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Efficient calculation of crossing symmetric BCJ tree numerators

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ABSTRACT: In this paper, we develop an improved method for directly calculating double-copy-compatible tree numerators in (super-)Yang-Mills and Yang-Mills-scalar theories. Our new scheme gets rid of any explicit dependence on *reference orderings*, restoring a form of crossing symmetry to the numerators. This in turn improves the computational efficiency of the algorithm, allowing us to go well beyond the number of external particles accessible with the reference order based methods. Motivated by a parallel study of one-loop BCJ numerators from forward limits, we explore the generalization to include a pair of fermions. To improve the accessibility of the new algorithm, we provide a MATHEMATICA package that implements the numerator construction. The structure of the computation also provides for a straightforward introduction of minimally-coupled massive particles potentially useful for future computations in both classical and quantum gravity.

KEYWORDS: Scattering Amplitudes, Gauge Symmetry, Superstrings and Heterotic Strings

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Contents

1	Introduction	1
2	CHY integrands and baseline expansion	5
2.1	Super-Yang-Mills (sYM) theories	5
2.2	Yang-Mills-scalar	7
3	DDM basis numerators from CHY integrands	11
3.1	Spanning trees and reference orderings	11
3.2	Recursive constructions without reference orderings for pure gluon and two-fermion numerators	13
3.3	Four-point YM example	18
3.4	Recursive constructions for multi-trace	19
3.5	Multi-trace example	21
4	Introducing masses	23
5	Conclusion and outlook	24
A	CHY integrands and scattering amplitudes	25
B	MATHEMATICA package	28
B.1	Constituent objects	28
B.2	Construction functions	29

1 Introduction

The efficient calculation of scattering amplitudes at high precision relies on the discovery of novel methods and structures, many of which are shared between disparate theories. While these structures are often highly obscured by traditional Lagrangian/Feynman diagram approaches, they tend to lead to new formulations of amplitudes in quantum field theories.

One powerful structure in tree amplitudes is the Bern-Carrasco-Johansson (BCJ) color-kinematics duality [1–3], which allows a large class of theories to be constructed as the double copy of simpler theories. This duality has been used to push the envelop on accessible calculations in various theories of quantum gravity [4–7] by “squaring” (super-)Yang-Mills (sYM) theory. Additionally, BCJ-compatible trees have been used in the blossoming application of scattering amplitudes techniques to black hole physics in classical gravity [8–23]. Unfortunately, most textbook methods for calculating tree amplitudes do not directly generate color-kinematics dual representations.

The Cachazo-He-Yuan (CHY) formalism [24–26] is a powerful method for studying the properties of tree-level scattering amplitudes in many theories, in part because it manifests double-copy properties. The formalism explicitly constructs tree amplitudes from the product of two half-integrands. There is now a well-developed program for exploring the types of objects needed in each half-integrand to generate theories of interest [27–34], which can be derived from world-sheet models based on ambitwistor strings [35–39] and their deformation [40, 41]. The half-integrands are known to be expressible completely in Del Duca-Dixon-Maltoni half-ladder basis [42] on the support of scattering equations [26],¹

$$\mathcal{I}_n^{\text{SE}} = \sum_{\beta \in S_{n-2}} \text{PT}(1, \beta, n) N(1, \beta, n), \text{ where } \text{PT}(\rho) \equiv \frac{1}{\sigma_{\rho_1 \rho_2} \sigma_{\rho_2 \rho_3} \cdots \sigma_{\rho_{n-1} \rho_n} \sigma_{\rho_n \rho_1}}. \quad (1.1)$$

In this basis, the kinematic coefficients $N(1, \beta, n)$ of the worldsheet *Parke-Taylor factors* $\text{PT}(1, \beta, n)$ can be seen as master numerators, since all non-ladder kinematic numerators can be built using BCJ numerator relations on the half-ladders [49]. Due to this fact, the half-ladder numerators fully specify tree-level amplitudes. This can be explicitly seen when considering sYM through the use of the *doubly-color-ordered biadjoint scalar amplitudes* $m(\alpha|\beta)$ [26, 50–52], which combine the half-ladder numerators N and the DDM color structures c [42] with the appropriate signs and propagators to generate amplitudes via

$$\mathcal{A}_n^{\text{tree}} = \mathcal{N} \sum_{\alpha, \beta \in S_{n-2}} c(1, \alpha, n) m(1, \alpha, n | 1, \beta, n) N(1, \beta, n). \quad (1.2)$$

These numerators can be directly double-copied with appropriate partners $\tilde{N}$ to yield various gravitational theories,

$$\mathcal{M}_n^{\text{tree}} = \mathcal{N}_{\text{GR}} \sum_{\alpha, \beta \in S_{n-2}} \tilde{N}(1, \alpha, n) m(1, \alpha, n | 1, \beta, n) N(1, \beta, n). \quad (1.3)$$

As amplitudes for bosonic theories, both $\mathcal{A}$ and $\mathcal{M}$ are *invariant* under particle exchange. The permutation invariance of $2, \dots, n-1$ is manifest if

$$N(1, \beta, n) \equiv N(1, 2, \dots, n) \Big|_{2 \rightarrow \beta_2, 3 \rightarrow \beta_3, \dots, n-1 \rightarrow \beta_{n-1}}, \quad (1.4)$$

which we will take as our definition of *crossing symmetry* acting on numerators.² The resemblance of eq. (1.3) to the Kawai-Lewellen-Tye (KLT) double-copy relation [53] provides a direct construction of $N(1, \beta, n)$ using color-ordered amplitudes and KLT momentum kernel [54, 55]. However, the result is non-local as the numerator is a rational function of Mandelstam variables, rather than polynomial.

¹This expansion also holds for worldsheet correlators in generic string theories [43–48].

²The crossing symmetry in 1 and n are due in part to the BCJ relations. Since the N are for half-ladders, one might expect that exchanging either $1 \leftrightarrow \beta_2$ or $n \leftrightarrow \beta_{n-1}$ in $N(1, \beta, n)$ would yield a minus sign. However, the interplay between color and kinematic jacobis allows this property of the half-ladders to be broken or restored as needed. Of course, imposing these additional crossing symmetry will further eliminate the gauge freedom in these numerators. On the other hand, when 1 and n are a different species from the rest, crossing symmetry involving them is necessarily absent.

On the other hand, CHY formalism provides a way to compute *local* numerators in the DDM basis through eq. (1.1). Although the possibility was recognized shortly after the discovery of CHY [26], the explicit construction was highly nontrivial and came much later.³ The computation for up to three gluons and two traces in Yang-Mills-scalar (YMS) theory [57] was done in [58–61]. Shortly after, a recursive expansion for the single-trace sector of YMS was proposed based on gauge invariance [62] and a direct ansatz construction [63], which was soon proved and refined by expanding the CHY integrand recursively [64]. Follow-up works further extended the algorithm to pure Yang-Mills [65] and multi-trace sector [66]. A crucial feature common to all these constructions is that the DDM basis numerators are given by assigning kinematic factors to *spanning trees* on n vertices — connected graphs on n vertices with no cycles.

While the spanning tree construction is generic, the need for a *reference ordering* (RO) causes the numerators to be a function of more than just the particle ordering, and thus not manifest crossing symmetry,

$$N_{\text{RO}}(1, \beta, n) \neq N_{\text{RO}}(1, 2, \dots, n) \Big|_{2 \rightarrow \beta_2, 3 \rightarrow \beta_3, \dots, n-1 \rightarrow \beta_{n-1}}. \quad (1.5)$$

Although this is not in line with eq. (1.4), the full amplitude remains permutation invariant due to the nontrivial kernel of $m(1, \alpha, n | 1, \beta, n)$, i.e. the BCJ amplitude relations. Crossing symmetry can be recovered by *averaging over all the ROs* (RO-average). However, this prescription introduces a factorial computational complexity, which is highly undesired. We note that the mathematical origin of this RO dependence roots in the Laplace expansion of determinants and Pfaffians: when one performs the expansion along a chosen row, the manifest exchange symmetry between this row and the rest is broken. Different choices lead to different patterns of breaking the symmetry. Alternatively, the prescriptions presented in [62, 64–66] can be derived from conventional string theory [32, 34, 61] and differential operators [67]. The stringy interpretation of the RO is a priority list for breaking the subcycles formed by world-sheet variables using integration-by-parts relations.

In section 2 and part of section 3, we will review the RO-based construction of sYM and multi-trace YMS numerators. While the technique summarized in algorithm 1 is mostly equivalent to those in refs. [34, 62, 64–66], we streamline the algorithm with new notations that are more suitable to describe the novel crossing symmetric construction discussed later. In addition, we propose a new baseline expansion (2.16) for YMS that works for any choice of particle 1 and n (gluon, scalars in the same or different traces).⁴

While the lack of crossing symmetry does not pose a problem when studying tree-level amplitudes, it can be challenging to deal with when using trees as part of loop calculations. The inability to relabel numerators prevents systematic approaches using graph isomorphisms that might generalize to loops. However, the CHY formalism has been extended to handle one-loop directly. This representation of loop amplitudes can be derived from ambitwistor string models based on a nodal Riemann sphere [37, 68–70], which in turn allows direct calculation from an $(n+2)$ -point tree-level amplitude through a forward limit [71, 72].

³See ref. [56] for a generic construction of *non-local* BCJ numerators using CHY integration rules.

⁴It is a natural generalization of ref. [66], where 1 and n are fixed to be scalars.

Following this approach, one can avoid the difficulty of dealing with contributions from different spin structures at one-loop level. The resultant loop integrands have the prominent feature that they contain linearized propagators rather than the standard Feynman-type ones. Moreover, kinematic numerators satisfying BCJ relations will automatically lead to such numerators at one-loop, which can be directly fed into a double-copy construction for gravity theories [73, 74]. These method of obtaining BCJ-compatible one-loop numerators have been explored by various works in literature [73–76].

Thus, high multiplicity loops in many theories are accessible from high multiplicity trees via CHY and forward limits. The first technical difficulty to overcome towards this goal is the need to efficiently produce tree-level BCJ numerators with very high multiplicity. As the main goal of this paper, we develop an improved algorithm to arrive at crossing symmetric numerators *without* performing the explicit after-the-fact RO-average. In other words, the new method gets rid of RO-dependence completely. The procedure, presented in algorithm 2, works by processing the structure of a spanning tree *recursively* rather than sequentially over arbitrarily split paths. The recursive structure allows kinematic weights to be assigned at each node which locally incorporate the appropriate tensor structures and normalizations such that a spanning tree processed in the new algorithm gives the same result as performing the RO-average after the old construction [34, 62, 64–66]. The bulk of this paper is dedicated to examining the combinatorics and kinematic factors needed at each stage of the recursion. In section 3.2, we derive the prescription for sYM amplitudes containing at most two fermions.⁵ We then generalize it to multi-trace sector of YMS theory in section 3.4. Our algorithm will automatically respect the crossing symmetry among gluons and scalar traces, all of which are not manifest when a RO is required. We have checked that our algorithm can provide complete ten-point results within minutes on a laptop. In a forthcoming paper [78], we will feed these tree-level numerators into the forward-limit machinery to study one-loop numerators with various amount of supersymmetry.

Additionally, since our formulation is dimensionally agnostic, it is possible to embed the numerators in higher dimensions in such a way as to introduce mass to specific particles via dimensional compactification. In particular, it is straightforward to perform this embedding such that only particles 1 and n obtain a mass. These types of diagrams are related to those needed for the recent 3PM (third post-Minkowski) calculation of ref. [21] as input to the unitarity cuts used for constructing the two-loop classical amplitude.

Our paper is organized as follows. In section 2, we review the general structure of the CHY representation of integrands. This section specifically focuses on the initial expansion of Pf' into pure Pf weighted by kinematics and Parke-Taylor factors. We cover pure gluon, two-fermion, and multi-trace YMS. Section 3 begins by reviewing the spanning tree method for CHY (half-)integrand construction. It then presents our derivation of a recursive restructuring of the spanning tree calculation which implicitly performs an average over reference ordering, as well as providing some examples of the new technique. Finally, in section 4, we briefly discuss introducing masses to our numerators. Appendix A provides

⁵Numerators with more fermions can be obtained from the pure-spinor formalism of sYM [43, 77].

our explicit conventions for the CHY matrices and normalizations. Appendix B describes our MATHEMATICAPackage that implements the spanning tree numerator construction.

2 CHY integrands and baseline expansion

In the CHY representation, n -point tree-level color-ordered amplitudes are expressed as an integral over the moduli space of n -punctured Riemann sphere,

$$A_n^{\text{tree}}(\rho) = \mathcal{N} \int d\mu_n^{\text{tree}} \text{PT}(\rho) \mathcal{I}_n^{\text{tree}}, \quad d\mu_n^{\text{tree}} \equiv \frac{d^n \sigma}{\text{Vol}[\text{SL}(2, \mathbb{C})]} \prod_{i=1}^n{}' \delta(E_i), \quad (2.1)$$

where the delta functions in the measure localize the integral to the $(n-3)!$ solutions of the scattering equations [24–26],

$$E_i \equiv \sum_{\substack{j=1 \\ j \neq i}}^n \frac{k_i \cdot k_j}{\sigma_{ij}} = 0, \quad \sigma_{ij} \equiv \sigma_i - \sigma_j. \quad (2.2)$$

The color ordering is encoded in the Parke-Taylor factor $\text{PT}(\rho)$, which is a weight-two function under the $\text{SL}(2, \mathbb{C})$ transformation $\sigma_i \rightarrow \frac{a\sigma_i + b}{c\sigma_i + d}$ with $ad - bc = 1$. The (half-)integrand $\mathcal{I}_n^{\text{tree}}$ depends on the kinematic data (momentum and polarization). It is constructed as an $\text{SL}(2, \mathbb{C})$ weight-two function as well such that the full integration form is $\text{SL}(2, \mathbb{C})$ invariant. The $\text{SL}(2, \mathbb{C})$ gauge redundancy, combined with the $\frac{1}{\text{Vol}[\text{SL}(2, \mathbb{C})]}$, can be removed by fixing the positions of three punctures and dropping three redundant scattering equations. For example, we can choose

$$d\mu_n^{\text{tree}} = (\sigma_{1,n-1} \sigma_{n-1,n} \sigma_{n1})^2 \prod_{i=2}^{n-2} \left[d\sigma_i \delta(E_i) \right], \quad (2.3)$$

where $(\sigma_1, \sigma_{n-1}, \sigma_n) = (0, 1, \infty)$. The normalization $\mathcal{N}$ contains the coupling constants of the theory, which we match against the explicit Lagrangian in section A.

2.1 Super-Yang-Mills (sYM) theories

We first focus on sYM theories. The CHY integrand for pure gluon amplitudes is [25]

$$\mathcal{I}_n^{\text{gluon}} = \text{Pf}'(\Psi). \quad (2.4)$$

The full definition of the *reduced Pfaffian* $\text{Pf}'(\Psi)$ is given in section A. One can rewrite $\text{Pf}'(\Psi)$ as a *baseline expansion* on the support of the scattering equations [62, 65, 79],

$$\mathcal{I}_n^{\text{gluon}} \stackrel{\text{SE}}{=} \sum_{(G,B) \in \mathbf{Part}_2(2,3,\dots,n-1)} \text{Pf}(\Psi_G) \sum_{\rho \in S_{|B|}} \text{PT}(1, \rho, n) W_{\text{gluon}}(1, \rho, n), \quad (2.5)$$

where the first summation is over all the bi-partitions of the set $\{2, 3, \dots, n-1\}$, and the second summation is over all the permutations of the set B .⁶ The matrix Ψ_G is a submatrix

⁶This sum is often written in the literature as over $\{2, \dots, n-1\} = A \cup B$, but we wish to avoid abuse of notation since A and B are required to be disjoint.

of Ψ in which the rows and columns are restricted to the set G (see section A for more details). We note that both G and B can be empty. In particular, when $G = \emptyset$, $\text{Pf}(\emptyset) = 1$. We call $(1, \rho, n)$ a *baseline* and the *baseline factor* W_{gluon} is given by

$$W_{\text{gluon}}(1, \rho, n) = (-1)^{|\rho|} \epsilon_1 \cdot f_{\rho_1} \cdot f_{\rho_2} \cdots f_{\rho_{|B|}} \cdot \epsilon_n, \quad (2.6)$$

where $f_i^{\mu\nu} \equiv k_i^\mu \epsilon_i^\nu - \epsilon_i^\mu k_i^\nu$ is the linearized field strength for gluon i . Our nomenclature is inspired by the *spanning tree* expansion scheme for Pf' , which will be discussed in section 3. For the derivation of eq. (2.5), we refer the interested readers to the aforementioned references. From the ambitwistor string point of view, singling out gluon 1 and n amounts to assigning ghost picture -1 to the vertex operators in the RNS formalism [35]. The gauge invariance of these two particles are not manifest, but can be recovered by using the scattering equations.

Interestingly, the CHY integrand with exactly two external fermions takes a similar baseline expansion form. If we fix the fermions to be particle 1 and n , the only difference from eq. (2.5) is in the baseline factor [78],

$$\mathcal{I}_n^{\text{2f}}(1_f, 2, \dots, n-1, n_f) = \sum_{(G,B) \in \mathbf{Part}_2(2,3,\dots,n-1)} \text{Pf}(\Psi_G) \sum_{\rho \in S_{|B|}} \text{PT}(1, \rho, n) W_{\text{2f}}(1_f, \rho, n_f). \quad (2.7)$$

There exist two forms of the baseline factor W_{2f} , equivalent on the support of the scattering equations and gauge transformations. In $D = 10$, they are given as⁷

$$W_{\text{2f}}^{(1)}(1_f, \rho, n_f) = \begin{cases} \frac{(-1)^{|\rho|}}{2} (\chi_1 f_{\rho_1} \cdots f_{\rho_{i-1}} \not{\epsilon}_\ell f_{\rho_{i+1}} \cdots f_{\rho_{|B|}} \chi_n) & \rho_i = \ell, \\ 0 & \ell \notin \rho \end{cases}, \quad (2.8a)$$

$$W_{\text{2f}}^{(2)}(1_f, \rho, n_f) = (-1)^{|\rho|} (\chi_1 f_{\rho_1} f_{\rho_2} \cdots f_{\rho_{|B|}} \xi_n), \quad (2.8b)$$

where in the first equation, ℓ is an arbitrary gluon and the baseline factor vanishes if $\ell \notin B$. Eq. (2.8b) has the benefit of being manifestly crossing symmetric and gauge invariant in all of the gluons. In both expressions, the gamma-matrix convention is $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$, and

$$\not{\epsilon}_\ell \equiv \epsilon_\ell^\mu \gamma_\mu, \quad f_i \equiv \frac{1}{8} f_i^{\mu\nu} [\gamma_\mu, \gamma_\nu] = \frac{1}{2} \not{k}_i \not{\epsilon}_i. \quad (2.9)$$

The Weyl-Majorana spinor χ_i is the solution to the equation of motion $\not{k}_i \chi_i = 0$, and the spinor ξ_i is related to χ_i through $\not{k}_i \xi_i = \chi_i$. The expressions for $D < 10$ can be obtained through a dimensional reduction. In particular, for $D = 4$ we can use

$$\chi_i \rightarrow \langle i |^{\alpha, I} \oplus [i]_{\dot{\alpha}, I}, \quad \xi_i \rightarrow \frac{|q\rangle_{\alpha, I}}{\langle iq\rangle} \oplus \frac{|q]^{\dot{\alpha}, I}}{[iq]} \quad (2.10)$$

⁷The two expressions originate from the two ways to assign ghost picture to the vertex operators in the RNS formalism of ambitwistor string: eq. (2.8a) comes from assigning ghost picture $-1/2$ to both fermion vertex operators and ghost picture -1 to the gluon vertex operator m , while eq. (2.8b) comes from assigning ghost picture $-1/2$ and $-3/2$ to the fermion vertex operators and ghost picture zero to all the gluons. We have found the two expressions useful for studying different aspects of forward limits, see ref. [78] for details.

where $\alpha, \dot{\alpha} = 1, 2$ are left- and right-handed Weyl spinor indices, $I = 1, 2, 3, 4$ is the $SU(4)$ R-symmetry index and the $\oplus$ is a formal sum over the representation spaces. In $D = 6$, we can similarly construct χ and ξ using 6D spinor helicity variables [80]

$$\chi_i \rightarrow |i_a\rangle^A \oplus |i_{\dot{a}}]_A, \quad \xi_i \rightarrow \frac{|q_{\dot{b}}]_A [q^{\dot{b}} i_a\rangle}{2q \cdot k_i} \oplus \frac{|q^b\rangle^A \langle q_b i_{\dot{a}}|}{2q \cdot k_i} \quad (2.11)$$

where the spinors $|i\rangle^A$ and $|i]_A$ are in the fundamental and anti-fundamental representation of $SU(4)$, and $a, \dot{a} = 1, 2$ are the $SU(2) \times SU(2)$ little group indices. In both eqs. (2.10) and (2.11), q is a reference vector that obeys $q \cdot k_i \neq 0$ but otherwise is arbitrary. The final amplitudes do not depend on the choice of q .

The baseline expansion brings the pure gluon and two-fermion CHY integrand into a unified form. In order to obtain the DDM basis numerators $N(1, \beta, n)$, we need a systematic way to expand $\text{Pf}(\Psi_G)$. In section 3, we will discuss how this can be done algorithmically using spanning trees.

2.2 Yang-Mills-scalar

Both eqs. (2.5) and (2.7) can be interpreted as expanding the pure-gluon/two-fermion integrand in terms of those for single-trace YMS amplitudes. If G is the set of gluons and $(1, \tau, n)$ is the (second) color ordering of the bi-adjoint scalars, we can write the half-integrand as [27]

$$\mathcal{I}_n^{\text{YMS}}((1, \tau, n)|G) = \text{PT}(1, \tau, n) \text{Pf}(\Psi_G). \quad (2.12)$$

which corresponds to choosing the baseline factor to be $W(1, \tau, n) = \delta_{\tau, \rho}$ in eq. (2.5).

In fact, there exists a similar baseline expansion for the multi-trace YMS integrand [28],

$$\mathcal{I}_n^{\text{YMS}}(\tau_1, \tau_2, \dots, \tau_m|G) = \text{PT}(\tau_1) \text{PT}(\tau_2) \dots \text{PT}(\tau_m) \text{Pf}'\Pi(\tau_1, \tau_2, \dots, \tau_m|G), \quad (2.13)$$

where G denotes the set of gluons and $\{\tau_1, \dots, \tau_m\}$ the scalar color traces. Since in many of our future manipulations, scalar traces are treated as a single object and have similar behavior as a gluon, we thus define for convenience the set

$$\{\mathbf{t}_1, \mathbf{t}_2, \dots, \mathbf{t}_N\} = \{\tau_1, \tau_2, \dots, \tau_m\} \cup G, \quad N = m + |G|, \quad (2.14)$$

where each entry $\mathbf{t}_i$ is either a scalar trace or a single gluon. The matrix Π is $2N$ dimensional, in which the entries are labeled by gluons and scalar traces. The explicit form of Π is given in section A. When $m = 0$ and 1, we have

$$\text{Pf}'\Pi(\tau_1|G) = \text{Pf}(\Psi_G), \quad \text{Pf}'\Pi(\emptyset|G) = \text{Pf}'(\Psi), \quad (2.15)$$

such that eq. (2.13) reduces to the single trace and pure gluon integrand respectively.

For the multitrace integrand eq. (2.13), the baseline expansion involves integrands with total number of gluons and trace reduced. It applies when the end points of the baseline,

chosen as 1 and n , do not belong to the same scalar trace. We can schematically write the expansion of $\mathcal{I}_n^{\text{YMS}}(\tau_1, \tau_2, \dots, \tau_m | G) = \mathcal{I}_n^{\text{YMS}}(\mathbf{t}_1, \mathbf{t}_2, \dots, \mathbf{t}_N)$ as

$$\begin{aligned} \mathcal{I}_n^{\text{YMS}}(\mathbf{t}_1, \mathbf{t}_2, \dots, \mathbf{t}_N) \stackrel{\text{SE}}{=} & \sum_{(A,B) \in \mathbf{Part}_2(\mathbf{t}_2, \dots, \mathbf{t}_{N-1})} \text{Pf}' \Pi(\{\mathbf{t}_1, B, \mathbf{t}_N\}, \overbrace{\alpha_1, \dots, \alpha_s}^A | G') \\ & \times \text{PT}(\alpha_1) \dots \text{PT}(\alpha_s) \sum_{\rho \in S_{\{\mathbf{t}_1, B, \mathbf{t}_N\} \setminus \{1, n\}}} \text{PT}(1, \rho, n) W_{\text{MT}}(1, \rho, n), \end{aligned} \quad (2.16)$$

where we assume that $1 \in \mathbf{t}_1$ and $n \in \mathbf{t}_N$. At this point, each term in the summation of eq. (2.16) is a multi-trace integrand with 1 and n in the same trace $(1, \rho, n)$, where ρ is a permutations of $\{\mathbf{t}_1, B, \mathbf{t}_N\}$ with $1 \in \mathbf{t}_1$ and $n \in \mathbf{t}_N$ fixed at the ends. Meanwhile, the set A is given by

$$A = \{\mathbf{t}_{a_1}, \dots, \mathbf{t}_{a_{|A|}}\} = \{\alpha_1, \dots, \alpha_s\} \cup G', \quad (2.17)$$

where α_i 's are scalar traces and G' a subset of gluons. In section 3.4, we will discuss the spanning tree algorithm to evaluate such integrands. Before moving into more details, we note that eq. (2.16) was first derived for pure scalar integrands in ref. [66]. However, the most generic form as given here is new. We have explicitly checked it for various gluon and trace configurations against direct numeric evaluations of eq. (2.1) up to very high multiplicities. It can be proved in a similar manner as appendix C and D of ref. [66], but we omit such a proof here for conciseness.

We now introduce some essential tools to specify the baseline factor $W_{\text{MT}}(1, \rho, n)$ in the second row of eq. (2.16). Treating each scalar trace in $\{\mathbf{t}_1, B, \mathbf{t}_N\}$ as a single object, we can define a sub-ordering (a_i, ω_i, b_i) for each $\mathbf{t}_i \in \{\mathbf{t}_1, B, \mathbf{t}_N\}$ by simply restricting the baseline $(1, \rho, n)$ to the elements of $\mathbf{t}_i$,

$$(a_i, \omega_i, b_i) \equiv (1, \rho, n) \Big|_{\mathbf{t}_i}, \quad (2.18)$$

where a_i and b_i are respectively the first and last element of $\mathbf{t}_i$ that appear in the baseline. If $\mathbf{t}_i$ is a scalar trace, then (a_i, ω_i, b_i) is a permutation of $\mathbf{t}_i$, while for $\mathbf{t}_i$ being a gluon, we simply define $a_i = b_i = \mathbf{t}_i$ and $\omega_i = \emptyset$. Given such a pair $(a_i, b_i) \subset \mathbf{t}_i$, we define the collection of *Kleiss-Kuijf-compatible* (*KK-compatible*) permutations as

$$\text{KK}[\mathbf{t}_i, a_i, b_i] = \begin{cases} \left\{ (a_i, \omega_i, b_i) \Big| \omega_i \in X_i \sqcup Y_i^T \right\} & \text{if } \mathbf{t}_i = (a_i, X_i, b_i, Y_i) \text{ is a trace} \\ (\mathbf{t}_i) & \text{if } \mathbf{t}_i \text{ is a gluon} \end{cases}, \quad (2.19)$$

where the set X_i and Y_i can be obtained by cyclically rotating a_i to the first element. They are exactly the orderings generated by a Kleiss-Kuijf relation [81], and hence the name. The sign generated by this operation

$$\text{sgn}_{a_i, \omega_i, b_i}^{\mathbf{t}_i} \equiv \begin{cases} (-1)^{|Y_i|} & \text{if } (a_i, \omega_i, b_i) \in \text{KK}[\mathbf{t}_i, a_i, b_i] \\ 0 & \text{otherwise} \end{cases}, \quad (2.20)$$

will be included in the baseline factor $W_{\text{MT}}(1, \rho, n)$. Note that for a gluon, we always have $\text{sgn}_{a_i, b_i}^{t_i} = \text{sgn}_{t_i, t_i}^{t_i} = 1$ since $|Y_i| = 0$ by definition. This function is nonzero only for KK-compatible permutations of t_i .

In addition, we define the ordering function $\mathcal{O}$ that rearranges the baseline $(1, \rho, n)$ by comparing the first element of the $|B| + 2$ sub-orderings (a_i, ω_i, b_i) : the triplet (a_i, ω_i, b_i) precedes (a_j, ω_j, b_j) if a_i precedes a_j in $(1, \rho, n)$, namely,

$$\mathcal{O}_{(1, \rho, n)} = (\overbrace{a_0, \omega_0, b_0}^{\lambda_0}, \overbrace{a_1, \omega_1, b_1}^{\lambda_1}, \dots, \overbrace{a_{|B|}, \omega_{|B|}, b_{|B|}}^{\lambda_{|B|}}, \overbrace{a_{|B|+1}, \omega_{|B|+1}, b_{|B|+1}}^{\lambda_{|B|+1}}). \quad (2.21)$$

We can also define the *coarse-grained* ordering function

$$\mathcal{O}_{(1, \rho, n)}^{\text{cg}} = (\lambda_0, \lambda_1, \dots, \lambda_{|B|}, \lambda_{|B|+1}) \quad (2.22)$$

that extracts a permutation formed by the $|B| + 2$ elements in $\{t_1, B, t_N\}$, which will be useful later in section 3.4. In fact, we must have $\lambda_0 = t_1$ and $a_0 = 1$ by construction. In particular, we will be interested in those baselines that are $\mathcal{O}$ -invariant,

$$(1, \rho, n) = \mathcal{O}_{(1, \rho, n)}. \quad (2.23)$$

For these cases, we must also have $\lambda_{|B|+1} = t_N$ and $b_{|B|+1} = n$. In other words, in $\mathcal{O}$ -invariant baselines every scalar trace is *consecutive* in the sense that between a_i and b_i there are no elements of other traces or gluons. To characterize these permutations, we introduces a new object

$$\mathcal{B}_{(1, \rho, n)}(\lambda_i | \lambda_j) = \{x \in \lambda_i | x \text{ before } a_j \text{ in } (1, \rho, n)\}, \quad (2.24)$$

that measures the “mixing level” between (a_i, ω_i, b_i) and (a_j, ω_j, b_j) ,

$$\mathcal{B}_{(1, \rho, n)}(\lambda_i | \lambda_j) = \begin{cases} \lambda_i & \text{all of } \lambda_i \text{ come before } a_j \text{ in } (1, \rho, n) \\ \tilde{\lambda}_i \subsetneq \lambda_i & \text{otherwise} \end{cases}, \quad (2.25)$$

where $\tilde{\lambda}_i$ is a *proper* subset of λ_i and it can be empty. Thus the $\mathcal{O}$ -invariant baselines have the form

$$(\overbrace{1, \theta_1, j_1}^{t_1}, \overbrace{a_1, \omega_1, b_1}^{\lambda_1}, \overbrace{a_2, \omega_2, b_2}^{\lambda_2}, \dots, \overbrace{a_{|B|}, \omega_{|B|}, b_{|B|}}^{\lambda_{|B|}}, \overbrace{i_N, \theta_N, n}^{t_N}), \quad (2.26)$$

and are characterized by $\mathcal{B}_{(1, \rho, n)}(\lambda_i | \lambda_{i+1}) = \lambda_i$ for $\lambda_i \in \{\lambda_0, \lambda_1, \dots, \lambda_{|B|}\}$. We must have t_N as the last consecutive block because $n \in t_N$ is the endpoint of the baseline. If there exists a λ_j after t_N , then we must have $\mathcal{B}_{(1, \rho, n)}(t_N | \lambda_j) \subsetneq t_N$ and thus the $\mathcal{O}$ -invariant condition is violated.

With the above preparations, we now reach the punchline of the multi-trace baseline expansion: the baseline factor $W_{\text{MT}}(1, \rho, n)$ is nonzero *if and only if*

- the baseline $(1, \rho, n)$ is $\mathcal{O}$ -invariant: $(1, \rho, n) = \mathcal{O}_{(1, \rho, n)}$;
- the restriction of $(1, \rho, n)$ to every $\lambda_i \in \{t_1, B, t_N\}$, denoted as (a_i, ω_i, b_i) , is KK-compatible.

For such permutations, $W_{\text{MT}}(1, \rho, n) = W_{\text{MT}}(\mathcal{O}_{(1, \rho, n)})$ is given by

$$\begin{aligned} W_{\text{MT}}(1, \rho, n) &= W_{\text{MT}}(\overbrace{1, \theta_1, j_1}^{\mathbf{t}_1}, \overbrace{a_1, \omega_1, b_1}^{\lambda_1}, \overbrace{a_2, \omega_2, b_2}^{\lambda_2}, \dots, \overbrace{a_{|B|}, \omega_{|B|}, b_{|B|}}^{\lambda_{|B|}}, \overbrace{i_N, \theta_N, n}^{\mathbf{t}_N}) \\ &= (-1)^{|B_g|} \mathcal{E}_{1, \theta_1, j_1}^{\mathbf{t}_1} \cdot \mathcal{T}_{a_1, \omega_1, b_1}^{\lambda_1} \cdot \mathcal{T}_{a_2, \omega_2, b_2}^{\lambda_2} \cdots \mathcal{T}_{a_{|B|}, \omega_{|B|}, b_{|B|}}^{\lambda_{|B|}} \cdot \tilde{\mathcal{E}}_{i_N, \theta_N, n}^{\mathbf{t}_N}. \end{aligned} \quad (2.27)$$

where $|B_g|$ is the number of gluons on the baseline, and

$$(\mathcal{T}_{a_i, \omega_i, b_i}^{\lambda_i})^{\mu\nu} = \begin{cases} \text{sgn}_{a_i, \omega_i, b_i}^{\lambda_i} (-k_{a_i}^\mu k_{b_i}^\nu) & \lambda_i \text{ is a trace and } \mathcal{B}_{(1, \rho, n)}(\lambda_i | \lambda_{i+1}) = \lambda_i \\ 0 & \lambda_i \text{ is a trace and } \mathcal{B}_{(1, \rho, n)}(\lambda_i | \lambda_{i+1}) \subsetneq \lambda_i \\ f_{\lambda_i}^{\mu\nu} & \lambda_i \text{ is a gluon} \end{cases}, \quad (2.28a)$$

$$(\mathcal{E}_{a_i, \omega_i, b_i}^{\lambda_i})^\mu = \begin{cases} \text{sgn}_{a_i, \omega_i, b_i}^{\lambda_i} (-k_{b_i}^\mu) & \lambda_i \text{ is a trace} \\ \epsilon_{\lambda_i}^\mu & \lambda_i \text{ is a gluon} \end{cases}, \quad (2.28b)$$

$$(\tilde{\mathcal{E}}_{a_i, \omega_i, b_i}^{\lambda_i})^\mu = \begin{cases} \text{sgn}_{a_i, \omega_i, b_i}^{\lambda_i} k_{a_i}^\mu & \lambda_i \text{ is a trace} \\ \epsilon_{\lambda_i}^\mu & \lambda_i \text{ is a gluon} \end{cases}. \quad (2.28c)$$

Thus, the baseline factor for a generic permutation can be written as

$$W_{\text{MT}}(1, \rho, n) = (-1)^{|B_g|} \mathcal{E}_{a_0, \omega_0, b_0}^{\lambda_0} \cdot \mathcal{T}_{a_1, \omega_1, b_1}^{\lambda_1} \cdot \mathcal{T}_{a_2, \omega_2, b_2}^{\lambda_2} \cdots \mathcal{T}_{a_{|B|}, \omega_{|B|}, b_{|B|}}^{\lambda_{|B|}} \cdot \tilde{\mathcal{E}}_{a_{|B|+1}, \omega_{|B|+1}, b_{|B|+1}}^{\lambda_{|B|+1}},$$

where $(\lambda_0, \lambda_1, \dots, \lambda_{|B|}, \lambda_{|B|+1}) = \mathcal{O}_{(1, \rho, n)}^{\text{cg}}$ is the coarse-grained baseline. The $\mathcal{B}_{(1, \rho, n)}$ and sgn in eq. (2.28) ensure any permutations that are not of the form of eq. (2.27) are immediately set to 0. With the help of this zeroing, the summation in the second line of eq. (2.16) automatically reduces to

$$\sum_{\rho \in S_{\{\mathbf{t}_1, B, \mathbf{t}_N\} \setminus \{1, n\}}}} \text{PT}(1, \rho, n) W_{\text{MT}}(1, \rho, n) = \sum_{\substack{\mathcal{O}\text{-invariant} \\ \text{KK-compatible}}} \text{PT}(1, \rho, n) W_{\text{MT}}(1, \rho, n). \quad (2.29)$$

Finally, we remark on the case that both the baseline endpoints 1 and n belong to the same trace, say τ_1 , which is left out by eq. (2.16). Starting from eq. (2.13), we simply write

$$\begin{aligned} \mathcal{I}_n^{\text{YMS}}(\tau_1, \tau_2, \dots, \tau_m | G) &= \text{PT}(\tau_2) \cdots \text{PT}(\tau_m) \text{Pf}' \Pi(\tau_1, \tau_2, \dots, \tau_m | G) \\ &\quad \times \sum_{\rho \in \text{KK}[\tau_1, 1, n]} \text{PT}(1, \rho, n) W_{\text{MT}}(1, \rho, n) \end{aligned} \quad (2.30)$$

by using Kleiss-Kuijff relations. Here, we can naturally reduce the definition of W_{MT} given in eq. (2.27) to the single-trace baseline,

$$W_{\text{MT}}(\overbrace{1, \rho, n}^{\tau_1}) = W_{\text{MT}}(\mathcal{O}_{(1, \rho, n)}) = \text{sgn}_{1, \rho, n}^{\tau_1}. \quad (2.31)$$

We note that since the sgn function is only nonzero for KK-compatible baselines, we can trivially extend the summation range in eq. (2.30) to $S_{\tau_1 \setminus \{1, n\}}$. Although this rewriting does not reduce the total number of scalar traces and gluons, it puts the CHY integrand into the same form as eq. (2.16), from which the half-ladder numerators can be obtained through a unified spanning-tree algorithm given in section 3.4. We note that the expansion (2.30) is first proved at the amplitude level in ref. [66].

3 DDM basis numerators from CHY integrands

The baseline expansion is the first step of our systematic approach to obtain the DDM basis numerators $N(1, \beta, n)$ from CHY integrands. It arranges the sYM and YMS integrands into an expansion of YMS integrands in which the total number of gluons and traces is reduced. Our next step is to further expand these YMS integrands in terms of the DDM basis Parke-Taylor factors. This is realized through a *spanning tree expansion*, originally developed in refs. [62, 64, 65, 82] using *reference orderings*. We review the original formulation before deriving a new approach to the construction.

3.1 Spanning trees and reference orderings

We first focus on eqs. (2.5) and (2.7), the baseline expansion for the pure gluon and two-fermion integrand, both of which are a linear combination of single trace integrands $\text{PT}(1, \rho, n) \text{Pf}(\Psi_G)$. Using a different type of recursive expansion, we can write them in terms of the single trace integrands involving fewer gluons [62, 64],

$$\text{PT}(1, \rho, n) \text{Pf}(\Psi_G) \stackrel{\text{SE}}{=} \sum_{(A,B) \in \mathbf{Part}_2(G \setminus l)} \text{Pf}(\Psi_A) \sum_{\alpha \in S_B} \sum_{i=0}^{|\rho|} \mathcal{P}(l, \alpha, \rho_i) \mathcal{C}_{(1, \rho, n)}(\rho_i, \alpha^T, l). \quad (3.1)$$

On the right hand side, we pick an arbitrary gluon $l \in G$ and sum over all the bi-partitions of the rest. As before, both the set A and B can be empty. We call $\mathcal{P}(l, \alpha, \rho_i)$ a *path factor* starting from l and terminating at ρ_i on the baseline (we define $\rho_0 = 1$), with the intermediate points a permutation α of the set B ,

$$\mathcal{P}(l, \alpha, \rho_i) = \epsilon_l \cdot f_{\alpha_1} \cdot f_{\alpha_2} \cdots f_{\alpha_{|B|}} \cdot k_{\rho_i}. \quad (3.2)$$

The nomenclature will be clear in the following section in which we represent this expansion in terms of spanning trees. Finally, $\mathcal{C}_{(1, \rho, n)}(\rho_i, \alpha^T, l)$ is the Cayley function of the path $(1, \rho)$ and (ρ_i, α^T, l) [82],

$$\mathcal{C}_{(1, \rho, n)}(\rho_i, \alpha^T, l) = \sum_{\lambda \in (\rho_{i+1}, \dots, \rho_{|\rho|}) \sqcup (\alpha^T, l)} \text{PT}(1, \rho_1, \dots, \rho_i, \lambda, n), \quad (3.3)$$

which is a linear combination of Parke-Taylor factors of length $n - |A|$. The final result of this recursive expansion is the Cayley functions of all the $(n - 1)$ -point spanning trees with a common path $(1, \rho)$, each of which is a linear combination of DDM basis Parke-Taylor factors [82]. It is important to note that in eq. (3.1) the gauge invariance and the crossing symmetry involving the leg l is not manifest. Since one needs to make such choices at each step of the recursion, the final result in general will not display any explicit gauge invariance or crossing symmetry.

With the help of eq. (3.1), one can further expand eqs. (2.5) and (2.7) by Parke-Taylor factors in the DDM basis,

$$\begin{aligned} \mathcal{I}_n^{\text{gluon/2f}} &\stackrel{\text{SE}}{=} \sum_{(G,B) \in \mathbf{Part}_2(2,3,\dots,n-1)} \text{Pf}(\Psi_G) \sum_{\rho \in S_{|B|}} \text{PT}(1, \rho, n) W_{\text{gluon/2f}}(1, \rho, n) \\ &\stackrel{\text{SE}}{=} \sum_{\beta \in S_{n-2}} \text{PT}(1, \beta, n) N_{\text{gluon/2f}}(1, \beta, n), \end{aligned} \quad (3.4)$$

Figure 1. Examples of trees with various IT memberships: (a) a tree that *only* belongs to $\text{IT}(1, 2, 3, 4, 5)$, (b) a tree that belongs to all $\text{IT}(1, (2) \sqcup (3, 4), 5)$, (c) a tree that belongs to $\text{IT}(1, \beta, 5)$ for all $\beta \in S_{\{2,3,4\}}$. The baselines are shown in red here and in all future occurrences.

where the coefficients $N(1, \beta, n)$ are the DDM basis numerators. They are associated with half-ladder diagrams, and are the master numerators for the BCJ tree relations. A given numerator $N(1, \beta, n)$ can be constructed by assigning kinematic factors to all the $(n-1)!$ increasing trees consistent with the label ordering [65],

$$N(1, \beta, n) = \sum_{T \in \text{IT}(1, \beta, n)} N(T), \quad (3.5)$$

where the increasing trees are defined as

$$\text{IT}(1, \beta, n) = \left\{ \text{tree } T \mid \begin{array}{l} \text{any edge } j \rightarrow i \in T \\ \text{satisfies that } i \text{ is before } j \text{ in } (1, \beta, n) \end{array} \right\}. \quad (3.6)$$

Thus by construction 1 is always the root of T while n is always a leaf. Note that a tree T can belong to several $\text{IT}(1, \rho, n)$. Some examples of trees and their IT membership can be seen in figure 1.

Then, for a given increasing tree T , the kinematic factor $N(T)$ is evaluated by the following algorithm [65].

Algorithm 1 (Reference Order Evaluation). Construction of the kinematic factor for a spanning tree T , using a given *reference ordering* $R \in S_{n-2}$:

1. Identify the baseline $(1, \rho, n)$, which is the path $n \rightarrow \rho_{|\rho|} \rightarrow \cdots \rightarrow \rho_1 \rightarrow 1$ in T . Note that $\rho = \emptyset$ is allowed.
2. Split the rest of T into *ordered splitting paths* $\text{OS}_R(T)$ based on a *reference ordering* $R \in S_{n-2}$: (a) draw a path from the first element of R towards the baseline, which will either end on the baseline or a previously identified ordered splitting path. (b) move to the next element in R that is not traversed yet, repeat the process until all vertices of T are traversed. For example,

$$\begin{array}{c} \text{Tree } T \\ \xrightarrow{R=(1,2,3,4,5)} \left\{ \begin{array}{c} \text{Baseline } 1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 5 \\ \text{Path } 1 \rightarrow 3 \\ \text{Path } 3 \rightarrow 4 \end{array} \right\}. \end{array} \quad (3.7)$$

3. Assign kinematic factors to each path

$$\begin{array}{ll} \text{baseline } n \rightarrow \rho_{|\rho|} \rightarrow \cdots \rightarrow \rho_1 \rightarrow 1 : & W(1, \rho, n) \\ l \rightarrow \alpha_1 \rightarrow \cdots \rightarrow \alpha_{|\alpha|} \rightarrow i \in \text{OS}_R(T) : & \mathcal{P}(l, \alpha, i). \end{array} \quad (3.8)$$

4. $N(T)$ is the product of all of these paths,

$$N(T) = W(1, \rho, n) \prod_{(l, \alpha, i) \in \text{OS}_R(T)} \mathcal{P}(l, \alpha, i). \quad (3.9)$$

Continuing the example, the ordered splitting in eq. (3.7) leads to

$$W_{\text{gluon}}(1, 2, 5) \mathcal{P}(3, 1) \mathcal{P}(4, 3) = W_{\text{gluon}}(1, 2, 5) (\epsilon_3 \cdot k_1) (\epsilon_4 \cdot k_3). \quad (3.10)$$

The reference ordering R is just a priority list of choosing the special gluon in the recursive expansion (3.1) when multiple choices are present. For simplicity we keep it the same for the evaluation of all the DDM basis numerators. We note that splitting a spanning tree into paths can be viewed as a graphic way to derive the ordered splitting based on a reference order; the algebraic method is first given in [62].

While this approach is a fully constructive method of building BCJ-compatible numerators, any crossing symmetry is broken in each of the DDM basis numerators, since the OS_R of different color ordering are not crossing symmetric. This leads to one of two computation difficulties when calculating the full integrand, either:

- each half-ladder numerator in the DDM basis must be computed individually, or
- a given half-ladder numerator must be *averaged over all the reference orderings* (RO-average) to make it crossing symmetric in $\{2, \dots, n-1\}$, which can then be freely relabeled to obtain the other half-ladder numerators.

Both approaches require an $\mathcal{O}((n-2)!)$ computation (assuming that relabeling takes negligible time comparing with computing a numerator) that must be performed after constructing a half-ladder numerator, which itself requires an $\mathcal{O}((n-1)!)$ computation, to arrive at the full DDM basis. Thus the computational complexity for the full DDM basis numerator is $\mathcal{O}[(n-1)! \times (n-2)!]$ for the RO method.

3.2 Recursive constructions without reference orderings for pure gluon and two-fermion numerators

The computational complexity of the reference-ordering approach urges a more efficient construction of the kinematic factors. The fact that the crossing symmetry among $\{2, \dots, n-1\}$ is recovered after RO-average hints that there should exist a new approach that solely depends on the structure of spanning trees and does not involve reference orderings at all. In this section, we present such an improved algorithm. The pure gluon and two-fermion half-ladder numerators built through our new approach will be naturally crossing symmetric in the legs $\{2, \dots, n-1\}$, removing the need for the RO-average and reducing the complexity of construction:⁸

$$\mathcal{O}[(n-1)! \times (n-2)!] \xrightarrow{\text{new approach}} \mathcal{O}[(n-1)!]. \quad (3.11)$$

⁸This is a very schematic comparison, and is specifically ignoring the complexity of operations that are very similar between the two procedures, e.g. processing an individual tree.

The final result is equivalent to the RO-average, but we only need to evaluate each spanning tree once. We begin by considering how to build the RO-average into simple building blocks of the increasing trees, and then present the full, recursive approach.

The first important observation is that the baseline $n \rightarrow \rho_{|\rho|} \rightarrow \cdots \rightarrow \rho_1 \rightarrow 1$ is already independent of the reference ordering. As such we will carry its definition over to our new construction. We can then proceed to considering paths of various lengths, and eventually more complicated trees.

The easiest paths to work with are length one. They are also reference-ordering independent, so their contribution can be carried over from the traditional approach,

$$\begin{array}{c} \bullet \\ \longleftarrow \\ i \end{array} \begin{array}{c} \bullet \\ \longleftarrow \\ l \end{array} : \quad \phi_{l \rightarrow i} = \mathcal{P}(l, i) = \epsilon_l \cdot k_i \quad (3.12)$$

for each such path. Length two paths are the first object that have reference-ordering dependence. However, the two inequivalent contributions dictated by the choice of reference orderings always enter with the same weight in the average,

$$\begin{array}{c} \bullet \\ \longleftarrow \\ i \end{array} \begin{array}{c} \bullet \\ \longleftarrow \\ j \end{array} \begin{array}{c} \bullet \\ \longleftarrow \\ l \end{array} : \quad \phi_{l \rightarrow j \rightarrow i} = \frac{1}{2} [\mathcal{P}(l, j) \mathcal{P}(j, i) + \mathcal{P}(l, j, i)] = \frac{1}{2} [(\epsilon_l \cdot k_j)(\epsilon_j \cdot k_i) + \epsilon_l \cdot f_j \cdot k_i]. \quad (3.13)$$

For this particular path factor only the relative ordering between l and j matters: the first term $\mathcal{P}(l, j) \mathcal{P}(j, i)$ is contributed by those reference orderings in which j is before l while the second term $\mathcal{P}(l, j, i)$ is from those with l before j . Thus averaging over reference orderings assign them the same weight.

For longer paths, it becomes more computationally difficult to directly calculate the RO-average. Interestingly, there exists a *recursive* structure that implicitly reproduces the full RO-average. The first important insight comes from stripping off the k_i from eqs. (3.12) and (3.13) such that each becomes a Lorentz vector $\mathcal{V}_A^\mu$

$$\phi_{l \rightarrow i} = \mathcal{V}_{\{l\}} \cdot k_i \quad \longrightarrow \quad \mathcal{V}_{\{l\}}^\mu = \epsilon_l^\mu \quad (3.14)$$

$$\phi_{l \rightarrow j \rightarrow i} = \mathcal{V}_{\{l \rightarrow j\}} \cdot k_i \quad \longrightarrow \quad \mathcal{V}_{\{l \rightarrow j\}}^\mu = \frac{1}{2} [(\epsilon_l \cdot k_j) \epsilon_j^\mu + \epsilon_l^\nu f_j^{\nu\mu}] . \quad (3.15)$$

The subscript on $\mathcal{V}$ denotes the set of subtrees rooted on i . We can then make use of the renaming of eq. (3.14) in eq. (3.15) to see the first hints of the recursion

$$\mathcal{V}_{\{l \rightarrow j\}}^\mu = \frac{1}{2} [(\mathcal{V}_{\{l\}} \cdot k_j) \epsilon_j^\mu + \mathcal{V}_{\{l\}}^\nu f_j^{\nu\mu}] . \quad (3.16)$$

Note that we have left the linearized field strength $f_j^{\nu\mu}$ unexpanded in order to more easily compare with the RO-average method.

While the two-vertex case is rather trivial in the RO-average, longer chains are not. Luckily, the recursive structure continues to be simple for arbitrary length chains of vertices. To see this, consider the problem of finding $\mathcal{V}_{I \rightarrow j}^\mu$, the partial contribution of a path $I_1 \rightarrow \cdots \rightarrow I_n \rightarrow j \rightarrow \cdots$ with the tensor structure “downstream” of j stripped off. Inspired by eq. (3.16), we make an ansatz for the recursive vertex,

$$\begin{array}{c} \bullet \\ \longleftarrow \\ j \end{array} \begin{array}{c} \bullet \\ \longleftarrow \\ \vdots \end{array} \begin{array}{c} \bullet \\ \longleftarrow \\ \vdots \end{array} \begin{array}{c} \bullet \\ \longleftarrow \\ \vdots \end{array} \begin{array}{c} \bullet \\ \longleftarrow \\ \vdots \end{array} \begin{array}{c} \bullet \\ \longleftarrow \\ \vdots \end{array} : \quad \mathcal{V}_{I \rightarrow j}^\mu = \mathbf{a}(I) (\mathcal{V}_I \cdot k_j) \epsilon_j^\mu + \mathbf{b}(I) \mathcal{V}_I^\nu f_j^{\nu\mu} , \quad (3.17)$$

Figure 2. Merging of branches at the vertex j .

where I is the set vertices above j in the increasing tree, and $\mathbf{a}, \mathbf{b}$ are combinatoric coefficients that we will fix using induction. First, assume that $\mathcal{V}_I$ has already been properly calculated. Then, we want to incorporate the new information from j in such a way that we implicitly reproduce the RO average along the path. In the RO prescription, terms of the form $(\mathcal{V}_I \cdot k_j) \epsilon_j^\mu$ will only occur for reference orderings in which j precedes *every* element of I . Schematically, the ordered splitting paths are of the form

$$(\mathcal{V}_I \cdot k_j) \epsilon_j^\mu \sim \left\{ \overleftarrow{\quad} \cdot j \quad , \quad j \cdot \overleftarrow{\quad} \cdot \underbrace{\quad \quad \quad}_I \quad \right\} + \dots \quad (3.18)$$

Similarly, $\mathcal{V}_I^\nu f_j^{\nu\mu}$ comes from the ROs in which *at least one* element of I precedes j ,

$$\mathcal{V}_I^\nu f_j^{\nu\mu} \sim \left\{ \overleftarrow{\quad} \cdot j \cdot \underbrace{\quad \quad \quad}_I \cdot \dots \cdot \overset{I_i}{\bullet} \quad , \quad \overset{I_i}{\bullet} \cdot \overleftarrow{\quad} \cdot \dots \cdot \overset{I_1}{\bullet} \quad \right\} + \dots \quad (3.19)$$

A careful counting of these permutations yields $\mathbf{a}(I) = p|I|!$ and $\mathbf{b}(I) = p|I||I|!$ with $|I|$ the number of vertices in I , and p the joint proportionality constant. We fix this constant by demanding that, in the final result, the term involving only a product of $\epsilon \cdot k$ always carries $+1$. Since this term is common to all the reference orderings, the average should not change its coefficient. As a result, $\mathbf{a}(I)$ and $\mathbf{b}(I)$ need to satisfy

$$\mathbf{a}(I) + \mathbf{b}(I) = 1 \quad \longrightarrow \quad \mathbf{a}(I) = \frac{1}{|I|+1}, \quad \mathbf{b}(I) = \frac{|I|}{|I|+1}. \quad (3.20)$$

Thus, the complete expression for the $\mathcal{V}$ factor of an arbitrary length path is given by the recursive definition

$$\mathcal{V}_{I \rightarrow j}^\mu = \frac{1}{|I|+1} \left[(\mathcal{V}_I \cdot k_j) \epsilon_j^\mu + |I| \mathcal{V}_I^\nu f_j^{\nu\mu} \right] \quad (3.21)$$

with the termination case given in eq. (3.14). Since we have constructed these terms using inductive matching to the RO contributions, the final result is guaranteed to match the RO-average calculation.

The other generalization necessary to evaluate an arbitrary tree is the merger of multiple paths at a vertex, schematic examples of which are shown in figure 2. The reference-ordering approach provides an explicit splitting of the tree at such vertices, but the goal

is to avoid such a choice. Luckily, the recursive structure can be continued to this case. The simple example is the merger of two length-one paths at a vertex. The result of RO-average is

$$\begin{aligned}
 \phi_{\{m,l\} \rightarrow j \rightarrow i} &= \mathcal{V}_{\{m,l\} \rightarrow j} \cdot k_i = \frac{1}{3} \left[(\epsilon_m \cdot k_j)(\epsilon_l \cdot k_j)(\epsilon_j \cdot k_i) \right. \\
 &\quad \left. + \epsilon_m \cdot f_j \cdot k_i (\epsilon_l \cdot k_j) + \epsilon_l \cdot f_j \cdot k_i (\epsilon_m \cdot k_j) \right], \quad (3.22)
 \end{aligned}$$

in which there are three different contributions: one each from when m or l comes before j in the reference orderings, and a third when j is first among the three. Using the knowledge we have already gained from analyzing the simpler paths in eqs. (3.13) and (3.15), we can rewrite $\mathcal{V}_{\{m,l\} \rightarrow j}^\mu$ as

$$\begin{aligned}
 \mathcal{V}_{\{m,l\} \rightarrow j}^\mu &= \frac{1}{3} \left[\epsilon_j^\mu \prod_{p \in \{m,l\}} \epsilon_p \cdot k_j + \sum_{p \in \{m,l\}} \epsilon_p^\nu f_j^{\nu\mu} \prod_{\substack{q \in \{m,l\} \\ q \neq p}} \epsilon_q \cdot k_j \right] \\
 &= \frac{1}{3} \left[\epsilon_j^\mu \prod_{p \in \{m,l\}} \mathcal{V}_{\{p\}} \cdot k_j + \sum_{p \in \{m,l\}} \mathcal{V}_{\{p\}}^\nu f_j^{\nu\mu} \prod_{\substack{q \in \{m,l\} \\ q \neq p}} \mathcal{V}_{\{q\}} \cdot k_j \right]. \quad (3.23)
 \end{aligned}$$

The use of $\prod$ and $\sum$ in eq. (3.23) makes the generalization to more incoming length-one paths straightforward: for incoming paths from vertices $M = \{m_1, \dots, m_r\}$ we have

$$\text{figure 2a: } \mathcal{V}_{M \rightarrow j}^\mu = \frac{1}{|M|+1} \left[\epsilon_j^\mu \prod_{m_a \in M} \mathcal{V}_{\{m_a\}} \cdot k_j + \sum_{m_a \in M} \mathcal{V}_{\{m_a\}}^\nu f_j^{\nu\mu} \prod_{\substack{m_b \in M \\ b \neq a}} \mathcal{V}_{\{m_b\}} \cdot k_j \right]. \quad (3.24)$$

Finally, it is necessary to generalize each $m_a \in M$ to an arbitrary length chain I_a , see figure 2b. Now M becomes $M = \{I_1, \dots, I_r\}$, where each I_a is a chain as in eq. (3.21). Due to the recursive structure, most of the work was already done above in eq. (3.21): we can simply replace each $\mathcal{V}_{\{m_a\}}^\mu$ by $\mathcal{V}_{I_a}^\mu$ and adjust the relative coefficients,

$$\begin{aligned}
 \text{figure 2b: } \mathcal{V}_{M \rightarrow j}^\mu &= \mathfrak{a} \epsilon_j^\mu \prod_{I_a \in M} \mathcal{V}_{I_a} \cdot k_j + \sum_{I_a \in M} \mathfrak{b}_a \mathcal{V}_{I_a}^\nu f_j^{\nu\mu} \prod_{\substack{I_b \in M \\ b \neq a}} \mathcal{V}_{I_b} \cdot k_j \quad (3.25) \\
 &= \frac{1}{1 + \sum_{I_a \in M} |I_a|} \left[\epsilon_j^\mu \prod_{I_a \in M} \mathcal{V}_{I_a} \cdot k_j + \sum_{I_a \in M} |I_a| \mathcal{V}_{I_a}^\nu f_j^{\nu\mu} \prod_{\substack{I_b \in M \\ b \neq a}} \mathcal{V}_{I_b} \cdot k_j \right],
 \end{aligned}$$

where $\mathfrak{a}$ is the (normalized) number of reference orderings in which j is before M , and similarly $\mathfrak{b}_a$ is the (normalized) number of reference orderings in which the first element in the sub-ordering of $M \cup \{j\}$ belongs to I_a .

Figure 3. A typical spanning tree with the baseline shown in red, where $T_a \in M_{\rho_i}$ are the sub-trees merging at ρ_i on the baseline.

It is important to notice that the coefficients $\mathbf{a}$ and $\mathbf{b}_a$ depend only on the number vertices in each sub-tree but not the structure. Therefore, this expression can be used to calculate any sub-trees merging at one vertex, see figure 2c, where $M = \{T_1, \dots, T_r\}$ and each T_a is an arbitrary sub-tree,

$$\text{figure 2c: } \mathcal{V}_{M \rightarrow j}^\mu = \frac{1}{1 + \sum_{T_a \in M} |T_a|} \left[\epsilon_j^\mu \prod_{T_a \in M} \mathcal{V}_{T_a} \cdot k_j + \sum_{T_a \in M} |T_a| \mathcal{V}_{T_a}^\nu f_j^{\nu\mu} \prod_{\substack{T_b \in M \\ b \neq a}} \mathcal{V}_{T_b} \cdot k_j \right], \quad (3.26)$$

due to the recursive assumption that each $\mathcal{V}_{T_a}$ has dressed its sub-tree correctly. Thus for the most generic sub-tree structure one only needs to correctly merge the information from each T_a via the relative normalizations, which are not sensitive to the structure of the sub-trees.

With these new kinematic dressings, we can completely get rid of item 2 in the previous algorithm since explicit ordered splitting computations are no longer necessary. For completeness, our improved algorithm for calculating the numerator contribution $N(T)$ for a given increasing tree T is:

Algorithm 2 (Recursive Evaluation). Construction of the kinematic factor for a spanning tree T recursively.

1. For the path $n \rightarrow \rho_{|\rho|} \rightarrow \dots \rho_1 \rightarrow 1$, assign a baseline factor $W(1, \rho, n)$. Note that $n \rightarrow 1$ and thus $\rho = \emptyset$ is allowed, which contributes $W(1, n)$.
2. For each vertex $\{1, \rho_1, \dots, \rho_{|\rho|}\}$ on the baseline, identify M_{ρ_i} as the set of sub-trees merging at ρ_i , see figure 3. For notational convenience we define $\rho_0 = 1$.
3. Calculate $\mathcal{V}$ from each sub-tree $T_a \in M_{\rho_i}$ using eq. (3.26) and assemble the result:

$$N(T) = W(1, \rho, n) \prod_{i=0}^{|\rho|} \prod_{T_a \in M_{\rho_i}} \mathcal{V}_{T_a} \cdot k_{\rho_i}. \quad (3.27)$$

This algorithm only needs to process a single spanning tree T once, and directly generates the crossing symmetric numerator in $\{2, \dots, n-1\}$, thus eliminating the $\mathcal{O}[(n-2)!]$

workload due to the RO-average, as advertised at the beginning of this subsection. We have explicitly checked agreement with numerators generated by RO-average through seven points, and numerically checked that they reproduce correct amplitudes through nine points.⁹ We note that in our numerators k_n does not appear by construction, and neither does $\epsilon_n \cdot k_1$. Thus our numerators are naturally in a basis of momentum conservation.

While the procedure was designed with pure Yang-Mills trees in mind, we can recall from eqs. (2.5) and (2.7) that all terms that care about 1 and n being fermions are localized to the baseline function W_{2f} . The rest of the kinematic structure is exactly identical to the pure Yang-Mills case. Thus, this algorithm *also* constructs the half-ladder numerators with two fermions.

3.3 Four-point YM example

With the general principle under control, we'll now turn to an explicit example, and demonstrate some of the functionality of the MATHEMATICApacage. We'll construct the DDM basis numerator $N(1, 2, 3, 4)$ using our new method, and then show that it indeed gives the correct numerator for $N(1, 3, 2, 4)$ via relabeling.

The first step for building the numerators is to enumerate $IT(1, 2, 3, 4)$, which is implemented by the function `increasingTrees`:

$$\text{increasingTrees}[\{1, 2, 3, 4\}] = \left\{ \begin{array}{cc} \begin{array}{c} \text{Diagram 1: } 1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \end{array} & \begin{array}{c} \text{Diagram 2: } 1 \rightarrow 2 \rightarrow 3 \text{ (gluon)} \end{array} \\ \begin{array}{c} \text{Diagram 3: } 1 \rightarrow 2 \text{ (gluon)} \rightarrow 3 \rightarrow 4 \end{array} & \begin{array}{c} \text{Diagram 4: } 1 \rightarrow 2 \text{ (gluon)} \rightarrow 3 \rightarrow 4 \end{array} \\ \begin{array}{c} \text{Diagram 5: } 1 \rightarrow 2 \text{ (gluon)} \rightarrow 3 \rightarrow 4 \end{array} & \begin{array}{c} \text{Diagram 6: } 1 \rightarrow 2 \text{ (gluon)} \rightarrow 3 \rightarrow 4 \end{array} \end{array} \right\}. \quad (3.28)$$

Then we need to dress each of the trees with their appropriate kinematic factors. This is handled by the function `itNumer`. For the simplest example, we have

$$\text{itNumer}[\text{Diagram 1}, 4] = W_{\text{gluon}}(1, 2, 3, 4). \quad (3.29)$$

The most complicated case is actually entirely covered by eq. (3.16):

$$\text{itNumer}[\text{Diagram 2}, 4] = \frac{1}{2} ((\epsilon_3 \cdot k_2)(\epsilon_2 \cdot k_1) + \epsilon_3 \cdot f_2 \cdot k_1) W_{\text{gluon}}(1, 4). \quad (3.30)$$

⁹The new technique can compute a nine-point half-ladder numerator in a few minutes, while the old algorithm given in section 3.1 struggles at seven points. Our new approach is used extensively for construction and checks in a parallel study of one-loop integrands from forward limits [78].

Combining all of the contributions together gives us

$$\begin{aligned}
 N(1, 2, 3, 4) &= W_{\text{gluon}}(1, 2, 3, 4) + W_{\text{gluon}}(1, 2, 4) (\epsilon_3 \cdot k_2 + \epsilon_3 \cdot k_1) + W_{\text{gluon}}(1, 3, 4) \epsilon_2 \cdot k_1 \\
 &\quad + W_{\text{gluon}}(1, 4) \left(\frac{1}{2} ((\epsilon_3 \cdot k_2)(\epsilon_2 \cdot k_1) + \epsilon_3 \cdot f_2 \cdot k_1) + (\epsilon_2 \cdot k_1)(\epsilon_3 \cdot k_1) \right). \quad (3.31)
 \end{aligned}$$

We can do a similar calculation to find the $(1, 3, 2, 4)$ ordering

$$\begin{aligned}
 N(1, 3, 2, 4) &= W_{\text{gluon}}(1, 3, 2, 4) + W_{\text{gluon}}(1, 3, 4) (\epsilon_2 \cdot k_3 + \epsilon_2 \cdot k_1) + W_{\text{gluon}}(1, 2, 4) \epsilon_3 \cdot k_1 \\
 &\quad + W_{\text{gluon}}(1, 4) \left(\frac{1}{2} ((\epsilon_2 \cdot k_3)(\epsilon_3 \cdot k_1) + \epsilon_2 \cdot f_3 \cdot k_1) + (\epsilon_3 \cdot k_1)(\epsilon_2 \cdot k_1) \right), \quad (3.32)
 \end{aligned}$$

which shows that, as claimed, the two numerators are just relabelings of each other under $(2 \leftrightarrow 3)$. The function `ptTreeNumer` will fully calculate an n -point tree numerator, and relabel and dress the numerator with the appropriate Parke-Taylor factor

$$\text{ptTreeNumer}[4] = N(1, 2, 3, 4) \text{PT}(1, 2, 3, 4) + N(1, 3, 2, 4) \text{PT}(1, 3, 2, 4). \quad (3.33)$$

As discussed in section 2.1, this also builds the correct numerator for two fermions, N_{2f} , by replacing $W_{\text{gluon}} \rightarrow W_{2f}$.

3.4 Recursive constructions for multi-trace

The recursive construction we described in section 3.2 and algorithm 2 can be extended to also handle multi-trace YMS numerators. We will work out how to build numerators of the form $N_{\text{MT}}(1, \beta, n)$, with an additional set of specified scalar traces τ_i . First using the baseline expansion discussed in section 2.2, we can put the legs 1 and n to the endpoints of a single scalar trace, see eqs. (2.16) and (2.30). Then we can work out the numerators following the process first proposed in ref. [66], in particular, a spanning-tree algorithm given in section 10.3 (see also appendix A of ref. [34]). However, this algorithm is RO-based and thus crossing symmetry is not manifest. In this section, we develop additional refinements that restore crossing symmetry to the numerators, in line with what we did for the pure Yang-Mills numerators in section 3.2. Similar to the pure Yang-Mills case, the process involves two steps, constructing spanning trees and dressing with kinematic factors. As discussed in section 2.2, the gluons and scalar traces are generally treated on the same footing.

First, we discuss the structure of the increasing trees to dress, and more generally the interplay between trace orderings and numerator ordering. The analog of the increasing-tree ordering used in eq. (3.6) is obtained by applying the coarse-grained $\mathcal{O}$ function (2.22) to the entire numerator ordering $(1, \beta, n)$,

$$\mathcal{O}_{(1, \beta, n)}^{\text{cg}} = (\lambda_1, \lambda_2, \dots, \lambda_N) \in S_{\{\mathbf{t}_1, \mathbf{t}_2, \dots, \mathbf{t}_N\}}, \quad (3.34)$$

where each λ_i represents a scalar trace or a gluon. It is an ordering of our particle inventory $\{\mathbf{t}_1, \mathbf{t}_2, \dots, \mathbf{t}_N\} = \{\tau_1, \tau_2, \dots, \tau_m\} \cup G$ with $N = m + |G|$, where scalar traces τ_i are treated on the same footing as a gluon in G . In addition, the first entry λ_1 must be the trace that

leg 1 belongs to, or the leg 1 itself if it is a gluon. We can then construct all the increasing trees according to eq. (3.6) and write

$$N_{\text{MT}}(1, \beta, n) = \sum_{T \in \text{IT}(\mathcal{O}_{(1, \beta, n)}^{\text{cg}})} N_{\text{MT}}(T). \quad (3.35)$$

The vertices of each tree T are $\mathbf{t}_i$'s, which can either be a scalar trace or a gluon.

Now that the increasing trees are laid out, we can begin assigning their kinematic dressings. To evaluate $N_{\text{MT}}(T)$, the first step is to extract the baseline from a tree T . Suppose $1 \in \mathbf{t}_1$ and $n \in \mathbf{t}_m$, we first extract the path $(\mathbf{t}_1, \lambda_1, \dots, \lambda_{|B|}, \mathbf{t}_m)$ that connects them in T . If n is a gluon, then $\mathbf{t}_m = n$ must be a leaf, while this is not necessary when $\mathbf{t}_m$ is a scalar trace. Moreover, $\mathbf{t}_m$ can coincide with $\mathbf{t}_1$. This happens when 1 and n belong to the same scalar trace to begin with. For this case, the path consists of just a single vertex $\mathbf{t}_1$. The baseline ordering $(1, \rho, n)$ is then obtained by restricting the numerator color ordering $(1, \beta, n)$ to $(\mathbf{t}_1, \lambda_1, \dots, \lambda_{|B|}, \mathbf{t}_m)$,

$$(1, \rho, n) = (1, \beta, n) \Big|_{(\mathbf{t}_1, \lambda_1, \dots, \lambda_{|B|}, \mathbf{t}_m)}, \quad (3.36)$$

which is exactly the baseline discussed in section 2.2. We can thus dress it with the kinematic factor $W_{\text{MT}}(1, \rho, n)$ defined in eq. (2.27). On the other hand, given a baseline $(1, \rho, n)$, we can recover the path $(\mathbf{t}_1, \lambda_1, \dots, \lambda_{|B|}, \mathbf{t}_m)$, which we call the *coarse-grained baseline*, by using the coarse-grained $\mathcal{O}$ function,

$$(\mathbf{t}_1, \lambda_1, \dots, \lambda_{|B|}, \mathbf{t}_m) = \mathcal{O}_{(1, \rho, n)}^{\text{cg}}. \quad (3.37)$$

Eq. (3.51) in the following section provides explicit examples of this baseline identification. Now similar to eq. (3.27), we can write $N_{\text{MT}}(T)$ as

$$N_{\text{MT}}(T) = W_{\text{MT}}(1, \rho, n) \prod_{i \in \mathcal{O}_{(1, \rho, n)}^{\text{cg}}} \prod_{T_a \in M_i} \mathcal{V}_{T_a} \cdot \mathcal{K}_{i, T_a}, \quad (3.38)$$

where M_i is the set of sub-trees merging at the vertex i on the coarse-grained baseline $\mathcal{O}_{(1, \rho, n)}^{\text{cg}}$. The kinematic factor associated to each sub-tree $T_a \in M_i$ is $\mathcal{V}_{T_a} \cdot \mathcal{K}_{i, T_a}$, where $\mathcal{K}$ will be defined below in eq. (3.43). The vertex function $\mathcal{V}$ can be evaluated by RO-based methods [34, 66], but in the following, we will propose an improved algorithm that automatically recovers the crossing symmetry.

We denote $M = \{T_1, T_2, \dots, T_r\}$ as the set of sub-trees merging at the vertex $\mathbf{t}_j$. By construction the vertices of our spanning trees are labeled by $\{\mathbf{t}_1, \dots, \mathbf{t}_N\}$. The recursive structure of $\mathcal{V}$ when including the vertex $\mathbf{t}_j$ takes a form very similar to eq. (3.26),

$$\mathcal{V}_{M \rightarrow \mathbf{t}_j}^\mu = \frac{1}{1 + \sum_{T_a \in M} |T_a|} \left[b_{\mathbf{t}_j}^\mu \prod_{T_a \in M} \mathcal{V}_{T_a} \cdot \mathcal{K}_{\mathbf{t}_j, T_a} + \sum_{T_a \in M} |T_a| \mathcal{V}_{T_a}^\nu \mathcal{T}_{\mathbf{t}_j, T_a}^{\nu\mu} \prod_{\substack{T_b \in M \\ b \neq a}} \mathcal{V}_{T_b} \cdot \mathcal{K}_{\mathbf{t}_j, T_b} \right], \quad (3.39)$$

where $|T_a|$ counts the number of vertices in T_a . The combinatoric coefficients here are the same as eq. (3.26) for pure gluon cases since scalar traces are treated as a single object on the same footing as gluons. The path-begin factor $b_{\mathbf{t}_j}^\mu$ is given by

$$b_{\mathbf{t}_j}^\mu = \begin{cases} \frac{1}{|\mathbf{t}_j|} \text{sgn}_{a_j, \omega_j, b_j}^{\mathbf{t}_j}(-k_{a_j}^\mu) & \mathbf{t}_j \text{ a trace} \\ \epsilon_{\mathbf{t}_j}^\mu & \mathbf{t}_j \text{ is a gluon} \end{cases}, \quad (3.40)$$

where (a_j, ω_j, b_j) the sub-ordering of the elements in $\mathbf{t}_j$ in the numerator ordering $(1, \beta, n)$. The effect of RO-average is encoded in the prefactor $\frac{1}{|\mathbf{t}_j|}$. This prescription recovers the crossing symmetry within the trace $\mathbf{t}_j$. We note that in the RO approach given in [34, 66], one has to pick the same a_j across when $\mathbf{t}_j$ appears in the path-begin factor for all the numerator ordering, which breaks the manifest cyclicity of the scalar traces. Next, the through factor $\mathcal{T}_{\mathbf{t}_j, T_a}^{\mu\nu}$. At this point, it is useful to generalize $\mathcal{B}$ to take trees (multiple traces) after the vertical bar via

$$\mathcal{B}_{(1, \beta, n)}(\mathbf{t}_i | T_a) \equiv \bigcap_{\mathbf{t}_l \in T_a} \mathcal{B}_{(1, \beta, n)}(\mathbf{t}_j | \mathbf{t}_l), \quad (3.41)$$

that is, $\mathcal{B}_{(1, \beta, n)}(\mathbf{t}_i | T_a)$ consists of the elements in $\mathbf{t}_i$ that come before everything in T_a in the numerator ordering $(1, \beta, n)$. Then $\mathcal{T}_{\mathbf{t}_j, T_a}$ is given by

$$\mathcal{T}_{\mathbf{t}_j, T_a}^{\mu\nu} = \begin{cases} \text{sgn}_{a_j, \omega_j, b_j}^{\mathbf{t}_j}(-k_{b_j}^\mu k_{a_j}^\nu) & \mathbf{t}_j \text{ is a trace and } \mathcal{B}_{(1, \beta, n)}(\mathbf{t}_j | T_a) = \mathbf{t}_j \\ 0 & \mathbf{t}_j \text{ is a trace and } \mathcal{B}_{(1, \beta, n)}(\mathbf{t}_j | T_a) \subsetneq \mathbf{t}_j \\ f_{\mathbf{t}_j}^{\mu\nu} & \mathbf{t}_j \text{ is a gluon} \end{cases}. \quad (3.42)$$

Note that it takes a similar form to the $\mathcal{T}$ factor used in the baseline factor $W_{\text{MT}}(1, \rho, n)$, see eq. (2.28a). The sgn function appearing in eqs. (3.40) and (3.42) ensures that $N_{\text{MT}}(1, \beta, n)$ is nonzero if and only if the numerator ordering $(1, \beta, n)$ is KK-compatible to every scalar trace. Finally, the path-end factor $\mathcal{K}_{\mathbf{t}_j, T_a}^\mu$ is given by

$$\mathcal{K}_{\mathbf{t}_j, T_a}^\mu = \sum_{m \in \mathcal{B}_{(1, \beta, n)}(\mathbf{t}_j | T_a)} k_m^\mu, \quad (3.43)$$

the total momentum from $\mathbf{t}_j$ that comes before everything in T_a . The factor $\mathcal{K}$ is actually the same as the one used in the RO approach [34, 66]. Finally, we note that the proof of eq. (3.39) resembles that of section 3.2, which we will not repeat here.

3.5 Multi-trace example

We'll now walk through a few examples of putting the multi-trace expansion into practice. First, we'll calculate $N(1, 2, 3, 4, 5, 6, 7, 8)$ where the traces are decomposed and color-ordered as

$$\tau_1 = (1, 8) \quad \tau_2 = (2, 3) \quad \tau_3 = (4, 6) \quad \tau_4 = (5, 7). \quad (3.44)$$

Since 1 and 8 are already in the same trace, we don't need to use the full power of the baseline recursion in this case, and can instead focus on the kinematic factors immediately.

The increasing trees that contribute to this numerator will have τ_1 as the baseline, and the remaining τ s as vertices along the trees. In this particle ordering, the traces are shuffled via

$$(1, \beta, 8) = (1, 2, 3, \overbrace{4, 5, 6}^{\tau_3}, \overbrace{7, 8}^{\tau_4}). \quad (3.45)$$

so the increasing trees are ordered via τ_2 before τ_3 before τ_4 . The six increasing trees that build this numerator can be generated using

$$\text{increasingTrees}[\tau_1, \tau_2, \tau_3, \tau_4] =$$

$$(3.46)$$

Since $\mathcal{B}(\tau_3|\tau_4) = (4) \neq \tau_3$, the two trees with edge $\tau_4 \rightarrow \tau_3$ are in the $\mathcal{T}^{\mu\nu} = 0$ case of eq. (3.42). Other than that important caveat, the last five of these trees can be evaluated by directly adapting various examples we covered in the YM construction. The first tree will be a good example application of the recursive structure. Using the kinematic dressings from section 3.4 in the recursion, we have

$$\begin{aligned} N(\tau_1 \rightarrow \tau_2 \rightarrow \tau_3 \rightarrow \tau_4) &= W_{\text{MT}}(\tau_1) k_1 \cdot \mathcal{V}_{\tau_2 \tau_3 \tau_4} \\ &= -W_{\text{MT}}(\tau_1) \frac{1}{3} k_1^\mu \left(\frac{1}{2} k_2^\mu (k_{23} \cdot \mathcal{V}_{\tau_3 \tau_4}) + 2k_2^\mu (k_3 \cdot \mathcal{V}_{\tau_3 \tau_4}) \right) \\ &= W_{\text{MT}}(\tau_1) (k_1 \cdot k_2) \frac{1}{6} \left[(k_{23} + 4k_3)^\nu \frac{1}{2} \left(\frac{1}{2} k_4^\nu (k_4 \cdot \mathcal{V}_{\tau_4}) + k_4^\nu(0) \right) \right] \\ &= -\frac{1}{48} (k_1 \cdot k_2) (k_4 \cdot k_5) (k_{23} \cdot k_4 + 4k_3 \cdot k_4), \end{aligned} \quad (3.47)$$

where $W_{\text{MT}}(\tau_1) = 1$. We do not directly provide a function to calculate individual tree dressings for YMS due to the need to calculate $\mathcal{B}$ and $\mathcal{O}$ from the full numerator ordering. However, the full result from combining all of the increasing trees can be calculated via the package using

$$\begin{aligned} \text{multiTraceNumber}[\{1, 2, 3, 4, 5, 6, 7, 8\}, \{\{1, 8\}, \{2, 3\}, \{4, 6\}, \{5, 7\}\}] &= \\ -\frac{1}{48} (k_1 \cdot k_2) [2(k_2 \cdot k_4)(k_2 \cdot k_5) + 6(k_2 \cdot k_5)(k_3 \cdot k_4) + 3(k_1 \cdot k_5)(k_2 \cdot k_4 + 3k_3 \cdot k_4) \\ + 6(k_2 \cdot k_4)(k_3 \cdot k_5) + 10(k_3 \cdot k_4)(k_3 \cdot k_5) + (k_2 \cdot k_4)(k_4 \cdot k_5) + 5(k_3 \cdot k_4)(k_4 \cdot k_5) \\ + 3(k_1 \cdot k_4)(2(k_1 \cdot k_5) + (k_2 \cdot k_5) + 3(k_3 \cdot k_5) + (k_4 \cdot k_5))] . \end{aligned} \quad (3.48)$$

Next, we look at a case involving gluons that demonstrates the baseline expansion. The goal is to calculate $N(1, 2, 3, 4, 5, 6)$ with

$$\tau_1 = (1, 2) \quad \tau_2 = (3, 4) \quad G = \{5, 6\}. \quad (3.49)$$

Since we don't have a scalar trace of the form $(1 \dots n)$, we need to perform the sum over pushing τ_2 and 5 into the baseline with τ_1 and 6. The various expansions of the baseline are given by

$$\begin{aligned} W_{\text{MT}}(\tau_1, 6) &= k_2 \cdot \epsilon_6 & W_{\text{MT}}(\tau_1, 5, 6) &= -k_2 \cdot f_5 \cdot \epsilon_6 \\ W_{\text{MT}}(\tau_1, \tau_2, 6) &= -(k_2 \cdot k_3)(k_4 \cdot \epsilon_6) & W_{\text{MT}}(\tau_1, \tau_2, 5, 6) &= (k_2 \cdot k_3)(k_4 \cdot f_5 \cdot \epsilon_6) \end{aligned} \quad (3.50)$$

which correspond to the increasing trees

$$\begin{aligned} W_{\text{MT}}(\tau_1, 6) : & \quad \begin{array}{c} \tau_2 \quad 5 \\ \diagup \quad \diagdown \\ \tau_1 \quad \bullet \quad 6 \end{array} & \quad \begin{array}{c} \tau_2 \\ \diagup \quad \diagdown \\ \tau_1 \quad \bullet \quad 5 \\ \quad \quad \diagdown \\ \quad \quad 6 \end{array} & \quad W_{\text{MT}}(\tau_1, 5, 6) : & \quad \begin{array}{c} \tau_2 \\ \diagup \quad \diagdown \\ \tau_1 \quad \bullet \quad 5 \\ \quad \quad \diagdown \\ \quad \quad 6 \end{array} \\ W_{\text{MT}}(\tau_1, \tau_2, 6) : & \quad \begin{array}{c} 5 \\ \diagup \quad \diagdown \\ \tau_1 \quad \bullet \quad \tau_2 \quad 6 \end{array} & \quad \begin{array}{c} 5 \\ \diagup \quad \diagdown \\ \tau_1 \quad \bullet \quad \tau_2 \\ \quad \quad \diagdown \\ \quad \quad 6 \end{array} & \quad W_{\text{MT}}(\tau_1, \tau_2, 5, 6) : & \quad \begin{array}{c} \tau_1 \quad \tau_2 \quad 5 \quad 6 \end{array} \end{aligned} \quad (3.51)$$

We can get the combined result by again using `multiTraceNumer`, dropping the gluons from the trace specification

$$\begin{aligned} \text{multiTraceNumer}[\{1, 2, 3, 4, 5, 6\}, \{\{1, 2\}, \{3, 4\}\}] = \\ \frac{1}{2} W_{\text{MT}}(\tau_1, 6) (2\epsilon_5 \cdot k_1 + 2\epsilon_5 \cdot k_2 + \epsilon_5 \cdot k_3 + 3\epsilon_5 \cdot k_4) (k_1 \cdot k_3 + k_2 \cdot k_3) \\ - W_{\text{MT}}(\tau_1, \tau_2, 6) (\epsilon_5 \cdot k_{1234}) - \frac{1}{2} W_{\text{MT}}(\tau_1, 5, 6) (k_1 \cdot k_3 + k_2 \cdot k_3) + W_{\text{MT}}(\tau_1, \tau_2, 5, 6). \end{aligned} \quad (3.52)$$

4 Introducing masses

Since the numerator construction we have described is dimensionally agnostic (up to the choice of fermion wavefunctions), it is straightforward to obtain numerators of minimally-coupled massive theories through a dimensional compactification. Such a construction is equivalent to a spontaneous symmetry breaking [83]. In this section, we will work with the mostly minus metric.

In particular, it is very convenient to introduce a mass to particles 1 and n that sit at the end points of the baseline. To construct the desired massive numerators in 4D, we start by picking special kinematics in 6D,

$$K_1 = \begin{pmatrix} k_1 \\ 0 \\ -m \end{pmatrix}, \quad K_2 = \begin{pmatrix} k_2 \\ 0 \\ 0 \end{pmatrix}, \quad \dots \quad K_n = \begin{pmatrix} k_n \\ 0 \\ m \end{pmatrix}. \quad (4.1)$$

where K_i are 6D vectors, and k_i are 4D vectors. The 6D massless conditions $K_1^2 = K_n^2 = 0$ impose the 4D massive conditions $k_1^2 = k_n^2 = m^2$. For scalars, no modifications are needed as their external wavefunctions are trivial. The gluon polarization vectors are all kept in 4D, $\mathcal{E}_i = (\epsilon_i, 0, 0)$, and transverse to the momentum, $K_i \cdot \mathcal{E}_i = k_i \cdot \epsilon_i = 0$.

To build up the explicit embedding of massive 4D spinors into massless 6D spinors, one can follow the technique of [4, 84],

$$|1_a\rangle^A = \begin{pmatrix} |1^+\rangle & |1^-\rangle \\ |1^+] & |1^-] \end{pmatrix} = \left(u^+(k_1), u^-(k_1)\right), \quad (4.2a)$$

$$|1_{\dot{a}}]_A = \begin{pmatrix} -\langle 1^+| & \langle 1^-| \\ -[1^+| & [1^-| \end{pmatrix} = \left(-\bar{v}^+(k_1), \bar{v}^-(k_1)\right), \quad (4.2b)$$

$$|n_a\rangle^A = \begin{pmatrix} -|n^+\rangle & |n^-\rangle \\ |n^+] & -|n^-] \end{pmatrix} = \left(v^+(k_n), -v^-(k_n)\right), \quad (4.2c)$$

$$|n_{\dot{a}}]_A = \begin{pmatrix} \langle n^+| & \langle n^-| \\ -[n^+| & -[n^-| \end{pmatrix} = \left(-\bar{u}^+(k_n), \bar{u}^-(k_n)\right), \quad (4.2d)$$

where $|i^\pm\rangle$ and $|i^\pm]$ are massive spinor helicity variables [85], and u and v are usual momentum-space Dirac spinors. The explicit representation of 6D spinors depends on the choice of 6D gamma matrices, and here we use the one given in the appendix A of ref. [80].¹⁰ Our definition of Dirac spinors in terms of massive spinor helicity variables follows ref. [86], where one can also find explicit formulas for these quantities.

We note that all the formal expressions for our numerators are unchanged under the uplift (4.1), except that one needs to use massive kinematics when evaluating. In fact, the only way to introduce an explicit mass term is via $K_1 \cdot K_n = k_1 \cdot k_n + m^2$ while $K_i \cdot K_j = k_i \cdot k_j$ for all the rest. Our numerators are constructed to only contain the wavefunction of particle n but not the momentum, such that the product $K_1 \cdot K_n$ will not naturally appear unless one deliberately swaps it in via momentum conservation.

5 Conclusion and outlook

In this paper we presented a review of the current state-of-the-art for calculating YM tree amplitudes in the CHY formalism. We identified a computational complexity that results from the requirement of the previous methods, summarized in algorithm 1, to define an *ordered splitting of paths* in order to build kinematic factors, which in turn builds numerators that are not crossing symmetric. To overcome this complexity, we developed a new method of assigning kinematic factors to increasing trees that incorporates all contributions from different *ordered splittings* implicitly, rather than via an explicit after-the-fact average. This new method, summarized in algorithm 2, is built around a recursive processing of the underlying spanning trees, allowing the required kinematic factors to be determined at each vertex by using a small amount of information from the previous steps in the recursion. One of the benefits of the new approach is the restoration of crossing symmetry at every step in the calculation, without needing to process the same tree multiple times. Our new numerator construction is now *only* a function of external particle ordering, and has no dependence on reference orderings. The new method was also straightforward to generalize to include full multi-trace YMS numerators or two-fermion numerators. Since

¹⁰This is our motivation for putting the mass into the sixth slot instead of the fifth in eq. (4.1), as this choice will lead to a simpler embedding of fermion wavefunctions.

the numerators we construct are in the DDM basis, they can be double-copied immediately to yield the corresponding numerators for Einstein-Yang-Mills.

A related benefit of the new method is a significant improvement in the accessible multiplicity of tree amplitudes. While the RO based methods require processing all of the spanning trees again for each numerator in eqs. (1.2) and (1.3), the new algorithm can simply relabel a prototype numerator. As such our new method is efficient enough that we can easily calculate 8 point YM trees, where two of the external legs can be swapped out for fermions or scalars instead; we have been able to calculate individual crossing symmetric tree numerators up to 10 points without significant difficulty. For 7 and fewer external particles, we have explicitly compared our results with the RO average. At 8 points and beyond, we have verified our results by comparison to a direct numerical calculation of eq. (2.1) on certain solutions of the scattering equations. This ability to calculate high-point trees with super-partners has been incredibly useful in the study of forward limits, which we leave to its own paper [78].

The fermionic baseline functions are the only piece of the calculation that depend on the dimension of interest. Thus, we described one method of using dimensional compactification to introduce masses to the 2 singled out particles labeled 1 and n . We hope that the combination of double-copy compatibility, potentially massive particles, and calculation efficiency will be of interest to the growing application of amplitudes techniques to classical gravity, for instance as input into the unitarity cuts approach of constructing relevant loop amplitudes [13, 15, 16, 19–21].

Finally, we provide a MATHEMATICAPackage that implements the recursive numerator construction. Appendix B and the examples in sections 3.3 and 3.5 have more details about the package and its usage.

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A CHY integrands and scattering amplitudes

In this appendix, we provide the explicit forms of the CHY integrands used in the main text and the normalizations used to match Feynman diagram results. Throughout the paper,

we use the following definition for the Pfaffian of a $(2n) \times (2n)$ antisymmetric matrix X ,

$$\text{Pf}(X) \equiv (-1)^{\frac{n(n+1)}{2}} \frac{1}{2^n n!} \sum_{\sigma \in S_{2n}} \text{sign}(\sigma) \prod_{i=1}^n X_{\sigma_{2i-1} \sigma_{2i}}. \quad (\text{A.1})$$

The sign factor $(-1)^{\frac{n(n+1)}{2}}$ is not present in the usual definition of a Pfaffian [88], but convenient for our purposes.

In the pure gluon integrand (2.4), the reduced Pfaffian is defined as

$$\text{Pf}'(\Psi) \equiv \frac{(-1)^{i+j+n-1}}{\sigma_{ij}} \text{Pf}(\Psi_{ij}^{ij}), \quad 1 \leq i < j \leq n. \quad (\text{A.2})$$

where the $2n \times 2n$ antisymmetric matrix Ψ is given in the block form,

$$\Psi = \begin{pmatrix} \mathbf{A} & -\mathbf{C}^T \\ \mathbf{C} & \mathbf{B} \end{pmatrix}. \quad (\text{A.3})$$

The three $n \times n$ blocks $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are given by

$$\begin{aligned} \mathbf{A}_{ij} &= \frac{k_i \cdot k_j}{\sigma_{ij}}, & \mathbf{B}_{ij} &= \frac{\epsilon_i \cdot \epsilon_j}{\sigma_{ij}}, & \mathbf{C}_{ij} &= \frac{\epsilon_i \cdot k_j}{\sigma_{ij}}, \\ \mathbf{A}_{ii} &= 0, & \mathbf{B}_{ii} &= 0, & \mathbf{C}_{ii} &= -\sum_{\substack{j=1 \\ j \neq i}}^n \mathbf{C}_{ij}. \end{aligned} \quad (\text{A.4})$$

The polarization vectors are normalized as $\epsilon_i \cdot \epsilon_i^* = 1$. The matrix Ψ_{ij}^{ij} is obtained from Ψ by deleting the i -th and j -th row and column. On the support of the scattering equations, $\text{Pf}'(\Psi)$ is independent of the choice of i and j . The Ψ_G that appears in the baseline expansion (2.5) and the single-trace integrand (2.12) is a $2|G| \times 2|G|$ antisymmetric submatrix of Ψ , obtained by restricting the rows and columns in the $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ blocks to the set G ,

$$\Psi_G = \begin{pmatrix} \mathbf{A}_G & -(\mathbf{C}_G)^T \\ \mathbf{C}_G & \mathbf{B}_G \end{pmatrix}, \quad (\text{A.5})$$

where $(\mathbf{X}_G)_{ij} = \mathbf{X}_{ij}$ for $\mathbf{X} = \mathbf{A}, \mathbf{B}, \mathbf{C}$ and $i, j \in G$.

The multi-trace integrand $\mathcal{I}_n(\beta_1, \dots, \beta_m | G)$ in eq. (2.13) contains the reduced Pfaffian of the $2(m + |G|) \times 2(m + |G|)$ antisymmetric matrix Π , which can also be written in a block form,

$$\Pi(\tau_1, \tau_2, \dots, \tau_m | G) = \begin{pmatrix} \mathbf{A}_G & \mathbf{A}_{G,\text{tr}} & -(\mathbf{C}_G)^T & -(\mathbf{C}_{\text{tr},G})^T \\ \mathbf{A}_{\text{tr},G} & \mathbf{A}_{\text{tr}} & -(\mathbf{C}_{G,\text{tr}})^T & -(\mathbf{C}_{\text{tr}})^T \\ \mathbf{C}_G & \mathbf{C}_{G,\text{tr}} & \mathbf{B}_G & \mathbf{B}_{G,\text{tr}} \\ \mathbf{C}_{\text{tr},G} & \mathbf{C}_{\text{tr}} & \mathbf{B}_{\text{tr},G} & \mathbf{B}_{\text{tr}} \end{pmatrix}, \quad (\text{A.6})$$

where the blocks are defined as follows,

- The $|G| \times |G|$ matrix $\mathbf{A}_G$, $\mathbf{B}_G$ and $\mathbf{C}_G$ are just those appeared in Ψ_G of eq. (A.5).
- The $m \times |G|$ matrix $\mathbf{A}_{\text{tr},G}$, $\mathbf{B}_{\text{tr},G}$ and $\mathbf{C}_{\text{tr},G}$ are given by

$$(\mathbf{A}_{\text{tr},G})_{ib} = \sum_{c \in \tau_i} \frac{k_c \cdot k_b}{\sigma_{cb}}, \quad (\mathbf{B}_{\text{tr},G})_{ib} = \sum_{c \in \tau_i} \frac{\sigma_c k_c \cdot \epsilon_b}{\sigma_{cb}}, \quad (\mathbf{C}_{\text{tr},G})_{ib} = \sum_{c \in \tau_i} \frac{\sigma_c k_c \cdot k_b}{\sigma_{cb}}. \quad (\text{A.7})$$

- The $|G| \times m$ matrix $A_{G,\text{tr}}$, $B_{G,\text{tr}}$ and $C_{G,\text{tr}}$ are given by

$$A_{G,\text{tr}} = -(A_{\text{tr},G})^T, \quad B_{G,\text{tr}} = -(B_{\text{tr},G})^T, \quad (C_{G,\text{tr}})_{ai} = \sum_{c \in \tau_i} \frac{\epsilon_a \cdot k_c}{\sigma_{ac}}. \quad (\text{A.8})$$

- The $m \times m$ matrix A_{tr} , B_{tr} and C_{tr} are given by

$$(A_{\text{tr}})_{ij} = \sum_{\substack{c \in \tau_i \\ d \in \tau_j}} \frac{k_c \cdot k_d}{\sigma_{cd}}, \quad (B_{\text{tr}})_{ij} = \sum_{\substack{c \in \tau_i \\ d \in \tau_j}} \frac{\sigma_c \sigma_d k_c \cdot k_d}{\sigma_{cd}}, \quad (C_{\text{tr}})_{ij} = \sum_{\substack{c \in \tau_i \\ d \in \tau_j}} \frac{\sigma_c k_c \cdot k_d}{\sigma_{cd}},$$

$$(A_{\text{tr}})_{ii} = 0, \quad (B_{\text{tr}})_{ii} = 0, \quad (C_{\text{tr}})_{ii} = \frac{1}{2}(k_{\tau_i})^2, \quad (\text{A.9})$$

where $k_{\tau_i} = \sum_{c \in \tau_i} k_c$ is the total momentum of the trace τ_i .

The reduced Pfaffian $\text{Pf}'(\Pi)$ is defined as

$$\begin{aligned} \text{Pf}'(\Pi) &= (-1)^{i+j} \text{Pf} \left(\Pi_{|G|+i,2|G|+m+j}^{[G|+i,2|G|+m+j]} \right) & 1 \leq i, j \leq m \\ &= (-1)^{|G|+m-1} \frac{(-1)^{a+b}}{\sigma_{ab}} \text{Pf}(\Pi_{ab}^{ab}) & a, b \in G \text{ and } a < b. \end{aligned} \quad (\text{A.10})$$

The two expressions are equivalent on the support of the scattering equations when both gluons and traces are present, while the first line applies for the pure scalar case and the second line applies for the pure gluon case. For these two special cases, one can then easily check that eq. (2.15) holds.

To explicitly calculate amplitudes from the CHY formalism, one needs to perform the moduli space integral in eq. (2.1). It turns the delta functions into a Jacobian and results in a summation over the $(n-3)!$ solutions of the scattering equations. Starting with the gauge fixing (2.3), we get

$$A_n(1, 2, \dots, n) = \mathcal{N} \sum_{\text{solutions}} \frac{(\sigma_{1,n-1} \sigma_{n-1,n} \sigma_{n,1})^2}{\det(\Phi_{1,n-1,n}^{1,n-1,n})} \text{PT}(1, 2, \dots, n) \mathcal{I}_n(\beta_1, \dots, \beta_m | G), \quad (\text{A.11})$$

where the matrix Φ is given by

$$\Phi_{ij} = \frac{k_i \cdot k_j}{\sigma_{ij}^2}, \quad \Phi_{ii} = - \sum_{\substack{j=1 \\ j \neq i}}^n \Phi_{ij}. \quad (\text{A.12})$$

To obtain the Jacobian $\det(\Phi_{1,n-1,n}^{1,n-1,n})$, one needs to delete three rows and columns from Φ following the gauge fixing. The normalization $\mathcal{N}$ is given by

$$\mathcal{N} = (-2i) \left(\frac{g}{\sqrt{2}} \right)^{|G|+2m-2} \left(-\frac{\lambda}{4} \right)^{|S|-2m} (-1)^m, \quad (\text{A.13})$$

where $|S| = n - |G|$ is the total number of scalars. This normalization is fixed by comparing with the color-ordered amplitudes from the YMS Lagrangian [57]

$$\mathcal{L} = -\frac{1}{4}(F_{\mu\nu}^a)^2 + \frac{1}{2}(D_\mu \phi^{aA})^2 + \frac{\lambda}{3!} f^{abc} \hat{f}^{ABC} \phi^{aA} \phi^{bB} \phi^{cC} - \frac{g^2}{4} f^{ace} f^{ebd} \phi^{aA} \phi^{bA} \phi^{cB} \phi^{dB}, \quad (\text{A.14})$$

with the gauge coupling g and ϕ^3 coupling λ . The scalar ϕ^{aA} is in the bi-adjoint representation of the gauge group and another global symmetry group, whose structure constants are f^{abc} and $\hat{f}^{ABC}$ respectively. In particular, the normalization for the pure gluon cases is

$$\mathcal{N} = (-2i) \left(\frac{g}{\sqrt{2}} \right)^{n-2}, \quad (\text{A.15})$$

which is shared by the two-fermion integrand (2.7) since the two amplitudes are related by a SUSY Ward identity. The outstanding factor 2 can be absorbed into the definition of the reduced Pfaffian, as pointed out in the original CHY paper [25], while the rest simply originate from rescaling color factors. More specifically, we define partial amplitudes in the color trace bases with the normalization $\text{Tr}(T^a T^b) = \delta^{ab}$ and $\text{Tr}(T^A T^B) = \delta^{AB}$. Both traces are taken in the fundamental representation of the gauge group.

B MATHEMATICA package

This appendix gives a more in-depth description for the functions provided by our MATHEMATICA package, `IncreasingTrees.m`. The package can be found by following this link to our GitLab:

IncreasingTrees GitLab.¹¹

There are no external dependencies for the package, although it does rely on the `Graph` functionality of MATHEMATICA and thus may depend on the version in use. We have tested the package on MATHEMATICA versions 11.2 and 12.0.

The package defines a set of objects from which it builds the kinematic numerators, as well as defining construction routines to build the various numerators. It localizes all variables within either its own context `IncreasingTrees`` or its private context `IncreasingTrees`Private``, and thus is generally safe to load along other packages. However, it does provide definitions for the single-character variables `k`, `e`, `W`, `d`, `F`, which may shadow definitions from other packages.

B.1 Constituent objects

The following objects are used to represent the numerators:

- `k[i__]`: Momentum vector for particle(s) (`i`). Multi-label `k` are automatically converted to sums: `k[1,2] = k1 + k2`.
- `e[i_]`: Polarization vector for particle (`i`).
- `d[x_,y_]`: Lorentz dot product between two vectors. Orderless, bilinear, and enforces transversality and onshellness of `k` and `e`.
- `W[___]`: Generic baseline factor. Can be swapped out directly using replacement rules for `WScalar`, `WGluon`, `WFermion2`, which are all described in section 2. `WL` and `WR` are additionally defined to allow for differentiating theories in the double copy.

¹¹<https://gitlab.com/aedison/increasingtrees>.

- `WFermion1[i_][__]`: Baseline factor for the $\left(-\frac{1}{2}, -\frac{1}{2}\right)$ ghost picture fermions in eq. (2.8a). (*i*) is the gluon in ghost picture -1 . For example, replacing `W→WFermion1[2]` in a numerator will apply this baseline with gluon 2 in the alternate ghost picture.
- `spinChain[__]`: Abstract spin index contraction generated by `WFermions`.
- `χ[i_]`: Fermionic wavefunction placeholder for particle (*i*).
- `ξ[i_]`: Ghost picture $-\frac{3}{2}$ fermionic wavefunction for particle (*i*).
- `F[i_]`: Linearized gluon field-strength. Only appears explicitly in the fermion baseline functions eqs. (2.8a) and (2.8b). We give explicit γ contractions in eq. (2.9).
- `PT[__]`: Abstract Parke-Taylor factor used as part of representing (half-)integrands.
- `cf[__]`: Abstract color factor for color-dressed amplitude construction, following eq. (1.2).

B.2 Construction functions

The package exposes the following functions used in assembling half-ladder numerators:

- `increasingTrees[labels_List]`: Constructs all ordered trees where the vertex labels are taken from (*labels*) and the ordering is consistent with the ordering of (*labels*). Thus, each tree spans the vertices and is only composed of `DirectedEdges` of the form (*labels*)[[*j*]] $\rightarrow$ (*labels*)[[*i*]] with $i < j$.
- `itNumer[g_Graph, maxV_]`: Processes (*g*) to extract the baseline information and runs the recursive construction (detailed in section 3.2) starting from each vertex along the baseline to generate a kinematic polynomial. (*maxV*) specifies which vertex should be treated as the end of the baseline.
- `ptTreeNumer[points_Integer, onlyFuncQ_:False]`: This function has a different return scheme based on whether (*onlyFuncQ*) is `True` or `False`:
 - `True`: generates a `Function` with (*points*) slots that in turn produces a half-ladder numerator with the specified external labels.
 - `False`(default): generates *all* half-ladder numerators with (*points*) number of external particles, dressed with appropriate PT factors. Explicitly uses the crossing symmetry of labels $(2, \dots, (\text{points}) - 1)$.
- `multiTraceNumer[labels_List, traces_List]`: Computes the kinematic numerator for a mix of scalars and gluons carrying labels specified and ordered in (*labels*), grouped into scalar traces specified by (*traces*). Handles the construction and insertion of the baseline factor directly. Traces should be given as a `List of Lists`, where each sublist specifies a color-ordered trace of scalar particles. Gluons can either be specified as length 1 sublists in (*traces*), or omitted from the trace specification entirely. Expects each particle to be labeled the same way in (*labels*) and (*traces*).

- `multiTraceIntegrand[points_Integer, traces_List]`: Computes the explicit permutation sum of `multiTraceNumer` over the permutations of labels ranging from $(2, \dots, (\text{points})-1)$, using the trace configurations given in `(traces)`, and dressed with appropriate PT factors. The particle labels in `(traces)` should be given as integers $1, \dots, (\text{points})$.

Unfortunately, since the symmetries of the multi-trace numerators are significantly more nuanced than in the case of only gluons (or gluons with two scalars or two fermions), we don't make direct use of them in `multiTraceIntegrand` like we do in `ptTreeNumer`. As such, while `multiTraceIntegrand` and `ptTreeNumer` produce identical results for pure gluons, we strongly recommend to use `ptTreeNumer` in these situations, especially for larger particle number.

In addition to the half-ladder numerator constructors, we also provide the tools to assemble the numerators into amplitudes. However, these computations run into the $\mathcal{O}((n-2)!)^2$ complexity of calculating the full matrix of $m(\alpha|\beta)$.

- `mab[α_List, β_List]`: Computes the biadjoint scalar amplitude for orderings (α) and (β) . The orderings should be in the same KK basis: $\alpha_1 = \beta_1$ and $\alpha_n = \beta_n$. The momentum k_{α_n} is implicitly removed via momentum conservation. Uses the algorithm proposed by ref. [89].
- `Aamp[labels_List]`: Constructs the Yang-Mills tree amplitude according to eq. (1.2), where the external particle labels are drawn from `(labels)`. Includes the explicit normalization factor eq. (A.13). The baseline factors `W[...]` are left unevaluated to allow for choice of special particles.
- `Aamp[n_Integer]`: Constructs the YM tree amplitude using $1, \dots, n$ as the labels.
- `Mamp[labels_List]`: Constructs the gravity tree amplitude according to eq. (1.3), where the external particle labels are drawn from `(labels)`. The baseline factors for the two copies are left unevaluated, with separate labels `WL` and `WR` to allow for double copies between different types of special particles. No explicit normalization is used.
- `Mamp[n_Integer]`: Constructs the gravity tree amplitude using $1, \dots, n$ as the labels.

It is important to note that *only* `Aamp` includes explicit normalizations. However, we make the YMS normalization accessible via

- `ymsNorm[traces_List]`: Computes the YMS normalization $\mathcal{N}$ from eq. (A.13), with the trace groupings specified as separate `Lists` in `(traces)`. Gluons/fermions are specified via length-one `Lists`. `ymCoup` and `scCoup` are used to represent the SYM coupling g and scalar coupling λ , respectively.

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