Seeking SUSY fixed points in the $4 - \epsilon$ expansion

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ABSTRACT: We use the $4 - \epsilon$ expansion to search for fixed points corresponding to $2 + 1$ dimensional $\mathcal{N}=1$ Wess-Zumino models of N_Φ scalar superfields interacting through a cubic superpotential. In the $N_\Phi = 3$ case we classify all SUSY fixed points that are perturbatively unitary. In the $N_\Phi = 4$ and $N_\Phi = 5$ cases, we focus on fixed points where the scalar superfields form a single irreducible representation of the symmetry group (irreducible fixed points). For $N_\Phi = 4$ we show that the S5 invariant super Potts model is the only irreducible fixed point where the four scalar superfields are fully interacting. For $N_\Phi = 5$, we go through all Lie subgroups of $O(5)$ and use the GAP system for computational discrete algebra to study finite subgroups of $O(5)$ up to order 800. This analysis gives us three fully interacting irreducible fixed points. Of particular interest is a subgroup of $O(5)$ that exhibits $O(3)/\mathbb{Z}_2$ symmetry. It turns out this fixed point can be generalized to a new family of models, with $N_\Phi = \frac{N(N-1)}{2} - 1$ and $O(N)/\mathbb{Z}_2$ symmetry, that exists for arbitrary integer $N \geq 3$.

KEYWORDS: Conformal Field Theory, Discrete Symmetries, Renormalization Group

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1 Introduction

In recent years, $2 + 1$ dimensional superconformal field theories (SCFTs) with minimal ($\mathcal{N} = 1$) supersymmetry have received significant attention [1–10]. Perhaps the simplest model with this symmetry is the so-called $\mathcal{N} = 1$ super-Ising model, which was proposed in [1] to describe a quantum critical point at the boundary of a $3 + 1$ dimensional topological superconductor. Because the super-Ising model preserves time reversal symmetry, its

spectrum contains only one relevant operator which is time-reversal invariant. This means that a generic renormalization group flow (not necessarily supersymmetric) can reach the fixed point by tuning a single parameter. This property is called emergent supersymmetry and might be realizable in experiment.

Apart from the super-Ising, a whole zoo of theories have been identified and they are sometimes related by dualities [4–7]. In particular, a certain super QED was shown to be dual to an $\mathcal{N} = 1$ Wess-Zumino model [5, 6].¹ This duality has many properties that resemble the self duality of non-supersymmetric QED coupled to two complex scalars, which describes the famous de-confinement quantum critical point [12, 13].

The study of $2 + 1$ dimensional $\mathcal{N} = 1$ SCFT has also been boosted by renewed interest on standard techniques. For example, it was observed in [3] that if one sets the number of Dirac fermions to $N_f = \frac{1}{4}$ in four dimensions, after analytic continuation to $4 - \epsilon$ dimensions, one ends up with a single three-dimensional Majorana fermion when $\epsilon = 1$. By studying the Gross-Neveu-Yukawa model with $N_f = \frac{1}{4}$ Dirac fermions and one real scalar, one gets a perturbative fixed point which is in good agreement with the $\mathcal{N} = 1$ super-Ising model in $2 + 1$ dimensions. This technique can be easily generalized to more general $2 + 1$ dimensional $\mathcal{N} = 1$ Wess-Zumino models with superpotential²

$$\mathcal{W} = \frac{1}{6} h_{ijk} \Phi^i \Phi^j \Phi^k. \quad (1.1)$$

In other words, one can study the corresponding Gross-Neveu-Yukawa model

$$\mathcal{L}_{\mathcal{N}=1} = \frac{1}{2} (\partial \phi^i)^2 + \frac{1}{2} \bar{\psi}^i \partial \cdot \Gamma \psi^i + \frac{1}{2} h_{ijk} \phi^i \bar{\psi}^j \psi^k + \frac{1}{8} h_{ijm} h_{klm} \phi^i \phi^j \phi^k \phi^l, \quad (1.2)$$

with $N_f = \frac{N_\Phi}{4}$.

Parallel to these analytical developments, progress has also been made using the highly successful numerical bootstrap program [14, 15]. Recent works that studied $2 + 1$ dimensional $\mathcal{N} = 1$ SCFTs using numerical techniques include [2, 8–10]. In particular, when applied to the $\mathcal{N} = 1$ super-Ising model, the scaling dimension of the superfield Φ can be determined to very high precision [8].

In this work we would like to find an organizational scheme for three-dimensional models with $\mathcal{N} = 1$ supersymmetry. Our analysis will be perturbative and rely on the standard ϵ -expansion. The logic being that a one-loop analysis should give us at least a qualitative understanding of potential fixed points. Once we know a fixed point exists, and have a basic idea of its spectrum, we can use more powerful techniques like the modern numerical bootstrap in order to solve it to high precision. This work is then similar to [16–18], where scalar models in several dimensions were studied using a similar method.

¹We denote these models as “Wess-Zumino” models here, however we should point out that they are different from the original four dimensional Wess-Zumino models considered in [11]. The models considered in this paper have minimal super-symmetry in three dimensions. Comparing to the four dimensional Wess-Zumino models, they have half the number of supercharges. See appendix A of [7] for the formulation of the Lagrangian of these models in terms of real superfields.

²Quadratic terms break time-reversal symmetry and cannot be added to the superpotential (see for example [6]).

Our approach is then the following, we use the $4 - \epsilon$ expansion to search for $\mathcal{N} = 1$ SCFTs with a small number of scalar superfields interacting through the superpotential (1.1). We systematically study all the cases when the number of scalar superfields N_Φ is less or equal than 5. In section 3, we start with the $N_\Phi = 3$ case, which will also include all the possible solutions for $N_\Phi = 1, 2$. For $N_\Phi = 3$ the superpotential has 10 independent couplings h_{ijk} , by solving the one-loop beta functions we list in table 1 all the possible fixed points with $N_\Phi \leq 3$ that are perturbatively unitary. We then increase the number of scalar to $N_\Phi = 4$ in section 4 and to $N_\Phi = 5$ in section 5. We focus on the SUSY fixed points where the scalar superfields form a single irreducible representation of the symmetry group. We call these fixed points “irreducible fixed points”. This means the coupling constants $h_{ijk} = \sum_a g_a d_{ijk}^a$ are a linear superposition of the degree three invariant tensors of the symmetry group. The invariant tensors are traceless and also satisfy

$$d_{imn}^a d_{jmn}^b \propto \delta_{ij}. \quad (1.3)$$

This is essentially the trace condition used in [19], and has important physical consequences. It implies there is only one boson mass operator $\phi_i \phi_i$ that is invariant under the symmetry group of the fixed points. For the most stable fixed point of the beta function, this operator is also the only relevant operator that preserves both time-reversal symmetry and the flavor symmetry of the SCFT. Similar to the super-Ising model, non-supersymmetric RG flows will reach the supersymmetric fixed point as long as one tunes the coupling of the boson mass term to zero. In other words, these systems have emergent supersymmetry. The results of the classification are listed in table 2. In the $N_\Phi = 4$ case, it turns out that there is only one group that gives us a fixed point where the four superfields are fully interacting. This result is based on the analysis of all the subgroups of $O(4)$ made in [20]. In the $N_\Phi = 5$ case, we focus on Lie subgroups and finite subgroups of $O(5)$ with orders less or equal than 800. In section 6, we observe that some fixed points for low values of N_Φ can be considered the first entry in a family of fixed points that can be generalized to higher values of N_Φ . For example, the $O(3)/\mathbb{Z}_2$ SCFT presented in table 2 can be generalized to a new family of SCFTs preserving an $O(N)/\mathbb{Z}_2$ symmetry. This family complements the three families (the super Potts models, the $SU(N)$ invariant SCFTs and the F4-family of SCFTs) of Wess-Zumino models already studied in [10].

2 General properties of the $\mathcal{N} = 1$ SCFTs

The one-loop beta function of the $\mathcal{N} = 1$ Wess-Zumino model with N_Φ scalar superfields coupled through the superpotential (1.1) is given by [3, 7],

$$\beta(h) = -\frac{\epsilon}{2} h_{ijk} + \frac{1}{16\pi^2} \left(\frac{1}{2} (h_{ijm} h_{kpq} h_{mpq} + h_{ikm} h_{jpq} h_{mpq} + h_{jkm} h_{ipq} h_{mpq}) + 2 h_{imp} h_{jpq} h_{kqm} \right), \quad (2.1)$$

and the anomalous dimension matrix (whose eigenvalues give us the anomalous dimension) reads

$$(\gamma_\Phi)_{ij} = \frac{1}{16\pi^2} \frac{1}{2} h_{imn} h_{jmn}. \quad (2.2)$$

N_Φ	v and λ 's	γ_Φ	Symmetry	A/ϵ^2
1	(3.3)	$\frac{1}{14}\epsilon$		−19.7392
2	(3.4)	$\frac{1}{6}\epsilon, \frac{1}{6}\epsilon$	S3	−92.1163
	(3.5)	$\frac{9}{109}\epsilon, \frac{13}{218}\epsilon$	Z2	−39.2973
3	(3.6)	$\frac{1}{10}\epsilon, \frac{1}{10}\epsilon, \frac{1}{10}\epsilon$	S4	−82.9047
	(3.7)	$0.232069\epsilon, 0.146209\epsilon, 0.146209\epsilon$	O(2)	−144.941
	(3.8)	$\frac{9}{190}\epsilon, \frac{47}{570}\epsilon, \frac{47}{570}\epsilon$	S3	−58.6635
	(3.9)	$\frac{49}{290}\epsilon, \frac{1}{290}(\sqrt{265} + 29)\epsilon, \frac{1}{290}(29 - \sqrt{265})\epsilon$	K4=Z2× Z2	−110.647
	(3.10)	$0.171857\epsilon, 0.16459\epsilon, 0.063942\epsilon$	Z2	−58.8549
	(3.11)	$0.0871871\epsilon, 0.0710876\epsilon, 0.0546982\epsilon$	Z2	−101.963

Table 1. All perturbatively unitary $\mathcal{N}=1$ fixed points with three or less interacting scalar superfields.

N_Φ	γ_Φ	Symmetry	A/ϵ^2	
4	$\frac{3}{34}\epsilon$	S5	−97.5349	Section 4.1
5	$\frac{1}{12}\epsilon$	S6	−115.145	Section 5.3
	$\frac{3}{20}\epsilon$	S5	−207.262	
	$\frac{7}{30}\epsilon$	O(3)/Z2	−322.407	

Table 2. Irreducible fixed points of $N_\Phi=4$ and $N_\Phi=5$. The $N=4$ fixed point listed here is the only irreducible fixed point of four fully interacting real scalars superfields. The $N_\Phi=5$ fixed points listed here are the only irreducible fixed points of five fully interacting real scalars superfields that have a finite symmetry whose order is less than 800.

Like the scalar theory in $6 - \epsilon$ dimensions, this is a gradient flow [16, 17, 21]. In other words,

$$\beta_{ijk}(h) = \partial A / \partial h_{ijk} \quad (2.3)$$

with

$$A = -\epsilon h_{ijk} h_{ijk} + \frac{1}{16\pi^2} \left(\frac{3}{8} h_{ijm} h_{kpq} h_{mpq} h_{ijk} + \frac{1}{2} h_{imp} h_{jpq} h_{kqm} h_{ijk} \right). \quad (2.4)$$

Along the RG flow to the IR, A decreases and at the fixed points,

$$A = -\frac{7\epsilon}{8} h_{ijk} h_{ijk} = -28\pi^2 \epsilon \sum_i \gamma_i. \quad (2.5)$$

Here γ_i are the anomalous dimensions.

3 $N_\Phi = 3$ reducible fixed points

The strategy we employ in this section is identical to the one described in [17] to study scalar ϕ^3 theory in $6 - \epsilon$ dimensions. Notice that the coupling constant h_{ijk} is a fully

symmetric tensor and can be characterized as

$$h_{ijk} = v_{(i}\delta_{jk)} + \lambda_{ijk}. \quad (3.1)$$

Here λ_{ijk} is symmetric and traceless. We parameterize the potential as

$$\begin{aligned} \mathcal{W}(\Phi) = & (v_j \Phi_j)(\Phi_i \Phi_i) - \frac{\lambda_1 \Phi_1 (\Phi_1^2 - 3\Phi_2^2)}{2\sqrt{2}} + \frac{\lambda_2 \Phi_2 (\Phi_2^2 - 3\Phi_1^2)}{2\sqrt{2}} + \frac{1}{2}\sqrt{3}\lambda_3 (\Phi_2^2 - \Phi_1^2) \Phi_3 \\ & + \sqrt{3}\lambda_4 \Phi_1 \Phi_2 \Phi_3 - \frac{1}{2}\sqrt{\frac{3}{10}}\lambda_5 \Phi_1 (\Phi_1^2 + \Phi_2^2 - 4\Phi_3^2) - \frac{1}{2}\sqrt{\frac{3}{10}}\lambda_6 \Phi_2 (\Phi_1^2 + \Phi_2^2 - 4\Phi_3^2) \\ & + \frac{\lambda_7 \Phi_3 (-3\Phi_1^2 - 3\Phi_2^2 + 2\Phi_3^2)}{\sqrt{10}}. \end{aligned} \quad (3.2)$$

The parameterization relies on the representation theory of $O(3)$. The couplings

$$\lambda_1 + i\lambda_2, \lambda_3 + i\lambda_4, \lambda_5 + i\lambda_6, \lambda_7, \lambda_5 - i\lambda_6, \lambda_3 - i\lambda_4, \lambda_1 - i\lambda_2,$$

form a $j = 3$ representation of $O(3)$ with $m = +3, +2, +1, 0, -1, -2, -3$ respectively. Clearly, v_j forms a vector representation of $O(3)$. We can use the J_1 and J_2 rotation to set $v_1 = v_2 = 0$, and after that we can use J_3 to set $\lambda_2 = 0$. After fixing the redundant $O(3)$ rotations solutions of $\beta(v, \lambda) = 0$ are discrete. The super Ising model plus two decoupled free scalar superfields is located at

$$v_3 = -\frac{2\pi}{5\sqrt{7}}, \quad \lambda_7 = \frac{-2}{3}\sqrt{\frac{2}{35}}\pi, \quad (3.3)$$

with all other v 's and λ 's vanishing.

The S_3 invariant fixed point (S_3 super Potts model) of two scalars plus a decoupled scalar superfield is located at

$$\lambda_1 = \frac{4\pi}{3\sqrt{3}}, \quad (3.4)$$

with all other v 's and λ 's vanishing. The Lagrangian of the S_3 super Potts model can in fact be re-written as an $\mathcal{N}=2$ supersymmetric Lagrangian with a single complex chiral superfield [8]. This explains the anomalous dimension $\gamma_\Phi = \frac{1}{6}\epsilon$.

The Z_2 fixed point with two coupled scalars plus one decoupled scalar superfield is located at

$$v_3 = \frac{2\pi}{\sqrt{109}}, \quad \lambda_4 = 2\sqrt{\frac{3}{109}}\pi, \quad \lambda_7 = -\frac{1}{3}\sqrt{\frac{10}{109}}\pi, \quad (3.5)$$

with all other v 's and λ 's vanishing.

The S_4 invariant super Potts model is located at

$$\lambda_3 = 2\sqrt{\frac{2}{15}}\pi, \quad (3.6)$$

with all other v 's and λ 's vanishing.

The $O(2)$ fixed point with the three scalars fully coupled is located at

$$v_3 = 0.440267, \quad \lambda_7 = -2.06839, \quad (3.7)$$

with all other v 's and λ 's vanishing.

The S3 invariant fixed point with all three scalars fully coupled is located at

$$v_3 = \frac{2\pi}{\sqrt{95}}, \quad \lambda_1 = \frac{4}{3}\sqrt{\frac{23}{285}}\pi, \quad \lambda_7 = \frac{-2}{3}\sqrt{\frac{2}{19}}\pi, \quad (3.8)$$

with all other v 's and λ 's vanishing.

The K4 invariant fixed point with all three scalars fully coupled is located at

$$v_3 = -\frac{2\pi}{5\sqrt{29}}, \quad \lambda_4 = -2\sqrt{\frac{53}{435}}, \quad \lambda_7 = \frac{-7}{3}\sqrt{\frac{2}{145}}\pi, \quad (3.9)$$

with all other v 's and λ 's vanishing.

The Z2 invariant fixed point with all three scalars fully coupled is located at

$$v_3 = 0.417419, \quad \lambda_1 = -2.19206, \quad \lambda_3 = 0.962964, \quad \lambda_5 = 0.401921, \quad \lambda_7 = 0.524619. \quad (3.10)$$

with all other v 's and λ 's vanishing.

Finally, another Z2 invariant fixed point with all three scalars fully coupled is located at

$$\begin{aligned} v_3 &= 0.708567, & \lambda_1 &= -1.08699, & \lambda_3 &= 0.0194906, \\ \lambda_4 &= 0.0337588, & \lambda_5 &= 0.13039, & \lambda_6 &= 0.225843, & \lambda_7 &= -0.63725. \end{aligned} \quad (3.11)$$

A summary of the previous analysis is presented in table 1.³ We can compare our results with the perturbative fixed points of scalar ϕ^3 theory in $6 - \epsilon$ dimension, which were studied in [17]. Our entries of table 1 turn out to be in one-to-one correspondence with the fixed points of scalar ϕ^3 theory in $6 - \epsilon$ dimensions. In $6 - \epsilon$ dimensions, only the O(2) fixed point and the K4 fixed point are perturbatively unitary. In our case, it seems that supersymmetry improves the situation significantly, and all the fixed points become perturbatively unitary.

4 $N_{\Phi} = 4$ irreducible fixed points

We now discuss the case when the scalar superfield transforms as an irreducible representation of the symmetry group. To construct the superpotential (1.1), the symmetry group needs to preserve a rank-3 fully symmetric invariant tensor d_{ijk}^a . The tensors d_{ijk}^a are necessarily traceless, such that the representation is reducible. Another important condition is that the invariant tensors satisfy⁴

$$d_{imn}^a d_{jmn}^b \propto \delta_{ij}. \quad (4.1)$$

³After submitting to the arxiv, we learned from O. Zanusso that the same table was also generated independently in [22].

⁴Otherwise one can build projectors of the form

$$d_{imn} d_{jmn} - e_1 \delta_{ij},$$

which projects the vector representation into invariant subspace. Here e_1 is the first eigenvalue of $d_{imn} d_{jmn}$.

This is analogous to the trace condition used in [19]. We can now normalize d_{ijk}^a to satisfy

$$d_{imn}^a d_{jmn}^b = \delta_{ab} \delta_{ij}. \quad (4.2)$$

If the symmetry group preserves a single d_{ijk} , the beta function and the anomalous dimension become

$$\begin{aligned} \beta_g &= -\frac{g}{2}\epsilon + \frac{4T_3 + 3}{32\pi^2}g^3, \\ \gamma_\Phi &= \frac{\epsilon}{8T_3 + 6}. \end{aligned} \quad (4.3)$$

Here T_3 is defined through

$$d_{imn}d_{jmp}d_{kpm} = T_3 d_{ijk}, \quad (4.4)$$

assuming d_{ijk} satisfies the normalization (4.2). It was shown in [16] that the constant satisfies⁵

$$-\frac{N_\Phi - 2}{2(N_\Phi + 2)} \leq T_3 \leq \frac{N_\Phi - 2}{N_\Phi - 1}. \quad (4.5)$$

For $N_\Phi \geq 1$, this implies $T_3 \geq -\frac{1}{2}$, and from the beta function (4.3) we know that the one-loop fixed point exists if and only if

$$T_3 > -\frac{3}{4}. \quad (4.6)$$

We therefore conclude that as long as there is a group that preserves a single invariant tensor d_{ijk} , there is a corresponding one-loop SUSY fixed point. At the interacting fixed point, the derivative of the beta function gives us

$$\gamma_g = \frac{\partial \beta_g}{\partial g} \Big|_{g=g_*} = \epsilon. \quad (4.7)$$

This means that the fixed point is RG stable.

The irreducible subgroups of $O(4)$ were studied extensively in [20]. It was discovered that only five of the groups have a four dimensional irreducible real representation that preserves a degree three polynomial.⁶ The result is recalled in table 3. The five polynomials are

$$\begin{aligned} I_1 &= x(x^2 - 3y^2) + z(z^2 - 3t^2), \\ I_2 &= y(y^2 - 3x^2) + t(t^2 - 3z^2), \\ I_3 &= t^3 - t(x^2 + y^2 + z^2) + 10/\sqrt{5}xyz, \\ I_4 &= x(z^2 - t^2) + t(x^2 - y^2) - 2yz(x + t), \\ I_5 &= x(x^2 - 3y^2) - t(t^2 - 3z^2). \end{aligned} \quad (4.8)$$

The corresponding fully symmetric invariant tensor can be easily constructed from these

⁵We thank H. Osborn for pointing out this bound.

⁶The representation is irreducible when the fields are real, but can be reducible when they are complex. In that case, the representation equals the sum of two complex conjugate representations.

SmallGroup Id	subgroup	order	invariant polynomial
[120,34]	S5	120	I_3
[72,40]	$(S3 \times S3) \rtimes Z2$	72	I_1
[36,10]	$S3 \times S3$	36	I_5
[20,3]	$Z5 \rtimes Z4$	20	I_4
[18,4]	$(Z3 \times Z3) \rtimes Z2$	18	I_1 and I_2

Table 3. Irreducible subgroups of $O(4)$ and the invariant polynomials preserved.

polynomials

$$d_{ijk}^a = \frac{\partial^3 I_a}{\partial x_i \partial x_j \partial x_k}, \quad \text{with } \vec{x} = \{x, y, z, t\}. \quad (4.9)$$

Of these five polynomials only two of them are independent. We can pick these two independent polynomials to be I_3 and I_1 . The polynomial I_4 is related to I_3 by an $O(3)$ rotation. The group that preserves I_4 is $Z5 \rtimes Z4$, which is, in fact, a subgroup of $S5$ (which preserves I_3). The two groups have the same degree three polynomial but different degree four polynomials. This means that to break the symmetry from $S5$ to $Z5 \rtimes Z4$, it is necessary to include quartic terms in the superpotential (1.1). Since we include only cubic terms (the quartic terms are not only irrelevant, but also break time-reversal symmetry), the symmetry of our Lagrangian will be $S5$. Also, I_5 is related to I_1 by an $O(3)$ rotation. Including either I_1 or I_5 in the superpotential (1.1) will preserve the symmetry $(S3 \times S3) \rtimes Z2$. The subgroup $(Z3 \times Z3) \rtimes Z2$ preserves two polynomials I_1 and I_2 . However, any combination of the two polynomial $aI_1 + bI_2$ can be brought back to the form cI_1 by an $O(3)$ rotation (the rotation parameters depend on a and b). This means a superpotential of the form $\mathcal{W} = g_1 I_1 + g_2 I_2$ in fact preserves the symmetry $(S3 \times S3) \rtimes Z2$ at any of point of the (g_1, g_2) -plane. This of course is also true for the SCFT fixed point.

4.1 The fixed points and anomalous dimensions

We can now use the explicit form of the polynomial I_3 to study the $S5$ invariant fixed point. Plugging I_3 into (4.9), rescaling it to satisfy the normalization (4.2), and then plugging it in (4.4) we get

$$T_3 = \frac{2}{3}. \quad (4.10)$$

Since $T_3 > -\frac{3}{4}$, the fixed point exists, and from (4.3) we know

$$\gamma_\Phi = \frac{3}{34}\epsilon. \quad (4.11)$$

This model belongs to the family of $\mathcal{N}=1$ Potts models studied in [10].

Similarly, we can also use the polynomial I_1 to study the $(S3 \times S3) \rtimes Z2$ invariant fixed point, we get

$$T_3 = 0. \quad (4.12)$$

So that

$$\gamma_{\Phi} = \frac{1}{6}\epsilon. \quad (4.13)$$

Consider a superpotential $\mathcal{W} = gI_1$, we know that

$$\mathcal{W} = g(\phi^3 + \chi^3 + c.c.), \quad (4.14)$$

with

$$\phi = x + iy, \quad \chi = z + it. \quad (4.15)$$

So that the model is simply two decoupled copies of the S3 super Potts model.

5 $N_{\Phi} = 5$ irreducible fixed points

5.1 The Lie group O(3)

Among the Lie subgroups of O(5), there is only one subgroup where the 5 dimensional vector irrep remains irreducible. This is the group O(3), under this embedding, the 5 of O(5) becomes the T irrep of O(3).⁷ The invariant tensor of such a representation can be constructed using the δ_{ij} of O(3), we will postpone the details of the construction to section 6.2, where we discuss the generalization of this fixed point to a series of SCFTs with the scalar superfields transforming in the T irrep of O(N). We note down here the T_3 constant,

$$T_3 = -\frac{3}{14}. \quad (5.1)$$

The anomalous dimension is

$$\gamma_{\phi}/\epsilon = \frac{7}{30}, \quad (5.2)$$

and the corresponding A function reads

$$A/\epsilon^2 = -\frac{1}{3}(98\pi^2). \quad (5.3)$$

Notice the Z2 transformation

$$Z = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad (5.4)$$

acts trivially on the T irrep of O(3), so that the symmetry of the fixed point is

$$O(3)/Z_2. \quad (5.5)$$

5.2 Finite subgroups of O(5) up to order 800 with 5 dimensional faithful irreps

We now use the GAP system [23] for computational discrete algebra and the Small Groups library [24] to search for finite groups with 5 dimensional faithful irreducible representations, similar to the study done in [25]. Since all representations of finite groups are unitary representations, these finite groups are subgroups of U(5). There are two theorems that are especially useful in seeking faithful irreducible representations [25].

- **Theorem 1** Suppose a finite group G has a p dimensional irreducible representation, then $\text{Ord}(G)/p$ is an integer.

⁷We denote the symmetric traceless, antisymmetric and singlet irrep of O(N) by T, A and S respectively.

- **Theorem 2** Suppose a finite group G has a p dimensional faithful irreducible representation, then this irreducible representation has a single character p in the character table.

The proof of **Theorem 1** can be found in [26] Page 288. The proof of **Theorem 2** was given in [25] appendix A.1. The SmallGroup library allows one to specify a finite group using two integers $[p, q]$, the first integer indicates the order of the group, while the second integer enumerates all finite groups with order p . We first select finite groups that are not Abelian. For example, the following command

$$l := \text{Filtered}(\text{AllSmallGroups}(60), x \rightarrow \text{IsAbelian}(x) = \text{false}); \quad (5.6)$$

gives all non-Abelian groups with order 60. The GAP system also allows us to calculate the character table easily

$$\text{Display}(\text{CharacterTable}(\text{SmallGroup}(60, 5))); \quad (5.7)$$

To select 5 dimensional irreps, we can use

$$\text{psi} := \text{Filtered}(\text{Irr}(\text{CharacterTable}(\text{SmallGroup}(60, 5))), x \rightarrow \text{Degree}(x) = 5); \quad (5.8)$$

The symbol psi is now a list which contains all the characters of the 5 dimensional irreps:

$$\begin{aligned} \text{gap} > \text{psi}[1]; \\ > \text{Character}(\text{CharacterTable}(\text{Alt}([1..5])), [5, -1, 1, 0, 0]) \end{aligned} \quad (5.9)$$

The characters contain a single 5, so according to Theorem 2 this is a faithful irrep. It is sometimes useful to check the structure description of the finite group

$$\text{StructureDescription}(\text{SmallGroup}(60, 5)); \quad (5.10)$$

From the output, we learn that $\text{SmallGroup}(60, 5)$ is the alternating group of five elements, or “A5”. Since our goal is to search for degree 3 invariant polynomials, we can also use the command

$$\text{MolienSeries}(\text{psi}[1]); \quad (5.11)$$

to obtain the Molien Series of the corresponding irrep. The Molien series

$$M(z) = \frac{1}{|G|} \sum_{g \in G} \frac{1}{\det[\mathbb{I} - z\rho(g)]} \quad (5.12)$$

is a generating function which counts the number of invariant polynomials of a certain degree. Here $|G|$ is the order of the group G , $\rho(g)$ is a representation of G . Doing a series expansion of $M(z)$, the coefficient of z^n counts the number of invariant polynomials of degree n . The Molien series of the five-dimensional irrep of A5 is

$$\frac{z^{10} - z^8 + z^6 + z^5 + z^4 - z^2 + 1}{(z^2 - 1)^2 (z^3 - 1)^2 (1 - z^5)} = 1 + z^2 + 2z^3 + 2z^4 + O(z^5). \quad (5.13)$$

ID	Group	Irrep	Molien Series	Molien Series (expansion)
[60, 5]	A5	5	$\frac{z^{10}-z^8+z^6+z^5+z^4-z^2+1}{(1-z^2)^2(1-z^3)^2(1-z^5)}$	$1+z^2+2z^3+2z^4+O(z^5)$
[120, 34]	S5	5 _a	$\frac{z^{12}+z^{10}+z^9+z^8+z^6+1}{(1-z^2)(1-z^3)(1-z^4)(1-z^5)}$	$1+z^2+z^3+2z^4+O(z^5)$
		5 _b	$\frac{z^{14}-z^{13}+z^{12}-z^{11}+z^{10}-z^9+2z^8-z^7+2z^6-z^5+z^4-z^3+z^2-z+1}{(1-z)(1-z^3)(1-z^4)(1-z^5)(1-z^6)}$	$1+z^2+z^3+2z^4+O(z^5)$
[360,188]	A6	5 _{std}	$\frac{z^{12}-z^9+z^6-z^3+1}{(1-z^2)(1-z^3)^2(1-z^4)(1-z^5)}$	$1+z^2+z^3+2z^4+O(z^5)$
		5'	$\frac{z^{12}-z^9+z^6-z^3+1}{(1-z^2)(1-z^3)^2(1-z^4)(1-z^5)}$	$1+z^2+z^3+2z^4+O(z^5)$
[720,763]	S6	5 _{std}	$\frac{1}{(1-z^2)(1-z^3)(1-z^4)(1-z^5)(1-z^6)}$	$1+z^2+z^3+2z^4+O(z^5)$
		5'	$\frac{1}{(1-z^2)(1-z^3)(1-z^4)(1-z^5)(1-z^6)}$	$1+z^2+z^3+2z^4+O(z^5)$

Table 4. Finite groups and their irreducible representation that preserves a degree two invariant polynomial and at least one degree three invariant polynomial.

So that it has one degree two invariant polynomial, two degree three polynomials and two degree four polynomials.

There are more than 10^7 finite groups of order less or equal than 800. Among them, we found only 109 groups that have at least one 5 dimensional faithful irrep. They are listed in table 5. Among these 109 finite groups, it turns out 13 of them are also subgroups of $O(5)$. That is, the finite groups preserve a bilinear form, or in other words, their Molien series takes the form

$$1 + z^2 + O(z^3). \quad (5.14)$$

These groups are marked with an “*” in table 5. Among the 13 finite subgroups of $O(5)$, only four of them have irreducible representations that preserve at least one degree three invariant polynomial. The results are listed in table 4. For a fixed N_Φ , bigger groups tend to preserve less invariant polynomials, which leads us to conjecture that this list is complete.⁸

5.2.1 The group A5

The `SmallGroup([60,5])` is the alternating group of five elements. As is explicit from the Molien series, it has a five dimensional irrep, which has a degree two invariant polynomial and two degree five polynomials. To work out the explicit form of the polynomial, after running command (5.8), we can use

$$\text{IrreducibleRepresentationsDixon}(\text{SmallGroup}(60,5), \text{psi}[1]); \quad (5.15)$$

to calculate a matrix representation of the generators of the group

$$\tilde{g}_1 = (1, 2, 3, 4, 5) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & -1 \\ 1 & -1 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \quad \tilde{g}_2 = (1, 2, 3, 4, 5) = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & -1 & 0 & -1 \\ 0 & -1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \quad (5.16)$$

⁸We actually extended the analysis up to order 1000 and our final result does not change.

Here (1,2,3,4,5) and (1,2,3) denote the cyclic permutations that generate the alternating group. We can now calculate the invariant tensor using these matrices, for example, we can use the Mathematica function “NullSpace[]”. Notice that the generators from “IrreducibleRepresentationsDixon” are not necessarily orthogonal. The bi-linear form

$$\Omega = \begin{pmatrix} 1 & \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ \frac{1}{3} & 1 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & 1 & \frac{1}{3} & -\frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} & 1 & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 1 \end{pmatrix}, \quad (5.17)$$

that is preserved by g_1 and g_2 is not proportional to the identity matrix. To fix this, we perform the Cholesky Decomposition of $\Omega = L^T L$. Then the new representation of the generators in terms of the orthogonal matrices is

$$g_1 = (L^T)^{-1} \tilde{g}_1 L, \quad g_2 = (L^T)^{-1} \tilde{g}_2 L. \quad (5.18)$$

In this basis, we found the two degree three invariant polynomials to be

$$\begin{aligned} K_1 = & -\frac{3}{10}x \left(y \left(-5t - \sqrt{15}w + 2\sqrt{15}z \right) + (2w + z) \left(\sqrt{15}t - w + 2z \right) \right) \\ & + \frac{\sqrt{2}}{200} \left(50t^3 + 30tw^2 + 12\sqrt{15}w^3 - 12wz \left(\sqrt{15}w - 15t \right) - 21z^2 \left(5t + \sqrt{15}w \right) \right. \\ & \left. - 15y^2 \left(-5t - \sqrt{15}w + 2\sqrt{15}z \right) - 30y(2w + z) \left(\sqrt{15}t - w + 2z \right) + 6\sqrt{15}z^3 \right), \end{aligned} \quad (5.19)$$

and

$$\begin{aligned} K_2 = & \frac{1}{10}x \left(-15t^2 + w^2 + 16wz + 5y^2 - 11z^2 \right) + \frac{2}{3}x^3 \\ & + \frac{\sqrt{2}}{600} \left(250y^3 - 15y \left(-15t^2 + 4z \left(7w - 3\sqrt{15}t \right) + 6\sqrt{15}tw + 43w^2 + 22z^2 \right) \right. \\ & \left. + 9(2w + z) \left(-5\sqrt{15}t^2 - 4z \left(\sqrt{15}w - 5t \right) - 10tw + \sqrt{15}w^2 + 4\sqrt{15}z^2 \right) \right). \end{aligned} \quad (5.20)$$

5.2.2 The group S5

The permutation group of five elements has two five dimensional irreps denoted by the Young diagram and . For simplicity, we will denote them as 5_a and 5_b . The

group A5 can be embedded in S5 through the standard way. That is, S5 contains all the permutations of five elements, while A5 contains even permutations that permute the five elements. Under this embedding, both 5_a and 5_b branch into the 5 dimensional irrep of A5.

$$\begin{aligned} 5_a & \rightarrow 5, \\ 5_b & \rightarrow 5. \end{aligned} \quad (5.21)$$

One can similarly work out the invariant polynomials by using the generators from “IrreducibleRepresentationsDixon”. It turns out that the invariant polynomial of 5_a is related to K_1 by an O(5) rotation, while the invariant polynomial of 5_b is related to K_2 by an O(5) rotation.

5.2.3 The group A6 and S6

The group S6 contains all permutations of six elements. There are two 5 dimensional irreps that preserve a degree three polynomial. One of them is the standard irrep of permutation groups “ 5_{std} ”. We will denote the other irrep as “ $5'$ ”. As is clear from table 4, the two irreps in fact have the same Molien series. This is only possible if the two irreps have exactly the same invariant polynomials. Physically, this means a potential $V(\phi)$ with ϕ transforming in the 5_{std} irrep can be also interpreted as a potential with ϕ transforming in the $5'$ irrep.

The symmetric group S5 can be embedded in S6 in the standard way. While S6 contains all the permutations that permute six elements, S5 contains all the permutations that permute five of the six elements. Under this embedding

$$\begin{aligned} 5_{std} &\rightarrow 4_{std} + 1, \\ 5' &\rightarrow 5_a. \end{aligned} \tag{5.22}$$

An invariant polynomial of the parent group S6 should also be an invariant polynomial of the subgroup S5. We therefore know the degree three invariant polynomial of $5'$ is simply K_1 . Since 5_{std} and $5'$ have the same Molien series, the invariant polynomial of 5_{std} should also be K_1 .

From table 4, we know the Molien series of the group A6 and S6 are the same up to z^4 terms. Physically, this means we can not break the symmetry from S6 to A5 until we introduce Φ^5 terms in the (super-)potential. Since we consider only cubic terms of the superpotential, we should consider only the group S6.

5.3 The beta functions and fixed points

We can now consider the beta function of the $\mathcal{N}=1$ Lagrangian with superpotential

$$\mathcal{W} = g_1 K_1 + g_2 K_2, \tag{5.23}$$

with K_1 and K_2 defined in (5.19) and (5.20). The beta function is

$$\begin{aligned} \beta_1(g_1, g_2) &= -\frac{\epsilon}{2}g_1 + \frac{1}{16\pi^2}3g_1^3, \\ \beta_2(g_1, g_2) &= -\frac{\epsilon}{2}g_2 + \frac{1}{16\pi^2}\frac{5g_2^3}{3}. \end{aligned} \tag{5.24}$$

and the corresponding anomalous dimension is

$$\gamma_\Phi/\epsilon = \frac{1}{2}(g_1^2 + g_2^2). \tag{5.25}$$

We find the following fixed points:

A fixed point at $(\frac{g_1}{4\pi\sqrt{\epsilon}} = 0, \frac{g_2}{4\pi\sqrt{\epsilon}} = \pm\sqrt{\frac{3}{10}})$ that preserves S5 symmetry. It has $\gamma_\Phi/\epsilon = \frac{3}{20}$ and $A/\epsilon = -21\pi^2$. The eigenvalues of matrix $\frac{\partial\beta_i}{\partial g_j}$ turn out to be $\gamma_1 = -\frac{\epsilon}{2}$ and $\gamma_2 = \frac{\epsilon}{25}$, so that this fixed point is unstable.

A fixed point at $(\frac{g_1}{4\pi\sqrt{\epsilon}} = \pm\frac{1}{\sqrt{6}}, \frac{g_2}{4\pi\sqrt{\epsilon}} = 0)$ that preserves S6 symmetry. It has $\gamma_\Phi/\epsilon = \frac{1}{12}$ and $A/\epsilon^2 = -\frac{1}{3}(35\pi^2)$. The eigenvalues of matrix $\frac{\partial\beta_i}{\partial g_j}$ turn out to be $\gamma_1 = \epsilon$ and $\gamma_2 = -\frac{\epsilon}{2}$, so that this fixed point is also unstable.

A fixed point at $(\frac{g_1}{4\pi\sqrt{\epsilon}} = \pm\frac{1}{\sqrt{6}}, \frac{g_2}{4\pi\sqrt{\epsilon}} = \pm\sqrt{\frac{3}{10}})$ that instead of the A5 symmetry, preserves $O(3)/Z_2$ symmetry. The superpotential takes the form

$$\mathcal{W} = \sqrt{\frac{7}{15}} \left(\sqrt{\frac{5}{14}} K_1 + \frac{3}{\sqrt{14}} K_2 \right), \quad (5.26)$$

The polynomial

$$\sqrt{\frac{5}{14}} K_1 + \frac{3}{\sqrt{14}} K_2, \quad (5.27)$$

is related to the invariant polynomial of $O(3)/Z_2$ by an $O(5)$ rotation. It has $\gamma_\Phi/\epsilon = \frac{7}{30}$ and $A/\epsilon^2 = -\frac{1}{3}(98\pi^2)$. The eigenvalues of matrix $\frac{\partial\beta_i}{\partial g_j}$ turn out to be $\gamma_1 = \epsilon$ and $\gamma_2 = \frac{\epsilon}{25}$, so that this fixed point is a stable fixed point. These A5 invariant RG flows have an IR fixed point with emergent $O(3)/Z_2$ symmetry.

6 Generalization to higher N_Φ

6.1 $SN \times SN$ bi-standard fixed points

We discuss here the irreducible fixed points first. The decoupled S3 Potts model studied in section 4.1 can also be understood as an SCFT with the four scalar superfields Φ_i transforming in the bi-standard representation of $S_3 \times S_3$. This SCFT can be easily generalized to a higher number of scalar superfields, if we take the scalars to transform in the $(r_1, \dots, r_n)$ representation of $G_1 \times \dots \times G_n$. The constant T_3 defined in (4.4) is simply the product of the T_3 's of the individual G 's,

$$T_3 = T_3^{(G_1)} \times \dots \times T_3^{(G_n)}. \quad (6.1)$$

Take the group to be $SN \times SN$, we have then $T_3 = (\frac{N-3}{N-2})^2$, so that

$$\gamma_\Phi = \frac{(N-2)^2}{2N(7N-36)+96} \epsilon, \quad (6.2)$$

and the A-function is

$$A/\epsilon^2 = -28\pi^2 \frac{(N-2)^2(N-1)^2}{2N(7N-36)+96}. \quad (6.3)$$

6.2 $O(N)/Z_2$ fixed points

We now try to generalize the $O(3)/Z_2$ fixed point studied in section 5.1 to a family of fixed points with $O(N)/Z_2$ symmetry. The scalar superfields Φ_{ij} carry indices in the T irrep of $O(N)$, such that these fixed points have $N_\Phi = \frac{N(N+1)}{2} - 1$ interacting scalar superfields. The product of two vector irreps of $O(N)$ can be decomposed into the S, A and T irreps:

$$\text{---} = \frac{1}{N} \text{---} \text{---} + \text{---} \text{---} + \text{---} P_T \text{---}.$$

The projector to the T irrep is

$$\text{Diagram with } P_T \text{ box} = \text{Diagram with empty box} - \frac{1}{N} \text{Diagram of two lines meeting and splitting} \quad (6.4)$$

Here we use the birdtrack notation [27]. Solid lines mean the Kronecker δ_{ij} , the unfilled box means symmetrization, and the filled box means anti-symmetrization. The invariant tensor for the T irrep of $O(N)$ is simply,⁹

$$\text{Y-junction} = \text{Diagram with three } P_T \text{ boxes} \quad (6.5)$$

We rescale the invariant tensor to satisfy the normalization (4.2), which can now be represented as

$$\text{Loop of two lines} = \text{Double line} \quad (6.6)$$

After contracting the Kronecker delta's, we get

$$\text{Triangle diagram} = T_3 \text{ Y-junction} \quad (6.7)$$

with

$$T_3 = \frac{N^2 + 4N - 24}{2(N^2 + 2N - 8)} \quad (6.8)$$

The anomalous dimension is then

$$\gamma_\Phi = \frac{N^2 + 2N - 8}{2(5N^2 + 14N - 72)} \epsilon \quad (6.9)$$

and the A-function is

$$A/\epsilon^2 = -\frac{7\pi^2(N-1)N(N^2 + 2N - 8)}{5N^2 + 14N - 72} \quad (6.10)$$

As already mentioned in section 5.1, since the scalar superfields Φ_{ij} carry an $O(N)$ T index, the central element of $O(N)$

$$Z = \text{Diag}\{-1, \dots, -1\}, \quad (6.11)$$

acts trivially on Φ_{ij} . The flavor symmetry of the SCFTs is then

$$O(N)/Z_2 \quad (6.12)$$

⁹For $O(3)$ a similar superpotential was considered in [28] in the context of $\mathcal{N} = 2$ SCFTs.

6.3 Generalization of A5

The group A5 is very interesting, as far as we know, this is the first example of a group that has an irrep that satisfies the trace condition (preserves a single degree two polynomial) and preserves more than one degree three polynomial. To generalize this to the higher-component case we use the birdtrack technique [27]. To be more explicit, we search for subgroups of SN groups that have an irrep with the Molien series

$$1 + z^2 + 2z^3 + 2z^4 + O(z^5). \quad (6.13)$$

Our calculation closely resembles the birdtrack classification of F4 family of Lie groups given in chapter 19 of [27]. We leave the details of the calculation to appendix A.

The birdtrack constrains allow only two solutions. One solution has $N_\phi = 5$, which is the group A5. Another solution has $N_\phi = 26$. We are, however, not able to identify the corresponding symmetry group. Let us assume that such a group exists and calculate the beta function and anomalous dimensions. Take the superpotential to be

$$\mathcal{W} = g_1 \text{---} \text{Y} + g_2 \text{---} \text{X} . \quad (6.14)$$

(For the definition of the diagrams, see appendix A.) We get

$$\begin{aligned} \beta_1(g_1, g_2) &= -\frac{\epsilon}{2}g_1 + \frac{1}{16\pi^2} \frac{171}{50} g_1^3, \\ \beta_2(g_1, g_2) &= -\frac{\epsilon}{2}g_2 + \frac{1}{16\pi^2} \frac{19g_2^3}{24}, \end{aligned} \quad (6.15)$$

and the corresponding anomalous dimension is

$$\gamma_\Phi/\epsilon = \frac{1}{2} \frac{1}{4\pi^2} (g_1^2 + g_2^2). \quad (6.16)$$

We find the following fixed points:

A fixed point at $(\frac{g_1}{4\pi\sqrt{\epsilon}} = 0, \frac{g_2}{4\pi\sqrt{\epsilon}} = \pm 2\sqrt{\frac{3}{19}})$ has $\gamma_\Phi/\epsilon = \frac{6}{19}$ and $A/\epsilon^2 = -\frac{1}{19} (4368\pi^2)$. The eigenvalues of the matrix $\partial\beta_i/\partial g_j$ turn out to be $\gamma_1 = -\epsilon/2$ and $\gamma_2 = 121\epsilon/190$, so that this fixed point is unstable.

A fixed point at $(\frac{g_1}{4\pi\sqrt{\epsilon}} = \pm \frac{5}{3\sqrt{19}}, \frac{g_2}{4\pi\sqrt{\epsilon}} = 0)$ preserves S27 symmetry. It has $\gamma_\Phi/\epsilon = \frac{25}{342}$ and $A/\epsilon^2 = -\frac{1}{171} (9100\pi^2)$. The eigenvalues of the matrix $\partial\beta_i/\partial g_j$ turn out to be $\gamma_1 = 31\epsilon/38$ and $\gamma_2 = -\epsilon/2$, so that this fixed point is also unstable.

A fixed point at $(\frac{g_1}{4\pi\sqrt{\epsilon}} = \pm \frac{5}{3\sqrt{19}}, \frac{g_2}{4\pi\sqrt{\epsilon}} = \pm 2\sqrt{\frac{3}{19}})$ has $\gamma_\Phi/\epsilon = \frac{7}{18}$ and $A/\epsilon^2 = -\frac{1}{9} (2548\pi^2)$. The eigenvalues of the matrix $\partial\beta_i/\partial g_j$ turn out to be $\gamma_1 = 31\epsilon/38$ and $\gamma_2 = 121\epsilon/190$. This fixed point is stable.

As already mentioned, our calculation closely resembles the birdtrack classification of F4 family of Lie groups [27]. Notice the birdtrack rules (A.7) and (A.8) are very similar to the rules that lead to the classification of the F4-family of Lie groups, see equation (19.16) of [27]. Other birdtrack conditions lead to the classification of E6, E7 and E8 family of

Lie Groups. Deligne conjectured that there exist categories interpolating these exceptional groups [29, 30]. See [31] on how to use the Deligne category to make sense of $O(N)$ invariant quantum field theories at non-integer N . It is tempting to conjecture that (A.7) and (A.8) also lead to a new Deligne category. We leave this for future work.

6.4 The $\mathcal{N}=1$ Potts models

The $\mathcal{N}=1$ Potts models were already studied in [10]. We recall here the procedure to construct the invariant tensor d_{ijk} according to [32]. In [10], the d_{ijk} were constructed explicitly and used to calculate γ_Φ to two loops in ϵ . The result was shown to be consistent with the result from the non-perturbative bootstrap. Take the scalar superfield Φ_i to transform in the $N-1$ dimensional standard representation of SN . First, we need to construct e_i^α with $\alpha = 1 \dots N$ and $i = 1 \dots N-1$. The e^α is a vector in the $N-1$ dimensional Euclidean space $\mathbb{R}^{N-1}$. The index α labels the N vertices of a hypertetrahedron. The e_i^α can be constructed through a recursion relation,

$$\begin{aligned} e_i^\alpha &= \sin(\theta_{N-1}) \tilde{e}_i^\alpha & \text{for } \alpha = 1 \dots N-1; \quad i = 1 \dots N-2, \\ e_N^\alpha &= \cos(\theta_{N-1}) & \text{for } \alpha = 1 \dots N-1, \\ e_i^N &= \delta_{N-1,i}, \end{aligned} \tag{6.17}$$

with $\cos(\theta_{N-1}) = -1/(N-1)$. Here $\tilde{e}_i^\alpha$ is the set of $N-1$ vectors in $\mathbb{R}^{N-2}$. The recursion starts with $e^1 = \{1\}$ and $e^2 = \{-1\}$. The invariant tensor d_{ijk} is defined as,

$$d_{ijk} = \sqrt{\frac{(N-1)^3}{(N-2)N^2}} \sum_{\alpha=1}^N e_i^\alpha e_j^\alpha e_k^\alpha. \tag{6.18}$$

The overall constant is chosen so that the invariant tensor satisfies the normalization (4.2). From the above definition of d_{ijk} , we get

$$\begin{aligned} T_3 &= \frac{N-3}{N-2}, \\ \gamma_\Phi &= \frac{N-2}{14N-36} \epsilon, \end{aligned} \tag{6.19}$$

and

$$A/\epsilon^2 = -\frac{28\pi^2(N-2)(N-1)}{14N-36}. \tag{6.20}$$

6.5 Generalization of reducible fixed points

Take the superpotential to be

$$W = \lambda_1 H \sum_{i=1}^{N-1} \Phi_i \Phi_i + \lambda_2 d_{ijk} \Phi_i \Phi_j \Phi_k + \lambda_3 H^3. \tag{6.21}$$

The β functions for the three couplings turn out to be

$$\begin{aligned}\beta_{\lambda_1} &= -\frac{\epsilon}{2}\lambda_1 + \frac{1}{(4\pi)^2} \frac{1}{18} \left(7\lambda_1^3 + 12\lambda_3\lambda_1^2 + 54\lambda_2^2\lambda_1 + 9\lambda_3^2\lambda_1 + \lambda_1^3N \right), \\ \beta_{\lambda_2} &= -\frac{\epsilon}{2}\lambda_2 + \frac{1}{(4\pi)^2} \frac{1}{2} \left(3\lambda_2^3 + 2\lambda_1^2\lambda_2 + 4\lambda_2^3T_3 \right), \\ \beta_{\lambda_3} &= -\frac{\epsilon}{2}\lambda_3 + \frac{1}{(4\pi)^2} \frac{1}{54} \left(-4\lambda_1^3 - 9\lambda_3\lambda_1^2 + 189\lambda_3^3 + 4\lambda_1^3N + 9\lambda_3\lambda_1^2N \right).\end{aligned}\quad (6.22)$$

The anomalous dimensions are given by

$$\begin{aligned}\gamma_H &= \frac{1}{(4\pi)^2} \frac{1}{18} \left(9\lambda_3^2 + \lambda_1^2(N-1) \right), \\ \gamma_\Phi &= \frac{1}{(4\pi)^2} \left(\frac{\lambda_1^2}{9} + \frac{\lambda_2^2}{2} \right).\end{aligned}\quad (6.23)$$

The constant T_3 is defined in (4.4). Take d_{ijk} to be the invariant tensor of the standard irrep of SN (6.18). At

$$\frac{\lambda_1}{4\pi\sqrt{\epsilon}} = -\frac{3}{\sqrt{N(7N-11)}}, \quad (6.24)$$

$$\frac{\lambda_2}{4\pi\sqrt{\epsilon}} = -\frac{\sqrt{(N-2)(N+1)(7N-18)}}{\sqrt{N(7N-18)(7N-11)}}, \quad (6.25)$$

$$\frac{\lambda_3}{4\pi\sqrt{\epsilon}} = \frac{N-1}{\sqrt{N(7N-11)}}, \quad (6.26)$$

the scalar superfield H and Φ_i together form the N dimensional standard representation of $SN+1$. This corresponds to the $SN+1$ Potts model studied in the previous subsection.

6.5.1 The $O(N-1)$ fixed points with $\gamma_H \neq \gamma_\Phi$

When $\lambda_2 = 0$, for any value of N , there exists only one perturbatively unitary $O(N-1)$ invariant fixed point with real coupling constant. This has been studied in [5]. The scalar superfields Φ_i transform in the vector representation of $O(N-1)$ while H is invariant under an $O(N-1)$ transformation. The value of the coupling at the fixed point can be worked out analytically. It is however rather complicated and not very illuminating. Let us record here the large N result,

$$\begin{aligned}\frac{\lambda_1}{4\pi\sqrt{\epsilon}} &= \frac{1}{\sqrt{N}} \left(-3 + \frac{9}{2N} - \frac{81}{8N^2} + \mathcal{O}\left(\frac{1}{N^3}\right) \right), \\ \frac{\lambda_3}{4\pi\sqrt{\epsilon}} &= \frac{1}{\sqrt{N}} \left(2 - \frac{27}{N} + \frac{3915}{4N^2} + \mathcal{O}\left(\frac{1}{N^3}\right) \right).\end{aligned}\quad (6.27)$$

The anomalous dimension is given by

$$\begin{aligned}\gamma_H/\epsilon &= \frac{1}{2} - \frac{48}{N^2} + \frac{36999}{16N^3} + \mathcal{O}\left(\frac{1}{N^3}\right), \\ \gamma_\Phi/\epsilon &= \frac{1}{N} - \frac{3}{N^2} + \frac{9}{N^3} + \mathcal{O}\left(\frac{1}{N^3}\right),\end{aligned}\quad (6.28)$$

and the A-function is

$$A/\epsilon^2 = -42\pi^2 + \frac{112\pi^2}{N} + \frac{1008\pi^2}{N^2} - \frac{257985\pi^2}{4N^3} + \mathcal{O}\left(\frac{1}{N^3}\right). \quad (6.29)$$

In three dimensions, the large N limit of the fixed points is dual to $\mathcal{N}=1$ higher-spin theory on AdS_4 through AdS/CFT correspondence [33, 34].

6.5.2 SN fixed point with $\gamma_H \neq \gamma_\Phi$

The fixed point is located at

$$\frac{\lambda_1}{4\pi\sqrt{\epsilon}} = -\frac{18\sqrt{N-6}}{\sqrt{N(N(7N-18)-864)}}, \quad (6.30)$$

$$\frac{\lambda_2}{4\pi\sqrt{\epsilon}} = \frac{\sqrt{(N-2)(7N-18)(N^2-216)}}{\sqrt{N(7N-18)(N(7N-18)-864)}}, \quad (6.31)$$

$$\frac{\lambda_3}{4\pi\sqrt{\epsilon}} = \frac{(N-6)^{3/2}}{\sqrt{N(N(7N-18)-864)}}. \quad (6.32)$$

(There are three equivalent fixed points related to the above one by field re-definition.) The scalar superfields Φ_i transform in the $N-1$ dimensional standard representation of SN while H is invariant under SN transformation. The anomalous dimensions are given by

$$\gamma_H = \frac{(N-6)(N+24)}{2(N(7N-18)-864)}\epsilon, \quad \gamma_\Phi = \frac{(N-2)N-144}{2(N(7N-18)-864)}\epsilon, \quad (6.33)$$

and the A-function is

$$A/\epsilon^2 = -\frac{14\pi^2 N((N-2)N-124)}{N(7N-18)-864}. \quad (6.34)$$

Notice there exists a non-conformal window given by

$$6 < N < 14.7, \quad (6.35)$$

within which the fixed point is non-unitary. The upper edge of the window is $6\sqrt{6} \approx 14.7$. At the edges of the window

$$\begin{aligned} N_{c,\text{lower}} = 6 : \quad & \lambda_1 = 0, \quad \lambda_2 = \frac{1}{\sqrt{6}}, \quad \lambda_3 = 0, \\ N_{c,\text{upper}} = 6\sqrt{6} : \quad & \lambda_1 = \frac{1}{\sqrt{2}}, \quad \lambda_2 = 0, \quad \lambda_3 = \frac{1}{6}(\sqrt{2} - 2\sqrt{3}). \end{aligned} \quad (6.36)$$

Clearly, at the lower edge of the non-conformal window, the fixed point collides with the decoupled SN Potts models and one free scalar theory. At the upper edge of the non-conformal window the fixed point collides with the $O(N-1)$ invariant fixed point.

The beta functions (6.22) have a few other fixed points. They are, however, decoupled combinations of either the super-Ising model, SN Potts model, or the free theory. We will not list them here. In figure 1, we plot the A-function of the fixed points to demonstrate their collisions.

Figure 1. A function for the $O(N-1)$ fixed point (red), the SN invariant Potts model+free superfield (green), and the new SN invariant fixed point (blue). The new SN invariant fixed point goes to the complex plane after collision with the other fixed points. The RG flow is therefore not conformal in the IR, leading to a non-conformal window.

7 Discussion

A natural generalization of the $\mathcal{N}=1$ analysis of this paper is to study $\mathcal{N}=2$ SCFTs. Consider the following $\mathcal{N}=2$ superpotential

$$\mathcal{W} = \frac{1}{6} h^{ijk} \Phi_i \Phi_j \Phi_k. \quad (7.1)$$

The β function is given by

$$\beta_{abc} = -\frac{\epsilon}{2} h^{ijk} + \gamma^i_m h^{mjk} + \gamma^j_m h^{mik} + \gamma^k_m h^{ijm}, \quad (7.2)$$

and the anomalous dimension matrix reads

$$\gamma^i_m = \frac{1}{2} h^{iab} h_{mab}. \quad (7.3)$$

The scalar superfield Φ is complex so that the coupling h^{ijk} transform in the fully symmetric representation of $U(N)$. By studying subgroups of $U(N)$ and analyzing their degree three invariant polynomials we can study the corresponding $\mathcal{N}=2$ superconformal fixed points. Among the 109 finite subgroups of $U(5)$ with 5 dimensional faithful irreducible representations, we found a single group whose Molien Series takes the form

$$1 + 2z^3 + O(z^4). \quad (7.4)$$

The group has a SmallGroup id [180,19] and structure description $GL(2,4)$. This is the general linear group of degree two over the field of four elements. Notice that the corresponding irrep has no degree two invariant polynomials, it is in fact a complex irrep. This irrep has two degree three polynomials. Using these two polynomials to construct an

$\mathcal{N}=2$ superpotential as in (7.1), and analyzing the beta function, we get a full conformal manifold. The group $GL(2,4)$ is in fact the direct product of A_5 and the cyclic group of order three Z_3 . The invariant polynomials of $GL(2,4)$ are simply K_1 and K_2 as defined in (5.19) and (5.20). The five variables $\{x, y, z, w, t\}$ are now complex. The action of Z_3 is

$$\{x, y, z, w, t\} \rightarrow e^{\frac{2\pi i}{3}} \{x, y, z, w, t\}. \quad (7.5)$$

It might be interesting to study this conformal manifold in more detail. A similar study of a conformal manifold containing the XYZ model was performed in [35].

Notice the degree three invariant polynomials studied in this paper can be used to construct Landau theories with cubic terms. The Landau criterion says that a second-order phase transition is possible only if the irrep of the order parameter has no degree three invariant polynomials. This however is known to fail in two dimensions. For example, the 3 state Potts model is known to go through a second order phase transition at the critical temperature. The corresponding Landau theory with a cubic term has an infrared conformal fixed point. It will be interesting to understand whether the Landau theories analogous to the SCFTs studied in this paper lead to unitary conformal field theories in two dimensions. The problem perhaps can be studied using conformal bootstrap techniques.

Due to the special form of the superpotential (1.1), we focus here on subgroups of $O(4)$ and $O(5)$ which preserve degree three invariant polynomials. It will also be interesting to study degree four polynomials, which can then be used to construct $\lambda\phi^4$ theories in $4 - \epsilon$ dimensions. The early works in the 80's focused on $\lambda\phi^4$ theories where the scalars form a single irreducible representation of the symmetry group. In [20, 36], all the subgroups of $O(4)$ and the corresponding $\lambda\phi^4$ theories were studied extensively. The work of [37–40] studied the $\lambda\phi^4$ theories with $N=6$ and $N=8$ scalars forming irreducible representations of the 230 crystallographic space groups (see also [41]). Recently, there was also revived interest on reducible fixed points of $\lambda\phi^4$ theories where the scalars form more than one irrep of the symmetry groups [16, 18, 42, 43]. Some of these perturbative fixed points were shown to survive in three dimensions using the numerical bootstrap [44–50]. A full classification of the irreducible $\lambda\phi^4$ theories with N scalars will need the classification of subgroups of $O(N)$, this can be difficult when N is large. The method used in section 5.2 based on the GAP system can be easily generalized to study $\lambda\phi^4$ theories. It will be interesting to see whether we can discover new fixed points in the $4 - \epsilon$ expansion.

As mentioned in the introduction, the numerical bootstrap method has proven to be very powerful in studying $2 + 1$ dimensional $\mathcal{N}=1$ SCFTs. In particular, the work of [10] studied three infinite families of Wess-Zumino models. It will be interesting to apply the numerical bootstrap to the new $O(N)/Z_2$ family discovered in this work. As usual, due to the non-perturbative nature of the numerical bootstrap, we expect rigorous numerical results that can then be compared to other methods such as the ϵ -expansion. We leave this interesting analysis for future work.

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A Classification of A5 family of groups

In this section, we search for subgroups of SN groups that have an irrep with the Molien series

$$1 + z^2 + 2z^3 + 2z^4 + O(z^5) . \quad (\text{A.1})$$

We decompose two vector irreps of $O(n)$ as S, A and T. If a subgroup of $O(n)$ preserves a rank three symmetric traceless tensor, the T irrep of $O(n)$ gets further decomposed into two irreps:

$$\begin{aligned}
 \text{Two parallel horizontal lines} &= \frac{1}{n} \left(\text{Two curved lines facing each other} \right) \\
 &+ \text{A thick vertical bar between two horizontal lines} \\
 &+ \text{A horizontal line with four lines branching out at the ends} \\
 &+ \left(\text{A thin vertical bar between two horizontal lines} - \frac{1}{n} \left(\text{Two curved lines facing each other} - \text{A horizontal line with four lines branching out at the ends} \right) \right) . \quad (\text{A.2})
 \end{aligned}$$

The invariant tensor of the standard irrep of SN satisfies a special condition. That is, the A irrep of $O(n)$ does not decompose. In term of birdtracks, it implies the following identity:

$$\text{A thick vertical bar between two horizontal lines with a vertical line in the middle} = -\frac{1}{n-1} \text{A thick vertical bar between two horizontal lines} . \quad (\text{A.3})$$

Here

$$n = N - 1 ,$$

[55, 1], $C_{11} \rtimes C_5$	*[60, 5], A_5
*[80, 49], $(C_2 \times C_2 \times C_2 \times C_2) \rtimes C_5$	[110, 2], $C_2 \times (C_{11} \rtimes C_5)$
*[120, 34], S_5	*[120, 35], $C_2 \times A_5$
[125, 3], $(C_5 \times C_5) \rtimes C_5$	[125, 4], $C_{25} \rtimes C_5$
[155, 1], $C_{31} \rtimes C_5$	*[160, 234], $((C_2 \times C_2 \times C_2 \times C_2) \rtimes C_5) \rtimes C_2$
*[160, 235], $C_2 \times ((C_2 \times C_2 \times C_2 \times C_2) \rtimes C_5)$	[165, 1], $C_3 \times (C_{11} \rtimes C_5)$
[180, 19], $GL(2,4)$	[205, 1], $C_{41} \rtimes C_5$
[220, 2], $C_4 \times (C_{11} \rtimes C_5)$	[240, 91], $A_5 \rtimes C_4$
[240, 92], $C_4 \times A_5$	*[240, 189], $C_2 \times S_5$
[240, 199], $C_3 \times ((C_2 \times C_2 \times C_2 \times C_2) \rtimes C_5)$	[250, 8], $((C_5 \times C_5) \rtimes C_5) \rtimes C_2$
[250, 10], $C_2 \times ((C_5 \times C_5) \rtimes C_5)$	[250, 11], $C_2 \times (C_{25} \rtimes C_5)$
[275, 1], $C_{11} \rtimes C_{25}$	[275, 3], $C_5 \times (C_{11} \rtimes C_5)$
[300, 22], $C_5 \times A_5$	[305, 1], $C_{61} \rtimes C_5$
[310, 2], $C_2 \times (C_{31} \rtimes C_5)$	[320, 1583], $((C_2 \times C_2 \times C_2 \times C_2) \rtimes C_5) \rtimes C_4$
[320, 1584], $C_4 \times ((C_2 \times C_2 \times C_2 \times C_2) \rtimes C_5)$	*[320, 1635], $((C_2 \times C_2 \times C_2 \times C_2) \rtimes C_5) \rtimes C_4$
*[320, 1636], $C_2 \times (((C_2 \times C_2 \times C_2 \times C_2) \rtimes C_5) \rtimes C_2)$	[330, 4], $C_6 \times (C_{11} \rtimes C_5)$
[355, 1], $C_{71} \rtimes C_5$	*[360, 118], A_6
[360, 119], $C_3 \times S_5$	[360, 122], $C_6 \times A_5$
[375, 2], $((C_5 \times C_5) \rtimes C_5) \rtimes C_3$	[375, 4], $C_3 \times ((C_5 \times C_5) \rtimes C_5)$
[375, 5], $C_3 \times (C_{25} \rtimes C_5)$	[385, 1], $C_7 \times (C_{11} \rtimes C_5)$
[400, 52], $(C_2 \times C_2 \times C_2 \times C_2) \rtimes C_{25}$	[400, 213], $C_5 \times ((C_2 \times C_2 \times C_2 \times C_2) \rtimes C_5)$
[405, 15], $(C_3 \times C_3 \times C_3 \times C_3) \rtimes C_5$	[410, 2], $C_2 \times (C_{41} \rtimes C_5)$
[420, 13], $C_7 \times A_5$	[440, 2], $C_8 \times (C_{11} \rtimes C_5)$
[465, 2], $C_3 \times (C_{31} \rtimes C_5)$	[480, 217], $A_5 \rtimes C_8$
[480, 220], $C_8 \times A_5$	[480, 943], $C_4 \times S_5$
[480, 1194], $C_3 \times (((C_2 \times C_2 \times C_2 \times C_2) \rtimes C_5) \rtimes C_2)$	[480, 1204], $C_6 \times ((C_2 \times C_2 \times C_2 \times C_2) \rtimes C_5)$
[495, 1], $C_9 \times (C_{11} \rtimes C_5)$	[500, 11], $((C_5 \times C_5) \rtimes C_5) \rtimes C_4$
[500, 13], $C_4 \times ((C_5 \times C_5) \rtimes C_5)$	[500, 14], $C_4 \times (C_{25} \rtimes C_5)$
[500, 25], $((C_5 \times C_5) \rtimes C_5) \rtimes C_4$	[500, 33], $C_2 \times (((C_5 \times C_5) \rtimes C_5) \rtimes C_2)$
[505, 1], $C_{101} \rtimes C_5$	[540, 31], $C_9 \times A_5$
[550, 2], $C_2 \times (C_{11} \rtimes C_{25})$	[550, 10], $C_{10} \times (C_{11} \rtimes C_5)$
[560, 173], $C_7 \times ((C_2 \times C_2 \times C_2 \times C_2) \rtimes C_5)$	[600, 144], $C_5 \times S_5$
[600, 147], $C_{10} \times A_5$	[605, 1], $C_{121} \rtimes C_5$
[605, 3], $C_{11} \times (C_{11} \rtimes C_5)$	[605, 5], $(C_{11} \times C_{11}) \rtimes C_5$
[605, 6], $(C_{11} \times C_{11}) \rtimes C_5$	[610, 2], $C_2 \times (C_{61} \rtimes C_5)$
[615, 1], $C_3 \times (C_{41} \rtimes C_5)$	[620, 2], $C_4 \times (C_{31} \rtimes C_5)$
[625, 6], $C_{125} \rtimes C_5$	[625, 7], $(C_5 \times C_5 \times C_5) \rtimes C_5$
[625, 8], $(C_{25} \rtimes C_5) \rtimes C_5$	[625, 9], $(C_{25} \times C_5) \rtimes C_5$
[625, 10], $(C_{25} \times C_5) \rtimes C_5$	[625, 14], $(C_{25} \times C_5) \rtimes C_5$
[640, 19097], $((C_2 \times C_2 \times C_2 \times C_2) \rtimes C_5) \rtimes C_8$	[640, 19103], $C_8 \times ((C_2 \times C_2 \times C_2 \times C_2) \rtimes C_5)$
[640, 21456], $((C_2 \times C_2 \times C_2 \times C_2) \rtimes C_5) \rtimes C_8$	[640, 21458], $C_4 \times (((C_2 \times C_2 \times C_2 \times C_2) \rtimes C_5) \rtimes C_2)$
*[640, 21536], $C_2 \times (((C_2 \times C_2 \times C_2 \times C_2) \rtimes C_5) \rtimes C_4)$	[655, 1], $C_{131} \rtimes C_5$
[660, 4], $C_{12} \times (C_{11} \rtimes C_5)$	[660, 13], $PSL(2,11)$
[660, 14], $C_{11} \times A_5$	[710, 2], $C_2 \times (C_{71} \rtimes C_5)$
[715, 1], $C_{13} \times (C_{11} \rtimes C_5)$	[720, 399], $C_9 \times ((C_2 \times C_2 \times C_2 \times C_2) \rtimes C_5)$
[720, 412], $C_3 \times (A_5 \rtimes C_4)$	[720, 419], $C_{12} \times A_5$
*[720, 763], S_6	*[720, 766], $C_2 \times A_6$
[720, 769], $C_6 \times S_5$	[750, 6], $((C_5 \times C_5) \rtimes C_5) \rtimes C_6$
[750, 7], $C_2 \times (((C_5 \times C_5) \rtimes C_5) \rtimes C_3)$	[750, 13], $C_3 \times (((C_5 \times C_5) \rtimes C_5) \rtimes C_2)$
[750, 24], $C_6 \times ((C_5 \times C_5) \rtimes C_5)$	[750, 25], $C_6 \times (C_{25} \rtimes C_5)$
[755, 1], $C_{151} \rtimes C_5$	[770, 4], $C_{14} \times (C_{11} \rtimes C_5)$
[775, 1], $C_{31} \rtimes C_{25}$	[780, 13], $C_{13} \times A_5$
[800, 384], $C_2 \times ((C_2 \times C_2 \times C_2 \times C_2) \rtimes C_{25})$	[800, 1195], $C_5 \times (((C_2 \times C_2 \times C_2 \times C_2) \rtimes C_5) \rtimes C_2)$
[800, 1203], $C_{10} \times ((C_2 \times C_2 \times C_2 \times C_2) \rtimes C_5)$	

Table 5. Finite subgroups of $U(5)$ of order up to 800 with faithful 5-dimensional irreducible representations. Groups marked with “*” are also subgroups of $O(5)$. The two integers [p,q] denote their id in Small Groups library, and p is the order of the finite group.

is the dimension of the standard irrep of SN. The coefficient is fixed by contracting the top two legs. From the above relation we can show that the invariant tensor of SN satisfies

$$\text{triangle diagram} = T_3 \text{ Y-junction}, \quad (\text{A.4})$$

with

$$T_3 = \frac{n-2}{n-1}. \quad (\text{A.5})$$

We now want to generalize the group A5 to a group with higher n . According to the Molien series, we first introduce another invariant tensor

$$\text{Y-junction with central dot} \quad (\text{A.6})$$

The Molien series also tells us that the irrep has only two degree four invariant polynomials. This means the two invariant tensors satisfy

$$\text{box with two loops} = A \text{ box with one loop} + A' \text{ box with two peaks}, \quad (\text{A.7})$$

and

$$\text{box with one loop and dot} = B \text{ box with one loop} + B' \text{ box with two peaks}. \quad (\text{A.8})$$

Contracting two legs with δ_{ij} , we get

$$A' = \frac{2(1-A)}{n+2}, \quad B' = \frac{2(-B)}{n+2}. \quad (\text{A.9})$$

Where we have used the normalization conditions,

$$\begin{aligned} \text{line with loop} &= \text{line}, \\ \text{line with loop and dot} &= \text{line}, \\ \text{line with loop and two dots} &= 0. \end{aligned} \quad (\text{A.10})$$

Contracting two legs of (A.7) with Y-junction , we get

$$\text{triangle with two dots} = M \text{ Y-junction}, \quad (\text{A.11})$$

with

$$M = \frac{An - 2A + 4}{2n + 4} + AT_3. \quad (\text{A.12})$$

From this we can prove

$$\text{Diagram 1} = M \text{Diagram 2}. \quad (\text{A.13})$$

The proof is given in appendix B. Contracting two legs of (A.7) with , we get

$$\text{Diagram 3} = P \text{Diagram 2}, \quad (\text{A.14})$$

with

$$P = \frac{(A^2 - 1)(n - 2)}{2(n + 2)} + A^2 T_3. \quad (\text{A.15})$$

Contracting two legs of (A.8) with , we obtain

$$\frac{2B}{3(n + 2)} + \frac{1}{6}(4M + 1) \text{Diagram 4} = (1/3)(1 + 2T_3)B \text{Diagram 2}. \quad (\text{A.16})$$

Since the two tensors are independent, we get

$$M = -\frac{1}{4} \quad \text{and} \quad B = 0. \quad (\text{A.17})$$

According to (A.12), this leads to

$$A = -\frac{(n - 1)(n + 10)}{6(n - 2)(n + 1)}. \quad (\text{A.18})$$

Using (A.7) and (A.8), after a long calculation, we get

$$\text{Diagram 5} = \frac{5n^2 - 71n + 86}{12(n - 2)(n + 1)} \text{Diagram 6} + \frac{66 - 15n}{-8n^2 + 8n + 16} \text{Diagram 7}. \quad (\text{A.19})$$

From this we know how the anti-symmetric subspace can be decomposed using the projectors

$$P_1 = \frac{(n + 1)(5n - 22)}{(n + 10)(5n - 13)} \text{Diagram 8} + \frac{12(-n^2 + n + 2)}{(n + 10)(5n - 13)} \text{Diagram 9}, \quad (\text{A.20})$$

and

$$P_2 = \frac{54(n-2)}{(n+10)(5n-13)} \text{ (diagram: two horizontal lines with a thick black vertical bar between them)} + \frac{12(n-2)(n+1)}{(n+10)(5n-13)} \text{ (diagram: two horizontal lines with a vertical line segment between them, each endpoint has a small circle)} . \quad (\text{A.21})$$

They satisfy

$$P_1 P_1 = P_1, \quad \text{and} \quad P_2 P_2 = P_2 . \quad (\text{A.22})$$

The dimension of the corresponding irrep is

$$d_1 = \frac{n(5n^3 - 10n^2 - 17n - 2)}{10n^2 + 74n - 260}, \quad d_2 = \frac{3n(7n^2 - 25n + 22)}{(n+10)(5n-13)} . \quad (\text{A.23})$$

We can calculate the diagram

$$\text{(diagram: a dashed circle with two dots on a horizontal line)} = \text{(diagram: a circle with two dots on a horizontal line, two arcs labeled } P_2 \text{ connect the dots)} = -\frac{n(7n-11)(55n-98)((n-43)n+118)}{16(13-5n)^2(n+10)^2} . \quad (\text{A.24})$$

Here we use to denote the Clebsch–Gordan coefficients $(T_i)_{ab}$ of $P_2 \times P_2 \rightarrow n$.

This diagram is a sum of squares,

$$\sum_{i,a,b} (T_i)_{ab} (T_i)_{ab} , \quad (\text{A.25})$$

and is therefore non-negative. This puts a constraint on the values that n can take. Notice that n is a positive integer, so that

$$3 \leq n \leq 40 . \quad (\text{A.26})$$

The dimensions of the irreps d_1 and d_2 also need to be positive integers, there are only two solutions of these Diophantine conditions, they are

$$\begin{aligned} n = 5, \quad d_1 = 4, \quad d_2 = 6, \\ n = 26, \quad d_1 = 249, \quad d_2 = 76. \end{aligned} \quad (\text{A.27})$$

The $n = 5$ solution corresponds to the group A5 we have studied, while the $n = 26$ solution might correspond to a subgroup of S27.

B Proof of (A.14)

Assume

$$\text{(diagram: a triangle with a dot at the left vertex and lines extending from the other two)} = a \text{ (diagram: a Y-junction)} + b \text{ (diagram: a Y-junction with a dot at the top vertex)} . \quad (\text{B.1})$$

Contracting from the right with and use (A.10), we get $a = 0$. So that we have

$$\text{Diagram 1} = b \cdot \text{Diagram 2} \quad (\text{B.2})$$

Contracting from the right with and use (A.11), we get

$$\text{Diagram 3} = M \cdot \text{Diagram 4} \quad (\text{B.3})$$

This completes the proof.

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