3|$. We consider a cubic vertex for the following three superfields:

$$\begin{aligned} & \Psi_{0, \frac{1}{2}, u_1} - \Psi_{m_2, s_2 + \frac{1}{2}, u_2} - \Phi_{m_3, s_3, u_3}, \quad s_2, s_3 \in \mathbb{N}_0, \\ & m_1 = 0, \quad m_2 \neq 0, \quad m_3 \neq 0, \quad |m_2| \neq |m_3|. \end{aligned} \quad (7.24)$$

Solution for the V_κ entering the vertices V and V' (6.36), (6.37) takes the form

$$V_\kappa = K_\kappa L_{\kappa,2}^{s_2} L_{\kappa,3}^{s_3}, \quad K_\kappa \equiv \frac{1}{\beta_1 \beta_2} (\kappa - m_2 \beta_1), \quad (7.25)$$

$$L_{\kappa,2} \equiv \frac{\kappa}{\beta_2} + \frac{\check{\beta}_2}{2\beta_2} m_2 + \frac{m_3^2}{2m_2}, \quad L_{\kappa,3} \equiv \frac{\kappa}{\beta_3} + \frac{\check{\beta}_3}{2\beta_3} m_3 - \frac{m_2^2}{2m_3}, \quad (7.26)$$

$$\kappa^2 = -\beta_1 \beta_3 m_2^2 - \beta_1 \beta_2 m_3^2. \quad (7.27)$$

$\Phi\Phi\Psi$ vertex. Two arbitrary spin massive superfields Φ with masses $|m_1| \neq |m_2|$ and one massless superfield Ψ . We consider a cubic vertex for the following three superfields:

$$\begin{aligned} & \Phi_{m_1, s_1, u_1} - \Phi_{m_2, s_2, u_2} - \Psi_{0, \frac{1}{2}, u_3}, \quad s_1, s_2 \in \mathbb{N}_0, \\ & m_1 \neq 0, \quad m_2 \neq 0, \quad m_3 = 0, \quad |m_1| \neq |m_2|. \end{aligned} \quad (7.28)$$

Solution for the $V_{a,\kappa}^{(1),(2)}$ entering the vertices V_a in (6.38) takes the form

$$\begin{aligned} V_{1,\kappa}^{(1)} &= \frac{m_2}{\kappa} B_\kappa, & V_{2,\kappa}^{(1)} &= -\frac{m_1}{\kappa} B_\kappa, & V_{3,\kappa}^{(1)} &= 0, \\ V_{1,\kappa}^{(2)} &= 0, & V_{2,\kappa}^{(2)} &= 0, & V_{3,\kappa}^{(2)} &= \frac{1}{\beta_3} B_\kappa, & B_\kappa &\equiv L_{\kappa,1}^{s_1} L_{\kappa,2}^{s_2}; \end{aligned} \quad (7.29)$$

where $L_{\kappa,1}$, $L_{\kappa,2}$, and κ take the same form as in (7.20), (7.21).

$\Phi\Phi\Psi$ vertex. One massless superfield Φ , one arbitrary spin massive superfield Φ and one arbitrary spin massive superfield Ψ with masses $|m_2| \neq |m_3|$. We consider a cubic vertex for the following three superfields:

$$\begin{aligned} \Phi_{0,0,u_1} - \Phi_{m_2,s_2,u_2} - \Psi_{m_3,s_3+\frac{1}{2},u_3}, \quad s_2, s_3 \in \mathbb{N}_0, \\ m_1 = 0, \quad m_2 \neq 0, \quad m_3 \neq 0, \quad |m_2| \neq |m_3|. \end{aligned} \quad (7.30)$$

Solution for the $V_{a,\kappa}^{(1),(2)}$ entering the vertices V_a in (6.38) takes the form

$$\begin{aligned} V_{1,\kappa}^{(1)} &= \frac{1}{\beta_1\beta_3} F_\kappa, & V_{2,\kappa}^{(1)} &= 0, & V_{3,\kappa}^{(1)} &= 0; \\ V_{1,\kappa}^{(2)} &= 0, & V_{2,\kappa}^{(2)} &= \frac{m_3}{\beta_3\kappa} F_\kappa, & V_{3,\kappa}^{(2)} &= -\frac{m_2}{\beta_3\kappa} F_\kappa, \\ F_\kappa &\equiv (\kappa + m_3\beta_1)L_{\kappa,2}^{s_2}L_{\kappa,3}^{s_3}, \end{aligned} \quad (7.31)$$

where $L_{\kappa,2}$, $L_{\kappa,3}$, and κ take the same form as in (7.26), (7.27).

$\Psi\Psi\Psi$ vertex. Two arbitrary spin massive superfields Ψ with masses $|m_1| \neq |m_2|$ and one massless superfield Ψ . We consider a cubic vertex for the following three superfields:

$$\begin{aligned} \Psi_{m_1,s_1+\frac{1}{2},u_1} - \Psi_{m_2,s_2+\frac{1}{2},u_2} - \Psi_{0,\frac{1}{2},u_3}, \quad s_1, s_2 \in \mathbb{N}_0, \\ m_1 \neq 0, \quad m_2 \neq 0, \quad m_3 = 0, \quad |m_1| \neq |m_2|. \end{aligned} \quad (7.32)$$

Solution for the $V_{a,\kappa}^{(1),(2)}$ entering vertices V_a in (6.38) takes the form

$$\begin{aligned} V_{1,\kappa}^{(1)} &= \frac{m_2}{\kappa\beta_1\beta_2} F_\kappa, & V_{2,\kappa}^{(1)} &= -\frac{m_1}{\kappa\beta_1\beta_2} F_\kappa, & V_{3,\kappa}^{(1)} &= 0, \\ V_{1,\kappa}^{(2)} &= 0, & V_{2,\kappa}^{(2)} &= 0, & V_{3,\kappa}^{(2)} &= \frac{1}{\beta_1\beta_2\beta_3} F_\kappa, \\ F_\kappa &\equiv (\kappa + m_1\beta_2 - m_2\beta_1)L_{\kappa,1}^{s_1}L_{\kappa,2}^{s_2}, \end{aligned} \quad (7.33)$$

where $L_{\kappa,1}$, $L_{\kappa,2}$, and κ take the same form as in (7.20), (7.21).

7.3 V_κ -vertices for two massless superfields and one massive superfield

$\Phi\Phi\Phi$ vertex. Two massless superfields Φ and one arbitrary spin massive superfield Φ . We consider a cubic vertex for the following three superfields:

$$\begin{aligned} \Phi_{0,0,u_1} - \Phi_{0,0,u_2} - \Phi_{m_3,s_3,u_3}, \quad s_3 \in \mathbb{N}_0, \\ m_1 = 0, \quad m_2 = 0, \quad m_3 \neq 0. \end{aligned} \quad (7.34)$$

Solution for the V_κ entering the vertices V and V' (6.36), (6.37) takes the form

$$V_\kappa = L_{\kappa,3}^{s_3}, \quad L_{\kappa,3} \equiv \frac{\kappa}{\beta_3} + \frac{\check{\beta}_3}{2\beta_3} m_3, \quad \kappa^2 = -\beta_1\beta_2 m_3^2. \quad (7.35)$$

$\Psi\Psi\Phi$ vertex. Two massless superfields Ψ and one arbitrary spin massive superfield Φ . We consider a cubic vertex for the following three superfields:

$$\begin{aligned} \Psi_{0,\frac{1}{2},u_1} - \Psi_{0,\frac{1}{2},u_2} - \Phi_{m_3,s_3,u_3}, \quad s_3 \in \mathbb{N}_0, \\ m_1 = 0, \quad m_2 = 0, \quad m_3 \neq 0. \end{aligned} \quad (7.36)$$

Solution for the V_κ entering the vertices V and V' (6.36), (6.37) takes the form

$$V_\kappa = \frac{\kappa}{\beta_1\beta_2} L_{\kappa,3}^{s_3}, \quad L_{\kappa,3} = \frac{\kappa}{\beta_3} + \frac{\check{\beta}_3}{2\beta_3} m_3, \quad \kappa^2 = -\beta_1\beta_2 m_3^2. \quad (7.37)$$

$\Psi\Psi\Phi$ vertex. One arbitrary spin massive superfield Ψ , one massless superfield Ψ and one massless superfield Φ . We consider a cubic vertex for the following three superfields:

$$\begin{aligned} \Psi_{m_1,s_1+\frac{1}{2},u_1} - \Psi_{0,\frac{1}{2},u_2} - \Phi_{0,0,u_3}, \quad s_1 \in \mathbb{N}_0, \\ m_1 \neq 0, \quad m_2 = 0, \quad m_3 = 0. \end{aligned} \quad (7.38)$$

Solution for the V_κ entering the vertices V and V' (6.36), (6.37) takes the form

$$V_\kappa = K_\kappa L_{\kappa,1}^{s_1}, \quad L_{\kappa,1} = \frac{\kappa}{\beta_1} + \frac{\check{\beta}_1}{2\beta_1} m_1, \quad K_\kappa = \frac{1}{\beta_1\beta_2} (\kappa + m_1\beta_2), \quad \kappa^2 = -\beta_2\beta_3 m_1^2. \quad (7.39)$$

$\Phi\Phi\Psi$ vertex. Two massless superfields Φ and one arbitrary spin massive superfield Ψ . We consider a cubic vertex for the following three superfields:

$$\begin{aligned} \Phi_{0,0,u_1} - \Phi_{0,0,u_2} - \Psi_{m_3,s_3+\frac{1}{2},u_3}, \quad s_3 \in \mathbb{N}_0, \\ m_1 = 0, \quad m_2 = 0, \quad m_3 \neq 0. \end{aligned} \quad (7.40)$$

Solution for the $V_{a,\kappa}^{(1),(2)}$ entering the vertices V_a (6.38) takes the form

$$\begin{aligned} V_{1,\kappa}^{(1)} &= \frac{1}{\beta_1\beta_3} (\kappa + m_3\beta_1) L_{\kappa,3}^{s_3}, & V_{2,\kappa}^{(1)} &= 0, & V_{3,\kappa}^{(1)} &= 0; \\ V_{1,\kappa}^{(2)} &= 0, & V_{2,\kappa}^{(2)} &= \frac{m_3}{\beta_3\kappa} (\kappa + m_3\beta_1) L_{\kappa,3}^{s_3}, & V_{3,\kappa}^{(2)} &= 0; \end{aligned} \quad (7.41)$$

where $L_{\kappa,3}$ and κ take the same form as in (7.35).

$\Phi\Phi\Psi$ vertex. One massless superfield Φ , one arbitrary spin massive superfield Φ and one massless superfield Ψ . We consider a cubic vertex for the following three superfields:

$$\begin{aligned} \Phi_{0,0,u_1} - \Phi_{m_2,s_2,u_2} - \Psi_{0,\frac{1}{2},u_3}, \quad s_2 \in \mathbb{N}_0, \\ m_1 = 0, \quad m_2 \neq 0, \quad m_3 = 0. \end{aligned} \quad (7.42)$$

Solution for the $V_{a,\kappa}^{(1),(2)}$ entering the vertices V_a (6.38) takes the form

$$\begin{aligned} V_{1,\kappa}^{(1)} &= \frac{m_2}{\kappa} L_{\kappa,2}^{s_2}, & V_{2,\kappa}^{(1)} &= 0, & V_{3,\kappa}^{(1)} &= 0; \\ V_{1,\kappa}^{(2)} &= 0, & V_{2,\kappa}^{(2)} &= 0, & V_{3,\kappa}^{(2)} &= \frac{1}{\beta_3} L_{\kappa,2}^{s_2}; \\ L_{\kappa,2} &\equiv \frac{\kappa}{\beta_2} + \frac{\check{\beta}_2}{2\beta_2} m_2, & \kappa^2 &= -\beta_3\beta_1 m_2^2. \end{aligned} \quad (7.43)$$

$\Psi\Psi\Psi$ vertex. Two massless superfields Ψ and one arbitrary spin massive superfield Ψ . We consider a cubic vertex for the following three superfields:

$$\begin{aligned} \Psi_{0,\frac{1}{2},u_1} - \Psi_{0,\frac{1}{2},u_2} - \Psi_{m_3,s_3+\frac{1}{2},u_3}, \quad s_3 \in \mathbb{N}_0, \\ m_1 = 0, \quad m_2 = 0, \quad m_3 \neq 0. \end{aligned} \quad (7.44)$$

Solution for the $V_{a,\kappa}^{(1),(2)}$ entering the vertices V_a (6.38) takes the form

$$\begin{aligned} V_{1,\kappa}^{(1)} &= \frac{m_3}{\kappa\beta_1\beta_3}(\kappa + m_3\beta_1)L_{\kappa,3}^{s_3}, \quad V_{2,\kappa}^{(1)} = 0, \quad V_{3,\kappa}^{(1)} = 0, \\ V_{1,\kappa}^{(2)} &= 0, \quad V_{2,\kappa}^{(2)} = -\frac{1}{\beta_1\beta_2\beta_3}(\kappa + m_3\beta_1)L_{\kappa,3}^{s_3}, \quad V_{3,\kappa}^{(2)} = 0, \end{aligned} \quad (7.45)$$

where $L_{\kappa,3}$ and κ take the same form as in (7.35).

8 V_κ -vertices for superfields with critical masses

In this section, we deal with the critical masses defined by relations (6.35). Taking into account that some masses may be equal to zero, we will consider the following cases in turn:

$$m_1 \neq 0, \quad m_2 \neq 0, \quad m_3 \neq 0, \quad \mathbf{P}_{em} = 0; \quad (8.1)$$

$$m_1 \neq 0, \quad m_2 \neq 0, \quad m_3 = 0, \quad |m_1| = |m_2|, \quad (8.2)$$

where $\mathbf{P}_{em}$ is defined in (6.31), (6.32). Note that, for $m_3 = 0$, the relation $\mathbf{P}_{em} = 0$ amounts to the relation $|m_1| = |m_2|$ in (8.2).

General expressions for the vertices we deal in this section are given in (6.41)–(6.43). From these expressions, we see that all that is required is to provide explicit expressions for the functions V_ϵ and $V_{a,\epsilon}^{(1),(2)}$. Such expressions for the V_ϵ and $V_{a,\epsilon}^{(1),(2)}$ are given below in sections 8.1, 8.2. Throughout this section we use $\mathbf{P}_{em}$ and $\epsilon_1, \epsilon_2, \epsilon_3$ defined in (6.31), (6.32).

8.1 V_κ -vertices for three massive superfields

$\Phi\Phi\Phi$ vertex. Three arbitrary spin massive superfields Φ . We consider a cubic vertex for the following three massive superfields:

$$\begin{aligned} \Phi_{m_1,s_1,u_1} - \Phi_{m_2,s_2,u_2} - \Phi_{m_3,s_3,u_3}, \quad s_1, s_2, s_3 \in \mathbb{N}_0, \\ m_1 \neq 0, \quad m_2 \neq 0, \quad m_3 \neq 0; \quad \mathbf{P}_{em} = 0. \end{aligned} \quad (8.3)$$

The V_ϵ entering the vertices V and V' (6.41), (6.42) is given by

$$V_\epsilon = \mathbb{P}_{em}^{\mathbf{S}_\epsilon} \prod_{a=1,2,3} \beta_a^{-\epsilon_a s_a}, \quad (8.4)$$

$$\mathbb{P}_{em} \equiv \frac{1}{3} \sum_{a=1,2,3} \check{\beta}_a \epsilon_a m_a, \quad \mathbf{S}_\epsilon \equiv \sum_{a=1,2,3} \epsilon_a s_a, \quad (8.5)$$

where $\check{\beta}_a$ is defined in (5.3). We recall that the $V_{-\epsilon}$ entering the vertices $\bar{V}, \bar{V}'$ (6.41), (6.42) is obtained from the V_ϵ (8.4) by using the replacement $\epsilon_1, \epsilon_2, \epsilon_3 \rightarrow -\epsilon_1, -\epsilon_2, -\epsilon_3$.

$\Psi\Psi\Phi$ vertex. Two arbitrary spin massive superfields Ψ and one arbitrary spin massive superfield Φ . We consider a cubic vertex for the following three massive superfields:

$$\begin{aligned} &\Psi_{m_1, s_1 + \frac{1}{2}, u_1} - \Psi_{m_2, s_2 + \frac{1}{2}, u_2} - \Phi_{m_3, s_3, u_3}, \quad s_1, s_2, s_3 \in \mathbb{N}_0, \\ &m_1 \neq 0, \quad m_2 \neq 0, \quad m_3 \neq 0; \quad \mathbf{P}_{em} = 0. \end{aligned} \quad (8.6)$$

Solution for the V_ϵ entering the vertices V and V' (6.41), (6.42) takes the form

$$V_\epsilon = K_\epsilon \mathbb{P}_{em}^{\mathbf{S}_\epsilon} \prod_{a=1,2,3} \beta_a^{-\epsilon_a s_a}, \quad K_\epsilon \equiv \mathbb{P}_{em}^{\frac{1}{2}(\epsilon_1 + \epsilon_2)} \beta_1^{-\frac{1}{2}(1 + \epsilon_1)} \beta_2^{-\frac{1}{2}(1 + \epsilon_2)}, \quad (8.7)$$

where we use the notation as in (8.5).

$\Phi\Phi\Psi$ vertex. Two arbitrary spin massive superfields Φ and one arbitrary spin massive superfield Ψ . We consider a cubic vertex for the following three massive superfields:

$$\begin{aligned} &\Phi_{m_1, s_1, u_1} - \Phi_{m_2, s_2, u_2} - \Psi_{m_3, s_3 + \frac{1}{2}, u_3}, \quad s_1, s_2, s_3 \in \mathbb{N}_0, \\ &m_1 \neq 0, \quad m_2 \neq 0, \quad m_3 \neq 0; \quad \mathbf{P}_{em} = 0. \end{aligned} \quad (8.8)$$

Solution for the $V_{a,\epsilon}^{(1),(2)}$ entering the vertices V_a (6.43) takes the form

$$V_{a,\epsilon}^{(1),(2)} = \mathcal{V}_{a,\epsilon}^{(1),(2)} \mathbb{P}_{em}^{\mathbf{S}_\epsilon} \prod_{a=1,2,3} \beta_a^{-\epsilon_a s_a}, \quad (8.9)$$

where we use the notation as in (8.5), while $\mathcal{V}_{a,\epsilon}^{(1),(2)}$ are defined as

$$\mathcal{V}_{1,\epsilon}^{(1)} = \frac{1}{\mathbb{P}_{em}} \left(\epsilon_1 \frac{\mathbb{P}_{em}^{\epsilon_3}}{\beta_3^{\epsilon_3}} - \frac{\epsilon_2 m_3}{m_1 m_2} \pi_{3-} \right), \quad (8.10)$$

$$\mathcal{V}_{2,\epsilon}^{(1)} = \frac{1}{\mathbb{P}_{em}} \left(\epsilon_2 \frac{\mathbb{P}_{em}^{\epsilon_3}}{\beta_3^{\epsilon_3}} + \frac{\epsilon_1 m_3}{m_1 m_2} \pi_{3-} \right), \quad (8.11)$$

$$\mathcal{V}_{3,\epsilon}^{(1)} = \frac{1}{\mathbb{P}_{em}} \left(\epsilon_3 \frac{\mathbb{P}_{em}^{\epsilon_3}}{\beta_3^{\epsilon_3}} + \frac{\epsilon_2 m_1 - \epsilon_1 m_2}{m_1 m_2} \pi_{3-} \right), \quad (8.12)$$

$$\mathcal{V}_{1,\epsilon}^{(2)} = \frac{1}{\mathbb{P}_{em}} \left(\epsilon_1 + \frac{\epsilon_2 m_3}{m_1 m_2} \frac{\mathbb{P}_{em}}{\beta_3} \pi_{3+} \right), \quad (8.13)$$

$$\mathcal{V}_{2,\epsilon}^{(2)} = \frac{1}{\mathbb{P}_{em}} \left(\epsilon_2 - \frac{\epsilon_1 m_3}{m_1 m_2} \frac{\mathbb{P}_{em}}{\beta_3} \pi_{3+} \right), \quad (8.14)$$

$$\mathcal{V}_{3,\epsilon}^{(2)} = \frac{1}{\mathbb{P}_{em}} \left(\epsilon_3 - \frac{\epsilon_2 m_1 - \epsilon_1 m_2}{m_1 m_2} \frac{\mathbb{P}_{em}}{\beta_3} \pi_{3+} \right), \quad (8.15)$$

$$\pi_{3\pm} \equiv \frac{1 \pm \epsilon_3}{2}. \quad (8.16)$$

Requiring hermicity of the Hamiltonian, we get the restrictions on the coupling constants,

$$\begin{aligned} C^{(1)*} &= \epsilon_1 \epsilon_2 I_u C^{(1)}, \quad C^{(2)*} = -\epsilon_1 \epsilon_2 I_u C^{(2)}, \\ \bar{C}^{(1)*} &= \epsilon_1 \epsilon_2 I_u \bar{C}^{(1)}, \quad \bar{C}^{(2)*} = -\epsilon_1 \epsilon_2 I_u \bar{C}^{(2)}. \end{aligned} \quad (8.17)$$

$\Psi\Psi\Psi$ vertex. Three arbitrary spin massive superfields Ψ . We consider a cubic vertex for the following three massive superfields:

$$\begin{aligned} & \Psi_{m_1, s_1 + \frac{1}{2}, u_1} - \Psi_{m_2, s_2 + \frac{1}{2}, u_2} - \Psi_{m_3, s_3 + \frac{1}{2}, u_3}, \quad s_1, s_2, s_3 \in \mathbb{N}_0, \\ & m_1 \neq 0, \quad m_2 \neq 0, \quad m_3 \neq 0; \quad \mathbf{P}_{em} = 0. \end{aligned} \quad (8.18)$$

Solution for the $V_{a,\epsilon}^{(1),(2)}$ entering the vertices V_a (6.43) takes the form

$$V_{a,\epsilon}^{(1),(2)} = K_\epsilon \mathcal{V}_{a,\epsilon}^{(1),(2)} \mathbb{P}_{em}^{\mathbf{S}_\epsilon} \prod_{a=1,2,3} \beta_a^{-\epsilon_a s_a}, \quad K_\epsilon = \beta_1^{-\frac{1}{2}(1+\epsilon_1)} \beta_2^{-\frac{1}{2}(1+\epsilon_2)} \mathbb{P}_{em}^{\frac{1}{2}(\epsilon_1+\epsilon_2)}, \quad (8.19)$$

where $\mathbb{P}_{em}$, $\mathbf{S}_\epsilon$ are defined as in (8.5), while $\mathcal{V}_{a,\epsilon}^{(1),(2)}$ are defined as in (8.10)–(8.16). To ensure hermicity of the Hamiltonian the coupling constants should satisfy the relations (8.17).

8.2 V_κ -vertices for two massive superfields and one massless superfield

$\Phi\Phi\Phi$ vertex. Two arbitrary spin massive superfields Φ and one massless superfield Φ . We consider a cubic vertex for the following three superfields:

$$\begin{aligned} & \Phi_{m_1, s_1, u_1} - \Phi_{m_2, s_2, u_2} - \Phi_{0,0,u_3}, \quad s_1, s_2 \in \mathbb{N}_0, \\ & m_1 \neq 0, \quad m_2 \neq 0, \quad m_3 = 0, \quad \epsilon_1 m_1 + \epsilon_2 m_2 = 0, \quad \epsilon_1^2 = 1, \quad \epsilon_2^2 = 1. \end{aligned} \quad (8.20)$$

Solution for the V_ϵ entering the vertices V and V' (6.41), (6.42) takes the form

$$V_\epsilon = \mathbb{P}_{em}^{\epsilon_1 s_1 + \epsilon_2 s_2} \beta_1^{-\epsilon_1 s_1} \beta_2^{-\epsilon_2 s_2}, \quad \mathbb{P}_{em} = \epsilon_2 m_2 \beta_3. \quad (8.21)$$

$\Psi\Psi\Phi$ vertex. Two arbitrary spin massive superfields Ψ and one massless superfield Φ . We consider a cubic vertex for the following three superfields:

$$\begin{aligned} & \Psi_{m_1, s_1 + \frac{1}{2}, u_1} - \Psi_{m_2, s_2 + \frac{1}{2}, u_2} - \Phi_{0,0,u_3}, \quad s_1, s_2 \in \mathbb{N}_0, \\ & m_1 \neq 0, \quad m_2 \neq 0, \quad m_3 = 0, \quad \epsilon_1 m_1 + \epsilon_2 m_2 = 0, \quad \epsilon_1^2 = 1, \quad \epsilon_2^2 = 1. \end{aligned} \quad (8.22)$$

Solution for the V_ϵ entering the vertices V and V' (6.41), (6.42) takes the form

$$\begin{aligned} V_\epsilon &= K_\epsilon \mathbb{P}_{em}^{\epsilon_1 s_1 + \epsilon_2 s_2} \beta_1^{-\epsilon_1 s_1} \beta_2^{-\epsilon_2 s_2}, \\ K_\epsilon &= \beta_1^{-\frac{1}{2}(1+\epsilon_1)} \beta_2^{-\frac{1}{2}(1+\epsilon_2)} \beta_3^{\frac{1}{2}(\epsilon_1+\epsilon_2)}, \quad \mathbb{P}_{em} = \epsilon_2 m_2 \beta_3. \end{aligned} \quad (8.23)$$

$\Psi\Psi\Phi$ vertex. One massless superfield Ψ , one arbitrary spin massive superfield Ψ and one arbitrary spin massive superfield Φ . We consider a cubic vertex for the following superfields:

$$\begin{aligned} & \Psi_{0, \frac{1}{2}, u_1} - \Psi_{m_2, s_2 + \frac{1}{2}, u_2} - \Phi_{m_3, s_3, u_3}, \quad s_1, s_2 \in \mathbb{N}_0, \\ & m_1 = 0, \quad m_2 \neq 0, \quad m_3 \neq 0, \quad \epsilon_2 m_2 + \epsilon_3 m_3 = 0, \quad \epsilon_2^2 = 1, \quad \epsilon_3^2 = 1. \end{aligned} \quad (8.24)$$

Solution for the V_ϵ entering the vertices V and V' (6.41), (6.42) takes the form

$$\begin{aligned} V_\epsilon &= K_\epsilon \mathbb{P}_{em}^{\epsilon_2 s_2 + \epsilon_3 s_3} \beta_2^{-\epsilon_2 s_2} \beta_3^{-\epsilon_3 s_3}, \\ K_\epsilon &= \beta_1^{-\frac{1}{2}(1-\epsilon_2)} \beta_2^{-\frac{1}{2}(1+\epsilon_2)}, \quad \mathbb{P}_{em} = \epsilon_3 m_3 \beta_1. \end{aligned} \quad (8.25)$$

$\Phi\Phi\Psi$ vertex. Two arbitrary spin massive superfields Φ and one massless superfield Ψ . We consider a cubic vertex for the following three superfields:

$$\begin{aligned} \Phi_{m_1, s_1, u_1} - \Phi_{m_2, s_2, u_2} - \Psi_{0, \frac{1}{2}, u_3}, \quad s_1, s_2 \in \mathbb{N}_0, \\ m_1 \neq 0, \quad m_2 \neq 0, \quad m_3 = 0, \quad \epsilon_1 m_1 + \epsilon_2 m_2 = 0, \quad \epsilon_1^2 = 1, \quad \epsilon_2^2 = 1. \end{aligned} \quad (8.26)$$

Solution for the $V_{a, \epsilon}^{(1), (2)}$ entering the vertices V_a (6.43) takes the form

$$\begin{aligned} V_{1, \epsilon}^{(1)} &= \epsilon_1 F_\epsilon, & V_{2, \epsilon}^{(1)} &= \epsilon_2 F_\epsilon, & V_{3, \epsilon}^{(1)} &= 0; \\ V_{1, \epsilon}^{(2)} &= 0, & V_{2, \epsilon}^{(2)} &= 0, & V_{3, \epsilon}^{(2)} &= \epsilon_1 \epsilon_2 F_\epsilon; \\ F_\epsilon &\equiv \frac{1}{\beta_3} \mathbb{P}_{\epsilon m}^{\epsilon_1 s_1 + \epsilon_2 s_2} \beta_1^{-\epsilon_1 s_1} \beta_2^{-\epsilon_2 s_2}, & \mathbb{P}_{\epsilon m} &= \epsilon_2 m_2 \beta_3. \end{aligned} \quad (8.27)$$

$\Phi\Phi\Psi$ vertex. One massless superfield Φ , one arbitrary spin massive superfield Φ and one arbitrary spin massive superfield Ψ . We consider a cubic vertex for the following three superfields:

$$\begin{aligned} \Phi_{0, 0, u_1} - \Phi_{m_2, s_2, u_2} - \Psi_{m_3, s_3 + \frac{1}{2}, u_3}, \quad s_2, s_3 \in \mathbb{N}_0, \\ m_1 = 0, \quad m_2 \neq 0, \quad m_3 \neq 0, \quad \epsilon_2 m_2 + \epsilon_3 m_3 = 0, \quad \epsilon_2^2 = 1, \quad \epsilon_3^2 = 1. \end{aligned} \quad (8.28)$$

Solution for the $V_{a, \epsilon}^{(1), (2)}$ entering the vertices V_a (6.43) takes the form

$$\begin{aligned} V_{1, \epsilon}^{(1)} &= \epsilon_2 \epsilon_3 F_\epsilon, & V_{2, \epsilon}^{(1)} &= 0, & V_{3, \epsilon}^{(1)} &= 0; \\ V_{1, \epsilon}^{(2)} &= 0, & V_{2, \epsilon}^{(2)} &= \epsilon_2 F_\epsilon, & V_{3, \epsilon}^{(2)} &= \epsilon_3 F_\epsilon; \\ F_\epsilon &\equiv \beta_1^{-\frac{1}{2}(1-\epsilon_3)} \beta_3^{-\frac{1}{2}(1+\epsilon_3)} \mathbb{P}_{\epsilon m}^{\epsilon_2 s_2 + \epsilon_3 s_3} \beta_2^{-\epsilon_2 s_2} \beta_3^{-\epsilon_3 s_3}, & \mathbb{P}_{\epsilon m} &= \epsilon_3 m_3 \beta_1. \end{aligned} \quad (8.29)$$

$\Psi\Psi\Psi$ vertex. Two arbitrary spin massive superfields Ψ and one massless superfield Ψ . We consider a cubic vertex for the following three superfields:

$$\begin{aligned} \Psi_{m_1, s_1 + \frac{1}{2}, u_1} - \Psi_{m_2, s_2 + \frac{1}{2}, u_2} - \Psi_{0, \frac{1}{2}, u_3}, \quad s_1, s_2 \in \mathbb{N}_0, \\ m_1 \neq 0, \quad m_2 \neq 0, \quad m_3 = 0, \quad \epsilon_1 m_1 + \epsilon_2 m_2 = 0, \quad \epsilon_1^2 = 1, \quad \epsilon_2^2 = 1. \end{aligned} \quad (8.30)$$

Solution for the $V_{a, \epsilon}^{(1), (2)}$ entering the vertices V_a (6.43) takes the form

$$\begin{aligned} V_{1, \epsilon}^{(1)} &= \epsilon_1 F_\epsilon, & V_{2, \epsilon}^{(1)} &= \epsilon_2 F_\epsilon, & V_{3, \kappa}^{(1)} &= 0; \\ V_{1, \kappa}^{(2)} &= 0, & V_{2, \kappa}^{(2)} &= 0, & V_{3, \epsilon}^{(2)} &= \epsilon_1 \epsilon_2 F_\epsilon; \\ F_\epsilon &\equiv \beta_1^{-\frac{1}{2}(1+\epsilon_1)} \beta_2^{-\frac{1}{2}(1+\epsilon_2)} \beta_3^{\frac{1}{2}(\epsilon_1 + \epsilon_2) - 1} \mathbb{P}_{\epsilon m}^{\epsilon_1 s_1 + \epsilon_2 s_2} \beta_1^{-\epsilon_1 s_1} \beta_2^{-\epsilon_2 s_2}, & \mathbb{P}_{\epsilon m} &= \epsilon_2 m_2 \beta_3. \end{aligned} \quad (8.31)$$

9 Superspace first-derivative cubic vertices for massless fields

We now present cubic vertices of three massless superfields, $m_1 = 0, m_2 = 0, m_3 = 0$. We find that, for the three massless superfields, equations (5.14)–(5.16) lead to four types of cubic vertices which we refer as type- k cubic vertices, where $k = 1, 2, 3, 4$. By definition,

the type- k cubic vertex is realized as homogeneous degree- k polynomial in the Grassmann momenta $\mathbb{P}_\theta$ and p_{η_a} .

Let us first present relations for the $p_{[3]}^-$ and $j_{[3]}^{-1}$ which are valid for all types of cubic vertices. Namely, all cubic vertices $p_{[3]}^-$ and the corresponding densities $j_{[3]}^{-1}$ can be presented as

$$p_{[3]}^- = i\mathbb{P}V_{\mathbb{P}}, \quad (9.1)$$

$$j_{[3]}^{-1} = 2i\mathbb{N}_\beta^{\eta E} V_{\mathbb{P}}, \quad (9.2)$$

where $V_{\mathbb{P}}$ is a vertex that depends only on the momenta $\beta_1, \beta_2, \beta_3$ and the Grassmann momenta $\mathbb{P}_\theta, p_{\eta_1}, p_{\eta_2}, p_{\eta_3}$, while the operator is given in $\mathbb{N}_\beta^{\eta E}$ is defined in (5.19). From (9.1), (9.2), we see that in order to get the cubic densities we should fix expressions for the $V_{\mathbb{P}}$ and the supercharge densities $q_{[3]}^{-R,L}$. The expressions for $V_{\mathbb{P}}$ and $q_{[3]}^{-R,L}$ we find are presented below in (9.3)–(9.11).¹⁰ Outline of the derivation of relations in (9.3)–(9.11) may be found at the end of appendix C.

Type-1 $\Phi\Phi\Phi$ and $\Psi\Psi\Phi$ vertices:

$$V_{\mathbb{P}} = \mathbb{P}_\theta X + \sum_{a=1,2,3} \frac{p_{\eta_a}}{\beta_a} X_a, \quad (9.3)$$

$$q_{[3]}^{-R} = -i\varepsilon\sqrt{2}\beta X, \quad q_{[3]}^{-L} = i\varepsilon\sqrt{2}\mathbb{P}_\theta \sum_{a=1,2,3} \frac{p_{\eta_a}}{\beta_a} X_a; \quad (9.4)$$

Type-3 $\Phi\Phi\Phi$ and $\Psi\Psi\Phi$ vertices:

$$V_{\mathbb{P}} = \frac{1}{\beta}\mathbb{P}_\theta \sum_{a=1,2,3} p_{\eta_{a+1}}p_{\eta_{a+2}}\bar{X}_a + p_{\eta_1}p_{\eta_2}p_{\eta_3}\bar{X}, \quad (9.5)$$

$$q_{[3]}^{-R} = -i\varepsilon\sqrt{2} \sum_{a=1,2,3} p_{\eta_{a+1}}p_{\eta_{a+2}}X_a, \quad q_{[3]}^{-L} = i\varepsilon\sqrt{2}\mathbb{P}_\theta \sum_{a=1,2,3} p_{\eta_1}p_{\eta_2}p_{\eta_3}\bar{X}; \quad (9.6)$$

Type-2 $\Phi\Phi\Psi$ and $\Psi\Psi\Psi$ vertices:

$$V_{\mathbb{P}} = \mathbb{P}_\theta \sum_{a=1,2,3} \frac{p_{\eta_a}}{\beta_a} Y_a + \sum_{a=1,2,3} p_{\eta_{a+1}}p_{\eta_{a+2}}\bar{Y}_a, \quad (9.7)$$

$$q_{[3]}^{-R} = -i\varepsilon\sqrt{2}\beta \sum_{a=1,2,3} \frac{p_{\eta_a}}{\beta_a} Y_a, \quad q_{[3]}^{-L} = i\varepsilon\sqrt{2}\mathbb{P}_\theta \sum_{a=1,2,3} p_{\eta_{a+1}}p_{\eta_{a+2}}\bar{Y}_a; \quad (9.8)$$

Type-0 $\Phi\Phi\Psi$ and $\Psi\Psi\Psi$ vertices:

$$V_{\mathbb{P}} = Y, \quad q_{[3]}^{-R} = 0, \quad q_{[3]}^{-L} = i\varepsilon\sqrt{2}\mathbb{P}_\theta Y, \quad (9.9)$$

Type-4 $\Phi\Phi\Psi$ and $\Psi\Psi\Psi$ vertices:

$$V_{\mathbb{P}} = \frac{1}{\beta}\mathbb{P}_\theta p_{\eta_1}p_{\eta_2}p_{\eta_3}\bar{Y}, \quad (9.10)$$

$$q_{[3]}^{-R} = -i\varepsilon\sqrt{2}p_{\eta_1}p_{\eta_2}p_{\eta_3}\bar{Y}, \quad q_{[3]}^{-L} = 0. \quad (9.11)$$

¹⁰As compared to massive fields in the section 6, we note the appearance of the additional type-0 and type-4 vertices for massless fields in (9.9)–(9.11).

We recall that β and ε are defined in (5.21), (5.22). In (9.3)–(9.11), the quantities X , Y , X_a , Y_a and their barred cousins are functions of the momenta β_1 , β_2 , β_3 . We find the following β -homogeneity equations

$$\mathbf{J}_\beta X = 0, \quad (-2 + \mathbf{J}_\beta) X_a = 0, \quad (9.12)$$

$$\left(-\frac{3}{2} + \mathbf{J}_\beta\right) Y = 0, \quad \left(-\frac{1}{2} + \mathbf{J}_\beta\right) Y_a = 0, \quad (9.13)$$

and identical equations for $\bar{X}, \bar{Y}, \bar{X}_a, \bar{Y}_a$. Solution to β -homogeneity equations (9.12), (9.13) can easily be presented, for example, in the following way:

$$X = v_{\phi\phi\phi} \left(\frac{\beta_1}{\beta_2}\right), \quad X_a = \frac{1}{\beta_1\beta_2} v_{a,\phi\phi\phi} \left(\frac{\beta_1}{\beta_2}\right), \quad \text{for type-1 } \Phi\Phi\Phi \text{ vertex;} \quad (9.14)$$

$$X = \frac{1}{\beta_3} v_{\psi\psi\phi} \left(\frac{\beta_1}{\beta_2}\right), \quad X_a = \beta_3 v_{a,\psi\psi\phi} \left(\frac{\beta_1}{\beta_2}\right), \quad \text{for type-1 } \Psi\Psi\Phi \text{ vertex;} \quad (9.15)$$

$$Y = \beta_3 v_{\phi\phi\psi} \left(\frac{\beta_1}{\beta_2}\right), \quad \text{for type-0 } \Phi\Phi\Psi \text{ vertex;} \quad (9.16)$$

$$Y = v_{\psi\psi\psi} \left(\frac{\beta_1}{\beta_2}\right), \quad \text{for type-0 } \Psi\Psi\Psi \text{ vertex;} \quad (9.17)$$

$$Y_a = v_{a,\phi\phi\psi} \left(\frac{\beta_1}{\beta_2}\right), \quad \text{for type-2 } \Phi\Phi\Psi \text{ vertex;} \quad (9.18)$$

$$Y_a = \frac{1}{\beta_3} v_{a,\psi\psi\psi} \left(\frac{\beta_1}{\beta_2}\right), \quad \text{for type-2 } \Psi\Psi\Psi \text{ vertex;} \quad (9.19)$$

where quantities $v_{\phi\phi\phi}, \dots, v_{a,\psi\psi\psi}$ appearing in (9.14)–(9.19) are functions of the ratio β_1/β_2 . These functions, which we refer to as v -functions, are taken to be β -analytic. Solution for $\bar{X}, \bar{Y}, \bar{X}_a, \bar{Y}_a$ takes the same form as (9.14)–(9.19), where we should replace the v -functions by their barred cousins $\bar{v}_{\phi\phi\phi}, \dots, \bar{v}_{a,\psi\psi\psi}$, which we refer to as $\bar{v}$ -functions. If we ignore the hermicity of the Hamiltonian $P_{[3]}^-$, then the v and $\bar{v}$ -functions are independent. Requiring that the Hamiltonian $P_{[3]}^-$ be hermitian, we find that the v and $\bar{v}$ -functions are related by the hermitian conjugation rules. In terms of the quantities X, Y, X_a, Y_a , and their barred cousins, those hermitian conjugation rules take the form

$$X^* = -I_\beta I_u \bar{X}, \quad X_a^* = -I_\beta I_u \bar{X}_a, \quad \text{for } \Phi\Phi\Phi \text{ vertex;} \quad (9.20)$$

$$X^* = I_\beta I_u \bar{X}, \quad X_a^* = I_\beta I_u \bar{X}_a, \quad \text{for } \Psi\Psi\Phi \text{ vertex;} \quad (9.21)$$

$$Y^* = -I_\beta I_u \bar{Y}, \quad Y_a^* = -I_\beta I_u \bar{Y}_a, \quad \text{for } \Phi\Phi\Psi \text{ vertex;} \quad (9.22)$$

$$Y^* = I_\beta I_u \bar{Y}, \quad Y_a^* = I_\beta I_u \bar{Y}_a, \quad \text{for } \Psi\Psi\Psi \text{ vertex;} \quad (9.23)$$

where, in (9.20)–(9.23), the asterisk stands for the complex conjugation, while the definition of the operator I_x is given in (A.1) in appendix A.

U(1) symmetry restrictions for type- k vertices. Using U(1) symmetry equations in (5.9), (5.26), we can straightforwardly obtain constraints on the labels u_1, u_2, u_3 of the cubic vertex $p_{[3]}^-$. Namely, using (5.9), we find that the u_1, u_2, u_3 of the cubic vertex $p_{[3]}^-$

should satisfy the restrictions:

$$\mathbf{U} = 1 \quad \text{for type-1 vertices}; \quad \mathbf{U} = -1 \quad \text{for type-3 vertices}; \quad (9.24)$$

$$\mathbf{U} = 2 \quad \text{for type-0 vertices}; \quad \mathbf{U} = -2 \quad \text{for type-4 vertices}; \quad (9.25)$$

$$\mathbf{U} = 0 \quad \text{for type-2 vertices}; \quad (9.26)$$

where $\mathbf{U}$ is defined as in (6.50).

In conclusion of this section, we recall that, for the massive and massless fields having $\kappa \neq 0$, we found a finite number of the cubic vertices in sections 6, 7, 8. In contrast to this, for the three massless fields, we obtained infinite number of light-cone gauge cubic vertices (9.3)–(9.11) which are parametrized by the v - and $\bar{v}$ -functions. To our knowledge, an infinite number of Lorentz covariant vertices associated with our infinite number light-cone gauge vertices are not available in the literature at the present time. This is to say that we encounter a mismatch between classifications of Lorentz covariant and light-cone gauge cubic vertices. This mismatch for the massless fields in $3d$ was noticed in ref. [12]. We recall that, for the case of massless fields in $4d$, the mismatch between classifications of Lorentz covariant cubic vertices and light-cone gauge cubic vertices is well known (see, e.g., ref. [25]).¹¹

10 Conclusions

In this paper, we applied a superfield light-cone gauge approach for the study of the interacting $N = 2$ massive arbitrary (integer and half-integer) spin supermultiplets and massless (spin-0 and spin- $\frac{1}{2}$) supermultiplets in $3d$ flat space. Using the light-cone momentum superspace, we developed the light-cone gauge formulation in terms of the unconstrained superfields. Using our superfield formulation, we built all cubic vertices that describe interactions of the massive and massless supermultiplets under consideration. We believe that development made in this paper might be helpful for the following applications and generalizations.

- i) In this paper, we studied cubic vertices for the $N = 2$ massive and massless supermultiplets in *the flat 3d space*. Light-cone gauge action for arbitrary spin free massive fields in AdS_3 space was obtained in ref. [32], while the method for studying cubic vertices of light-cone gauge higher-spin massless AdS fields was developed in ref. [33].¹² Generalization of our studies in this paper and in refs. [32, 33] to the case of light-cone gauge interacting $N = 2$ massive and massless supermultiplets in AdS_3 space could be of great interest. For the reader convenience, we note the

¹¹Recent study of new Lorentz covariant vertices in the framework of $4d$ chiral higher-spin gravity may be found in refs. [26–28]. We expect that adaptation of the twistor methods in these references to $3d$ may be helpful for the study of yet unknown new Lorentz covariant vertices in $3d$. Recent discussion of twistors in $3d$ may be found in ref. [29]. For interesting application of twistors to the study of a superparticle in $\text{AdS}_5 \times S^5$, see refs. [30, 31].

¹²We note the ref. [34] where it was shown that the light-cone gauge action turns out to be helpful for studying boundary 3-point correlation functions.

recent ref. [35] devoted to studying cubic vertices for fields in AdS_3 space by using covariant frame-like formulation. For discussion of methods for the investigation of supersymmetric theories in $3d$, see refs. [36–40] (and references therein). Various studies of supersymmetric theories of massless fields in $4d$ which could be helpful for the investigation of interacting massive supermultiplets in $3d$ may be found, e.g., in refs. [41–46].¹³

- ii) Conformal higher-spin theories in $3d$ have actively been studied in the recent time (see, e.g., refs. [61–67] and references therein). We note that the ordinary-derivative formulation of conformal fields in refs. [68, 69] and Stueckelberg formulation of massive fields share some common features. The light-cone gauge approach considerably simplifies the investigation of interacting massive fields. We expect therefore that the light-cone gauge formulation of conformal fields in ref. [70] will also be helpful for the study of interacting light-cone gauge conformal fields in $3d$.
- iii) $4d$ chiral higher-spin gravity (CHSG) proposed in ref. [71], upon dimensional reduction to $3d$, leads to $3d$ massive CHSG discussed in refs. [12, 72]. We note that $4d$, $N = 1$ supersymmetric CHSG discussed in ref. [20], upon dimensional reduction to $3d$, leads to $N = 2$ supersymmetric massive CHSG in $3d$ with central charges. In this paper, we studied $N = 2$ supersymmetric theories in $3d$ without central charges. It would be then interesting to understand whether exist or not $N = 2$ massive supersymmetric CHSG in $3d$ without central charges. In this respect, the light-cone gauge method discussed in refs. [73, 74] could be helpful.

Acknowledgments

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A Notation and conventions

For any quantity A that depends on three variables x_1, x_2, x_3 , we introduce an operation I_x which is defined by the relations

$$A \equiv A(x_1, x_2, x_3), \quad I_x A \equiv A(-x_1, -x_2, -x_3). \quad (\text{A.1})$$

We use Grassmann momenta denoted as p_θ, p_η , while left derivatives of the p_θ and p_η are denoted as ∂_{p_θ} and ∂_{p_η} respectively. Integrals over p_θ and p_η are defined to be $\int dp_\theta p_\theta = 1$ and $\int dp_\eta p_\eta = 1$. The basic integral is normalized to be

$$\int dp_\eta dp_\theta p_\theta p_\eta = 1. \quad (\text{A.2})$$

¹³For $d > 3$, we mention various methods of building cubic vertices in refs. [47–53] and BRST method in refs. [54–57]. For the studying supersymmetric interactions in AdS_5 , see refs. [58, 59]. The use of the BRST method in $3d$ may be found, e.g., in ref. [60].

We note the following helpful Grassmann integrals

$$\int dp_\eta^\dagger dp_\theta^\dagger e^{\frac{1}{\beta}(p_\theta^\dagger p_\theta + p_\eta^\dagger p_\eta)} = -\frac{1}{\beta^2} p_\theta p_\eta, \quad (\text{A.3})$$

$$\int dp_\eta^\dagger dp_\theta^\dagger e^{\frac{1}{\beta}(p_\theta^\dagger p_\theta + p_\eta^\dagger p_\eta)} p_\theta^\dagger = \frac{1}{\beta} p_\eta, \quad (\text{A.4})$$

$$\int dp_\eta^\dagger dp_\theta^\dagger e^{\frac{1}{\beta}(p_\theta^\dagger p_\theta + p_\eta^\dagger p_\eta)} p_\eta^\dagger = -\frac{1}{\beta} p_\theta, \quad (\text{A.5})$$

$$\int dp_\eta^\dagger dp_\theta^\dagger e^{\frac{1}{\beta}(p_\theta^\dagger p_\theta + p_\eta^\dagger p_\eta)} p_\theta^\dagger p_\eta^\dagger = 1, \quad (\text{A.6})$$

where β in (A.3)–(A.6) should not be confused with the β in (5.21).

Ghost parities of p_θ , p_η and dp_θ , dp_η are given by

$$\text{GP}(p_\theta) = 1, \quad \text{GP}(p_\eta) = 1, \quad \text{GP}(dp_\theta) = 1, \quad \text{GP}(dp_\eta) = 1, \quad (\text{A.7})$$

and we assume the conventions $AB = (-)^{\varepsilon_A \varepsilon_B} BA$, $\varepsilon_X \equiv \text{GP}(X)$. Hermitian conjugation is defined as $(AB)^\dagger = B^\dagger A^\dagger$, where two quantities A , B have arbitrary ghost parity.

Berezin integrals for 3-point vertices. For any quantity A that depends on the Grassmann momenta p_{θ_a} , p_{η_a} , $A = A(p_{\theta_a}, p_{\eta_a})$, we introduce a quantity $\langle A(p_\theta, p_\eta) \rangle$ defined by the relation

$$\langle A(p_\theta, p_\eta) \rangle \equiv \int \prod_{a=1,2,3} dp_{\eta_a}^\dagger dp_{\theta_a}^\dagger \delta(\mathbf{P}_\theta^\dagger) \exp \left(\sum_{b=1,2,3} \frac{1}{\beta_b} (p_{\theta_b}^\dagger p_{\theta b} + p_{\eta b}^\dagger p_{\eta b}) \right) A(p_\theta^\dagger, p_\eta^\dagger), \quad (\text{A.8})$$

where $\delta(\mathbf{P}_\theta^\dagger) = p_{\theta_1}^\dagger + p_{\theta_2}^\dagger + p_{\theta_3}^\dagger$. Using (A.3)–(A.6), we get then straightforwardly the following basic relations

$$\langle 1 \rangle = -\frac{1}{\beta^2} \delta(\mathbf{P}_\theta) \mathbb{P}_\theta p_{\eta 1} p_{\eta 2} p_{\eta 3}, \quad (\text{A.9})$$

$$\langle \mathbb{P}_\theta \rangle = -\frac{1}{\beta} \delta(\mathbf{P}_\theta) p_{\eta 1} p_{\eta 2} p_{\eta 3}, \quad (\text{A.10})$$

$$\langle p_{\eta a} \rangle = -\frac{\beta_a}{\beta^2} \delta(\mathbf{P}_\theta) \mathbb{P}_\theta p_{\eta a+1} p_{\eta a+2}, \quad (\text{A.11})$$

$$\langle \mathbb{P}_\theta p_{\eta a} \rangle = -\frac{\beta_a}{\beta} \delta(\mathbf{P}_\theta) p_{\eta a+1} p_{\eta a+2}, \quad (\text{A.12})$$

$$\langle p_{\eta a+1} p_{\eta a+2} \rangle = \frac{1}{\beta} \delta(\mathbf{P}_\theta) \mathbb{P}_\theta \frac{p_{\eta a}}{\beta_a}, \quad (\text{A.13})$$

$$\langle \mathbb{P}_\theta p_{\eta a+1} p_{\eta a+2} \rangle = \delta(\mathbf{P}_\theta) \frac{p_{\eta a}}{\beta_a}, \quad (\text{A.14})$$

$$\langle p_{\eta 1} p_{\eta 2} p_{\eta 3} \rangle = \frac{1}{\beta} \delta(\mathbf{P}_\theta) \mathbb{P}_\theta, \quad (\text{A.15})$$

$$\langle \mathbb{P}_\theta p_{\eta 1} p_{\eta 2} p_{\eta 3} \rangle = \delta(\mathbf{P}_\theta), \quad (\text{A.16})$$

where $\beta \equiv \beta_1 \beta_2 \beta_3$, while $\mathbb{P}_\theta$ is defined in (5.3). In turn, by using relations (A.9)–(A.16), we derive the relations which are helpful for the studying hermicity properties of the cubic

vertices

$$\begin{aligned} & \left\langle \left(\kappa \mathbb{P}_\theta - \beta \sum_{a=1,2,3} \frac{m_a}{\beta_a} p_{\eta a} \right) \right\rangle \\ &= \frac{1}{\kappa \beta} \delta(\mathbf{P}_\theta) \left(\kappa \mathbb{P}_\theta + \beta \sum_{a=1,2,3} \frac{m_a}{\beta_a} p_{\eta a} \right) \sum_{b=1,2,3} m_b p_{\eta b+1} p_{\eta b+2}, \end{aligned} \quad (\text{A.17})$$

$$\begin{aligned} & \left\langle \left(\kappa \mathbb{P}_\theta - \beta \sum_{a=1,2,3} \frac{m_a}{\beta_a} p_{\eta a} \right) \sum_{b=1,2,3} p_{\eta b} V_b \right\rangle \\ &= -\frac{\kappa}{\beta^2} \delta(\mathbf{P}_\theta) \left(\kappa \mathbb{P}_\theta + \beta \sum_{a=1,2,3} \frac{m_a}{\beta_a} p_{\eta a} \right) \sum_{b=1,2,3} p_{\eta b} f_b, \end{aligned} \quad (\text{A.18})$$

where quantities f_a appearing in (A.18) are defined in terms of the vertices V_a as

$$f_a \equiv \frac{\beta}{\kappa^2} (m_{a+1} \beta_{a+2} V_{a+2} - m_{a+2} \beta_{a+1} V_{a+1}), \quad a = 1, 2, 3. \quad (\text{A.19})$$

Inverse relations to the ones in (A.19) are given by

$$V_a = \frac{1}{\beta} (m_{a+1} \beta_{a+2} f_{a+2} - m_{a+2} \beta_{a+1} f_{a+1}), \quad a = 1, 2, 3. \quad (\text{A.20})$$

B Properties of superfields $\Theta_{m,\lambda,u}^*$

We find it convenient to use superfields $\Theta_{m,\lambda,u}^*$ for building interaction vertices. These superfields are defined in (3.28)–(3.33).

Differential operators realization of $N = 2$ Poincaré superalgebra on superfield $\Theta_{m,\lambda,u}^*$. The realization is given by

$$P^1 = -p, \quad P^+ = -\beta, \quad P^- = -p^-, \quad p^- = -\frac{p^2 + m^2}{2\beta}, \quad (\text{B.1})$$

$$J^{+1} = ix^+ P^1 + \partial_p \beta, \quad J^{+-} = ix^+ P^- + \partial_\beta \beta + M^{+-}, \quad (\text{B.2})$$

$$J^{-1} = -\partial_\beta p + \partial_p p^- - \frac{im}{\beta} \left(\lambda - \frac{1}{2} p_\theta \partial_{p_\eta} - \frac{1}{2} p_\eta \partial_{p_\theta} \right) - \frac{p}{\beta} M^{+-}, \quad (\text{B.3})$$

$$U = 1 + u - p_\theta \partial_{p_\theta} - p_\eta \partial_{p_\eta}, \quad (\text{B.4})$$

$$M^{+-} \equiv \frac{1}{2} p_\theta \partial_{p_\theta} + \frac{1}{2} p_\eta \partial_{p_\eta} - \frac{1}{2} e_\lambda, \quad (\text{B.5})$$

$$Q^{+R} = (-)^{e_\lambda} \beta \partial_{p_\theta}, \quad Q^{+L} = (-)^{e_\lambda + \frac{1}{2}} p_\theta, \quad (\text{B.6})$$

$$Q^{-R} = \frac{(-)^{e_\lambda}}{\sqrt{2}} (p \partial_{p_\theta} + im \partial_{p_\eta}), \quad Q^{-L} = \frac{(-)^{e_\lambda + \frac{1}{2}}}{\sqrt{2} \beta} (p p_\theta - im p_\eta), \quad (\text{B.7})$$

where the e_λ is defined in (2.28). Note that superfields $\Theta_{m,\lambda,u}^*$ and $\Theta_{m,\lambda,u}$ are related as

$$\beta \Theta_{m,\lambda,-u}^* (-p, -p_\theta, p_\eta) = \Theta_{m,\lambda,u} (p, p_\theta, p_\eta). \quad (\text{B.8})$$

Making use of the relations (B.1)–(B.8) and (3.35), (3.36), it is easy to check the standard equal-time (anti)commutator between the generators and the superfields $\Theta_{m,\lambda,u}^*$,

$$[\Theta_{m,\lambda,u}^*, G_{[2]}]_{\pm} = G_{\text{diff}} \Theta_{m,\lambda,u}^*, \quad (\text{B.9})$$

where G_{diff} may be found in (B.1)–(B.7). If $\Theta_{m,\lambda,u}^{*\dagger}$ is a hermitian conjugate of the superfield $\Theta_{m,\lambda,u}^*$, then one has the relation

$$\Theta_{m,\lambda,u}^*(p, p_\theta, p_\eta)^\dagger = \beta \int dp_\eta dp_\theta e^{\frac{1}{\beta}(p_\theta p_\theta^\dagger + p_\eta p_\eta^\dagger)} \Theta_{m,\lambda,-u}^*(-p, p_\theta, -p_\eta). \quad (\text{B.10})$$

Hermitian conjugation rules for vertices. Using relation (B.10) and requiring the Hamiltonian $P_{[3]}^-$ to be hermitian, we find that the type-1 $\Phi\Phi\Phi$ and $\Psi\Psi\Phi$ vertices $V, \bar{V}$ in (6.7) are related to the type-3 $\Phi\Phi\Phi$ and $\Psi\Psi\Phi$ vertices $V', \bar{V}'$ (6.12) in the following way:

$$V^* = \varepsilon_{\Theta_1\Theta_2\Theta_3} I_u I_\beta \bar{V}', \quad \bar{V}^* = \varepsilon_{\Theta_1\Theta_2\Theta_3} I_u I_\beta V', \quad \text{for non-critical masses;} \quad (\text{B.11})$$

$$V^* = \varepsilon_{\Theta_1\Theta_2\Theta_3} I_u I_\beta V', \quad \bar{V}^* = \varepsilon_{\Theta_1\Theta_2\Theta_3} I_u I_\beta \bar{V}', \quad \text{for critical masses;} \quad (\text{B.12})$$

while, for the type-2 $\Psi\Psi\Phi$ and $\Psi\Psi\Psi$ vertices $V_a, \bar{V}_a$ in (6.7), we find the relations

$$\frac{1}{\kappa} (m_{a+2}\beta_{a+1}V_{a+1}^* - m_{a+1}\beta_{a+2}V_{a+2}^*) = \varepsilon_{\Theta_1\Theta_2\Theta_3} I_\beta I_u \bar{V}_a, \quad \text{for non-critical masses;} \quad (\text{B.13})$$

$$\frac{1}{\kappa} (m_{a+2}\beta_{a+1}V_{a+1}^* - m_{a+1}\beta_{a+2}V_{a+2}^*) = \varepsilon_{\Theta_1\Theta_2\Theta_3} I_\beta I_u V_a, \quad \text{for critical masses;} \quad (\text{B.14})$$

$$\frac{1}{\kappa} (m_{a+2}\beta_{a+1}\bar{V}_{a+1}^* - m_{a+1}\beta_{a+2}\bar{V}_{a+2}^*) = \varepsilon_{\Theta_1\Theta_2\Theta_3} I_\beta I_u \bar{V}_a, \quad \text{for critical masses;} \quad (\text{B.15})$$

$$\varepsilon_{\Theta_1\Theta_2\Theta_3} \equiv (-)^{e_{\lambda_1}e_{\lambda_2}+e_{\lambda_2}e_{\lambda_3}+e_{\lambda_3}e_{\lambda_1}}, \quad (\text{B.16})$$

where, in (B.13)–(B.15), $a = 1, 2, 3$. In (B.11)–(B.15), the asterisk stands for the complex conjugation, while the operators I_u, I_β are defined as in (A.1). We recall that, for the non-critical masses, the κ is defined in (6.8), while, for the critical masses, we use κ given in (6.47). We recall also that, for the superfield Φ , $e_\lambda = 0$, while, for the superfield Ψ , $e_\lambda = 1$. This gives the relations

$$\varepsilon_{\Phi_1\Phi_2\Phi_3} = 1, \quad \varepsilon_{\Phi_1\Phi_2\Psi_3} = 1, \quad \varepsilon_{\Psi_1\Psi_2\Phi_3} = -1, \quad \varepsilon_{\Psi_1\Psi_2\Psi_3} = -1. \quad (\text{B.17})$$

C Derivation of superspace representation for cubic vertex $p_{[3]}^-$ and densities $j_{[3]}^{-1}, q_{[3]}^{-R,L}$

In this appendix, firstly, we outline a procedure of the derivation of the superspace representation for the vertices $p_{[3]}^-$ given in (6.7), (6.12), (6.15) and equations for the V -vertices given in (6.9), (6.10), (6.13), (6.14), and (6.16)–(6.19). Secondly, we outline the derivation of the densities $j_{[3]}^{-1}, q_{[3]}^{-R,L}$ given in (6.20)–(6.29). Thirdly, we outline a procedure of the derivation of the cubic densities for the three massless fields in section 9. Our procedure of the derivation of the cubic densities in section 6 is realized in the following six steps.

Step 1. By definition, the vertex $p_{[3]}^-$ is degree-1 polynomial in $\mathbb{P}$. Expanding then the $p_{[3]}^-$ into the $\mathbb{P}$ and the Grassmann momentum $\mathbb{P}_\theta$, we get

$$p_{[3]}^- = i\mathbb{P}V_{\mathbb{P}} + V_0, \quad (\text{C.1})$$

$$V_{\mathbb{P}} = \mathbb{P}_\theta V_{\mathbb{P}\mathbb{P}_\theta} + V_{\mathbb{P}0}, \quad V_0 = \mathbb{P}_\theta V_{0\mathbb{P}_\theta} + V_{00}, \quad (\text{C.2})$$

where vertices $V_{\mathbb{P}\mathbb{P}_\theta}$, $V_{\mathbb{P}0}$, $V_{0\mathbb{P}_\theta}$, V_{00} depend on the Grassmann momenta p_{η_1} , p_{η_2} , p_{η_3} and the momenta β_1 , β_2 , β_3 .

Step 2. At this step we consider restrictions imposed by the equations (5.14), (5.15). Plugging (C.1) into (5.14), (5.15) and comparing terms of various powers of the momenta $\mathbb{P}$ and $\mathbb{P}_\theta$, we get expressions for the supercharge densities given by

$$q_{[3]}^{-R} = -i\varepsilon\sqrt{2}\beta V_{\mathbb{P}\mathbb{P}_\theta}, \quad q_{[3]}^{-L} = i\varepsilon\sqrt{2}\mathbb{P}_\theta V_{\mathbb{P}0}, \quad (\text{C.3})$$

and the following restrictions on the vertices appearing in (C.2),

$$V_{0\mathbb{P}_\theta} - \sum_{a=1,2,3} m_a \partial_{p_{\eta a}} V_{\mathbb{P}0} = 0, \quad \sum_{a=1,2,3} m_a \partial_{p_{\eta a}} V_{\mathbb{P}\mathbb{P}_\theta} = 0, \quad (\text{C.4})$$

$$\sum_{a=1,2,3} m_a \partial_{p_{\eta a}} V_{00} + \beta \sum_{a=1,2,3} \frac{m_a^2}{\beta_a} V_{\mathbb{P}\mathbb{P}_\theta} = 0, \quad \sum_{a=1,2,3} m_a \partial_{p_{\eta a}} V_{0\mathbb{P}_\theta} = 0, \quad (\text{C.5})$$

$$\frac{1}{\beta} V_{00} + \sum_{a=1,2,3} \frac{m_a}{\beta_a} p_{\eta a} V_{\mathbb{P}\mathbb{P}_\theta} = 0, \quad \sum_{a=1,2,3} \frac{m_a}{\beta_a} p_{\eta a} V_{\mathbb{P}0} = 0, \quad (\text{C.6})$$

$$\sum_{a=1,2,3} \frac{m_a}{\beta_a} p_{\eta a} V_{0\mathbb{P}_\theta} - \sum_{a=1,2,3} \frac{m_a^2}{\beta_a} V_{\mathbb{P}0} = 0, \quad \sum_{a=1,2,3} \frac{m_a}{\beta_a} p_{\eta a} V_{00} = 0. \quad (\text{C.7})$$

We recall that β and ε are defined in (5.21), (5.22). Expanding the vertices $V_{\mathbb{P}\mathbb{P}_\theta}$, $V_{\mathbb{P}0}$, $V_{0\mathbb{P}_\theta}$, and V_{00} into the Grassmann momenta $p_{\eta a}$, we find that the general solution to equations (C.4)–(C.7) can be presented as

$$V_{\mathbb{P}\mathbb{P}_\theta} = X + \sum_{a=1,2,3} X_a p_{\eta a} + Y' \sum_{a=1,2,3} m_a p_{\eta a+1} p_{\eta a+2}, \quad (\text{C.8})$$

$$V_{00} = -\beta X \sum_{b=1,2,3} \frac{m_b}{\beta_b} p_{\eta b} - \beta \sum_{b=1,2,3} \frac{m_b}{\beta_b} p_{\eta b} \sum_{a=1,2,3} X_a p_{\eta a} + \kappa^2 Y' p_{\eta 1} p_{\eta 2} p_{\eta 3}, \quad (\text{C.9})$$

$$V_{0\mathbb{P}_\theta} = \kappa^2 Y + \kappa^2 \sum_{a=1,2,3} Y_a p_{\eta a} + X' \sum_{a=1,2,3} m_a p_{\eta a+1} p_{\eta a+2}, \quad (\text{C.10})$$

$$V_{\mathbb{P}0} = -\beta Y \sum_{b=1,2,3} \frac{m_b}{\beta_b} p_{\eta b} - \beta \sum_{b=1,2,3} \frac{m_b}{\beta_b} p_{\eta b} \sum_{a=1,2,3} Y_a p_{\eta a} + X' p_{\eta 1} p_{\eta 2} p_{\eta 3}, \quad (\text{C.11})$$

where vertices X_a , Y_a satisfy the algebraic constraints

$$\sum_{a=1,2,3} m_a X_a = 0, \quad \sum_{a=1,2,3} m_a Y_a = 0, \quad (\text{C.12})$$

and vertices X , Y , X' , Y' , X_a , Y_a , $a = 1, 2, 3$, depend only on the momenta β_1 , β_2 , β_3 . We recall that the κ appearing in (C.9), (C.10) is defined in (6.8).

Step 3. At this step we consider restrictions imposed by the equations (5.16). Plugging (C.1), (C.2) into (5.16), we find relation (6.20) and the following equations

$$\sum_{a=1,2,3} \left(\frac{m_a^2}{\beta_a} \mathbb{N}_\beta^{\eta E} - \frac{\check{\beta}_a m_a^2}{6\beta_a} \right) V_{\mathbb{P}\mathbb{P}_\theta} - i\mathcal{M}V_{0\mathbb{P}_\theta} - \sum_{a=1,2,3} \frac{\check{\beta}_a}{6\beta} m_a \partial_{p_{\eta a}} V_{00} = 0, \quad (\text{C.13})$$

$$\sum_{a=1,2,3} \left(\frac{m_a^2}{\beta_a} \mathbb{N}_\beta^{\eta E} - \frac{\check{\beta}_a m_a^2}{6\beta_a} \right) V_{\mathbb{P}0} - i\mathcal{M}V_{00} + \sum_{a=1,2,3} \frac{\check{\beta}_a}{6\beta_a} m_a p_{\eta a} V_{0\mathbb{P}_\theta} = 0, \quad (\text{C.14})$$

$$\frac{1}{\beta} \mathbb{N}_\beta^{\eta E} V_{0\mathbb{P}_\theta} + i\mathcal{M}V_{\mathbb{P}\mathbb{P}_\theta} + \sum_{a=1,2,3} \frac{\check{\beta}_a}{6\beta} m_a \partial_{p_{\eta a}} V_{\mathbb{P}0} = 0, \quad (\text{C.15})$$

$$\frac{1}{\beta} \mathbb{N}_\beta^{\eta E} V_{00} + i\mathcal{M}V_{\mathbb{P}0} - \sum_{a=1,2,3} \frac{\check{\beta}_a}{6\beta_a} m_a p_{\eta a} V_{\mathbb{P}\mathbb{P}_\theta} = 0. \quad (\text{C.16})$$

Plugging (C.8)–(C.11) into (C.13)–(C.16) and introducing V -vertices by the relations

$$X = \frac{1}{2}(V + \bar{V}), \quad Y = \frac{1}{2\kappa}(V - \bar{V}), \quad (\text{C.17})$$

$$X' = \frac{1}{2}(V' + \bar{V}'), \quad Y' = \frac{1}{2\kappa}(V' - \bar{V}'), \quad (\text{C.18})$$

$$X_a = \frac{1}{2}(V_a + \bar{V}_a), \quad Y_a = \frac{1}{2\kappa}(V_a - \bar{V}_a), \quad (\text{C.19})$$

we find that equations (C.13)–(C.16) amount to the 1st equations for the V -vertices given in (6.9), (6.10), (6.13), (6.14) and equations given in (6.16), (6.18).

Step 4. Using J^{+-} symmetry equation for $p_{[3]}^-$ in (5.6) and relations (C.1), (C.2), (C.8)–(C.11), (C.17)–(C.19), we obtain the 2nd equations for V vertices in (6.9), (6.10), (6.13), (6.14), and equations in (6.17), (6.19).

Step 5. Plugging expressions $V_{\mathbb{P}\mathbb{P}_\theta}$ (C.8), $V_{\mathbb{P}0}$ (C.11) into $V_{\mathbb{P}}$ (C.2) and using (C.17)–(C.19), we find expressions for $V_{\mathbb{P}}$ given in (6.21), (6.24), (6.27).

Step 6. Plugging expressions $V_{\mathbb{P}\mathbb{P}_\theta}$ (C.8), $V_{\mathbb{P}0}$ (C.11) into (C.3) and using (C.17)–(C.19), we find expressions for the supercharge densities $q_{[3]}^{-R,L}$ given in (6.22), (6.23), (6.25), (6.26), (6.28), and (6.29).

Cubic densities for massless fields. For this case, as above, we start with the general expressions given in (C.1), (C.2). Setting $m_1 = 0$, $m_2 = 0$, $m_3 = 0$ in (C.4), (C.6), we then find

$$V_{0\mathbb{P}_\theta} = 0, \quad V_{00} = 0. \quad (\text{C.20})$$

Using (C.20) in (C.1), we get $p_{[3]}^-$ given in (9.1). Using $p_{[3]}^-$ (9.1) in (5.16), we get $j_{[3]}^{-1}$ given in (9.2). After that we find that, for $m_1 = 0$, $m_2 = 0$, $m_3 = 0$, the relations in (C.4)–(C.7) do not impose constraints for the dependence of the $V_{\mathbb{P}\mathbb{P}_\theta}$, $V_{\mathbb{P}0}$ on the Grassmann momenta $p_{\eta a}$. This implies that the Grassmann momenta expansion for the $V_{\mathbb{P}\mathbb{P}_\theta}$, $V_{\mathbb{P}0}$ takes the most

general form given by

$$V_{\mathbb{PP}_\theta} = X + \sum_{a=1,2,3} \frac{p_{\eta a}}{\beta_a} Y_a + \frac{1}{\beta} \sum_{a=1,2,3} p_{\eta a+1} p_{\eta a+2} \bar{X}_a + \frac{1}{\beta} p_{\eta_1} p_{\eta_2} p_{\eta_3} \bar{Y}, \quad (\text{C.21})$$

$$V_{\mathbb{P}0} = Y + \sum_{a=1,2,3} \frac{p_{\eta a}}{\beta_a} X_a + \sum_{a=1,2,3} p_{\eta a+1} p_{\eta a+2} \bar{Y}_a + p_{\eta_1} p_{\eta_2} p_{\eta_3} \bar{X}. \quad (\text{C.22})$$

Using (C.21), (C.22) in (C.3), we get expressions for the supercharge densities given in (9.4), (9.6), (9.8), (9.9), and (9.11). Comparing (C.21) and (C.8), we see appearance of the additional term of degree-3 in the $p_{\eta a}$ in (C.21), while, comparing (C.22) and (C.11), we see appearance of the additional term of degree-0 in the $p_{\eta a}$ in (C.22). It is those additional terms that lead to the type-0 vertices (9.9) and the type-4 vertices (9.10) for massless fields in section 9.

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