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Lattice $\mathcal{N} = 4$ super Yang-Mills at strong coupling

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ABSTRACT: In this paper we present results from numerical simulations of $\mathcal{N} = 4$ super Yang-Mills for two color gauge theory over a wide range of 't Hooft coupling $0 < \lambda \leq 30$ using a supersymmetric lattice action [1]. Numerical study of this lattice theory has been stymied until recently by both sign problems and the occurrence of lattice artifact phases at strong coupling. We have recently developed a new action that appears capable of solving both problems. The resulting action possesses just SU(2) rather than U(2) gauge symmetry. By explicit computations of the fermion Pfaffian we present evidence that the theory possesses no sign problem and exists in a single phase out to arbitrarily strong coupling. Furthermore, preliminary work shows that the logarithm of the supersymmetric Wilson loop varies as the square root of the 't Hooft coupling λ for large λ in agreement with holographic predictions.

KEYWORDS: AdS-CFT Correspondence, Extended Supersymmetry, Lattice Quantum Field Theory

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1 Introduction

In this paper we use numerical simulation to explore the phase structure and Wilson loops of a lattice formulation of $\mathcal{N} = 4$ super Yang-Mills. The lattice action is a generalization of the formulation described in [1]. The theory preserves both $SU(N)$ gauge invariance, a S^4 point group symmetry associated with the underlying A_4^* lattice and most importantly a single exact supersymmetry.

The original supersymmetric lattice formulation of $\mathcal{N} = 4$ SYM has been the subject of a great deal of both numerical and analytical work [2–5]. General arguments have been put forward that the theory should approach the continuum $\mathcal{N} = 4$ theory after tuning a single marginal operator. However, after some initial successes the numerical work has been handicapped by two problems: the existence of a chirally broken phase for 't Hooft couplings $\lambda > 4$ and the observation of a sign problem which develops in a similar region of coupling [6]. While these problems are not present in dimensionally reduced versions of the theory [7–15], they have prevented the systematic investigation of the four dimensional theory. The chirally broken phase has been linked to the condensation of monopoles associated with the $U(1)$ sector of the theory [16].

In this paper we show that the situation is markedly improved if one adds a new operator to the lattice action which preserves the S^4 symmetry and exact supersymmetry but explicitly breaks the $U(N)$ gauge symmetry down to $SU(N)$.

2 Review of the old supersymmetric construction

We start from the supersymmetric lattice action appearing in [1].

$$S = \frac{N}{4\lambda} \mathcal{Q} \sum_x \text{Tr} \left(\chi_{ab} \mathcal{F}_{ab} + \eta \bar{\mathcal{D}}_a \mathcal{U}_a + \frac{1}{2} \eta d \right) + S_{\text{closed}} \quad (2.1)$$

where the lattice field strength

$$\mathcal{F}_{ab}(x) = \mathcal{U}_a(x)\mathcal{U}_b(x + \hat{a}) - \mathcal{U}_b(x)\mathcal{U}_a(x + \hat{b}) \quad (2.2)$$

where $\mathcal{U}_a(x)$ denotes the complexified gauge field living on the lattice link running from $x \rightarrow x + \hat{a}$ where $\hat{a}$ denotes one of the five basis vectors of the underlying A_4^* lattice. Similarly

$$\overline{\mathcal{D}}_a \mathcal{U}_a = \mathcal{U}_a(x)\overline{\mathcal{U}}_a(x) - \overline{\mathcal{U}}_a(x - \hat{a})\mathcal{U}_a(x - \hat{a}). \quad (2.3)$$

The five fermion fields ψ_a , being superpartners of the (complex) gauge fields, live on the corresponding links, while the ten fermion fields $\chi_{ab}(x)$ are associated with new face links running from $x + \hat{a} + \hat{b} \rightarrow x$. The scalar fermion $\eta(x)$ lives on the lattice site x and is associated with the conserved supercharge $\mathcal{Q}$ which acts on the fields in the following way

$$\begin{aligned} \mathcal{Q}\mathcal{U}_a &\rightarrow \psi_a \\ \mathcal{Q}\psi_a &\rightarrow 0 \\ \mathcal{Q}\eta &\rightarrow d \\ \mathcal{Q}d &\rightarrow 0 \\ \mathcal{Q}\chi_{ab} &\rightarrow \overline{\mathcal{F}}_{ab} \\ \mathcal{Q}\overline{\mathcal{U}}_a &\rightarrow 0 \end{aligned} \quad (2.4)$$

Notice that $\mathcal{Q}^2 = 0$ which guarantees the supersymmetric invariance of the first part of the lattice action. The auxiliary site field $d(x)$ is needed for nilpotency of $\mathcal{Q}$ offshell. The second term S_{closed} is given by

$$S_{\text{closed}} = -\frac{N}{16\lambda} \sum_x \text{Tr} \epsilon_{abcde} \chi_{ab} \overline{\mathcal{D}}_c \chi_{de} \quad (2.5)$$

where the covariant difference operator acting on the fermion field χ_{de} takes the form

$$\overline{\mathcal{D}}_c \chi_{de}(x) = \overline{\mathcal{U}}_c(x - \hat{c}) \chi_{de}(x + \hat{a} + \hat{b}) - \chi_{de}(x - \hat{d} - \hat{e}) \overline{\mathcal{U}}_c(x + \hat{a} + \hat{b}) \quad (2.6)$$

The latter term can be shown to be supersymmetric via an exact lattice Bianchi identity $\epsilon_{abcde} \overline{\mathcal{D}}_c \chi_{de} = 0$. Carrying out the $\mathcal{Q}$ variation and integrating out the auxiliary field d we obtain the supersymmetric lattice action $S = S_b + S_f$ where

$$S_b = \frac{N}{4\lambda} \sum_x \text{Tr} \left(\mathcal{F}_{ab} \overline{\mathcal{F}}_{ab} + \frac{1}{2} \text{Tr} (\overline{\mathcal{D}}_a \mathcal{U}_a)^2 \right) \quad (2.7)$$

and

$$S_f = -\frac{N}{4\lambda} \sum_x \left(\text{Tr} \chi_{ab} \mathcal{D}_{[a} \psi_{b]} + \text{Tr} \eta \overline{\mathcal{D}}_a \psi_a \right) \quad (2.8)$$

In the continuum this action can be obtained by discretization of the Marcus or GL twist of $\mathcal{N} = 4$ Yang-Mills but in flat space is completely equivalent to it. In the continuum the twist is done as a prelude to the construction of a topological quantum field theory but in the context of lattice supersymmetry it is merely used as a change of variables that allows

for discretization while preserving a single exact supersymmetry. The twisting removes the spinors from the theory replacing them by the antisymmetric tensor fields η, ψ_a, χ_{ab} which appears as components of a Kähler-Dirac field. The latter is equivalent at zero coupling to a (reduced) staggered field and hence describes four physical Majorana fermions in the continuum limit — as required for $\mathcal{N} = 4$ Yang-Mills. The twisting procedure also packs the six scalar fields of the continuum theory together with the four gauge fields into five complex gauge fields corresponding to the lattice fields $\mathcal{U}_a$.

As described above, the discrete theory is defined on a somewhat exotic lattice — A_4^* . This admits a larger set of rotational symmetries than a hypercubic lattice and this fact plays a role in controlling the renormalization of the theory. Finally, to retain exact supersymmetry all fields reside in the algebra of the gauge group — taking their values in the adjoint representation of $U(N)$: $f(x) = \sum_{A=1}^{N^2} T^A f^A(x)$ with $\text{Tr}(T^A T^B) = -\delta^{AB}$.

Ordinarily this would be incompatible with lattice gauge invariance because the measure would not be gauge invariant for link based fields. However, in this $\mathcal{N} = 4$ construction the problem is evaded since the fields are complexified which ensures that the Jacobians that arise after gauge transformation of $\mathcal{U}$ and $\bar{\mathcal{U}}$ cancel.¹

However this restriction to the algebra does pose a further problem. Ordinarily the naive continuum limit is obtained by expanding the group elements about the identity $U_a(x) = I + aA_a(x) + \dots$. The presence of the unit matrix in this expansion is what gives rise to hopping terms in the lattice theory and derivative operators in the continuum limit. If the gauge fields live in the group the unit matrix arises naturally on expanding the exponential but with fields valued in the algebra it is less clear how such an expansion arises. The saving grace is to notice that the gauge fields take their values in $GL(N, C)$ so that this term can arise by giving a vacuum expectation value to the *imaginary* part of the trace mode of the field. Typically this is accomplished by adding to the supersymmetric action a new term of the form

$$S_{\text{mass}} = \mu^2 \sum_x \text{Tr} \left(\bar{\mathcal{U}}_a(x) \mathcal{U}_a(x) - I \right)^2 \quad (2.9)$$

While this breaks the exact supersymmetry softly all counter terms induced by this breaking will have couplings that are multiplicative in μ^2 and hence vanishing as $\mu^2 \rightarrow 0$. Notice also that this term also generates masses for the scalar fields in the theory and hence also regulates the usual flat directions of SYM theory.

3 The new action

It has been observed that for couplings $\lambda > 2$ the action described in the previous section undergoes a phase transition to a regime in which both the Polyakov line and the Wilson loop fall abruptly toward zero. Associated with this is a growth in the density of lattice

¹Actually one should qualify this statement. While the complexified bosonic measure is invariant under lattice $U(N)$ gauge transformations it is more subtle to show that the fermion measure is invariant when the fermions reside on links. We shall show that this issue is completely evaded in the theory with $SU(N)$ gauge invariance

U(1) monopoles [5]. These features are inconsistent with the expected superconformal phase of $\mathcal{N} = 4$ Yang-Mills. Actually, in pure compact QED in four dimensions, this monopole transition is a well known lattice artifact. Various efforts have been made over the intervening years to remove this monopole phase — typically this has been done by adding supersymmetric or non-supersymmetric terms to the action that force the determinant of the plaquette operator to unity. Such a procedure retains the full $U(N)$ gauge symmetry but restricts the fluctuations of the field strength in the U(1) directions. The supersymmetric plaquette term introduced in [17] represents the best of these approaches but can only allow simulation up to $\lambda \sim 6.0$. It also suffers from a sign problem for $\lambda > 4$ [6] — that is, the Pfaffian arising after fermion integration, exhibits strong phase fluctuations which prohibit Monte Carlo sampling.

Here we explore an approach in which a new supersymmetric term is introduced which drives the determinant of each individual gauge link to unity. The new term takes the form

$$\frac{N}{4\lambda} \kappa \mathcal{Q} \sum_{x,a} \text{Tr}(\eta) (\text{Re det}(U_a(x) - 1)) \quad (3.1)$$

After $\mathcal{Q}$ variation and integration over d this modifies the second term in the bosonic action S_b to:

$$\frac{N}{4\lambda} \sum_{x,a} \frac{1}{2} \text{Tr} \left(\bar{\mathcal{D}}_a \mathcal{U}_a(x) + \kappa \text{Re det}(U_a(x)) I_n \right)^2 \quad (3.2)$$

where I_N denotes the $N \times N$ unit matrix. A corresponding new fermion term is generated

$$\delta S_f = -\frac{N}{8\lambda} \kappa \sum_{x,a} \text{Tr}(\eta) \text{det}(U_a(x)) \text{Tr}(U_a^{-1}(x) \psi_a(x)) \quad (3.3)$$

The new term has the effect of suppressing the U(1) phase fluctuations of the complex gauge links that were the origin of the monopole problem. Of course this term explicitly breaks the U(1) gauge symmetry. However since the U(1) is simply a decoupled free theory in the continuum limit this should cause no real harm since $SU(N)$ gauge invariance is preserved. Indeed, close to the continuum limit, it should be apparent that the new terms merely generate mass terms for the trace components of the fields.

In the original theory the gauge links were valued in $GL(N, C)$. After this term is added the moduli space of the theory is reduced to $SL(N, C)$. Notice that since any matrix in $SL(N, C)$ can be written as the exponential of a traceless matrix the presence of this term *guarantees* that gauge links can be expanded about the unit matrix for vanishing values of the lattice spacing. In this light the remaining rationale for keeping S_{mass} is simply to lift the usual $SU(N)$ flat directions. Indeed, as the reader will see, for most of our results μ^2 is taken very small.

The breaking of U(1) gauge invariance also clarifies a delicate issue concerning the invariance of the fermion measure in the original formulation. Consider the integration measure for the five link fermions $\prod_{x,a} d\psi_a(x)$ in the $U(N)$ theory. Under a gauge transformation $\psi_a(x) \rightarrow G(x) \psi_a(x) G^\dagger(x + \hat{a})$ this measure transforms by a non-trivial Jacobian corresponding to the product of the determinants of the gauge factors $G(x)$ and $G^\dagger(x + \hat{a})$. On the torus one can arrange an ordering of the fermion fields in the path integral measure

Figure 1. Expectation value of the determinant vs λ for 8^4 lattices at $\mu = 0.1, 0.05, 0.01$.

such that these factors will cancel out along closed loops but this will not be possible for all lattice topologies. Thus the question of the invariance of the measure under the full $U(N)$ group is a delicate one. However these problems are completely avoided if G is restricted to lie in just $SU(N)$ as in the new action and the fermion measure is then unambiguously defined for an arbitrary lattice.

Of course the main question is whether such a term is effective at eliminating the monopole phase seen at strong coupling. In the next section we shall show evidence that this is true and at least in the case of 2 colors we see no sign of phase transitions out to arbitrarily large 't Hooft coupling.

4 Phase structure

Our simulations utilize the rational hybrid Monte Carlo (HMC) algorithm where the Pfaffian resulting from the fermion integration is replaced by

$$\text{Pf}(M) = \left(\det M^\dagger M\right)^{\frac{1}{4}} \quad (4.1)$$

where M is the fermion operator. Notice that this representation neglects any Pfaffian phase which is a key issue which we will return to later. Typical ensembles used in our analysis consist of 5000 HMC trajectories with 1000 – 2000 discarded for thermalization. Errors are assessed using a jackknife procedure using 20 – 40 bins.

As a test of the new action we first plot the expectation value of the link determinant as a function of 't Hooft coupling. We show results in fig 1 for 8^4 lattices at $\mu = 0.1, 0.05, 0.01$. Clearly the expectation value is close to unity out to very large λ provided μ^2 is small enough confirming that we have effectively reduced the gauge fields to $SU(2)$. We note that we scan out to $\lambda = 30$ in order to go beyond the self-dual point $\lambda_{SD} = 4\pi N = 8\pi$.

In figure 2 we plot the expectation value of the bosonic action as a function of λ for 8^4 lattices at $\mu = 0.1, 0.05, 0.01$. This expectation value can be calculated exactly by exploit-

Figure 2. Expectation value of the bosonic action vs λ for 8^4 lattices at $\mu = 0.1, 0.05, 0.01$.

ing the (almost) $\mathcal{Q}$ -exact nature of the lattice action and yields $\frac{1}{V}\langle S_b \rangle = \frac{9N^2}{2}$ for an N color theory on a system with (lattice) volume V *independent* of coupling λ . For $SU(2)$ this implies $S_B = 18.0$ for all λ . The results are clearly consistent with this prediction to a fraction of a percent as $\mu^2 \rightarrow 0$ even for very large values of the coupling confirming the presence of an exact supersymmetry. Even more important there is no sign of the phase transition that had been seen before in the $U(2)$ theory. Indeed all the observables we have looked at show smooth dependence on λ providing evidence that the lattice theory possesses only a single phase out to arbitrarily strong coupling. It is interesting to note that the bosonic action is proportional to N^2 and not $N^2 - 1$ even though we suppress the $U(1)$ modes. That is because they are still present in this formulation; rather than being removed, they are being tamed. The new terms added to the action mostly affect the vacuum of these fields — which is why they still contribute to the counting of degrees of freedom.

Further confidence in this finding comes from studying a simple bilinear Ward identity given by $\langle \mathcal{Q} \text{Tr} (\eta \mathcal{U}_a \bar{\mathcal{U}}_a) \rangle = 0$. Figure 3 shows this quantity as a function of λ for several μ at $L = 8$. It falls slowly with λ and decreases more quickly with decreasing μ . To clarify its dependence on lattice size we plot the Ward identity for $\lambda = 10.0$ vs L for two values of μ in figure 4. This plot makes it clear that the Ward identity decreases with increasing L . Indeed, comparing $L = 6$ with $L = 12$ at $\mu = 0.005$ the change is consistent with a $1/L^2$ dependence on lattice size.

5 Absence of a sign problem

Of course these results are derived from simulations of a model in which the phase of the Pfaffian that results from fermion integration is neglected. To check for the presence of such a phase we have computed it using the ensemble of configurations generated in our phase quenched Monte Carlo. Writing the Pfaffian phase as $e^{i\alpha(\lambda, U)}$ we plot the quantity $1 - \cos \alpha$ as a function of μ at $\lambda = 10.0$ and $\kappa = 1.0$ in figure 5. The different data points correspond

Figure 3. Bilinear Q -susy ward identity vs λ for 8^4 lattices for $\mu = 0.1, 0.05, 0.01$.

Figure 4. Bilinear Q -susy ward identity vs L at $\lambda = 10.0$ for $\mu = 0.025, 0.005$.

to lattices of size 2^4 , $3^2 \times 4^2$, $3^3 \times 4$ and 3×4^3 respectively. When measuring the phase of the Pfaffian we set $\kappa = 0$ in the fermion operator. Clearly the phase angle is driven towards very small values for small enough μ . We have observed this for all values of λ — the analogous plot figure 11 for $\lambda = 30$ is shown in the appendix. Of course the lattices used in these tests are quite small and one should worry whether the sign problem returns on larger volumes. Our results suggest that this is not the case — the average phase appears to saturate as the volume increases. Systems with sign problems typically exhibit phase fluctuations that increase exponentially with volume. This lattice model seems very different in this regard.

Retaining the new $U(1)$ breaking fermion term in the evolution but neglecting it when measuring the phase is clearly a questionable procedure. However, the modification that is neglected relates to the trace modes, which decouple from the $SU(N)$ theory in the contin-

Figure 5. Pfaffian phase vs μ at $\lambda = 10.0$.

Figure 6. Pfaffian phase vs μ at $\lambda = 10.0$. Ensembles were generated without including the new fermionic term.

uum in any case. So, in some sense we are discarding an irrelevant piece. Nevertheless, we have also generated ensembles in which the new fermion term is dropped from the fermion action in both the evolution and the measurement of the phase. A typical plot of the resultant phase versus for μ at $\lambda = 10.0$ is shown in figure 6 for several lattice volumes. The observed behavior is very similar to that seen in figure 5 and lends confidence to the assertion that the system does not suffer from a sign problem. Since this procedure breaks $\mathcal{Q}$ -symmetry softly (proportional to κ) it leads to larger deviations in the Ward identities and so we have reinstated the new fermion term in our later simulations used for studying Wilson loops. The fact that eliminating the new fermion term from both the Pfaffian measurement and the simulation still preserves the good behavior can be understood as

the new bosonic term accomplishing the most important task: stabilizing and suppressing the $U(1)$ modes of the link fields in a $\mathcal{Q}$ -symmetric way that is only softly broken.

It is interesting to try and understand theoretically why the observed phase fluctuations are so small. We start by writing the expectation value of the phase measured in the phase quenched ensemble as

$$\langle e^{i\alpha(\lambda, \kappa, U)} \rangle_{\text{phase quenched}} = \int D\mathcal{U} D\bar{\mathcal{U}} e^{i\alpha(\kappa, \lambda, \mathcal{U})} |\text{Pf}(\mathcal{U})| e^{-S_B(\lambda, \kappa, \mathcal{U})} = 1 \quad (5.1)$$

where we have chosen the normalization of the measure so that the full partition with susy preserving periodic boundary conditions (the Witten index) is unity. Furthermore, $\mathcal{Q}$ -invariance ensures that this expectation value of the phase factor is independent of κ and can be computed for $\kappa \rightarrow \infty$ where the partition function is saturated by configurations with unit determinant — the $SU(2)$ theory. Finally, the topological character of this partition function can be exploited to localize the integral to configurations which are constant over the lattice — the integral reducing to a Yang-Mills matrix model integral. The resultant Pfaffian for the $SU(2)$ matrix model is known to be real, positive definite [18]. Of course our simulations are performed at finite κ , and use a thermal boundary condition, but the numerical results strongly suggest that as a practical matter the phase fluctuations are small for the relevant range of parameters.

The encouraging results for the phase of the Pfaffian may also be related to the fact that out to very large λ the center symmetry is unbroken, so that Eguchi-Kawai reduction [19] may be valid. In that case the theory is equivalent to a single-site lattice, where the gauge theory is in fact just the matrix model that has been indicated in the previous paragraph. This may also explain why we are able to obtain results consistent with large N predictions (below), since the fact that we are in volumes larger than a single site may in fact translate into larger N in the reduced model.

6 Supersymmetric Wilson loops

The previous results provide strong evidence that the lattice theory exists in a single phase with unbroken supersymmetry out to very large values of the gauge coupling and that the model can be simulated with a Monte Carlo algorithm without encountering a sign problem. With this in hand we turn to whether the lattice simulations can provide confirmation of known results for $\mathcal{N} = 4$ Yang-Mills at strong coupling. Most of these analytic results were obtained by exploiting the AdS/CFT correspondence which allows strong coupling results in the gauge theory to be obtained by solving a classical gravity problem in anti-de Sitter space. Using this duality a variety of results for *supersymmetric* Wilson loops have been obtained over the last twenty years. Such Wilson loops generalize the usual Wilson loops by including contributions from the scalars and are realized in the twisted construction by forming path ordered products of the *complexified* lattice gauge fields $\mathcal{U}_a$. In the continuum the generic feature of such Wilson loops is that for strong coupling they depend not on λ as one would expect from perturbation theory but instead vary like $\sqrt{\lambda}$. In figure 7 we show the logarithm of the $n \times n$ supersymmetric Wilson loops $W(n, n)$ for a 12^4 lattice at $\kappa = 1.0$ plotted as a function of $\sqrt{\lambda}$. The straight lines correspond to fits

Figure 7. Supersymmetric $n \times n$ Wilson loops on 12^4 lattice at $\mu = 0.025$.

Figure 8. Estimated string tension vs λ for 12^4 lattice at $\mu = 0.025$ using $\chi(6, 6)$.

with $\sqrt{\lambda} \geq 3$. It is clear that all the loops show a $\sqrt{\lambda}$ dependence at strong coupling in agreement with the holographic prediction. This is encouraging. It is also clear that the fits show a linear dependence on the length of the perimeter of the loop. If we parametrize the static potential defined by $W(R, T) = e^{-V(R)T}$ in the form

$$V(R) = \sigma(\lambda)R + \alpha(\lambda)/R + M(\lambda) \quad (6.1)$$

The presence of the constant term $M(\lambda)$ will yield the observed perimeter scaling provided the string tension is small or zero. Such a perimeter term also occurs in continuum treatments where it corresponds to the energy of a static probe source in the fundamental representation and has to be explicitly subtracted out to see the non-abelian Coulomb behavior hidden in $\alpha(\lambda)$ [20].

Figure 9. Renormalized supersymmetric 4×4 and 2×2 Wilson loops on 8^4 lattice at $\mu = 0.025$.

One way to remove the perimeter dependence is to consider Creutz ratios defined by

$$\chi(R, T) = \frac{W(R, T)W(R-1, T-1)}{W(R, T-1)W(R-1, T)} \quad (6.2)$$

For a theory with Wilson loops containing both perimeter, area and Coulomb behaviors one finds

$$\ln \chi(R, R) \sim -\sigma(\lambda) + \alpha(\lambda)/R^2 \quad (6.3)$$

Thus we can read off the string tension by examining the large R behavior of $\ln \chi(R, R)$. In figure 8 we plot $\ln \chi(6, 6) = -\sigma$ versus λ for a 12^4 lattice at $\lambda = 10.0$ and $\mu = 0.025$.

Clearly, the string tension is very small even at strong coupling which is consistent with the existence of a single superconformal phase in the theory in the IR. Of course the most interesting question is whether we can see evidence for a non-abelian Coulomb potential at small R . Direct fits to the Creutz ratio are consistent with the presence of such a term but the errors in $\alpha(\lambda)$ are large.

An alternative way to probe for this is to divide the original Wilson loops by an appropriate power of the measured Polyakov line P which is given by product of gauge links along a thermal cycle. The (logarithm of the) Polyakov line also picks up a term linear in the length of the lattice due to a massive source and hence can be used to subtract the linear divergence in the rectangular Wilson loop. We thus define a renormalized Wilson loop on a L^4 lattice of the form

$$W^R(R, R) = \frac{W(R, R)}{P^{\frac{2R}{L}}} \quad (6.4)$$

These are shown in figure 9 for a 8^4 lattice. Notice that the 2×2 and 4×4 loops now lie near to each other which is consistent with conformal invariance and the presence of a non-abelian Coulomb term while the strong coupling behavior still exhibits a dependence on $\sqrt{\lambda}$. This result can also be seen on the larger 12^4 lattice shown in figure 10. Notice that

Figure 10. Renormalized supersymmetric 6×6 and 3×3 Wilson loops on 12^4 lattice at $\mu = 0.025$.

Loop Size	$a\sqrt{\lambda}+b$	Reduced- χ^2
4×4	$0.6(1)\sqrt{\lambda} + 8.0(4)$	8.11
2×2	$0.59(4)\sqrt{\lambda} + 8.8(2)$	2.25

Table 1. Normalized Supersymmetric Wilson loop fits on 8^4 lattice at $\mu = 0.025$ for $f(\lambda) = a\sqrt{\lambda}+b$.

Loop Size	$a\sqrt{\lambda}+b$	Reduced- χ^2
6×6	$0.88(7)\sqrt{\lambda} + 12.4(3)$	6.58
3×3	$0.86(2)\sqrt{\lambda} + 12.94(9)$	0.90

Table 2. Normalized Supersymmetric Wilson loop fits on 12^4 lattice at $\mu = 0.025$ for $f(\lambda) = a\sqrt{\lambda} + b$.

the average slope in this case is somewhat larger than the data on 8^4 . This presumably reflects the residual breaking of conformal invariance due to finite volume as well as finite lattice spacing. However it may also indicate that our definition of a renormalized Wilson loop does not do a perfect job of subtracting all the linear divergences needed to reveal an underlying Coulombic term. Further work is needed on larger lattices to clarify this issue.

Details of the fits for the different Wilson loops and lattice are shown in tables 1, 2.

The square root behavior at large λ is consistent with the result for circular Wilson loops in $\mathcal{N} = 4$ SYM derived by Gross and Drukker [21] and Maldacena’s holographic argument [22]. There are also explicit calculations using holography for the rectangular Wilson loop in [20]. The strange $\sqrt{\lambda}$ dependence *cannot* be seen in perturbation theory and this (admittedly) very preliminary result is a very non-trivial test of the correctness of the lattice approach in a non-perturbative regime.

Figure 11. Pfaffian phase vs μ at $\lambda = 30.0$.

7 Conclusions

We have found that a supersymmetric modification of the lattice action enables us to extend our simulations to what seem to be arbitrarily large values of the 't Hooft coupling without encountering difficulties that had previously limited our studies to modest λ . This seems to be attributable to stabilizing the potential for the U(1) modes in a way that preserves the essential $\mathcal{Q}$ supersymmetry of the construction. The current study has been limited to gauge group SU(2). It is natural to inquire what occurs for this construction for other SU(N). We will investigate this in future studies; however, we expect that a sign problem will reemerge since in the zero-dimensional matrix models for $N > 2$ the Pfaffian is no longer strictly positive.

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