Chaos and pole skipping in CFT_2

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ABSTRACT: Recent work has suggested an intriguing relation between quantum chaos and energy density correlations, known as pole skipping. We investigate this relationship in two dimensional conformal field theories on a finite size spatial circle by studying the thermal energy density retarded two-point function on a torus. We find that the location $\omega_* = i\lambda$ of pole skipping in the complex frequency plane is determined by the central charge and the stress energy one-point function $\langle T \rangle$ on the torus. In addition, we find a bound on λ in $c > 1$ compact, unitary CFT_2 s identical to the chaos bound, $\lambda \leq 2\pi T$. This bound is saturated in large c CFT_2 s with a sparse light spectrum, as quantified by [1], for all temperatures above the dual Hawking-Page transition temperature.

KEYWORDS: AdS-CFT Correspondence, Holography and condensed matter physics (AdS/CMT), Conformal Field Theory

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1 Introduction

Recent years have seen tremendous progress in understanding the chaotic properties of many body quantum systems. While long believed to play a role in microscopic mechanisms for fundamental properties such as transport and thermalization, chaos has traditionally been difficult to quantify in general many body systems. Exciting developments following the study of chaos in gravitational contexts have illustrated the role of out-of-time-ordered correlation functions (OTOCs) in diagnosing chaotic properties in quantum systems [2–9]. The simplest and best studied example is the OTO four point function, given by the real time thermal correlation function

$$C(t, x) \equiv \langle W(t, x) V(0) W(t, x) V(0) \rangle_\beta, \quad (1.1)$$

with the subscript denoting a thermal expectation value computed at inverse temperature $\beta = T^{-1}$. Here V and W are simple local operators containing a small number of degrees of freedom, e.g. a primary of $\mathcal{O}(1)$ conformal weight in a conformal field theory (CFT). This correlation function governs the non-trivial time dependence of a commutator norm $\langle [W(t, x), V(0)]^2 \rangle$ and can be used to probe a quantum version of the butterfly effect, demonstrating how a change in initial conditions affects later measurements. It has been argued that, at least for some classes of quantum systems, $C(t, x)$ behaves as

$$C(t, x) \approx c_0 + c_1 e^{\lambda_L(t-x/v_B)}, \quad (1.2)$$

where λ_L is the quantum Lyapunov exponent and v_B is the speed characterizing how quickly the Lyapunov growth propagates through the system, referred to as the butterfly velocity.

Part of the excitement regarding chaos and the Lyapunov exponent arises from holographic considerations. General properties of thermal quantum correlation functions lead

to an upper bound on λ_L , $\lambda_L \leq \frac{2\pi}{\beta}$, with the bound saturated in holographic systems with black holes present in the bulk [7]. Since this maximal Lyapunov growth occurs rather generically in holographic systems in black hole backgrounds, chaotic properties can be used as a diagnostic of whether a given conformal field theory admits a simple holographic gravitational description [8, 10], especially in combination with other known criteria such as a large number of degrees of freedom and a sparse spectrum of low-lying operators [11, 12]. It is currently unknown what the list of necessary and sufficient conditions a CFT must satisfy in order to have a semiclassical Einstein gravity dual, so any hints along these lines are welcome.

The downside of the OTOC as a diagnostic of chaos is that it is a somewhat unfamiliar observable, as traditional field theory observables tend to be either time-ordered or response functions. In the context of conformal field theories, the four-point function is famously the first correlation function whose form is not completely specified by conformal symmetry, instead requiring a conformal block decomposition depending explicitly on the OPE coefficients for the operators under consideration. In addition, the OTOC of interest is a thermal correlation function, which further exacerbates computational difficulties (especially in higher dimensions). Controlled calculations of such thermal conformal block expansions, which must then be analytically continued to real time, present considerable technical challenges, even in two dimensions [13–16].

A very intriguing recent development has been evidence, first discovered in [17] and discussed further in [18, 19], of a relation between chaotic and hydrodynamic properties, referred to as pole skipping, in the context of maximally chaotic systems such as holographic models and SYK models. It has been observed in these and related models that the thermal energy density two-point functions possess subtle analyticity properties in the complex frequency and momentum planes that permit one to extract the Lyapunov exponent and butterfly velocity. While energy density response functions outside the hydrodynamic regime are still very complicated observables, the possible connection between chaos and hydrodynamics is very enticing and has been further explored in, e.g. [20–44]. More precisely, the pole skipping behavior is revealed by writing the energy density retarded two-point function as

$$G_{T^{tt}T^{tt}}^R(\omega, k) = \frac{N(\omega, k)}{D(\omega, k)}, \quad (1.3)$$

with both $N(\omega, k)$ and $D(\omega, k)$ possessing a line of zeroes that passes through the point

$$\omega_* = i\lambda_L, \quad k_* = ik_0 = i\frac{\lambda_L}{v_B}. \quad (1.4)$$

While the small ω, k , behavior of the denominator $D(\omega, k)$ is determined by hydrodynamic considerations, the conjecture is that this line of hydrodynamic zeroes analytically continues into the complex frequency and momentum plane precisely through the ‘chaotic’ point (1.4). This would be pole is then cancelled by a corresponding zero in the numerator. A simple consequence of this scenario is that the Green’s function directly at the pole skipping point (ω_*, k_*) is not uniquely defined and depends on the direction of approach. This can be seen

by expanding the numerator and denominator near (ω_*, k_*) , yielding

$$G^R(\omega_* + \delta\omega, k_* + \delta k) \approx \frac{\partial_\omega N(\omega_*, k_*)(\delta\omega/\delta k) + \partial_k N(\omega_*, k_*)}{\partial_\omega D(\omega_*, k_*)(\delta\omega/\delta k) + \partial_k D(\omega_*, k_*)}. \quad (1.5)$$

This ambiguity has been seen to emerge in general holographic settings from the near-horizon behavior of the relevant wave equations [19, 21], as we briefly review below. In addition to holographic models, pole skipping has also been observed in SYK chains and sheds some light on previous observations connecting chaos and energy diffusion [45–47].

In this article, we investigate pole skipping in two dimensional conformal field theories by considering the behavior of stress tensor response functions. While traditional hydrodynamics breaks down in two dimensions, it has been observed in [48] that the real time, thermal stress tensor Green’s functions on an infinite spatial manifold also exhibit pole skipping features analogous to those present in higher dimensional gravitational systems. However, as noted in [48], the result is surprising in the sense that the pole skipping location is universal with any two dimensional CFT having a ‘skipped pole’ at the position $\omega_* = 2\pi i T$. Identifying this location with the Lyapunov exponent via $\omega_* = i\lambda_L$, one would conclude that every two dimensional CFT is maximally chaotic, which would indeed be quite surprising. We revisit this situation and note that by considering a CFT on a compact spatial manifold, this tension is partially resolved. The location ω_* depends on the spectrum of the CFT through the stress tensor one point function on the torus. Explicitly, we compute $\langle TT \rangle_{T^2}$ on a torus T^2 with modular parameter $\tau = \frac{i\beta}{2\pi R}$ (corresponding to inverse temperature $\beta = T^{-1}$ and spatial circle radius R) and analytically continue to find the retarded Green’s function. Fourier transforming, we show that pole skipping in the holomorphic stress tensor retarded Green’s function occurs at

$$\omega_*^2 = k_*^2 = \frac{24}{c} \langle T \rangle_{T^2} = -\frac{12}{c} \frac{E}{R}. \quad (1.6)$$

Here E is the thermodynamic energy, $E = -\partial_\beta \log Z$, or alternatively obtained from the local energy density operator via $\langle T^{tt} \rangle = \frac{E}{2\pi R}$. Comparing to (1.4), we see that $v_B = \omega_*/k_* = 1$. Upon taking the high temperature limit, or equivalently the noncompact limit $R \rightarrow \infty$, (1.6) reproduces the answer obtained in [48] on spatial slice $\mathbb{R}$, as $\frac{E}{R} \rightarrow \frac{c}{12} (\frac{2\pi}{\beta})^2$ by the universal Cardy asymptotics of unitary two dimensional CFTs.

As a corollary of our results, we show that, for compact unitary CFTs with $c > 1$, modular invariance implies an upper bound on the energy density E/R for all temperatures. This immediately gives a bound on the pole skipping location, which we can write (assuming ω_* is purely imaginary, i.e. $E > 0$) as a bound on $\lambda = \text{Im } \omega_*$

$$\lambda \leq 2\pi T. \quad (1.7)$$

If we identify the pole skipping location with the Lyapunov exponent as suggested in [18], $\lambda = \lambda_L$, then we recognize the bound on chaos of [7]. Here we find the bound as a consequence of modular invariance in compact, unitary two dimensional CFTs with $c > 1$. Furthermore, we note that for any large c CFT₂ with a light sparse spectrum, as quantified by Hartman, Keller, and Stoica (HKS) in [1], the stress tensor one point function

is fixed by the free energy of a BTZ black hole, yielding $\frac{E}{R} = \frac{c}{12} \left(\frac{2\pi}{\beta}\right)^2$ for all $\beta < 2\pi R$, and the corresponding pole skipping location $\omega_* = i\lambda$ is precisely the maximal Lyapunov exponent, $\lambda = 2\pi T$, for all temperatures above the Hawking-Page transition. We note that there are known large c CFTs with a sparse light spectrum in this sense that are not expected to have semiclassical Einstein gravity duals or maximal chaos, such as some permutation orbifolds [1, 10, 49], suggesting that the relationship between λ and λ_L may be more complicated. Nevertheless, we find it intriguing that, at least in the context of two dimensional CFTs, both obey the same bound.

The paper is organized as follows. In section 2, we review some relevant background, including the gravitational origins of pole skipping as well as the HKS spectrum constraints required for a CFT₂ to reproduce gravitational thermodynamics in AdS₃. Following this, in section 3, we turn to purely CFT evaluations of the stress tensor response functions, first on the cylinder, and then moving to the torus. With these results in hand, we combine the holomorphic and anti-holomorphic results to obtain the energy density Green's functions and prove the bound (1.7). Finally, we conclude and discuss some potential future directions in section 4.

2 Diagnostics of holographic CFTs

In this section, we review some salient features of holographic conformal field theories that will be relevant for our analysis. First, we sketch the gravitational arguments for pole skipping in holographic response functions, where the peculiar non-analyticities emerge as a consequence of the near-horizon geometry in black hole backgrounds. Following this, we turn to two-dimensional conformal field theories and review the constraints imposed on the spectrum by bulk thermodynamic considerations.

2.1 Pole skipping

Pole skipping is a feature of holographic response functions that arises due to the nature of the relevant wave equations in the presence of a black hole horizon. Here we will briefly sketch the argument for AdS-Schwarzschild black holes in $d \geq 3$ Einstein-Hilbert gravity with negative cosmological constant. Note however, that due to the lack of propagating gravitational waves in three dimensions, the analysis is not directly relevant for the BTZ geometries dual to finite temperature two-dimensional CFTs; in a sense, the results of section 3 will be to demonstrate that pole skipping does arise in this setting as well.

We consider the real time stress tensor response function $G_{T_{tt}T_{tt}}^R(\omega, k)$ in an AdS-Schwarzschild black hole in $d+1 \geq 4$ dimensions. In ingoing Eddington-Finkelstein coordinates, the metric for such a planar black hole reads

$$ds^2 = \frac{L^2}{r^2} \left[-f(r)dv^2 + 2dvdr + dx_i dx^i \right]. \quad (2.1)$$

Here the conformal boundary is located at $r = 0$, the emblackening factor is $f(r) = 1 - (r/r_+)^d$, $v = t - r_*$ is the ingoing Eddington-Finkelstein coordinate (with r_* determined

by $dr_* = dr/f(r)$, and the boundary spatial coordinate indices run over $i = 1, \dots, d-1$. The temperature of the solution is then $T = \frac{d}{4\pi r_+}$.

For our purposes, we do not need to obtain the full Green's function $G_{TttTtt}^R(\omega, k)$, and it will suffice to illustrate that, for a particular value of ω and k , the holographic calculation of such a response function exhibits the novel features mentioned in the introduction. Recall that to obtain such a response function, the holographic dictionary instructs us to consider perturbations of the background (2.1) that solve the linearized Einstein equations and satisfy ingoing boundary conditions at the horizon [50]. The response function can then be read off from this linearized solution by taking the ratio of the coefficients in an expansion near the conformal boundary ($r = 0$). Our strategy, following very closely the discussion in [19], is to demonstrate how, in constructing a solution to the linearized Einstein equation via an expansion about the horizon $r = r_+$, one finds that at the particular point

$$\omega_* = 2\pi i T, \quad k_*^2 = -\frac{8\pi^2 T^2 (d-1)}{d}, \quad (2.2)$$

an additional ingoing solution emerges and leads to ambiguities in the response function at this point. More examples of pole skipping in holographic contexts and their implications can be found in [21].

To see this, we consider the Einstein equation

$$0 = R_{ab} + \frac{d}{L^2} g_{ab}, \quad (2.3)$$

linearized about the solution (2.1). Decomposing the perturbations in Fourier modes, $\delta g_{ab}(v, r, x^i) = e^{-i(\omega v - kx)} \delta g_{ab}(r)$, where we take the wave vector to point along $x^1 \equiv x$, we obtain a set of coupled second order differential equations for the $\delta g_{ab}(r)$. For generic ω and k , these equations admit two independent solutions, one of which is ingoing and the other outgoing. We distinguish between the two solutions by imposing regularity at the horizon. In particular, we can construct the generic solutions via series expansions about the horizon, and the ingoing solution will have a regular series expansion about the singular point $r = r_+$

$$\delta g_{ab}(r) = \delta g_{ab}(r_+) + \mathcal{O}(r - r_+). \quad (2.4)$$

In principle, the wave equation will fix all of the coefficients in such an expansion in terms of data on the horizon. To see why (2.2) is a special point, we consider the behavior of the vv component of (2.3) near the horizon, which reads

$$0 = \left(k^2 - \frac{4\pi T(d-1)}{d} i\omega \right) \delta g_{vv}(r_+) + (\omega - 2\pi i T) [2k \delta g_{vx}(r_+) + \omega \delta g_{x^i x^i}(r_+)] + \mathcal{O}(r - r_+). \quad (2.5)$$

For generic ω and k , this equation imposes a non-trivial constraint on the initial data $\delta g_{ab}(r_+)$. However, precisely when $\omega = \omega_* = 2\pi i T$, this equation simplifies dramatically and reduces, for generic k , to the constraint $\delta g_{vv}(r_+) = 0$. If we further tune $k \rightarrow k_*$, then $\delta g_{vv}(r_+)$ is completely unconstrained, and it appears the generic ingoing solution will have an additional free parameter at this point. Further analysis of the remaining components

of the linearized equations do not resolve this ambiguity and confirms that indeed at this point there is an additional ingoing solution with no outgoing solution.

This additional ingoing solution is the gravitational origin of the pole skipping phenomena advertised above. While we won't explicitly go through the details, having this additional solution allows one to independently tune the coefficients of the near boundary expansion and hence obtain any value desired for the dual response function. We have only considered the simplest setting for this behavior, namely pure Einstein-Hilbert gravity without matter, but the mechanism holds far more generally [19]; furthermore, one can find additional isolated locations in the complex (ω, k) plane that admit an extra ingoing solution, both for the stress-tensor correlators considered here as well as more general scalar and current response functions [21]. All of these locations present non-trivial constraints on the behavior of the dual response functions, and the crucial ingredient in their emergence is the presence of a black hole horizon. Therefore, one expects that such behavior should be considered as a necessary ingredient in a holographic CFT, and indeed our goal will be to formulate a sharp constraint in CFT₂ for pole skipping precisely at (2.2) with $d = 2$.

2.2 HKS sparse spectrum

We now briefly review the constraints gravitational thermodynamics imposes on a holographic CFT₂, first elucidated in [1]. The fundamental physical input is the universal free energy of three-dimensional gravity with a negative cosmological constant as a function of temperature, with a Hawking-Page transition from a thermal AdS solution to the BTZ black hole as the temperature is increased [51]. The upshot of the analysis is a sharp formulation of the notion of a sparse spectrum of light operators that has long been suspected to be necessary for a conformal field theory to have a simple holographic dual [11, 12]. We will see in section 3.2 how this universal thermodynamics also gives rise to pole skipping at a frequency corresponding to a maximal Lyapunov exponent for high temperatures.

The thermodynamics of three-dimensional gravity in the semiclassical limit ($c = \frac{3\ell}{2G_N} \rightarrow \infty$) can be seen by comparing the saddle points of the Einstein-Hilbert action in the canonical ensemble. The saddle points of interest are obtained by modular transformations of the thermal gas solution in AdS₃, which corresponds to a partition function given solely by the vacuum character of the dual CFT:¹

$$Z_{tAdS}(\tau, \bar{\tau}) = \chi_0(\tau) \bar{\chi}_0(\bar{\tau}). \quad (2.6)$$

Here $\chi_0(\tau)$ is the Virasoro vacuum character

$$\chi_0(\tau) = \frac{(1-q)q^{-\frac{c-1}{24}}}{\eta(\tau)} = \frac{q^{-\frac{c}{24}}}{\prod_{n=2}^{\infty} (1-q^n)}, \quad (2.7)$$

where $\eta(\tau)$ is Dedekind η function and $q = e^{2\pi i \tau}$. For a CFT at inverse temperature $\beta = T^{-1}$ and on a spatial circle of radius R , the modular parameter τ is given by $\tau = \frac{i\beta}{2\pi R}$,

¹Here we've included the contribution of the boundary gravitons generated by the two asymptotic Virasoro algebras [52]. A slogan to remember these results is that the gravitational answers are often 'vacuum dominated', i.e. given solely by the vacuum block (in an appropriate channel) in the dual CFT. For the case at hand, the relevant conformal blocks are the Virasoro characters.

with $\bar{\tau} = \tau^*$. This partition function leads to a free energy of

$$-\beta F_{tAdS} = \frac{c\beta}{12R} + \mathcal{O}(1), \quad (2.8)$$

in the $c \rightarrow \infty$ limit. The BTZ free energy is obtained similarly, by starting with a partition function given by the modular S transform of the vacuum:

$$Z_{BTZ}(\tau, \bar{\tau}) = \chi_0(-1/\tau) \chi_0(-1/\bar{\tau}), \quad (2.9)$$

which yields the free energy

$$-\beta F_{BTZ} = \frac{\pi^2 c R}{3\beta} + \mathcal{O}(1). \quad (2.10)$$

Comparing these two free energies, we find a phase transition at $\beta = 2\pi R$, with the black hole solution dominating at high temperatures. All other modular images turn out to be subleading and we will not consider them here.

As demonstrated in [1], general unitary large c CFT₂s need not exhibit this thermodynamic behavior. However, by studying modular invariance, [1] found that a bound on the density of light states

$$\rho(h, \bar{h}) \lesssim e^{4\pi\sqrt{h\bar{h}}} \quad (2.11)$$

is necessary and sufficient condition to reproduce the aforementioned gravitational thermodynamics. Here h and $\bar{h}$ are the left and right conformal weights (eigenvalues of L_0 and $\bar{L}_0$), with a ‘light state’ being quantified as those satisfying $h, \bar{h} < \frac{c}{24} + \epsilon$ for some small positive ϵ .

We emphasize that (2.11) is sufficient only for matching the gravitational thermodynamic behavior, and in particular is not sufficient to guarantee a simple gravitational dual. There are examples that satisfy (2.11) and yet are not expected to have Einstein duals, e.g. permutation orbifolds [1, 53–56].

For our purposes, we will primarily be interested in the high temperature/BTZ regime $\beta < 2\pi R$, where, assuming (2.11) is satisfied, any putative dual geometry should contain a black hole in the large c limit. As the presence of a black hole horizon is crucial both for a non-trivial Lyapunov exponent [4, 5] as well as the pole skipping discussed in the previous subsection, this is the regime where one expects to find chaotic behavior in the dual CFT. The goal of the remainder of this note is to verify this expectation in the energy density response functions.

3 Pole skipping in CFT₂

We finally turn to the analysis of the stress tensor response functions in CFT₂.² We first present a calculation of the relevant Green’s function when the spatial manifold is noncompact, before turning to the more interesting case of a compact spatial slice. The

²For the entirety of this paper, we only consider conformal field theories with equal holomorphic and anti-holomorphic central charges $c = \bar{c}$.

strategy is to start with the Euclidean two point function $G^E(\tau_E, x)$, and then analytically continue the result to obtain the retarded real time two point function $G^R(t, x)$, via

$$G^R(t, x) = -i\theta(t) \left[G^E(\epsilon + it, x) - G^E(-\epsilon + it, x) \right], \quad (3.1)$$

where $\epsilon > 0$ is an infinitesimal positive real number. It is then a matter of Fourier transforming to determine the location of pole skipping.

3.1 $\langle TT \rangle$ on $\mathbb{R} \times \mathbb{S}^1$

We start with a noncompact spatial slice, so the Euclidean geometry is $\mathbb{R} \times \mathbb{S}_\beta^1$, where the radius of the Euclidean time circle is β . This result has been previously obtained and discussed at length in [48], but we include a presentation here that easily generalizes to the case of a compact spatial manifold. As is well known, correlation functions on the cylinder are determined by the corresponding correlation functions on the complex plane $\mathbb{C}$, as they are related by a conformal transformation. Therefore, we first start on the plane, with standard metric $ds^2 = dzd\bar{z}$, where the holomorphic stress tensor has the two point function

$$\langle T(z)T(0) \rangle_{\mathbb{C}} = \frac{c}{2z^4}. \quad (3.2)$$

To obtain the correlation function on $\mathbb{R} \times \mathbb{S}_\beta^1$, we use the coordinate transformation $z = e^{\frac{2\pi}{\beta}w}$, where $w = x + i\tau$ is a complex coordinate on $\mathbb{R} \times \mathbb{S}_\beta^1$. Using the transformation law for the stress tensor, we have

$$T_{\mathbb{R} \times \mathbb{S}^1}(w) = \left(\frac{2\pi}{\beta} \right)^2 \left[z^2 T_{\mathbb{C}}(z) - \frac{c}{24} \right], \quad (3.3)$$

and so the connected correlation function on the cylinder reads

$$\begin{aligned} G_{\mathbb{R} \times \mathbb{S}^1}^E(\tau, x) &\equiv \langle T(w)T(0) \rangle_{\mathbb{R} \times \mathbb{S}^1} - \langle T \rangle_{\mathbb{R} \times \mathbb{S}^1}^2 = \left(\frac{2\pi}{\beta} \right)^4 e^{\frac{4\pi}{\beta}w} \langle T_{\mathbb{C}}(e^{\frac{2\pi}{\beta}w}) T_{\mathbb{C}}(1) \rangle \\ &= \frac{c}{32} \left(\frac{2\pi}{\beta} \right)^4 \sinh^{-4}(\pi w / \beta). \end{aligned} \quad (3.4)$$

We now analytically continue to find the real time Green's function. This amounts to setting $\tau = \epsilon + it$, where the sign of ϵ determines the operator ordering.³ The retarded Green's function is defined as the commutator

$$G^R(t, x) = -i\theta(t) \langle [\mathcal{O}(t, x), \mathcal{O}(0, 0)] \rangle, \quad (3.5)$$

where $\theta(t)$ is the Heaviside theta function. For the case at hand we find that the stress tensor response function is

$$\begin{aligned} G_{\mathbb{R} \times \mathbb{S}^1}^R(t, x) &= -i\theta(t) \left[G_{\mathbb{R} \times \mathbb{S}^1}^E(\epsilon + it, x) - G_{\mathbb{R} \times \mathbb{S}^1}^E(-\epsilon + it, x) \right] \\ &= -\frac{ic}{32} \left(\frac{2\pi}{\beta} \right)^4 \theta(t) \left[\sinh^{-4} \left(\frac{\pi}{\beta}(x - t + i\epsilon) \right) - \sinh^{-4} \left(\frac{\pi}{\beta}(x - t - i\epsilon) \right) \right]. \end{aligned} \quad (3.6)$$

³For a discussion of the appropriate ϵ prescriptions for Lorentzian correlators, see e.g. the recent works [13, 57] or the textbook accounts in [58, 59].

Figure 1. Contour $\mathcal{C}$ in the complex (Lorentzian) time plane used in evaluating the Fourier transform of $G_{\mathbb{R} \times S^1}^R(t, x)$, shown with $x > 0$. The two branches, infinitesimally close to the real line, correspond to the two terms in the commutator defining G^R . The Heaviside function in the definition of G^R restricts the contour to positive real times, and hence the singularity at $t = x$ is not enclosed by the contour of integration for $x < 0$.

All that remains is to Fourier transform this result. In fact, the detailed form the retarded two point function is not terribly important; we can rewrite the Fourier transform of G^R as a contour integral of the Euclidean Green's function, requiring only knowledge of its poles and their residues. Performing the integral over time first, we see that

$$\begin{aligned} G_{\mathbb{R} \times S^1}^R(\omega, x) &= \int_{-\infty}^{\infty} dt e^{i\omega t} G_{\mathbb{R} \times S^1}^R(t, x) \\ &= -i \int_0^{\infty} dt e^{i\omega t} \left[G_{\mathbb{R} \times S^1}^E(\epsilon + it, x) - G_{\mathbb{R} \times S^1}^E(-\epsilon + it, x) \right] \\ &= -i \int_{\mathcal{C}} dt e^{i\omega t} G_{\mathbb{R} \times S^1}^E(it, x), \end{aligned} \quad (3.7)$$

where $\mathcal{C}$ is a contour that wraps the positive real t -axis counterclockwise. To see this, note that we can shift the integration variable for the first term in the square brackets to $t \rightarrow t - i\epsilon$, and similarly shift the second term $t \rightarrow t + i\epsilon$ to obtain the integral

$$\left\{ e^{-\omega\epsilon} \int_{-i\epsilon}^{\infty - i\epsilon} dt - e^{\omega\epsilon} \int_{i\epsilon}^{\infty + i\epsilon} dt \right\} e^{i\omega t} G_{\mathbb{R} \times S^1}^E(it, x). \quad (3.8)$$

As we send $\epsilon \rightarrow 0$, we land on the contour $\mathcal{C}$ shown in figure (1).

From this contour manipulation, we have reduced the calculation to the evaluation of a residue. From (3.4), we see that $G_{\mathbb{R} \times S^1}^E(it, x)$ has a pole when $x = t$, which is enclosed by $\mathcal{C}$ when $x > 0$, and we find a residue of

$$2\pi i \operatorname{Res}_{t=x} \left[e^{i\omega t} G_{\mathbb{R} \times S^1}^E(it, x) \right] = \frac{\pi^2 c}{6} \theta(x) e^{i\omega x} \omega \left[\omega^2 + \left(\frac{2\pi}{\beta} \right)^2 \right]. \quad (3.9)$$

Finally we can Fourier transform in x (including an infinitesimal positive imaginary component to ω for convergence purposes) to find [48]

$$G_{\mathbb{R} \times S^1}^R(\omega, k) = \frac{\pi c}{6} \frac{\omega \left[\omega^2 + \left(\frac{2\pi}{\beta} \right)^2 \right]}{\omega - k}. \quad (3.10)$$

With this result, we see that any conformal field theory on the cylinder in two dimensions exhibits pole skipping at $\omega_* = k_* = 2\pi i T$, in agreement with the gravitational result (2.2) in higher dimensions continued to $d = 2$. Identifying the pole skipping location with Lyapunov exponent, one arrives at the conclusion that all two dimensional CFTs are maximally chaotic, which seems counterintuitive. We next turn to the analogous calculation on a torus, with a compact spatial manifold, where we will see that this situation is at least partially remedied.

3.2 $\langle TT \rangle$ on T^2

To perform the same calculation on the torus, we use the fact that the stress tensor two-point function on the torus is also completely fixed by Virasoro Ward identities, up to the value of the one point function $\langle T \rangle_{T^2}$. More explicitly, the two-point function of interest is given in terms of the Weierstrass elliptic function $\wp(z)$ by [60]

$$\begin{aligned} G_{T^2}^E(z-w) &\equiv \langle T(z)T(w) \rangle_{T^2} - \langle T \rangle_{T^2}^2 \\ &= \frac{c}{12} \wp''(z-w) + 2[\wp(z-w) + 2\eta_1] \langle T \rangle_{T^2} + 2\pi i \partial_\tau \langle T \rangle_{T^2}. \end{aligned} \quad (3.11)$$

The Weierstrass elliptic function is defined as

$$\wp(z) = \frac{1}{z^2} + \sum_{(m,n) \neq (0,0)} \left[\frac{1}{(z + 2\pi R m + i n \beta)^2} - \frac{1}{(2\pi R m + i n \beta)^2} \right], \quad (3.12)$$

and $\eta_1 = \zeta(1/2)$ is a constant, where $\zeta'(z) = -\wp(z)$. The modular parameter of the torus τ is given by $\tau = \frac{i\beta}{2\pi R}$, with R the radius of our spatial circle. A simple consistency check of (3.11) is to note that it reproduces the correct behavior in the OPE limit $z \rightarrow w$, i.e. $\frac{c}{2(z-w)^4} + \frac{2\langle T \rangle}{(z-w)^2}$. The stress tensor one point function is determined by the partition function using⁴

$$\langle T \rangle_{T^2} = \frac{i}{2\pi R^2} \partial_\tau \log Z. \quad (3.13)$$

We can now follow the same strategy as the previous subsection, writing $z = x + i\tau_E$, using a subscript E to distinguish Euclidean time from the torus modular parameter, and evaluating the real time retarded Green function via

$$G_{T^2}^R(t, x) = -i\theta(t) \left[G^E(x, \epsilon + it) - G^E(x, -\epsilon + it) \right], \quad (3.14)$$

with $G^E(x, \tau_E) = G^E(x + i\tau_E)$. Again, the two terms give the two terms in a commutator $[T(x, t), T(0)]$ thanks to the ϵ prescription. Conveniently, many of the constant terms in the Euclidean Green's function drop out in the difference, and so we are left with

$$G_{T^2}^R(t, x) = -i\theta(t) \left[\frac{c}{12} \wp''(x - t + i\epsilon) + 2\langle T \rangle_{T^2} \wp(x - t + i\epsilon) - (\epsilon \rightarrow -\epsilon) \right]. \quad (3.15)$$

⁴Note that the factor of R^2 and a sign in this expression result from a different map from the plane to the cylinder compared to the previous subsection. Here we take the plane coordinate to be given by $z = e^{-i(x+i\tau_E)/R}$ with $x + i\tau_E$ the complex coordinate on this more standard radial quantization cylinder. This choice also leads to a relative sign between the energy density $T_{\tau_E \tau_E}$ and $T + \bar{T}$.

Figure 2. Contour used in evaluating the Fourier transform of $G_{T^2}^R$. Here the singularity at $t = x$ has doubly periodic images, separated by $i\beta$ and $2\pi R$, as required by the torus geometry.

To perform the Fourier transform, we perform the same manipulations we did on the cylinder, ending up with a contour integral of the form

$$G_{T^2}^R(\omega, x) = -i \int_C dt e^{i\omega t} \left[\frac{c}{12} \wp''(x-t) + 2\langle T \rangle_{T^2} \wp(x-t) \right]. \quad (3.16)$$

We see that the only difference between the torus and the cylinder is that now an infinite series of poles contribute, thanks to the images of the OPE singularities on the compact spatial slice. See figure (2) for an illustration. The remaining integrals are straightforward to evaluate using (3.12), and one finds

$$\int_C dt e^{i\omega t} \wp(x-t) = -2\pi\omega \sum_{x+2\pi mR > 0} e^{i\omega(x+2\pi mR)}, \quad (3.17)$$

$$\int_C dt e^{i\omega t} \wp''(x-t) = 2\pi\omega^3 \sum_{x+2\pi mR > 0} e^{i\omega(x+2\pi mR)}. \quad (3.18)$$

These are geometric series and, provided we add an infinitesimal positive imaginary part to ω , we can take $0 < x < 2\pi R$ and evaluate the sums over $m \geq 0$ to find

$$G_{T^2}^R(\omega, x) = -\frac{ic}{12} \frac{2\pi\omega e^{i\omega x}}{1 - e^{2\pi i R\omega}} \left[\omega^2 - \frac{\langle T \rangle_{T^2}}{c/24} \right]. \quad (3.19)$$

Performing the last Fourier transform over x , we obtain

$$G_{T^2}^R(\omega, k_n) = \frac{\pi c}{6} \frac{\omega}{\omega - k_n} \left[\omega^2 - \frac{\langle T \rangle_{T^2}}{c/24} \right]. \quad (3.20)$$

Here $k_n = n/R$ is the discrete Fourier momentum on a compact spatial circle.

With this result, we see that the location of the zero in the numerator of the response function is determined by $\langle T \rangle$, and so the pole skipping location moves to $\omega_*^2 = k_*^2 = \frac{\langle T \rangle}{(c/24)}$. Note that this result reduces to that obtained in the previous subsection as we send $R \rightarrow \infty$. This is because in the non-compact limit, which is equivalent to the high-temperature limit

$\beta \rightarrow 0$ by modular invariance, the partition function, and hence $\langle T \rangle$, for every unitary CFT₂ will be dominated by the modular S transform of the Virasoro vacuum character, leading to a value of $\langle T \rangle = -c/24(2\pi/\beta)^2$.

If we now consider CFTs that satisfy the HKS spectrum condition (2.11), then the universal high temperature free energy leads to

$$\langle T \rangle_{HKS} = -\frac{c}{24} \left(\frac{2\pi}{\beta} \right)^2, \quad (3.21)$$

for all $\beta < 2\pi R$, so that the pole skipping occurs at $\omega_* = k_* = 2\pi i T$ for all sufficiently high temperatures. Thus we conclude that imposing the HKS condition is sufficient to guarantee that pole skipping occurs at the ‘maximal chaos’ location for all temperatures above the Hawking-Page transition in the gravitational dual, as expected by the presence of a black hole.

3.3 Energy density pole skipping and a bound on ω_*

Thus far we have focused exclusively on the behavior of the holomorphic stress tensor response functions. For the honest energy density response functions, we use $-2\pi T_{\tau\tau} = T + \bar{T}$, and so the connected Euclidean Green’s function schematically reads

$$\langle T_{\tau\tau} T_{\tau\tau} \rangle^c = \langle TT \rangle^c + \langle \bar{T}\bar{T} \rangle^c. \quad (3.22)$$

The antiholomorphic contribution can be derived by exactly the same procedure as $\langle T \rangle$ above, and results in the simple substitution $k \rightarrow -k$. The end result after combining these two pieces is

$$G_{T^{tt}T^{tt}}^R(\omega, k) = \frac{c}{12\pi} \frac{\omega^2}{\omega^2 - k^2} \left[\omega^2 + \frac{2\pi \langle T^{tt} \rangle}{c/12} \right] = \frac{c}{12\pi} \frac{\omega^2}{\omega^2 - k^2} \left[\omega^2 + \frac{12}{c} \frac{E}{R} \right]. \quad (3.23)$$

Since we have $\bar{\tau} = \tau^*$ and no angular potential, $\bar{\tau} = -\tau$, we have dropped a term proportional to $\langle T - \bar{T} \rangle$, as $\langle T \rangle = \langle \bar{T} \rangle$. We have also used the fact that T^{tt} is the local energy density, and so $2\pi \langle T^{tt} \rangle = \frac{E}{R}$, where $E = -\partial_\beta \log Z$ is the thermodynamic energy. Thus we see that the energy density also exhibits pole skipping with the location set by $\langle T^{tt} \rangle$, $\omega_*^2 = -\frac{12}{c} \frac{E}{R}$. Evaluating for a CFT with a sparse light spectrum in the high temperature/BTZ regime, where $\frac{E}{R} = \frac{\pi^2 c}{3\beta^2}$, again yields $\omega_*^2 = -(2\pi/\beta)^2$.

As a corollary of this result, we note that the pole skipping location, assuming ω_* is purely imaginary, monotonically moves upward in the upper half plane as the temperature is increased, since $\partial_T E \geq 0$. This is simply a consequence of thermodynamics, as $\partial_T E$ is the heat capacity of the system and must be positive for stability of the ensemble.

In fact, we can learn more using modular invariance. We claim that the energy density is bounded for all temperatures,

$$\frac{E}{R} \leq \frac{\pi^2 c}{3\beta^2}. \quad (3.24)$$

in any unitary two dimensional CFT with a discrete spectrum and $c > 1$. This directly leads to a bound on $\lambda = \text{Im } \omega_*$

$$\lambda \leq 2\pi T. \quad (3.25)$$

We note that while Cardy asymptotics imply that $\frac{E}{R} \rightarrow \frac{\pi^2 c}{3\beta^2}$ as $\beta \rightarrow 0$, it is not immediately obvious to us that such a bound must hold for all T . Fortunately, the bound is a simple consequence of modular invariance, at least for $c > 1$. To see this, we write the partition function in terms of the Virasoro characters

$$\chi_0(\tau) = \frac{(1-q)q^{-\frac{c-1}{24}}}{\eta(\tau)} = \frac{q^{-\frac{c}{24}}}{\prod_{n=2}(1-q^n)}, \quad \chi_h(\tau) = \frac{q^{h-\frac{c-1}{24}}}{\eta(\tau)} = \frac{q^{h-\frac{c}{24}}}{\prod_{n=1}(1-q^n)}. \quad (3.26)$$

Here it is important we are working at $c > 1$ in order to ensure the absence of null states. Evaluating the partition function at $\tau = -\bar{\tau} = \frac{i\beta}{2\pi R}$ yields

$$Z(\beta/R) = \frac{e^{\frac{\beta c}{12R}}}{\prod_{n=2}^{\infty}(1 - e^{\frac{-n\beta}{R}})^2} \left[1 + \sum_p \frac{n_p e^{\frac{-\beta \Delta_p}{R}}}{(1 - e^{\frac{-\beta}{R}})^2} \right], \quad (3.27)$$

where the sum runs over all Virasoro primary operators in the CFT and n_p is the degeneracy of primaries with dimension $\Delta_p = h_p + \bar{h}_p$. Modular invariance tells us that

$$Z(\beta/R) = Z(4\pi^2 R/\beta). \quad (3.28)$$

Computing the energy $E = -\partial_\beta \log Z$ after using this modular transformation, one finds

$$\frac{\beta^2}{4\pi^2 R} E = \frac{c}{12} - \frac{\sum_p n_p e^{\frac{-4\pi^2 R \Delta_p}{\beta}} \left(\Delta_p + e^{\frac{-2\pi^2 R}{\beta}} \operatorname{csch} \frac{2\pi^2 R}{\beta} \right)}{(1 - e^{\frac{-4\pi^2 R}{\beta}})^2 + \sum_q n_q e^{\frac{-4\pi^2 R \Delta_q}{\beta}}} - \sum_{n=2}^{\infty} \frac{2n}{e^{\frac{4\pi^2 n R}{\beta}} - 1}. \quad (3.29)$$

While this expression is a bit complicated, all we need is the fact that the last two terms on the right hand side are always negative, since the degeneracies n_p are positive integers and the conformal weights are always positive in a unitary theory, $\Delta_p > 0$. Rearranging, we have $\frac{c}{12} - \frac{\beta^2}{4\pi^2} \frac{E}{R} \geq 0$, or

$$\frac{12}{c} \frac{E}{R} \leq \left(\frac{2\pi}{\beta} \right)^2. \quad (3.30)$$

Combining this with our pole skipping result above, we immediately find a bound on pole skipping for any compact, unitary CFT₂ with $c > 1$: writing $\omega_* = i\lambda$, assuming $E > 0$, we have

$$\lambda \leq 2\pi T. \quad (3.31)$$

Identifying λ with the Lyapunov exponent λ_L , we see that this is none other than the chaos bound of [7]. As mentioned at the end of the previous subsection, a sparse light spectrum fixes the value of $\langle T \rangle$, and hence the energy density, which leads to

$$E_{HKS} = 2\pi R \langle T^{tt} \rangle_{HKS} = -2R \langle T \rangle_{HKS} = \frac{cR}{12} \left(\frac{2\pi}{\beta} \right)^2. \quad (3.32)$$

Thus the bound (3.30) is saturated and we have maximal pole skipping for all temperatures T above the self dual point $2\pi RT = 1$.

For simplicity, here we have only considered the bound (3.30) in theories with $c > 1$, to avoid complications due to null states. We have numerically checked that the bound also

holds for the Ising minimal model, but we leave a systematic check of all Virasoro minimal models to future work. In addition, we have only considered a system with no angular potential, i.e. a purely imaginary modular parameter $\text{Re}\tau = 0$. It is straightforward to determine the pole skipping location when $\text{Re}\tau \neq 0$, which introduces a dependence on $\langle T - \bar{T} \rangle \sim P/R$, where P is the expectation value of the total momentum on the circle. It seems possible that a more sophisticated analysis of modular invariance in this setup could lead to a more general bound on λ , such as that of [61], but we leave this for future study.⁵

The fact that all CFTs exhibit maximal pole skipping on the cylinder is a consequence of the fact that we are considering a CFT on an infinite spatial manifold, which is equivalent by modular invariance to infinite temperature. Hence the maximal pole skipping on the cylinder is a consequence of the famous Cardy asymptotics governing every unitary CFT₂. In this section, we've seen that on finite spatial slices pole skipping is sensitive to the spectrum of the theory, and that the pole skipping location is bounded by the temperature in the same manner as the Lyapunov exponent. The gravitational result, namely that this upper bound is then saturated for all sufficiently high temperatures on finite spatial slices, further requires a sparse light spectrum at large central charge, in the manner of HKS.

4 Discussion

In this note, we've revisited pole skipping in CFT₂. By considering the behavior of a generic CFT on the torus, we have shown that the pole skipping location depends on the spectrum of the CFT through the value of the stress tensor one point function $\langle T \rangle$. This indicates that at least on a compact spatial manifold the pole skipping behavior is not universal as suggested by considerations on a spatial line $\mathbb{R}$. A corollary of this result, following from a bound on the energy density due to modular invariance, indicates that the pole skipping location for a generic compact unitary CFT₂ with $c > 1$ is bounded, $\lambda \leq 2\pi T$, where $\omega_* = i\lambda$. Identifying this location with the Lyapunov exponent, $\lambda = \lambda_L$, this bound reproduces the MSS chaos bound of [7]. Furthermore the spectrum constraints necessary to match the CFT thermodynamics at large central charge with that of a putative gravitational dual suffice to saturate this bound, fixing the pole skipping location to that expected from gravitational calculations in a black hole background, corresponding to a maximal Lyapunov exponent.

In this work, we have only focussed on the simplest example of pole skipping, and the work is dramatically simplified by the fact that the stress tensor correlators are essentially fixed by Virasoro symmetry. However, pole skipping is believed to be a very general feature of holographic response functions, and furthermore any given response function can have many different pole skipping locations, typically lying in the lower half of the complex frequency plane [21, 23]. While these additional pole skipping locations are absent for the two dimensional stress tensor case considered in this work, they are present for more general holographic retarded two-point functions in two dimensions, e.g. for a scalar with arbitrary conformal dimension. Unfortunately, two point functions on the torus generically require non-trivial conformal block decompositions, which poses technical challenges. Matching

⁵We thank Mukund Rangamani for comments on this point.

to gravitational answers, which are much less sensitive to compact vs. non-compact spatial manifolds (cf. the emergence of BTZ quasinormal modes via CFT_2 on a Euclidean cylinder [62]), will likely require constraints on OPE coefficients, perhaps along the lines of those imposed by higher genus modular bootstrap considerations [63]. It would be very interesting to see if progress can be made along these directions.

Furthermore, pole skipping has been conjectured to play an important role in effective field theories for quantum chaos [18, 48]. For instance, [48] has used the stress tensor cylinder two-point function in CFT_2 to construct an effective action for chaos (with some aspects generalized to higher dimensions in [64]), where the pole skipping in the upper half plane produces exponentially growing terms in the propagators of soft modes in the effective field theory. It is conjectured that the exchange of these soft modes are responsible for the behavior of OTOCs at large central charge, with the maximal pole skipping location giving rise to a maximal Lyapunov exponent in the OTOCs. As we have seen that the location of pole skipping is in fact sensitive to the spectrum of the theory, this opens up a window to possibly explore such effective field theories in a scenario with non-maximal chaos.

Pole skipping for non-maximally chaotic theories has also recently been considered in [37], where the energy density two-point function for the large- q SYK chain [45] is studied. In [37], the authors advocate isolating the stress tensor contribution to the (velocity-dependent) Lyapunov exponent $\lambda_L^{(T)}$ and butterfly velocity $v_B^{(T)}$, and they conjecture that this stress tensor contribution is responsible for pole skipping in the energy density two-point function. Our results explicitly demonstrate the privileged role played by the stress tensor in energy density pole skipping,⁶ although the computational techniques are distinct and specific to CFT_2 . These two results suggest that there could be a relationship between $\langle T \rangle$ and $\lambda_L^{(T)}$ in finite volume, which would be intriguing and is worth investigating. To probe the dependence on the appropriate butterfly velocity, it is likely necessary to move away from the conformal limit, since $v_B = v_B^{(T)} = 1$ in CFT_2 [65]. Such questions may be within reach of conformal perturbation theory and could therefore provide additional insight into the conjecture, which we leave to future work.

Finally, having an independent expectation for the location of pole skipping suggests that more tests of the relation between pole skipping and Lyapunov growth in OTOCs may be possible. For instance, perhaps progress can be made in computing OTOCs in sufficiently simple CFTs and comparing Lyapunov growth with the corresponding stress tensor expectation values.

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⁶We note that [37] worked in the infinite volume limit, and therefore the value $\lambda_L^{(T)} = \frac{2\pi}{\beta}$ appearing in their conjecture may be modified upon working in finite volume or at lower temperatures. For instance, the 2D Ising minimal model on a torus with finite modular parameter exhibits pole-skipping at a frequency with $|\omega_*| < \frac{2\pi}{\beta}$.

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