Compatible cycles and CHY integrals

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ABSTRACT: The CHY construction naturally associates a vector in $\mathbb{R}^{(n-3)!}$ to every 2-regular graph with n vertices. Partial amplitudes in the biadjoint scalar theory are given by the inner product of vectors associated with a pair of cycles. In this work we study the problem of extending the computation to pairs of arbitrary 2-regular graphs. This requires the construction of *compatible* cycles, i.e. cycles such that their union with a 2-regular graph admits a Hamiltonian decomposition. We prove that there are at least $(n-2)!/4$ such cycles for any 2-regular graph. We also find a connection to breakpoint graphs when the initial 2-regular graph only has double edges. We end with a comparison of the lower bound on the number of randomly selected cycles needed to generate a basis of $\mathbb{R}^{(n-3)!}$, using the super Catalan numbers and our lower bound for compatible cycles.

KEYWORDS: Scattering Amplitudes, Field Theories in Higher Dimensions

Contents

1 Introduction

Scattering amplitudes of massless particles are very constrained by physical requirements such as locality and unitarity (see e.g. [1, 2]). In 2013, He, Yuan and one of the authors, introduced the CHY formalism which encodes locality and unitarity into the structure of the moduli space of punctured Riemann spheres [3–5]. The CHY formula has become a powerful tool for producing amplitudes of a variety of theories, including gravity, in arbitrary dimensions [6–10]. Moreover, it leads to ways of combining amplitudes of two theories to produce new ones [5] generalizing the Kaway-Lewellen-Tye (KLT) construction [11] discovered in the 80's. The key ingredient in the CHY reformulation of KLT-like relations is the set of amplitudes of a cubic scalar theory with $\mathrm{U}(N) \times \mathrm{U}(\tilde{N})$ flavor group. The Lagrangian of the theory is given by

$$\mathcal{L} = \partial_\mu \Phi_{a\tilde{a}} \partial^\mu \Phi^{a\tilde{a}} + g f^{abc} \tilde{f}^{\tilde{a}\tilde{b}\tilde{c}} \Phi_{a\tilde{a}} \Phi_{b\tilde{b}} \Phi_{c\tilde{c}} \tag{1.1}$$

where f^{abc} and $\tilde{f}^{\tilde{a}\tilde{b}\tilde{c}}$ are the structure constants of the flavor group [5].

It is well-known that scattering amplitudes of n particles in the adjoint representation of a unitary group can be decomposed into partial amplitudes labeled by a cycle, i.e., a connected 2-regular graph on n vertices [12–14]. The theory defined by (1.1) has two unitary groups and therefore its partial amplitudes are labeled by two cycles C_α and C_β on n vertices and usually denoted by $m_n(\alpha|\beta)$. Here we choose to make the dependence on the cycles explicit when necessary by writing $m_n(C_\alpha|C_\beta)$.

We attempt to use graph language in a way that is broadly both consistent with graph theory and previous work in the CHY formalism, as will be summarized at the end of the

introduction. The CHY formulation starts by defining a map from the set of 2-regular loopless graphs, including multigraphs, to an $(n-3)!$-dimensional real vector space

$$\phi : \mathcal{G}^{\text{2-reg}} \to \mathbb{R}^{(n-3)!}. \tag{1.2}$$

We will refer to graphs in the set $\mathcal{G}^{\text{2-reg}}$ simply as 2-regular graphs. The map has the following crucial property: given any pair of 2-regular graphs, G_1 and G_2, on the same vertex set, the inner product $\phi(G_1)\cdot\phi(G_2)$ only depends on the 4-regular graph obtained by the edge-disjoint union $G_1 \cup G_2$. More explicitly, if $G_1 \cup G_2$ admits a different decomposition in terms of a pair of 2-regular graphs, i.e., $G_1 \cup G_2 = G_3 \cup G_4$ then $\phi(G_1) \cdot \phi(G_2) = \phi(G_3) \cdot \phi(G_4)$. The amplitudes of the biadjoint theory are then given by $m_n(C_\alpha|C_\beta) = \phi(C_\alpha) \cdot \phi(C_\beta)$.

In [15], Gomez and one of the authors noticed that the natural extension $m_n(G_1|G_2) = \phi(G_1) \cdot \phi(G_2)$ can be expressed completely in terms of $m_n(\alpha|\beta)$ if a certain condition is satisfied.

In order to state the condition a definition is needed.

Definition 1.1. *Given a 2-regular graph G, a* compatible cycle *to G is a cycle C such that the 4-regular graph obtained by the edge-disjoint union $G \cup C$ admits a hamiltonian decomposition, i.e., $G \cup C = C_1 \cup C_2$ where C_1 and C_2 are both cycles on the same vertex set as G.*

See the end of the section for the general definition of a hamiltonian decomposition. The construction of $m_n(G_1|G_2)$ in terms of $m_n(C_\alpha|C_\beta)$ requires solving the following:

Problem 1.2. *Given a 2-regular graph G on n vertices, find at least $(n-3)!$ compatible cycles such that under ϕ they form a basis of $\mathbb{R}^{(n-3)!}$.*

The reason is that if such a basis is found then the vector $\phi(G)$ can be expanded in terms of any basis of cycles, already known to exist, but with coefficients which can be computed entirely in terms of $m_n(C_\alpha|C_\beta)$, by using $\phi(G) \cdot \phi(C)$ with C compatible to G to produce linear equations for the coefficients. In section 2 we provide details on this construction.

In this work we study the combinatorial part of the problem and prove the following theorem.

Theorem 1.3. *Given a 2-regular graph G on n vertices, there are at least $(n-2)!/4$ compatible cycles for G. In the case that G has only even cycles then there are at least $(n-2)!/2$ compatible cycles for G.*

The proof is constructive and provides an algorithm for finding the compatible cycles. Note that $(n-2)!/4 \geq (n-3)!$ for $n \geq 6$ and so while we do not solve problem 1.2 as we do not have a combinatorial handle on the linear independence, the number of compatible cycles is favorable. For $n < 6$ there are also many compatible cycles as computed exactly by one of us with Gomez in [15]; in particular the explicit computation gives a basis of $\mathbb{R}^{(n-3)!}$ for all $n \leq 6$ cases.

Another reason to be optimistic about the future resolution of the linear independence problem is the work of Bjerrum-Bohr, Bourjaily, Damgaard, and Feng, [16], in which monodromy relations expressed in terms of cross ratios were used to find an algorithm for the expansion of $\phi(G)$ in term of a basis of cycles, although the coefficients are not manifestly given in terms of $m_n(C_\alpha|C_\beta)$. We give more details on their construction in section 2.

The paper starts in section 2 with a brief review of the Feynman diagram definition of $m_n(\alpha|\beta)$ and the formula for defining $m_n(G_1|G_2)$ which uses the compatible cycles. This section can be skipped in a first reading of the paper in case the reader is only interested in the proof of the result for 2-regular graphs. In section 3, we provide a simple construction which not only gives a lower bound for the number of compatible cycles which is larger than $(n-3)!$ but also an algorithm to find them. In section 4 we establish a connection to breakpoint graphs. We end in section 5 with a short discussion on the issue of finding a basis of $\mathbb{R}^{(n-3)!}$ by using super Catalan numbers to give a lower bound on the number of randomly selected cycles needed to generate a basis of $\mathbb{R}^{(n-3)!}$. This counting indicates that the larger the n the harder it is to find a linear independence basis. We discuss some modifications to the original algorithm of [15] and give an outlook with future directions.

1.1 Review of graph theory terminology

We end the introduction with a short review of graph theory terminology. Readers are encourage to skip this in a first reading and only use it if needed.

A graph is loopless if it has no edge with both ends at the same vertex.

For us graphs may have multiple edges (hence being multigraphs in the usual graph theoretic sense), but must be loopless.

Definition 1.4. *A graph is k-regular if all vertices have degree k, that is, have k edges ending on them.*

We are particularly interested in 2-regular graphs, which are simply a collection of cycles.

As used above given two graphs G_1 and G_2 on the same vertex set we will write $G_1 \cup G_2$ for the graph whose edges are the disjoint union of the edges of G_1 and the edges of G_2. In particular if the same edge appears in G_1 and G_2 then that edge will be a double edge in $G_1 \cup G_2$.

Definition 1.5. *A hamiltonian cycle in a graph G is a subgraph of G which is a cycle and which uses each vertex of G exactly once.*

Given a 2k-regular graph G, a hamiltonian decomposition of G, when it exists, is a decomposition of the edges of G into k disjoint hamiltonian cycles: $G = C_1 \cup C_2 \cup \cdots \cup C_k$ with each C_j a cycle on the same vertex set as G.

For the main argument we also need the notion of a perfect matching.

Definition 1.6. *A matching in a graph G is a 1-regular subgraph, that is, a subset of edges of the graph where no two edges of the subset share a vertex.*

A perfect matching in a graph G is a matching that uses all vertices of the graph. We will also use the notion of perfect matching on a vertex set (without the requirement of being a subgraph of some G), meaning simply a 1-regular graph on that vertex set.

Given a perfect matching M in a graph G, and a vertex v of G, the M-neighbour of v is the vertex connected to v by an edge of M.

For more graph theory background the reader is referred to [17] or [18].

2 Biadjoint scalar amplitudes and extension to general 2-regular graphs

In this work we are interested in tree-level scattering amplitudes of a quantum field theory of massless scalars interacting via cubic couplings controlled by the structure constants of the algebra of $\mathrm{U}(N) \times \mathrm{U}(\tilde{N})$. The lagrangian presented in (1.1) produces Feynman diagrams which can be decomposed according to the algebra structure leading to what is known as a color decomposition of amplitudes into partial amplitudes. Consider the scattering of n particles carrying $\mathrm{U}(N) \times \mathrm{U}(\tilde{N})$ labels $\{a_1, \tilde{a}_1\}, \{a_2, \tilde{a}_2\}, \ldots, \{a_n, \tilde{a}_n\}$, then the amplitude can be written as

$$A_n(\{a_i, \tilde{a}_i\}) = \sum_{\alpha,\beta \in S_n/\mathbb{Z}_n} \mathrm{Tr}\left(T^{a_{\alpha(1)}} T^{a_{\alpha(2)}} \cdots T^{a_{\alpha(n)}}\right) \mathrm{Tr}(\tilde{T}^{\tilde{a}_{\beta(1)}} \tilde{T}^{\tilde{a}_{\beta(2)}} \cdots \tilde{T}^{\tilde{a}_{\beta(n)}})\, m_n(\alpha|\beta).$$
(2.1)

Here T^a and $\tilde{T}^{\tilde{a}}$ are the generators of the Lie algebra of $\mathrm{U}(N)$ and $\mathrm{U}(\tilde{N})$ respectively, i.e., they form a basis of the space of $N \times N$ (or $\tilde{N} \times \tilde{N}$) hermitian matrices.

Each particle carries a momentum vector k_a^μ and $m_n(\alpha|\beta)$ is only a function of Mandelstam invariants $s_{ab} := 2k_a \cdot k_b$. These invariants form a real $n \times n$ symmetric matrix satisfying the following properties

$$s_{aa} = 0 \quad \text{and} \quad \sum_{b=1}^n s_{ab} = 0 \quad \forall\, a \in \{1, 2, \ldots, n\}.$$
(2.2)

The space of kinematic invariants is $n(n-3)/2$ dimensional.

A tree-level Feynman diagram in a cubic scalar theory is defined as a tree with n leaves and $n-2$ trivalent vertices. We will assume our Feynman diagrams are tree-level from here on out. To each Feynman diagram Γ one associates a rational function of s_{ab} as follows. Let E_Γ be the set of edges connecting two trivalent vertices. Removing $e \in E_\Gamma$ divides Γ into two disconnected graphs with a corresponding partition of the leaves into two sets $L_e \cup R_e = \{1, 2, \ldots, n\}$. The conditions (2.2) imply that

$$\sum_{a,b \in L_e} s_{ab} = \sum_{c,d \in R_e} s_{cd}$$
(2.3)

and therefore it is a quantity that can be associated with the edge e.

The rational function associated with Γ is then

$$R_\Gamma(S) := \prod_{e \in E_\Gamma} \left(\sum_{a,b \in L_e} s_{ab}\right)^{-1}.$$
(2.4)

There are trivial factors of 2 generated from the symmetric way the sums in the denominator were defined and can be eliminated if desired.

Any Feynman diagram Γ admits several planar embeddings. A planar embedding is a drawing of Γ on a disk such that no lines cross and all leaves are attached to the boundary of the disk. Since we are working with trees, any given planar embedding is uniquely specified by the (cyclic) ordering of the labels $\{1, 2, \ldots, n\}$ on the boundary of the disk.

There are $(n-1)!$ possible cyclic orderings, i.e. distributions of n labels on the boundary of a disk. However, it is convenient to identify two orderings if they are related by a reflection. This means that there are only $(n-1)!/2$ inequivalent ones. Let $\mathcal{O}$ denote the set of all $(n-1)!/2$ orderings. More precisely,

$$\mathcal{O} := \{\omega \in S_n : \omega(1) = 1,\ \omega(2) < \omega(n)\}. \tag{2.5}$$

The first condition reduces the $n!$ permutations to $(n-1)!$ by using cyclicity to fix 1 while the second condition selects one of the two permutations related by a reflection that fixes 1.

Definition 2.1. *Let $\Omega(\omega)$ be the set of all Feynman diagrams with n leaves that a admit a planar embedding defined by $\omega \in \mathcal{O}$.*

Now we are ready to give a formula for partial amplitudes in terms of Feynman diagrams

$$m_n(\alpha|\beta) := (-1)^{w(\alpha,\beta)} \sum_{\Gamma \in \Omega(\alpha) \bigcap \Omega(\beta)} R_\Gamma(S). \tag{2.6}$$

In this formula the sum is over all Feynman diagrams that admit both a planar embedding defined by α and one defined by β. The overall sign is not is important for the purposes of this work so we refer the reader to [5] for its definition.

In a nutshell, the CHY formulation of $m_n(\alpha|\beta)$ requires finding the critical points of

$$\mathcal{S}(x_1, x_2, \ldots, x_n) := \sum_{1 \le a < b \le n} s_{ab} \log(x_a - x_b). \tag{2.7}$$

There are $(n-3)!$ critical points obtained as solutions to what are known as the scattering equations [3–5]

$$\frac{\partial \mathcal{S}}{\partial x_a} = \sum_{b=1, b \ne a} \frac{s_{ab}}{x_a - x_b} = 0 \quad \forall\, a \in \{1, 2, \ldots, n\}. \tag{2.8}$$

Let's denote the $(n-3)!$ solutions as x_a^I. In general the solutions are complex but when the s_{ab}'s are chosen in what is known as the positive region all solutions are real [19]. Given any cycle C_α, one constructs a vector $\phi(C_\alpha) \in \mathbb{R}^{(n-3)!}$ whose components are given by

$$\phi(C_\alpha)_I := \frac{K_I}{(x_{\alpha_1}^I - x_{\alpha_2}^I)(x_{\alpha_2}^I - x_{\alpha_3}^I) \cdots (x_{\alpha_n}^I - x_{\alpha_1}^I)}, \tag{2.9}$$

where K_I is a function obtained from second derivatives of $\mathcal{S}$ and is invariant under permutations of labels and hence α independent. Therefore K_I is not relevant to our discussion and we refer the reader to [5] for details.

Finally, partial amplitudes are computed as

$$m_n(\alpha|\beta) = \sum_{I=1}^{(n-3)!} \phi(C_\alpha)_I \, \phi(C_\beta)_I. \tag{2.10}$$

We will also use the notation $\phi(C_\alpha) \cdot \phi(C_\beta)$ for the inner product in (2.10).

Now it is clear how to generalize ϕ to a map that assigns to any 2-regular graph a vector in $\mathbb{R}^{(n-3)!}$. Let G be any 2-regular graph with edge set E then

$$\phi(G)_I := K_I \prod_{e \in E} \frac{1}{x^I_{e_i} - x^I_{e_f}}. \tag{2.11}$$

Given any two 2-regular graphs G_1 and G_2 one also defines

$$m_n(G_1|G_2) := \phi(G_1) \cdot \phi(G_2). \tag{2.12}$$

As mentioned in the introduction the map ϕ has the property, which is clear from its definition, that the value of $m_n(G_1|G_2)$ is only a function of the 4-regular graph obtained as the union $G_1 \cup G_2$.

The scattering equations (2.8) are polynomial equations and are difficult to solve for generic values of s_{ab}. This is why it is useful to try and express $m_n(G_1|G_2)$ in terms of $m_n(\alpha|\beta)$, which are known rational functions of s_{ab}. One way to achieve this was proposed by Gomez and one of the authors in [15]. The first step is to choose any basis of $\mathbb{R}^{(n-3)!}$ made out of vectors corresponding to cycles, not necessarily compatible to any G_i. For example, it is known that by fixing the position of three labels and permuting the rest one has $(n-3)!$ cycles that generate a basis (see e.g. [15]). Consider one such sets $\mathcal{A} = \{(\gamma, n-2, n-1, n) : \gamma \in S_{n-3}\}$ and expand $\phi(G_i)$ in the corresponding basis

$$\phi(G_i) = \sum_{\alpha \in \mathcal{A}} c_{i,\alpha} \phi(C_\alpha). \tag{2.13}$$

Now, if a basis $\mathcal{B}_i$ of $\mathbb{R}^{(n-3)!}$ is found using compatible cycles to G_i then it is possible to compute the coefficients $c_{i,\alpha}$ by solving the system of equations

$$\phi(G_i) \cdot \phi(C_\beta) = \sum_{\alpha \in \mathcal{A}} c_{i,\alpha} \, \phi(C_\alpha) \cdot \phi(C_\beta) \tag{2.14}$$

with C_β in $\mathcal{B}_i$. Therefore $\phi(G_i) \cdot \phi(C_\beta) = \phi(C) \cdot \phi(C')$ for some cycles C and C'.

Using (2.13) one finds that

$$m_n(G_1|G_2) = \sum_{\alpha,\beta \in \mathcal{A}} c_{1,\alpha} c_{2,\beta} \, m_n(\alpha, \beta) \tag{2.15}$$

and since all coefficients $c_{i,\alpha}$ are known using (2.14) we have achieved the desired formula.

Let us end this section with a short description of the algorithm from [16] mentioned in the introduction which also achieves an expansion of the form (2.13) with coefficients given in terms of the invariants s_{ab}. The main tool is the monodromy relations expressed

in terms of cross ratios [20]: for any subset $A \subset \{1, 2, \ldots, n\}$ with $2 \leq |A| \leq n - 2$ and for any $a \in A$ and $b \in A^c = \{1, 2, \ldots, n\} \setminus A$,

$$1 = - \sum_{c \in A, d \in A^c} s_{cd} \frac{(x_a - x_c)(x_d - x_b)}{(x_b - x_c)(x_a - x_d)}. \tag{2.16}$$

In [16] the identity (2.16), which holds on the support of the scattering equations, is used to write $\phi(G)$ as a linear combination of 2-regular graphs with less cycles. In other words, (2.16) can be used to fuse cycles. Iterating the procedure until all graphs involved are single cycles gives rise to an expansion of the form (2.13), although the coefficients are not manifestly given in terms of $m_n(C_\alpha | C_\beta)$. It would be interesting to try and find a connection between the two kinds of expansions.

3 Lower bounds on the number of compatible cycles

In the following arguments we will use the notion of perfect matching in a slightly different way than what is typical in graph theory. Given a graph G, when we refer to a perfect matching on $V(G)$, the vertex set of G, we mean any 1-regular graph on the vertices $V(G)$. So this is not a perfect matching *of* G as it is not a subgraph of G. It is simply a perfect matching of the complete graph on $V(G)$, in other words a 1-regular graph on $V(G)$.

Additionally, to reiterate what was mentioned in the introduction, for us graphs are loopless but can have multiple edges. Furthermore, when we take the union of two graphs on the same vertex set, this denotes the disjoint union on the edge sets. That is, if G_1 and G_2 are graphs on the same vertex set V, and both G_1 and G_2 have one edge between v_1 and v_2, $v_1, v_2 \in V$, then $G_1 \cup G_2$ has two edges between v_1 and v_2.

With this in mind we are ready to count compatible cycles. We begin by counting compatible cycles to graphs with only even length cycles.

Theorem 3.1. *Let G be a 2-regular graph on n vertices which consists of only even cycles. There are at least $(n - 2)!/2$ compatible cycles for G.*

For G as in the statement of the theorem, we will fix a decomposition of G as $G = A \cup B$ with A and B perfect matchings on G, so that G consists of AB-alternating cycles. With this decomposition in mind we will prove the theorem with the help of two lemmas, as follows. The first lemma simply counts how many ways there are to complete a perfect matching into a cycle.

Lemma 3.2. *Let A be a perfect matching on n vertices. There are $(n - 2)!!$ perfect matchings P on the same set of vertices such that $P \cup A$ is a cycle.*

Proof. Pick a vertex $v \in V(A)$. Let its neighbour in A be a. Starting from v, there are $n - 2$ choices for a vertex p which is distinct from a and v. Let vp be an edge in P. Now let a' be the neighbour of p in A, and there are $n - 4$ choices for a vertex p' which is distinct from v, a, p, a' which we also add to P. Continuing likewise, we can extend the path $v, a, p, a', p', \ldots$. For a final edge of P take the edge from the end of the path
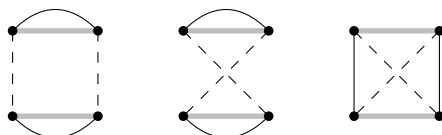
Figure 1. The base case for the lemma. Thick grey edges are B, thin black edges are P, and thin dashed edges are the possibilities for Q.

(which by construction will not yet have an incident P-edge) to v. Then P is a perfect matching, $P \cup A$ is a cycle and there are $(n-2)!!$ choices for P constructed in this manner. Furthermore, all P as in the statement can be constructed in this manner, as the cycle $P \cup A$ determines the choices. $\qquad\square$

Another way to prove the previous lemma is to contract A, pick a cycle on the remaining $n/2$ vertices, and then note that this cycle can be expanded back to the original vertex set to give a P as in the statement in $2^{n/2-1}$ ways, because after inserting the first edge of A into the cycle, each remaining edge of A can be inserted into the cycle in one of two ways. Then since $(n-2)!! = 2^{n/2-1}(n/2-1)!$ for even n we obtain the same result.

Lemma 3.3. *Let B and P be two perfect matchings on the same set of n vertices. Then there are at least $(n-3)!!$ choices for a perfect matching Q on this vertex set with the property that both $Q \cup B$ and $P \cup Q$ are cycles.*

Proof. This proof is the main part of the whole argument for the general result. We proceed by induction. The base case is $n = 4$ and the result follows by checking several cases: either $B = P$ on 4 vertices or $B \cup P$ is a cycle on 4 vertices. Since $n = 4$ we only need to find $1 = (4-3)!!$ perfect matching Q with the desired properties.

In either case, we can simply draw at least one Q no matter the choice of P, as illustrated in figure 1.

For the induction, let V be the vertex set. Pick a vertex v. Label its B-neighbour b and its P neighbour p. Pick a vertex q not in $\{v, b, p\}$ and draw a Q-edge vq. There are at least $n-3$ choices for this q (there may be more than $n-3$ choices as v, b, p may not all be distinct). Given such a choice of q, label its B-neighbour β and its P-neighbour π. From here, we create two matchings on the vertex set $V' = V \setminus \{v, q\}$, namely $B' = (B|_{V'}) \cup \{b\beta\}$ and $P' = (P|_{V'}) \cup \{p\pi\}$. It should be noted that these are indeed matchings, since none of b, p, β, π are saturated in the restrictions of their respective matchings to V', and all of these matchings are also perfect. Now, B' and P' satisfy the induction hypothesis, and so give rise to $(n-2-3)!!$ choices of Q' with the property that both $C'_B := Q' \cup B'$ and $C'_P := P' \cup Q'$ are cycles.

The goal from here is to lift Q' up to the perfect matching $Q := Q' \cup \{vq\}$ on V and show that Q satisfies the lemma. To this end, note that $C'_B \setminus \{b\beta\}$ and $C'_P \setminus \{p\pi\}$ induce paths S_B and S_P on V which hit all of the vertices except v and q. Therefore $S_B \cup \{bv, vq, q\beta\}$ is a cycle consisting of all of the edges of B and Q, and $S_P \cup \{pv, vq, q\pi\}$ is a cycle consisting of all the edges of P and Q. This means Q satisfies the lemma.
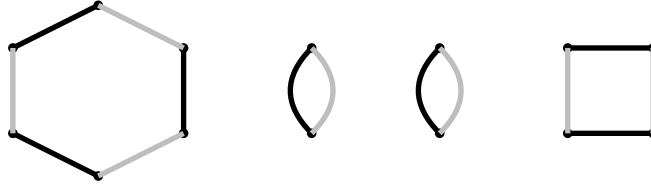
Figure 2. An example graph G with all even cycles decomposed as $A \cup B$ where thick black edges are A and thick grey edges are B.

Now, there were at least $n-3$ choices for the edge vq and at least $(n-5)!!$ choices for the matching Q'. If there is no repetition here, we will have at least $(n-3)!!$ choices for Q and the claim will be proven. To see that there is indeed no repetition, note that two different choices of q cannot lead to the same cycle, and given the same choice of q, the paths S_B and S_P will depend only on the (already distinct) choices of Q'. This completes the proof. $\qquad\square$

Proof of theorem 3.1. Decompose G as $G = A \cup B$ with A and B perfect matchings on G, so that G consists of AB-alternating cycles.

By the first lemma we have $(n-2)!!$ perfect matchings P such that $P \cup A$ is a cycle. For each such P then apply the second lemma to obtain $(n-3)!!$ perfect matchings Q such that $Q \cup B$ and $P \cup Q$ are also cycles.

The compatible cycle thus constructed is $P \cup Q$, but each such compatible cycle can potentially appear twice as either of the two perfect matchings making it up could have been constructed first. The result, then, is at least

$$\frac{(n-2)!!(n-3)!!}{2} = \frac{(n-2)!}{2}$$

compatible cycles as desired. $\qquad\square$

To illustrate how this theorem can be used algorithmically to construct compatible cycles consider the example graph in figure 2. By the first lemma we can construct the perfect matching P by beginning at a vertex, say the upper of the two leftmost vertices in the figure, following A, in this case to the top vertex, and then choosing any vertex other than the two already mentioned to join to the top vertex making an edge for P. Suppose we choose the lower of the two vertices to the right in the same cycle of G. Then we follow A again and pick any vertex not already seen to add a new edge to P and so on. Continuing in this way one possible P we could obtain is as illustrated in figure 3.

Next we follow the second lemma. Beginning again at the upper of the two leftmost vertices, we pick any vertex other than this vertex's neighbours in B and P to make an edge for Q. In this case say we pick the lower vertex of the leftmost bubble. This is illustrated in figure 4.

From this choice of edge the second lemma tells us to construct G' (along with P') as illustrated in figure 5, with vertex labels as in the lemma.
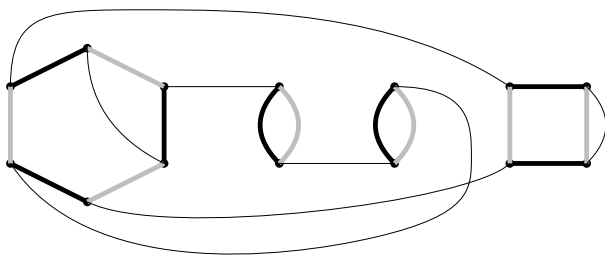
Figure 3. The example graph G along with a perfect matching P (thin black lines) so that $P \cup A$ is a cycle.
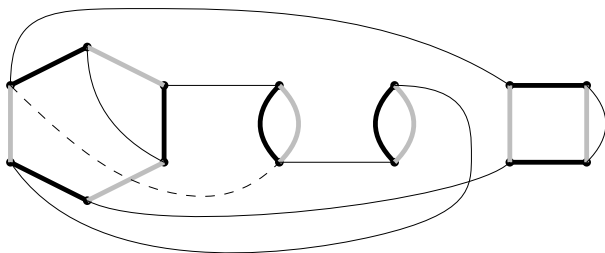

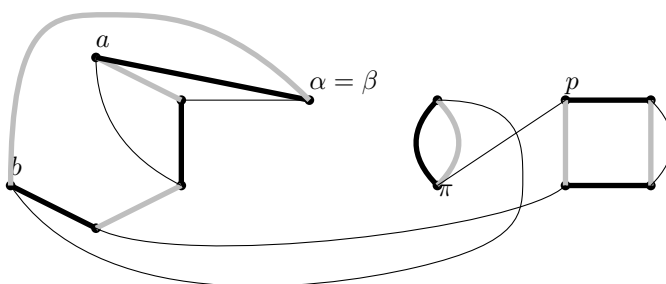**Figure 4**. G and P along with a first edge for the construction of Q (dashed line).

Figure 5. The graph $G' = A' \cup B'$ with P'.
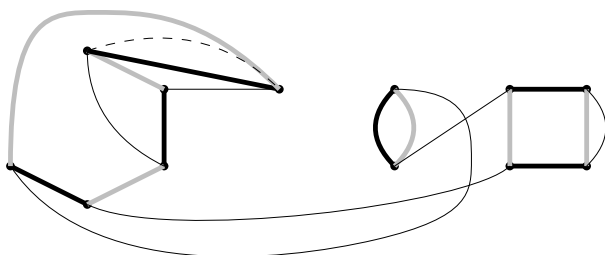

Figure 6. The graph $G' = A' \cup B'$ with P' and a first choice of edge for Q'.
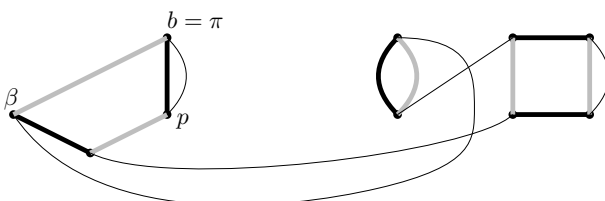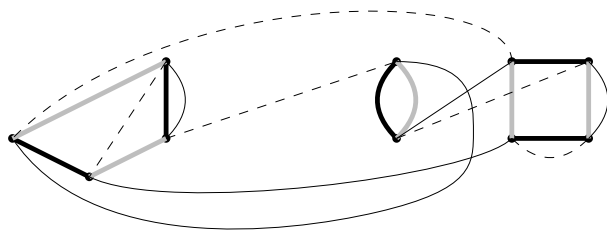

Figure 7. The graph $G'' = A'' \cup B''$ with P''.
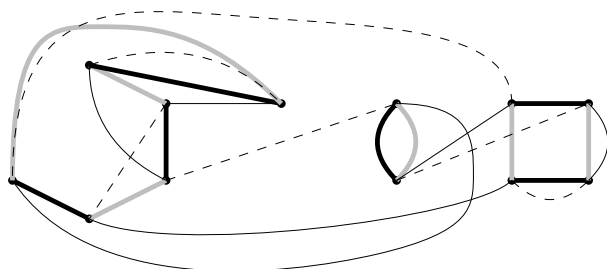
Figure 8. The graph $G'' = A'' \cup B''$ with P'' and Q''.

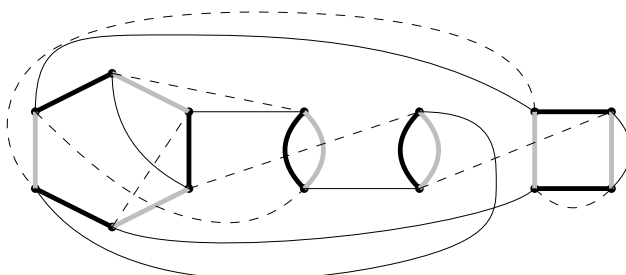**Figure 9**. The graph $G' = A' \cup B'$ with P' and Q'.

Figure 10. The graph $G = A \cup B$ with P and Q. The thick black edges are A, the thick grey edges are B, the thin black edges are P and the dashed edges are Q.

The process now continues. Let's progress one more step explicitly, choosing the first edge of Q' as shown in figure 6. This results in the graph G'' as illustrated in figure 7. Continuing the process we can construct Q''; one possibility for Q'' is illustrated in figure 8

Bringing Q'' up to Q' on G' we obtain the situation illustrated in figure 9, and bringing Q' up to Q on G we obtain our compatible cycle $P \cup Q$ as illustrated in figure 10. Observe that in this last figure, $A \cup P$, $B \cup Q$ and $P \cup Q$ are all cycles as expected.

As this example illustrates, the theorem in fact gives an algorithm to generate at least $(n-2)!/2$ compatible cycles for any 2-regular G with all cycles even.

Theorem 3.4. *For an arbitrary* 2-*regular graph* G, *there are at least* $(n-2)!/4$ *compatible cycles* C *for* G. *In the special case where* G *has only even cycles then there are at least* $(n-2)!/2$ *compatible cycles.*

Proof. If G has only even cycles, then apply the previous result. Now assume G has at least one odd cycle.

Let $O_1, \ldots, O_k$ be the odd cycles of G. Pick a vertex v_i from each O_i. Now we will 'bandage' these cycles at the v_i in the sense that the v_i will be treated just as a point

along the 'single edge' between their neighbours. Formally, define G', the bandaged graph, to be the graph obtained from G by contracting one of the edges incident to v_i for each $1 \leq i \leq k$; we no longer use the vertex labels v_i in G', as we think of the contracted vertices as having come from their other vertex in G while v_i is gone as it has been bandaged up. Additionally let e_i be the edge in G' which came from the non-contracted incident edge to v_i for each $1 \leq i \leq k$. We can then obtain G from G' by in each edge e_i putting a new vertex v_i.

Applying the previous theorem we have $(n-k-2)!/2$ choices for C on G'. Now we must extend C to G. We would like to do this by, for each v_i in turn, picking an edge ww' of C and replacing it with the edges wv_i and v_iw'. As we do so we increase the number of edges in C and so increase the number of ways to continue this process in subsequent steps.

However, not every choice will preserve that C is a compatible cycle. Consider then a compatible cycle C for G' given by the previous theorem. From that construction we have that $G' = A \cup B$ where A and B are perfect matchings (alternating along the cycles of G'), and $C = P \cup Q$ where P and Q are perfect matchings such that $P \cup A$ and $Q \cup B$ are also cycles. Consider now e_1. If $e_1 \in A$ then for any ww' in Q we can add v_1 to e_1, letting both of the resulting edges be in A, and we can replace ww' in Q by wv_1 and v_1w'. Then with these changes we still have $P \cup Q$, $P \cup A$ and $Q \cup B$ cycles. Note that if $e_1 \in A$ but ww' were in P then the same construction would result in $P \cup A$ not being a cycle. However, $e_1 \in B$ and $ww' \in P$ also results in $P \cup Q$, $P \cup A$ and $P \cup B$ remaining cycles. Consequently we have $(n-k)/2$ choices for ww'.

Continuing with $e_2, e_3, \ldots$, the argument above did not require that A, B, P, Q were matchings, and so whenever $e_i \in A$ we take $ww' \in Q$ and whenever $e_i \in B$ we take $ww' \in P$. Also, whenever $ww' \in Q$ then Q has one more edge after that step of the construction and whenever $ww' \in P$ then P has one more edge after that step of the construction. All together, then, we have

$$\underbrace{\left(\frac{n-k}{2}\right)\left(\frac{n-k}{2}+1\right)\left(\frac{n-k}{2}+2\right)\cdots}_{\text{as many factors as } e_i \in A} \underbrace{\left(\frac{n-k}{2}\right)\left(\frac{n-k}{2}+1\right)\left(\frac{n-k}{2}+2\right)\cdots}_{\text{as many factors as } e_i \in B}$$

choices to extend C to a compatible cycle on G. The expression above is bounded below by

$$\frac{1}{2^k}\underbrace{(n-k)(n-k)(n-k+2)(n-k+2)\cdots}_{k \text{ times}}$$

$$\geq \frac{1}{2^k}(n-k-1)(n-k)(n-k+1)(n-k+2)\cdots(n-2)$$

Combining this with the number of choices for C we get a total of at least

$$\frac{1}{2^k}(n-k-1)(n-k)(n-k+1)(n-k+2)\cdots(n-2)\frac{(n-k-2)!}{2} = \frac{1}{2^{k+1}}(n-2)!$$

compatible cycles for G.

To take care of the powers of 2, we need to more closely analyze the freedom we had in the initial choices. Keeping the v_i fixed, note that in the initial choice of decomposition

of G' into A and B, each v_i is either in A or in B. Let us fix a choice of A and B for G' with $e_1 \in A$. Suppose we have a compatible cycle C for G constructed as above based on this choice of A and B. Then the edges of C alternate between P and Q except at the v_i where two edges in the same set occur consecutively. Since we know $e_1 \in A$ in G', the construction above gives that v_1 is between two Q edges in C. Following the alternation of edges around C, starting with the Q-edges around v_1 we can determine for each v_i whether it is surrounded by P-edges or Q-edges. If a v_i is surrounded by P-edges in C then e_i is a B edge in G' and if v_i is surrounded by Q edges in C then e_i is an A edge in G'.

The argument of the previous paragraph implies that knowing a compatible cycle C constructed as described above and knowing that $e_1 \in A$ in G' is enough to determine which e_i are in A and which in B as edges in G'. However, which e_i are in A and which are in B comes from our initial choice of decomposition of G' into A and B. Consequently, different choices of how the e_i are assigned to A and B must give different compatible cycles C. Since the argument required us to fix $e_1 \in A$, it remains to choose which of A or B for the e_i for $2 \leq i \leq k$. That is, there remain $k-1$ binary choices.

Together with the construction given above for C, this means that we obtain a total of at least

$$\frac{2^{k-1}}{2^{k+1}}(n-2)! = \frac{(n-2)!}{4}$$

compatible cycles for G. $\qquad\square$

4 Connection to breakpoint graphs

Counting compatible cycles is closely related to counting breakpoint graphs, which are certain graphs used in studying genomic rearrangements. We will not need the definition of a breakpoint graph here (originally due to Bafna and Pevzner [21]), rather we consider the set up of Grusea and Labarre [22] which contains a reformualtion of the notion of breakpoint graph that already puts the problem closer to compatible cycle enumeration.

We need the following

Definition 4.1. • *([22] definition 5.2) Given vertices $\{0, 1, \ldots, 2m, 2m+1\}$, a configuration is the union of two perfect matchings on those vertices, δ_B and δ_G where $\delta_G = \{\{2i, 2i+1\} : 0 \leq i \leq m\}$*

  • *([22] definition 5.3) Given a configuration $\delta_B \cup \delta_G$, write $\overline{\delta}_G$ for the perfect matching $\overline{\delta}_G = \{\{2i-1, 2i\} : 1 \leq i \leq m\} \cup \{2m+1, 0\}$ and let the complement of the configuration $\delta_B \cup \delta_G$ be $\delta_B \cup \overline{\delta}_G$.*

  • *([22] definition 3.1) The signed Hultman number $\mathcal{S}_H^{\pm}(m, k)$ is the number of signed permutations on m elements whose breakpoint graph consists of k disjoint cycles.*

Then in place of the definition of a breakpoint graph for a signed permutation, we can use the following lemma.

Lemma 4.2 ([22] lemma 5.1). *A configuration $\delta_B \cup \delta_G$ is the breakpoint graph of a signed permutation π if and only if $\delta_B \cup \overline{\delta}_G$ is a cycle.*

We also don't need the definition of a signed permutation, but merely the observations from [22] that a signed permutation on m elements leads to a breakpoint graph on $\{0, 1, \ldots, 2m, 2m+1\}$, and the the map between signed permutations and breakpoint graphs is bijective.

With this set-up, consider the all-bubbles case of our problem from the previous sections. That is suppose G consists of $n/2$ double edges which are vertex disjoint. Label the vertices of G as $\{0, 1, \ldots, 2m, 2m+1\}$ (where $m = (n-2)/2$) so that the double edges of G run between $2i-1$ and $2i$ for $1 \leq i \leq m$ and between $2m+1$ and 0. Then with notation as above $G = \overline{\delta}_G \cup \overline{\delta}_G$, where as usual the union denotes an edge-disjoint union, so taking the union of two copies of $\overline{\delta}_G$ gives double edges.

Now take any breakpoint graph with one cycle and call it C. By Grusea and Labarre's lemma C can be written as $\delta_B \cup \delta_G$ and $\delta_B \cup \overline{\delta}_G$ is a cycle. In fact C is a compatible cycle for G. To see this note that we have $\delta_B \cup \overline{\delta}_G$ and $\delta_G \cup \overline{\delta}_G$ are both cycles, as is $\delta_B \cup \delta_G$ since we took a breakpoint graph with one cycle.

Furthermore, all compatible cycles when G consists of only bubbles can be obtained in this way because Grusea and Labarre's lemma says that a breakpoint graph with one cycle is exactly a graph where the above unions of matchings are cycles.

According to Grusea and Labarre's results the number of such breakpoint graphs is $\mathcal{S}_H^{\pm}(n/2-1, 1)$. Finally, we need to consider how many different labellings of G would result in different families of breakpoint graphs. This is asking, given $\overline{\delta}_G$ how many different δ_G could it correspond to. This is exactly the problem solved in Lemma 3.2, so there are $(n-2)!!$ choices. As in the compatible cycle construction, this counts each compatible cycle twice since either of the two matchings making it up could be δ_G. All together this tells us that the number of compatible cycles to a graph G consisting of $n/2$ bubbles is

$$\frac{1}{2}(n-2)!! \, \mathcal{S}_H^{\pm}(n/2-1, 1). \tag{4.1}$$

In [15] this formula was guessed based on explicit computations of initial terms along with the sequence A001171 in the OEIS [23], but now, by the above, it is proven.

Note that this is better than our results of the previous section for the all bubbles case because it gives an exact enumeration. For more general 2-regular G with only even cycles there remains a connection to breakpoint graph enumeration, but it does not capture all possible compatible cycles.

To explore this more general situation, let G be a 2-regular graph on n vertices with k cycles, all of even length. Fix a decomposition of G into two matchings. By Grusea and Labarre's lemma, labelling G so that it is a breakpoint graph is equivalent to finding a third matching which gives a cycle when combined with either matching from G. By Lemma 3.3 there are at least $(n-3)!!$ ways to do this. Fix a labelling of G so that G is a breakpoint graph, and write $G = \delta_B \cup \delta_G$. Now consider any breakpoint graph H with one cycle (relative to the same labelling). Then $H = \delta_{B'} \cup \delta_G$ and $\delta_{B'} \cup \overline{\delta}_G$ is a cycle. Furthermore $\delta_B \cup \overline{\delta}_G$ is a cycle since G is a breakpoint graph and $\delta_{B'} \cup \delta_G$ is a cycle since H was chosen to have only one cycle. So $C = \delta_{B'} \cup \overline{\delta}_G$ is a compatible cycle for G.

Not all compatible cycles for G arise by breakpoint graphs. However, all the ones constructed by the techniques of the previous section do arise from breakpoint graphs. Despite this, we do not obtain an improved bound from Grusea and Labarre's results on signed Hultman numbers because when working with a fixed G, we are fixing not just k, the number of cycles, but also the lengths of each cycle. This suggests enumerating a refined version of the signed Hultman numbers which keeps track of the cycle structure rather than just the number of cycles. This could be interesting as combinatorics, and might yield better bounds on compatible cycles, or perhaps applications in breakpoint graphs.

4.1 Lower bound vs. exact count

A natural question is to get an approximate notion of how close our bound is to the actual number of compatible cycles. While finding the exact number seems to be a difficult problem, our lower bound was obtained by using very simple constructions. Luckily, returning to the all bubble case for the moment, a formula for computing the number of breakpoint graphs with one cycle is given in [22] and therefore we can use it to compare it to our lower bound. Let $s = n/2$ be the number of double edges, i.e., bubbles. The formula for the number of breakpoint graphs $\mathcal{S}_H^{\pm}(s-1, 1)$ given by

$$\mathcal{S}_H^{\pm}(s-1, 1) = \frac{2^{(3s-2)}s!(s-1)!^2}{(2s)!} + \sum_{a=1}^{s-1}(-1)^{s+1}s\sum_{b=1}^{\min(a,s-a)}(-1)^{a-b}T_{a,b,s} \qquad (4.2)$$

where

$$T_{a,b,s} := \frac{2^{3(a-b)-1}(2a-2b+1)(a-1)!\left((2b)!(a-1)!(s-a-b+1)!\right)^2}{(s^2-(a-b+1)^2)(s^2-(a-b)^2)(s-a-b)!(2a-1)!(b-1)!\left((2b-1)b!\right)^2}. \qquad (4.3)$$

In [15], the asymptotic behavior of $\mathcal{S}_H^{\pm}(n/2-1, 1)$ was numerically studied and found to give the following number of compatible cycles

$$\frac{1}{2}(n-2)!!\mathcal{S}_H^{\pm}(n/2-1, 1) \sim \frac{\pi}{4}n(n-3)!. \qquad (4.4)$$

This means that for the all bubbles case the ratio of the exact count in the asymptotic regime to our lower bound, i.e. $(n-2)!/2$, is only $\pi/2 \sim 1.57$.

5 Discussion

In this work we presented a constructive proof of the existence of $(n-2)!/4$ compatible cycles to any 2-regular graph G. Moreover, when G possesses only even cycles our lower bound becomes $(n-2)!/2$. Our construction has important applications in the computation of CHY integrals, which give rise to the map ϕ as review in section 2. While integrations involving two cycles C_α and C_β compute amplitudes $m_n(\alpha|\beta)$ in a biadjoint scalar theory, more general CHY integrals are known to compute amplitudes in many other theories such as Yang-Mills and Einstein gravity [4]. These more general amplitudes require the integration of functions which are not of the simple form $m_n(\alpha|\beta)$. The more general integrals are associated with arbitrary 2-regular graphs, say G_1 and G_2, and the corresponding integration, $m_n(G_1|G_2)$, has to be performed.

Several techniques have been proposed in the literature for computing CHY integrals as the ones arising from $m_n(G_1|G_2)$. Some of them use the global residue theorem [24], cross ratio identities [25], deformations of the scattering equations [26], etc. The technique relevant to our work expresses $m_n(G_1|G_2)$ directly in terms of the simple objects $m_n(\alpha|\beta)$ and requires finding $(n-3)!$ compatible cycles to G_i such that under the map ϕ they generate a basis of $\mathbb{R}^{(n-3)!}$.

We have concentrated on the combinatorial part of the problem leaving the question of linear independence for the future. However there are some comments that can be made which follow from the Feynman diagram interpretation of $m_n(\alpha|\beta)$ and which show that the problem of independence is non-trivial even though our lower bound shows that for large n the number of compatible cycles to any 2-regular graph is, at least, $n/4$ times larger than the size of the required basis.

5.1 Linear independence

In order to show that the problem is non-trivial, let us consider a given cycle; without loss of generality choose the one defined by the canonical order, $C_{\mathbb{I}}$. We want to determine the total number of cycles such that the corresponding vectors under the map ϕ are orthogonal to $\phi(C_{\mathbb{I}})$. Let us denote the set of such cycles by $\mathsf{O}(\mathbb{I})$. More explicitly,

$$\mathsf{O}(\mathbb{I}) := \{C_\alpha \in \mathcal{O} \,:\, \phi(C_\alpha) \cdot \phi(\mathbb{I}) = 0\}. \tag{5.1}$$

Recall that $\mathcal{O}$, defined in (2.5), denotes the set of all cycles. The reason $\mathsf{O}(\mathbb{I})$ is interesting is that no subset of cycles in $\mathsf{O}(\mathbb{I})$, including the whole set, can possibly give a basis of $\mathbb{R}^{(n-3)!}$. This is clear as they would not be able to generate the vector $\phi(\mathbb{I})$ which is orthogonal to that space.

Let us determine the size of $\mathsf{O}(\mathbb{I})$. Start by recalling that there are $(n-1)!/2$ cycles for n labels and that $m_n(\mathbb{I}|\alpha) = \phi(\mathbb{I}) \cdot \phi(C_\alpha)$ can be computed using Feynman diagrams, (2.6), i.e.

$$m_n(\mathbb{I}|\alpha) := (-1)^{w(\mathbb{I}|\alpha)} \sum_{\Gamma \in \Omega(\mathbb{I}) \bigcap \Omega(\alpha)} R_\Gamma(S). \tag{5.2}$$

It is known that there are C_{n-2} Feynman diagrams which are planar with respect to a given order, where the C_m are the standard Catalan numbers. One way to see this is that there is a bijection between planar cubic Feynman diagrams and triangulations of an n-gon. These can also be thought of as the vertices of an associahedron. Finding the diagrams that are shared by two orderings is equivalent to finding the intersection of two associahedra. The set of all such intersections with the canonical order associahedron corresponds to all possible subdivisions of an n-gon.

Luckily, the number of all such subdivisions is also well-known and it is given by the super Catalan or Schröder–Hipparchus numbers S_n. The first few corresponding to $n = 4, 5, 6, 7, 8$ are $3, 11, 45, 197, 903, 4279$ respectively (see e.g. the sequence A001003 in the OEIS [23]).

Having found the number of cycles that give a non-trivial intersection with the canonical order, the complement, i.e., the number of orthogonal cycles is then given by

$$|\mathsf{O}(\mathbb{I}_n)| = \frac{(n-1)!}{2} - \mathsf{S}_n. \tag{5.3}$$

The number $|\mathsf{O}(\mathbb{I}_n)|$ gives a lower bound on the number of cycles that can be chosen without succeeding to construct a basis of $\mathbb{R}^{(n-3)!}$. Let us see how this compares to $(n-2)!/2$, our lower bound on the number of compatible cycles when G only has even cycles.

Let us consider the asymptotic behavior of the super Catalan numbers,

$$\log(\mathsf{S}_n) \sim n \log\left(3 + \sqrt{8}\right) - \frac{3}{2}\log(n) + \mathcal{O}(n^0). \tag{5.4}$$

This number is very small compared to the total number of cycles $\log((n-1)!/2) \sim n \log(n) - n - 1/2 \log(n) + \mathcal{O}(n^0)$.

Clearly, $|\mathsf{O}(\mathbb{I}_n)| = |\mathsf{O}(\alpha)|$ for any ordering α as can be seen from the definition of the map ϕ reviewed in section 2 and applied to cycles in (2.9). This means that the $(n-1)!/2 \times (n-1)!/2$ matrix $m_n(\alpha|\beta)$ is very sparse when n is large and it is increasingly difficult to find a basis of the space.

This sparsity is even stronger than that expected from the block diagonal shape of the so-called KLT kernel [27]. Consider a basis for α of the form $(1, \omega, n-1, n)$ and one for β of the form $(1, \gamma_1, n, \gamma_2, n-1)$ with $|\gamma_1| - |\gamma_2| = (n+1 \mod 2)$. In this case the $(n-3)! \times (n-3)!$ matrix $\mathcal{S}^{\mathrm{KLT}}_{\alpha,\beta} := m_n^{-1}(\alpha|\beta)$ is known to be block diagonal with blocks of size $d \times d$ with $d = ([n/2] - 1)!([n/2] - 2)!$ if n is even and $d = (([n/2] - 2)!([n/2] - 2)!)$ if n is odd. The blocks are completely solid, i.e., they do not possess any vanishing entries. Of course, the $(n-3)! \times (n-3)!$ matrix $m_n(\alpha|\beta)$ is also block diagonal. However, somewhat unexpectedly each block becomes sparse already for $n \sim 40$. Moreover, the sparsity increases as n does since the ratio of S_n to the size of a single block tends to zero as n grows.

The behavior of the linear relations among the vectors $\phi(C_\alpha)$ as n grows can have important consequences not only for the construction considered in this work but also for the KLT procedure which connects theories such as Yang-Mills and gravity. The study of linear dependencies is an area in mathematics known as matriod theory [28]. The collection of all vectors $\phi(C_\alpha)$ defines a matroid of rank $(n-3)!$ on a ground set of $(n-1)!/2$ elements. For $n = 4, 5$ one has what is known as the uniform matroids $U_{1,3}$ and $U_{2,12}$ respectively. For $n > 5$ the matroids have much more structure. For example, for $n = 6$ we have found that there are $126,820$ bases for the submatroid defined by the 24 orderings $\alpha = (1, 2, \omega)$ with ω a permutation of $\{3, 4, 5, 6\}$ which form what is known as the Kleiss-Kuijf set of orderings [29].

We leave a more in depth study of linear independence, asymptotic structure of the matrix $m_n(\alpha, \beta)$ and the matroids defined by the map ϕ to future work.

5.2 Outlook

According to the numerical data gathered in [15], when the number of vertices n is fixed, graphs with the largest number of cycles always have the smallest number of compatible cycles. When n is even, such graphs are those with $n/2$ cycles and the number of compatible cycles was determined in section 4 from the connection to breakpoint graphs. If the behavior found in [15] is correct, then it is clear that studying 2-regular graphs with only two cycles should be a natural starting point for the construction of a basis of compatible cycles, i.e., a set of $(n-3)!$ linearly independent vectors.

This observation suggests a natural generalization to the proposal of [15] for computing $m_n(G_1|G_2)$ in which the procedure is carried out in steps determined by the number of cycles in G_i.

Start with the set of all 2-regular graphs with only two cycles $\mathcal{G}_{2\,\text{cycles}}^{2\text{-reg}}$ and then compute all possible vectors $\phi(G)$ with $G \in \mathcal{G}_{2\,\text{cycles}}^{2\text{-reg}}$ as a linear combination of vectors $\phi(C_\alpha)$ using their basis of compatible cycles, assuming it exists. Once this is done one can extend the set of compatible cycles to include "compatible graphs" with 2 cycles.

Definition 5.1. *Given a 2-regular graph G, a* compatible graph *to G is a 2-regular graph B with a single or two cycles such that the 4-regular graph obtained by edge-disjoint union $G \cup B$ admits a decomposition of the form $G \cup B = B_1 \cup B_2$ where B_1 and B_2 are both graphs with a single or two cycles.*

This means that the main problem can also be modified accordingly.

Problem 5.2. *Given a 2-regular graph G on n vertices, find at least $(n-3)!$ compatible graphs such that under ϕ they form a basis of $\mathbb{R}^{(n-3)!}$.*

Clearly the set of compatible graphs to a given 2-regular graph is larger than the number of compatible cycles. Therefore, even if finding a set of $(n-3)!$ linearly independent vectors gets harder as n increases, as suggested by the discussion above, one can compensate by enlarging the set to compatible graphs.

This notion can be further extended to recursively include graphs with three, four cycles, etc. It would be very interesting to explore this further and the connection of this more general notion of compatibility with breakpoint graphs with more cycles.

Acknowledgments

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