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Spin-locality of η^2 and $\bar{\eta}^2$ quartic higher-spin vertices

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ABSTRACT: Higher-spin theory contains a complex coupling parameter η . Different higher-spin vertices are associated with different powers of η and its complex conjugate $\bar{\eta}$. Using Z -dominance Lemma of [1], that controls spin-locality of the higher-spin equations, we show that the third-order contribution to the zero-form $B(Z; Y; K)$ admits a Z -dominated form that leads to spin-local vertices in the η^2 and $\bar{\eta}^2$ sectors of the higher-spin equations. These vertices include, in particular, the η^2 and $\bar{\eta}^2$ parts of the ϕ^4 scalar field vertex.

KEYWORDS: Higher Spin Gravity, Higher Spin Symmetry

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In memory of Dima Polyakov.

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1 Introduction

Higher-spin (HS) gauge theory is a theory of an infinite set of gauge fields of all spins. Since gauge invariant HS interaction vertices contain higher derivatives of degrees increasing with spin [2–5], HS gauge theory is not a local field theory in the usual sense. Some arguments that HS gauge theory has to be essentially non-local were given in [6] based on the holographic correspondence with the boundary (critical) sigma-model, conjectured by Klebanov and Polyakov [7] (see also [8]).

To be free from the assumptions of holographic correspondence it is important to analyze the issue of (non)locality of HS gauge theory directly in the bulk. Based on the nonlinear HS equations of [9, 10] such analysis was performed in [11–15] in different sectors of the theory at some lowest orders. All vertices derived in these papers turned

out to be spin-local¹ including some of the quintic vertices in the Lagrangian counting. Also somewhat different arguments pointing out at locality of the HS gauge theory were presented in a recent paper [17].

The obtained vertices agree with holographic prediction at cubic order [18, 19]. However, the bulk vertices derived from nonlinear HS equations so far did not contain the ϕ^4 vertex for the spin-zero field ϕ . On the other hand, it is this vertex [20] that was argued to be highly non-local [6] in the HS theory holographically dual to the boundary sigma-model [7]. The degree of non-locality prescribed by the analysis of [6] led the authors to a conclusion of a fundamental failure for HS holographic reconstruction programme beyond cubic order. Still, the same vertex was also analyzed in [21] concluding that the non-locality if present is of a very special form.

The aim of this paper is to carry out holographically independent approach to the locality problem. We do it by extending the analysis of locality of [1, 13–16] to the vertices of order C^3 in the sector of equations on the zero-forms C that contain in particular the ϕ^4 vertex of interest in the form of ϕ^3 contribution to the field equations. Note that the vertices studied in this paper include an AdS_4 extension of those obtained by Metsaev in [22].

In this paper we give general arguments based on the so-called Z -dominance Lemma of [1] that the holomorphic, i.e., η^2 , and antiholomorphic $\bar{\eta}^2$ vertices in HS gauge theory must be spin-local, where η is a complex parameter in the HS equations. We explicitly demonstrate by direct calculation that every individual contribution to the (anti)holomorphic part of a quartic vertex acquires a form that results in a complete spin-locality of the entire piece. While our analysis is sufficient to see that the result is local it does not give directly the manifestly local form of the remaining local vertex. The derivation of the latter, which uses partial integrations and Schouten identity, we leave for the future. The analysis of the mixed $\eta\bar{\eta}$ vertex that is the only remaining sector in the analysis of locality of the ϕ^4 sector, needs somewhat different tools and is beyond the scope of this paper.

The paper is organized as follows. In section 2, the necessary background on HS equations is presented. Section 3 contains brief recollection on the so called limiting shifted homotopy and the interpretation of the Z -dominance lemma via space $\mathcal{H}^+$ of star-product functions. In section 4, we collect expressions for the holomorphic vertices obtained from the generating equations. Discussion of the obtained results and problems yet to be solved is placed in section 5. Some useful formulas are collected in appendix A. Appendices B and C contain the detailed derivation of the third-order contribution $B_3^{\eta\eta}$ to the zero-form and second-order contribution $W_2^{\eta\eta}$ to the one-form fields, respectively.

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¹Roughly speaking, spin-locality implies that the vertices are local for any finite subset of fields of different spins. More precisely, this is literally the case at the lowest interaction order but may need some further elaboration at higher orders. For more detail on these issues we refer the reader to [16].

2 Recollection of higher-spin equations

In the frame-like formalism [23], unfolded equations for interacting HS fields in AdS_4 can be schematically put into the form [24]

$$d_x \omega + \omega * \omega = \Upsilon(\omega, \omega, C) + \Upsilon(\omega, \omega, C, C) + \dots, \quad (2.1)$$

$$d_x C + \omega * C - C * \omega = \Upsilon(\omega, C, C) + \Upsilon(\omega, C, C, C) + \dots, \quad (2.2)$$

where HS fields are encoded in two generating functions, the one-form

$$\omega(Y, x) = dx^\mu \omega_\mu(Y, x) = \sum_{n, m} dx^\mu \omega_{\mu \alpha_1 \dots \alpha_n, \dot{\alpha}_1 \dots \dot{\alpha}_m}(x) y^{\alpha_1} \dots y^{\alpha_n} \bar{y}^{\dot{\alpha}_1} \dots \bar{y}^{\dot{\alpha}_m}; \quad m + n = 2(s - 1), \quad (2.3)$$

and zero-form

$$C(Y, x) = \sum_{n, m} C_{\alpha_1 \dots \alpha_n, \dot{\alpha}_1 \dots \dot{\alpha}_m}(x) y^{\alpha_1} \dots y^{\alpha_n} \bar{y}^{\dot{\alpha}_1} \dots \bar{y}^{\dot{\alpha}_m}; \quad |m - n| = 2s \quad (2.4)$$

with two-component indices $\alpha, \dot{\alpha} = 1, 2$. $Y^A = (y^\alpha, \bar{y}^{\dot{\alpha}})$ is $sp(4)$ spinor. Field components of definite s are associated with spin- s massless fields, encoding the original Fronsdal field along with all its on-shell nontrivial space-time derivatives. In (2.1), (2.2) and in the sequel all products of the fields are wedge products which is implicit. $d_x = dx^\mu \frac{\partial}{\partial x^\mu}$ is space-time De Rham differential.

Star product is defined as follows

$$f(y, \bar{y}) * g(y, \bar{y}) = \int \frac{d^2 u d^2 v}{(2\pi)^2} \frac{d^2 \bar{u} d^2 \bar{v}}{(2\pi)^2} e^{iu_\alpha v^\alpha + i\bar{u}_{\dot{\alpha}} \bar{v}^{\dot{\alpha}}} f(y + u, \bar{y} + \bar{u}) g(y + v, \bar{y} + \bar{v}). \quad (2.5)$$

The form of the vertices on r.h.s. of (2.1) and (2.2) is determined by the consistency condition with $d_x^2 = 0$. This determines the vertices up to field redefinitions

$$\omega' = F(\omega, C), \quad C' = G(C), \quad (2.6)$$

where $F(\omega, C)$ is linear in the one-form ω , while both $F(\omega, C)$ and $G(C)$ can be nonlinear in C . Indeed, a field redefinition in the consistent system produces another consistent system. Since ω and C contain all on-shell nontrivial derivatives of the Fronsdal fields, non-linear field redefinitions (2.6) may contain infinite tails of higher derivatives thus being non-local though having particular quasi-local form expandable in power series in terms of components of (2.3) and (2.4). So, if system (2.1), (2.2) is local or spin-local (for more detail see [16]) in some specific choice of variables it may lose this property in other variables. Other way around, if system (2.1), (2.2) is non-local in some set of variables, this does not necessarily imply that there is no other set of variables making the system spin-local.

Direct computation of vertices consistent with a given locality requirement from compatibility conditions is technically involved. Indeed, deriving HS equations from formal consistency may result in non-localities due to an inherent natural field redefinition freedom (see [25] for an earlier account). To avoid such non-localities one has to impose extra conditions on field variables that are *a priori* unknown. An alternative scheme making

the derivation of vertices much easier and setting control on field variables is based on the generating system of [10], that has the form

$$d_x W + W * W = 0, \quad (2.7)$$

$$d_x S + W * S + S * W = 0, \quad (2.8)$$

$$d_x B + [W, B]_* = 0, \quad (2.9)$$

$$S * S = i(\theta^A \theta_A + B * \Gamma), \quad \Gamma = \eta \gamma + \bar{\eta} \bar{\gamma}, \quad (2.10)$$

$$[S, B]_* = 0, \quad (2.11)$$

where master fields W , S and B depend on space-time coordinates x and commuting spinor coordinates Y_A and $Z_A = (z_\alpha, \bar{z}_{\dot{\alpha}})$. In what follows the x -dependence is implicit. In addition there is also a dependence on discrete involutive Klein elements $K = (k, \bar{k})$ such that (to simplify formulae, in the sequel the star-product symbol $*$ is implicit in the products with Klein elements, i.e., $AK \equiv A * K$)

$$\{k, y_\alpha\} = \{k, z_\alpha\} = 0, \quad [k, \bar{y}_{\dot{\alpha}}] = [k, \bar{z}_{\dot{\alpha}}] = 0, \quad k^2 = 1, \quad [k, \bar{k}] = 0 \quad (2.12)$$

and analogously for $\bar{k}$.

Star product $*$ acts on functions of Y and Z according to

$$(f * g)(Z, Y) = \frac{1}{(2\pi)^4} \int d^4 U d^4 V f(Z + U; Y + U) g(Z - V; Y + V) \exp(i U_A V^A), \quad (2.13)$$

where $sp(4)$ indices $A, B, \dots$ are raised and lowered by the antisymmetric form $\epsilon_{AB} = -\epsilon_{BA}$ as follows $X^A = \epsilon^{AB} X_B$ and $X_A = X^B \epsilon_{BA}$. Master fields are differential forms with respect to space-time differential dx^ν and auxiliary spinor differential $\theta_A = (\theta_\alpha, \bar{\theta}_{\dot{\alpha}})$ satisfying

$$\{\theta_A, \theta_B\} = 0, \quad \{\theta_\alpha, k\} = \{\bar{\theta}_{\dot{\alpha}}, \bar{k}\} = 0, \quad [\theta_\alpha, \bar{k}] = [\bar{\theta}_{\dot{\alpha}}, k] = 0. \quad (2.14)$$

$B(Z, Y; K|x)$ is a zero-form, while $W(Z, Y; K|x) = dx^\mu W_\mu(Z, Y; K|x)$ is a one-form in space-time differential and $S(Z, Y; K|x) = \theta^\alpha S_\alpha(Z, Y; K|x) + \bar{\theta}^{\dot{\alpha}} \bar{S}_{\dot{\alpha}}(Z, Y; K|x)$ is a one-form in auxiliary spinor differentials. Finally, γ and $\bar{\gamma}$ are central two-forms attributed to the Klein operators

$$\gamma = \exp(iz_\alpha y^\alpha) k \theta^\alpha \theta_\alpha, \quad \bar{\gamma} = \exp(i\bar{z}_{\dot{\alpha}} \bar{y}^{\dot{\alpha}}) \bar{k} \bar{\theta}^{\dot{\alpha}} \bar{\theta}_{\dot{\alpha}}. \quad (2.15)$$

To see that they are central [10], one should use that $\theta^3 = \bar{\theta}^3 = 0$ and that the star-product elements $\kappa := \exp(iz_\alpha y^\alpha)$ and $\bar{\kappa} := \exp(i\bar{z}_{\dot{\alpha}} \bar{y}^{\dot{\alpha}})$ have the properties analogous to (2.12) with respect to star products with $f(Y, Z)$ but commute with the Klein operators and differentials θ^α and $\bar{\theta}^{\dot{\alpha}}$.

2.1 Perturbation theory

A proper HS vacuum is the following exact solution of (2.7)–(2.11)

$$B_0 = 0, \quad (2.16)$$

$$S_0 = \theta^\alpha z_\alpha + \bar{\theta}^{\dot{\alpha}} \bar{z}_{\dot{\alpha}}, \quad (2.17)$$

$$W_0 = \omega(Y|x), \quad d_x \omega + \omega * \omega = 0. \quad (2.18)$$

The flat connection ω can be chosen to describe AdS_4 . Since vacuum value of S_0 is non-trivial it is going to generate via star (anti)commutators PDEs in Z for master fields. Indeed, consider equation (2.11) in the first order

$$[S_0, B_1]_* + [S_1, B_0]_* = 0. \quad (2.19)$$

Using star product (2.13) one can check that

$$[Z_A, f(Z, Y; \theta)]_* = -2i \frac{\partial}{\partial Z^A} f(Z, Y; \theta). \quad (2.20)$$

It means that B_1 field is Z -independent

$$B_1(Z, Y) = C(Y), \quad (2.21)$$

which is the generating function for HS curvatures (2.4). Lower index in B_1 , shows the order of the expression in the C -field. Correspondingly, B_2 is of second order in C and so on

$$B = C + B_2(C, C) + B_3(C, C, C) + \dots \quad (2.22)$$

The same rule applies to S and W , i.e.,

$$S = S_0 + S_1(C) + S_2(C, C) + \dots, \quad W = \omega + W_1(\omega, C) + W_2(\omega, C, C) + \dots \quad (2.23)$$

To solve for Z -dependence of master fields at each perturbation order one has to solve equation of the form

$$d_Z f(Z; Y; \theta) = J(Z; Y; \theta), \quad d_Z := \theta^A \frac{\partial}{\partial Z^A}, \quad (2.24)$$

where J originates from the lower-order terms and f is either B , S or W . Let us note that it is vacuum solution of the auxiliary field S_0 (2.17) manifesting itself in operator d_Z that allows one to capture the field redefinition ambiguity as its kernel. Different solutions can be obtained by one or another contracting homotopy operator Δ

$$f = \Delta J \quad (2.25)$$

resulting from the standard homotopy trick. For an operator ∂ , that should be also nilpotent, $\partial^2 = 0$, one considers operator

$$N = d_Z \partial + \partial d_Z. \quad (2.26)$$

If N is diagonalizable one can introduce the almost inverse operator N^* . Since all the d_Z -cohomologies are in the kernel of N one rewrites the solution to (2.24) as

$$f = \partial N^* J = \Delta J. \quad (2.27)$$

The simplest choice for ∂ is

$$\partial = Z^A \frac{\partial}{\partial \theta^A}. \quad (2.28)$$

Then N turns out to be an Euler operator and thus easily invertible

$$N = Z^A \frac{\partial}{\partial Z^A} + \theta^A \frac{\partial}{\partial \theta^A}, \quad N^* J(Z; Y; \theta) = \int_0^1 \frac{dt}{t} J(tZ; Y; t\theta). \quad (2.29)$$

This choice of ∂ leads to contracting homotopy operator

$$\Delta_0 J(Z; Y; \theta) = Z^A \frac{\partial}{\partial \theta^A} \int_0^1 dt \frac{1}{t} J(tZ; Y; t\theta), \quad (2.30)$$

referred to as the *conventional* homotopy operator in [14].

Using (2.10) and (2.17) one finds in particular

$$-2\text{id}_z S_1^\eta = i\eta C * \gamma = i\eta \theta^\alpha \theta_\alpha e^{iz_\alpha y^\alpha} C(-z, \bar{y}) k. \quad (2.31)$$

Using now (2.30) we find solution for S_1^η in the form

$$S_1^\eta = \eta \theta^\alpha z_\alpha \int_0^1 dt t \exp\{itz_\alpha y^\alpha\} C(-tz, \bar{y}) k. \quad (2.32)$$

Different choices of homotopy operators represent gauge and field redefinition ambiguity. A particular class of the so-called shifted homotopies can be defined by considering $Z^A - Q^A$ instead of Z^A with some Z -independent Q^A . Local properties of HS vertices crucially depend on properties of the chosen homotopy operators. A class of homotopy operators consistent with locality requirement based on shifted homotopies at non-trivial interaction level was proposed in [15]. Let us also note that another way of fixing the Z -dependence based on the so-called gauge function method is reviewed in [26] (see also [27] for applications to various backgrounds).

2.1.1 Notation

Let us set up our notation. Derivative with respect to holomorphic argument of the C -fields is denoted as $\partial_{i\alpha}$ where index i indicates position of the C -field in expression that contains several C 's as seen from left to right. Derivative with respect to holomorphic argument of ω -field is denoted as $\partial_{\omega\alpha}$.

Whenever arguments of C 's or ω are not written explicitly we assume the *exponential* form. This means the following: suppose one has ωCCC . Then it should be understood as

$$\omega(y_\omega, \bar{y}) \bar{*} C(y_1, \bar{y}) \bar{*} C(y_2, \bar{y}) \bar{*} C(y_3, \bar{y}), \quad (2.33)$$

where $\bar{*}$ denotes the star product with respect to the barred variables. Derivatives ∂_ω and ∂_i act as

$$\partial_{\omega\alpha} = \frac{\partial}{\partial y_\omega^\alpha}, \quad \partial_{i\alpha} = \frac{\partial}{\partial y_i^\alpha} \quad (2.34)$$

followed by all the auxiliary variables set to zero, i.e.,

$$y_\omega = y_i = 0. \quad (2.35)$$

To make contact with the p, t notation of [14] and [15] note that

$$t_\alpha = -i\partial_{\omega\alpha}, \quad p_{i\alpha} = -i\partial_{i\alpha}. \quad (2.36)$$

3 Limiting homotopy procedure, subspace $\mathcal{H}^+$ and Z -dominance lemma

The limiting shifted contracting homotopy was introduced in [15] as the generalization of the shifted homotopy introduced in [14]

$$\begin{aligned} \Delta_{q,\beta} f(z, y|\theta) &= \\ &= \int \frac{d^2 u d^2 v}{(2\pi)^2} e^{iu_\alpha v^\alpha} (z^\alpha + q^\alpha + u^\alpha) \frac{\partial}{\partial \theta^\alpha} \int_0^1 \frac{dt}{t} f(tz - (1-t)(q+u), y + \beta v|t\theta), \end{aligned} \quad (3.1)$$

where q^α is a z -independent spinorial shift parameter while $\beta \in (-\infty, 1)$ is a free parameter. For simplicity we confine ourselves to the holomorphic sector of undotted spinors which is of most interest in this paper. (Antiholomorphic sector of dotted spinors is analysed analogously.)

This operator satisfies the following resolution of identity

$$d_z \Delta_{q,\beta} + \Delta_{q,\beta} d_z = 1 - h_{q,\beta}, \quad (3.2)$$

where

$$h_{q,\beta} f(z, y|\theta) = \int \frac{d^2 u d^2 v}{(2\pi)^2} e^{iu_\alpha v^\alpha} f(-q - u, y + \beta v|0) \quad (3.3)$$

is the projector on d_z -cohomology.

We say that function $f(z, y)$ of the form

$$f(z, y|\theta) = \int_0^1 d\mathcal{T} e^{i\mathcal{T} z_\alpha y^\alpha} \phi(\mathcal{T} z, y|\mathcal{T}\theta, \mathcal{T}) \quad (3.4)$$

belongs to the space $\mathcal{H}^+$ if there exists such real $\varepsilon > 0$, that

$$\lim_{\mathcal{T} \rightarrow 0} \mathcal{T}^{1-\varepsilon} \phi(w, u|\theta, \mathcal{T}) = 0. \quad (3.5)$$

Note that the definition of space $\mathcal{H}^+$ is relaxed compared to that of space $\mathcal{H}^{+0}$ of [16] because it does not require any specific behaviour of the ϕ at $\mathcal{T} \rightarrow 1$. Nevertheless in our calculations sometimes it is convenient to use specific form degree relations of [16] that describe star products for the forms of specific degrees p, p' , belonged to spaces $\mathcal{H}_p^{0+}$ and $\mathcal{H}_{p'}^{+0}$.

There are two main options that appear in the computations below to satisfy (3.5):

$$\phi_1(\mathcal{T} z, y|\mathcal{T}\theta, \mathcal{T}) = \frac{\mathcal{T}^{\delta_1}}{\mathcal{T}} \tilde{\phi}_1(\mathcal{T} z, y|\mathcal{T}\theta), \quad \phi_2(\mathcal{T} z, y|\mathcal{T}\theta, \mathcal{T}) = \theta(\mathcal{T} - \delta_2) \frac{1}{\mathcal{T}} \tilde{\phi}_2(\mathcal{T} z, y|\mathcal{T}\theta) \quad (3.6)$$

with some $\delta_{1,2} > 0$ and step-function $\theta(x)$. (Note that according to [16] the poles in $\mathcal{T}$ in (3.6) are fictitious being cancelled by the $\mathcal{T}$ -dependence of z - and θ -dependent terms in (3.4)). Functions $\tilde{\phi}_i(\mathcal{T} z, y|\mathcal{T}\theta)$ are formal power series in variables $\mathcal{T} z$ and y as they are realized as star products of various powers of C 's and ω 's. Functions of the form ϕ_2 result from the decomposition $1 = \theta(a - \epsilon) + \theta(\epsilon - a)$.

Space $\mathcal{H}^+$ can be represented as the direct sum

$$\mathcal{H}^+ = \mathcal{H}_0^+ \oplus \mathcal{H}_1^+ \oplus \mathcal{H}_2^+, \quad (3.7)$$

where $\mathcal{H}_p^+$ are spanned by the degree- p functions in θ with kernels that satisfy (3.5).

Equations of motion (2.1), (2.2) resulting from nonlinear system (2.7)–(2.11) have r.h.s.'s independent of Z^A and θ^A since they belong to the sector of zero-forms in θ and are d_Z -closed as a consequence of equations (2.7)–(2.11) resolved at the previous stages. On the other hand, various terms contributing to the r.h.s.'s of equations (2.1), (2.2) as a result of solution of equations (2.7)–(2.11) are of the form (3.4). In particular, each of these terms is usually Z -dependent. While r.h.s.'s of (2.1), (2.2) are Z -independent as a consequence of equations (2.7)–(2.11), the fact that the sum of all of them is Z -independent is not obvious, demanding an appropriate partial integrations over homotopy parameters that appear at various stages of the order-by-order analysis of nonlinear HS equations. After all, functions (3.4) can be Z -independent only if they have a distributional measure supported at $\mathcal{T} = 0$, i.e., after appropriate partial integrations the measure contains a factor of $\delta(\mathcal{T})$. Such a measure has dimension -1 in $\mathcal{T}$. If a function contains an additional factor of $\mathcal{T}^\varepsilon$, it cannot contribute to the Z -independent answer. This just means that functions of the class $\mathcal{H}_0^+$ cannot contribute to the Z -independent equations (2.1), (2.2). This is the content of Z -dominance Lemma of [1]: any terms in $\phi(w, u|\theta, \mathcal{T})$ dominated by a positive power of $\mathcal{T}$ do not contribute to the dynamical equations (2.1), (2.2). Application of this fact to locality is straightforward once it is shown that all terms containing infinite towers of higher derivatives in the vertices of interest belong to $\mathcal{H}_0^+$ and, therefore, do not contribute to HS equations (2.2). This is what is shown in this paper.

A related fact is that, as shown in [15], the space $\mathcal{H}^+$ exhibits special properties under the action of the limiting shifted homotopy $\Delta_{q,\beta}$ at $\beta \rightarrow -\infty$ leading to local HS interactions. Namely, it maps $\mathcal{H}_1^+$ to $\mathcal{H}_0^+$ [16],

$$\lim_{\beta \rightarrow -\infty} \Delta_{q,\beta} f_1(z, y|\theta) = f_0(z, y|0), \quad \forall f_1 \in \mathcal{H}_1^+, \quad f_0 \in \mathcal{H}_0^+. \quad (3.8)$$

Since elements of $\mathcal{H}_0^+$ do not contribute to Z -independent physical vertex by Z -dominance lemma, this property allows us to discard all terms from $\mathcal{H}^+$ in the analysis of the ωC^3 vertex in equation (2.2).

More in detail, to prove that the vertex is spin-local we find it most efficient to represent the r.h.s. of equations in the d_Z -exact spin-local form modulo terms from $\mathcal{H}^+$ that do not contribute to the final vertex by Z -dominance lemma. In other words, to solve equation (2.24) we represent J on its r.h.s. in the form

$$J = d_Z \hat{f} + J^+, \quad J^+ \in \mathcal{H}^+, \quad (3.9)$$

where $\hat{f}$ is spin-local. Since $d_Z J = 0$ and $d_Z^2 = 0$, we conclude that

$$d_Z J^+ = 0. \quad (3.10)$$

We now note that for the sum of two d_Z -closed terms one can use independent contracting homotopies for each of the summands. Hence, we can write for f on the l.h.s. of (2.24)

$$f = \hat{f} + \lim_{\beta \rightarrow -\infty} \Delta_{q,\beta} J^+ \quad (3.11)$$

with some q . Since $J^+ \in \mathcal{H}^+$, with this definition the contribution of J^+ will give zero to the dynamical field equations for any q while $\hat{f}$ will give a spin-local contribution. Details

of the derivation of the decomposition (3.9) in the B and W sectors are given in appendices B and C, respectively.

To put it differently, to eliminate Z -dependence of a seemingly Z -dependent expression we manage to show that the vertices $\Upsilon^{\eta\eta}(\omega, C, C, C)$ and $\Upsilon^{\bar{\eta}\bar{\eta}}(\omega, C, C, C)$ are spin-local modulo terms from $\mathcal{H}^+$ that vanish by Z -dominance lemma, i.e.,

$$\Upsilon(Y) = \hat{\Upsilon}(Z, Y) + \Upsilon_+(Z, Y), \quad \Upsilon_+ \in \mathcal{H}^+, \quad (3.12)$$

where $\hat{\Upsilon}$ is spin-local but Z -dependent expression. Now we observe that since the physical vertex $\Upsilon(Y)$ is θ , Z -independent it can be written in the form

$$\Upsilon(Y) = \hat{\Upsilon}(Z, Y) + \Upsilon_+(Z, Y) = h_{q,\beta}(\hat{\Upsilon}(Z, Y) + \Upsilon_+(Z, Y)) \quad (3.13)$$

where $h_{q,\beta}$ is the cohomology projector with any shift parameters q and β (recall that, as is obvious from (3.3), cohomology projectors leave θ , Z -independent functions invariant [15]). Taking the limit $\beta \rightarrow -\infty$ we find that, for any q , components of $\hat{\Upsilon}(Z, Y)$ can contribute to the resulting vertex $\Upsilon(Y)$ while $\Upsilon_+(Z, Y)$ cannot since the limiting projector $h_{q,-\infty}$ (3.3) acts trivially on $\mathcal{H}_0^+$ [16],

$$\lim_{\beta \rightarrow -\infty} h_{q,\beta} f_0(Z, Y|0) = 0, \quad f_0 \in \mathcal{H}_0^+. \quad (3.14)$$

This implies that

$$h_{q,-\infty}(\hat{\Upsilon}(Z, Y) + \Upsilon_+(Z, Y)) = h_{q,-\infty}(\hat{\Upsilon}(Z, Y)). \quad (3.15)$$

This formula provides an alternative interpretation of Z -dominance Lemma stating that elements of $\mathcal{H}_0^+$ do not contribute to the physical Z -independent vertices though practically, its application with any q may not be useful since in most cases the result has a seemingly non-local form.

On the other hand, by Z -dominance lemma, a spin-local vertex $\hat{\Upsilon}(Z, Y)$ must be decomposable into a sum

$$\hat{\Upsilon}(Z, Y) = \Upsilon^{loc}(Y) + \Upsilon_+^{loc}(Z, Y), \quad \Upsilon_+^{loc} \in \mathcal{H}^+ \quad (3.16)$$

with a spin-local Z -independent $\Upsilon^{loc}(Y)$. Hence one can conclude that $\Upsilon_+^{loc}(Z, Y) + \Upsilon_+(Z, Y) = 0$ which fact is not at all manifest being a consequence of Schouten identity and various relations between integrals over homotopy parameters.

Of course, once $\Upsilon^{loc}(Y)$ is found, the application of $h_{q,-\infty}$ with any q to the r.h.s. of (3.12) gives a spin-local vertex $\Upsilon(Y) = \Upsilon^{loc}(Y)$. It should be stressed again however that, though one can formally obtain the final result via application of any cohomology projector, it will be spin-local, but not manifestly spin-local, containing non-local contributions that will all cancel by virtue of partial integrations and Schouten identity used in the derivation of the decomposition (3.16) which is practically easier to find. As will be demonstrated in the forthcoming paper [28], being technically involved, this approach makes it possible to compute explicit form of the physical spin-local vertices.

4 Final results

4.1 General structure of equations

Dynamical equations up to the third order in the zero-forms C can be schematically put into the form

$$\begin{aligned} d_x C + [\omega, C]_* = & \Upsilon^\eta(\omega, C, C) + \Upsilon^{\bar{\eta}}(\omega, C, C) + \Upsilon^{\eta\eta}(\omega, C, C, C) \\ & + \Upsilon^{\bar{\eta}\bar{\eta}}(\omega, C, C, C) + \Upsilon^{\eta\bar{\eta}}(\omega, C, C, C) + \dots \end{aligned} \quad (4.1)$$

The vertex $\Upsilon^{\eta\eta}(\omega, C, C, C)$ resulting from system (2.7)–(2.11) has the form

$$\Upsilon^{\eta\eta}(\omega, C, C, C) = -d_x B_3^{\eta\eta} - d_x B_2^\eta - [\omega, B_3^{\eta\eta}]_* - [W_1^\eta, B_2^\eta]_* - [W_2^{\eta\eta}, C]_* . \quad (4.2)$$

Here $W_1^\eta, W_2^{\eta\eta}$ and $B_2^\eta, B_3^{\eta\eta}$ are master fields of the corresponding orders from expansions (2.22), (2.23) which are to be obtained from the generating system via solving equation of the type (2.24). Each term on the r.h.s. of this equation depends both on Y and on Z . These vertices can be decomposed into two parts

$$\Upsilon^{\eta\eta}(\omega, C, C, C) = \hat{\Upsilon}^{\eta\eta}(\omega, C, C, C) + \Upsilon_+^{\eta\eta}(\omega, C, C, C), \quad \Upsilon_+^{\eta\eta}(\omega, C, C, C) \in \mathcal{H}_0^+, \quad (4.3)$$

where unlike the whole $\Upsilon^{\eta\eta}(\omega, C, C, C)$ its two contributions on the right of (4.3) can be z -dependent.

In this paper we compute the $\hat{\Upsilon}^{\eta\eta}(\omega, C, C, C)$ part of the vertices. This part turns out to be free from contractions between holomorphic variables of the C -fields because such terms belong to $\mathcal{H}_0^+$. Consistency of equations (2.7)–(2.11) guarantees that $\Upsilon^{\eta\eta}(\omega, C, C, C)$ is Z -independent and, according to Z -dominance Lemma, it can be realized only as $\delta(\mathcal{T})$ in the kernel. Hence Z -independent expression for $\Upsilon^{\eta\eta}(\omega, C, C, C)$ must be free from infinite tower of contractions between holomorphic variables which implies spin-locality of the resulting HS equations.

In this section we present final expression for $\hat{\Upsilon}^{\eta\eta}(\omega, C, C, C)$

$$\hat{\Upsilon}^{\eta\eta}(\omega, C, C, C) = \hat{\Upsilon}_{\omega CCC}^{\eta\eta} + \hat{\Upsilon}_{C\omega CC}^{\eta\eta} + \hat{\Upsilon}_{CC\omega C}^{\eta\eta} + \hat{\Upsilon}_{CC C\omega}^{\eta\eta} \quad (4.4)$$

obtained from the generating system (2.7)–(2.11) using the perturbation scheme up to the third order in C -field. Details of their derivation are presented in appendices B and C.

The vertices in (4.4) are composed from the following terms

$$\hat{\Upsilon}_{\omega CCC}^{\eta\eta} \approx -d_x \hat{B}_3^{\eta\eta} \Big|_{\omega CCC} - \omega * \hat{B}_3^{\eta\eta} - d_x B_2^{\eta loc} \Big|_{\omega CCC} - W_{1\omega C}^\eta * B_2^{\eta loc} - \widehat{W}_{2\omega CC}^{\eta\eta} * C, \quad (4.5)$$

$$\hat{\Upsilon}_{C\omega CC}^{\eta\eta} \approx -d_x \hat{B}_3^{\eta\eta} \Big|_{C\omega CC} - d_x B_2^{\eta loc} \Big|_{C\omega CC} - W_{1C\omega}^\eta * B_2^{\eta loc} - \widehat{W}_{2C\omega C}^{\eta\eta} * C + C * \widehat{W}_{2\omega CC}^{\eta\eta}, \quad (4.6)$$

$$\hat{\Upsilon}_{CC\omega C}^{\eta\eta} \approx -d_x \hat{B}_3^{\eta\eta} \Big|_{CC\omega C} - d_x B_2^{\eta loc} \Big|_{CC\omega C} + B_2^{\eta loc} * W_{1\omega C}^\eta - W_{2CC\omega}^{\eta\eta} * C + C * W_{2C\omega C}^{\eta\eta}, \quad (4.7)$$

$$\hat{\Upsilon}_{CC C\omega}^{\eta\eta} \approx -d_x \hat{B}_3^{\eta\eta} \Big|_{CC C\omega} + \hat{B}_3^{\eta\eta} * \omega - d_x B_2^{\eta loc} \Big|_{CC C\omega} + B_2^{\eta loc} * W_{1C\omega}^\eta + C * \widehat{W}_{2CC\omega}^{\eta\eta}. \quad (4.8)$$

The expression for $B_2^{\eta loc}$ has the form [13]

$$\begin{aligned} B_2^{\eta loc} = & \frac{\eta}{2} \int d^3 \tau_+ \left[\delta' \left(1 - \sum_{i=1}^3 \tau_i \right) - i z_\alpha y^\alpha \delta \left(1 - \sum_{i=1}^3 \tau_i \right) \right] e^{i\tau_1 z_\alpha y^\alpha + i\tau_1 \partial_{1\alpha} \partial_2^\alpha} \\ & \times C(-\tau_1 z + \tau_2 y, \bar{y}) * C(-\tau_1 z - \tau_3 y, \bar{y}) k, \end{aligned} \quad (4.9)$$

where we use a short-hand notation

$$d^3\tau_+ := d\tau_1 d\tau_2 d\tau_3 \theta(\tau_1) \theta(\tau_2) \theta(\tau_3). \quad (4.10)$$

Whenever this notation is used there is always a delta-function in a corresponding expression $\delta(1 - \tau_1 - \tau_2 - \tau_3)$ that bounds the τ_i variables from above. Note that $B_2^{\eta loc}$ is a sum of B_2^η obtained in [14] and local cohomology (i.e., Z -independent) shift δB_2^η

$$\begin{aligned} B_2^{\eta loc} &= B_2^\eta + \delta B_2^\eta, \\ \delta B_2^\eta &= \frac{\eta}{2} \int d^2\tau_+ \delta(1 - \tau_1 - \tau_2) C(\tau_1 y, \bar{y}) \bar{*} C(-\tau_2 y, \bar{y}) k. \end{aligned} \quad (4.11)$$

The expressions $W_{1\omega C}^\eta$ and $W_{1C\omega}^\eta$ were obtained in [14], having the form

$$\begin{aligned} W_{1\omega C}^\eta &= -\frac{\eta}{2} \int_0^1 d\tau_1 \int_0^1 d\sigma (1 - \tau_1) (z^\alpha \partial_{\omega\alpha}) \exp \left\{ i\tau_1 z_\alpha y^\alpha + i(1 - (1 - \tau_1)\sigma) \partial_{\omega\alpha} \partial_1^\alpha \right\} \\ &\quad \times \omega(-\tau_1 z + (1 - \tau_1)\sigma y, \bar{y}) \bar{*} C(-\tau_1 z, \bar{y}) k, \end{aligned} \quad (4.12)$$

$$\begin{aligned} W_{1C\omega}^\eta &= -\frac{\eta}{2} \int_0^1 d\tau_1 \int_0^1 d\sigma (1 - \tau_1) (z^\alpha \partial_{\omega\alpha}) \exp \left\{ i\tau_1 z_\alpha y^\alpha + i(1 - (1 - \tau_1)\sigma) \partial_{1\alpha} \partial_\omega^\alpha \right\} \\ &\quad \times C(-\tau_1 z, \bar{y}) \bar{*} \omega(-\tau_1 z - (1 - \tau_1)\sigma y, \bar{y}) k. \end{aligned} \quad (4.13)$$

Note that the terms with d_x contribute to the third order via the second-order contribution to $d_x C$

$$d_x C = C * \omega - \omega * C + \Upsilon_{\omega CC}^\eta + \delta \Upsilon_{\omega CC}^\eta + \Upsilon_{C\omega C}^\eta + \delta \Upsilon_{C\omega C}^\eta + \Upsilon_{CC\omega}^\eta + \delta \Upsilon_{CC\omega}^\eta. \quad (4.14)$$

Here $\Upsilon_{\omega CC}^\eta$, $\Upsilon_{C\omega C}^\eta$ and $\Upsilon_{CC\omega}^\eta$ are vertices obtained in [14] and $\delta \Upsilon_{\omega CC}^\eta$, $\delta \Upsilon_{C\omega C}^\eta$, $\delta \Upsilon_{CC\omega}^\eta$ result from the local field redefinition of B_2^η (4.11) giving

$$\begin{aligned} \Upsilon_{\omega CC}^\eta + \delta \Upsilon_{\omega CC}^\eta &= -\frac{\eta}{2} \int d^3\tau_+ \delta \left(1 - \sum_{i=1}^3 \tau_i \right) (y^\alpha \partial_{\omega\alpha}) \exp \left\{ i(1 - \tau_2) \partial_{\omega\alpha} \partial_1^\alpha - i\tau_2 \partial_{\omega\alpha} \partial_2^\alpha \right\} \\ &\quad \times \omega((1 - \tau_3)y, \bar{y}) \bar{*} C(\tau_1 y, \bar{y}) \bar{*} C((\tau_1 - 1)y, \bar{y}) k, \end{aligned} \quad (4.15)$$

$$\begin{aligned} \Upsilon_{CC\omega}^\eta + \delta \Upsilon_{CC\omega}^\eta &= -\frac{\eta}{2} \int d^3\tau_+ \delta \left(1 - \sum_{i=1}^3 \tau_i \right) (y^\alpha \partial_{\omega\alpha}) \exp \left\{ i(1 - \tau_2) \partial_{2\alpha} \partial_\omega^\alpha - i\tau_2 \partial_{1\alpha} \partial_\omega^\alpha \right\} \\ &\quad \times C((1 - \tau_1)y, \bar{y}) \bar{*} C(-\tau_1 y, \bar{y}) \bar{*} \omega((\tau_3 - 1)y, \bar{y}) k, \end{aligned} \quad (4.16)$$

$$\begin{aligned} \Upsilon_{C\omega C}^\eta + \delta \Upsilon_{C\omega C}^\eta &= -\frac{\eta}{2} \int d^3\tau_+ \delta \left(1 - \sum_{i=1}^3 \tau_i \right) (y^\alpha \partial_{\omega\alpha}) \exp \left\{ i\tau_2 \partial_{1\alpha} \partial_\omega^\alpha + i(1 - \tau_2) \partial_{\omega\alpha} \partial_2^\alpha \right\} \\ &\quad \times C(\tau_3 y, \bar{y}) \bar{*} \omega(-\tau_1 y, \bar{y}) \bar{*} C((\tau_3 - 1)y, \bar{y}) k \\ &\quad - \frac{\eta}{2} \int d^3\tau_+ \delta \left(1 - \sum_{i=1}^3 \tau_i \right) (y^\alpha \partial_{\omega\alpha}) \exp \left\{ i(1 - \tau_2) \partial_{1\alpha} \partial_\omega^\alpha + i\tau_2 \partial_{\omega\alpha} \partial_2^\alpha \right\} \\ &\quad \times C((1 - \tau_3)y, \bar{y}) \bar{*} \omega(\tau_1 y, \bar{y}) \bar{*} C(-\tau_3 y, \bar{y}) k. \end{aligned} \quad (4.17)$$

4.2 The fields

The expressions for $\widehat{B}_3^{\eta\eta}$ and $\widehat{W}_2^{\eta\eta}$ derived in appendices B and C are

$$\widehat{B}_3^{\eta\eta} = -\frac{\eta^2}{4} \int_0^1 d\mathcal{T} \mathcal{T} \int d^3\rho_+ \delta \left(1 - \sum_{i=1}^3 \rho_i \right) \int_0^1 d\xi \frac{\rho_1 (z_\alpha y^\alpha)^2}{(\rho_1 + \rho_2)(\rho_1 + \rho_3)} \quad (4.18)$$

$$\times \exp \left\{ i\mathcal{T} z_\alpha y^\alpha + \mathcal{T} z^\alpha \left(-(\rho_1 + \rho_3) \partial_{1\alpha} + (\rho_2 - \rho_3) \partial_{2\alpha} + (\rho_1 + \rho_2) \partial_{3\alpha} \right) \right. \\ \left. + (1 - \xi) y^\alpha \left(\frac{\rho_1}{\rho_1 + \rho_2} \partial_{1\alpha} - \frac{\rho_2}{\rho_1 + \rho_2} \partial_{2\alpha} \right) + \xi y^\alpha \left(\frac{\rho_1}{\rho_1 + \rho_3} \partial_{3\alpha} - \frac{\rho_3}{\rho_1 + \rho_3} \partial_{2\alpha} \right) \right\} CCC,$$

$$\widehat{W}_{2CC\omega}^{\eta\eta} = \frac{\eta^2}{4} \int_0^1 d\mathcal{T} \mathcal{T} (z^\gamma \partial_{\omega\gamma})^2 \int d^4\rho_+ \delta \left(1 - \sum_{i=1}^4 \rho_i \right) \frac{\rho_1}{(\rho_1 + \rho_2)(\rho_3 + \rho_4)} \quad (4.19)$$

$$\times \exp \left\{ i\mathcal{T} z_\alpha y^\alpha + \mathcal{T} z^\alpha \left(-(\rho_1 + \rho_2) \partial_{1\alpha} + (\rho_3 + \rho_4) \partial_{2\alpha} + (1 - \rho_2) \partial_{\omega\alpha} \right) \right. \\ \left. + \frac{\rho_1 \rho_3}{(\rho_1 + \rho_2)(\rho_3 + \rho_4)} y^\alpha \partial_{\omega\alpha} + i \left(\frac{(1 - \rho_4) \rho_2}{\rho_1 + \rho_2} + \rho_4 \right) \partial_{2\alpha} \partial_\omega^\alpha - i \frac{\rho_1 \rho_4}{\rho_3 + \rho_4} \partial_{1\alpha} \partial_\omega^\alpha \right\} CC\omega,$$

$$\widehat{W}_{2\omega CC}^{\eta\eta} = \frac{\eta^2}{4} \int_0^1 d\mathcal{T} \mathcal{T} (z^\gamma \partial_{\omega\gamma})^2 \int d^4\rho_+ \delta \left(1 - \sum_{i=1}^4 \rho_i \right) \frac{\rho_1}{(\rho_1 + \rho_2)(\rho_3 + \rho_4)} \quad (4.20)$$

$$\times \exp \left\{ i\mathcal{T} z_\alpha y^\alpha + \mathcal{T} z^\alpha \left((1 - \rho_2) \partial_{\omega\alpha} - (\rho_3 + \rho_4) \partial_{1\alpha} + (\rho_1 + \rho_2) \partial_{2\alpha} \right) \right. \\ \left. + \frac{\rho_1 \rho_3}{(\rho_1 + \rho_2)(\rho_3 + \rho_4)} y^\alpha \partial_{\omega\alpha} + i \left(\frac{(1 - \rho_4) \rho_2}{\rho_1 + \rho_2} + \rho_4 \right) \partial_{\omega\alpha} \partial_1^\alpha - i \frac{\rho_4 \rho_1}{\rho_3 + \rho_4} \partial_{\omega\alpha} \partial_2^\alpha \right\} \omega CC,$$

$$\widehat{W}_{2\omega C\omega}^{\eta\eta} = -\frac{\eta^2}{2} \int_0^1 d\mathcal{T} \mathcal{T} (z^\gamma \partial_{\omega\gamma})^2 \int d^4\rho_+ \delta \left(1 - \sum_{i=1}^4 \rho_i \right) \frac{\rho_1 + \rho_3}{(\rho_1 + \rho_2)(\rho_3 + \rho_4)} \quad (4.21)$$

$$\times \exp \left\{ i\mathcal{T} z_\alpha y^\alpha + \mathcal{T} z^\alpha \left(-(\rho_3 + \rho_4) \partial_{1\alpha} + (\rho_1 - \rho_3) \partial_{\omega\alpha} + (\rho_1 + \rho_2) \partial_{2\alpha} \right) \right. \\ \left. - \frac{\rho_3 \rho_1}{(\rho_1 + \rho_2)(\rho_3 + \rho_4)} y^\alpha \partial_{\omega\alpha} + i \frac{\rho_4 (1 - \rho_2)}{\rho_3 + \rho_4} \partial_{\omega\alpha} \partial_2^\alpha + i \frac{\rho_2 (1 - \rho_4)}{\rho_1 + \rho_2} \partial_{1\alpha} \partial_\omega^\alpha \right\} C\omega C.$$

From now on we skip antiholomorphic (barred) variables for brevity. More precisely, CCC on the r.h.s. of (4.18) is to be understood as $C(y_1, \bar{y}) \bar{*} C(y_2, \bar{y}) \bar{*} C(y_3, \bar{y})|_{y_i=0}$, ωCC on the r.h.s. of (4.20) as $\omega(y_\omega, \bar{y}) \bar{*} C(y_1, \bar{y}) \bar{*} C(y_2, \bar{y})|_{y_\omega=y_i=0}$ etc.

Expressions (4.18)–(4.21) are spin-local because the exponential factors in all of them are free from terms $\partial_{i\alpha} \partial_j^\alpha$ describing contractions between higher components of the zero-forms $C(Y)$ bringing higher-derivative vertices for fields of particular spins. So are the terms induced by these expressions in vertices (4.5)–(4.8). Indeed, differentiating $\widehat{B}_3^{\eta\eta}$ one should use only first-order part from r.h.s. of (4.14) which does not bring contractions between C -fields, similarly star product with ω does not bring contractions due to (A.3), (A.4). On the other hand, though star product of $\widehat{W}_2^{\eta\eta}$ with C does bring contractions between the

fields C , all of them result from the z -dependent terms in the exponentials (4.18)–(4.21) that carry at least one power of $\mathcal{T}$. Such terms contain an additional factor of $\mathcal{T}$ in front of the contraction terms $\partial_{i\alpha}\partial_j^\alpha$ thus belonging to $\mathcal{H}^+$. Hence all the contributions to the vertex (4.2) induced from $B_3^{\eta\eta}$ and $W_2^{\eta\eta}$ are spin-local modulo terms in $\mathcal{H}^+$.

4.3 Equations

4.3.1 B_3 driven terms

Direct computation of the B_3 induced terms using (4.14), (4.18) and (A.3), (A.4) gives

$$d_x \hat{B}_3^{\eta\eta}|_{\omega CCC} \approx \frac{\eta^2}{4} \int_0^1 d\mathcal{T} \mathcal{T} \int d^3 \rho_+ \delta \left(1 - \sum_{i=1}^3 \rho_i \right) \int_0^1 d\xi \frac{\rho_1 (z_\alpha y^\alpha)^2 e^{i\mathcal{T} z_\alpha y^\alpha}}{(\rho_1 + \rho_2)(\rho_1 + \rho_3)} \times \quad (4.22)$$

$$\times \exp \left\{ \mathcal{T} z^\alpha \left(-(\rho_1 + \rho_3)(\partial_{\omega\alpha} + \partial_{1\alpha}) + (\rho_2 - \rho_3)\partial_{2\alpha} + (\rho_1 + \rho_2)\partial_{3\alpha} \right) + i\partial_{\omega\alpha}\partial_1^\alpha \right. \\ \left. + (1 - \xi)y^\alpha \left(\frac{\rho_1}{\rho_1 + \rho_2}(\partial_{\omega\alpha} + \partial_{1\alpha}) - \frac{\rho_2}{\rho_1 + \rho_2}\partial_{2\alpha} \right) + \xi y^\alpha \left(\frac{\rho_1}{\rho_1 + \rho_3}\partial_{3\alpha} - \frac{\rho_3}{\rho_1 + \rho_3}\partial_{2\alpha} \right) \right\} \omega CCC,$$

$$d_x \hat{B}_3^{\eta\eta}|_{C\omega CC} \approx -\frac{\eta^2}{4} \int_0^1 d\mathcal{T} \mathcal{T} \int d^3 \rho_+ \delta \left(1 - \sum_{i=1}^3 \rho_i \right) \int_0^1 d\xi \frac{\rho_1 (z_\alpha y^\alpha)^2 e^{i\mathcal{T} z_\alpha y^\alpha}}{(\rho_1 + \rho_2)(\rho_1 + \rho_3)} \times \quad (4.23)$$

$$\times \exp \left\{ \mathcal{T} z^\alpha \left(-(\rho_1 + \rho_3)(\partial_{\omega\alpha} + \partial_{1\alpha}) + (\rho_2 - \rho_3)\partial_{2\alpha} + (\rho_1 + \rho_2)\partial_{3\alpha} \right) + i\partial_{1\alpha}\partial_\omega^\alpha \right. \\ \left. + (1 - \xi)y^\alpha \left(\frac{\rho_1}{\rho_1 + \rho_2}(\partial_{\omega\alpha} + \partial_{1\alpha}) - \frac{\rho_2}{\rho_1 + \rho_2}\partial_{2\alpha} \right) + \xi y^\alpha \left(\frac{\rho_1}{\rho_1 + \rho_3}\partial_{3\alpha} - \frac{\rho_3}{\rho_1 + \rho_3}\partial_{2\alpha} \right) \right\} C\omega CC$$

$$+ \frac{\eta^2}{4} \int_0^1 d\mathcal{T} \mathcal{T} \int d^3 \rho_+ \delta \left(1 - \sum_{i=1}^3 \rho_i \right) \int_0^1 d\xi \frac{\rho_1 (z_\alpha y^\alpha)^2 e^{i\mathcal{T} z_\alpha y^\alpha}}{(\rho_1 + \rho_2)(\rho_1 + \rho_3)} \times \\ \times \exp \left\{ \mathcal{T} z^\alpha \left(-(\rho_1 + \rho_3)\partial_{1\alpha} + (\rho_2 - \rho_3)(\partial_{\omega\alpha} + \partial_{2\alpha}) + (\rho_1 + \rho_2)\partial_{3\alpha} \right) + i\partial_{\omega\alpha}\partial_2^\alpha \right. \\ \left. + (1 - \xi)y^\alpha \left(\frac{\rho_1}{\rho_1 + \rho_2}\partial_{1\alpha} - \frac{\rho_2}{\rho_1 + \rho_2}(\partial_{\omega\alpha} + \partial_{2\alpha}) \right) \right. \\ \left. + \xi y^\alpha \left(\frac{\rho_1}{\rho_1 + \rho_3}\partial_{3\alpha} - \frac{\rho_3}{\rho_1 + \rho_3}(\partial_{\omega\alpha} + \partial_{2\alpha}) \right) \right\} C\omega CC,$$

$$d_x \hat{B}_3^{\eta\eta}|_{CC\omega C} \approx -\frac{\eta^2}{4} \int_0^1 d\mathcal{T} \mathcal{T} \int d^3 \rho_+ \delta \left(1 - \sum_{i=1}^3 \rho_i \right) \int_0^1 d\xi \frac{\rho_1 (z_\alpha y^\alpha)^2 e^{i\mathcal{T} z_\alpha y^\alpha}}{(\rho_1 + \rho_2)(\rho_1 + \rho_3)} \times \quad (4.24)$$

$$\times \exp \left\{ \mathcal{T} z^\alpha \left(-(\rho_1 + \rho_3)\partial_{1\alpha} + (\rho_2 - \rho_3)(\partial_{\omega\alpha} + \partial_{2\alpha}) + (\rho_1 + \rho_2)\partial_{3\alpha} \right) + i\partial_{2\alpha}\partial_\omega^\alpha \right. \\ \left. + (1 - \xi)y^\alpha \left(\frac{\rho_1}{\rho_1 + \rho_2}\partial_{1\alpha} - \frac{\rho_2}{\rho_1 + \rho_2}(\partial_{\omega\alpha} + \partial_{2\alpha}) \right) \right. \\ \left. + \xi y^\alpha \left(\frac{\rho_1}{\rho_1 + \rho_3}\partial_{3\alpha} - \frac{\rho_3}{\rho_1 + \rho_3}(\partial_{\omega\alpha} + \partial_{2\alpha}) \right) \right\} CC\omega C$$

$$\begin{aligned}
 & + \frac{\eta^2}{4} \int_0^1 d\mathcal{T} \mathcal{T} \int d^3 \rho_+ \delta \left(1 - \sum_{i=1}^3 \rho_i \right) \int_0^1 d\xi \frac{\rho_1 (z_\alpha y^\alpha)^2 e^{i\mathcal{T} z_\alpha y^\alpha}}{(\rho_1 + \rho_2)(\rho_1 + \rho_3)} \times \\
 & \times \exp \left\{ \mathcal{T} z^\alpha \left(-(\rho_1 + \rho_3) \partial_{1\alpha} + (\rho_2 - \rho_3) \partial_{2\alpha} + (\rho_1 + \rho_2) (\partial_{\omega\alpha} + \partial_{3\alpha}) \right) + i \partial_{\omega\alpha} \partial_3^\alpha \right. \\
 & \left. + (1 - \xi) y^\alpha \left(\frac{\rho_1}{\rho_1 + \rho_2} \partial_{1\alpha} - \frac{\rho_2}{\rho_1 + \rho_2} \partial_{2\alpha} \right) + \xi y^\alpha \left(\frac{\rho_1}{\rho_1 + \rho_3} (\partial_{\omega\alpha} + \partial_{3\alpha}) - \frac{\rho_3}{\rho_1 + \rho_3} \partial_{2\alpha} \right) \right\} CC\omega C, \\
 d_x \hat{B}_3^{\eta\eta} |_{CCC\omega} & \approx - \frac{\eta^2}{4} \int_0^1 d\mathcal{T} \mathcal{T} \int d^3 \rho_+ \delta \left(1 - \sum_{i=1}^3 \rho_i \right) \int_0^1 d\xi \frac{\rho_1 (z_\alpha y^\alpha)^2 e^{i\mathcal{T} z_\alpha y^\alpha}}{(\rho_1 + \rho_2)(\rho_1 + \rho_3)} \times \quad (4.25) \\
 & \times \exp \left\{ \mathcal{T} z^\alpha \left(-(\rho_1 + \rho_3) \partial_{1\alpha} + (\rho_2 - \rho_3) \partial_{2\alpha} + (\rho_1 + \rho_2) (\partial_{\omega\alpha} + \partial_{3\alpha}) \right) + i \partial_{3\alpha} \partial_\omega^\alpha \right. \\
 & \left. + (1 - \xi) y^\alpha \left(\frac{\rho_1}{\rho_1 + \rho_2} \partial_{1\alpha} - \frac{\rho_2}{\rho_1 + \rho_2} \partial_{2\alpha} \right) + \xi y^\alpha \left(\frac{\rho_1}{\rho_1 + \rho_3} (\partial_{\omega\alpha} + \partial_{3\alpha}) - \frac{\rho_3}{\rho_1 + \rho_3} \partial_{2\alpha} \right) \right\} CCC\omega.
 \end{aligned}$$

$$\begin{aligned}
 \omega * \hat{B}_3^{\eta\eta} & \approx - \frac{\eta^2}{4} \int_0^1 d\mathcal{T} \mathcal{T} \int d^3 \rho_+ \delta \left(1 - \sum_{i=1}^3 \rho_i \right) \int_0^1 d\xi \frac{\rho_1 [z_\alpha (y^\alpha - i \partial_\omega^\alpha)]^2 e^{i\mathcal{T} z_\alpha y^\alpha}}{(\rho_1 + \rho_2)(\rho_1 + \rho_3)} \times \quad (4.26) \\
 & \times \exp \left\{ \mathcal{T} z^\alpha \left(-\partial_{\omega\alpha} - (\rho_1 + \rho_3) \partial_{1\alpha} + (\rho_2 - \rho_3) \partial_{2\alpha} + (\rho_1 + \rho_2) \partial_{3\alpha} \right) + y^\alpha \partial_{\omega\alpha} \right. \\
 & + (1 - \xi) y^\alpha \left(\frac{\rho_1}{\rho_1 + \rho_2} \partial_{1\alpha} - \frac{\rho_2}{\rho_1 + \rho_2} \partial_{2\alpha} \right) + \xi y^\alpha \left(\frac{\rho_1}{\rho_1 + \rho_3} \partial_{3\alpha} - \frac{\rho_3}{\rho_1 + \rho_3} \partial_{2\alpha} \right) \\
 & \left. + i \frac{(1 - \xi) \rho_1}{\rho_1 + \rho_2} \partial_{\omega\alpha} \partial_1^\alpha - i \left(\frac{(1 - \xi) \rho_2}{\rho_1 + \rho_2} + \frac{\xi \rho_3}{\rho_1 + \rho_3} \right) \partial_{\omega\alpha} \partial_2^\alpha + i \frac{\xi \rho_1}{\rho_1 + \rho_3} \partial_{\omega\alpha} \partial_3^\alpha \right\} \omega CCC, \\
 \hat{B}_3^{\eta\eta} * \omega & \approx - \frac{\eta^2}{4} \int_0^1 d\mathcal{T} \mathcal{T} \int d^3 \rho_+ \delta \left(1 - \sum_{i=1}^3 \rho_i \right) \int_0^1 d\xi \frac{\rho_1 [z_\alpha (y^\alpha + i \partial_\omega^\alpha)]^2 e^{i\mathcal{T} z_\alpha y^\alpha}}{(\rho_1 + \rho_2)(\rho_1 + \rho_3)} \times \quad (4.27) \\
 & \times \exp \left\{ \mathcal{T} z^\alpha \left(\partial_{\omega\alpha} - (\rho_1 + \rho_3) \partial_{1\alpha} + (\rho_2 - \rho_3) \partial_{2\alpha} + (\rho_1 + \rho_2) \partial_{3\alpha} \right) + y^\alpha \partial_{\omega\alpha} \right. \\
 & + (1 - \xi) y^\alpha \left(\frac{\rho_1}{\rho_1 + \rho_2} \partial_{1\alpha} - \frac{\rho_2}{\rho_1 + \rho_2} \partial_{2\alpha} \right) + \xi y^\alpha \left(\frac{\rho_1}{\rho_1 + \rho_3} \partial_{3\alpha} - \frac{\rho_3}{\rho_1 + \rho_3} \partial_{2\alpha} \right) \\
 & \left. + i \frac{(1 - \xi) \rho_1}{\rho_1 + \rho_2} \partial_{1\alpha} \partial_\omega^\alpha - i \left(\frac{(1 - \xi) \rho_2}{\rho_1 + \rho_2} + \frac{\xi \rho_3}{\rho_1 + \rho_3} \right) \partial_{2\alpha} \partial_\omega^\alpha + i \frac{\xi \rho_1}{\rho_1 + \rho_3} \partial_{3\alpha} \partial_\omega^\alpha \right\} CCC\omega.
 \end{aligned}$$

4.3.2 B_2 driven terms

The terms resulting from d_x differentiation of B_2 and its multiplication with W_1 by virtue of (4.12), (4.13), (4.14) and (A.2) give

$$\begin{aligned}
 d_x B_2^{\eta loc} |_{\omega CCC} & \approx - \frac{i\eta^2}{4} \int_0^1 d\mathcal{T} \int_0^1 d\xi \int d^3 \rho_+ \delta \left(1 - \sum_{i=1}^3 \rho_i \right) (z_\alpha y^\alpha) \left[(\mathcal{T} z^\alpha - \xi y^\alpha) \partial_{\omega\alpha} \right] \times \\
 & \times \exp \left\{ i\mathcal{T} z_\alpha y^\alpha + i(1 - \rho_2) \partial_{\omega\alpha} \partial_1^\alpha - i\rho_2 \partial_{\omega\alpha} \partial_2^\alpha \right.
 \end{aligned}$$

$$\begin{aligned}
 & +\mathcal{T}z^\alpha\left(-(\rho_1+\rho_2)\partial_{\omega\alpha}-\rho_1\partial_{1\alpha}+(\rho_2+\rho_3)\partial_{2\alpha}+\partial_{3\alpha}\right) \\
 & +y^\alpha\left(\xi(\rho_1+\rho_2)\partial_{\omega\alpha}+\xi\rho_1\partial_{1\alpha}-\xi(\rho_2+\rho_3)\partial_{2\alpha}+(1-\xi)\partial_{3\alpha}\right)\}\omega CCC, \tag{4.28}
 \end{aligned}$$

$$\begin{aligned}
 d_x B_2^{\eta loc}|_{C\omega CC} & \approx -\frac{i\eta^2}{4}\int_0^1 d\mathcal{T}\int_0^1 d\xi\int d^3\rho_+\delta\left(1-\sum_{i=1}^3\rho_i\right)(z_\alpha y^\alpha)\left[(\mathcal{T}z^\alpha-\xi y^\alpha)\partial_{\omega\alpha}\right]\times \\
 & \times \exp\left\{i\mathcal{T}z_\alpha y^\alpha+i\rho_2\partial_{1\alpha}\partial_\omega^\alpha+i(1-\rho_2)\partial_{\omega\alpha}\partial_2^\alpha\right. \\
 & +\mathcal{T}z^\alpha\left(-\rho_3\partial_{1\alpha}+\rho_1\partial_{\omega\alpha}+(\rho_1+\rho_2)\partial_{2\alpha}+\partial_{3\alpha}\right) \\
 & \left.+y^\alpha\left(\xi\rho_3\partial_{1\alpha}-\xi\rho_1\partial_{\omega\alpha}-\xi(\rho_1+\rho_2)\partial_{2\alpha}+(1-\xi)\partial_{3\alpha}\right)\right\}C\omega CC \\
 & -\frac{i\eta^2}{4}\int_0^1 d\mathcal{T}\int_0^1 d\xi\int d^3\rho_+\delta\left(1-\sum_{i=1}^3\rho_i\right)(z_\alpha y^\alpha)\left[(\mathcal{T}z^\alpha-\xi y^\alpha)\partial_{\omega\alpha}\right]\times \\
 & \times \exp\left\{i\mathcal{T}z_\alpha y^\alpha+i(1-\rho_2)\partial_{1\alpha}\partial_\omega^\alpha+i\rho_2\partial_{\omega\alpha}\partial_\omega^\alpha\right. \\
 & +\mathcal{T}z^\alpha\left(-(\rho_1+\rho_2)\partial_{1\alpha}-\rho_1\partial_{\omega\alpha}+\rho_3\partial_{2\alpha}+\partial_{3\alpha}\right) \\
 & \left.+y^\alpha\left(\xi(\rho_1+\rho_2)\partial_{1\alpha}+\xi\rho_1\partial_{\omega\alpha}-\xi\rho_3\partial_{2\alpha}+(1-\xi)\partial_{3\alpha}\right)\right\}C\omega CC \\
 & -\frac{i\eta^2}{4}\int_0^1 d\mathcal{T}\int_0^1 d\xi\int d^3\rho_+\delta\left(1-\sum_{i=1}^3\rho_i\right)(z_\alpha y^\alpha)\left[(\mathcal{T}z^\alpha+(1-\xi)y^\alpha)\partial_{\omega\alpha}\right]\times \\
 & \times \exp\left\{i\mathcal{T}z_\alpha y^\alpha+i(1-\rho_+\rho_2)\partial_{\omega\alpha}\partial_2^\alpha-i\rho_2\partial_{\omega\alpha}\partial_3^\alpha\right. \\
 & +\mathcal{T}z^\alpha\left(-\partial_{1\alpha}-(\rho_1+\rho_2)\partial_{\omega\alpha}-\rho_1\partial_{2\alpha}+(\rho_2+\rho_3)\partial_{3\alpha}\right) \\
 & \left.+y^\alpha\left(\xi\partial_{1\alpha}-(1-\xi)(\rho_1+\rho_2)\partial_{\omega\alpha}-(1-\xi)\rho_1\partial_{2\alpha}+(1-\xi)(\rho_2+\rho_3)\partial_{3\alpha}\right)\right\}C\omega CC, \tag{4.29}
 \end{aligned}$$

$$\begin{aligned}
 d_x B_2^{\eta loc}|_{CC\omega C} & \approx -\frac{i\eta^2}{4}\int_0^1 d\mathcal{T}\int_0^1 d\xi\int d^3\rho_+\delta\left(1-\sum_{i=1}^3\rho_i\right)(z_\alpha y^\alpha)\left[(\mathcal{T}z^\alpha-\xi y^\alpha)\partial_{\omega\alpha}\right]\times \\
 & \times \exp\left\{i\mathcal{T}z_\alpha y^\alpha+i(1-\rho_2)\partial_{2\alpha}\partial_\omega^\alpha-i\rho_2\partial_{1\alpha}\partial_\omega^\alpha\right. \\
 & +\mathcal{T}z^\alpha\left(-(\rho_1+\rho_2)\partial_{1\alpha}+\rho_3\partial_{2\alpha}+(\rho_2+\rho_3)\partial_{\omega\alpha}+\partial_{3\alpha}\right) \\
 & \left.+y^\alpha\left(\xi(\rho_1+\rho_2)\partial_{1\alpha}-\xi\rho_3\partial_{2\alpha}-\xi(\rho_2+\rho_3)\partial_{\omega\alpha}+(1-\xi)\partial_{3\alpha}\right)\right\}CC\omega C \\
 & -\frac{i\eta^2}{4}\int_0^1 d\mathcal{T}\int_0^1 d\xi\int d^3\rho_+\delta\left(1-\sum_{i=1}^3\rho_i\right)(z_\alpha y^\alpha)\left[(\mathcal{T}z^\alpha+(1-\xi)y^\alpha)\partial_{\omega\alpha}\right]\times \\
 & \times \exp\left\{i\mathcal{T}z_\alpha y^\alpha+i\rho_2\partial_{2\alpha}\partial_\omega^\alpha+i(1-\rho_2)\partial_{\omega\alpha}\partial_3^\alpha\right. \\
 & +\mathcal{T}z^\alpha\left(-\partial_{1\alpha}-\rho_3\partial_{2\alpha}+\rho_1\partial_{\omega\alpha}+(\rho_1+\rho_2)\partial_{3\alpha}\right) \\
 & \left.+y^\alpha\left(\xi\partial_{1\alpha}-(1-\xi)\rho_3\partial_{2\alpha}+(1-\xi)\rho_1\partial_{\omega\alpha}+(1-\xi)(\rho_1+\rho_2)\partial_{3\alpha}\right)\right\}CC\omega C \\
 & -\frac{i\eta^2}{4}\int_0^1 d\mathcal{T}\int_0^1 d\xi\int d^3\rho_+\delta\left(1-\sum_{i=1}^3\rho_i\right)(z_\alpha y^\alpha)\left[(\mathcal{T}z^\alpha+(1-\xi)y^\alpha)\partial_{\omega\alpha}\right]\times \\
 & \times \exp\left\{i\mathcal{T}z_\alpha y^\alpha+i(1-\rho_2)\partial_{2\alpha}\partial_\omega^\alpha+i\rho_2\partial_{\omega\alpha}\partial_3^\alpha\right.
 \end{aligned}$$

$$\begin{aligned}
 & +\mathcal{T}z^\alpha\left(-\partial_{1\alpha}-(\rho_1+\rho_2)\partial_{2\alpha}-\rho_1\partial_{\omega\alpha}+\rho_3\partial_{3\alpha}\right) \\
 & +y^\alpha\left(\xi\partial_{1\alpha}-(1-\xi)(\rho_1+\rho_2)\partial_{2\alpha}-(1-\xi)\rho_1\partial_{\omega\alpha}+(1-\xi)\rho_3\partial_{3\alpha}\right)\}CC\omega C, \tag{4.30}
 \end{aligned}$$

$$\begin{aligned}
 d_x B_2^{\eta loc}|_{CC\omega} & \approx -\frac{i\eta^2}{4}\int_0^1 d\mathcal{T}\int_0^1 d\xi\int d^3\rho_+\delta\left(1-\sum_{i=1}^3\rho_i\right)(z_\alpha y^\alpha)\left[(\mathcal{T}z^\alpha+(1-\xi)y^\alpha)\partial_{\omega\alpha}\right]\times \\
 & \times \exp\left\{i\mathcal{T}z_\alpha y^\alpha+i(1-\rho_2)\partial_{3\alpha}\partial_\omega^\alpha-i\rho_2\partial_{2\alpha}\partial_\omega^\alpha\right. \\
 & +\mathcal{T}z^\alpha\left(-\partial_{1\alpha}-(\rho_2+\rho_3)\partial_{2\alpha}+\rho_1\partial_{3\alpha}+(\rho_1+\rho_2)\partial_{\omega\alpha}\right) \\
 & \left.+y^\alpha\left(\xi\partial_{1\alpha}-(1-\xi)(\rho_2+\rho_3)\partial_{2\alpha}+(1-\xi)\rho_1\partial_{3\alpha}+(1-\xi)(\rho_1+\rho_2)\partial_{\omega\alpha}\right)\right\}CCC\omega, \tag{4.31}
 \end{aligned}$$

$$\begin{aligned}
 W_{1\omega C}^\eta*B_2^{\eta loc} & \approx \frac{i\eta^2}{4}\int_0^1 d\mathcal{T}\int_0^1 d\sigma\int d^3\rho_+\frac{\delta\left(1-\sum_{i=1}^3\rho_i\right)}{\rho_1+\rho_2}\mathcal{T}(z^\gamma\partial_{\omega\gamma})\left[z_\alpha y^\alpha+i\sigma z^\alpha\partial_{\omega\alpha}\right]\times \\
 & \times \exp\left\{i\mathcal{T}z_\alpha y^\alpha+i(1-\sigma)\partial_{\omega\alpha}\partial_1^\alpha-i\frac{\rho_1\sigma}{\rho_1+\rho_2}\partial_{\omega\alpha}\partial_2^\alpha+i\frac{\rho_2\sigma}{\rho_1+\rho_2}\partial_{\omega\alpha}\partial_3^\alpha\right. \\
 & +\mathcal{T}z^\alpha\left(-(\rho_1+\rho_2+\sigma\rho_3)\partial_{\omega\alpha}-(\rho_1+\rho_2)\partial_{1\alpha}+(\rho_3-\rho_1)\partial_{2\alpha}+(\rho_3+\rho_2)\partial_{3\alpha}\right) \\
 & \left.+y^\alpha\left(\sigma\partial_{\omega\alpha}-\frac{\rho_1}{\rho_1+\rho_2}\partial_{2\alpha}+\frac{\rho_2}{\rho_1+\rho_2}\partial_{3\alpha}\right)\right\}\omega CCC, \tag{4.32}
 \end{aligned}$$

$$\begin{aligned}
 W_{1C\omega}^\eta*B_2^{\eta loc} & \approx \frac{i\eta^2}{4}\int_0^1 d\mathcal{T}\mathcal{T}\int_0^1 d\sigma\int d^3\rho_+\frac{\delta\left(1-\sum_{i=1}^3\rho_i\right)}{\rho_1+\rho_2}(z^\gamma\partial_{\omega\gamma})\left[z_\alpha y^\alpha+i\sigma z^\alpha\partial_{\omega\alpha}\right]\times \\
 & \times \exp\left\{i\mathcal{T}z_\alpha y^\alpha+i(1-\sigma)\partial_{1\alpha}\partial_\omega^\alpha-i\frac{\rho_1\sigma}{\rho_1+\rho_2}\partial_{\omega\alpha}\partial_2^\alpha+i\frac{\rho_2\sigma}{\rho_1+\rho_2}\partial_{\omega\alpha}\partial_3^\alpha\right. \\
 & +\mathcal{T}z^\alpha\left(-(\rho_1+\rho_2+\sigma\rho_3)\partial_{\omega\alpha}-(\rho_1+\rho_2)\partial_{1\alpha}+(\rho_3-\rho_1)\partial_{2\alpha}+(\rho_3+\rho_2)\partial_{3\alpha}\right) \\
 & \left.+y^\alpha\left(\sigma\partial_{\omega\alpha}-\frac{\rho_1}{\rho_1+\rho_2}\partial_{2\alpha}+\frac{\rho_2}{\rho_1+\rho_2}\partial_{3\alpha}\right)\right\}C\omega CC, \tag{4.33}
 \end{aligned}$$

$$\begin{aligned}
 B_2^{\eta loc}*W_{1\omega C}^\eta & \approx \frac{i\eta^2}{4}\int_0^1 d\mathcal{T}\mathcal{T}\int_0^1 d\sigma\int d^3\rho_+\frac{\delta\left(1-\sum_{i=1}^3\rho_i\right)}{\rho_1+\rho_2}\left[z_\alpha y^\alpha-i\sigma z^\alpha\partial_{\omega\alpha}\right](z^\gamma\partial_{\omega\gamma})\times \\
 & \times \exp\left\{i\mathcal{T}z_\alpha y^\alpha+i(1-\sigma)\partial_{\omega\alpha}\partial_3^\alpha-i\frac{\rho_1\sigma}{\rho_1+\rho_2}\partial_{1\alpha}\partial_\omega^\alpha+i\frac{\rho_2\sigma}{\rho_1+\rho_2}\partial_{2\alpha}\partial_\omega^\alpha\right. \\
 & +\mathcal{T}z^\alpha\left(-(\rho_3+\rho_1)\partial_{1\alpha}-(\rho_3-\rho_2)\partial_{2\alpha}+(\rho_1+\rho_2-\sigma\rho_3)\partial_{\omega\alpha}+(\rho_1+\rho_2)\partial_{3\alpha}\right) \\
 & \left.+y^\alpha\left(\frac{\rho_1}{\rho_1+\rho_2}\partial_{1\alpha}-\frac{\rho_2}{\rho_1+\rho_2}\partial_{2\alpha}-\sigma\partial_{\omega\alpha}\right)\right\}CC\omega C, \tag{4.34}
 \end{aligned}$$

$$\begin{aligned}
 B_2^{\eta loc}*W_{1C\omega}^\eta & \approx \frac{i\eta^2}{4}\int_0^1 d\mathcal{T}\mathcal{T}\int_0^1 d\sigma\int d^3\rho_+\frac{\delta\left(1-\sum_{i=1}^3\rho_i\right)}{\rho_1+\rho_2}\left[z_\alpha y^\alpha-i\sigma z^\alpha\partial_{\omega\alpha}\right](z^\gamma\partial_{\omega\gamma})\times \\
 & \times \exp\left\{i\mathcal{T}z_\alpha y^\alpha+i(1-\sigma)\partial_{3\alpha}\partial_\omega^\alpha-i\frac{\rho_1\sigma}{\rho_1+\rho_2}\partial_{1\alpha}\partial_\omega^\alpha+i\frac{\rho_2\sigma}{\rho_1+\rho_2}\partial_{2\alpha}\partial_\omega^\alpha\right. \\
 & +\mathcal{T}z^\alpha\left(-(\rho_3+\rho_1)\partial_{1\alpha}-(\rho_3-\rho_2)\partial_{2\alpha}+(\rho_1+\rho_2-\sigma\rho_3)\partial_{\omega\alpha}+(\rho_1+\rho_2)\partial_{3\alpha}\right) \\
 & \left.+y^\alpha\left(\frac{\rho_1}{\rho_1+\rho_2}\partial_{1\alpha}-\frac{\rho_2}{\rho_1+\rho_2}\partial_{2\alpha}-\sigma\partial_{\omega\alpha}\right)\right\}CCC\omega. \tag{4.35}
 \end{aligned}$$

4.3.3 W_2 driven terms

Terms resulting from star product with $W_2^{\eta\eta}$ are

$$C * \widehat{W}_{2\omega CC}^{\eta\eta} \approx \frac{\eta^2}{4} \int_0^1 d\mathcal{T} \mathcal{T} (z^\gamma \partial_{\omega\gamma})^2 \int d^4 \rho_+ \delta \left(1 - \sum_{i=1}^4 \rho_i \right) \frac{\rho_1}{(\rho_1 + \rho_2)(\rho_3 + \rho_4)} \times \quad (4.36)$$

$$\begin{aligned} & \times \exp \left\{ i\mathcal{T} z_\alpha y^\alpha + \mathcal{T} z^\alpha \left(-\partial_{1\alpha} + (1 - \rho_2) \partial_{\omega\alpha} - (\rho_3 + \rho_4) \partial_{2\alpha} + (\rho_1 + \rho_2) \partial_{3\alpha} \right) + y^\alpha \partial_{1\alpha} \right. \\ & + \frac{\rho_1 \rho_3}{(\rho_1 + \rho_2)(\rho_3 + \rho_4)} (y^\alpha \partial_{\omega\alpha} + i \partial_{1\alpha} \partial_\omega^\alpha) \\ & \left. + i \left(\frac{(1 - \rho_4) \rho_2}{\rho_1 + \rho_2} + \rho_4 \right) \partial_{\omega\alpha} \partial_1^\alpha - i \frac{\rho_4 \rho_1}{\rho_3 + \rho_4} \partial_{\omega\alpha} \partial_2^\alpha \right\} C \omega C C, \end{aligned}$$

$$\widehat{W}_{2\omega CC}^{\eta\eta} * C \approx \frac{\eta^2}{4} \int_0^1 d\mathcal{T} \mathcal{T} (z^\gamma \partial_{\omega\gamma})^2 \int d^4 \rho_+ \delta \left(1 - \sum_{i=1}^4 \rho_i \right) \frac{\rho_1}{(\rho_1 + \rho_2)(\rho_3 + \rho_4)} \times \quad (4.37)$$

$$\begin{aligned} & \times \exp \left\{ i\mathcal{T} z_\alpha y^\alpha + \mathcal{T} z^\alpha \left((1 - \rho_2) \partial_{\omega\alpha} - (\rho_3 + \rho_4) \partial_{1\alpha} + (\rho_1 + \rho_2) \partial_{2\alpha} + \partial_{3\alpha} \right) + y^\alpha \partial_{3\alpha} \right. \\ & + \frac{\rho_1 \rho_3}{(\rho_1 + \rho_2)(\rho_3 + \rho_4)} (y^\alpha \partial_{\omega\alpha} + i \partial_{\omega\alpha} \partial_3^\alpha) \\ & \left. + i \left(\frac{(1 - \rho_4) \rho_2}{\rho_1 + \rho_2} + \rho_4 \right) \partial_{\omega\alpha} \partial_1^\alpha - i \frac{\rho_4 \rho_1}{\rho_3 + \rho_4} \partial_{\omega\alpha} \partial_2^\alpha \right\} \omega C C C, \end{aligned}$$

$$C * \widehat{W}_{2CC\omega}^{\eta\eta} \approx \frac{\eta^2}{4} \int_0^1 d\mathcal{T} \mathcal{T} (z^\gamma \partial_{\omega\gamma})^2 \int d^4 \rho_+ \delta \left(1 - \sum_{i=1}^4 \rho_i \right) \frac{\rho_1}{(\rho_1 + \rho_2)(\rho_3 + \rho_4)} \times \quad (4.38)$$

$$\begin{aligned} & \times \exp \left\{ i\mathcal{T} z_\alpha y^\alpha + \mathcal{T} z^\alpha \left(-\partial_{1\alpha} - (\rho_1 + \rho_2) \partial_{2\alpha} + (\rho_3 + \rho_4) \partial_{3\alpha} + (1 - \rho_2) \partial_{\omega\alpha} \right) + y^\alpha \partial_{1\alpha} \right. \\ & + \frac{\rho_1 \rho_3}{(\rho_1 + \rho_2)(\rho_3 + \rho_4)} (y^\alpha \partial_{\omega\alpha} + i \partial_{1\alpha} \partial_\omega^\alpha) \\ & \left. + i \left(\frac{(1 - \rho_4) \rho_2}{\rho_1 + \rho_2} + \rho_4 \right) \partial_{2\alpha} \partial_\omega^\alpha - i \frac{\rho_1 \rho_4}{\rho_3 + \rho_4} \partial_{1\alpha} \partial_\omega^\alpha \right\} C C C \omega, \end{aligned}$$

$$\widehat{W}_{2CC\omega}^{\eta\eta} * C \approx \frac{\eta^2}{4} \int_0^1 d\mathcal{T} \mathcal{T} (z^\gamma \partial_{\omega\gamma})^2 \int d^4 \rho_+ \delta \left(1 - \sum_{i=1}^4 \rho_i \right) \frac{\rho_1}{(\rho_1 + \rho_2)(\rho_3 + \rho_4)} \times \quad (4.39)$$

$$\begin{aligned} & \times \exp \left\{ i\mathcal{T} z_\alpha y^\alpha + \mathcal{T} z^\alpha \left(-(\rho_1 + \rho_2) \partial_{1\alpha} + (\rho_3 + \rho_4) \partial_{2\alpha} + (1 - \rho_2) \partial_{\omega\alpha} + \partial_{3\alpha} \right) + y^\alpha \partial_{3\alpha} \right. \\ & + \frac{\rho_1 \rho_3}{(\rho_1 + \rho_2)(\rho_3 + \rho_4)} (y^\alpha \partial_{\omega\alpha} + i \partial_{\omega\alpha} \partial_3^\alpha) \\ & \left. + i \left(\frac{(1 - \rho_4) \rho_2}{\rho_1 + \rho_2} + \rho_4 \right) \partial_{2\alpha} \partial_\omega^\alpha - i \frac{\rho_1 \rho_4}{\rho_3 + \rho_4} \partial_{1\alpha} \partial_\omega^\alpha \right\} C C \omega C, \end{aligned}$$

$$C * \widehat{W}_{2C\omega C}^{\eta\eta} \approx -\frac{\eta^2}{2} \int_0^1 d\mathcal{T} \mathcal{T} (z^\gamma \partial_{\omega\gamma})^2 \int d^4\rho_+ \delta\left(1 - \sum_{i=1}^4 \rho_i\right) \frac{\rho_1 + \rho_3}{(\rho_1 + \rho_2)(\rho_3 + \rho_4)} \times \quad (4.40)$$

$$\begin{aligned} & \times \exp \left\{ i\mathcal{T} z_\alpha y^\alpha + \mathcal{T} z^\alpha \left(-\partial_{1\alpha} - (\rho_3 + \rho_4) \partial_{2\alpha} + (\rho_1 - \rho_3) \partial_{\omega\alpha} + (\rho_1 + \rho_2) \partial_{3\alpha} \right) + y^\alpha \partial_{1\alpha} \right. \\ & \left. - \frac{\rho_3 \rho_1}{(\rho_1 + \rho_2)(\rho_3 + \rho_4)} (y^\alpha \partial_{\omega\alpha} + i \partial_{1\alpha} \partial_\omega^\alpha) + i \frac{\rho_4(1 - \rho_2)}{\rho_3 + \rho_4} \partial_{\omega\alpha} \partial_2^\alpha + i \frac{\rho_2(1 - \rho_4)}{\rho_1 + \rho_2} \partial_{1\alpha} \partial_\omega^\alpha \right\} C C \omega C, \\ \widehat{W}_{2C\omega C}^{\eta\eta} * C & \approx -\frac{\eta^2}{2} \int_0^1 d\mathcal{T} \mathcal{T} (z^\gamma \partial_{\omega\gamma})^2 \int d^4\rho_+ \delta\left(1 - \sum_{i=1}^4 \rho_i\right) \frac{\rho_1 + \rho_3}{(\rho_1 + \rho_2)(\rho_3 + \rho_4)} \times \quad (4.41) \\ & \times \exp \left\{ i\mathcal{T} z_\alpha y^\alpha + \mathcal{T} z^\alpha \left(-(\rho_3 + \rho_4) \partial_{1\alpha} + (\rho_1 - \rho_3) \partial_{\omega\alpha} + (\rho_1 + \rho_2) \partial_{2\alpha} + \partial_{3\alpha} \right) + y^\alpha \partial_{3\alpha} \right. \\ & \left. - \frac{\rho_3 \rho_1}{(\rho_1 + \rho_2)(\rho_3 + \rho_4)} (y^\alpha \partial_{\omega\alpha} + i \partial_{\omega\alpha} \partial_3^\alpha) + i \frac{\rho_4(1 - \rho_2)}{\rho_3 + \rho_4} \partial_{\omega\alpha} \partial_2^\alpha + i \frac{\rho_2(1 - \rho_4)}{\rho_1 + \rho_2} \partial_{1\alpha} \partial_\omega^\alpha \right\} C \omega C C. \end{aligned}$$

In the end of this section let us stress again that all terms on the r.h.s. of vertex (4.2) are free from C -field contractions $\partial_{i\alpha} \partial_j^\alpha$ in the exponentials, hence being spin-local. This is the central result of this paper.

5 Conclusion

In this paper we have analyzed the ωC^3 vertices in the equation for the zero-form C (2.2) in the holomorphic η^2 sector, showing that these vertices are spin-local in the terminology of [16]. In particular, they contain the holomorphic part of the ϕ^4 vertex in the Lagrangian nomenclature for a spin-zero scalar field ϕ . This is another step in the analysis of locality of HS gauge theory performed in [14, 15]. To complete the analysis of spin-locality of the HS gauge theory at quartic order it remains to extend these results to the mixed $\eta\bar{\eta}$ sector. This problem differs in some respects from the (anti)holomorphic one and will be analyzed elsewhere.

On the other hand, there are remaining problems even in the holomorphic sector left unsolved. The most important one is to find explicit Z -independent local form of the holomorphic vertex ωC^3 . The naive attempt to set $Z = 0$ in the vertex obtained in this paper does not necessarily lead to correct result since the omitted terms in $\mathcal{H}_0^+$ are needed for consistency of the equations and may contribute to the sector of equations. Indeed, setting $Z = 0$ corresponds to the application of the conventional homotopy projector which does not eliminate the part of the vertex in $\mathcal{H}^+$. Let us stress again in this regard that the elaborated technique based on dropping off terms from $\mathcal{H}_0^+$ turns out to be highly efficient for checking out spin-locality. To obtain explicit form of these vertices there are two alternative ways of the analysis.

One is to eliminate the Z -dependence from the vertex by direct partial integration. Being technically involved and not at all obvious due to the need of using Schouten identity and partial integrations, this program is realised at least for a particular vertex in the forthcoming paper [28].

Another is to apply the limiting shifted homotopy procedure with appropriately chosen shift when solving for HS fields. Note that the Z -dependence for HS master fields has been found using no shifted homotopies in our paper. It would be interesting to understand if the locality is reached within well elaborated contracting homotopy approach. Since the choice of homotopy shift and hence cohomology projector via resolution of identity (3.2), (3.3) affects field redefinitions that can themselves be non-local the art is to find a shift that makes the result manifestly spin-local. This is an interesting problem for the future.

To summarize, the results of this paper indicate that equations of motion of HS gauge theory have a tendency of being spin-local. At this stage it is crucially important to see whether this property extends to the mixed $\eta\bar{\eta}\omega C^3$ sector of equation (2.2) which is the most urgent problem on the agenda.

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A Useful formulas

Useful multiplication formula for the star product of functions of the form

$$f_j(z, y) = \int_0^1 d\tau_j \exp i(\tau_j z_\alpha y^\alpha) \phi_j(\tau_j z, (1 - \tau_j)y | \tau_j \theta, \tau_j) \quad (\text{A.1})$$

is [29]

$$\begin{aligned} f_1 * f_2(z, y) &= \int_0^1 d\tau_1 \int_0^1 d\tau_2 \int e^{iu_\alpha v^\alpha} \exp i(\tau_1 \circ \tau_2 z_\alpha y^\alpha) \\ &\times \phi_1\left(\tau_1[(1 - \tau_2)z - \tau_2 y + u], (1 - \tau_1)[(1 - \tau_2)y - \tau_2 z + u] | \tau_1 \theta, \tau_1\right) \\ &\times \phi_2\left(\tau_2[(1 - \tau_1)z + \tau_1 y - v], (1 - \tau_2)[(1 - \tau_1)y + \tau_1 z + v] | \tau_2 \theta, \tau_2\right). \end{aligned} \quad (\text{A.2})$$

For instance, if one function is z -independent the following formulas are handy in star-product computation

$$f(y) * \Gamma(z, y) = f(y) \Gamma(z + i \overleftarrow{\partial}_f, y - i \overleftarrow{\partial}_f), \quad (\text{A.3})$$

$$\Gamma(z, y) * f(y) = \Gamma(z + i \partial_f, y + i \partial_f) f(y). \quad (\text{A.4})$$

Second-order zero-form vertices are [14]

$$\begin{aligned} \Upsilon_{\omega CC}^\eta &= -\frac{i\eta}{2} \int d^3\tau_+ \delta\left(1 - \sum_{i=1}^3 \tau_i\right) \partial_{\omega\alpha} (\partial_1^\alpha + \partial_2^\alpha) \exp\left\{i(1 - \tau_2) \partial_{\omega\alpha} \partial_1^\alpha - i\tau_2 \partial_{\omega\alpha} \partial_2^\alpha\right\} \\ &\times \omega((1 - \tau_3)y, \bar{y}) \bar{*} C(\tau_1 y, \bar{y}) \bar{*} C((\tau_1 - 1)y, \bar{y}) k, \end{aligned} \quad (\text{A.5})$$

$$\begin{aligned} \Upsilon_{CC\omega}^\eta &= -\frac{i\eta}{2} \int d^3\tau_+ \delta\left(1 - \sum_{i=1}^3 \tau_i\right) \partial_{\omega\alpha} (\partial_1^\alpha + \partial_2^\alpha) \exp\left\{i(1 - \tau_1) \partial_{2\alpha} \partial_\omega^\alpha - i\tau_1 \partial_{1\alpha} \partial_\omega^\alpha\right\} \\ &\times C((1 - \tau_2)y, \bar{y}) \bar{*} C(-\tau_2 y, \bar{y}) \bar{*} \omega((\tau_3 - 1)y, \bar{y}) k, \end{aligned} \quad (\text{A.6})$$

$$\begin{aligned}
 \Upsilon_{C\omega C}^\eta = & -\frac{i\eta}{2} \int d^3\tau_+ \delta\left(1 - \sum_{i=1}^3 \tau_i\right) \partial_{\omega\alpha} (\partial_1^\alpha + \partial_2^\alpha) \exp\left\{i\tau_3 \partial_{1\alpha} \partial_\omega^\alpha + i(1 - \tau_3) \partial_{\omega\alpha} \partial_2^\alpha\right\} \\
 & \times C(\tau_2 y, \bar{y}) \bar{*} \omega(-\tau_1 y, \bar{y}) \bar{*} C((\tau_2 - 1)y, \bar{y}) k \\
 & -\frac{i\eta}{2} \int d^3\tau_+ \delta\left(1 - \sum_{i=1}^3 \tau_i\right) \partial_{\omega\alpha} (\partial_1^\alpha + \partial_2^\alpha) \exp\left\{i(1 - \tau_2) \partial_{1\alpha} \partial_\omega^\alpha + i\tau_2 \partial_{\omega\alpha} \partial_2^\alpha\right\} \\
 & \times C((1 - \tau_3)y, \bar{y}) \bar{*} \omega(\tau_1 y, \bar{y}) \bar{*} C(-\tau_3 y, \bar{y}) k.
 \end{aligned} \tag{A.7}$$

B $B_3^{\eta\eta}$

Computation of $B_3^{\eta\eta}$ goes as follows. Equation for $B_3^{\eta\eta}$ from (2.11) is

$$2i d_z B_3^{\eta\eta} = [S_1^\eta, B_2]_* + [S_2^{\eta\eta}, C]_* . \tag{B.1}$$

An important observation of section 6.2 of [15] based on the technique of re-ordering operators $O_\beta f(z, y)$ was that if $S_2^{\eta\eta}$ is computed using $B_2^{\eta loc}$ (4.9), then the contribution to the vertices $\Upsilon^{\eta\eta}(\omega, \omega, C, C)$ from $S_2^{\eta\eta}$ vanishes at $\beta \rightarrow -\infty$. Proceeding analogously one can see that contribution to the vertices $\Upsilon^{\eta\eta}(\omega, C, C, C)$ from such $S_2^{\eta\eta}$ also vanishes at $\beta \rightarrow -\infty$.

Hence to find the part of $B_3^{\eta\eta}$ that contributes to $\hat{\Upsilon}^{\eta\eta}$ one has to solve the equation

$$d_z \hat{B}_3^{\eta\eta} = \frac{i}{2} [B_2^{\eta loc}, S_1^\eta]_* . \tag{B.2}$$

S_1^η is given by (2.32). Replacing the integration over simplex in (4.9) by the integration over unit square

$$\int d^3\tau_+ \delta(1 - \sum_{i=1}^3 \tau_i) = \int_0^1 d\tau_1 \int_0^1 d\sigma (1 - \tau_1), \tag{B.3}$$

by changing the variables as follows

$$\tau_2 = (1 - \tau_1)\sigma, \quad \tau_3 = (1 - \tau_1)(1 - \sigma) \tag{B.4}$$

and then performing partial integration with respect to τ_1 using star-product formula (A.2) and dropping the terms from $\mathcal{H}_1^+$ we obtain

$$\begin{aligned}
 [B_2^{\eta loc}, S_1^\eta]_* \approx & -\frac{i\eta^2}{2} \theta^\beta \int_0^1 d\tau_1 \int_0^1 dt \int_0^1 d\sigma t(1 - t)(1 - \tau_1) e^{i\tau_1 \circ t z_\alpha y^\alpha} (z_\alpha y^\alpha)_{z\beta} \\
 & \times \left\{ C\left([- \tau_1(1 - t) - \sigma t(1 - \tau_1)]z + \sigma y\right) \right. \\
 & \times C\left([- \tau_1(1 - t) + t(1 - \tau_1)(1 - \sigma)]z - (1 - \sigma)y\right) C\left(t(1 - \tau_1)z\right) \\
 & - C\left(-t(1 - \tau_1)z\right) C\left([\tau_1(1 - t) - \sigma t(1 - \tau_1)]z - \sigma y\right) \\
 & \left. \times C\left([\tau_1(1 - t) + (1 - \sigma)(1 - \tau_1)t\right]z + (1 - \sigma)y\right\}.
 \end{aligned} \tag{B.5}$$

Recall that we use notation with hidden $\bar{y}$ variables:

$$\begin{aligned} C(-\tau_1 z + \sigma(1 - \tau_1)y)C(-\tau_1 z - (1 - \sigma)(1 - \tau_1)y) &\equiv \\ &\equiv C(-\tau_1 z + \sigma(1 - \tau_1)y, \bar{y}) \bar{*} C(-\tau_1 z - (1 - \sigma)(1 - \tau_1)y, \bar{y}). \end{aligned} \quad (\text{B.6})$$

Since only small values of

$$\mathcal{T} := \tau_1 \circ t = \tau_1(1 - t) + t(1 - \tau_1) \quad (\text{B.7})$$

contribute to $\widehat{\Upsilon}^{\eta\eta}$ one needs to consider two triangle regions of the init square in (τ_1, t) coordinates. Only the lower triangle with small t and τ_1 contributes because the upper-one gives $\mathcal{T}^3$ in the pre-exponential thus belonging to $\mathcal{H}_1^+$. The following change of variables is handy in the further analysis

$$\begin{aligned} \int d\mathcal{T} \int d\tau_1 dt \theta(\tau_1) \theta(t) \theta(\varepsilon - \tau_1 - t) \delta(\mathcal{T} - \tau_1 - t) f(\tau_1, t) &= \\ = \int_0^\varepsilon d\mathcal{T} \int_0^\mathcal{T} dt f(\mathcal{T} - t, t) &= \int_0^\varepsilon d\mathcal{T} \int_0^1 dt' \mathcal{T} f(\mathcal{T}(1 - t'), \mathcal{T}t'). \end{aligned} \quad (\text{B.8})$$

Adding the terms from $\mathcal{H}^+$, which do not affect the HS field equations, one can reach further simplifications. For instance, one can add $\int_\varepsilon^1 d\mathcal{T} \int_0^1 dt' f(\mathcal{T}(1 - t'), \mathcal{T}t')$ to (B.8), i.e.,

$$\int d\mathcal{T} \int d\tau_1 dt \theta(\tau_1) \theta(t) \theta(\varepsilon - \tau_1 - t) \delta(\mathcal{T} - \tau_1 - t) f(\tau_1, t) \approx \int_0^1 d\mathcal{T} \int_0^1 dt' \mathcal{T} f(\mathcal{T}(1 - t'), \mathcal{T}t'). \quad (\text{B.9})$$

(Recall that sign $\approx$ means that equality is up to terms from $\mathcal{H}^+$.)

In (B.5) it is convenient to introduce new variables

$$\rho_1 = t'\sigma, \quad \rho_2 = t'(1 - \sigma), \quad \rho_3 = 1 - t'. \quad (\text{B.10})$$

They form a simplex since $\rho_1 + \rho_2 + \rho_3 = 1$. The inverse formulas are

$$\sigma = \frac{\rho_1}{\rho_1 + \rho_2}, \quad (1 - \sigma) = \frac{\rho_2}{\rho_1 + \rho_2}, \quad t' = 1 - \rho_3 = \rho_1 + \rho_2. \quad (\text{B.11})$$

In these new variables, the commutator takes the form

$$\begin{aligned} [B_2^{\eta loc}, S_1^\eta]_* &\approx -\frac{i\eta^2}{2} \theta^\beta z_\beta (z_\alpha y^\alpha) \int_0^1 d\mathcal{T} \mathcal{T}^2 \int d^3 \rho_+ \delta \left(1 - \sum_{i=1}^3 \rho_i \right) e^{i\mathcal{T} z_\alpha y^\alpha} \\ &\times \left\{ C \left(-\mathcal{T}(\rho_1 + \rho_3)z + \frac{\rho_1}{\rho_1 + \rho_2}y \right) C \left(\mathcal{T}(\rho_2 - \rho_3)z - \frac{\rho_2}{\rho_1 + \rho_2}y \right) C \left(\mathcal{T}(\rho_1 + \rho_2)z \right) \right. \\ &\left. - C \left(-\mathcal{T}(\rho_1 + \rho_3)z \right) C \left(\mathcal{T}(\rho_2 - \rho_3)z - \frac{\rho_3}{\rho_1 + \rho_3}y \right) C \left(\mathcal{T}(\rho_1 + \rho_2)z + \frac{\rho_1}{\rho_1 + \rho_3}y \right) \right\}. \end{aligned} \quad (\text{B.12})$$

Note that z -dependence is the same in the both terms. Introducing an additional integration parameter ξ , the y -dependence can be uniformized as follows

$$[B_2^{\eta loc}, S_1^\eta]_* \approx \frac{i\eta^2}{2} \theta^\beta z_\beta (z_\alpha y^\alpha) \int_0^1 d\mathcal{T} \mathcal{T}^2 \int d^3 \rho_+ \delta \left(1 - \sum_{i=1}^3 \rho_i \right) \int_0^1 d\xi \frac{\partial}{\partial \xi} \exp \{ \mathcal{Z} \} CCC, \quad (\text{B.13})$$

where the following notations are used

$$D_\alpha = -(\rho_1 + \rho_3)\partial_{1\alpha} + (\rho_2 - \rho_3)\partial_{2\alpha} + (\rho_1 + \rho_2)\partial_{3\alpha}, \quad (\text{B.14})$$

$$\begin{aligned} \mathcal{Z} = & i\mathcal{T}z_\alpha y^\alpha + \mathcal{T}z^\alpha D_\alpha + (1 - \xi)y^\alpha \left(\frac{\rho_1}{\rho_1 + \rho_2}\partial_{1\alpha} - \frac{\rho_2}{\rho_1 + \rho_2}\partial_{2\alpha} \right) \\ & + \xi y^\alpha \left(\frac{\rho_1}{\rho_1 + \rho_3}\partial_{3\alpha} - \frac{\rho_3}{\rho_1 + \rho_3}\partial_{2\alpha} \right). \end{aligned} \quad (\text{B.15})$$

Evaluating the derivative with respect to ξ taking into account that

$$\frac{\partial \mathcal{Z}}{\partial \xi} = \frac{\rho_1}{\mathcal{T}(\rho_1 + \rho_2)(\rho_1 + \rho_3)} y^\alpha \frac{\partial \mathcal{Z}}{\partial z^\alpha} \quad (\text{B.16})$$

along with the Schouten identity

$$\begin{aligned} (\theta^\beta z_\beta) \left(y^\alpha \frac{\partial \mathcal{Z}}{\partial z^\alpha} \right) &= (\theta^\beta y_\beta) \left(z^\alpha \frac{\partial \mathcal{Z}}{\partial z^\alpha} \right) + (z_\alpha y^\alpha) \left(\theta^\beta \frac{\partial \mathcal{Z}}{\partial z^\beta} \right) \\ &= (\theta^\beta y_\beta) \left(\mathcal{T} \frac{\partial \mathcal{Z}}{\partial \mathcal{T}} \right) + (z_\alpha y^\alpha) d_z \mathcal{Z}, \end{aligned} \quad (\text{B.17})$$

the expression for the commutator can be rewritten in the form

$$\begin{aligned} [B_2^{\eta loc}, S_1^\eta]_* &\approx d_z \left[-\frac{\eta^2 (z_\alpha y^\alpha)^2}{2} \int_0^1 d\mathcal{T} \mathcal{T} \int d^3 \rho_+ \delta \left(1 - \sum_{i=1}^3 \rho_i \right) \right. \\ &\quad \times \int_0^1 d\xi \frac{\rho_1}{(\rho_1 + \rho_2)(\rho_1 + \rho_3)} \exp \{ \mathcal{Z} \} CCC \Big] \\ &\quad - \frac{\eta^2}{2} \theta^\beta y_\beta \int d\mathcal{T} \delta(1 - \mathcal{T}) \mathcal{T}^2 \int d^3 \rho_+ \delta \left(1 - \sum_{i=1}^3 \rho_i \right) \\ &\quad \times \int_0^1 d\xi \frac{\rho_1 (z_\alpha y^\alpha) \exp \{ \mathcal{Z} \}}{(\rho_1 + \rho_2)(\rho_1 + \rho_3)} CCC. \end{aligned} \quad (\text{B.18})$$

Since the second (boundary) term belongs to $\mathcal{H}_1^+$ and thus contributes to $\Upsilon_+^{\eta\eta}$ the part of $B_3^{\eta\eta}$ that contributes to $\hat{\Upsilon}^{\eta\eta}$ can be chosen in the form (4.18).

C $W_2^{\eta\eta}$

A particular solution for $W_2^{\eta\eta}$ was found in [15] where it was used in the computation of vertices $\Upsilon^{\eta\eta}(\omega, \omega, C, C)$. However, this solution turns out to be technically inconvenient for the analysis modulo $\mathcal{H}^+$ subspace. In this section we apply the approach proposed in the previous section that allows us to single out the $\mathcal{H}_0^+$ part from $W_2^{\eta\eta}$. Vertices $\Upsilon^{\eta\eta}(\omega, \omega, C, C)$ computed with $W_2^{\eta\eta}$ given by (4.19)–(4.21) may differ from those of [15] by a local field redefinition.

To compute $W_2^{\eta\eta}$ consider the equation

$$d_x S_1^\eta + W_1^\eta * S_1^\eta + S_1^\eta * W_1^\eta + d_x S_2^{\eta\eta} + \omega * S_2^{\eta\eta} + S_2^{\eta\eta} * \omega + S_0 * W_2^{\eta\eta} + W_2^{\eta\eta} * S_0 = 0. \quad (\text{C.1})$$

As mentioned in appendix B contribution to the vertices $\Upsilon^{\eta\eta}(\omega, C, C, C)$ from $S_2^{\eta\eta}$ vanishes at $\beta \rightarrow -\infty$. The remaining equation to be solved is

$$2id_z \widehat{W}_2 \approx d_x S_1^\eta + W_1^\eta * S_1^\eta + S_1^\eta * W_1^\eta. \quad (C.2)$$

Here S_1^η is given by (2.32) while W_1^η consists of two parts (4.12) and (4.13).

To calculate $d_x S_1^\eta$ one needs second-order zero-form vertices $\Upsilon^\eta(\omega, C, C)$ obtained in [14] with additional shifts $\delta \Upsilon^\eta(\omega, C, C)$ generated by the local shift δB_2^η (4.11). These are given by (4.15)–(4.17).

C.1 $W_{2CC\omega}^{\eta\eta}$

Equation for $W_{2CC\omega}^{\eta\eta}$ has the form

$$2id_z \widehat{W}_{2CC\omega}^{\eta\eta} \approx d_x S_1^\eta|_{CC\omega} + S_1^\eta * W_{1C\omega}^\eta. \quad (C.3)$$

Computing $S_1^\eta * W_{1C\omega}^\eta$ and dropping terms from $\mathcal{H}_1^+$ one obtains discarding barred variables as in (B.6)

$$\begin{aligned} S_1^\eta * W_{1C\omega}^\eta &\approx \frac{\eta^2}{2} \int_0^1 dt \int_0^1 d\tau_1 \int_0^1 d\sigma t(1-t)(1-\tau_1)^2 (\theta^\beta z_\beta) (z^\alpha \partial_{\omega\alpha}) \\ &\quad \times \exp \left\{ i\tau_1 \circ tz_\alpha y^\alpha + i(1-\sigma)\partial_{2\alpha}\partial_\omega^\alpha \right\} \\ &\quad \times C(-t(1-\tau_1)z) C(\tau_1(1-t)z) \omega([\tau_1(1-t) + \sigma t(1-\tau_1)]z + \sigma y). \end{aligned} \quad (C.4)$$

Taking into account that only small values of $\tau_1 \circ t$ contribute to $\widehat{\Upsilon}^{\eta\eta}$ one can change integration variables as in (B.9) to obtain

$$\begin{aligned} S_1^\eta * W_{1C\omega}^\eta &\approx \frac{\eta^2}{2} \int_0^1 d\mathcal{T} \mathcal{T}^2 \int_0^1 dt' \int_0^1 d\sigma t' (\theta^\beta z_\beta) (z^\alpha \partial_{\omega\alpha}) \exp \left\{ i\mathcal{T} z_\alpha y^\alpha + i(1-\sigma)\partial_{2\alpha}\partial_\omega^\alpha \right\} \\ &\quad \times C(-\mathcal{T}t'z) C(\mathcal{T}(1-t')z) \omega(\mathcal{T}[(1-t') + t'\sigma]z + \sigma y). \end{aligned} \quad (C.5)$$

In the simplex variables (B.10), (B.11) this expression can be rewritten as

$$\begin{aligned} S_1^\eta * W_{1C\omega}^\eta &\approx \frac{\eta^2}{2} \int_0^1 d\mathcal{T} \mathcal{T}^2 \int d^3\rho_+ \delta\left(1 - \sum_{i=1}^3 \rho_i\right) (\theta^\beta z_\beta) (z^\alpha \partial_{\omega\alpha}) \\ &\quad \times \exp \left\{ i\mathcal{T} z_\alpha y^\alpha + i\frac{\rho_2}{\rho_1 + \rho_2} \partial_{2\alpha}\partial_\omega^\alpha \right\} \\ &\quad \times C(-\mathcal{T}(\rho_1 + \rho_2)z) C(\mathcal{T}\rho_3 z) \omega\left(\mathcal{T}(\rho_1 + \rho_3)z + \frac{\rho_1}{\rho_1 + \rho_2}y\right). \end{aligned} \quad (C.6)$$

To compute $d_x S_1^\eta$ one has to use vertex (4.16)

$$\begin{aligned} d_x S_1^\eta|_{CC\omega} &\approx -\frac{\eta^2}{2} \int_0^1 d\mathcal{T} \mathcal{T}^2 \int d^3\rho_+ \delta\left(1 - \sum_{i=1}^3 \rho_i\right) (\theta^\beta z_\beta) (z^\alpha \partial_{\omega\alpha}) \\ &\quad \times \exp \left\{ i\mathcal{T} z_\alpha y^\alpha + i(1-\rho_1)\partial_{2\alpha}\partial_\omega^\alpha - i\rho_1\partial_{1\alpha}\partial_\omega^\alpha \right\} \\ &\quad \times C(-\mathcal{T}(\rho_1 + \rho_2)z) C(\mathcal{T}\rho_3 z) \omega(\mathcal{T}(\rho_1 + \rho_3)z). \end{aligned} \quad (C.7)$$

As in appendix B, y -dependence in (C.3) can be uniformized with the help of the new integration parameter ξ . Using new notation for brevity

$$\begin{aligned} Z = & i\mathcal{T}z_\alpha y^\alpha + \mathcal{T}z^\alpha \left(-(\rho_1 + \rho_2)\partial_{1\alpha} + \rho_3\partial_{2\alpha} + (\rho_1 + \rho_3)\partial_{\omega\alpha} \right) \\ & + \xi \left(i\frac{\rho_2}{\rho_1 + \rho_2}\partial_{2\alpha}\partial_\omega^\alpha + \frac{\rho_1}{\rho_1 + \rho_2}y^\alpha\partial_{\omega\alpha} \right) + (1 - \xi) \left(i(1 - \rho_1)\partial_{2\alpha}\partial_\omega^\alpha - i\rho_1\partial_{1\alpha}\partial_\omega^\alpha \right) \end{aligned} \quad (\text{C.8})$$

one has from (C.3) taking into account (C.6) and (C.7)

$$2i\text{d}_z \widehat{W}_{2CC\omega}^{\eta\eta} = \frac{\eta^2}{2} \int_0^1 d\mathcal{T} \mathcal{T}^2 \int d^3\rho_+ \delta \left(1 - \sum_{i=1}^3 \rho_i \right) (\theta^\beta z_\beta) (z^\alpha \partial_{\omega\alpha}) \int_0^1 d\xi \frac{\partial}{\partial \xi} e^Z CC\omega. \quad (\text{C.9})$$

Evaluating the derivative over ξ

$$\frac{\partial Z}{\partial \xi} = \frac{\rho_1}{\rho_1 + \rho_2} \left(-i\rho_3\partial_{2\alpha}\partial_\omega^\alpha + i(\rho_1 + \rho_2)\partial_{1\alpha}\partial_\omega^\alpha + y^\alpha\partial_{\omega\alpha} \right) \quad (\text{C.10})$$

and taking into account that

$$\frac{\partial Z}{\partial \xi} = \frac{-i\rho_1}{\mathcal{T}(\rho_1 + \rho_2)} \partial_\omega^\alpha \frac{\partial Z}{\partial z^\alpha} \quad (\text{C.11})$$

along with the Schouten identity

$$(\theta^\beta z_\beta) \left(\partial_\omega^\alpha \frac{\partial Z}{\partial z^\alpha} \right) = (\theta^\beta \partial_{\omega\beta}) \left(z^\alpha \frac{\partial Z}{\partial z^\alpha} \right) - \left(\theta^\beta \frac{\partial Z}{\partial z^\beta} \right) (z^\alpha \partial_{\omega\alpha}), \quad (\text{C.12})$$

the pre-exponential part can be written in the form

$$(\theta^\beta z_\beta) \left(\partial_\omega^\alpha \frac{\partial Z}{\partial z^\alpha} \right) = (\theta^\beta \partial_{\omega\beta}) \left(\mathcal{T} \frac{\partial Z}{\partial \mathcal{T}} \right) - (z^\alpha \partial_{\omega\alpha}) \text{d}_z Z \quad (\text{C.13})$$

and thus r.h.s. of (C.3) can be put into the form

$$\begin{aligned} 2i\text{d}_z \widehat{W}_{2CC\omega}^{\eta\eta} = & -\frac{i\eta^2}{2} \int_0^1 d\mathcal{T} \frac{\rho_1 \mathcal{T}^2}{\rho_1 + \rho_2} \int d^3\rho_+ \delta \left(1 - \sum_{i=1}^3 \rho_i \right) \int_0^1 d\xi (\theta^\beta \partial_{\omega\beta}) (z^\gamma \partial_{\omega\gamma}) \frac{\partial}{\partial \mathcal{T}} e^Z CC\omega \\ & + \frac{i\eta^2}{2} \int_0^1 d\mathcal{T} \frac{\rho_1 \mathcal{T}^2}{\rho_1 + \rho_2} \int d^3\rho_+ \delta \left(1 - \sum_{i=1}^3 \rho_i \right) \int_0^1 d\xi (z^\gamma \partial_{\omega\gamma})^2 \text{d}_z e^Z CC\omega. \end{aligned} \quad (\text{C.14})$$

After integrating by parts with respect to $\mathcal{T}$ in the first term, the resulting boundary term belongs to $\mathcal{H}^+$ (cf. the second case of (3.6)) and hence can be discarded. This brings equation for $\widehat{W}_{2CC\omega}^{\eta\eta}$ to the form

$$2i\text{d}_z \widehat{W}_{2CC\omega}^{\eta\eta} \approx \text{d}_z \left\{ \frac{i\eta^2}{2} \int_0^1 d\mathcal{T} \int_0^1 d\xi \int d^3\rho_+ \delta \left(1 - \sum_{i=1}^3 \rho_i \right) \frac{\mathcal{T} \rho_1 (z^\gamma \partial_{\omega\gamma})^2}{\rho_1 + \rho_2} e^Z CC\omega \right\}. \quad (\text{C.15})$$

This allows us to choose the part of $W_{2CC\omega}^{\eta\eta}$, that contributes to $\hat{\Upsilon}^{\eta\eta}$, in the form

$$\begin{aligned}\widehat{W}_{2CC\omega}^{\eta\eta} = & \frac{\eta^2}{4} \int_0^1 d\mathcal{T} \int_0^1 d\xi \int d^3\rho_+ \delta\left(1 - \sum_{i=1}^3 \rho_i\right) \frac{\mathcal{T}\rho_1(z^\gamma \partial_{\omega\gamma})^2}{\rho_1 + \rho_2} \\ & \times \exp\left\{i\mathcal{T}z_\alpha y^\alpha + \mathcal{T}z^\alpha\left(-(\rho_1 + \rho_2)\partial_{1\alpha} + \rho_3\partial_{2\alpha} + (\rho_1 + \rho_3)\partial_{\omega\alpha}\right)\right. \\ & + \xi\left(\frac{i\rho_2}{\rho_1 + \rho_2}\partial_{2\alpha}\partial_\omega^\alpha + \frac{\rho_1}{\rho_1 + \rho_2}y^\alpha\partial_{\omega\alpha}\right) \\ & \left. + (1 - \xi)\left(i(1 - \rho_1)\partial_{2\alpha}\partial_\omega^\alpha - \rho_1\partial_{1\alpha}\partial_\omega^\alpha\right)\right\} CC\omega.\end{aligned}\quad (\text{C.16})$$

Finally, one can change the integration variables to rewrite $\widehat{W}_{2CC\omega}^{\eta\eta}$ in the form of the integral over a four-dimensional simplex according to

$$\begin{aligned}& \int_0^1 d\xi \int d^3\rho_+ \delta\left(1 - \sum_{i=1}^3 \rho_i\right) f(\xi, 1 - \xi; \rho_1, \rho_2, \rho_3) \\ &= \int d^2\xi_+ \delta(1 - \xi_1 - \xi_2) \int d^3\rho_+ \delta\left(1 - \sum_{i=1}^3 \rho_i\right) f(\xi_1, \xi_2; \rho_1, \rho_2, \rho_3) \\ &= \int d\zeta_1 \int d\zeta_2 \int d^2\xi_+ \delta(1 - \xi_1 - \xi_2) \int d^3\rho_+ \delta\left(1 - \sum_{i=1}^3 \rho_i\right) \\ & \quad \times \delta(\zeta_1 - \rho_3\xi_1) \delta(\zeta_2 - \rho_3\xi_2) f(\xi_1, \xi_2; \rho_1, \rho_2, \rho_3) \\ &= \int d^2\zeta_+ \int d^3\rho_+ \frac{1}{\rho_3^2} \delta\left(1 - \frac{\zeta_1}{\rho_3} - \frac{\zeta_2}{\rho_3}\right) \delta\left(1 - \sum_{i=1}^3 \rho_i\right) f\left(\frac{\zeta_1}{\rho_3}, \frac{\zeta_2}{\rho_3}; \rho_1, \rho_2, \rho_3\right) \\ &= \int d^2\zeta_+ \int d^2\rho_+ \frac{\delta(1 - \rho_1 - \rho_2 - \zeta_1 - \zeta_2)}{1 - \rho_1 - \rho_2} f\left(\frac{\zeta_1}{1 - \rho_1 - \rho_2}, \frac{\zeta_2}{1 - \rho_1 - \rho_2}; \rho_1, \rho_2, 1 - \rho_1 - \rho_2\right) \\ &= \int d^4\rho_+ \delta\left(1 - \sum_{i=1}^4 \rho_i\right) \frac{1}{1 - \rho_1 - \rho_2} f\left(\frac{\rho_3}{1 - \rho_1 - \rho_2}, \frac{\rho_4}{1 - \rho_1 - \rho_2}; \rho_1, \rho_2, 1 - \rho_1 - \rho_2\right).\end{aligned}\quad (\text{C.17})$$

In these variables $\widehat{W}_{2CC\omega}^{\eta\eta}$ acquires the form (4.19).

C.2 $W_{2C\omega C}^{\eta\eta}$

Equation for this part of the connection is

$$2id_z \widehat{W}_{2C\omega C}^{\eta\eta} \approx d_x S_1^\eta \Big|_{C\omega C} + S_1^\eta * W_{1\omega C}^\eta + W_{1C\omega}^\eta * S_1^\eta. \quad (\text{C.18})$$

Star product $S_1^\eta * W_{1\omega C}^\eta$ can be computed by (A.2). Discarding terms in $\Upsilon_+^{\eta\eta}$ and omitting the barred variables, $S_1^\eta * W_{1\omega C}^\eta$ takes the form

$$\begin{aligned}S_1^\eta * W_{1\omega C}^\eta \approx & \frac{\eta^2}{2} \int_0^1 dt \int_0^1 d\tau_1 \int_0^1 d\sigma t(1-t)(1-\tau_1)^2 \left(\theta^\beta z_\beta\right) (z^\alpha \partial_{\omega\alpha}) \\ & \times \exp\left\{i\tau_1 \circ t z_\alpha y^\alpha + i(1-\sigma)\partial_{\omega\alpha}\partial_2^\alpha\right\} \\ & \times C\left(-t(1-\tau_1)z\right)\omega\left([\tau_1(1-t) - \sigma t(1-\tau_1)]z - \sigma y\right)C\left(\tau_1(1-t)z\right).\end{aligned}\quad (\text{C.19})$$

Since only small values of $\mathcal{T}$ contribute to $\hat{\Upsilon}^{\eta\eta}$ we consider only lower triangle of the unit square in (τ_1, t) and perform the same change of variables as in (B.9). Using the simplex variables (B.10), (B.11) the result can be re-written as

$$\begin{aligned} S_1^\eta * W_{1\omega C}^\eta &\approx \frac{\eta^2}{2} \int_0^1 d\mathcal{T} \mathcal{T}^2 \int d^3\rho_+ \delta\left(1 - \sum_{i=1}^3 \rho_i\right) (\theta^\beta z_\beta) (z^\alpha \partial_{\omega\alpha}) \\ &\times \exp\left\{i\mathcal{T} z_\alpha y^\alpha + i\frac{\rho_2}{\rho_1 + \rho_2} \partial_{\omega\alpha} \partial_2^\alpha\right\} \\ &\times C\left(-\mathcal{T}(1 - \rho_3)z\right) \omega\left(-\mathcal{T}\rho_1 z + \mathcal{T}\rho_3 z - \frac{\rho_1}{\rho_1 + \rho_2} y\right) C(\mathcal{T}\rho_3 z). \end{aligned} \quad (\text{C.20})$$

Analogously, for star product $W_{1C\omega}^\eta * S_1^\eta$

$$\begin{aligned} W_{1C\omega}^\eta * S_1^\eta &\approx \frac{\eta^2}{2} \int_0^1 d\mathcal{T} \mathcal{T}^2 \int d^3\rho_+ \delta\left(1 - \sum_{i=1}^3 \rho_i\right) (\theta^\beta z_\beta) (z^\alpha \partial_{\omega\alpha}) \\ &\times \exp\left\{i\mathcal{T} z_\alpha y^\alpha + i\frac{\rho_2}{\rho_1 + \rho_2} \partial_{\omega\alpha} \partial_2^\alpha\right\} \\ &\times C\left(-\mathcal{T}\rho_3 z\right) \omega\left(\mathcal{T}\rho_1 z - \mathcal{T}\rho_3 z - \frac{\rho_1}{\rho_1 + \rho_2} y\right) C(\mathcal{T}(1 - \rho_3)z). \end{aligned} \quad (\text{C.21})$$

The $d_x S_1^\eta$ part computed using vertex (4.17) is

$$\begin{aligned} d_x S_1^\eta|_{C\omega C} &\approx -\frac{\eta^2}{2} \int_0^1 d\mathcal{T} \mathcal{T}^2 \int d^3\rho_+ \delta\left(1 - \sum_{i=1}^3 \rho_i\right) (\theta^\beta z_\beta) (z^\alpha \partial_{\omega\alpha}) \\ &\times \exp\left\{i\mathcal{T} z_\alpha y^\alpha + i\rho_2 \partial_{1\alpha} \partial_\omega^\alpha + i(1 - \rho_2) \partial_{\omega\alpha} \partial_2^\alpha\right\} \\ &\times C\left(-\mathcal{T}\rho_3 z\right) \omega\left(\mathcal{T}\rho_1 z\right) C(\mathcal{T}(1 - \rho_3)z) \\ &- \frac{\eta^2}{2} \int_0^1 d\mathcal{T} \mathcal{T}^2 \int d^3\rho_+ \delta\left(1 - \sum_{i=1}^3 \rho_i\right) (\theta^\beta z_\beta) (z^\alpha \partial_{\omega\alpha}) \\ &\times \exp\left\{i\mathcal{T} z_\alpha y^\alpha + i(1 - \rho_2) \partial_{1\alpha} \partial_\omega^\alpha + i\rho_2 \partial_{\omega\alpha} \partial_2^\alpha\right\} \\ &\times C\left(-\mathcal{T}(1 - \rho_3)z\right) \omega\left(-\mathcal{T}\rho_1 z\right) C(\mathcal{T}\rho_3 z). \end{aligned} \quad (\text{C.22})$$

It is natural to group the r.h.s. of (C.18) in the following way

$$\begin{aligned} d_x S_1^\eta|_{C\omega C} + S_1^\eta * W_{1\omega C}^\eta + W_{1C\omega}^\eta * S_1^\eta &\approx \\ &\approx \frac{\eta^2}{2} \int_0^1 d\mathcal{T} \mathcal{T}^2 \int d^3\rho_+ \delta\left(1 - \sum_{i=1}^3 \rho_i\right) (\theta^\beta z_\beta) (z^\alpha \partial_{\omega\alpha}) \exp\{i\mathcal{T} z_\alpha y^\alpha\} \times \\ &\times \left[-\exp\{i\rho_2 \partial_{1\alpha} \partial_\omega^\alpha + i(1 - \rho_2) \partial_{\omega\alpha} \partial_2^\alpha\} C\left(-\mathcal{T}\rho_3 z\right) \omega\left(\mathcal{T}\rho_1 z\right) C(\mathcal{T}(1 - \rho_3)z) \right. \\ &+ \exp\left\{i\frac{\rho_2}{\rho_1 + \rho_2} \partial_{\omega\alpha} \partial_2^\alpha\right\} C\left(-\mathcal{T}\rho_3 z\right) \omega\left(\mathcal{T}\rho_1 z - \mathcal{T}\rho_3 z - \frac{\rho_1}{\rho_1 + \rho_2} y\right) C(\mathcal{T}(1 - \rho_3)z) \\ &- \exp\left\{i(1 - \rho_2) \partial_{1\alpha} \partial_\omega^\alpha + i\rho_2 \partial_{\omega\alpha} \partial_2^\alpha\right\} C\left(-\mathcal{T}(1 - \rho_3)z\right) \omega\left(-\mathcal{T}\rho_1 z\right) C(\mathcal{T}\rho_3 z) \\ &\left. + \exp\left\{i\frac{\rho_2}{\rho_1 + \rho_2} \partial_{\omega\alpha} \partial_2^\alpha\right\} C\left(-\mathcal{T}\rho_3 z\right) \omega\left(\mathcal{T}\rho_1 z - \mathcal{T}\rho_3 z - \frac{\rho_1}{\rho_1 + \rho_2} y\right) C(\mathcal{T}(1 - \rho_3)z) \right]. \end{aligned}$$

Introducing new notations for brevity

$$\begin{aligned}
 Z_1 &= i\mathcal{T}z_\alpha y^\alpha + \mathcal{T}z^\alpha \left(-\rho_3 \partial_{1\alpha} + \rho_1 \partial_{\omega\alpha} + (1-\rho_3) \partial_{2\alpha} \right) \\
 &\quad + \xi \left(-\mathcal{T} \rho_3 z^\alpha \partial_{\omega\alpha} - \frac{\rho_1}{\rho_1 + \rho_2} y^\alpha \partial_{\omega\alpha} + i \frac{\rho_2}{\rho_1 + \rho_2} \partial_{1\alpha} \partial_\omega^\alpha \right) \\
 &\quad + (1-\xi) \left(i \rho_2 \partial_{1\alpha} \partial_\omega^\alpha + i(1-\rho_2) \partial_{\omega\alpha} \partial_2^\alpha \right), \\
 Z_2 &= i\mathcal{T}z_\alpha y^\alpha + \mathcal{T}z^\alpha \left(-(1-\rho_3) \partial_{1\alpha} - \rho_1 \partial_{\omega\alpha} + \rho_3 \partial_{2\alpha} \right) \\
 &\quad + \xi \left(\mathcal{T} \rho_3 z^\alpha \partial_{\omega\alpha} - \frac{\rho_1}{\rho_1 + \rho_2} y^\alpha \partial_{\omega\alpha} + i \frac{\rho_2}{\rho_1 + \rho_2} \partial_{\omega\alpha} \partial_2^\alpha \right) \\
 &\quad + (1-\xi) \left(i(1-\rho_2) \partial_{1\alpha} \partial_\omega^\alpha + i \rho_2 \partial_{\omega\alpha} \partial_2^\alpha \right),
 \end{aligned} \tag{C.23}$$

the r.h.s. of (C.18) can be written as an integral of a total derivative

$$2i d_z \widehat{W}_{2C\omega C}^{\eta\eta} \approx \frac{\eta^2}{2} \int_0^1 d\mathcal{T} \mathcal{T}^2 \int d^3 \rho_+ \delta \left(1 - \sum_{i=1}^3 \rho_i \right) (\theta^\beta z_\beta) (z^\alpha \partial_{\omega\alpha}) \int_0^1 d\xi \frac{\partial}{\partial \xi} (e^{Z_1} + e^{Z_2}) C \omega C. \tag{C.24}$$

Since

$$\begin{aligned}
 \frac{\partial Z_1}{\partial \xi} &= -\rho_3 (\mathcal{T}z + i\partial_1 + i\partial_2)^\alpha \partial_{\omega\alpha} - \frac{\rho_1}{\rho_1 + \rho_2} \left[y^\alpha \partial_{\omega\alpha} + i \rho_3 \partial_{1\alpha} \partial_\omega^\alpha + i(1-\rho_3) \partial_{\omega\alpha} \partial_2^\alpha \right] \\
 &= -\rho_3 (\mathcal{T}z + i\partial_1 + i\partial_2)^\alpha \partial_{\omega\alpha} + i \frac{\rho_1}{\mathcal{T}(\rho_1 + \rho_2)} \partial_\omega^\alpha \frac{\partial Z_1}{\partial z^\alpha},
 \end{aligned} \tag{C.25}$$

analogously to (C.12) by virtue of Schouten identity one has

$$(\theta^\beta z_\beta) \frac{\partial Z_1}{\partial \xi} = -(\theta^\beta z_\beta) \rho_3 (\mathcal{T}z + i\partial_1 + i\partial_2)^\alpha \partial_{\omega\alpha} - i \frac{\rho_1 (z^\gamma \partial_{\omega\gamma})}{\mathcal{T}(\rho_1 + \rho_2)} d_z Z_1 + i \frac{\rho_1 (\theta^\beta \partial_{\omega\beta})}{\rho_1 + \rho_2} \frac{\partial Z_1}{\partial \mathcal{T}}. \tag{C.26}$$

Therefore the Z_1 -dependent part from the r.h.s. of (C.24) can be rewritten in the form

$$\begin{aligned}
 &\frac{\eta^2}{2} \int_0^1 d\mathcal{T} \mathcal{T}^2 \int d^3 \rho_+ \delta \left(1 - \sum_{i=1}^3 \rho_i \right) (\theta^\beta z_\beta) (z^\alpha \partial_{\omega\alpha}) \int_0^1 d\xi \frac{\partial}{\partial \xi} e^{Z_1} C \omega C \\
 &\approx -\frac{\eta^2}{2} (\theta^\beta z_\beta) (z^\gamma \partial_{\omega\gamma}) \int_0^1 d\mathcal{T} \mathcal{T}^2 \int_0^1 d\xi \int d^3 \rho_+ \delta \left(1 - \sum_{i=1}^3 \rho_i \right) \rho_3 (\mathcal{T}z + i\partial_1 + i\partial_2)^\alpha \partial_{\omega\alpha} e^{Z_1} C \omega C \\
 &\quad - \frac{i\eta^2}{2} (z^\gamma \partial_{\omega\gamma})^2 \int_0^1 d\mathcal{T} \mathcal{T} \int_0^1 d\xi \int d^3 \rho_+ \delta \left(1 - \sum_{i=1}^3 \rho_i \right) \frac{\rho_1}{\rho_1 + \rho_2} d_z e^{Z_1} C \omega C \\
 &\quad + \frac{i\eta^2}{2} (\theta^\beta \partial_{\omega\beta}) (z^\gamma \partial_{\omega\gamma}) \int_0^1 d\mathcal{T} \mathcal{T}^2 \int_0^1 d\xi \int d^3 \rho_+ \delta \left(1 - \sum_{i=1}^3 \rho_i \right) \frac{\rho_1}{\rho_1 + \rho_2} \frac{\partial}{\partial \mathcal{T}} e^{Z_1} C \omega C.
 \end{aligned} \tag{C.27}$$

Integrating by parts in the last term, the resulting boundary term contributes to $\Upsilon_+^{\eta\eta}$.

Hence

$$\begin{aligned}
 & \frac{\eta^2}{2} \int_0^1 d\mathcal{T} \mathcal{T}^2 \int d^3 \rho_+ \delta \left(1 - \sum_{i=1}^3 \rho_i \right) (\theta^\beta z_\beta) (z^\alpha \partial_{\omega\alpha}) \int_0^1 d\xi \frac{\partial}{\partial \xi} e^{Z_1} C \omega C \\
 & \approx -\frac{\eta^2}{2} (\theta^\beta z_\beta) (z^\gamma \partial_{\omega\gamma}) \int_0^1 d\mathcal{T} \mathcal{T}^2 \int_0^1 d\xi \int d^3 \rho_+ \delta \left(1 - \sum_{i=1}^3 \rho_i \right) \rho_3 (\mathcal{T} z + i\partial_1 + i\partial_2)^\alpha \partial_{\omega\alpha} e^{Z_1} C \omega C \\
 & \quad + \text{d}_z \left\{ -\frac{i\eta^2}{2} (z^\gamma \partial_{\omega\gamma})^2 \int_0^1 d\mathcal{T} \mathcal{T} \int_0^1 d\xi \int d^3 \rho_+ \delta \left(1 - \sum_{i=1}^3 \rho_i \right) \frac{\rho_1}{\rho_1 + \rho_2} e^{Z_1} C \omega C \right\}.
 \end{aligned} \tag{C.28}$$

Analogously, since

$$\begin{aligned}
 \frac{\partial Z_2}{\partial \xi} &= \rho_3 (\mathcal{T} z + i\partial_1 + i\partial_2)^\alpha \partial_{\omega\alpha} + i \frac{\rho_1}{\mathcal{T}(\rho_1 + \rho_2)} \partial_\omega^\alpha \frac{\partial Z_2}{\partial z^\alpha}, \\
 (\theta^\beta z_\beta) \frac{\partial Z_2}{\partial \xi} &= (\theta^\beta z_\beta) \rho_3 (\mathcal{T} z + i\partial_1 + i\partial_2)^\alpha \partial_{\omega\alpha} - i \frac{\rho_1 (z^\gamma \partial_{\omega\gamma})}{\mathcal{T}(\rho_1 + \rho_2)} \text{d}_z Z_2 + i \frac{\rho_1 (\theta^\beta \partial_{\omega\beta})}{\rho_1 + \rho_2} \frac{\partial Z_2}{\partial \mathcal{T}}
 \end{aligned} \tag{C.29}$$

and, therefore, the Z_2 -dependent part of the r.h.s. of (C.24) yields

$$\begin{aligned}
 & \frac{\eta^2}{2} \int_0^1 d\mathcal{T} \mathcal{T}^2 \int d^3 \rho_+ \delta \left(1 - \sum_{i=1}^3 \rho_i \right) (\theta^\beta z_\beta) (z^\alpha \partial_{\omega\alpha}) \int_0^1 d\xi \frac{\partial}{\partial \xi} e^{Z_2} C \omega C \\
 & \approx \frac{\eta^2}{2} (\theta^\beta z_\beta) (z^\gamma \partial_{\omega\gamma}) \int_0^1 d\mathcal{T} \mathcal{T}^2 \int_0^1 d\xi \int d^3 \rho_+ \delta \left(1 - \sum_{i=1}^3 \rho_i \right) \rho_3 (\mathcal{T} z + i\partial_1 + i\partial_2)^\alpha \partial_{\omega\alpha} e^{Z_2} C \omega C \\
 & \quad - \frac{i\eta^2}{2} (z^\gamma \partial_{\omega\gamma})^2 \int_0^1 d\mathcal{T} \mathcal{T} \int_0^1 d\xi \int d^3 \rho_+ \delta \left(1 - \sum_{i=1}^3 \rho_i \right) \frac{\rho_1}{\rho_1 + \rho_2} \text{d}_z e^{Z_2} C \omega C \\
 & \quad + \frac{i\eta^2}{2} (\theta^\beta \partial_{\omega\beta}) (z^\gamma \partial_{\omega\gamma}) \int_0^1 d\mathcal{T} \mathcal{T}^2 \int_0^1 d\xi \int d^3 \rho_+ \delta \left(1 - \sum_{i=1}^3 \rho_i \right) \frac{\rho_1}{\rho_1 + \rho_2} \frac{\partial}{\partial \mathcal{T}} e^{Z_2} C \omega C.
 \end{aligned} \tag{C.30}$$

Modulo terms from $\mathcal{H}^+$ this equals to

$$\begin{aligned}
 & \frac{\eta^2}{2} (\theta^\beta z_\beta) (z^\gamma \partial_{\omega\gamma}) \int_0^1 d\mathcal{T} \mathcal{T}^2 \int_0^1 d\xi \int d^3 \rho_+ \delta \left(1 - \sum_{i=1}^3 \rho_i \right) \rho_3 (\mathcal{T} z + i\partial_1 + i\partial_2)^\alpha \partial_{\omega\alpha} e^{Z_2} C \omega C \\
 & \quad + \text{d}_z \left\{ -\frac{i\eta^2}{2} (z^\gamma \partial_{\omega\gamma})^2 \int_0^1 d\mathcal{T} \mathcal{T} \int_0^1 d\xi \int d^3 \rho_+ \delta \left(1 - \sum_{i=1}^3 \rho_i \right) \frac{\rho_1}{\rho_1 + \rho_2} e^{Z_2} C \omega C \right\}.
 \end{aligned} \tag{C.31}$$

As a result, the r.h.s. of (C.24) acquires the form

$$\begin{aligned}
 2i \text{d}_z \widehat{W}_{2C\omega C}^{\eta\eta} & \approx \text{d}_z \left\{ -\frac{i\eta^2}{2} (z^\gamma \partial_{\omega\gamma})^2 \int_0^1 d\mathcal{T} \mathcal{T} \int_0^1 d\xi \int d^3 \rho_+ \right. \\
 & \quad \times \delta \left(1 - \sum_{i=1}^3 \rho_i \right) \frac{\rho_1}{\rho_1 + \rho_2} [e^{Z_1} + e^{Z_2}] C \omega C \Big\} \\
 & \quad + \frac{\eta^2}{2} (\theta^\beta z_\beta) (z^\gamma \partial_{\omega\gamma}) \int_0^1 d\mathcal{T} \mathcal{T}^2 \int_0^1 d\xi \int d^3 \rho_+ \\
 & \quad \times \delta \left(1 - \sum_{i=1}^3 \rho_i \right) \rho_3 (\mathcal{T} z + i\partial_1 + i\partial_2)^\alpha \partial_{\omega\alpha} [e^{Z_2} - e^{Z_1}] C \omega C.
 \end{aligned} \tag{C.32}$$

To see that the last term vanishes it is convenient to change integration variables to those of the four-dimensional simplex (C.17). In these four-dimensional simplicial variables

$$\begin{aligned} Z_1 = & i\mathcal{T}z_\alpha y^\alpha + \mathcal{T}z^\alpha \left(-(\rho_3 + \rho_4)\partial_{1\alpha} + (\rho_1 - \rho_3)\partial_{\omega\alpha} + (\rho_1 + \rho_2)\partial_{2\alpha} \right) - \frac{\rho_3\rho_1}{(\rho_1 + \rho_2)(\rho_3 + \rho_4)} y^\alpha \partial_{\omega\alpha} \\ & + i\frac{\rho_3\rho_2}{(\rho_1 + \rho_2)(\rho_3 + \rho_4)} \partial_{1\alpha} \partial_\omega^\alpha + i\frac{\rho_4\rho_2}{\rho_3 + \rho_4} \partial_{1\alpha} \partial_\omega^\alpha + i\frac{\rho_4(1 - \rho_2)}{\rho_3 + \rho_4} \partial_{\omega\alpha} \partial_2^\alpha, \end{aligned} \quad (\text{C.33})$$

$$\begin{aligned} Z_2 = & i\mathcal{T}z_\alpha y^\alpha + \mathcal{T}z^\alpha \left(-(\rho_1 + \rho_2)\partial_{1\alpha} + (\rho_3 - \rho_1)\partial_{\omega\alpha} + (\rho_3 + \rho_4)\partial_{2\alpha} \right) - \frac{\rho_3\rho_1}{(\rho_1 + \rho_2)(\rho_3 + \rho_4)} y^\alpha \partial_{\omega\alpha} \\ & + i\frac{\rho_3\rho_2}{(\rho_1 + \rho_2)(\rho_3 + \rho_4)} \partial_{\omega\alpha} \partial_2^\alpha + i\frac{\rho_4\rho_2}{\rho_3 + \rho_4} \partial_{\omega\alpha} \partial_2^\alpha + i\frac{\rho_4(1 - \rho_2)}{\rho_3 + \rho_4} \partial_{1\alpha} \partial_\omega^\alpha. \end{aligned} \quad (\text{C.34})$$

Shuffling the ρ -variables in Z_2

$$\rho_1 \longrightarrow \rho_3, \quad \rho_3 \longrightarrow \rho_1, \quad \rho_2 \longrightarrow \rho_4, \quad \rho_4 \longrightarrow \rho_2, \quad (\text{C.35})$$

taking into account that

$$\begin{aligned} \delta(1 - \rho_1 - \rho_2 - \rho_3 - \rho_4) \left(\frac{\rho_1\rho_4}{(\rho_1 + \rho_2)(\rho_3 + \rho_4)} + \frac{\rho_4\rho_2}{\rho_1 + \rho_2} \right) &= \delta(1 - \rho_1 - \rho_2 - \rho_3 - \rho_4) \frac{\rho_4(1 - \rho_2)}{\rho_3 + \rho_4}, \\ \delta(1 - \rho_1 - \rho_2 - \rho_3 - \rho_4) \left(\frac{\rho_3\rho_2}{(\rho_1 + \rho_2)(\rho_3 + \rho_4)} + \frac{\rho_4\rho_2}{\rho_3 + \rho_4} \right) &= \delta(1 - \rho_1 - \rho_2 - \rho_3 - \rho_4) \frac{\rho_2(1 - \rho_4)}{\rho_1 + \rho_2}, \end{aligned}$$

one can see that

$$\begin{aligned} Z_1 = Z_2 = & i\mathcal{T}z_\alpha y^\alpha + \mathcal{T}z^\alpha \left(-(\rho_3 + \rho_4)\partial_{1\alpha} + (\rho_1 - \rho_3)\partial_{\omega\alpha} + (\rho_1 + \rho_2)\partial_{2\alpha} \right) \\ & - \frac{\rho_3\rho_1}{(\rho_1 + \rho_2)(\rho_3 + \rho_4)} y^\alpha \partial_{\omega\alpha} + i\frac{\rho_4(1 - \rho_2)}{\rho_3 + \rho_4} \partial_{\omega\alpha} \partial_2^\alpha + i\frac{\rho_2(1 - \rho_4)}{\rho_1 + \rho_2} \partial_{1\alpha} \partial_\omega^\alpha. \end{aligned} \quad (\text{C.36})$$

The part of $W_{2\omega C}^{\eta\eta}$ that contributes to $\hat{\Upsilon}^{\eta\eta}$ thus has the form (4.21).

C.3 $W_{2\omega CC}^{\eta\eta}$

Equation for the part of $W_{2\omega CC}^{\eta\eta}$ that contributes to $\hat{\Upsilon}^{\eta\eta}$ is

$$2i\text{d}_z \widehat{W}_{2\omega CC}^{\eta\eta} \approx \text{d}_x S_1^\eta|_{\omega CC} + W_{1\omega C}^\eta * S_1^\eta. \quad (\text{C.37})$$

Since computation is analogous to that for $\widehat{W}_{2CC\omega}^{\eta\eta}$ we present only the final result

$$\begin{aligned} \widehat{W}_{2\omega CC}^{\eta\eta} = & \frac{\eta^2}{4} \int_0^1 d\mathcal{T} \mathcal{T} \int_0^1 d\xi \int d^3\rho_+ \delta \left(1 - \sum_{i=1}^3 \rho_i \right) \frac{\rho_1(z^\alpha \partial_{\omega\alpha})^2}{\rho_1 + \rho_2} \\ & \times \exp \left\{ i\mathcal{T}z_\alpha y^\alpha + \mathcal{T}z^\alpha \left((\rho_1 + \rho_3)\partial_{\omega\alpha} - \rho_3\partial_{1\alpha} + (\rho_1 + \rho_2)\partial_{2\alpha} \right) \right. \\ & \quad \left. + \xi \left(i(1 - \rho_1)\partial_{\omega\alpha} \partial_1^\alpha - i\rho_1\partial_{\omega\alpha} \partial_2^\alpha \right) \right. \\ & \quad \left. + (1 - \xi) \left(i\frac{\rho_2}{\rho_1 + \rho_2} \partial_{\omega\alpha} \partial_1^\alpha + \frac{\rho_1}{\rho_1 + \rho_2} y^\alpha \partial_{\omega\alpha} \right) \right\} \omega CC. \end{aligned} \quad (\text{C.38})$$

Equivalently, as an integral over a four-dimensional simplex it is given in (4.20).

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