Massive fishnets

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ABSTRACT: Recently, infinite families of massive Feynman integrals were found to feature an unexpected Yangian symmetry. In the massless case, similar integrability properties are understood via the interpretation of individual Feynman integrals as correlators in the massless fishnet theory introduced by Gürdoğan and Kazakov. Here we seek for an analogous interpretation of the integrability of massive Feynman integrals. We contrast two approaches to define simple massive quantum field theories in four dimensions. First, we discuss spontaneous symmetry breaking in the massless bi-scalar fishnet theory. We then propose an alternative route to a massive fishnet theory by taking a double-scaling limit of $\mathcal{N} = 4$ SYM theory on the Coulomb branch. Both approaches lead to a massive extension of the massless fishnet theory, differing in how masses enter into the propagators. In the latter theory, planar off-shell amplitudes are in one-to-one correspondence with precisely those massive Feynman integrals that were shown to be invariant under the Yangian. This suggests a re-investigation of Coulomb branch $\mathcal{N} = 4$ SYM theory with regard to integrability. Finally, we demonstrate that in the case of spontaneous symmetry breaking, the original conformal symmetry leads to soft theorems for scattering amplitudes in the broken phase.

KEYWORDS: $1/N$ Expansion, Integrable Field Theories, Scattering Amplitudes, Spontaneous Symmetry Breaking

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1 Introduction

The framework of quantum field theory (QFT) is arguably one of the most advanced tools of theoretical physics. At the same time computations quickly become untractable and are limited to a perturbative regime. This underlines the demand for toy models that are sufficiently simple to progress computations, while still featuring enough complexity to study generic properties of quantum field theory. More than 20 years ago the celebrated AdS/CFT duality opened a new door to unveil such toy models of QFT — first of all $\mathcal{N} = 4$ super Yang-Mills (SYM) theory [1]. In these models, simplicity goes hand in hand with a rich spectrum of symmetries. In particular, integrability has proven extremely powerful for computations in various instances of the AdS/CFT correspondence [2, 3]. The appearance of integrable structures in massless quantum field theories in four dimensions is strongly intertwined with the duality to two-dimensional worldsheet string models. Their self-duality results in two instances of conformal symmetry which in the most prominent

cases close into a Yangian algebra [4, 5]. This infinite-dimensional symmetry is highly constraining and provides the algebraic underpinning of rational integrable models [6–9].

The Yangian symmetry is inherited by large classes of fishnet Feynman integrals via their interpretation as scattering amplitudes or correlation functions in the so-called fishnet theories [10–13]. The massless fishnet theories have been defined as particular double-scaling limits of gamma-deformed $\mathcal{N} = 4$ SYM theory and represent inspiring toy models of quantum field theory in four dimensions, see [12–40] for progress in various directions. They have a simple field content, are believed to be integrable and require no supersymmetry. The price to pay for these features is the lack of unitarity. While non-unitarity may be considered a significant drawback, it is compensated by a one-to-one correspondence between correlation functions (or scattering amplitudes) and Feynman integrals. This correspondence allows to translate the symmetries of correlators directly to the fundamental building blocks of generic quantum field theories:

$$\mathcal{N} = 4 \text{ SYM} \quad \rightarrow \quad \gamma\text{-deformation} \quad \rightarrow \quad \text{fishnet theory} \quad \rightarrow \quad \text{Feynman graphs} \quad (1.1)$$

Hence, the integrability features of the AdS/CFT duality find their way to phenomenologically interesting quantities and can be used to fix certain Feynman integrals completely [41].

In particular with regard to phenomenology, the interest in Feynman integrals is not limited to the case of massless propagators. This raises the question of whether the above sequence of steps can be generalized to the massive case, i.e. to deduce constraints on massive Feynman integrals from AdS/CFT integrability. When mass is introduced into $\mathcal{N} = 4$ SYM theory via the Higgs mechanism, an extended massive version of dual (super)conformal symmetry was found to survive the symmetry breaking [42]. However, no such massive extension is known for the ordinary conformal symmetry and thus there is no known massive Yangian symmetry on the Coulomb branch of $\mathcal{N} = 4$ SYM theory. Hence, one may be tempted to believe that integrability is lost in the massive phase.

Recently, however, it was discovered that integrability also features in the building blocks of massive quantum field theories, i.e. on the very right of a potential massive version of the sequence (1.1) [43]. Large families of ubiquitous massive Feynman integrals were found to be invariant under the generators of an extended Yangian. Again, this massive Yangian is understood as the closure of a massive dual conformal symmetry and a novel generalization of the ordinary conformal symmetry. The integrands of those families of massive Feynman integrals can be considered as the Kaluza-Klein reductions of higher dimensional massless integrands. This suggests to interpret the observed integrability properties within the AdS/CFT duality.

In the present paper we seek the interpretation of the massive Yangian symmetry of Feynman integrals found in [43] as the integrability of a massive quantum field theory. Our guideline is the relation of massless integrability in $\mathcal{N} = 4$ SYM theory and Feynman integrals via the fishnet theory. In contrast to the massless case, we argue in the subsequent section 2 that there are (at least) two different ways to introduce interesting massive extensions of the fishnet theory. Explicitly, we distinguish the following cases:

- I. Spontaneous symmetry breaking in the massless fishnet theory.
- II. Double-scaling limit of Coulomb-branch $\mathcal{N} = 4$ SYM theory.

The theories resulting from the alternative routes I and II both have their own advantages. The spontaneously broken fishnet (SBF) theory obtained from taking the first route was recently shown to allow for a ‘natural’ breaking of conformal symmetry, where some of the symmetry breaking vacua survive in the quantum theory [31]. Moreover, in section 4 we explicitly demonstrate that scattering amplitudes in this theory are subject to conformal soft theorems. In section 3 we contrast the symmetries of scattering amplitudes and Feynman integrals arising in both theories. For option II we illustrate that there is a one-to-one correspondence¹ between off-shell amplitudes and those massive Feynman integrals that were found to feature a massive Yangian [43]. Due to the close similarity to the massless bi-scalar fishnet theory, we thus refer to the theory obtained via scenario II as the massive fishnet (MF) theory. We end the paper with a summary and outlook in section 5.

2 Routes to massive fishnets

In this section we outline two different approaches to define a massive extension of the fishnet theory. The basis for our considerations is the definition of the massless fishnet theory via a particular double-scaling limit of deformed $\mathcal{N} = 4$ SYM theory [10]. We thus start with a short review of this limit, followed by the discussion of the massive cases. At the end of this section we discuss different possibilities to choose a vacuum expectation value (VEV) and to take the large- N_c limit of the massive theories.

2.1 Review of massless fishnets

The massless fishnet theories arise as the planar limits of a certain double-scaled version of gamma-deformed $\mathcal{N} = 4$ SYM theory. On the level of the Lagrangian, the gamma-deformation [44–46] can be implemented by an operator $\mathcal{P}_\gamma$ that acts bi-locally on a matrix product of some generic fields Φ_j according to²

$$\mathcal{P}_\gamma(\Phi_1\Phi_2\ldots\Phi_n) = \exp\left(\frac{i}{2}\gamma_A\epsilon_{ABC}q^B\wedge q^C\right)\Phi_1\Phi_2\ldots\Phi_n. \quad (2.1)$$

Here the anti-symmetric wedge-product is defined as

$$q^A\wedge q^B = \sum_{k=1}^n\sum_{j=1}^{k-1}\left(q_k^Aq_j^B - q_j^Aq_k^B\right), \quad (2.2)$$

and q_j denotes the R-symmetry Cartan charge vector of the field Φ_j as given in appendix A. Under the deformation induced by the operator $\mathcal{P}_\gamma$, the terms in the Lagrangian of $\mathcal{N} = 4$ SYM theory acquire different phase factors depending on the three parameters $\gamma_{1,2,3}$. Hence, the set of parameters in the undeformed theory, i.e. the coupling g and the number of colors N_c , is enhanced by these three deformation parameters.

As the next step it was noticed in [10] that setting $\xi^2 := g^2N_ce^{-i\gamma_3}$ and taking the double-scaling limit

$$g \rightarrow 0, \quad \gamma_j \rightarrow i\infty, \quad \xi = \text{fix}, \quad (2.3)$$

¹As in the massless case, this one-to-one correspondence remains conjectural.

²This formulation is reminiscent of the bi-local Yangian level-one generators.

Figure 1. Routes to massive fishnets.

the complexity of the gamma-deformed model reduces drastically. The resulting bi-scalar fishnet theory is described by the Lagrangian

$$\mathcal{L}_F = N_c \text{tr} \left(-\partial_\mu \bar{X} \partial^\mu X - \partial_\mu \bar{Z} \partial^\mu Z + \xi^2 \bar{X} \bar{Z} X Z \right). \quad (2.4)$$

Here X and Z are complex scalar fields transforming in the adjoint representation of $SU(N_c)$ and ξ is the new scalar coupling. The integrability of the fishnet theory is inherited from planar $\mathcal{N} = 4$ SYM theory and materializes in its conformal and dual conformal symmetry, which combine into a Yangian [12, 13].

We note that at the quantum level the above massless fishnet theory is incomplete and requires the addition of further double-trace couplings [14, 17]. In the present paper we aim to indentify simple massive quantum field theories by investigating the symmetries of their scattering amplitudes. At leading order in the large- N_c limit the double-trace couplings do not contribute to the type of observables studied here. Therefore, we ignore them in the following.

As in the case of $\mathcal{N} = 4$ SYM theory, it is natural to ask what happens to the conformal and dual conformal symmetries if masses are introduced into the fishnet theory via the Higgs mechanism, cf. [42]. A controlled way to introduce a mass scale into a conformal theory is to spontaneously break conformal symmetry by having a scalar field acquire a non-zero vacuum expectation value (VEV). Starting from $\mathcal{N} = 4$ Super-Yang-Mills theory, there are at least two distinct routes to construct a massive fishnet theory. These are illustrated in figure 1 and discussed in detail below. These two routes differ in the order in which a scalar field acquires a vacuum expectation value and the gamma-twist is carried out. It turns out that the double-scaling limit and the expansion around a non-zero VEV commute, thus exchanging the order of the two operations does not give rise to a third candidate theory.

2.2 Spontaneous symmetry breaking in the fishnet theory

In this subsection we start from the double-scaled gamma-deformed theory defined by (2.4) and let one of its scalar fields acquire a vacuum expectation value. This situation has been investigated in the interesting letter [31]. There it was shown that e.g. for the case of a diagonal expectation value, under certain conditions conformal symmetry survives loop corrections in perturbation theory. Therefore, we study the case in which the field Z is expanded according to³

$$Z \rightarrow \langle Z \rangle + Z, \quad (2.5)$$

where the VEV of Z takes the diagonal form

$$\langle Z \rangle^a_b = \frac{z_a}{\xi} \delta^a_b, \quad (2.6)$$

with $z_a \in \mathbb{C}$. The Lagrangian for the resulting spontaneously broken fishnet (SBF) theory then reads

$$\mathcal{L}_{\text{SBF}} = \mathcal{L}_{\text{F}} + N_c z_a \bar{z}_b \bar{X}^a_b X^b_a + N_c \xi \left(\bar{z}_a X^a_b Z^b_c \bar{X}^c_a + z_a \bar{X}^a_b \bar{Z}^b_c X^c_a \right). \quad (2.7)$$

In [31] different constraints for the entries (2.6) of the VEV were identified in order for conformal symmetry to stay a spontaneously broken symmetry once quantum corrections are taken into account. Those constraints are expected to become important when extending the soft theorems of section 4 to loop level or when considering double-trace observables. Note that $\bar{z}_a$ does not necessarily have to be complex conjugate to z_a which is due to the non-unitarity of the Lagrangian and the resulting equations of motion, cf. [31].

Feynman rules and physical masses. The propagator for the massive X field is given by

$$\bar{X} \xrightarrow[a_1]{b_1} \xrightarrow[p]{\quad} \xrightarrow[b_2]{a_2} X = \frac{1}{N_c} \frac{\delta_{a_1}^{b_2} \delta_{a_2}^{b_1}}{p^2 + z_{a_1} \bar{z}_{b_1}}. \quad (2.8)$$

Importantly, the parameters z_a enter the propagator in the form of a product $z_{a_1} \bar{z}_{b_1}$. Also the three-point vertices are proportional to the z_a :

$$\begin{aligned} \bar{X} \xrightarrow[a_1]{b_1} \xrightarrow{\quad} \xrightarrow[b_3]{a_3} X & \quad \begin{array}{c} \bar{Z} \\ \downarrow a_2 b_2 \end{array} \\ & = N_c z_{a_1} \xi \delta_{a_1}^{b_3} \delta_{a_3}^{b_2} \delta_{a_2}^{b_1}, \\ \bar{X} \xrightarrow[a_1]{b_1} \xrightarrow{\quad} \xrightarrow[b_2]{a_2} X & \quad \begin{array}{c} \downarrow b_3 a_3 \\ Z \end{array} \\ & = N_c \bar{z}_{b_1} \xi \delta_{a_1}^{b_3} \delta_{a_3}^{b_2} \delta_{a_2}^{b_1}. \end{aligned} \quad (2.9)$$

³Note the slight abuse of notation where the excitation on the right hand side is denoted by the same letter as the field before symmetry breaking.

For completeness, we also give the four-point vertex which takes the form

$$\begin{array}{c}
 \bar{Z} \\
 \downarrow a_2 b_2 \\
 \bar{X} \xrightarrow[b_1]{a_1} \text{---} \text{---} \xrightarrow[b_3]{a_3} X \\
 \downarrow b_4 a_4 \\
 Z
 \end{array} = N_c \xi^2 \delta_{a_1}^{b_4} \delta_{a_4}^{b_3} \delta_{a_3}^{b_2} \delta_{a_2}^{b_1}. \quad (2.10)$$

It is instructive to study Feynman graphs in this theory in double-line notation. Consider for instance the example

$$\begin{array}{c}
 \bar{X} \xrightarrow[b_1]{a_1} \text{---} \text{---} \xrightarrow[b_2]{a_2} X \\
 \downarrow \text{---} \text{---} \downarrow \\
 \bar{X} \xrightarrow[b_4]{a_4} \text{---} \text{---} \xrightarrow[b_3]{a_3} X
 \end{array} = \delta_{a_1}^{b_4} \delta_{a_2}^{b_1} \delta_{a_3}^{b_2} \delta_{a_4}^{b_3} \sum_{a_0=1}^{N_c} \begin{array}{c} \bar{X} \text{---} \text{---} \text{---} X \\ \text{---} \text{---} \text{---} \\ \bar{X} \text{---} \text{---} \text{---} X \end{array}. \quad (2.11)$$

Here it becomes clear that there is a natural association between the color index carried by a line and the region in the Feynman diagram surrounded by that line. Since the mass parameters z_a also carry a color index, they correspond to the regions and we refer to them as region masses. The physical mass of a particle propagating between two regions can be read off from the propagator (2.8) and is given by the product of the two corresponding region masses:

$$m_{ab}^2 = z_a \bar{z}_b. \quad (2.12)$$

The spectrum of physical masses consists of all possible products of two region masses z_a . In contrast, on the Coulomb branch of $\mathcal{N} = 4$ SYM theory, the mass term in a propagator is given by the differences of neighboring region masses m_a and m_b :

$$(m_{ab}^{\mathcal{N}=4})^2 = (m_a - m_b)^2. \quad (2.13)$$

As opposed to z_a and $\bar{z}_a$, in the latter case the real parameters m_a on their own take the role of physical masses. It would be desirable to identify a fishnet-like theory with the latter propagator structure, since massive Feynman diagrams with such difference-mass propagators were recently found to be subject to a Yangian symmetry [43]. This motivates to search for an alternative massive extension of the fishnet theory. We will come back to the situation of spontaneous symmetry breaking in due course.

2.3 Double-scaling limit of Coulomb-branch $\mathcal{N} = 4$ SYM theory

In this subsection we take the second route to a massive fishnet theory that was indicated above. We show that in the simplest case, this route results in a bi-scalar theory which inherits the difference-mass propagators from the Coulomb branch of $\mathcal{N} = 4$ SYM theory. As discussed in more detail in the subsequent section 3, off-shell amplitudes in this theory are in one-to-one correspondence with massive fishnet Feynman integrals that have been found to be Yangian-invariant, cf. [43]. This motivates to refer to the resulting theory as

the massive fishnet (MF) theory. At the end we will also discuss a slightly more general double-scaling limit which yields a tri-scalar theory.

Our starting point is the Euclidean Lagrangian of $\mathcal{N} = 4$ SYM theory, see e.g. [47]:

$$\begin{aligned}\mathcal{L}_{\mathcal{N}=4} = N_c \text{tr} \bigg[& -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - D^\mu \phi_j^\dagger D_\mu \phi^j + i \bar{\psi}_A^\alpha D_\alpha^\mu \psi_A^\mu \\ & - \frac{1}{4} g^2 \{ \phi_i^\dagger, \phi^i \} \{ \phi_j^\dagger, \phi^j \} + g^2 \phi_i^\dagger \phi_j^\dagger \phi^i \phi^j \\ & + g \bar{\psi}_4 [\phi^j, \bar{\psi}_j] - g i \epsilon_{ijk} \psi^k \phi^i \psi^j + g \psi_j [\phi_j^\dagger, \psi_4] - g i \epsilon^{ijk} \bar{\psi}_k \phi_i^\dagger \bar{\psi}_j \bigg],\end{aligned}\quad (2.14)$$

where $i, j = 1, 2, 3$, $A = 1, 2, 3, 4$, $D_\alpha^\mu = D_\mu (\tilde{\sigma}^\mu)_\alpha^\alpha$ and $(\tilde{\sigma}^\mu)_\alpha^\alpha = (-i\sigma_2, i\sigma_3, 1, -i\sigma_1)_\alpha^\alpha$ and

$$D_\mu = \partial_\mu + \frac{ig}{\sqrt{2}} [A_\mu, \cdot], \quad F_{\mu\nu} = -\frac{i\sqrt{2}}{g} [D_\mu, D_\nu]. \quad (2.15)$$

Note that the fields in the above Lagrangian have been rescaled by a factor of $\sqrt{N_c}$ and we have $g^2 = g_{\text{YM}}^2 N_c$.⁴

Introducing a VEV. We give a *real* VEV to the complex scalar field ϕ_3 , i.e. we replace

$$\phi_3 \rightarrow \langle \phi_3 \rangle + \phi_3, \quad (2.16)$$

with the real VEV matrix $V := \langle \phi_3 \rangle = \langle \phi_3 \rangle^*$.⁵ Moreover, we choose the VEV to be diagonal:

$$V = g^{-1} \Lambda, \quad \Lambda = \text{diag}(m_1, \dots, m_{N_c}). \quad (2.17)$$

Here some of the m_a can and will eventually be set to zero, which is convenient to adjust the types of Feynman diagrams with massive and massless propagators that contribute to observables in the planar limit. Due to their different roles we denote the first two complex matrix fields by

$$X = \phi_1, \quad \bar{X} = \phi_1^\dagger, \quad Z = \phi_2, \quad \bar{Z} = \phi_2^\dagger, \quad (2.18)$$

and we introduce the Hermitian matrices

$$Y = \frac{1}{\sqrt{2}} (\phi_3 + \phi_3^\dagger), \quad W = -\frac{1}{\sqrt{2}} i (\phi_3 - \phi_3^\dagger). \quad (2.19)$$

Then we can write the Coulomb-branch Lagrangian of $\mathcal{N} = 4$ SYM theory as

$$\mathcal{L}_{\text{Coul}} = \mathcal{L}_{\mathcal{N}=4} + \mathcal{L}_{\text{VEV}}, \quad (2.20)$$

where we have defined

$$\begin{aligned}\mathcal{L}_{\text{VEV}} = N_c \text{tr} \bigg[& \frac{1}{2} g^2 [V, W]^2 + g^2 [V, \phi_{i'}] [V, \phi_{j'}^\dagger] + \frac{1}{2} g^2 [V, A^\mu]^2 + ig [V, A^\mu] \partial_\mu Y \\ & + \sqrt{2} g^2 \left(\frac{1}{2} [V, W] [Y, W] + \frac{1}{2} [V, A^\mu] [Y, A_\mu] - \frac{1}{2} \{V, Y\} \{ \phi_{j'}, \phi_{j'}^\dagger \} + V \phi_{j'} Y \phi_{j'}^\dagger \right. \\ & \left. + V \phi_{j'}^\dagger Y \phi_{j'} \right) + g \bar{\psi}_4 [V, \bar{\psi}_3] - g i \epsilon_{3j'k'} \psi^{k'} V \psi^{j'} + g \psi_3 [V, \psi_4] - g i \epsilon^{3j'k'} \bar{\psi}_{k'} V \bar{\psi}_{j'} \bigg].\end{aligned}\quad (2.21)$$

⁴Often one defines $\lambda = g_{\text{YM}}^2 N_c$ and $\tilde{g}^2 = g_{\text{YM}}^2 N_c / (4\pi^2)$, which may result in factors of 4π when the results are expressed in terms of $\tilde{g}$.

⁵This reality condition implies that only one of the six real scalars of $\mathcal{N} = 4$ SYM theory acquires a VEV, cf. [42]. Here we denote the VEV entries by m_a since they correspond to physical masses of the theory. In contrast, in section 2.2 only the products of the VEV entries z_a yield physical masses.

Here the indices i', j' are summed from 1 to 2. Next we remove the mixing between the scalar Y and the vector A^μ by adding the $R_{\xi=\alpha}$ gauge fixing term, cf. [42]⁶

$$\mathcal{L}_{\text{gauge}} = -\frac{1}{2}N_c \text{tr} G^2, \quad G = \frac{1}{\sqrt{\alpha}}(\partial_\mu A^\mu + ig\alpha[V, Y]), \quad (2.22)$$

and we add the ghost term

$$\mathcal{L}_{\text{ghost}} = \text{tr} \left(\bar{c} \partial^\mu D_\mu c - g^2 \alpha \bar{c} [V, [V + Y, c]] \right). \quad (2.23)$$

This yields the final VEV Lagrangian

$$\begin{aligned} \mathcal{L}_{\text{VEV}}^{\text{gg}} = N_c \text{tr} & \left[\frac{1}{2} g^2 [V, W]^2 + g^2 [V, \phi_{i'}][V, \phi_{j'}^\dagger] + \frac{1}{2} g^2 [V, A^\mu]^2 + \frac{1}{2} g^2 \alpha [V, Y]^2 \right. \\ & + \sqrt{2} g^2 \left(\frac{1}{2} [V, W][Y, W] + \frac{1}{2} [V, A^\mu][Y, A_\mu] - \frac{1}{2} \{V, Y\} \{ \phi_{j'}, \phi_{j'}^\dagger \} + V \phi_{j'} Y \phi_{j'}^\dagger \right. \\ & \left. \left. + V \phi_{j'}^\dagger Y \phi_{j'} \right) + g \bar{\psi}_4 [V, \bar{\psi}_3] - g i \epsilon_{3j'k'} \psi^{k'} V \psi^{j'} + g \psi_3 [V, \psi_4] - g i \epsilon^{3j'k'} \bar{\psi}_{k'} V \bar{\psi}_{j'} \right. \\ & \left. - \frac{1}{2\alpha} (\partial_\mu A^\mu)^2 + \bar{c} \partial^\mu D_\mu c - g^2 \alpha \bar{c} [V, [V + Y, c]] \right]. \quad (2.24) \end{aligned}$$

Note that the scalar field Y has acquired an α -dependent mass from the above gauge fixing contribution. In the following we will set $\alpha = 1$. Before moving on to deforming this Lagrangian, we note that in components it is easier to identify the mass terms of difference form. For $V^a_b = g^{-1} m_a \delta^a_b$ we have for instance

$$\frac{1}{2} g^2 \text{tr} [V, W]^2 = -\frac{1}{2} (m_a - m_b)^2 W^a_b W^b_a. \quad (2.25)$$

Gamma-deformation. We would like to mimic the derivation of the massless fishnet theory, i.e. to follow the sequence

$$\mathcal{N} = 4 \text{ SYM} \quad \rightarrow \quad \gamma\text{-deformation} \quad \rightarrow \quad \text{fishnet theory}. \quad (2.26)$$

Hence, the next step would be to introduce a gamma-deformation by acting on the Lagrangian with the bilocal phase operator (2.1):

$$\mathcal{P}_\gamma(\Phi_1 \Phi_2 \dots \Phi_n) = \exp \left(\frac{i}{2} \gamma_A \epsilon_{ABC} q^B \wedge q^C \right) \Phi_1 \Phi_2 \dots \Phi_n. \quad (2.27)$$

However, the action of this bi-local phase operator on a trace of fields is only well defined if this trace is invariant under R-symmetry. The traces contributing to the Lagrangian of massless $\mathcal{N} = 4$ SYM theory are R-symmetry singlets, such that the action of the operator $\mathcal{P}_\gamma$ is unique, cf. e.g. [48]. In contrast, this is generically not the case for the traces contributing to the above Coulomb-branch Lagrangian. We thus refine the procedure as

⁶This is particularly important for the double-scaling limit considered below, where Y only decouples if this mixing term is removed.

follows. First we introduce the partition operator Q_{a_j} that maps a trace of fields to a linear combination of non-cyclic monomials⁷

$$Q_{a_j} : \text{tr}(\Phi_1 \Phi_2 \dots \Phi_n) \mapsto a_1 \Phi_1 \Phi_2 \dots \Phi_n + a_2 \Phi_2 \Phi_3 \dots \Phi_1 + \dots + a_n \Phi_n \Phi_1 \dots \Phi_{n-1}, \quad (2.28)$$

with new parameters a_k which obey $\sum_{k=1}^n a_k = 1$. Here the only cyclically invariant choice that makes the above map well defined is $a_k = 1/n$ for $k = 1, \dots, n$, but it is useful to keep the parameters explicit for the moment to see that they drop out in certain limits. Acting on the monomials on the right hand side, we then define the linear map Q^{-1} such that

$$Q^{-1} : \Phi_1 \Phi_2 \dots \Phi_n \mapsto \text{tr}(\Phi_1 \Phi_2 \dots \Phi_n). \quad (2.29)$$

We thus have

$$Q^{-1} Q_{a_j} \text{tr}(\Phi_1 \Phi_2 \dots \Phi_n) = \text{tr}(\Phi_1 \Phi_2 \dots \Phi_n). \quad (2.30)$$

The phase operator P_γ is well defined on the above monomials and we define the twisted Coulomb-branch Lagrangian as

$$\mathcal{L}_{\text{Coul}}^{\gamma, a_j} = Q^{-1} \mathcal{P}_\gamma Q_{a_j} \mathcal{L}_{\text{Coul}}. \quad (2.31)$$

When acting on the Lagrangian of massless $\mathcal{N} = 4$ SYM theory, the insertion of the operator Q makes no difference and we have

$$\mathcal{L}_{\mathcal{N}=4}^\gamma = Q^{-1} \mathcal{P}_\gamma Q_{a_j} \mathcal{L}_{\mathcal{N}=4} = \mathcal{P}_\gamma \mathcal{L}_{\mathcal{N}=4}. \quad (2.32)$$

Hence, by construction this modification of the gamma-deformation only affects the massive VEV part (2.21) of (2.20):

$$\mathcal{L}_{\text{Coul}}^{\gamma, a_j} = \mathcal{L}_{\mathcal{N}=4}^\gamma + Q^{-1} \mathcal{P}_\gamma Q_{a_j} \mathcal{L}_{\text{VEV}}. \quad (2.33)$$

Explicitly, we have in the massless sector (see [11, 50])

$$\begin{aligned} \mathcal{L}_{\mathcal{N}=4}^\gamma = N_c \text{tr} \Bigg[& -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - D^\mu \phi_j^\dagger D_\mu \phi^j + i \bar{\psi}_A^\alpha D_\alpha^\alpha \psi_A^A \\ & - \frac{1}{4} g^2 \{ \phi_i^\dagger, \phi^i \} \{ \phi_j^\dagger, \phi^j \} + g^2 e^{-i\epsilon^{ijk} \gamma_k} \phi_i^\dagger \phi_j^\dagger \phi^i \phi^j \\ & - g e^{-\frac{i}{2} \gamma_j^-} \bar{\psi}_j \phi^j \bar{\psi}_4 + g e^{+\frac{i}{2} \gamma_j^-} \bar{\psi}_4 \phi^j \bar{\psi}_j - i g \epsilon_{ijk} e^{\frac{i}{2} \epsilon_{jkm} \gamma_m^+} \psi^k \phi^i \psi^j \\ & - g e^{+\frac{i}{2} \gamma_j^-} \psi_4 \phi_j^\dagger \psi_j + g e^{-\frac{i}{2} \gamma_j^-} \psi_j \phi_j^\dagger \psi_4 - i g \epsilon^{ijk} e^{\frac{i}{2} \epsilon_{jkm} \gamma_m^+} \bar{\psi}_k \phi_i^\dagger \bar{\psi}_j \Bigg], \end{aligned} \quad (2.34)$$

where one defines

$$\gamma_1^\pm = -\frac{\gamma_3 \pm \gamma_2}{2}, \quad \gamma_2^\pm = -\frac{\gamma_1 \pm \gamma_3}{2}, \quad \gamma_3^\pm = -\frac{\gamma_2 \pm \gamma_1}{2}. \quad (2.35)$$

⁷This is natural from the spin chain point of view, where field traces are considered as spin chain states (monomials) with an extra cyclicity condition. See [49] for similar notation. Introducing the parameters a_k may seem ad hoc at first sight. However, below we will see that these parameters drop out after taking the double-scaling limit that defines the massive fishnet theory. In principle, one should include a grading for fermionic fields into the definition of Q_{a_j} . Here we refrain from making this explicit since all fermions decouple in the limits we are interested in.

We note that at the quantum level completeness of the theory defined by the massless Lagrangian $\mathcal{L}_{\mathcal{N}=4}^\gamma$ requires additional double-trace counter terms, cf. [47, 50–52]. Here our focus lies on making the first step in identifying an integrable massive quantum field theory. We thus ignore these extra contributions, and leave an exploration of the completeness of the considered massive theories for future work.

Similarly to the massless sector, we may evaluate the gamma-deformation of the above VEV contribution

$$\mathcal{L}_{\text{VEV}}^\gamma = Q^{-1} \mathcal{P}_\gamma Q_{a_j} \mathcal{L}_{\text{VEV}}. \quad (2.36)$$

While it is straightforward to obtain $\mathcal{L}_{\text{VEV}}^\gamma$ using computer algebra, the explicit expression for this intermediate result is lengthy and not very illuminating. Keeping the parameters a_j in (2.28) explicit, in the first of the below limits, these parameters will drop out of the Lagrangian. We will then also consider a limit in which the respective parameters remain in the resulting theory.

Massive fishnet theory. Next we want to take a double-scaling limit in the spirit of the massless discussion of [10]. We define the new coupling

$$\xi^2 = g^2 e^{-i\gamma_3}, \quad (2.37)$$

and take the limit⁸

$$g \rightarrow 0, \quad \gamma_3 \rightarrow i\infty, \quad \xi = \text{fix}. \quad (2.38)$$

As shown in [10], in the massless sector this yields the bi-scalar fishnet Lagrangian (modulo kinetic terms for fields that decouple from the theory)

$$\mathcal{L}_F = N_c \text{tr} \left(-\partial_\mu \bar{X} \partial^\mu X - \partial_\mu \bar{Z} \partial^\mu Z + \xi^2 \bar{X} \bar{Z} X Z \right). \quad (2.39)$$

The full Lagrangian to which we refer as the *massive fishnet theory* Lagrangian takes the form

$$\mathcal{L}_{\text{MF}} = \mathcal{L}_F + \mathcal{L}_M, \quad (2.40)$$

where for the VEV part of the gamma-deformed Lagrangian (2.36) the above limit yields

$$\begin{aligned} \mathcal{L}_M &= N_c \text{tr} \left([\Lambda, X][\Lambda, \bar{X}] + [\Lambda, Z][\Lambda, \bar{Z}] \right) \\ &= -N_c (m_a - m_b)^2 X^a_b \bar{X}_a^b - N_c (m_a - m_b)^2 Z^a_b \bar{Z}_a^b. \end{aligned} \quad (2.41)$$

Here in going to the second line we used the explicit form of the diagonal VEV matrix $\Lambda^a_b = m_b \delta_b^a$. We remark that taking the planar limit of this theory corresponds to taking $N_c \rightarrow \infty$ while keeping ξ and the masses m_a fix. As opposed to the SBF Lagrangian (2.7), the above MF Lagrangian (2.40) features mass terms for the two scalars but no three-point vertices.

Note that if in addition we take $\gamma_{1,2} \rightarrow i\infty$ with $\xi_j = \xi^2 = g^2 e^{-i\gamma_j}$ fix (now also for $j = 1, 2$ in (2.38)), some contributions to the VEV Lagrangian scale as (finite)/ g and

⁸For $\gamma_{1,2}$ finite this implies $\xi_{1,2} \rightarrow 0$ with $\xi_j^2 = g^2 e^{-i\gamma_j}$.

diverge in the limit $g \rightarrow 0$. This differs from the massless case where the respective limit yields a finite theory. In the massive case, however, there are alternative ways to the above limit which keep these terms finite. We illustrate this below for a tri-scalar example.

For completeness, we also give the Feynman rules for the massive fishnet theory. The propagator, connecting a conjugate field $\bar{X}$ or $\bar{Z}$ on the left with a field X or Z on the right, respectively, reads

$$\frac{b_1}{a_1} \xrightarrow[p]{a_2} \frac{a_2}{b_2} = \frac{1}{N_c} \frac{\delta_{a_1}^{b_2} \delta_{a_2}^{b_1}}{p^2 + (m_{a_1} - m_{b_1})^2}. \quad (2.42)$$

Moreover, the four-point vertex takes the form

$$\begin{array}{c} \bar{Z} \\ a_2 \downarrow b_2 \\ \bar{X} \xrightarrow[b_1]{a_1} \text{---} \text{---} \xrightarrow[b_3]{a_3} X \\ \uparrow b_4 \quad \uparrow a_4 \\ Z \end{array} = N_c \xi^2 \delta_{a_1}^{b_4} \delta_{a_4}^{b_3} \delta_{a_3}^{b_2} \delta_{a_2}^{b_1}. \quad (2.43)$$

If we consider the VEV scenario described in section 2.4, only some of the masses m_a are non-vanishing such that also massless propagators contribute to the resulting observables.

Tri-scalar double-scaling limit. To take an alternative limit we define three couplings ξ_j differently and according to

$$\xi_1^2 = g e^{-i\gamma_1}, \quad \xi_2^2 = g e^{-i\gamma_2}, \quad \xi_3^2 = g^2 e^{-i\gamma_3}, \quad (2.44)$$

and then take the limit

$$g \rightarrow 0, \quad \gamma_j \rightarrow i\infty, \quad \xi_j = \text{fix}. \quad (2.45)$$

For the resulting Lagrangian $\mathcal{L}^{\text{Tri}} = \mathcal{L}_F^{\text{Tri}} + \mathcal{L}_M^{\text{Tri}}$, the massless sector takes the form

$$\mathcal{L}_F^{\text{Tri}} = N_c \text{tr} \left(-\partial_\mu \bar{X} \partial^\mu X - \partial_\mu \bar{Z} \partial^\mu Z - \frac{1}{2} \partial_\mu Y \partial^\mu Y + \xi_3^2 \bar{X} \bar{Z} X Z \right), \quad (2.46)$$

while the massive contribution becomes

$$\begin{aligned} \mathcal{L}_M^{\text{Tri}} = N_c \text{tr} & \left[[\Lambda, X][\Lambda, \bar{X}] + [\Lambda, Z][\Lambda, \bar{Z}] + \frac{1}{2} [\Lambda, Y]^2 \right. \\ & + \frac{\xi_1^2}{\sqrt{2}} (c_1 \Lambda Y \bar{Z} Z + c_2 \Lambda \bar{Z} Z Y + c_3 \Lambda Z Y \bar{Z}) \\ & \left. + \frac{\xi_2^2}{\sqrt{2}} (c_1 \Lambda Y X \bar{X} + c_2 \Lambda X \bar{X} Y + c_3 \Lambda \bar{X} Y X) \right]. \end{aligned} \quad (2.47)$$

Here the first line contains the difference-mass terms and the parameters c_k correspond to rescaled combinations of the parameters a_j introduced via the map (2.28). For a trace of length 4 the cyclic choice is $a_j = 1/4$ which implies

$$c_k = \frac{1}{4} (-1, -1, +4)_k, \quad k = 1, 2, 3. \quad (2.48)$$

Theory	4pt Vertex	3pt Vertex	Propagators
Spontaneously Broken Fishnets (SBF)	Yes	Yes	Product
Massive Fishnets (MF)	Yes	No	Difference
Tri-Scalar Theory	Yes	Yes	Difference

Table 1. Comparison of the different massive extensions of the fishnet theory discussed here.

We remark that taking the same limit in the purely massless case with $\Lambda = 0$, the scalar Y decouples from the theory whereas in the massive case considered here, it interacts with the scalars X and Z via three-point vertices.

We note that the ladder diagrams of Wick and Cutkosky, which have been known to feature a dual conformal symmetry for a long time [53, 54], are naturally embedded into this theory which includes a massless and two massive scalars. These ladder graphs have massive stringers and massless rungs taking the form:⁹

In conclusion, the massive extensions of the fishnet theory considered in this and in the previous subsection compare as in table 1.

2.4 Planar limit and VEV scenarios

In the massive phases introduced above, taking the large- N_c limit necessitates some care. In fact, while we will keep referring to this limit as the ‘planar limit’, the surviving diagrams at leading order of a $1/N_c$ expansion are not necessarily all (geometrically) planar diagrams of the theory. We consider a diagonal VEV matrix as discussed above:

$$V = \text{diag}(v_1, \dots, v_{N_c}). \quad (2.50)$$

Now there are different possibilities for the scaling of the VEV entries v_j with N_c . A priori, for every v_j , we can distinguish three different behaviors:

$$\lim_{N_c \rightarrow \infty} v_i(N_c) \rightarrow \begin{cases} 0 \\ \text{finite} \\ \infty \end{cases}. \quad (2.51)$$

The degrees of freedom related to diverging VEV entries will be non-dynamical and decouple from the theory. Hence, we are effectively limited to the first two scaling options. In principle, we can then consider arbitrary configurations of vanishing and non-vanishing region masses. However, here we are interested in configurations that inherit as much

⁹For a judicious choice of VEV configuration, as explained in the subsequent section 2.4, the Wick-Cutkosky ladders are the only contributions to the scattering amplitude $A(\bar{X}(m_1), \bar{Z}(m_2), Z(m_2), X(m_1))$ in the tri-scalar theory.

Figure 2. Replacing one external leg of a non-planar 12-point diagram by a VEV turns it into a planar 11-point diagram.

symmetry from the original massless planar theory as possible. This singles out a particular subset of configurations, which have finitely many non-vanishing and infinitely many vanishing VEV entries in the planar limit:

$$V = \text{diag}(v_1, \dots, v_n, 0, 0 \dots, 0). \quad (2.52)$$

SBF theory. To illustrate why this configuration is distinguished in the case of spontaneous symmetry breaking, consider the example of the 12-point diagram in figure 2 which contributes to the non-planar sector of the massless theory. While the left-hand diagram is non-planar, replacing the external leg 12 by a VEV turns it into a (geometrically) planar diagram, which for generic VEV configurations might contribute to observables in the large- N_c limit of the massive theory (spoiling e.g. the dual conformal symmetry). We refer to this class of diagrams as pseudo-planar, because they arise from Higgsing non-planar diagrams. To argue how the VEV configurations, which we singled out above, remove all pseudo-planar diagrams, let us first establish a canonical normalization for scattering amplitudes in the planar limit. We denote the canonically normalized n -point amplitude for the scattering of n particles of type Φ_a by¹⁰

$$\langle \Phi_1 \Phi_2 \dots \Phi_n \rangle = N_c^{-1} \text{out} \langle 0 | S | \Phi_1, \dots \Phi_n \rangle_{\text{in}}, \quad (2.53)$$

as well as its planar limit by

$$\langle \Phi_1 \Phi_2 \dots \Phi_n \rangle_{\text{planar}} = \lim_{\substack{N_c \rightarrow \infty \\ \xi^2 = \text{const.}}} \langle \Phi_1 \Phi_2 \dots \Phi_n \rangle. \quad (2.54)$$

Here we consider the scattering of individual component fields, i.e. the Φ_a carry fundamental and anti-fundamental $\text{SU}(N_c)$ indices. It is easy to see that in the massless phase, this reproduces the usual definition of the planar limit where only geometrically planar diagrams, i.e. diagrams which can be drawn on a plane without self-intersections, contribute to planar amplitudes.

¹⁰While this definition in terms of S -matrix elements of one-particle-states is valid in the massive phase, in the massless phase the right-hand-side is to be understood as a formal object constructed via the LSZ-reduction, cf. [55].

Now note that there are no pseudo-planar diagrams with all internal regions massless in the SBF theory. This is due to the fact that the three-point vertex (2.9) is proportional to the adjacent region mass and vanishes if that region is massless. If there are only finitely many non-vanishing region masses, the planar limit will be dominated by diagrams with vanishing region masses and therefore only proper planar diagrams will contribute. In particular, the canonical normalization (2.53) ensures that observables, whose leading contribution in a $1/N_c$ expansion consists of pseudo-planar diagrams only, vanish in the planar limit.

Massive fishnet theory. Also for the massive fishnet theory introduced in section 2.3 the above VEV scenario is preferred: finitely many region masses take non-vanishing values, while infinitely many vanish, at least in the planar limit. In the MF theory this is advantageous since for this choice only diagrams with massive propagators on the boundary of the Feynman graph contribute to planar scattering amplitudes. These are precisely the Feynman graphs that feature the massive Yangian symmetry [43]. In the following we will thus restrict to this VEV scenario and we will usually assume the first n VEV entries to be non-vanishing, cf. (2.52).

3 Integrability alias Yangian symmetry

It is well known that planar $\mathcal{N} = 4$ SYM theory owes many of its interesting properties to the interplay of its conformal and dual conformal symmetry, which combine to form an infinite dimensional symmetry of Yangian type. Here we aim at studying the symmetry properties of observables that are well defined in massless and massive theories, since we want to understand how integrability carries over from the massless to the massive situation. The objects we will consider are planar color-ordered amplitudes, i.e. the leading coefficients multiplying the different tensor structures in the expansion¹¹

$$\langle (\Phi_1)_{b_1}^{a_1} (\Phi_2)_{b_2}^{a_2} \dots (\Phi_n)_{b_n}^{a_n} \rangle = \sum_{\sigma \in S_n/Z_n} \delta_{b_{\sigma(1)}}^{a_{\sigma(2)}} \delta_{b_{\sigma(2)}}^{a_{\sigma(3)}} \dots \delta_{b_{\sigma(n)}}^{a_{\sigma(1)}} A_n(\Phi_{\sigma(1)}, \dots, \Phi_{\sigma(n)}), \quad (3.1)$$

where $\vec{\Phi} \in \{X, \bar{X}, Z, \bar{Z}\}^n$ and we suppressed the momentum dependence of the external fields. In massless theories (or theories where all fields have the same mass) a customary way of projecting ordinary amplitudes on one particular color-ordered amplitude is by taking a trace over the external color indices. In fact, from the above expansion it is easy to see that

$$A_n(\Phi_1, \dots, \Phi_n) = \frac{1}{N_c^n} \left(\langle \text{tr}(\Phi_1 \Phi_2 \dots \Phi_n) \rangle + \mathcal{O}\left(\frac{1}{N_c}\right) \right). \quad (3.2)$$

For the massive theories introduced above, this identification is not possible since the A_n in (3.1) still depend on the color indices through the region masses. That is, taking the trace leads to a sum over external particles with different masses, whereas every term in (3.1) has well defined external masses. In the present paper we will therefore not consider single-trace amplitudes but rather read off the color-ordered amplitudes from the expansion (3.1).

¹¹The form of this expansion follows from the structure of the double-line Feynman rules.

3.1 Review of massless fishnets

Before we discuss how Yangian symmetry works in the massive theories described above, let us briefly review the way it is realized in the massless fishnet theory. In [12] it was shown that in the planar limit the color-ordered amplitudes defined by (3.1) feature a (dual) conformal Yangian symmetry. Furthermore, since only a single Feynman diagram contributes to every amplitude, this symmetry extends to the diagrammatic level and can be used to bootstrap single Feynman diagrams without reference to a particular quantum field theory [41].

To exhibit the Yangian symmetry, the amplitudes are written in terms of dual position variables, the so-called region momenta x_j . These are related to the original momenta of the momentum space amplitude via

$$p_j = x_j - x_{j+1}. \quad (3.3)$$

In these variables, the Yangian symmetry of amplitudes corresponds to invariance under the action of a set of local and bilocal generators given by¹²

$$J^A = \sum_{k=1}^n J_k^A, \quad \hat{J}^A = \frac{1}{2} f^A_{BC} \sum_{j < k=1}^n J_k^C J_j^B + \sum_{j=1}^n s_j J_j^A, \quad (3.4)$$

where $J^A \in \{P^\mu, L^{\mu\nu}, D, K^\mu\}$ denotes the conformal Lie algebra generators with the local currents

$$\begin{aligned} P_i^\mu &= -i \frac{\partial}{\partial x_{i\mu}}, & K_i^\mu &= -2ix_i^\mu \left(x_i^\nu \frac{\partial}{\partial x_i^\nu} + \Delta_i \right) + ix_i^2 \frac{\partial}{\partial x_{i\mu}}, \\ L_i^{\mu\nu} &= i \left(x_i^\mu \frac{\partial}{\partial x_{i\nu}} - x_i^\nu \frac{\partial}{\partial x_{i\mu}} \right), & D_i &= -i \left(x_i^\mu \frac{\partial}{\partial x_i^\mu} + \Delta_i \right). \end{aligned} \quad (3.5)$$

In (3.4), the f^A_{BC} are the inverse structure constants of the conformal Lie algebra formed by the local currents. The s_j represent the so-called evaluation parameters and their rational values depend on the specific amplitude under consideration. The generators J^A and $\hat{J}^A$ are called level-zero and level-one dual conformal generators, respectively. For example, the level-one momentum generator of the conformal algebra explicitly reads

$$\hat{P}^\mu = \frac{i}{2} \sum_{j,k=1}^n \text{sign}(j-k) \left(D_j P_k^\mu + L_j^{\mu\nu} P_{k\nu} \right) + \sum_{j=1}^n s_j P_j^\mu. \quad (3.6)$$

The other level-one generators are detailed in appendix C. To summarize, the Yangian symmetry implies a set of differential equations that color-ordered amplitudes satisfy, namely

$$J^A A_n = 0, \quad \hat{J}^A A_n = 0. \quad (3.7)$$

Note that for on-shell legs conformal symmetries may be broken in a subtle way that requires further investigation beyond the scope of the present paper, cf. [56–61]. Here we

¹²Note that in addition the definition of the Yangian algebra requires the so-called Serre relations for nested commutators of these generators, cf. [6–9].

will focus on the properties of off-shell amplitudes as the criterion to single out a theory with distinguished symmetry properties.

In [42] it was found that even though its conformal symmetry is spontaneously broken, $\mathcal{N} = 4$ SYM theory on the Coulomb branch still features a massive representation of dual conformal symmetry. This representation can be understood as the ordinary massless representation of conformal symmetry in five spacetime dimensions with the fifth component of the vector x_j^μ playing the role of the mass m_j . Motivated by this result, in the following we study the dual conformal properties of two of the massive models we described above: the spontaneously broken fishnet theory and the massive fishnet theory.

3.2 Spontaneous symmetry breaking

We find that the level-zero dual conformal symmetry of the massless theory carries over directly to the theory with spontaneously broken conformal symmetry (the SBF theory) by simply modifying the scaling and special conformal generator to read

$$\begin{aligned} D_i &= -i \left(x_i^\nu \frac{\partial}{\partial x_i^\nu} + z_i \frac{\partial}{\partial z_i} + \bar{z}_i \frac{\partial}{\partial \bar{z}_i} + \Delta_i \right), \\ K_i^\mu &= -2ix_i^\mu \left(x_i^\nu \frac{\partial}{\partial x_i^\nu} + z_i \frac{\partial}{\partial z_i} + \bar{z}_i \frac{\partial}{\partial \bar{z}_i} + \Delta_i \right) + ix_i^2 \frac{\partial}{\partial x_{i\mu}}. \end{aligned} \quad (3.8)$$

Note that these generators differ from those in the five-dimensional formulation of Coulomb-branch $\mathcal{N} = 4$ SYM in [42].¹³ Still, these generators satisfy the same algebra, thereby forming a representation of dual conformal symmetry.

Of course, Poincaré symmetry, i.e. invariance under the action of P^μ and $L^{\mu\nu}$ as given in (3.5) is manifest. The key to conformal symmetry is the covariant behavior of the product mass propagator, i.e.

$$\begin{aligned} (D_1 + D_2) \frac{1}{x_{12}^2 + z_1 \bar{z}_2} &= \frac{i(2 - \Delta_1 - \Delta_2)}{x_{12}^2 + z_1 \bar{z}_2}, \\ (K_1^\mu + K_2^\mu) \frac{1}{x_{12}^2 + z_1 \bar{z}_2} &= \frac{2i((1 - \Delta_1)x_1^\mu + (1 - \Delta_2)x_2^\mu)}{x_{12}^2 + z_1 \bar{z}_2}, \end{aligned} \quad (3.9)$$

which can clearly be made to vanish by a judicious choice of scaling dimensions Δ_i . Since tree-level diagrams are simple products of single propagators, and since all the conformal generators are first order differential operators, the covariance factor of the whole amplitude is given by the sum of all single-propagator covariance factors. Again, for any diagram there is a choice of scaling dimensions for which the conformal generators annihilate the diagram. To lift this invariance to the level of amplitudes, note that due to the appearance of a three-point vertex in the Lagrangian (2.7),

$$\mathcal{L}_{\text{SBF}} = \mathcal{L}_{\text{F}} + N_c z_a \bar{z}_b \bar{X}^a{}_b X^b{}_a + N_c \xi \left(\bar{z}_a X^a{}_b Z^b{}_c \bar{X}^c{}_a + z_a \bar{X}^a{}_b \bar{Z}^b{}_c X^c{}_a \right), \quad (3.10)$$

the one-to-one correspondence between amplitudes and diagrams breaks down in the spontaneously broken phase. In particular, there can be different diagrams contributing to the

¹³The form of the dilatation operator agrees with the six-dimensional formulation of [62], where z and $\bar{z}$ map to the fifth and sixth components of an extended spacetime vector.

same color-ordered tree-level amplitude, which arise from breaking up a four-point vertex according to

$$\begin{array}{c} b \\ | \\ \text{---} \text{---} \text{---} \\ | \\ a \end{array} \begin{array}{c} c \\ | \\ \text{---} \text{---} \text{---} \\ | \\ d \end{array} + \begin{array}{c} b \\ | \\ \text{---} \text{---} \text{---} \\ | \\ a \end{array} \begin{array}{c} c \\ | \\ \text{---} \text{---} \text{---} \\ | \\ d \end{array} + \begin{array}{c} b \\ | \\ \text{---} \text{---} \text{---} \\ | \\ a \end{array} \begin{array}{c} c \\ | \\ \text{---} \text{---} \text{---} \\ | \\ d \end{array} \sim 1 + \frac{z_a \bar{z}_c}{x_{ac}^2 + z_a \bar{z}_c} + \frac{z_d \bar{z}_b}{x_{db}^2 + z_d \bar{z}_b}. \quad (3.11)$$

All of these contributions have the same covariance behavior under the action of D and K^μ . Therefore all diagrams contributing to a given tree-level amplitude are rendered invariant by the same choice of Δ_i , which implies that all tree-level amplitudes of the theory are dual conformal.

Note that for the VEV scenario distinguished in section 2.4, all integrals contributing to loop-level amplitudes in the SBF theory are identical to the ones appearing in the massless fishnet theory (up to massive tree-level pieces attached to massless loops). In fact, due to the product propagator and the necessity of massless internal regions to avoid pseudo-planar diagrams, all propagators that bound internal regions are massless. Hence, loop amplitudes in this theory are essentially the same as in the massless theory and thus feature the same symmetries, i.e. level-zero and level-one dual conformal alias Yangian symmetry (see appendix D for an example on the level-zero symmetry). These symmetries, however, do not provide us with any information about massive Feynman integrals, since the latter do simply not contribute to the planar limit described above.

3.3 Massive fishnet theory

In the massive fishnet theory, the situation differs significantly. Since the theory features difference-mass propagators, as does $\mathcal{N} = 4$ SYM theory on the Coulomb branch, color-ordered amplitudes are invariant under the action of the same dual conformal generators [42], i.e.

$$\begin{aligned} D_i &= -i \left(x_i^\mu \frac{\partial}{\partial x_i^\mu} + m_i \frac{\partial}{\partial m_i} + \Delta_i \right), \\ K_i^\mu &= -2i x_i^\mu \left(x_i^\nu \frac{\partial}{\partial x_i^\nu} + m_i \frac{\partial}{\partial m_i} + \Delta_i \right) + i(x_i^2 + m_i^2) \frac{\partial}{\partial x_{i\mu}}. \end{aligned} \quad (3.12)$$

Level-zero dual conformal symmetry. Again, level-zero dual conformal symmetry of any given tree-level diagram in the massless fishnet theory follows from covariance of the propagator:

$$\begin{aligned} (D_1 + D_2) \frac{1}{x_{12}^2 + (m_1 - m_2)^2} &= \frac{i(2 - \Delta_1 - \Delta_2)}{x_{12}^2 + (m_1 - m_2)^2}, \\ (K_1^\mu + K_2^\mu) \frac{1}{x_{12}^2 + (m_1 - m_2)^2} &= \frac{2i((1 - \Delta_1)x_1^\mu + (1 - \Delta_2)x_2^\mu)}{x_{12}^2 + (m_1 - m_2)^2}. \end{aligned} \quad (3.13)$$

Loop-level fishnet diagrams with massless internal regions are again invariant since the massless dual conformal generator is essentially a total derivative and can be inserted in loop integrals. Since there are no three-point vertices in the massive fishnet theory, there is

Figure 3. Momentum-space Feynman diagrams (black) in one-to-one correspondence with one-particle-irreducible color-ordered amplitudes in the planar limit of the massive fishnet theory. Solid (dashed) lines correspond to massive (massless) propagators. Gray lines indicate the dual graph corresponding to the same Feynman integral expressed in terms of region momenta.

a one-to-one mapping between color-ordered scattering amplitudes and Feynman diagrams, and this symmetry immediately carries over to the amplitude level.

Note that in contrast to the SBF theory discussed in section 3.2, due to the difference-mass propagators the internal massless regions do not render the entire loop integral massless. The diagrams surviving in the planar limit are fishnet diagrams with massless internal lines in the ‘bulk’ of the diagram and a surrounding massive line. Some low-loop examples are given in figure 3. Therefore the symmetries of this theory are inherited by individual massive Feynman integrals.

Note that the above generators can be interpreted as generating a subgroup of five-dimensional dual conformal symmetry where the mass variable m_i plays the role of the fifth component of the vector x_i^μ , i.e. $x_i^5 = m_i$. Hence, the massive level-zero generators given above are obtained from the massless generators by adding this extra-dimensional component:

$$D \rightarrow D + D_{\text{extra}}, \quad D_{\text{extra}} = -i \sum_{i=1}^n x_i^5 \frac{\partial}{\partial x_i^5}, \quad (3.14)$$

$$K^\mu \rightarrow K^\mu + K_{\text{extra}}^\mu, \quad K_{\text{extra}}^\mu = \sum_{i=1}^n \left(-2i x_i^\mu x_i^5 \frac{\partial}{\partial x_i^5} + i (x_i^5)^2 \frac{\partial}{\partial x_{i\mu}} \right). \quad (3.15)$$

Interestingly, tree-level diagrams are invariant under an even larger symmetry algebra. To be explicit, the algebra can be extended to the full five-dimensional dual conformal algebra by the following six extra operators:

$$\begin{aligned} P_i^5 &= -i \frac{\partial}{\partial m_i}, & L_i^{5\nu} &= i \left(m_i \frac{\partial}{\partial x_{i\nu}} - x_i^\nu \frac{\partial}{\partial m_i} \right), \\ K_i^5 &= 2im_i \left(x_i^\nu \frac{\partial}{\partial x_i^\nu} + m_i \frac{\partial}{\partial m_i} + \Delta_i \right) - i(x_i^2 + m_i^2) \frac{\partial}{\partial m_i}. \end{aligned} \quad (3.16)$$

Here again $m_i = x_i^5$. It is easy to see that P^5 is a symmetry of tree-level amplitudes, since it is the generator of translations in the mass parameters. Invariance under the other generators then follows from commuting P^5 with generators of the four-dimensional dual conformal algebra. However, this extra tree-level symmetry is broken at loop level, since the planar limit described above sets all internal region masses to zero, breaking the invariance under P^5 .

Level-one dual conformal symmetry at tree level and one loop. As for the level-zero symmetry, we also consider an extra-dimensional contribution to the level-one generators:

$$\hat{J}_{(y)}^A = \hat{J}^A + y \hat{J}_{\text{extra}}^A. \quad (3.17)$$

Here the term proportional to the parameter y arises from extending the sum over internal spacetime indices over $D = 5$ instead of $D = 4$ dimensions. For the level-one momentum generator for instance, we have

$$\hat{P}_{\text{extra}}^\mu = \frac{i}{2} \sum_{j,k=1}^n \text{sign}(k-j) P_{j,5} L_k^{\mu 5}, \quad (3.18)$$

which is understood as a contribution originating from a sum over spacetime indices $\hat{\nu}$ ranging now over five dimensions:

$$\hat{P}_{(y=1)}^\mu = \frac{i}{2} \sum_{j,k=1}^n \text{sign}(j-k) \left(D_j P_k^\mu + L_j^{\mu \hat{\nu}} P_{k\hat{\nu}} \right) + \sum_{j=1}^n s_j P_j^\mu. \quad (3.19)$$

It is useful to treat the extra generators à la (3.18) separately since for certain cases they annihilate Feynman integrals on their own and can thus be used to constrain these, cf. [43]. The evaluation parameter s_j entering the above level-one generator $\hat{J}_{(y)}^A$ can be constructed from the corresponding Feynman diagrams by the following rules adapted from [43] (in the present paper we have $D = 4$ and unit propagator weights):¹⁴

1. Start at point x_1 of the dual momentum graph and set $s_1 = (n-1)/2$.
2. To derive s_j , walk from point 1 to point j clockwise along the external boundary of the $(x\text{-space})$ graph and add the respective propagator weights,
3. where an external propagator contributes $-1/2$ while an internal propagator contributes $(D-2)/2 = 1$.
4. All unconnected external points x_j of the dual graph have $s_j = 0$.

¹⁴Note that due to the dual-conformal symmetry of the amplitudes, one is actually free to shift all evaluation parameters by a common offset. For consistency, we still stick to the conventions of [43], where non-dual-conformal integrals were considered as well.

To illustrate these rules, consider the ten-point amplitude

$$A(Z, \bar{X}, \bar{X}, \bar{Z}, \bar{Z}, \bar{Z}, X, X, Z, Z) = \text{Diagram} \quad (3.20)$$

The only contributing diagram is the two-loop double-box diagram in momentum space or the double-cross diagram in dual-momentum space, respectively. The evaluation parameters constructed from the above rules read

$$s_j = \frac{1}{2}(9, 0, 7, 0, 5, 5, 0, 3, 0, 1)_j. \quad (3.21)$$

Effectively, the 10-point level-one generator constructed in this way reduces to a 6-point operator that acts non-trivially only on the six external points of the dual (gray) Feynman graph, cf. [41].

In general, the level-one generators $\hat{J}_{(y=1)}^A$, including the extra-dimensional contribution, annihilate color-ordered amplitudes A_n in the massive fishnet theory at tree level and one loop:

$$\hat{J}_{(y=1)}^A A_n = 0. \quad (3.22)$$

We find that in order to have the full symmetry, some of the amplitudes¹⁵ require the inclusion of the extra-dimensional contributions to the generators in (3.17), while for others it is optional and represents a symmetry on its own, cf. [43]. The diagrams which require the 5D sum, i.e. $y = 1$, are those that feature external regions with more than one adjacent propagator, see figure 4. In terms of the dual graph, this special class of diagrams is classified by the fact that external region momenta are attached to multiple propagators. Moreover, while the level-zero extra symmetries (3.16) are broken by loop corrections, it was found in [43] that the corresponding extensions (e.g. (3.18)) to the level-one generators independently annihilate all one-particle-irreducible one-loop diagrams. Going beyond the scope of [43], we explicitly checked that these also annihilate tree amplitudes as well as one-loop amplitudes with additional tree-level parts glued to them.

Level-one dual conformal symmetry at higher loops. Starting from two loops the extra generators $\hat{J}_{\text{extra}}^A$ do not furnish symmetries. For most loop amplitudes the statement

$$\hat{J}^A A_n = 0 \quad (3.23)$$

is implied (up to two loops) or conjectured (beyond two loops) by the findings of [43]. This statement, however, does not hold for those graphs that feature external regions with more than one adjacent propagator, see figure 4. To turn the massive fishnet amplitudes into invariants under the generators with $y = 0$, we introduce a point-splitting procedure: if

¹⁵The corresponding Feynman diagrams were not considered in [43].

Figure 4. On the left hand side we have an example of a massive tree-level fishnet diagram with multiple propagators adjacent to the same external region 1 (or the dual diagram with multiple propagators connected to the same external region momentum). At tree level and one loop such diagrams are annihilated by the level-one symmetry with extra-dimensional contribution $\hat{J}^A + \hat{J}_{\text{extra}}^A$. At higher loops we introduce a point-splitting in region momentum space alias an additional leg in momentum space (right hand side), such that the diagram is annihilated by the generators $\hat{J}^A$ without extra term.

coincident external points are treated as separate, the generator $\hat{J}^A$ annihilates the amplitude. In momentum space this corresponds to introducing extra external legs between adjacent propagators. This modified amplitude then features the level-one symmetry with 4D sums over spacetime indices, and the actual massive fishnet amplitude can be recovered by taking soft limits in the extra legs. An example of this procedure is shown in figure 4. There is also an interest in not including extra-dimensional contributions to the level-one generators, i.e. in setting $y = 0$, since this simplifies the resulting operator when translated to momentum space (cf. (3.25) below).

Momentum space interpretation of level-one symmetry. In [43] the level-one symmetry $\hat{P}^\mu$ was also translated back to momentum space, where it leads to invariance under the action of a massive conformal algebra generated by the ordinary momentum and Lorentz generators

$$\bar{P}_i^\mu = p_i^\mu, \quad \bar{L}_i^{\mu\nu} = p_i^\mu \frac{\partial}{\partial p_{i\nu}} - p_i^\nu \frac{\partial}{\partial p_{i\mu}}, \quad (3.24)$$

as well as the extended conformal generators

$$\begin{aligned} \bar{D}_j &= p_{j\nu} \frac{\partial}{\partial p_{j\nu}} + \frac{1}{2} \left(m_j \frac{\partial}{\partial m_j} + m_{j+1} \frac{\partial}{\partial m_{j+1}} \right) + \bar{\Delta}_j, \\ \bar{K}_j^\mu &= p_j^\mu \frac{\partial^2}{\partial p_j^\nu \partial p_{j\nu}} - 2\bar{D}_j \frac{\partial}{\partial p_{j\mu}}. \end{aligned} \quad (3.25)$$

In the cases that require the point-splitting procedure described above, physical legs have $\bar{\Delta}_j = 1$, whereas the non-physical extra legs have $\bar{\Delta}_j = 0$.

4 Conformal soft theorems in the SBF theory

In the previous section we have discussed the status of dual conformal symmetry in both the SBF theory as well as the massive fishnet theory, and of ordinary conformal symmetry in the latter theory. To complete the discussion, let us demonstrate the imprints of ordinary conformal symmetry on amplitudes in the SBF theory. It is well known that in theories with

spontaneously broken conformal symmetry the conformal generators control the behavior of scattering amplitudes with soft Goldstone bosons [63, 64] (see also [65–69]). Due to the limited amount of diagrams, the SBF theory furnishes a pedagogical example of this mechanism. For the moment we restrict ourselves to tree-level amplitudes.

Reparametrization of SBF theory. To properly exhibit the soft behavior, the action of the SBF theory must be parametrized in terms of the Goldstone boson mode. The appropriate parametrization of the field Z is given by

$$Z^a{}_b = \left[\left(\frac{z_a}{\xi} + \frac{1}{\sqrt{2}} \sigma(x) \frac{z_a}{z} \right) \delta^a_b + \varphi^a{}_b \right] e^{i \frac{\xi}{\sqrt{2} z_a} \rho(x)}, \quad (4.1)$$

where σ is the Goldstone boson related to spontaneously broken scale invariance (dilaton), whereas ρ is the Goldstone boson related to the broken $U(1)$ symmetry. We furthermore define

$$z^2 = \sum_{a=1}^{N_c} z_a \bar{z}_a. \quad (4.2)$$

Note that φ takes values only in the subset of $\mathfrak{su}(N)$ which is orthogonal to $z_a \delta^a_b / z$ with respect to the scalar product $\langle \cdot, \cdot \rangle = \text{Tr}(\cdot \cdot)$. We then have

$$-N_c \text{Tr}(\partial_\mu \bar{Z} \partial^\mu Z) = \mathcal{L}_{\text{kin}} + \mathcal{L}_{\text{int}, \rho}, \quad (4.3)$$

where we have defined the kinetic part of the Lagrangian as

$$\mathcal{L}_{\text{kin}} = -N_c \left(\frac{1}{2} \partial_\mu \sigma \partial^\mu \sigma + \frac{1}{2} \partial_\mu \rho \partial^\mu \rho + \text{Tr}(\partial_\mu \bar{\varphi} \partial^\mu \varphi) \right), \quad (4.4)$$

as well as interaction part

$$\mathcal{L}_{\text{int}, \rho} = -i \frac{\xi}{\sqrt{2} z} \partial_\mu \rho \text{Tr}(\varphi \partial^\mu \bar{\varphi} - \partial^\mu \varphi \bar{\varphi}) - \frac{\xi}{\sqrt{2} z} \partial_\mu \rho \partial^\mu \rho \sigma \quad (4.5)$$

$$- \frac{\xi^2}{4 z^2} \partial_\mu \rho \partial^\mu \rho \sigma^2 - \frac{\xi^2}{2 z^2} \partial_\mu \rho \partial^\mu \rho \text{Tr}(\varphi \bar{\varphi}). \quad (4.6)$$

Upon expanding Z in the above way, the original interaction term becomes

$$\begin{aligned} \xi^2 \text{Tr}(\bar{X} \bar{Z} X Z) &= z_a \bar{z}_b \bar{X}^a{}_b X^b{}_a + \xi \left(\bar{z}_b \bar{X}^a{}_b X^b{}_c \varphi^c{}_a + z_a \bar{X}^a{}_b \bar{\varphi}^b{}_c X^c{}_a \right) \\ &+ \frac{\xi^2}{\sqrt{2} z} \sigma \left(\bar{z}_b \bar{X}^a{}_b X^b{}_c \varphi^c{}_a + z_b X^a{}_b \bar{X}^b{}_c \bar{\varphi}^c{}_a \right) + \sqrt{2} \frac{\xi}{z} \sigma z_a \bar{z}_b \bar{X}^a{}_b X^b{}_a \\ &+ \frac{\xi^2}{2 z^2} \sigma^2 z_a \bar{z}_b \bar{X}^a{}_b X^b{}_a + \xi^2 \bar{X}^a{}_b \bar{\varphi}^b{}_c X^c{}_d \varphi^d{}_a. \end{aligned} \quad (4.7)$$

Color ordering. Amplitudes in this theory can be color-ordered only with respect to their external X and φ legs. As an example, consider the six-point amplitude

$$\mathcal{A}_6 = \langle \bar{X}(p_1)^{a_1}{}_{b_1} \bar{X}(p_2)^{a_2}{}_{b_2} \varphi(p_3)^{a_3}{}_{b_3} X(p_4)^{a_4}{}_{b_4} X(p_5)^{a_5}{}_{b_5} \sigma(p_6) \rangle. \quad (4.8)$$

It decomposes into different color structures according to

$$\mathcal{A}_6 = \sum_{s \in S_5/Z_5} \delta_{a_s(2)}^{b_s(1)} \delta_{a_s(3)}^{b_s(2)} \delta_{a_s(4)}^{b_s(3)} \delta_{a_s(5)}^{b_s(4)} \delta_{a_s(1)}^{b_s(5)} A_6(\Phi_{s(1)}, \Phi_{s(2)}, \Phi_{s(3)}, \Phi_{s(4)}, \Phi_{s(5)} | \sigma(p_6)), \quad (4.9)$$

where $\vec{\Phi}$ is a vector containing all colored external fields in the amplitude, including their full kinematic, flavor and color information, i.e.

$$\vec{\Phi} = (\bar{X}(p_1)^{a_1}_{b_1}, \bar{X}(p_2)^{a_2}_{b_2}, \varphi(p_3)^{a_3}_{b_3}, X(p_4)^{a_4}_{b_4}, X(p_5)^{a_5}_{b_5}). \quad (4.10)$$

In this example, only a subset of the color structures is non-vanishing. Explicitly, the non-zero color-ordered tree-level amplitudes read

$$\begin{aligned} & A_6(\bar{X}(p_1)^{a_1}_{b_1}, \bar{X}(p_2)^{a_2}_{b_2}, \varphi(p_3)^{a_3}_{b_3}, X(p_4)^{a_4}_{b_4}, X(p_5)^{a_5}_{b_5} | \sigma(p_6)) \\ &= \begin{array}{c} \begin{array}{c} \text{Diagram 1: } \bar{X}(1) \text{ to } 2 \text{ and } 4, \varphi(3) \text{ to } 2 \text{ and } 4, X(4) \text{ to } 2 \text{ and } 4, X(5) \text{ to } 2 \text{ and } 4, \text{ wavy line } 6 \text{ to } 2 \text{ and } 4. \end{array} \\ + \begin{array}{c} \text{Diagram 2: } \bar{X}(1) \text{ to } 2 \text{ and } 4, \varphi(3) \text{ to } 2 \text{ and } 4, X(4) \text{ to } 2 \text{ and } 4, X(5) \text{ to } 2 \text{ and } 4, \text{ wavy line } 6 \text{ to } 2 \text{ and } 4. \end{array} \\ + \begin{array}{c} \text{Diagram 3: } \bar{X}(1) \text{ to } 2 \text{ and } 4, \varphi(3) \text{ to } 2 \text{ and } 4, X(4) \text{ to } 2 \text{ and } 4, X(5) \text{ to } 2 \text{ and } 4, \text{ wavy line } 6 \text{ to } 2 \text{ and } 4. \end{array} \\ + (5 \text{ others}) \end{array} \\ &+ \begin{array}{c} \begin{array}{c} \text{Diagram 4: } \bar{X}(1) \text{ to } 2 \text{ and } 4, \varphi(3) \text{ to } 2 \text{ and } 4, X(4) \text{ to } 2 \text{ and } 4, X(5) \text{ to } 2 \text{ and } 4, \text{ wavy line } 6 \text{ to } 2 \text{ and } 4. \end{array} \\ + \begin{array}{c} \text{Diagram 5: } \bar{X}(1) \text{ to } 2 \text{ and } 4, \varphi(3) \text{ to } 2 \text{ and } 4, X(4) \text{ to } 2 \text{ and } 4, X(5) \text{ to } 2 \text{ and } 4, \text{ wavy line } 6 \text{ to } 2 \text{ and } 4. \end{array} \\ + \begin{array}{c} \text{Diagram 6: } \bar{X}(1) \text{ to } 2 \text{ and } 4, \varphi(3) \text{ to } 2 \text{ and } 4, X(4) \text{ to } 2 \text{ and } 4, X(5) \text{ to } 2 \text{ and } 4, \text{ wavy line } 6 \text{ to } 2 \text{ and } 4. \end{array} \\ + (5 \text{ others}) \end{array} \\ &+ \begin{array}{c} \begin{array}{c} \text{Diagram 7: } \bar{X}(1) \text{ to } 2 \text{ and } 4, \varphi(3) \text{ to } 2 \text{ and } 4, X(4) \text{ to } 2 \text{ and } 4, X(5) \text{ to } 2 \text{ and } 4, \text{ wavy line } 6 \text{ to } 2 \text{ and } 4. \end{array} \\ + \begin{array}{c} \text{Diagram 8: } \bar{X}(1) \text{ to } 2 \text{ and } 4, \varphi(3) \text{ to } 2 \text{ and } 4, X(4) \text{ to } 2 \text{ and } 4, X(5) \text{ to } 2 \text{ and } 4, \text{ wavy line } 6 \text{ to } 2 \text{ and } 4. \end{array} \\ + \begin{array}{c} \text{Diagram 9: } \bar{X}(1) \text{ to } 2 \text{ and } 4, \varphi(3) \text{ to } 2 \text{ and } 4, X(4) \text{ to } 2 \text{ and } 4, X(5) \text{ to } 2 \text{ and } 4, \text{ wavy line } 6 \text{ to } 2 \text{ and } 4. \end{array} \\ + (2 \text{ others}), \end{array} \end{aligned}$$

as well as

$$A_6(\bar{X}(p_1)^{a_1}_{b_1}, \varphi(p_3)^{a_3}_{b_3}, \bar{X}(p_2)^{a_2}_{b_2}, X(p_4)^{a_4}_{b_4}, X(p_5)^{a_5}_{b_5} | \sigma(p_6)) \quad (4.11)$$

$$= \begin{array}{c} \begin{array}{c} \text{Diagram 10: } \bar{X}(1) \text{ to } 2 \text{ and } 4, \varphi(3) \text{ to } 2 \text{ and } 4, X(4) \text{ to } 2 \text{ and } 4, X(5) \text{ to } 2 \text{ and } 4, \text{ wavy line } 6 \text{ to } 2 \text{ and } 4. \end{array} \\ + \begin{array}{c} \text{Diagram 11: } \bar{X}(1) \text{ to } 2 \text{ and } 4, \varphi(3) \text{ to } 2 \text{ and } 4, X(4) \text{ to } 2 \text{ and } 4, X(5) \text{ to } 2 \text{ and } 4, \text{ wavy line } 6 \text{ to } 2 \text{ and } 4. \end{array} \\ + \begin{array}{c} \text{Diagram 12: } \bar{X}(1) \text{ to } 2 \text{ and } 4, \varphi(3) \text{ to } 2 \text{ and } 4, X(4) \text{ to } 2 \text{ and } 4, X(5) \text{ to } 2 \text{ and } 4, \text{ wavy line } 6 \text{ to } 2 \text{ and } 4. \end{array} \\ + (5 \text{ others}), \end{array}$$

and the amplitudes that are related to the above by the permutations of the external X and $\bar{X}$ fields, respectively. Obviously, partial amplitudes with different color-orderings may have different contributing graph topologies. Moreover the Goldstone boson carries no color and does therefore not respect the color-ordering but can attach to any massive line of the diagrams.

Soft theorems. Before the spontaneous symmetry breaking, the fishnet amplitudes featured a conformal symmetry,¹⁶ that is, the tree amplitudes were invariant under the tensor product representation

$$\bar{J}^A = \sum_{k=1}^n \bar{J}_k^A, \quad (4.12)$$

¹⁶We ignore potential conformal anomalies, cf. [56–61].

where the index A runs over the set of conformal generators:

$$\begin{aligned}\bar{P}_i^\mu &= p_i^\mu, & \bar{K}_i^\mu &= p_i^\mu \frac{\partial^2}{\partial p_i^\nu \partial p_{i\nu}} - 2 \left(p_i^\nu \frac{\partial}{\partial p_i^\nu} + \Delta_i \right) \frac{\partial}{\partial p_{i\mu}}, \\ \bar{D}_i &= p_i^\mu \frac{\partial}{\partial p_i^\mu} + \Delta_i, & \bar{L}_i^{\mu\nu} &= p_i^\mu \frac{\partial}{\partial p_{i\nu}} - p_i^\nu \frac{\partial}{\partial p_{i\mu}}.\end{aligned}\tag{4.13}$$

These generators satisfy the commutation relations given in appendix B. Note that on scalar amplitudes where $\Delta_i = 1$, $\bar{K}$ commutes with the on-shell constraint $p^2 = 0$, which is not true in general, cf. [70]. In the broken phase, this leads to the soft theorem for the color-ordered amplitudes [63, 64, 67]

$$\lim_{\tau \rightarrow 0} A_n(\dots | \sigma(\tau p_n)) = \frac{\xi}{\sqrt{2}z} \left[\frac{1}{\tau} S_m^{(-1)} + S_m^{(0)} + S^{(0)} + \tau(S_m^{(1)} + S^{(1)}) \right] A_{n-1}(\dots) + \mathcal{O}(\tau^2),\tag{4.14}$$

where we have rescaled $p_n \rightarrow \tau p_n$ such that $\tau \rightarrow 0$ corresponds to taking the momentum p_n soft. Moreover, we have defined

$$S_m^{(-1)} = \sum_{k=1}^{n-1} \frac{m_k^2}{p_k \cdot p_n} \quad S_m^{(0)} = \sum_{k=1}^{n-1} \frac{m_k^2}{p_n \cdot p_k} p_n \cdot \frac{\partial}{\partial p_k} \quad S_m^{(1)} = \sum_{k=1}^{n-1} \frac{1}{2} \frac{m_k^2}{p_n \cdot p_k} p_n^\mu p_n^\nu \frac{\partial^2}{\partial p_k^\mu \partial p_k^\nu},\tag{4.15}$$

as well as using the generators $\bar{D}$ and $\bar{K}$ of (4.13):

$$S^{(0)} = -\bar{D} + 4, \quad S^{(1)} = \frac{1}{2} p_n \cdot \bar{K}.\tag{4.16}$$

Here the m_k refer to the physical masses of the external legs k , i.e. we have

$$m(\varphi^a_b) = 0, \quad m(X^a_b) = z_a \bar{z}_b.\tag{4.17}$$

Since the soft operators only depend on the external kinematic data and not on the ordering of the legs, they are the same for differently color-ordered partial amplitudes contributing to the same full amplitude. Therefore, the soft theorem also holds at the level of the full amplitudes.

Examples. The soft theorem applies to amplitudes in the SBF theory. We checked it explicitly for the following examples:¹⁷

- a/b) the three-point amplitude with one massless and two massive legs (the T-amplitude),
- c/d) the four-point amplitudes with two massless and two massive legs (the cross and the table amplitude),
- e) the four-point amplitude with four massive legs (the H-amplitude),
- f) the four-point amplitude with one massless and two massive legs as well as one Goldstone boson (thus, in this case the soft limit of a two-Goldstone-boson amplitude is taken),

¹⁷Listed here are the $n - 1$ point amplitudes on the right-hand side of the soft-theorem.

Table 2. List of amplitudes for which the soft theorem has been checked. In particular, the list contains all tree-level amplitudes with four external legs or less. In the cases where multiple diagrams contribute, we have picked a single representative diagram. All other diagrams can be constructed by permuting external legs and splitting up internal four-point vertices.

- g) the six-point amplitude with six massive legs,
- h) the six-point amplitude with two massive and four massless legs (the double-cross amplitude).

As an explicit example, consider the second to last case. Up to cyclic permutations, there are three non-vanishing topologies contributing to color-ordered amplitudes, i.e.

$$\begin{aligned}
 A_6(X_1, X_2, X_3, \bar{X}_4, \bar{X}_5, \bar{X}_6) &= \text{diagram 1} + \text{diagram 2} + \text{diagram 3}, \\
 A_6(X_1, X_2, \bar{X}_4, X_3, \bar{X}_5, \bar{X}_6) &= \text{diagram 4}, \\
 A_6(X_1, X_2, \bar{X}_4, \bar{X}_5, X_3, \bar{X}_6) &= \text{diagram 5}.
 \end{aligned}
 \tag{4.18}$$

Acting on these amplitudes with the soft operators in (4.14), one recovers the soft limit expansion of the amplitude with an additional Goldstone boson attached to the above diagrams at all possible locations.

5 Conclusions and outlook

The most important results of this paper are summarized as follows. Generalizing the derivation of the massless fishnet theory of [10], in section 2.3 we defined a massive fishnet theory as a particular double-scaling limit of $\mathcal{N} = 4$ SYM theory on the Coulomb branch. In the resulting theory, planar scattering amplitudes are conjecturally in one-to-one correspondence¹⁸ with individual massive Feynman integrals of fishnet structure. The theory is

¹⁸As in the massless case, a general proof of this one-to-one correspondence is still lacking.

integrable in the sense that these off-shell amplitudes are annihilated by a massive extension of Yangian symmetry as recently shown in [43].¹⁹ Hence, the situation is very similar to the massless bi-scalar fishnet theory which justifies the name *massive fishnet theory* for the theory defined by the Lagrangian

$$\begin{aligned}\mathcal{L}_{\text{MF}} = & N_c \text{tr} \left(-\partial_\mu \bar{X} \partial^\mu X - \partial_\mu \bar{Z} \partial^\mu Z + \xi^2 \bar{X} \bar{Z} X Z \right) \\ & - N_c (m_a - m_b)^2 X^a{}_b \bar{X}_a^b - N_c (m_a - m_b)^2 Z^a{}_b \bar{Z}_a^b.\end{aligned}\quad (5.1)$$

We have contrasted this massive fishnet theory with the situation of spontaneous symmetry breaking (SBF theory) in the massless fishnet theory. A crucial difference lies in how the masses enter into propagators. While the above massive fishnet theory has massive propagators of difference form, spontaneous symmetry breaking leads to product-mass propagators. As a consequence, for the given VEV configuration, Feynman integrals with massive propagators drop out of scattering amplitudes in the planar limit of the latter theory. Moreover, it is not clear if the notion of Yangian symmetry can be extended to Feynman integrals with product-mass propagators. In section 4 we have explicitly demonstrated that the SBF theory furnishes a pedagogical example where scattering amplitudes obey conformal soft theorems which are controlled by the generators of the (ordinary) conformal algebra. It would certainly be interesting to extend these tests to higher loop orders and to see how the loop-conditions discussed in [31] enter.

In the massless case one can argue that Feynman integrals inherit their integrability from $\mathcal{N} = 4$ SYM theory via the fishnet theory. Here we followed the reverse logic and based our motivation on the Yangian symmetry of massive Feynman integrals, which we now understand as planar off-shell amplitudes of the massive fishnet theory defined through the Lagrangian (2.40). The derivation of this massive fishnet theory from Higgsed $\mathcal{N} = 4$ SYM theory raises the question whether integrability can also be formulated in the latter. In particular, it might be interesting to take a closer look at Coulomb-branch scattering amplitudes with regard to the massive Yangian symmetry discussed here. As a complementary approach one may wonder whether the methods of [49, 71, 72] can be extended to show a Yangian invariance of the action of Coulomb-branch $\mathcal{N} = 4$ SYM theory. That would close the remaining gap in the argument to interpret integrability of the massive fishnet theory as being inherited from a massive integrable mother theory. As a third approach into this direction it should be very enlightening to see if the holographic dual of the massless fishnet theory found in [27, 29, 33] can be extended to the massive case. This would shed light on the interpretation of our findings in the context of the AdS/CFT duality.

It is natural to ask for the properties and potential integrability of the zoo of theories obtained from other double-scaling limits, e.g. the massive tri-scalar theory discussed at the end of section 2.3. This tri-scalar theory shows that different field contents than in the massless case are reachable in the massive situation (cf. [11] for the massless case). A next step would be a systematic investigation of the symmetry properties of the tri-scalar theory

¹⁹Here we emphasize the word ‘off-shell’ since potential subtleties in the on-shell case have not been investigated yet, cf. [56–61].

and to see whether it shows similar integrability properties as the massive fishnet theory. Moreover, one might expect that the tri-scalar theory features a soft theorem similar to the one discussed in section 4 for the SBF theory.

At the more technical level we note that in the derivation of the massive fishnet theory in section 2.3 we have ignored double-trace counter-terms at the intermediate step of the massless gamma-deformed theory [47, 50–52]. Similar terms have later been added to the original definition of the massless fishnet theory [14, 17]. It would be important to investigate whether some observables in the massive fishnet theory require such extra terms for a consistent treatment. Furthermore, we note that the observation of a massive Yangian symmetry in [43] actually applies to D spacetime dimensions. It is thus natural to investigate massive fishnet theories in dimensions different from 4. Moreover, here we have focussed on symmetry properties of scattering amplitudes and Feynman integrals in order to single out a massive fishnet theory. It should be interesting to also look at other observables and to see if integrability facilitates their computation.

Finally, we note that the difference-mass propagators discussed in this paper can be understood as an integrability-preserving regulator for on-shell/collinear divergences of the respective Feynman integrals. It has been known for more than ten years that this mass-regulator preserves the dual conformal part of the Yangian algebra [42]. With the observation of [43] we now know that in fact this regulator preserves a massive extension of the whole Yangian. It is thus a natural next step to exploit this powerful symmetry for the computation of Feynman integrals with on-shell massless external particles as they arise from the massless limit of the diagrams discussed here.

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A Details on Gamma-deformation

The $u(1)^3$ charges q_j entering (2.1) read

	ψ_α^1	ψ_α^2	ψ_α^3	ψ_α^4	A_μ	ϕ_1	ϕ_2	ϕ_3
q_1	$+\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$+\frac{1}{2}$	0	1	0	0
q_2	$-\frac{1}{2}$	$+\frac{1}{2}$	$-\frac{1}{2}$	$+\frac{1}{2}$	0	0	1	0
q_3	$-\frac{1}{2}$	$-\frac{1}{2}$	$+\frac{1}{2}$	$+\frac{1}{2}$	0	0	0	1

To facilitate the comparison of different versions of the Lagrangian of $\mathcal{N} = 4$ SYM theory and its gamma-deformation given in the literature, we note the identity

$$\frac{1}{4} \text{tr}\{\phi_i^\dagger, \phi^i\}\{\phi_j^\dagger, \phi^j\} = \frac{1}{4} \text{tr}[\phi_i^\dagger, \phi^i][\phi_j^\dagger, \phi^j] + \text{tr} \phi_i^\dagger \phi_j^\dagger \phi^j \phi^i. \quad (\text{A.1})$$

B Commutators of conformal generators

The conformal generators (4.13) in momentum space satisfy the following commutation relations:

$$\begin{aligned} [\bar{D}, \bar{P}^\mu] &= \bar{P}^\mu, & [\bar{D}, \bar{K}^\mu] &= -\bar{K}^\mu, \\ [\bar{P}^\mu, \bar{L}^{\nu\rho}] &= \left(\eta^{\mu\nu} \bar{P}^\rho - \eta^{\mu\rho} \bar{P}^\nu\right), & [\bar{K}^\mu, \bar{L}^{\nu\rho}] &= \left(\eta^{\mu\nu} \bar{K}^\rho - \eta^{\mu\rho} \bar{K}^\nu\right), \\ [\bar{K}^\mu, \bar{P}^\nu] &= -2 \left(\eta^{\mu\nu} \bar{D} - \bar{L}^{\mu\nu}\right), & [\bar{L}^{\mu\nu}, \bar{L}^{\rho\sigma}] &= \left(\eta^{\mu\sigma} \bar{L}^{\rho\nu} + (3 \text{ more})\right). \end{aligned} \quad (\text{B.1})$$

All sets of dual conformal generators in section 3 satisfy the commutation relations

$$\begin{aligned} [D, P^\mu] &= i P^\mu, & [D, K^\mu] &= -i K^\mu, \\ [P^\mu, L^{\nu\rho}] &= i \left(\eta^{\mu\nu} P^\rho - \eta^{\mu\rho} P^\nu\right), & [K^\mu, L^{\nu\rho}] &= i \left(\eta^{\mu\nu} K^\rho - \eta^{\mu\rho} K^\nu\right), \\ [K^\mu, P^\nu] &= 2i \left(\eta^{\mu\nu} D - L^{\mu\nu}\right), & [L^{\mu\nu}, L^{\rho\sigma}] &= i \left(\eta^{\mu\sigma} L^{\rho\nu} + (3 \text{ more})\right). \end{aligned} \quad (\text{B.2})$$

Note that one could adopt a normalization in which both algebras are identical. We refrain from doing so to be in line with the existing literature.

C Dual conformal level-one generators

Explicitly, the conformal level-one generators are given by

$$\begin{aligned} \hat{P}^\mu &= \frac{i}{2} \sum_{j,k=1}^n \text{sign}(j-k) \left(D_j P_k^\mu + L_j^{\mu\nu} P_{k\nu} \right) + \sum_{j=1}^n s_j P_j^\mu, \\ \hat{L}^{\mu\nu} &= \frac{i}{2} \sum_{j,k=1}^n \text{sign}(j-k) \left(\frac{1}{2} (K_j^\nu P_k^\mu - K_j^\mu P_k^\nu) + L_j^{\mu\rho} L_{k\rho}{}^\nu \right) + \sum_{j=1}^n s_j L_j^{\mu\nu}, \\ \hat{D} &= \frac{i}{2} \sum_{j,k=1}^n \text{sign}(j-k) \frac{1}{2} K_j^\mu P_{k\mu} + \sum_{j=1}^n s_j D_j, \\ \hat{K}^\mu &= \frac{i}{2} \sum_{j,k=1}^n \text{sign}(j-k) \left(K_j^\mu D_k + L_j^{\mu\nu} K_{k\nu} \right) + \sum_{j=1}^n s_j K_j^\mu, \end{aligned} \quad (\text{C.1})$$

where sums of double indices run over $\mu, \nu, \rho \in \{1, 2, 3, 4\}$.

D Dual conformal symmetry of loop amplitudes in the SBF theory

Let us discuss the particular example of the box diagram, i.e. the amplitude

$$A_8(\bar{X} \bar{X} \bar{Z} \bar{Z} X X Z Z) = \text{Diagram} = \xi^8 \int \frac{dx_A^D}{x_{2A}^2 x_{4A}^2 x_{6A}^2 x_{8A}^2}. \quad (\text{D.1})$$

Within the refined planar limit described in section 2.4 the internal region is massless, leaving only a single diagram which is in fact identical to the corresponding massless amplitude. To prove that this amplitude (and all other loop amplitudes, even the ones that did not appear in the massless theory and have more than one contributing diagram) is covariant, note that the massless special conformal current K_A^μ acting on the internal region is essentially a total derivative and can be inserted under the integral sign,

$$(K_A^\mu - 2iDx_A^\mu)f(x_A) = -i\frac{\partial}{\partial x_A^\nu}\left((2x_A^\mu x_A^\nu - x_A^2\eta^{\mu\nu})f(x_A)\right). \quad (\text{D.2})$$

Therefore, acting with the special conformal generator with $\Delta_{2i} = 1, \Delta_{2i+1} = 0$ on the above amplitude, one finds

$$\begin{aligned} K^\mu A_8(\bar{X}\bar{X}\bar{Z}\bar{Z}XXZZ) &= \xi^8 \int dx_A^D (K_1^\mu + \dots + K_8^\mu + K_A^\mu - 2iDx_A^\mu) \frac{1}{x_{2A}^2 x_{4A}^2 x_{6A}^2 x_{8A}^2} \\ &= 2i\xi^8 \int dx_A^D \frac{(4-D)x_A^\mu}{x_{2A}^2 x_{4A}^2 x_{6A}^2 x_{8A}^2}, \end{aligned} \quad (\text{D.3})$$

which vanishes identically in $D = 4$ dimensions. In the same way the symmetry can be proved to hold for any loop diagram with an arbitrary number of loops: by inserting a massless dual conformal generator for every internal region and using the covariant behavior of every single propagator. Finally, the result will vanish if and only if every internal region is connected to exactly four other regions, i.e. for fishnet diagrams.

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