Limiting shifted homotopy in higher-spin theory and spin-locality

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ABSTRACT: Higher-spin vertices containing up to quintic interactions at the Lagrangian level are explicitly calculated in the one-form sector of the non-linear unfolded higher-spin equations using a $\beta \rightarrow -\infty$ -shifted contracting homotopy introduced in the paper. The problem is solved in a background independent way and for any value of the complex parameter η in the higher-spin equations. All obtained vertices are shown to be spin-local containing a finite number of derivatives in the spinor space for any given set of spins. The vertices proportional to η^2 and $\bar{\eta}^2$ are in addition ultra-local, i.e., zero-forms that enter into the vertex in question are free from the dependence on at least one of the spinor variables y or $\bar{y}$. Also the η^2 and $\bar{\eta}^2$ vertices are shown to vanish on any purely gravitational background hence not contributing to the higher-spin current interactions on AdS_4 . This implies in particular that the gravitational constant in front of the stress tensor is positive being proportional to $\eta\bar{\eta}$. It is shown that the β -shifted homotopy technique developed in this paper can be reinterpreted as the conventional one but in the β -dependent deformed star product.

KEYWORDS: Higher Spin Gravity, Higher Spin Symmetry, Gauge-gravity correspondence

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1 Introduction

In this paper we continue our analysis of non-linear corrections resulting from higher-spin (HS) equations of [1] and their locality properties. In the preceding papers [2, 3] a proper formalism based on the Pfaffian Locality Theorem (PLT) [2] that allows one extracting physical vertices in the local frame was proposed and set in motion. Using it we have checked some low-order vertices and found them local. Here we extend this program to higher orders.

One of the most convenient ways of studying HS interaction problem is by using the unfolded approach [4, 5] that makes HS symmetry manifest. The equations of motion schematically read

$$d_x \omega = -\omega * \omega + \Upsilon(\omega, \omega, C) + \Upsilon(\omega, \omega, C, C) + \dots, \quad (1.1)$$

$$d_x C = -[\omega, C]_* + \Upsilon(\omega, C, C) + \dots, \quad (1.2)$$

where master fields, differential one-form $\omega(Y|x)$ and zero-form $C(Y|x)$, depend on auxiliary spinor variables $Y^A = (y^\alpha, \bar{y}^{\dot{\alpha}})$ that encode HS physical fields along with their on-shell descendants. These span naturally representations of the HS algebra being realized as star-product operation $*$ acting on Y 's. By $\Upsilon(\omega, C, \dots)$ and $\Upsilon(\omega, \omega, C, \dots)$ we schematically denote higher order corrections to the interaction vertices. First vertices on the r.h.s.'s, $-\omega * \omega$ and $-\omega * C$, are fully determined by the global HS symmetry. This allows one extracting all of the remaining Υ 's where the HS symmetry gets deformed from the mere consistency requirement $d_x^2 = 0$. In practice this procedure gets increasingly complicated with the order of perturbation as can be seen for instance from [6]. Yet, at each order one faces cohomological ambiguity in the form of field redefinition that one should dispose one way or another. The freedom in field redefinitions is therefore a natural freedom in the form of Υ -vertices. Algebraically, the process is equivalent to the resolution of the Chevalley-Eilenberg cohomological problem for the HS algebra [7]. In [5] the machinery of Hochschild cohomology was used based on the observation that the HS field equations remain consistent with all fields valued in any associative algebra.

The deformation problem for (1.1), (1.2) was explicitly solved to all orders in [1] where a simple generating system for (1.1), (1.2) was proposed. This system proves that there is no obstruction for system (1.1), (1.2) and allows one extracting HS vertices at any order.

The same time it naturally captures field redefinition ambiguity. To be more specific, the generating equations of [1] are written in a bigger space large enough to include representative freedom for fields ω and C . As shown in [8] and in the forthcoming paper [9], the setup of [1] allows one to distinguish between different star-product functional classes being of great importance in the context of locality problem for HS equations including the analysis of this paper.

As was originally shown in [10–12] (see also [13, 14]) HS vertices contain higher derivatives of the order increasing with spin. HS equations (1.1), (1.2) are based on the infinite-dimensional HS symmetry [15–17] that mixes fields of any spin. The transformation law includes higher derivatives. It is therefore natural to expect some sort of non-locality resided in equations (1.1), (1.2). It has long been known however [10–14] that at least at cubic order (quadratic at the level of equations) interaction of three massless fields is local, i.e., any vertex $s_1 - s_2 - s_3$ contains at most finite number of space-time derivatives. Put it differently, while the number of derivatives in cubic interactions grows with spin and hence is unbounded for the infinite tower of massless fields, still for three given spins their interaction is local. This means that at least at lowest interaction order the structure of non-locality is very specific admitting local decomposition in terms of individual spin vertices. An optimistic point of view is that at any order HS vertex $s_1 - s_2 - \dots - s_n$ is local. More realistic option suggested in [2] and discussed in more detail below is that some sort of spin-locality can occur in the twistor variable space that controls space-time locality via the HS unfolded equations. Whether it is so or not remains to be seen but some indication that HS locality might break down at quartic level was given in [13] using light cone formalism in Minkowski space (see also [18]).

More recently, the holographic reconstruction of HS bulk interactions from free boundary theory in accordance with the original HS holographic conjecture [19, 20] was argued to point out certain non-locality of the quartic vertex [21–23]. Interpretation of these results however is not straightforward as the holographic reconstruction lends itself to a specific choice of field variables and it is not clear if the non-locality so observed is universal or an artifact of the formalism. A related comment is that the proper definition of locality in AdS background with non-commuting derivatives may require identification of a specific ordering prescription.

It is therefore important to study HS locality problem by *AdS/CFT*-independent means directly in the bulk. The only option of this kind available so far is to study nonlinear HS equations of [1]. As demonstrated, in particular, in this paper, our approach works not only for the parity-even HS theories holographically dual to free boundary theories but also for HS theories with the arbitrary complex parameter η conjectured to be holographically dual to the boundary models with Chern-Simons interactions in accordance with [24, 25]. Note that application of the idea of holographic reconstruction to the latter models is far more involved.

Our approach rests on the analysis of HS generating equations [1] that reproduce system (1.1), (1.2) along with its field redefinition freedom. It makes it very convenient to analyze its effect on HS locality. Technically, one or another field representation for r.h.s.'s of (1.1), (1.2) is accounted for by using one or another contracting homotopy operator in

the generating system [1]. From this perspective local (minimally non-local) interaction results from the appropriate choice of contracting homotopy in the generating system. Particularly, in [2] a class of the so-called shifted contracting homotopies¹ was introduced and it was shown to reduce the degree of non-locality in all orders of the theory governed by PLT. Properties of these contracting homotopies were studied in [3] where they were shown to reproduce local lower-order vertex $\Upsilon(\omega, C, C)$ found originally in [26, 27] for AdS_4 provided the PLT conditions are respected. The analysis of [3] alongside revealed an important concept of ultra-locality missing in other approaches. Namely, it was shown that while $\Upsilon(\omega, \omega, C)$ -vertex is always local it can be reduced to the form where it contains no dependence on the generating y or $\bar{y}$ variables in the zero-forms $C(Y|x)$. Such vertices were called ultra-local in [3]. The concept of ultra-locality plays great role in locality in general. Particularly, it can be shown that if one starts with some local but not ultra-local vertex $\Upsilon(\omega, \omega, C)$ then it entails non-locality of $\Upsilon(\omega, C, C)$.

A natural question is whether spin ultra-locality persists at higher order. To answer this question, in this paper we examine next-to-leading order vertex $\Upsilon(\omega, \omega, C, C)$ that captures part of the quintic interaction. The vertex has the following structure

$$\Upsilon(\omega, \omega, C, C) = \Upsilon^{\eta\eta} + \Upsilon^{\bar{\eta}\bar{\eta}} + \Upsilon^{\eta\bar{\eta}}, \quad (1.3)$$

where η is an arbitrary complex parameter of the $d = 4$ HS theory [1, 5] and the labels carried by the vertices refer to the $\eta, \bar{\eta}$ -dependent overall factor in the vertex in question. From the PLT perspective the vertices $\Upsilon^{\eta\eta}$, $\Upsilon^{\bar{\eta}\bar{\eta}}$ and $\Upsilon^{\eta\bar{\eta}}$ originate from different PLT classes and thus may have different locality properties. Remarkably, the homotopy technique developed in this paper implies that, in accordance with PLT, the most nontrivial part residing in $\Upsilon^{\eta\eta}$ and $\Upsilon^{\bar{\eta}\bar{\eta}}$ turns out to be ultra-local, the fact being of great importance for the locality properties at higher orders.

We show that, in agreement with [28], $\Upsilon^{\eta\eta}$ and $\Upsilon^{\bar{\eta}\bar{\eta}}$ vanish if ω is set to its AdS_4 vacuum value. Moreover they vanish if ω only contains spin $s \leq 2$ fields, particularly for any gravitational background. In this case, the only contribution is from the vertex $\Upsilon^{\eta\bar{\eta}}$ which is local at any ω . This implies in turn that the resulting currents on the r.h.s. of the HS equations are local and the coupling constants in front of them have definite signs independent of the phase of η . In particular, the gravitational constant as a coefficient in front of the stress tensor is positive.

To arrive at these results we generalize shifted contracting homotopy of [2] and adopt them for higher-order analysis adding one more $\beta \frac{\partial}{\partial y}$ shift with an arbitrary parameter β . We show that the β -dependent contracting homotopy operators are well defined for $-\infty < \beta < 1$. The limit $\beta \rightarrow -\infty$ is then conjectured to correspond to the local frame of HS theory. Using the introduced β -shifted contracting homotopy we calculate vertices $\Upsilon(\omega, \omega, C, C|\beta)$ and find their local limit at $\beta \rightarrow -\infty$. In [9], the effect of parameter β on HS generating equations is analyzed in the language of classes of functions introduced in the context of locality in [8].

One of the remarkable properties of the β -shifted contracting homotopy is that the lower-order vertices Υ^η and $\Upsilon^{\bar{\eta}}$ found in [3, 26] as well as the vertex $\Upsilon^{\eta\bar{\eta}}$ found in this

¹Called resolutions in that reference.

paper turn out to be β -independent. This explains in particular why the proposed modified scheme does not affect the lower-order results of [3, 26]. On the other hand, being spurious at lower orders, β becomes essential at higher orders so that the limit $\beta \rightarrow -\infty$ becomes crucial for finding local expressions for $\Upsilon^{\eta\eta}$ and $\Upsilon^{\bar{\eta}\bar{\eta}}$.

In calculation of (anti)holomorphic vertices we faced a remarkable cancellation. While our approach leads to the ultra-local $\Upsilon(\omega, \omega, C, C)$ its consistency condition involves some lower order vertices, particularly $\Upsilon(\omega, C, C)$. The latter is obtained using shifted homotopies. Though local it does not however have a form of a minimally derivative vertex. A local field redefinition can be carried out to make it the minimal one. This redefinition was explicitly found in [26]. Remarkably, it is exactly this field redefinition that leads to a dramatic cancellation in (anti)holomorphic structures, particularly making $\Upsilon^{\eta\eta}(\omega, \omega, C, C)$ to vanish on AdS background. This makes us argue that the maximally local vertices require maximal locality at each perturbation order.

Studying the properties of the β -shifted contracting homotopy we show that their effect is equivalent to certain β -induced reordering of the original HS star product.² An interesting remaining question is whether the local results for HS vertices can be equivalently obtained by virtue of the conventional homotopy with no β -shift but from the β -reordered HS equations at $\beta = -\infty$.

The paper is organized as follows: in section 2 we recollect necessary background on nonlinear HS equations. The new β -shifted contracting homotopy is defined in section 3 where also its properties are derived. The final results for vertices are presented in section 4 while technical details of their derivation are presented in sections 5 and 6.2. An alternative interpretation of the β -shifted contracting homotopy is presented in section 6.1 where it is shown to be equivalent to conventional homotopy in the appropriate β -dependent star product and then is used in section 6.2 to demonstrate cancellation of certain second-order contribution to HS vertices. Conclusion contains discussion of the obtained results and future directions. Further technical detail are presented in appendices A and B. Also appendices B and C collect some useful formulae.

2 Higher-spin generating equations

2.1 Nonlinear equations

Nonlinear HS equations (1.1), (1.2) originate from the following generating system [1]

$$d_x W + W * W = 0, \quad (2.1)$$

$$d_x S + W * S + S * W = 0, \quad (2.2)$$

$$d_x B + [W, B]_* = 0, \quad (2.3)$$

$$S * S = i(\theta^A \theta_A + B * \Gamma), \quad \Gamma = \eta\gamma + \bar{\eta}\bar{\gamma}, \quad (2.4)$$

$$[S, B]_* = 0. \quad (2.5)$$

²We are grateful to Carlo Iazeolla, David De Filippi and Per Sundell for the stimulating discussion of this issue.

Master fields $W(Z; Y; K|x)$, $S(Z; Y; K|x)$ and $B(Z; Y; K|x)$ depend on generating spinor variables $Z_A = (z_\alpha, \bar{z}_{\dot{\alpha}})$, $Y_A = (y_\alpha, \bar{y}_{\dot{\alpha}})$, where spinorial indices range over two values, and discrete involutive elements $K = (k, \bar{k})$

$$\{k, y_\alpha\} = \{k, z_\alpha\} = 0, \quad [k, \bar{y}_{\dot{\alpha}}] = [k, \bar{z}_{\dot{\alpha}}] = 0, \quad k^2 = 1, \quad [k, \bar{k}] = 0. \quad (2.6)$$

Similarly for $\bar{k}$.

Star product is defined as follows

$$(f * g)(Z, Y) = \frac{1}{(2\pi)^4} \int d^4U d^4V f(Z + U; Y + U) g(Z - V; Y + V) \exp(iU_A V^A). \quad (2.7)$$

Indices are raised and lowered with the aid of antisymmetric form $\epsilon_{AB} = -\epsilon_{BA}$ as follows $X^A = \epsilon^{AB} X_B$ and $X_A = X^B \epsilon_{BA}$. There are two types of anticommuting differentials in (2.1), (2.5), space-time dx^n and spinor differentials $\theta_A = (\theta_\alpha, \bar{\theta}_{\dot{\alpha}})$. Master fields belong to different gradings with respect to these differentials, $W = W_n dx^n$, $S = S_\alpha \theta^\alpha + \bar{S}_{\dot{\alpha}} \bar{\theta}^{\dot{\alpha}}$, and B is a zero-form. θ -differentials have the following commutation rules

$$\{\theta_A, \theta_B\} = 0, \quad \{\theta_\alpha, k\} = \{\bar{\theta}_{\dot{\alpha}}, \bar{k}\} = 0, \quad [\theta_\alpha, \bar{k}] = [\bar{\theta}_{\dot{\alpha}}, k] = 0. \quad (2.8)$$

Lastly, taking into account that $\theta^3 = \bar{\theta}^3 = 0$,

$$\gamma = \exp(iz_\alpha y^\alpha) k \theta^\alpha \theta_\alpha, \quad \bar{\gamma} = \exp(i\bar{z}_{\dot{\alpha}} \bar{y}^{\dot{\alpha}}) \bar{k} \bar{\theta}^{\dot{\alpha}} \bar{\theta}_{\dot{\alpha}} \quad (2.9)$$

turn out to be central with respect to the star product, i.e.,

$$\gamma * f = f * \gamma, \quad \bar{\gamma} * f = f * \bar{\gamma}, \quad \forall f = f(Z; Y; K; \theta). \quad (2.10)$$

2.2 Perturbative expansion

To set up perturbation theory one starts with vacuum solution

$$B_0 = 0, \quad (2.11)$$

$$S_0 = \theta^\alpha z_\alpha + \bar{\theta}^{\dot{\alpha}} \bar{z}_{\dot{\alpha}}. \quad (2.12)$$

Plugging it into (2.1)–(2.5) we find that W_0 should be Z -independent, $W_0 = \omega(Y; K|x)$, and satisfy (2.1). Similarly, at next order one gets $B_1 = C(Y; K|x)$ from $[S_0, B_1] = 0$ and C satisfies (2.3). This way we find the first terms on the r.h.s.'s of (1.1), (1.2). As in [5], our perturbative expansion is in powers of the zero-forms C with one-forms ω treated as being of zero order.

Since

$$[S_0,]_* = -2id_Z, \quad d_Z := \theta^A \frac{\partial}{\partial Z^A}, \quad (2.13)$$

typical equation that one has to solve to determine the field Z -dependence at any order is

$$d_Z f(Z; Y; \theta) = J(Z; Y; \theta), \quad (2.14)$$

where J originates from the lower-order terms. For example, if $f = S$, then (discarding the $\bar{\theta}, \bar{z}$ sector for brevity) most general form that follows from equations at order n is

$$J_n = \theta^2 \int d\tau d\sigma C(y_1) \dots C(y_n) \rho \exp \left(i\tau z_\alpha y^\alpha - A^j z^\alpha \partial_{j\alpha} - B^j y^\alpha \partial_{j\alpha} - \frac{i}{2} P^{ij} \partial_i^\alpha \partial_{j\alpha} \right) \Big|_{y_i=0} k, \quad (2.15)$$

where $\partial_{i\alpha} = \frac{\partial}{\partial y_i^\alpha}$, τ and σ are integration parameters over some compact domain, A^i, B^i, P^{ij} are some τ, σ -dependent coefficients (it is convenient to identify the coefficient in front of $z_\alpha y^\alpha$ with one of the integration variables τ) and ρ is some polynomial in $\partial_{i\alpha}$ with τ, σ -dependent coefficients. Note also that while $C(y, \bar{y})$ depends on $\bar{y}$ as well as on y , here being restricted to purely holomorphic sector we highlight only the y -dependence of C . For instance, it is easy to find condition on S_1 from (2.4)

$$-2i d_z S_1 = i\eta C * \gamma = i\eta \theta^2 \int d\tau \delta(1 - \tau) C(y_1) \exp(i\tau z_\alpha y^\alpha - \tau z^\alpha \partial_{1\alpha}) \Big|_{y_1=0} k. \quad (2.16)$$

Particular solution to (2.14) can be written as

$$f = \Delta J, \quad (2.17)$$

where Δ is some contracting homotopy operator. The challenge is to find such contracting homotopies that lead to HS vertices being as local as possible. Contracting homotopies used in [2, 3] with respect to z have the form

$$\Delta_q J(z; y; \theta) = (z + q)^\alpha \frac{\partial}{\partial \theta^\alpha} \int_0^1 d\tau \frac{1}{\tau} J(\tau z - (1 - \tau)q; y; \tau \theta), \quad (2.18)$$

where q in principle can be any z -independent spinor (operator). The form of (2.15) suggests the following natural choice for q

$$q_\alpha = -i \sum_j v^j \partial_{j\alpha} \quad (2.19)$$

with *a priori* arbitrary coefficients v^i . Conventional contracting homotopy of [1], that leads to local field equations at the linearized level, corresponds to $v^i = 0$ and is known to be inconsistent with locality at the interaction level [29, 30]. The role of field derivative shifts in (2.19) is to set some control over non-local contractions $\partial_i^\alpha \partial_{j\alpha}$ in vertices.

Let us clarify our notion of spin-locality attributed to spinor space rather than space-time, i.e., to derivatives in auxiliary spinor rather than space-time variables. The two are related via the unfolding routine that generally assumes non-linear and infinite derivative one-to-one map. Therefore, what we call local should be understood as spin-local. In the lowest order these two notions coincide.

The precise relation of spin-locality and the corresponding space-time derivative behavior can be worked out on AdS_4 background. In that case in (1.1), (1.2) one has

$$\omega = \Omega + \omega', \quad (2.20)$$

where Ω is the bilinear AdS_4 flat connection

$$\Omega = \frac{i}{4}(\omega_{\alpha\beta}y^\alpha y^\beta + \bar{\omega}_{\dot{\alpha}\dot{\beta}}\bar{y}^{\dot{\alpha}}\bar{y}^{\dot{\beta}} + 2e_{\alpha\dot{\alpha}}y^\alpha\bar{y}^{\dot{\alpha}}) \quad (2.21)$$

with $\omega_{\alpha\beta}$, $\bar{\omega}_{\dot{\alpha}\dot{\beta}}$ and $e_{\alpha\dot{\alpha}}$ being the AdS_4 background fields while $\omega'(Y|x)$ stands for perturbative fluctuations. Plugging Ω into HS equations (1.1), (1.2) using (2.7) results in

$$d_x\omega' - \omega^{\alpha\beta}y_\alpha\frac{\partial}{\partial y^\beta}\omega' - \bar{\omega}^{\dot{\alpha}\dot{\beta}}\bar{y}_{\dot{\alpha}}\frac{\partial}{\partial \bar{y}^{\dot{\beta}}}\omega' - e^{\alpha\dot{\beta}}\left(y_\alpha\frac{\partial}{\partial \bar{y}^{\dot{\beta}}} + \bar{y}_{\dot{\beta}}\frac{\partial}{\partial y^\alpha}\right)\omega' = -\omega' * \omega' \quad (2.22)$$

$$+ \Upsilon(\Omega, \Omega, C) + \dots,$$

$$d_x C - \omega^{\alpha\beta}y_\alpha\frac{\partial}{\partial y^\beta}C - \bar{\omega}^{\dot{\alpha}\dot{\beta}}\bar{y}_{\dot{\alpha}}\frac{\partial}{\partial \bar{y}^{\dot{\beta}}}C + ie^{\alpha\dot{\beta}}\left(y_\alpha\bar{y}_{\dot{\beta}} - \frac{\partial}{\partial y^\alpha\partial \bar{y}^{\dot{\beta}}}\right)C = -[\omega', C]_* \quad (2.23)$$

$$+ \Upsilon(\Omega, C, C) + \dots$$

Equations (2.22), (2.23) show that space-time derivatives entering via De Rham derivative d_x are related to spinor derivatives. Particularly, at lowest order, space-time derivative of the field $C(Y|x)$ is expressed via second derivative with respect to $\frac{\partial}{\partial y}$ and $\frac{\partial}{\partial \bar{y}}$. At higher orders the map is more involved but still available from (2.22), (2.23). As HS equations (2.1)–(2.5) are naturally formulated in spinor terms everywhere in this paper the locality is understood as spin-locality. On the other hand, as discussed in more detail in [9], spin-locality implies space-time locality of higher-order interactions rewritten in terms of the original constituent fields like ω and C and local currents of various ranks (degrees in fields) built from these fields.

Let us stress the important difference between the one-forms ω and zero-forms C manifested by equations (2.22), (2.23). The l.h.s. of (2.22) is homogeneous in Y . As a result, $\omega(Y)$ contains at most a finite number of derivatives of the dynamical frame-like component in ω for any given spin s (the degree of homogeneity in Y is $2(s-1)$; for more detail see e.g. [31]). That is why the presence of ω can only add a finite number of derivatives for a fixed spin, not affecting general aspects of the analysis of spin-locality. On the other hand, equation (2.23) relates infinitely many components of $C(y, \bar{y}; K|x)$ with the space-time derivatives for any given spin s because

$$2s = |n_y - n_{\bar{y}}|, \quad (2.24)$$

where n_y and $n_{\bar{y}}$ are the numbers of unbarred and barred variables Y in $C(y, \bar{y}; K|x)$ [31]. As a result, contractions between zero-forms can produce non-localities even for fixed spins. Correspondingly, in the analysis of this paper we will only control the form of contractions between zero-forms C neglecting contractions involving one-forms ω . From (2.24) it also follows that, if spins are fixed, to prove spin-locality it suffices to show that the number of contractions of either unbarred or barred $y, \bar{y}$ -variables is finite.

The important result of [2] is that the degree of non-locality of HS vertices which depends on field variable choice via contracting homotopy operators reduces provided a linear condition on v^i (2.19) from PLT is imposed. For the even sector that (anti)-holomorphic parts of W and S belong to the locality condition is

$$\sum_i (-)^i v_C^i = 0, \quad (2.25)$$

while for the odd one where (anti)holomorphic part of B resides it reads

$$\sum_i (-)^i v_C^i = 1, \quad (2.26)$$

where by v_C^i we label those coefficients in (2.19) that are attributed to zero-forms C derivatives only and index i runs over the values $i = 1, \dots, n$, where n is the perturbative order (the amount of C 's). Shifts q (2.19) with v_C^i satisfying (2.25) ((2.26)) will be referred to as *PLT-even(odd)*.

Shifted contracting homotopy with q (2.19) contains shifts that differentiate fields C and ω but not exponential kernel like in (2.15). It works perfectly fine at lowest interaction order $\Upsilon(\omega, \omega, C)$ and $\Upsilon(\omega, C, C)$ considered in [3] but when it comes to higher orders it becomes insufficient. One reason for this is that at higher orders homotopy field derivatives act on fields from lower orders that show up in combinations with y -dependent kernels. This suggests that the corresponding $\partial_{i\alpha}$ acts on the field along with explicit y -dependence of the kernel as well, demanding an extension of shifted contracting homotopy (2.19) to include explicit differentiation over the y -variable

$$q' = q + i\beta \frac{\partial}{\partial y}, \quad (2.27)$$

where β is a parameter. As shown below, the respective contracting homotopies are well defined for $-\infty < \beta < 1$. Since $\frac{\partial}{\partial y}$ and y do not commute the form of contracting homotopy operator (2.18) becomes ambiguous demanding an ordering choice. To make it well defined we will use the integral representation discussed in the next section.

3 β -shifted contracting homotopy

3.1 Definition

Contracting homotopy operator well suited for the higher-order analysis of HS equations has the form

$$\Delta_{q,\beta} J = \int \frac{d^2 u d^2 v}{(2\pi)^2} \exp(iu_\alpha v^\alpha) \int_0^1 \frac{d\tau}{\tau} (z + q - v)^\alpha \frac{\partial}{\partial \theta^\alpha} J(\tau z - (1 - \tau)(q - v); y + \beta u; \tau \theta), \quad (3.1)$$

where q is a y - and z -independent spinor. q can be an operator that acts on fields C and ω as in (2.19). For $\beta = 0$, operator $\Delta_{q,\beta}$ reduces down to (2.18), while for $\beta \neq 0$ it accounts for shift (2.27). Note that, by integration by parts, v effectively differentiates over u , i.e., y .

Contracting homotopy (3.1) gives resolution of identity allowing one solving (2.14):

$$\{d_z, \Delta_{q,\beta}\} = 1 - h_{q,\beta}, \quad (3.2)$$

where

$$h_{q,\beta} J(z; y; \theta) = \int \frac{d^2 u d^2 v}{(2\pi)^2} \exp(iu_\alpha v^\alpha) J(-q + v; y + \beta u; 0) \quad (3.3)$$

is the cohomology projector to the z, θ -independent part.

The proof of (3.2) is similar to the conventional resolution of identity. Consider

$$\begin{aligned}
 (\mathbf{d}_z \Delta_{q,\beta} + \Delta_{q,\beta} \mathbf{d}_z) J(z; y; \theta) &= \int \frac{d^2 u d^2 v}{(2\pi)^2} \exp(iu_\alpha v^\alpha) \int_0^1 \frac{d\tau}{\tau} \left(\theta^\alpha \frac{\partial}{\partial \theta^\alpha} + (z+q-v)^\alpha \theta^\beta \frac{\partial}{\partial \theta^\alpha} \frac{\partial}{\partial z^\beta} \right. \\
 &\quad \left. + (z+q-v)^\alpha \frac{\partial}{\partial \theta^\alpha} \theta^\beta \frac{\partial}{\partial z^\beta} \right) J(\tau z - (1-\tau)(q-v); y + \beta u; \tau \theta) \\
 &= \int \frac{d^2 u d^2 v}{(2\pi)^2} \exp(iu_\alpha v^\alpha) \int_0^1 \frac{d\tau}{\tau} \left(\theta^\alpha \frac{\partial}{\partial \theta^\alpha} + (z+q-v)^\alpha \frac{\partial}{\partial (z+q-v)^\alpha} \right) \\
 &\quad \times J(\tau z - (1-\tau)(q-v); y + \beta u; \tau \theta) \\
 &= \int \frac{d^2 u d^2 v}{(2\pi)^2} \exp(iu_\alpha v^\alpha) \int_0^1 d\tau \frac{d}{d\tau} J(\tau z - (1-\tau)(q-v); y + \beta u; \tau \theta) \\
 &= J(z; y; \theta) - \int \frac{d^2 u d^2 v}{(2\pi)^2} \exp(iu_\alpha v^\alpha) J(v-q; y + \beta u; 0). \tag{3.4}
 \end{aligned}$$

3.2 Properties

3.2.1 General

β -shifted contracting homotopies share standard properties derived in [3] at $\beta = 0$. First, they anticommute

$$\Delta_{q_1, \beta_1} \Delta_{q_2, \beta_2} = - \Delta_{q_2, \beta_2} \Delta_{q_1, \beta_1}, \quad \Delta_{q, \beta}^2 = 0 \tag{3.5}$$

and obey

$$h_{q, \beta} \Delta_{q, \beta} = 0. \tag{3.6}$$

Also they satisfy the following useful property

$$\Delta_{q_3, \beta_3} \Delta_{q_2, \beta_2} - \Delta_{q_3, \beta_3} \Delta_{q_1, \beta_1} + \Delta_{q_2, \beta_2} \Delta_{q_1, \beta_1} = h_{q_3, \beta_3} \Delta_{q_2, \beta_2} \Delta_{q_1, \beta_1} \tag{3.7}$$

for any q_i and β_i . Applying h_{q_4, β_4} to both sides of this relation and using that

$$h_{q_2, \beta_2} h_{q_1, \beta_1} = h_{q_1, \beta_1} \tag{3.8}$$

one recovers *triangle identity* of [32] in the form

$$h_{q_4, \beta_4} \Delta_{q_3, \beta_3} \Delta_{q_2, \beta_2} - h_{q_4, \beta_4} \Delta_{q_3, \beta_3} \Delta_{q_1, \beta_1} + h_{q_4, \beta_4} \Delta_{q_2, \beta_2} \Delta_{q_1, \beta_1} = h_{q_3, \beta_3} \Delta_{q_2, \beta_2} \Delta_{q_1, \beta_1}. \tag{3.9}$$

Though slightly modified, contracting homotopies (3.1) share star-exchange properties studied in [3]:

$$\Delta_{q, \beta} (f(y; k) * J(z; y; k; \theta)) = f(y; k) * \Delta_{q+(1-\beta)p, \beta} J(z; y; k; \theta), \tag{3.10}$$

$$h_{q, \beta} (f(y; k) * J(z; y; k; \theta)) = f(y; k) * h_{q+(1-\beta)p, \beta} J(z; y; k; \theta), \tag{3.11}$$

$$\Delta_{q, \beta} (J(z; y; \theta) * k^\nu * f(y; k)) = \Delta_{q+(-1)^\nu(1-\beta)p, \beta} \left(J(z; y; \theta) * k^\nu \right) * f(y, k), \tag{3.12}$$

$$h_{q, \beta} (J(z; y; \theta) * k^\nu * f(y; k)) = h_{q+(-1)^\nu(1-\beta)p, \beta} \left(J(z; y; \theta) * k^\nu \right) * f(y, k), \tag{3.13}$$

where we use the notation

$$p_\alpha f(y; k) \equiv f(y; k) p_\alpha := -i \frac{\partial}{\partial y^\alpha} (f_1(y) + f_2(y)k). \tag{3.14}$$

Note that star-exchange formulae (3.10)–(3.13) literally acquire the form of those studied in [3] for $\beta = 0$ upon the redefinition

$$\hat{q} = (1 - \beta)q, \quad \hat{p} = (1 - \beta)p, \quad (3.15)$$

which gives for example

$$\Delta_{\hat{q}, \beta} (f(y; k) * J(z; y; k; \theta)) = f(y; k) * \Delta_{\hat{q} + \hat{p}, \beta} J(z; y; k; \theta). \quad (3.16)$$

The class of $\beta = 0$ contracting homotopies was considered in [3] for which we adopt the simplified notation

$$\Delta_q := \Delta_{q, 0}, \quad h_q := h_{q, 0}. \quad (3.17)$$

The following important combination of these operators typically shows up at lower orders

$$h_c \Delta_b \Delta_a f(z, y) \theta^\beta \theta_\beta = 2 \int d^3_\Delta \tau (b - c)_\gamma (a - c)^\gamma f(-\tau_1 c - \tau_3 b - \tau_2 a, y), \quad (3.18)$$

where the τ_{1-3} integration is carried over a simplex

$$\int d^3_\Delta \tau := \int_{[0,1]^3} d\tau_1 d\tau_2 d\tau_3 \delta(1 - \tau_1 - \tau_2 - \tau_3). \quad (3.19)$$

Eq. (3.18) entails the scaling property

$$h_{\lambda a} \Delta_{\lambda b} \Delta_{\lambda c} f(z, y) = \lambda^2 h_a \Delta_b \Delta_c f(\lambda z, y) \quad \forall \lambda \neq 0. \quad (3.20)$$

3.2.2 Special property of $\Delta_{0, \beta}$

An important property of $\Delta_{0, \beta}$ is that it anticommutes with the space-time differential d_x

$$\{d_x, \Delta_{0, \beta}\} = 0. \quad (3.21)$$

Indeed, consider for example a one-form in θ

$$\int d\tau \theta^\alpha g_\alpha(\partial_1, \dots, \partial_N, y, z) \exp \left(i\tau z_\alpha y^\alpha - A^j(\tau) z^\alpha \partial_{j\alpha} - B^j(\tau) y^\alpha \partial_{j\alpha} - \frac{i}{2} P^{ij}(\tau) \partial_i^\alpha \partial_{j\alpha} \right) \underbrace{\Phi \dots \Phi}_N, \quad (3.22)$$

where Φ can be either C or ω . As a result of application of d_x each Φ will turn into some Υ from (1.1), (1.2) according to equations of motion. Corresponding derivative with respect to Y of Φ will be replaced by the derivative of Υ . Since operator $\Delta_{0, \beta}$ does not contain derivatives over arguments of Φ (the shift parameter $q = 0$) (3.21) follows.

As a consequence of resolution of identity (3.2) and (3.21) we also have

$$[d_x, h_{0, \beta}] = 0. \quad (3.23)$$

3.2.3 Action on γ

An important property of the β -shifted contracting homotopy is its action on the central element γ (2.9)

$$\Delta_{q,\beta} \gamma = \Delta_{(1-\beta)^{-1}q,0} \gamma. \quad (3.24)$$

That is when applied to γ it acts as shifted contracting homotopy (2.18) with a rescaled parameter q . The proof of (3.24) is straightforward. Applying $\Delta_{q,\beta}$ to γ (2.9) using (3.1) one finds that

$$\begin{aligned} \Delta_{q,\beta} \gamma &= 2 \left(z + q(1-\beta)^{-1} \right)^\alpha \theta_\alpha \int_0^1 d\tau \frac{(1-\beta)\tau}{(1-\beta(1-\tau))^3} \\ &\quad \times \exp \left(\frac{i}{1-\beta(1-\tau)} (\tau z - (1-\tau)q)_\alpha y^\alpha \right) k. \end{aligned} \quad (3.25)$$

Changing integration variable

$$\tau' = \tau(1-\beta(1-\tau))^{-1} \in [0, 1] \quad (3.26)$$

we obtain

$$\Delta_{q,\beta} \gamma = 2 \left(z + \frac{q}{1-\beta} \right)^\alpha \theta_\alpha \int_0^1 d\tau' \tau' \exp(i(\tau' z - (1-\tau')q)_\alpha y^\alpha) k = \Delta_{\frac{q}{1-\beta},0} \gamma. \quad (3.27)$$

Using the anticommutativity property (3.5) from here it follows also

$$\Delta_{q_1,\beta_1} \Delta_{q_2,\beta_2} \gamma = \Delta_{(1-\beta_1)^{-1}q_1,0} \Delta_{(1-\beta_2)^{-1}q_2,0} \gamma. \quad (3.28)$$

Recall that [3]

$$\begin{aligned} \Delta_q \gamma &\equiv \Delta_{q,0} \gamma = 2(z^\beta + q^\beta) \theta_\beta \int_0^1 d\tau \tau \exp(i(\tau z_\alpha - (1-\tau)q_\alpha) y^\alpha) k, \\ \Delta_{q_1} \Delta_{q_2} \gamma &\equiv \Delta_{q_1,0} \Delta_{q_2,0} \gamma = 2(z + q_1)_\gamma (z + q_2)^\gamma \int d_\Delta^3 \tau \exp(i(\tau_1 z - \tau_2 q_2 - \tau_3 q_1)_\alpha y^\alpha) k. \end{aligned} \quad (3.29)$$

Since, HS vertices eventually are driven by β -shifted contracting homotopies applied to γ one might think of β deformation as unnecessary just leading to some rescaling of the homotopy parameter. This turns out to be the case at lower orders but drastically departs at higher orders where structures like $\Delta (\Delta \gamma * \Delta \gamma)$ show up. Moreover, this fact explains why the necessity of the β -shifted contracting homotopy was not seen in the lower-order analysis of [3]. Property (3.24) plays crucial role in the β -independence of the lower-order HS vertices.

3.3 β dependence

3.3.1 Contracting homotopy operator and cohomology projector

To appreciate the role of the parameter β it is useful to apply the β -shifted contracting homotopy to the function of the form

$$f(z, y, \theta) = \int_{[0,1]^2} d^2 \tau \delta(1 - \tau_1 - \tau_2) \exp[i\tau_1 z_\alpha y^\alpha] \phi(\tau_1 z, \tau_2 y, \tau_1 \theta, \tau_1), \quad (3.30)$$

which naturally results from the perturbative analysis at higher orders [8]. In this case, formula (3.3) yields

$$h_{0,\beta}(f) = \int_0^1 d\tau \frac{1}{(1-\beta\tau)^2} \int \frac{d^2v d^2u}{(2\pi)^2} \exp i[v_\alpha u^\alpha] \phi\left(u, \frac{(1-\tau)}{(1-\tau\beta)}y + \frac{\tau(1-\tau)\beta}{(1-\tau\beta)}v, 0, \tau\right). \quad (3.31)$$

Derivation of the expression for contracting homotopy is more involved. Upon the change of homotopy integration variables described in appendix A it yields

$$\begin{aligned} \Delta_{0,\beta}(f) = & \frac{1}{(2\pi)^2} \int d^2u d^2v \int d^3\tau \left[\frac{(1-\beta)\tau_1}{1-\beta(1-\tau_2)} \right]^{p-1} \exp i[v_\alpha u^\alpha + \tau_1 z_\alpha y^\alpha] \\ & \times \frac{(1-\beta\tau_1)z^\beta - \beta\tau_3 u^\beta}{1-\beta(1-\tau_2)} \frac{\partial}{\partial \theta^\beta} \phi\left(\tau_1 z + \frac{\tau_2\tau_3\beta}{1-\beta(1-\tau_2)}u, v + \tau_3 y, \theta, \frac{1-\tau_3-\beta\tau_1}{1-\beta(1-\tau_2)}\right), \end{aligned} \quad (3.32)$$

where p is the degree of f in θ ,

$$f(w, u, \mu\theta, \tau) = \mu^p f(w, u, \theta, \tau). \quad (3.33)$$

The last argument of ϕ in (3.33) results from the change of integration variables.

Star product of the two functions of the form (3.30) has the form [8]

$$\begin{aligned} f_1 * f_2 = & \frac{1}{(2\pi)^2} \int_0^1 d\tau_1 \int_0^1 d\tau_2 \int d^2s d^2t \exp i[\tau_1 \circ \tau_2 z_\alpha y^\alpha + s_\alpha t^\alpha] \\ & \times \phi_1(\tau_1((1-\tau_2)z - \tau_2 y + s), (1-\tau_1)((1-\tau_2)y - \tau_2 z + s), \tau_1 \theta, \tau_1) \\ & \times \phi_2(\tau_2((1-\tau_1)z + \tau_1 y - t), (1-\tau_2)((1-\tau_1)y + \tau_1 z + t), \tau_2 \theta, \tau_2), \end{aligned} \quad (3.34)$$

where the product law

$$\tau_1 \circ \tau_2 = \tau_1(1-\tau_2) + \tau_2(1-\tau_1) \quad (3.35)$$

is commutative and associative. Note that $0 \leq \tau_1 \circ \tau_2 \leq 1$ as well as $1 - \tau_1 \circ \tau_2$,

$$1 - \tau_1 \circ \tau_2 = \tau_1 \tau_2 + (1-\tau_1)(1-\tau_2). \quad (3.36)$$

Formula (3.34) is heavily used in the computation of higher-order corrections to HS equations.

The following comments are now in order. Formulae (3.31), (3.33) contain prefactors and rational dependence on the integration homotopy parameters τ due to the Gaussian integration resulting from the substitution of the dependence on u and v into the exponential factor in (3.30). The resulting expressions are well defined for

$$-\infty < \beta < 1. \quad (3.37)$$

Beyond this region, they may contain divergencies due to the degeneracy of the quadratic form in the Gaussian integral. At $\beta = 0$, these formulae reproduce those of the conventional homotopy introduced in [1]. The presence of β in the denominators of (3.31) makes the limit $\beta \rightarrow -\infty$ nontrivial having the effect of suppressing many of the terms including those responsible for contractions between zero-forms C as well as the Y -dependence in their arguments.

3.3.2 Local limit

Let us briefly explain the idea of the limiting mechanism. Consider the following integral

$$\int_0^1 d\tau \frac{1}{(1 - \beta\tau)^{2+n}}. \quad (3.38)$$

Setting $\beta = -\varepsilon^{-1}$ we obtain

$$\int_0^1 \frac{d\tau}{(1 - \beta\tau)^{2+n}} = (n+1)^{-1} \varepsilon + O(\varepsilon^2). \quad (3.39)$$

This implies that the contributions to HS vertices containing such or further suppressed factors all vanish in the limit $\beta \rightarrow -\infty$. For instance,

$$\lim_{\beta \rightarrow -\infty} \int_0^1 d\tau \frac{(\beta\tau)^m}{(1 - \beta\tau)^{2+n}} = 0, \quad m \leq n. \quad (3.40)$$

On the other hand, expressions of the form

$$\lim_{\beta \rightarrow -\infty} \int_0^1 d\tau \frac{\beta(\beta\tau)^m}{(1 - \beta\tau)^{2+n}} \quad (3.41)$$

with $m \leq n$ remain finite in the limit $\beta \rightarrow -\infty$ (for more detail see appendix B3). This fact admits an important interpretation: addition of at least one power of τ to the integrand of an integral remaining finite in the limit $\beta \rightarrow -\infty$ sends the expression to zero in the limit. As will be illustrated in the next section (for more detail see [9]) this mechanism makes the limit $\beta \rightarrow -\infty$ appropriate for locality.

3.3.3 Example

Let us outline how the limit $\beta \rightarrow -\infty$ leads to local results considering as an example the $W_1 * W_1$ part of the HS vertex in the one-form sector (1.1). Here it is important that the process starts from functions (3.30) with a y -independent function ϕ since (see also appendix B)

$$C(y) * \gamma = C(-z) \exp(iz_\alpha y^\alpha) k \theta^\alpha \theta_\alpha. \quad (3.42)$$

Then [3]

$$S_1 = s_1 + \bar{s}_1 = -\frac{\eta}{2} \Delta_0 (C * \gamma) + h.c. \quad (3.43)$$

with

$$s_1 = \eta \int_0^1 d\tau \tau z_\alpha \theta^\alpha \exp(i\tau z_\alpha y^\alpha) C(-\tau z) k \quad (3.44)$$

and the convention that $h.c.$ (Hermitean conjugation) swaps barred and unbarred variables along with dotted and undotted indices.³ Then

$$W_1 = \frac{1}{2i} \Delta_0 (d_x S_1 + \omega * S_1 + S_1 * \omega) + h.c.. \quad (3.45)$$

³We avoid using symbol $c.c.$ since HS equations are invariant under the Hermitean conjugation that not only conjugates complex numbers but also reverses the order of product factors including the differentials θ that may give additional sign compensating the effect of complex conjugation of i (for more detail see e.g. [31]).

For instance, the term $\Delta_0 (\omega * s_1)$ has the structure (discarding ω along with all its derivatives)

$$\Delta_0 (\omega * s_1) = \eta \int_0^1 d\tau_1 (1 - \tau_1) \exp i(\tau_1 z_\alpha y^\alpha) C(-\tau_1 z + \dots) k. \quad (3.46)$$

The computation of $\Delta_0 (\omega * s_1) * \Delta_0 (\omega * s_1)$ is straightforward. Referring for the explicit final result to the next sections here we focus on the form of the arguments of zero-forms C .

By virtue of (3.34) this yields upon evaluation of the integration over s and t

$$\begin{aligned} \Delta_0 (\omega * s_1) * \Delta_0 (\omega * s_1) &\sim \eta^2 \int_0^1 d\tau_1 \int_0^1 d\sigma_1 (1 - \tau_1)(1 - \sigma_1) \exp i[\tau_1 \circ \sigma_1 z_\alpha y^\alpha + \tau_1 \sigma_1 \partial_{1\alpha} \partial_2^\alpha] \\ &\times C(y_1 + \tau_1 \sigma_1 y - \tau_1 (1 - \sigma_1) z + \dots) C(y_2 + \sigma_1 \tau_1 y + \sigma_1 (1 - \tau_1) z + \dots) \Big|_{y_1=y_2=0}, \end{aligned} \quad (3.47)$$

where ω as well as all contractions between all ω and C are discarded. The important feature of this formula is that the y -dependence of both factors of C as well as the term $\partial_{1\alpha} \partial_2^\alpha$ in the exponential are accompanied with the factor of $\tau_1 \sigma_1$. As a result, the dependence on y as well as all contracting terms between the two fields C disappear from $h_{-\infty}(\Delta_0 (\omega * s_1) * \Delta_0 (\omega * s_1))$.

Indeed, $\Delta_0 (\omega * s_1) * \Delta_0 (\omega * s_1)$ can be represented in the form

$$\Delta_0 (\omega * s_1) * \Delta_0 (\omega * s_1) = \Delta_0 (\omega * s_1) * \Delta_0 (\omega * s_1) \Big|_0 + \Delta_0 (\omega * s_1) * \Delta_0 (\omega * s_1) \Big|_1, \quad (3.48)$$

where

$$\begin{aligned} \Delta_0 (\omega * s_1) * \Delta_0 (\omega * s_1) \Big|_0 &\sim \eta^2 \int_0^1 d\tau_1 \int_0^1 d\sigma_1 (1 - \tau_1)(1 - \sigma_1) \exp i[\tau_1 \circ \sigma_1 z_\alpha y^\alpha] \times \\ &\times C(-\tau_1 z + \dots) C(\sigma_1 z + \dots) \end{aligned} \quad (3.49)$$

is a part of (3.47) with all terms containing a factor of $\tau_1 \sigma_1$ set to zero while

$$\Delta_0 (\omega * s_1) * \Delta_0 (\omega * s_1) \Big|_1 \sim \eta^2 \int_0^1 d\tau_1 \int_0^1 d\sigma_1 (1 - \tau_1)(1 - \sigma_1) \sigma_1 \tau_1 \exp i[\tau_1 \circ \sigma_1 z_\alpha y^\alpha] C C + \dots \quad (3.50)$$

contains the rest. The ultra-local term $\Delta_0 (\omega * s_1) * \Delta_0 (\omega * s_1) \Big|_0$, which is free from the y dependence in the arguments of C as well as from the contractions between the two factors of C , does contribute to the final result. On the other hand the term $\Delta_0 (\omega * s_1) * \Delta_0 (\omega * s_1) \Big|_1$ does not if the limiting projector $h_{0,-\infty}$ is used.

Indeed, using that

$$\tau_1 (1 - \sigma_1) \leq \tau_1 \circ \sigma_1, \quad \sigma_1 (1 - \tau_1) \leq \tau_1 \circ \sigma_1, \quad (3.51)$$

we observe that

$$\Delta_0 (\omega * s_1) * \Delta_0 (\omega * s_1) \Big|_1 \leq \int_0^1 d\tau_1 \int_0^1 d\sigma_1 (\sigma_1 \circ \tau_1)^2 \exp i[\tau_1 \circ \sigma_1 z_\alpha y^\alpha] C(\dots) C(\dots). \quad (3.52)$$

Using (3.31) and observing that any factor of $\frac{\tau_1 \circ \sigma_1}{(1 - \tau_1 \circ \sigma_1 \beta)} \leq \frac{1}{-\beta} \rightarrow 0$ at $\beta \rightarrow -\infty$ one can see that analogously to (3.40)

$$\lim_{\beta \rightarrow -\infty} h_{0,\beta}(\Delta_0 (\omega * s_1) * \Delta_0 (\omega * s_1)) \Big|_1 = 0. \quad (3.53)$$

Thus the expression $h_{0,-\infty}(\Delta_0(\omega * s_1) * \Delta_0(\omega * s_1))$ indeed turns out to be ultra-local i.e., local and with no y -dependence in the arguments of C . Let us stress the crucial role of the factor of $(1 - \tau_1)(1 - \sigma_1)$ in the integration measure as well as of the coefficient $\tau_1\sigma_1$ in front of the y -dependence in C and the contracting terms $\partial_1\partial_2$ in the exponential.

Note that one should not be confused that the projector $h_{0,-\infty}$ is applied to a β -independent expression $W_1 * W_1$. This is because the first-order expression for W_1 considered in this example is β -independent as a consequence of (3.24). Other contributions to equations (1.1) like those generated by W_2 are β -dependent. Some of them originating from S_2 disappear in the $\beta \rightarrow -\infty$ limit upon local field redefinition. The resolution of identity implies that a single cohomology projector $h_{0,\beta}$ should be applied to HS vertex generating equation (2.1) or (2.3) as a whole (i.e., it makes no sense to apply different projectors to different terms in these equations since each of them is not z -independent while the full expression is). Remarkably, a similar phenomenon of the suppression of non-local terms in the limit $\beta \rightarrow -\infty$ occurs in other contributions to the vertices in question as we explain now. That is why the limit $\beta \rightarrow -\infty$ plays a distinguished role in the locality analysis.

4 $C^2\omega^2$ vertices

4.1 Generalities

Our goal is to extract $\Upsilon(\omega, \omega, C, C)$ vertex in (1.1) from HS equations (2.1)–(2.5). To do so we need to have all master fields to be solved up to the order C^2 . We label them as W_2, S_2 and B_2 . Those that are linear in C are W_1, S_1 and C itself. In terms of these fields the desired vertex appears in parenthesis

$$d_x\omega + \omega * \omega = -\{\omega, W_1\}_* - (d_x W_1 + d_x W_2 + W_1 * W_1 + \{\omega, W_2\}_*) + O(C^3). \quad (4.1)$$

Note that $d_x W_1$ contributes to C^2 -terms too⁴ through the second-order corrections obtained in [3] (see eqs. (C.1), (C.2)). Now, in solving for W_1 and $W_2|_{\eta\eta}$ we will use contracting homotopy $\Delta_{q,\beta}$.

The local frame is prescribed by PLT [2]. It applies differently for different PLT field parity. Namely, (anti)holomorphic W belongs to even sector of HS equations and the locality requirement (2.25) is fulfilled for instance for $q = 0$. So, solving for holomorphic W and S we can use $\Delta_{0,\beta}$ with some β . In other words, $W_1 = \Delta_{0,\beta}(\dots)$ and $W_2|_{\eta\eta} = \Delta_{0,\beta}(\dots)$, where the precise expressions will be specified in what follows.

Generic ω vertex $\Upsilon(\omega, \omega, C)$ calculated in [3] and presented in appendix C turned out to be ultra-local. To proceed further we need explicit expressions for W_1 and W_2 to plug them into (4.1). Once it is done we take the limit $\beta \rightarrow -\infty$ to get the final result for the local vertex that takes form (1.3) with

$$\Upsilon^{\eta\eta}(\omega, \omega, C, C) = \Upsilon_{\omega\omega CC}^{\eta\eta} + \Upsilon_{\omega C\omega C}^{\eta\eta} + \Upsilon_{C\omega\omega C}^{\eta\eta} + \Upsilon_{C\omega C\omega}^{\eta\eta} + \Upsilon_{CC\omega\omega}^{\eta\eta} + \Upsilon_{\omega CC\omega}^{\eta\eta}, \quad (4.2)$$

$$\Upsilon^{\bar{\eta}\bar{\eta}}(\omega, \omega, C, C) = \Upsilon_{\omega\omega CC}^{\bar{\eta}\bar{\eta}} + \Upsilon_{\omega C\omega C}^{\bar{\eta}\bar{\eta}} + \Upsilon_{C\omega\omega C}^{\bar{\eta}\bar{\eta}} + \Upsilon_{C\omega C\omega}^{\bar{\eta}\bar{\eta}} + \Upsilon_{CC\omega\omega}^{\bar{\eta}\bar{\eta}} + \Upsilon_{\omega CC\omega}^{\bar{\eta}\bar{\eta}}, \quad (4.3)$$

$$\Upsilon^{\eta\bar{\eta}}(\omega, \omega, C, C) = \Upsilon_{\omega\omega CC}^{\eta\bar{\eta}} + \Upsilon_{\omega C\omega C}^{\eta\bar{\eta}} + \Upsilon_{C\omega\omega C}^{\eta\bar{\eta}} + \Upsilon_{C\omega C\omega}^{\eta\bar{\eta}} + \Upsilon_{CC\omega\omega}^{\eta\bar{\eta}} + \Upsilon_{\omega CC\omega}^{\eta\bar{\eta}}, \quad (4.4)$$

where the lower labels refer to different orderings of the factors ω and C in the vertex.

⁴In general $d_x W_1$ and $d_x W_2$ contain contribution of arbitrary high order.

Calculation of HS vertices is simplified by the following trick. Since for any z , θ -independent function $f(y)$, $h_{q,\beta}f(y) = f(y)$, one can apply any projector $h_{q,\beta}$ to each Z , θ^A -independent part on the r.h.s. of (4.1) like $\Upsilon_{C\omega C}^\eta$, $\Upsilon_{C\omega C\omega}^{\eta\eta}$, $\Upsilon_{\omega C\omega C}^{\eta\bar{\eta}}$, etc.

Applying $h_{0,\beta}$ to $(d_x W_1 + d_x W_2 + W_1 * W_1 + \{\omega, W_2\})|_{\eta\eta}$ one arrives at

$$d_x \omega + \omega * \omega = \Upsilon(\omega, \omega, C) + \Upsilon^{\eta\eta}(\omega, \omega, C, C) + \dots, \quad (4.5)$$

where

$$\Upsilon^{\eta\eta}(\omega, \omega, C, C) = -h_{0,\beta}(W_1 * W_1 + \{\omega, W_2\})|_{\eta\eta}. \quad (4.6)$$

The convenience of this particular projector is that it annihilates contributions from $d_x W_1$ and $d_x W_2$. Indeed, by virtue of (3.6) and (3.23) one has

$$h_{0,\beta}(d_x W_1)|_{\eta\eta} = h_{0,\beta}(d_x W_2)|_{\eta\eta} = 0. \quad (4.7)$$

Now we are in a position to present the final results for all vertices (4.2)–(4.4) leaving the explanation of the details of their derivation for sections 5.2 and 5.3.

4.2 η^2 vertices

Using notations [2, 3] with $\bar{*}$ denoting the star product for the barred variables $\bar{y}$ and

$$p_{j\alpha} := -i \frac{\partial}{\partial y^{j\alpha}}, \quad \bar{p}_{j\dot{\alpha}} := -i \frac{\partial}{\partial \bar{y}^{j\dot{\alpha}}}, \quad (4.8)$$

$$t_{j\alpha} := -i \frac{\partial}{\partial y^{j\alpha}}, \quad \bar{t}_{j\dot{\alpha}} := -i \frac{\partial}{\partial \bar{y}^{j\dot{\alpha}}}, \quad (4.9)$$

where $p_{j\alpha}$ and $t_{j\alpha}$ denote derivatives with respect to the unbarred arguments of zero-forms C and one-forms ω , respectively, counted from the left to the right (indices $j = 1, 2$) we have

$$\Upsilon_{\omega\omega CC}^{\eta\eta} = -\frac{\eta^2}{4} \int_{[0,1]^2} d\sigma d\sigma' \sigma\sigma' \int d_\Delta^3 \tau (t_{1\alpha} t_2^\alpha)^2 \quad (4.10)$$

$$\begin{aligned} & \times \tau_1 \exp \left[i(\tau_2 \sigma + \tau_3 \sigma' + \tau_1 \sigma \sigma') t_{1\alpha} t_2^\alpha \right] \omega(y - (1 - \tau_3) \sigma y, \bar{y}; K) \bar{*} \omega(\tau_3 \sigma' y, \bar{y}; K) \bar{*} \\ & \times \bar{*} C(\tau_2 \sigma t_1 + (1 - \sigma'(1 - \tau_2)) t_2, \bar{y}; K) \bar{*} C(-\tau_1 \sigma t_1 - \sigma' \tau_1 t_2, \bar{y}; K), \end{aligned}$$

$$\Upsilon_{CC\omega\omega}^{\eta\eta} = \frac{\eta^2}{4} \int_{[0,1]^2} d\sigma d\sigma' \sigma\sigma' \int d_\Delta^3 \tau (t_{1\alpha} t_2^\alpha)^2 \quad (4.11)$$

$$\begin{aligned} & \times \tau_1 \exp \left[-i(\tau_2 \sigma + \tau_3 \sigma' + \tau_1 \sigma \sigma') t_{1\alpha} t_2^\alpha \right] C(\tau_1 \sigma' t_1 + \sigma \tau_1 t_2, \bar{y}; K) \bar{*} \\ & \times \bar{*} C(-\tau_2 \sigma t_2 - (1 - \sigma'(1 - \tau_2)) t_1, \bar{y}; K) \bar{*} \omega(\tau_3 \sigma' y, \bar{y}; K) \bar{*} \omega(y - \sigma(1 - \tau_3) y, \bar{y}; K), \end{aligned}$$

$$\Upsilon_{C\omega C}^{\eta\eta} = \frac{\eta^2}{4} \int_{[0,1]^2} d\sigma d\sigma' \sigma \sigma' \int d_\Delta^3 \tau (t_{1\alpha} t_2^\alpha)^2 \quad (4.12)$$

$$\begin{aligned} & \times \tau_3 \exp \left[-i(\tau_2 \sigma + \tau_1 \sigma' + \tau_3 \sigma \sigma') t_{1\alpha} t_2^\alpha \right] C(-\tau_1 \sigma' t_2 - (1 - \sigma(1 - \tau_1)) t_1, \bar{y}; K) \bar{*} \omega(-\tau_3 \sigma y, \bar{y}; K) \bar{*} \\ & \times \bar{*} \omega(-\tau_3 \sigma' y, \bar{y}; K) \bar{*} C(\tau_2 \sigma t_1 + (1 - \sigma'(1 - \tau_2)) t_2, \bar{y}; K), \end{aligned} \quad (4.13)$$

$$\begin{aligned} & \Upsilon_{\omega C \omega}^{\eta\eta} = \frac{\eta^2}{4} \int_{[0,1]^2} d\sigma d\sigma' \sigma \sigma' \int d_\Delta^3 \tau (t_{1\alpha} t_2^\alpha)^2 \\ & \times \left\{ \tau_1 \exp \left[i(-\tau_2 \sigma + \tau_3 \sigma' + \tau_1 \sigma \sigma') t_{1\alpha} t_2^\alpha \right] \omega(y - (1 - \tau_3) \sigma y, \bar{y}; K) \bar{*} C(\tau_2 \sigma t_1 - (1 - \sigma'(1 - \tau_2)) t_2, \bar{y}; K) \bar{*} \right. \\ & \times \bar{*} \omega(-\tau_3 \sigma' y, \bar{y}; K) \bar{*} C(-\tau_1 \sigma t_1 + \sigma' \tau_1 t_2, \bar{y}; K) \\ & - \tau_1 \exp \left[i(\tau_2 \sigma + \tau_3 \sigma' - \tau_1 \sigma \sigma') t_{1\alpha} t_2^\alpha \right] \omega(y - (1 - \tau_3) \sigma y, \bar{y}; K) \bar{*} C(\tau_1 \sigma t_1 - \sigma' \tau_1 t_2, \bar{y}; K) \bar{*} \\ & \times \bar{*} \omega(-\tau_3 \sigma' y, \bar{y}; K) \bar{*} C(-\tau_2 \sigma t_1 + (1 - \sigma'(1 - \tau_2)) t_2, \bar{y}; K) \\ & - \tau_3 \exp \left[i(\tau_2 \sigma - \tau_1 \sigma' + \tau_3 \sigma \sigma') t_{1\alpha} t_2^\alpha \right] \omega(\tau_3 \sigma y, \bar{y}; K) \bar{*} C(-\sigma' \tau_1 t_2 + (1 - \sigma(1 - \tau_1)) t_1, \bar{y}; K) \bar{*} \\ & \left. \times \bar{*} \omega(-\tau_3 \sigma' y, \bar{y}; K) \bar{*} C(-\tau_2 \sigma t_1 + (1 - \sigma'(1 - \tau_2)) t_2, \bar{y}; K) \right\}, \end{aligned}$$

$$\Upsilon_{C\omega C\omega}^{\eta\eta} = \frac{\eta^2}{4} \int_{[0,1]^2} d\sigma d\sigma' \sigma \sigma' \int d_\Delta^3 \tau (t_{1\alpha} t_2^\alpha)^2 \quad (4.14)$$

$$\begin{aligned} & \times \left\{ \tau_1 \exp \left[i(\tau_2 \sigma + \tau_3 \sigma' - \tau_1 \sigma \sigma') t_{1\alpha} t_2^\alpha \right] C(\tau_2 \sigma t_2 - (1 - \sigma'(1 - \tau_2)) t_1, \bar{y}; K) \bar{*} \omega(-\tau_3 \sigma' y, \bar{y}; K) \bar{*} \right. \\ & \times \bar{*} C(-\tau_1 \sigma t_2 + \sigma' \tau_1 t_1, \bar{y}; K) \bar{*} \omega(y - (1 - \tau_3) \sigma y, \bar{y}; K) \\ & - \tau_1 \exp \left[-i(\tau_2 \sigma - \tau_3 \sigma' - \tau_1 \sigma \sigma') t_{1\alpha} t_2^\alpha \right] C(\tau_1 \sigma t_2 - \sigma' \tau_1 t_1, \bar{y}; K) \bar{*} \omega(-\tau_3 \sigma' y, \bar{y}; K) \bar{*} \\ & \times \bar{*} C(-\tau_2 \sigma t_2 + (1 - \sigma'(1 - \tau_2)) t_1, \bar{y}; K) \bar{*} \omega(y - (1 - \tau_3) \sigma y, \bar{y}; K) \\ & - \tau_3 \exp \left[i(-\tau_2 \sigma + \tau_1 \sigma' + \tau_3 \sigma \sigma') t_{1\alpha} t_2^\alpha \right] C(\sigma' \tau_1 t_2 - (1 - \sigma(1 - \tau_1)) t_1, \bar{y}; K) \bar{*} \omega(-\tau_3 \sigma y, \bar{y}; K) \bar{*} \\ & \left. \times \bar{*} C(\tau_2 \sigma t_1 - (1 - \sigma'(1 - \tau_2)) t_2, \bar{y}; K) \bar{*} \omega(\tau_3 \sigma' y, \bar{y}; K) \right\}, \end{aligned}$$

$$\Upsilon_{\omega C C \omega}^{\eta\eta} = \frac{\eta^2}{4} \int_{[0,1]^2} d\sigma d\sigma' \sigma \sigma' \int d_\Delta^3 \tau (t_{1\alpha} t_2^\alpha)^2 \quad (4.15)$$

$$\begin{aligned} & \times \left\{ \tau_1 \exp \left[-i(-\tau_2 \sigma + \tau_3 \sigma' - \tau_1 \sigma \sigma') t_{1\alpha} t_2^\alpha \right] \omega(y - (1 - \tau_3) \sigma y, \bar{y}; K) \bar{*} C(\tau_1 \sigma t_1 + \sigma' \tau_1 t_2, \bar{y}; K) \bar{*} \right. \\ & \times \bar{*} C(-\tau_2 \sigma t_1 - (1 - \sigma'(1 - \tau_2)) t_2, \bar{y}; K) \bar{*} \omega(\tau_3 \sigma' y, \bar{y}; K) \\ & - \tau_1 \exp \left[i(\tau_2 \sigma - \tau_3 \sigma' + \tau_1 \sigma \sigma') t_{1\alpha} t_2^\alpha \right] \omega(\tau_3 \sigma' y, \bar{y}; K) \bar{*} C(\tau_2 \sigma t_2 + (1 - \sigma'(1 - \tau_2)) t_1, \bar{y}; K) \bar{*} \\ & \times \bar{*} C(-\tau_1 \sigma t_2 - \sigma' \tau_1 t_1, \bar{y}; K) \bar{*} \omega(y - (1 - \tau_3) \sigma y, \bar{y}; K) \\ & + \tau_3 \exp \left[i(\tau_2 \sigma + \tau_1 \sigma' - \tau_3 \sigma \sigma') t_{1\alpha} t_2^\alpha \right] \omega(-\tau_3 \sigma y, \bar{y}; K) \bar{*} C(\sigma' \tau_1 t_2 + (1 - \sigma(1 - \tau_1)) t_1, \bar{y}; K) \bar{*} \\ & \left. \times \bar{*} C(-\tau_2 \sigma t_1 - (1 - \sigma'(1 - \tau_2)) t_2, \bar{y}; K) \bar{*} \omega(\tau_3 \sigma' y, \bar{y}; K) \right\}. \end{aligned}$$

All vertices (4.10)–(4.15) are local because neither exponentials nor arguments of C -fields contain C -derivatives $p_{j\alpha}$. As discussed in section 2.2, in this case the $\bar{*}$ product over the antiholomorphic variables $\bar{y}$ does not spoil the spin-locality of the vertices. Moreover all obtained vertices are ultra-local in the sense of [3]. Indeed fields C in (4.10)–(4.15) carry no dependence on y just as for central on-shell theorem that describes propagation of free fields or its generalization via vertex (C.3)–(C.5). This implies that at the next order they will not produce non-localities via star product.

Antiholomorphic vertex (4.3) results from (4.2) by swapping barred and unbarred sectors including η and $\bar{\eta}$.

Another property of the resulting η^2 and $\bar{\eta}^2$ vertices (4.2) and (4.3) is

$$\Upsilon^{\eta\eta} = \Upsilon^{\bar{\eta}\bar{\eta}} = 0 \quad \text{for} \quad \omega = \omega_{s \leq 2}, \quad (4.16)$$

where $\omega_{s \leq 2}$ denotes the part of ω that is at most bilinear in the Y -variables and commutative with respect to possible color indices to correspond to the gravitational spin-two sector. Indeed, since all vertices (4.10)–(4.15) contain the prefactor of four ω -derivatives $(t_1 \cdot t_2)^2$ it makes the whole vertex vanish for commutative ω 's at most bilinear in oscillators. Particularly, it vanishes for ω describing AdS_4 background. This fact is in agreement with the result shown in [28] that local HS current interactions are independent of the phase of η that guarantees in particular positivity of the gravitational constant identified with the coefficient in front of the stress tensor. It also agrees with the AdS_4/CFT_3 prediction on the parity dependence of three-point correlation functions [26, 33].

4.3 $\eta\bar{\eta}$ vertices

Using for brevity the notation (3.17) used in [3], the $\eta\bar{\eta}$ vertices are

$$\Upsilon_{\omega\omega CC}^{\eta\bar{\eta}} = -\frac{\eta\bar{\eta}}{16} \omega * \omega * C * C * h_{p_2} \Delta_{p_1+2p_2} \Delta_{p_1+2p_2+t_2} \gamma * \bar{h}_{\bar{p}_2} \bar{\Delta}_{\bar{p}_1+\bar{p}_2+\bar{t}_1+\bar{t}_2} \bar{\Delta}_{\bar{p}_1+\bar{p}_2+\bar{t}_2} \bar{\gamma} + \delta \Upsilon_{\omega\omega CC}^{\eta\bar{\eta}} + h.c., \quad (4.17)$$

$$\begin{aligned} \Upsilon_{C\omega\omega C}^{\eta\bar{\eta}} &= \frac{\eta\bar{\eta}}{16} C * \omega * \omega * C * \\ &\times \left[\frac{1}{2} h_{p_1+2p_2+t_1+t_2} \Delta_{p_1+2p_2+t_1+2t_2} \Delta_{p_1+2p_2+2t_1+2t_2} \gamma * \bar{h}_{\bar{p}_1+2\bar{p}_2+\bar{t}_1+\bar{t}_2} \bar{\Delta}_{\bar{p}_1+2\bar{p}_2+\bar{t}_1+2\bar{t}_2} \bar{\Delta}_{\bar{p}_1+2\bar{p}_2+2\bar{t}_1+2\bar{t}_2} \bar{\gamma} \right. \\ &- h_{p_1+2p_2+t_1+t_2} \Delta_{p_1+2p_2+t_1+2t_2} \Delta_{p_1+2p_2+2t_1+2t_2} \gamma * \bar{h}_{\bar{p}_1+\bar{p}_2+\bar{t}_1+\bar{t}_2} \bar{\Delta}_{\bar{p}_1+2\bar{p}_2+2\bar{t}_1+2\bar{t}_2} \bar{\Delta}_{\bar{p}_2} \bar{\gamma} \\ &\left. + (h_{p_2} - h_{p_1+2p_2+t_1+2t_2}) \Delta_{p_2+t_2} \Delta_{p_1+2p_2+t_1+t_2} \gamma * \bar{h}_{\bar{p}_1+\bar{p}_2+\bar{t}_1+\bar{t}_2} \bar{\Delta}_{\bar{p}_1+2\bar{p}_2+\bar{t}_1+2\bar{t}_2} \bar{\Delta}_{\bar{p}_1+2\bar{p}_2+2\bar{t}_1+2\bar{t}_2} \bar{\gamma} \right] + h.c., \end{aligned} \quad (4.18)$$

$$\begin{aligned} \Upsilon_{\omega C\omega C}^{\eta\bar{\eta}} &= \frac{\eta\bar{\eta}}{16} \omega * C * \omega * C * \\ &\times \left[(h_{p_2} - h_{p_1+2p_2+t_1+2t_2}) \Delta_{p_2+t_2} \Delta_{p_1+2p_2+t_1+t_2} \gamma * \bar{h}_{\bar{p}_1+\bar{p}_2+\bar{t}_1+\bar{t}_2} \bar{\Delta}_{\bar{p}_2} \bar{\Delta}_{\bar{p}_1+2\bar{p}_2+\bar{t}_1+2\bar{t}_2} \bar{\gamma} \right. \\ &- h_{p_1+2p_2+t_1+2t_2} \Delta_{p_2+t_2} \Delta_{p_1+2p_2+t_2} \gamma * \bar{h}_{\bar{p}_1+\bar{p}_2+\bar{t}_1+\bar{t}_2} \bar{\Delta}_{\bar{p}_2+\bar{t}_2} \bar{\Delta}_{\bar{p}_2} \bar{\gamma} \\ &+ (h_{p_2} - h_{p_1+2p_2+2t_2}) \Delta_{p_2+t_2} \Delta_{p_1+2p_2+t_2} \gamma * (\bar{h}_{\bar{p}_1+\bar{p}_2+\bar{t}_1+\bar{t}_2} - \bar{h}_{\bar{p}_1+2\bar{p}_2+2\bar{t}_2}) \bar{\Delta}_{\bar{p}_1+\bar{p}_2+\bar{t}_2} \bar{\Delta}_{\bar{p}_2} \bar{\gamma} \\ &- \frac{1}{2} h_{p_1+2p_2+t_1+t_2} \Delta_{p_1+2p_2+t_1+2t_2} \Delta_{p_2} \gamma * \bar{h}_{\bar{p}_1+2\bar{p}_2+\bar{t}_1+\bar{t}_2} \bar{\Delta}_{\bar{p}_1+2\bar{p}_2+t_1+2\bar{t}_2} \bar{\Delta}_{\bar{p}_2} \bar{\gamma} \\ &+ \frac{1}{2} h_{p_1+2p_2+t_2} \Delta_{p_1+2p_2+2t_2} \Delta_{p_2} \gamma * \bar{h}_{\bar{p}_1+2\bar{p}_2+\bar{t}_2} \bar{\Delta}_{\bar{p}_1+2\bar{p}_2+2\bar{t}_2} \bar{\Delta}_{\bar{p}_2} \bar{\gamma} \\ &+ \frac{1}{2} h_{p_1+2p_2+t_1+2t_2} \Delta_{p_2+t_2} \Delta_{p_2} \gamma * \bar{h}_{\bar{p}_1+2\bar{p}_2+\bar{t}_1+2\bar{t}_2} \bar{\Delta}_{\bar{p}_2+\bar{t}_2} \bar{\Delta}_{\bar{p}_2} \bar{\gamma} \\ &\left. - \frac{1}{2} h_{p_1+2p_2+2t_2} \Delta_{p_2+t_2} \Delta_{p_2} \gamma * \bar{h}_{\bar{p}_1+2\bar{p}_2+2\bar{t}_2} \bar{\Delta}_{\bar{p}_2+\bar{t}_2} \bar{\Delta}_{\bar{p}_2} \bar{\gamma} \right] + h.c., \end{aligned} \quad (4.19)$$

$$\begin{aligned} \Upsilon_{\omega CC\omega}^{\eta\bar{\eta}} &= \frac{\eta\bar{\eta}}{16} \omega * C * C * \omega * \\ &\times \left[\frac{1}{2} h_{p_2} \Delta_{p_2+2t_2} \Delta_{p_1+2p_2+2t_2} \gamma * \bar{h}_{\bar{p}_2+\bar{t}_2} \bar{\Delta}_{\bar{p}_2+2\bar{t}_2} \bar{\Delta}_{\bar{p}_1+2\bar{p}_2+\bar{t}_1+2\bar{t}_2} \bar{\gamma} \right. \\ &- \frac{1}{2} h_{p_1+2p_2+2t_2} \Delta_{p_2+t_2} \Delta_{p_2+2t_2} \gamma * \bar{h}_{\bar{p}_1+2\bar{p}_2+2\bar{t}_2} \bar{\Delta}_{\bar{p}_2+\bar{t}_2} \bar{\Delta}_{\bar{p}_2+2\bar{t}_2} \bar{\gamma} \\ &+ h_{p_2+2t_2} \Delta_{p_2+t_2} \Delta_{p_1+2p_2+t_1+2t_2} \gamma * \bar{h}_{\bar{p}_1+2\bar{p}_2+\bar{t}_1+2\bar{t}_2} \bar{\Delta}_{\bar{p}_2+\bar{t}_2} \bar{\Delta}_{\bar{p}_1+\bar{p}_2+\bar{t}_1+\bar{t}_2} \bar{\gamma} \\ &+ h_{p_2+2t_2} \Delta_{p_2+t_2} \Delta_{p_1+2p_2+2t_2} \gamma * (\bar{h}_{\bar{p}_1+\bar{p}_2+\bar{t}_2} - \bar{h}_{\bar{p}_2+\bar{t}_2}) \bar{\Delta}_{\bar{p}_1+\bar{p}_2+\bar{t}_1+\bar{t}_2} \bar{\Delta}_{\bar{p}_1+2\bar{p}_2+2\bar{t}_2} \bar{\gamma} \\ &\left. + h_{p_2+2t_2} \Delta_{p_1+2p_2+2t_2} \Delta_{p_1+2p_2+t_1+2t_2} \gamma * (\bar{h}_{\bar{p}_1+\bar{p}_2+\bar{t}_1+2\bar{t}_2} - \bar{h}_{\bar{p}_2+\bar{t}_2}) \bar{\Delta}_{\bar{p}_1+\bar{p}_2+\bar{t}_1+\bar{t}_2} \bar{\Delta}_{\bar{p}_2+2\bar{t}_2} \bar{\gamma} \right] + \delta \Upsilon_{\omega CC\omega}^{\eta\bar{\eta}} + h.c., \end{aligned} \quad (4.20)$$

$$\Upsilon_{C\omega C\omega}^{\eta\bar{\eta}} = \frac{\eta\bar{\eta}}{16} C * \omega * C * \omega * \quad (4.21)$$

$$\begin{aligned} & \times \left[(h_{p_1+2p_2+2t_1+2t_2} - h_{p_2+2t_2}) \Delta_{p_2+t_1+2t_2} \Delta_{p_1+2p_2+t_1+2t_2} \gamma * \bar{h}_{\bar{p}_1+\bar{p}_2+\bar{t}_1+2\bar{t}_2} \bar{\Delta}_{\bar{p}_1+\bar{p}_2+\bar{t}_1+\bar{t}_2} \bar{\Delta}_{\bar{p}_1+2\bar{p}_2+2\bar{t}_1+2\bar{t}_2} \bar{\gamma} \right. \\ & \left. - h_{p_2+2t_2} \Delta_{p_2+t_2} \Delta_{p_1+2p_2+t_1+2t_2} \gamma * \bar{h}_{\bar{p}_1+\bar{p}_2+\bar{t}_1+\bar{t}_2} \bar{\Delta}_{\bar{p}_1+2\bar{p}_2+\bar{t}_1+2\bar{t}_2} \bar{\Delta}_{\bar{p}_1+2\bar{p}_2+2\bar{t}_1+2\bar{t}_2} \bar{\gamma} \right] + h.c., \\ \Upsilon_{CC\omega\omega}^{\eta\bar{\eta}} &= \frac{\eta\bar{\eta}}{16} C * C * \omega * \omega * \quad (4.22) \\ & \times h_{p_2+2t_1+2t_2} \Delta_{p_2+t_1+2t_2} \Delta_{p_1+2p_2+2t_1+2t_2} \gamma * \bar{h}_{\bar{p}_1+\bar{p}_2+\bar{t}_1+2\bar{t}_2} \bar{\Delta}_{\bar{p}_1+2\bar{p}_2+2\bar{t}_1+2\bar{t}_2} \bar{\Delta}_{\bar{p}_1+\bar{p}_2+\bar{t}_1+\bar{t}_2} \bar{\gamma} + \delta\Upsilon_{CC\omega\omega}^{\eta\bar{\eta}} + h.c., \end{aligned}$$

where

$$\begin{aligned} \delta\Upsilon_{\omega\omega CC}^{\eta\bar{\eta}} &= \frac{i\eta\bar{\eta}}{8} \int_0^1 d\tau e^{i(t_1+t_2)\alpha} (\tau p_1 + (\tau-1)p_2 - t_1)_\alpha \omega(y, \bar{y}) \bar{*} \omega(y, \bar{y}) \bar{*} C(\tau y, \bar{y}) \bar{*} C((\tau-1)y, \bar{y}) k * \\ & \times * \bar{h}_{\bar{p}_1+\bar{p}_2+\bar{t}_1+\bar{t}_2} \bar{\Delta}_{\bar{p}_1+\bar{p}_2+\bar{t}_2} \bar{\Delta}_{\bar{p}_1+\bar{p}_2} \bar{\gamma} \quad (4.23) \end{aligned}$$

$$\begin{aligned} \delta\Upsilon_{\omega CC\omega}^{\eta\bar{\eta}} &= \frac{i\eta\bar{\eta}}{8} \int_0^1 d\tau e^{i(t_1+t_2)\alpha} (\tau p_1 + (\tau-1)p_2 + t_1)_\alpha \omega(y, \bar{y}) \bar{*} C(\tau y, \bar{y}) \bar{*} C((\tau-1)y, \bar{y}) \bar{*} \omega(-y, \bar{y}) k * \\ & \times * (\bar{h}_{\bar{p}_1+\bar{p}_2+\bar{t}_1+\bar{t}_2} - \bar{h}_{\bar{p}_1+\bar{p}_2+2\bar{t}_2}) \bar{\Delta}_{\bar{p}_1+\bar{p}_2+\bar{t}_1+2\bar{t}_2} \bar{\Delta}_{\bar{p}_1+\bar{p}_2+\bar{t}_2} \bar{\gamma} \quad (4.24) \end{aligned}$$

$$\begin{aligned} \delta\Upsilon_{CC\omega\omega}^{\eta\bar{\eta}} &= \frac{i\eta\bar{\eta}}{8} \int_0^1 d\tau e^{i(t_1+t_2)\alpha} (\tau p_1 + (\tau-1)p_2 - t_1)_\alpha C(\tau y, \bar{y}) \bar{*} C((\tau-1)y, \bar{y}) \bar{*} \omega(-y, \bar{y}) \bar{*} \omega(-y, \bar{y}) k * \\ & \times * \bar{h}_{\bar{p}_1+\bar{p}_2+\bar{t}_1+\bar{t}_2} \bar{\Delta}_{\bar{p}_1+\bar{p}_2+2\bar{t}_1+2\bar{t}_2} \bar{\Delta}_{\bar{p}_1+\bar{p}_2+\bar{t}_1+2\bar{t}_2} \bar{\gamma}, \quad (4.25) \end{aligned}$$

where $\delta\Upsilon^{\eta\bar{\eta}}$'s show up from specific local and z -independent field redefinition of field B designed to make lower order vertices to contain minimal number of derivative contractions. The redefinition of this type is introduced in section 5.2.1.

Using also that $f(y) * \bar{f}(\bar{y}) = f(y)\bar{f}(\bar{y})$ for any $f(y)$ and $\bar{f}(\bar{y})$ and, hence,

$$h_a \Delta_b \Delta_c \gamma * \bar{h}_{\bar{a}} \bar{\Delta}_{\bar{b}} \bar{\Delta}_{\bar{c}} \bar{\gamma} = h_a \Delta_b \Delta_c \gamma \bar{h}_{\bar{a}} \bar{\Delta}_{\bar{b}} \bar{\Delta}_{\bar{c}} \bar{\gamma} \quad (4.26)$$

we observe that the resulting $\eta\bar{\eta}$ vertices are free from contractions between spinor indices of the zero-forms C in either holomorphic or antiholomorphic sectors (or both). This can be easily seen from the fact shown in [3] that the expression

$$C * C * h_{a_1 p_1 + a_2 p_2} \Delta_{b_1 p_1 + b_2 p_2} \Delta_{c_1 p_1 + c_2 p_2} \gamma \quad (4.27)$$

is local provided that

$$a_2 - a_1 = b_2 - b_1 = c_2 - c_1 = 1. \quad (4.28)$$

One can see that all vertices (4.17)–(4.22) meet this condition.

5 Derivation details

It is natural to expect that calculation of vertices (4.2)–(4.4) should be quite involved. To compute W_2 one needs B_2 and S_2 which potentially leads to complicated analysis. Vertex $\Upsilon^{\eta\bar{\eta}}$ however comes from the cross-product of holomorphic and anti-holomorphic first-order fields and therefore is simpler. In particular, in this sector the analysis is β -independent because of the properties considered in section 3.2.3. The challenge is to calculate (4.2). Somewhat surprisingly the analysis in the holomorphic sector simplifies due to a remarkable

cancellation that takes place for the S_2 contribution to the vertices. The cancellation is related to a structure relation found in [9]. We will show that in the limit $\beta \rightarrow -\infty$ one can ignore (anti)holomorphic part of the S field to the second order as it turns out to give no contribution to (anti)holomorphic vertices provided one fixes B_2 corresponding to the minimal local couplings in $\Upsilon(\omega, C, C)$ found in [26].

5.1 First order

From (2.4) we have at first order

$$-2id_Z S_1 = i\eta C * \gamma + i\bar{\eta} C * \bar{\gamma}, \quad (5.1)$$

which is solved as

$$S_1 = -\frac{\eta}{2} \Delta_{0,\beta} (C * \gamma) + h.c. = -\frac{\eta}{2} C * \Delta_{p,0} \gamma + h.c., \quad (5.2)$$

where in the last line we made use of (3.10) and (3.24). We see that though contracting homotopy (3.1) depends on β the result for S_1 is β -independent thanks to special properties (3.24) of central elements (2.9). Thus obtained S_1 is therefore identical to the one resulting from the conventional contracting homotopy $\Delta_{0,0}$.

The situation with W_1 is analogous. From (2.2) at first order we have

$$2id_Z W_1 = d_x S_1 + \omega * S_1 + S_1 * \omega + h.c.. \quad (5.3)$$

Solving it using $\Delta_{0,\beta}$ (recall that due to (3.21) and (3.5), (5.2) $\Delta_{0,\beta} (d_x S_1) = 0$)

$$W_1 = \frac{1}{2i} \Delta_{0,\beta} (\omega * S_1 + S_1 * \omega) + h.c. \quad (5.4)$$

one finds upon using (3.10), (3.12) and (3.24) that

$$W_1 = -\frac{\eta}{4i} (C * \omega * \Delta_{p+t,0} \Delta_{p+2t,0} \gamma - \omega * C * \Delta_{p+t,0} \Delta_{p,0} \gamma) + h.c.. \quad (5.5)$$

Again, W_1 is β -independent and equal to the one obtained via conventional contracting homotopy. This makes in particular vertex $\Upsilon(\omega, \omega, C)$ β -independent. Note that for $\beta = 1$ the invariance of on-shell theorem was also demonstrated in [36].

5.2 Second order. η^2 sector

5.2.1 Solving for B_2

To find B_2 following [3] we use (2.5) up to the second order which amounts to

$$-2id_Z B_2 + [S_1, C]_* = 0. \quad (5.6)$$

Substituting S_1 from (5.2) and using (3.10) one gets (similarly in the $\bar{\eta}$ -sector)

$$-2id_Z B_2 = -\frac{\eta}{2} C * C * \Delta_{p_2,0} \gamma + \frac{\eta}{2} C * \Delta_{p_1,0} \gamma * C = -\frac{\eta}{2} C * C * (\Delta_{p_2,0} - \Delta_{p_1+2p_2,0}) \gamma, \quad (5.7)$$

where p_1 and p_2 are derivatives (4.8) of the first and second factors of C , respectively.

The field B belongs to the PLT-odd sector of HS equation in nomenclature of [2]. Therefore, contracting homotopy corresponding to the local frame should satisfy (2.26). At this order this leads us to use $\Delta_{(1-\beta)(v_1 p_1 + v_2 p_2), \beta}$ for solving B_2 , where $v_2 - v_1 = 1$. The overall prefactor of $(1 - \beta)$ appears due to β -induced rescaling (3.15) (see [9] for the corresponding PLT modification). One can choose any value for the remaining $v_1 + v_2$ parameter. In [3] it was shown that the resulting B_2 is independent of that free parameter. So, we can take $v_2 = 1$ and $v_1 = 0$ for convenience

$$B_2 = \frac{\eta}{4i} \Delta_{(1-\beta)p_2, \beta} (C * C * (\Delta_{p_2, 0} - \Delta_{p_1+2p_2, 0}) \gamma). \quad (5.8)$$

Again, using (3.10) and (3.24) we end up with the β -independent result from [3]

$$B_2 = \frac{\eta}{4i} C * C * \Delta_{p_1+2p_2, 0} \Delta_{p_2, 0} \gamma + h.c.. \quad (5.9)$$

This entails that vertex $\Upsilon(\omega, C, C)$ is also β independent (see (C.6)–(C.14)).

An important comment is as follows. While B_2 does reproduce local vertices in accordance with the PLT they do not carry minimal amount of derivatives. It was shown in [26] that the number of derivatives can be reduced for

$$B_2^{\min} = B_2 + \delta B_2(y), \quad (5.10)$$

$$\delta B_2(y) = \frac{\eta}{2} \int_0^1 d\tau C(\tau y, \bar{y}; K) \bar{*} C((\tau - 1)y, \bar{y}; K) k + h.c. \quad (5.11)$$

with B_2 (5.9). It is interesting to know if the field redefinition δB_2 admits any shifted homotopy representation. In section 6.2 it will be shown that some limiting representation does exist.

5.2.2 Solving for S_2 and W_2

Equation for S_2 resulting from (2.4) in the η^2 sector is

$$d_z S_2 = \frac{i}{2} (i\eta B_2 * \gamma - S_1 * S_1). \quad (5.12)$$

The η^2 part of S_2 belongs to PLT-even locality class. Therefore, it can be solved using $\Delta_{0, \beta}$

$$S_2 = \frac{i}{2} \Delta_{0, \beta} (i\eta B_2 * \gamma - S_1 * S_1). \quad (5.13)$$

In [9] it is shown, that so obtained S_2 (5.13) belongs to a specific class of functions $\mathcal{H}^{+0}$ which contribution to dynamical equations of motion (2.1) is ultra-local in the limit $\beta \rightarrow -\infty$ in the given order of perturbation theory. Moreover, we will show in section 6.2 that such contribution is just zero for B_2 chosen from (5.10). This implies in particular that in solving for holomorphic part of W_2 one can ignore the contribution due to S_2 .

Equation for W_2 has the form

$$d_z W_2 = \frac{1}{2i} (d_x S_1 + d_x S_2 + W_1 * S_1 + S_1 * W_1 + \omega * S_2 + S_2 * \omega). \quad (5.14)$$

Again, the η^2 -part of W_2 belongs to PLT-even class and can be solved using $\Delta_{0,\beta}$. This results in that $d_x S_1$ as well as $d_x S_2$ vanish after applying $\Delta_{0,\beta}$ because, as discussed in section 3.2, $\{d_x, \Delta_{0,\beta}\} = 0$ and $\Delta_{0,\beta} \Delta_{0,\beta} = 0$. As argued above, terms that result from S_2 can be omitted. Hence the part of W_2 to be taken into account analyzing the dynamical equations is

$$W_2 \simeq \frac{1}{2i} \Delta_{0,\beta} (W_1 * S_1 + S_1 * W_1), \quad (5.15)$$

where $\simeq$ means equality up to terms coming from S_2 , which we collectively denote by W_2'

$$W_2' = \frac{1}{2i} \Delta_{0,\beta} (\omega * S_2 + S_2 * \omega). \quad (5.16)$$

This piece will be shown in section 6.2 not to contribute to the final vertex provided S_2 itself is solved for using (5.10). Substituting W_1 from (5.5) and S_1 from (5.2) we obtain the final result which we leave for appendix B4.

Vertex $\Upsilon(\omega, \omega, C, C)$ comes about in two pieces (4.6). Each contains $\Upsilon^{\eta\eta}$, $\Upsilon^{\bar{\eta}\bar{\eta}}$ and $\Upsilon^{\eta\bar{\eta}}$. We start with $\Upsilon^{\eta\eta}$. From that $\Upsilon^{\bar{\eta}\bar{\eta}}$ can be obtained by swapping $y \leftrightarrow \bar{y}$, $\eta \leftrightarrow \bar{\eta}$.

5.2.3 η^2 vertex $\Upsilon^{\eta\eta}$

Vertex $\Upsilon^{\eta\eta}$ contains six structures (4.2). Computation of each of these is pretty similar. Some of the structures result from one of the two pieces (4.6). Namely, $\Upsilon_{CC\omega\omega}^{\eta\eta}$ and $\Upsilon_{\omega\omega CC}^{\eta\eta}$ belong to $h_{0,\beta}\{\omega, W_2\}_*$, while $\Upsilon_{C\omega\omega C}^{\eta\eta}$ is in $h_{0,\beta}(W_1 * W_1)$. The rest acquire contributions from the two simultaneously. To simplify presentation we calculate here $\Upsilon_{C\omega\omega C}^{\eta\eta}$ and $\Upsilon_{CC\omega\omega}^{\eta\eta}$ only each coming from a single piece of different origin.

We begin with $\Upsilon_{CC\omega\omega}^{\eta\eta}$. From (4.6) this one is given by

$$\Upsilon_{CC\omega\omega}^{\eta\eta}(\beta) = -h_{0,\beta}(W_2 * \omega) \Big|_{CC\omega\omega} \simeq -\frac{1}{2i} h_{0,\beta}(\Delta_{0,\beta} (S_1 * W_1) * \omega) \Big|_{CC\omega\omega}, \quad (5.17)$$

where $\simeq$ means that terms that do not contribute to the vertex at $\beta \rightarrow -\infty$ are discarded. For the exact equality one should add contribution from S_2 to W_2 , but that just vanishes in the vertex $\Upsilon(-\infty)$ as shown in section 6.2.

Using (B.1), (B.2) and (B.13) we find the following result⁵

$$\begin{aligned} \Upsilon_{CC\omega\omega}^{\eta\eta}(\beta) = & -\frac{\eta^2}{4} C \bar{*} C \bar{*} \omega \bar{*} \omega \int_0^1 d\sigma \int_0^1 d\tau_1 \int d_\Delta^3 \tau' \frac{1-\beta}{\tau_1' \xi^2} t_2^\alpha t_1^\beta \frac{\partial}{\partial p_1^\alpha} \frac{\partial}{\partial p_2^\beta} \times \\ & \times \exp i(y^\alpha a_\alpha + a_1 t_1^\alpha t_{2\alpha} + a_2 t_2^\alpha p_{1\alpha} + a_3 t_1^\alpha p_{2\alpha} + a_4 t_2^\alpha p_{2\alpha} + a_5 t_1^\alpha p_{1\alpha} + a_6 p_1^\alpha p_{2\alpha}), \end{aligned} \quad (5.18)$$

⁵Expressions containing $\bar{*}$ should be understood as follows. Star product $\bar{*}$ involves antiholomorphic oscillators $\bar{y}$ only, while fields ω and C upon being differentiated are taken at $y = 0$. For example below in (5.18) C 's and ω 's on the left should be understood as $C(0, \bar{y})$ and $\omega(0, \bar{y})$ while their y -dependence is placed into integrals by means of the identity $f(y) = f(0)e^{-ip^\alpha y_\alpha}$.

$$a^\alpha = \frac{1}{\xi} (\tau_1 \tau'_1 (p_1 + p_2 + t_1) + (1 - \tau_1) \tau'_3 t_1 + (1 - \sigma \tau_o) t_2 - \beta \tau_o (1 - \sigma) t_2)^\alpha, \quad (5.19)$$

$$a_1 = \frac{1}{\xi} (\tau_1 \tau'_1 + \tau'_3 (1 - \tau_1)) + \frac{1 - \beta}{\xi} \sigma (\tau'_1 (1 - \tau_1) + \tau_1 \tau'_3), \quad (5.20)$$

$$a_2 = \frac{1 - \beta}{\xi} \sigma \tau_1 (1 - \tau'_1) - \frac{1}{\xi} \tau_1 \tau'_1, \quad (5.21)$$

$$a_3 = \frac{\beta}{\xi} \tau'_1 \tau'_3 (1 - 2\tau_1) + \tau'_3 - 1, \quad (5.22)$$

$$a_4 = -\frac{\tau_1 \tau'_1}{\xi} - \frac{1 - \beta}{\xi} \sigma \tau'_1 (1 - \tau_1), \quad (5.23)$$

$$a_5 = -\frac{\tau_1 \tau'_1}{\xi} + \frac{1 - \beta}{\xi} \tau_1 \tau'_3, \quad (5.24)$$

$$a_6 = \frac{\tau_1 \tau'_1}{\xi}, \quad (5.25)$$

$$\tau_o := \tau_1 \circ \tau'_1 = \tau_1 + \tau'_1 - 2\tau_1 \tau'_1, \quad (5.26)$$

$$\xi = 1 - \beta \tau_o, \quad (5.27)$$

where p_i, t_j are defined in (4.8), (4.9). Homotopy integration variable σ appears from the application of $\Delta_{0,\beta}$ (3.1), τ_1 from S_1 and τ'_{1-3} from W_1 . Since we consider the vertex in the limit $\beta \rightarrow -\infty$ it was not necessary to carry out exact calculation (5.18) as lots of terms vanish in this limit. However we would like to demonstrate that the expression provided is perfectly well defined for any $\beta < 1$. For finite $\beta < 1$ the integrand has no poles and therefore the result is finite. For $\beta \rightarrow -\infty$ it is easy to see that each a_i in the exponential is bounded by a β -independent constant due to inequalities

$$\tau'_3 \leq 1 - \tau'_1, \quad \tau_1 (1 - \tau'_1) \leq \tau_o, \quad \tau'_1 (1 - \tau_1) \leq \tau_o. \quad (5.28)$$

Hence, the integration behavior depends on the pre-exponential that forms in (5.18) after differentiation over p_1 and p_2 . It acquires the following leading form at large β

$$\frac{\beta^3 \tau \tau_1 \tau'_3 (1 - \tau'_1) (1 - \tau_1)}{(1 + \beta \tau_o)^4} \leq \frac{\beta^2}{(1 - \beta \tau_o)^3}, \quad (5.29)$$

which is integrable and has finite limit at $\beta \rightarrow -\infty$.

To take the limit we first integrate (5.18) over τ'_2 which is only present in the measure $d^3_\Delta \tau'$. It gives the integration domain for $\tau'_3 \in [0, 1 - \tau'_1]$ which makes it convenient introducing $\tau'_3 = (1 - \tau'_1) \sigma'$, where $\sigma' \in [0, 1]$ is the new integration variable. The vertex then reduces to

$$\begin{aligned} \Upsilon_{CC\omega\omega}^{\eta\eta}(\beta) = & -\frac{1}{4} C^* \bar{C}^* \bar{\omega}^* \bar{\omega} \int_{[0,1]^2} d\sigma d\sigma' \sigma \sigma' \int_{[0,1]^2} d\tau_1 d\tau'_1 \frac{\beta^3 \tau_1 (1 - \tau'_1)^2 (1 - \tau_1)}{(1 - \beta \tau_o)^4} \times \\ & \times (t_1 t_2)^2 \exp i \left(-\frac{\beta}{\xi} \tau_1 (1 - \tau'_1) A - \frac{\beta}{\xi} \tau'_1 (1 - \tau_1) B + \frac{1}{\xi} C + D \right) + O(1/\beta), \end{aligned} \quad (5.30)$$

where

$$A = \sigma\sigma' t_1^\alpha t_{2\alpha} + \sigma t_2^\alpha p_{1\alpha} + \sigma' t_1^\alpha p_{1\alpha} - \sigma y^\alpha t_{2\alpha}, \quad (5.31)$$

$$B = \sigma t_1^\alpha t_{2\alpha} - \sigma' t_1^\alpha p_{2\alpha} - \sigma t_2^\alpha p_{2\alpha} - \sigma y^\alpha t_{2\alpha}, \quad (5.32)$$

$$C = \sigma' t_1^\alpha t_{2\alpha} + \sigma' y^\alpha t_{1\alpha}, \quad D = y^\alpha t_{2\alpha} + (\sigma' - 1)t_1^\alpha p_{2\alpha}. \quad (5.33)$$

The last step is to take limit $\beta \rightarrow -\infty$ which we do using (B.19). This gives

$$\Upsilon_{C\omega\omega}^{\eta\eta} = \frac{1}{4} C^* \bar{C}^* \omega^* \bar{\omega}^* (t_1 t_2)^2 \int_{[0,1]^2} d\sigma d\sigma' \sigma \sigma' \int d_\Delta^3 \tau \tau_1 \exp i(\tau_1 A + \tau_2 B + \tau_3 C + D). \quad (5.34)$$

Recalling the definition of p_i (4.8) one rewrites the result in the form (4.11).

Vertex $\Upsilon_{C\omega\omega C}^{\eta\eta}$ can be worked out analogously. One calculates

$$\Upsilon_{C\omega\omega C}^{\eta\eta}(\beta) = -h_{0,\beta}(W_1 * W_1) \Big|_{C\omega\omega C} \quad (5.35)$$

at finite β and then sends $\beta \rightarrow -\infty$. Again, direct computation using (B.2) and (3.3) gives well defined expression for any $\beta < 1$. Up to terms that vanish for large $-\beta$ the final result reads

$$\begin{aligned} \Upsilon_{C\omega\omega C}^{\eta\eta}(\beta) &= \frac{1}{4} C^* \bar{\omega}^* \omega^* \bar{C}^* (t_1 t_2)^2 \int_{[0,1]^2} d\sigma d\sigma' \sigma \sigma' \int_{[0,1]^2} d\tau_1 d\tau'_1 \frac{\beta^2 (1 - \tau_1)(1 - \tau'_1)}{(1 - \beta\tau_\circ)^4} \times \\ &\times \exp i \left(-\frac{\beta}{\xi} \tau_1 (1 - \tau'_1) \tilde{A} - \frac{\beta}{\xi} \tau'_1 (1 - \tau_1) \tilde{B} + \frac{1}{\xi} \tilde{C} + \tilde{D} \right) + O(1/\beta) \end{aligned} \quad (5.36)$$

with

$$\tilde{A} = -(\sigma t_1 + \sigma' t_2)^\alpha (p_1 + t_1 + \sigma' t_2)_\alpha, \quad \tilde{B} = (\sigma t_1 + \sigma' t_2)^\alpha (p_2 + t_2 + \sigma t_1)_\alpha, \quad (5.37)$$

$$\tilde{C} = (\sigma t_1 + \sigma' t_2)^\alpha y_\alpha, \quad \tilde{D} = p_1^\alpha t_{1\alpha} - p_2^\alpha t_{2\alpha} - t_2^\alpha t_{1\alpha} \sigma \sigma'. \quad (5.38)$$

Taking the limit $\beta \rightarrow -\infty$ using (B.20) we find

$$\Upsilon_{C\omega\omega C}^{\eta\eta} = \frac{1}{4} C^* \bar{\omega}^* \omega^* \bar{C}^* (t_{1\alpha} t_2^\alpha)^2 \int_{[0,1]^2} d\sigma d\sigma' \sigma \sigma' \int d_\Delta^3 \tau \tau_3 \exp i(\tau_1 \tilde{A} + \tau_2 \tilde{B} + \tau_3 \tilde{C} + \tilde{D}), \quad (5.39)$$

which can be rewritten as (4.12).

The rest of the structures in (4.2) and (4.3) are calculated analogously. In all these cases one encounters the following integrals that should be calculated at $\beta \rightarrow -\infty$

$$\begin{aligned} I_1(\beta) &= - \int_{[0,1]^2} d\tau d\tau' \frac{\beta^3 \tau (1 - \tau')^2 (1 - \tau)}{(1 - \beta\tau \circ \tau')^4} \\ &\times \exp i \left(-\frac{\beta}{\xi} \tau (1 - \tau') A - \frac{\beta}{\xi} \tau' (1 - \tau) B + \frac{1}{\xi} C + D \right), \end{aligned} \quad (5.40)$$

$$\begin{aligned} I_2(\beta) &= \int_{[0,1]^2} d\tau d\tau' \frac{\beta^2 (1 - \tau') (1 - \tau)}{(1 - \beta\tau \circ \tau')^4} \\ &\times \exp i \left(-\frac{\beta}{\xi} \tau (1 - \tau') A - \frac{\beta}{\xi} \tau' (1 - \tau) B + \frac{1}{\xi} C + D \right), \end{aligned} \quad (5.41)$$

where $\xi = 1 - \beta\tau \circ \tau'$ and A, B, C and D do not depend on τ and τ' . As proven in appendix B3, the limiting values of these are given in terms of integrals over a simplex

$$I_1(-\infty) = \int d_{\Delta}^3 \tau \tau_1 \exp i(\tau_1 A + \tau_2 B + \tau_3 C + D), \quad (5.42)$$

$$I_2(-\infty) = \int d_{\Delta}^3 \tau \tau_3 \exp i(\tau_1 A + \tau_2 B + \tau_3 C + D). \quad (5.43)$$

Let us stress that the resulting expressions turn out to be ultra-local because the coefficients A, B, C and D (5.31)–(5.33) and $\tilde{A}, \tilde{B}, \tilde{C}$ and $\tilde{D}$ (5.37), (5.38) do not contain $p_{1\alpha}p_2^\alpha$, that implies locality, and $p_{i\alpha}y^\alpha$, that implies that arguments of the zero-forms C are y -independent.

5.3 Second order. $\eta\bar{\eta}$ sector

Our strategy for extracting $\Upsilon^{\eta\bar{\eta}}$ will be as follows. We first solve equations for fields S_2 and W_2 that depend on B_2 using (5.9). Having expressions for W_1 and W_2 we then calculate the mixed vertices $\Upsilon^{\eta\bar{\eta}}$ via (4.1). This way we get the result that does not account for the local field redefinition (5.10). Its effect (5.11) leads to the change of $W_2 \rightarrow W_2 + \delta W_2$ in both holomorphic and mixed sectors and therefore changes vertices $\Upsilon^{\eta\bar{\eta}} \rightarrow \Upsilon^{\eta\bar{\eta}} + \delta\Upsilon^{\eta\bar{\eta}}$. That change can be easily taken into account provided δW_2 is solved by using conventional contracting homotopy Δ_0 . The choice of Δ_0 is just a matter of convenience since for any local field redefinitions any homotopy results in a local contribution to the final vertex at given order. This allows one using (3.21) to extract $\delta\Upsilon^{\eta\bar{\eta}}$ from (4.1) as

$$\delta\Upsilon^{\eta\bar{\eta}} = -h_0\{\omega, \delta W_2\}_*. \quad (5.44)$$

Recall also that since second order $\eta\bar{\eta}$ -sector originate from products of first order holomorphic $\times$ antiholomorphic contributions parameter β drops out and one can set $\beta = 0$ in that calculation.

5.3.1 Mixed contracting homotopy

To evaluate $\Upsilon^{\eta\bar{\eta}}$ one needs to solve equation of the type (2.14) with r.h.s. containing one-forms proportional to θ^α and $\bar{\theta}^{\dot{\alpha}}$ or two-forms proportional to $\theta^\alpha\bar{\theta}^{\dot{\alpha}}$. There are several approaches that can be used to solve the problem. One is to apply a total contracting homotopy Δ^{tot} with respect to $Z^A = (z^\alpha, \bar{z}^{\dot{\alpha}})$ and $\Theta^A = (\theta^\alpha, \bar{\theta}^{\dot{\alpha}})$. Another one is to use the spectral sequence approach with respect to the shifted contracting homotopy Δ_q (2.18), $\bar{\Delta}_{\bar{q}}$ and cohomology projectors $h_q \equiv h_{q,\beta}$ and $\bar{h}_{\bar{q}}$, introduced in [3]

$$h_q J(z, y; \theta) = J(-q, y; 0). \quad (5.45)$$

This means that starting from, say, holomorphic sector with no $\bar{\theta}_\alpha$ dependence one reconstructs $\theta^p\bar{\theta}^{\bar{p}}$ -forms by solving equation

$$d_z f^{p,\bar{p}} = g^{p+1,\bar{p}} - d_{\bar{z}} f^{p+1,\bar{p}-1} \quad (5.46)$$

with the help of some Δ_q step by step increasing $\bar{p}$ up to the last step in the anti-holomorphic sector with the equation

$$d_{\bar{z}} f^{0, \bar{p}-1} = g^{0, \bar{p}} \quad (5.47)$$

to be solved with the help of $\bar{\Delta}_{\bar{q}}$ for some $\bar{q}$.

Both of these approaches are inconvenient for one reason or another. The total contracting homotopy Δ_{tot} does not preserve the PLT structure with respect to holomorphic and anti-holomorphic variables separately. The spectral sequence approach (5.47) treats the holomorphic and anti-holomorphic sectors asymmetrically.

It is more convenient to use a mixed holomorphic-antiholomorphic symmetric operator

$$\tilde{\Delta}_{q\bar{q}} := \frac{1}{2}(\Delta_q + \bar{\Delta}_{\bar{q}}). \quad (5.48)$$

Using formulas from [3] we have

$$\{d_Z, \tilde{\Delta}_{q\bar{q}}\} = 1 - \tilde{h}_{q\bar{q}}, \quad (5.49)$$

where

$$\tilde{h}_{q\bar{q}} := \frac{1}{2}(h_q + \bar{h}_{\bar{q}}). \quad (5.50)$$

Here contracting homotopy $\bar{\Delta}_{\bar{q}}$ and respective projector $\bar{h}_{\bar{q}}$ can be obtained from Δ_q (2.18) h_q and (5.45) by swapping $z \rightarrow \bar{z}$, $q \rightarrow \bar{q}$.

As follows from its definition, $\tilde{h}_{q\bar{q}}$ is *not* a projector to d_Z -cohomology. Nevertheless, equations of the type (2.14) can be solved by (5.49) if the r.h.s. is annihilated by $\tilde{h}_{q\bar{q}}$. As we shall see, for this to be true, the contracting homotopy indices q and $\bar{q}$ should be chosen appropriately. Note that the analysis of the mixed sector is β -independent.

For the future convenience we present here the following useful formulae derived in [3]:

$$\Delta_b - \Delta_a = [d_z, \Delta_a \Delta_b] + h_a \Delta_b, \quad (5.51)$$

$$(\Delta_b - \Delta_a)\gamma = d_z \Delta_a \Delta_b \gamma, \quad (5.52)$$

$$(h_a \Delta_b \Delta_c - h_a \Delta_b \Delta_d - h_a \Delta_d \Delta_c + h_b \Delta_d \Delta_c)\gamma = 0, \quad (5.53)$$

$$(\Delta_a - \Delta_b)(\Delta_c - \Delta_d)\gamma = (h_a - h_b) \Delta_c \Delta_d \gamma. \quad (5.54)$$

5.3.2 Solving for S_2 and W_2

Equation for S_2 resulting from (2.4) in the mixed sector is

$$d_Z S_2|_{\eta\bar{\eta}} = \frac{i}{2}(i\eta B_2^\eta * \bar{\gamma} + i\eta B_2^{\bar{\eta}} * \gamma - S_1^{\bar{\eta}} * S_1^\eta - S_1^\eta * S_1^{\bar{\eta}}) \quad (5.55)$$

with S_1 (5.2) and B_2 (5.9). Using star-exchange formulae (3.10)–(3.13) with $\beta = 0$, equation for S_2 in the $\eta\bar{\eta}$ sector can be brought to the form

$$d_Z S_2|_{\eta\bar{\eta}} = -\frac{i\eta\bar{\eta}}{8} C * C * \left[\Delta_{p_2} \Delta_{p_1+2p_2} \gamma * \bar{\gamma} + \bar{\Delta}_{\bar{p}_2} \bar{\Delta}_{\bar{p}_1+2\bar{p}_2} \bar{\gamma} * \gamma \right. \\ \left. + \Delta_{p_1+2p_2} \gamma * \bar{\Delta}_{\bar{p}_2} \bar{\gamma} - \Delta_{p_2} \gamma * \bar{\Delta}_{\bar{p}_1+2\bar{p}_2} \bar{\gamma} \right]. \quad (5.56)$$

According to [2] the r.h.s. of (5.56) is *totally PLT-odd*, i.e., PLT-odd with respect to both holomorphic and anti-holomorphic variables. Hence, the proper choice demands both q and $\bar{q}$ be PLT-odd obeying (2.26) and the conjugated condition, respectively. Remarkably, the r.h.s. of eq. (5.56) is annihilated by $\tilde{h}_{q'\bar{q}'}$ if

$$q' = -\mu p_2 - (1 + \mu)p_1, \quad \bar{q}' = -\bar{\mu}\bar{p}_2 - (1 + \bar{\mu})\bar{p}_1, \quad \forall \mu, \bar{\mu} \in \mathbb{C} \quad (5.57)$$

as one can see using the following formula that holds for any μ [3]

$$h_{(1+\mu)q_2-\mu q_1} \Delta_{q_1} \Delta_{q_2} \gamma = 0 \quad (5.58)$$

along with the following corollary of (3.11)

$$\tilde{h}_{q'\bar{q}'} C * C * (\dots) = C * C * \tilde{h}_{(q'+p_1+p_2)(\bar{q}'+\bar{p}_1+\bar{p}_2)}(\dots). \quad (5.59)$$

Evidently q' and $\bar{q}'$ (5.57) are PLT-odd.

Hence one can solve (5.56) using $\tilde{\Delta}_{q'\bar{q}'}$, namely

$$\begin{aligned} S_2|_{\eta\bar{\eta}}^{\mu\bar{\mu}} = & -\frac{i\eta\bar{\eta}}{16} C * C * \left[\Delta_{(1-\mu)p_2-\mu p_1} + \bar{\Delta}_{(1-\bar{\mu})\bar{p}_2-\bar{\mu}\bar{p}_1} \right] \left[\Delta_{p_2} \Delta_{p_1+2p_2} \gamma * \bar{\gamma} + \bar{\Delta}_{\bar{p}_2} \bar{\Delta}_{\bar{p}_1+2\bar{p}_2} \bar{\gamma} * \gamma \right. \\ & \left. + \Delta_{p_1+2p_2} \gamma * \bar{\Delta}_{\bar{p}_2} \bar{\gamma} - \Delta_{p_2} \gamma * \bar{\Delta}_{\bar{p}_1+2\bar{p}_2} \bar{\gamma} \right]. \quad (5.60) \end{aligned}$$

We set $\mu = \bar{\mu} = -1$ since some of the formulas simplify for this choice. This yields

$$S_2|_{\eta\bar{\eta}} = -\frac{i\eta\bar{\eta}}{8} C * C * \left[\Delta_{p_2} \Delta_{p_1+2p_2} \gamma * \bar{\Delta}_{\bar{p}_1+2\bar{p}_2} \bar{\gamma} + \Delta_{p_1+2p_2} \gamma * \bar{\Delta}_{\bar{p}_2} \bar{\Delta}_{\bar{p}_1+2\bar{p}_2} \bar{\gamma} \right]. \quad (5.61)$$

Equation for W_2 resulting from (2.4) in the mixed sector is

$$d_Z W_2^{\eta\bar{\eta}} = -\frac{i}{2} (d_x S_1 + d_x S_2 + W_1 * S_1 + S_1 * W_1 + \omega * S_2 + S_2 * \omega)|_{\eta\bar{\eta}}. \quad (5.62)$$

$W_2^{\eta\bar{\eta}}$ contains three types of terms

$$W_2^{\eta\bar{\eta}} = W_{2\omega CC}^{\eta\bar{\eta}} + W_{2C\omega C}^{\eta\bar{\eta}} + W_{2CC\omega}^{\eta\bar{\eta}}, \quad (5.63)$$

where the lower label refers to the ordering of fields ω and C . From (5.62) it follows

$$d_Z W_{2CC}^{\eta\bar{\eta}} = \frac{\eta\bar{\eta}}{16} \omega * C * C * \left[\underbrace{h_{p_2} \Delta_{p_1+2p_2} \Delta_{p_1+2p_2+t} \gamma * \bar{\Delta}_{\bar{p}_1+\bar{p}_2+\bar{t}} \bar{\gamma}}_{d_x S_1} \right] \quad (5.64)$$

$$+ \underbrace{\Delta_{p_2} \Delta_{p_1+2p_2+t} \gamma * \bar{\Delta}_{\bar{p}_1+2\bar{p}_2+\bar{t}} \bar{\gamma}}_{d_x S_2} + \underbrace{\Delta_{p_1+2p_2+t} \Delta_{p_1+2p_2} \gamma * \bar{\Delta}_{\bar{p}_2} \bar{\gamma}}_{W_1 \omega C * S_1} - \underbrace{\Delta_{p_2} \Delta_{p_1+2p_2} \gamma * \bar{\Delta}_{\bar{p}_1+2\bar{p}_2} \bar{\gamma}}_{\omega * S_2} \Big] + h.c., \quad (5.65)$$

$$d_Z W_{2\omega C}^{\eta\bar{\eta}} = \frac{\eta\bar{\eta}}{16} C * \omega * C * \left[\underbrace{(h_{p_1+2p_2+2t} - h_{p_2}) \Delta_{p_2+t} \Delta_{p_1+2p_2+t} \gamma * \bar{\Delta}_{\bar{p}_1+\bar{p}_2+\bar{t}} \bar{\gamma}}_{d_x S_1} - \underbrace{\Delta_{p_2} \Delta_{p_1+t+2p_2} \gamma * \bar{\Delta}_{\bar{p}_1+2\bar{p}_2+\bar{t}} \bar{\gamma} + \Delta_{p_2+t} \Delta_{p_1+2p_2+2t} \gamma * \bar{\Delta}_{\bar{p}_1+2\bar{p}_2+2\bar{t}} \bar{\gamma}}_{d_x S_2} \right. \\ \left. - \underbrace{\Delta_{p_1+2p_2+t} \Delta_{p_1+2p_2+2t} \gamma * \bar{\Delta}_{\bar{p}_2} \bar{\gamma}}_{W_1 C \omega * S_1} - \underbrace{\Delta_{p_2+t} \Delta_{p_2} \gamma * \bar{\Delta}_{\bar{p}_1+2\bar{p}_2+2\bar{t}} \bar{\gamma}}_{S_1 * W_1 \omega C} \right] + h.c., \quad (5.66)$$

$$d_Z W_{2CC\omega}^{\eta\bar{\eta}} = \frac{\eta\bar{\eta}}{16} C * C * \omega * \left[\underbrace{h_{p_2+2t} \Delta_{p_2+t} \Delta_{p_1+2p_2+2t} \gamma * \bar{\Delta}_{\bar{p}_1+\bar{p}_2+\bar{t}} \bar{\gamma}}_{d_x S_1} - \underbrace{\Delta_{p_2+t} \Delta_{p_1+2p_2+2t} \gamma * \bar{\Delta}_{\bar{p}_1+2\bar{p}_2+2\bar{t}} \bar{\gamma}}_{d_x S_2} + \underbrace{\Delta_{p_2+t} \Delta_{p_2+2t} \gamma * \bar{\Delta}_{\bar{p}_1+2\bar{p}_2+2\bar{t}} \bar{\gamma}}_{S_1 * W_1 C \omega} \right. \\ \left. + \underbrace{\Delta_{p_2+2t} \Delta_{p_1+2p_2+2t} \gamma * \bar{\Delta}_{\bar{p}_1+2\bar{p}_2+2\bar{t}} \bar{\gamma}}_{S_2 * \omega} \right] + h.c.. \quad (5.67)$$

Notation $\underbrace{P}_Q$ specifies the part of P coming from Q .

Note that all terms of the form $\Delta_a \Delta_b (\gamma) * \bar{\Delta}_c \bar{\Delta}_d (\bar{\gamma})$ in (5.64)–(5.67) are totally PLT-odd while those of the form $h_a \Delta_b \Delta_c (\gamma) * \bar{\Delta}_c \bar{\Delta}_d (\bar{\gamma})$ are PLT-even with respect to the barred variables and PLT-odd with respect to the unbarred ones. From here it follows that for any $q^U, \bar{q}^U$

$$\tilde{h}_{q^U \bar{q}^U} d_Z W_{2U}^{\eta\bar{\eta}} \neq 0, \quad U = \{CC\omega, \omega CC, C\omega C\}. \quad (5.68)$$

However it turns out that each equation (5.64)–(5.67) can be rewritten as

$$d_Z W_{2U}^{\eta\bar{\eta}} = d_Z F_U + G_U, \quad (5.69)$$

where all G_U are totally PLT-odd and there exist such shifts q_U and $\bar{q}_U$ that

$$\tilde{h}_{q_U \bar{q}_U} G_U = 0. \quad (5.70)$$

Hence (5.69) is solved by

$$W_{2U}^{\eta\bar{\eta}} = F^U + \tilde{\Delta}_{q_U \bar{q}_U} G^U. \quad (5.71)$$

Consider firstly $W_{2\omega CC}^{\eta\bar{\eta}}$ (5.64). Eq. (5.52) yields for any $\bar{q}$

$$h_{p_2} \Delta_{p_1+2p_2} \Delta_{p_1+2p_2+t} \gamma * \bar{\Delta}_{\bar{p}_1+\bar{p}_2+\bar{t}} \bar{\gamma} = d_Z (h_{p_2} \Delta_{p_1+2p_2} \Delta_{p_1+2p_2+t} \gamma * \bar{\Delta}_{\bar{q}} \bar{\Delta}_{\bar{p}_1+\bar{p}_2+\bar{t}} \bar{\gamma}) \\ + h_{p_2} \Delta_{p_1+2p_2} \Delta_{p_1+2p_2+t} \gamma * \bar{\Delta}_{\bar{q}} \bar{\gamma}. \quad (5.72)$$

Denoting

$$F_{q,\bar{q}} \omega_{CC} = \frac{\eta\bar{\eta}}{16} \omega * C * C * h_{p_2} \Delta_{p_1+2p_2} \Delta_{p_1+2p_2+t} \gamma * \bar{\Delta}_{\bar{q}} \bar{\Delta}_{\bar{p}_1+\bar{p}_2+\bar{t}} \bar{\gamma} + h.c., \quad (5.73)$$

$$G_{q,\bar{q}} \omega_{CC} = d_Z W_{2\omega_{CC}}^{\eta\bar{\eta}} - d_Z F_{q,\bar{q}} \omega_{CC}$$

we have to find such $Q, \bar{Q}$ and $q, \bar{q}$ that

$$\tilde{h}_{Q-p_1-p_2-t, \bar{Q}-\bar{p}_1-\bar{p}_2-\bar{t}} G_{q,\bar{q}} \omega_{CC} = 0. \quad (5.74)$$

By virtue of (3.11) this equation demands in particular

$$\left[h_{p_2} \Delta_{p_1+2p_2} \Delta_{p_1+2p_2+t} \gamma * \bar{\Delta}_{\bar{q}} \bar{\gamma} + h_Q \Delta_{p_2} \Delta_{p_1+2p_2+t} \gamma * \bar{\Delta}_{\bar{p}_1+2\bar{p}_2+\bar{t}} \bar{\gamma} \right. \\ \left. + h_Q \Delta_{p_1+2p_2+t} \Delta_{p_1+2p_2} \gamma * \bar{\Delta}_{\bar{p}_2} \bar{\gamma} - h_Q \Delta_{p_2} \Delta_{p_1+2p_2} \gamma * \bar{\Delta}_{\bar{p}_1+2\bar{p}_2} \bar{\gamma} \right] = 0.$$

There are three evident solutions to this equation

$$\{Q = p_1 + 2p_2 + t, \bar{q} = \bar{p}_1 + 2\bar{p}_2\}, \quad \{Q = p_1 + 2p_2, \bar{q} = \bar{p}_1 + 2\bar{p}_2 + \bar{t}\}, \quad \{Q = p_2, \bar{q} = \bar{p}_2\}. \quad (5.75)$$

Choosing for simplicity $Q = q = p_2, \bar{q} = \bar{Q} = \bar{p}_2$ by virtue of eq. (5.52) one obtains

$$F_{p_2, \bar{p}_2} \omega_{CC} = \frac{\eta\bar{\eta}}{16} \omega * C * C * h_{p_2} \Delta_{p_1+2p_2} \Delta_{p_1+2p_2+t} \gamma * \bar{\Delta}_{\bar{p}_2} \bar{\Delta}_{\bar{p}_1+\bar{p}_2+\bar{t}} \bar{\gamma} + h.c., \\ \tilde{h}_{-p_1-t, -\bar{p}_1-\bar{t}} G_{p_2, \bar{p}_2} \omega_{CC} = 0. \quad (5.76)$$

Hence, by virtue of (5.71),

$$W_{2\omega_{CC}}^{\eta\bar{\eta}} = -\frac{\eta\bar{\eta}}{16} \omega * C * C * \left[h_{p_2} \Delta_{p_1+2p_2} \Delta_{p_1+2p_2+t} \gamma * \bar{\Delta}_{\bar{p}_2} \bar{\Delta}_{\bar{p}_1+\bar{p}_2+\bar{t}} \bar{\gamma} \right. \\ \left. + \frac{1}{2} \Delta_{p_2} \Delta_{p_1+2p_2+t} \gamma * \bar{\Delta}_{\bar{p}_2} \bar{\Delta}_{\bar{p}_1+2\bar{p}_2+\bar{t}} \bar{\gamma} + \frac{1}{2} \Delta_{p_2} \Delta_{p_1+2p_2} \gamma * \bar{\Delta}_{\bar{p}_1+2\bar{p}_2} \bar{\Delta}_{\bar{p}_2} \bar{\gamma} \right] + h.c. \quad (5.77)$$

solves (5.64).

Consideration of (5.66) is analogous. Setting

$$F_{q,\bar{q}} C \omega_C = \frac{\eta\bar{\eta}}{16} C * \omega * C * [(h_{p_1+2p_2+2t} - h_{p_2}) \Delta_{p_2+t} \Delta_{p_1+2p_2+t} \gamma * \bar{\Delta}_{\bar{q}} \bar{\Delta}_{\bar{p}_1+\bar{p}_2+\bar{t}} \bar{\gamma}] + h.c., \\ G_{q,\bar{q}} C \omega_C = d_Z W_{2C\omega_C}^{\eta\bar{\eta}} - d_Z F_{q,\bar{q}} C \omega_C \quad (5.78)$$

one can see that the equation

$$\tilde{h}_{Q-p_1-p_2-t, \bar{Q}-\bar{p}_1-\bar{p}_2-\bar{t}} G_{q,\bar{q}} C \omega_C = 0 \quad (5.79)$$

admits a solution $q = p_1 + 2p_2 + 2t, \bar{q} = \bar{p}_1 + 2\bar{p}_2 + 2\bar{t}, Q = p_1 + 2p_2 + t, \bar{Q} = \bar{p}_1 + 2\bar{p}_2 + \bar{t}$. The respective solution to (5.66) of the form (5.71) is

$$W_{2C\omega_C}^{\eta\bar{\eta}} = -\frac{\eta\bar{\eta}}{16} C * \omega * C * \left[(h_{p_1+2p_2+2t} - h_{p_2}) \Delta_{p_2+t} \Delta_{p_1+2p_2+t} \gamma * \bar{\Delta}_{\bar{p}_1+2\bar{p}_2+2\bar{t}} \bar{\Delta}_{\bar{p}_1+\bar{p}_2+\bar{t}} \bar{\gamma} \right. \\ - \frac{1}{2} \Delta_{p_2} \Delta_{p_1+2p_2+t} \gamma * \bar{\Delta}_{\bar{p}_1+2\bar{p}_2+2\bar{t}} \bar{\Delta}_{\bar{p}_1+2\bar{p}_2+\bar{t}} \bar{\gamma} - \frac{1}{2} \Delta_{p_1+2p_2+t} \Delta_{p_1+2p_2+2t} \gamma * \bar{\Delta}_{\bar{p}_1+2\bar{p}_2+2\bar{t}} \bar{\Delta}_{\bar{p}_2} \bar{\gamma} \\ \left. + \frac{1}{2} h_{p_1+2p_2+t} \Delta_{p_1+2p_2+2t} \Delta_{p_2} \gamma * \bar{\Delta}_{\bar{p}_1+2\bar{p}_2+t} \bar{\Delta}_{\bar{p}_1+2\bar{p}_2+2\bar{t}} \bar{\gamma} \right] + h.c.. \quad (5.80)$$

Finally, from (5.67) it follows that

$$d_Z W_{2CC\omega}^{\eta\bar{\eta}} = -\frac{\eta\bar{\eta}}{16} C * C * \omega * d_Z (h_{p_2+t} \Delta_{p_2+2t} \Delta_{p_1+2p_2+2t} \gamma * \bar{\Delta}_{\bar{p}_1+2\bar{p}_2+2\bar{t}} \bar{\Delta}_{\bar{p}_1+\bar{p}_2+\bar{t}} \bar{\gamma}) + h.c. . \quad (5.81)$$

Hence

$$W_{2CC\omega}^{\eta\bar{\eta}} = \frac{\eta\bar{\eta}}{16} C * C * \omega * h_{p_2+t} \Delta_{p_2+2t} \Delta_{p_1+2p_2+2t} \gamma * \bar{\Delta}_{\bar{p}_1+2\bar{p}_2+2\bar{t}} \bar{\Delta}_{\bar{p}_1+\bar{p}_2+\bar{t}} \bar{\gamma} + h.c. \quad (5.82)$$

solves (5.67).

The following comment is now in order. The choice of $\mu = \bar{\mu} = -1$ of the contracting homotopy parameters $\frac{1}{2} [\Delta_{(1-\mu)p_2-\mu p_1} + \bar{\Delta}_{(1-\bar{\mu})\bar{p}_2-\bar{\mu}\bar{p}_1}]$ in (5.60) leads to asymmetric result for $W_2^{\eta\bar{\eta}}$ as can be seen from the fact that $W_{2CC\omega}^{\eta\bar{\eta}}$ (5.82) is simpler than $W_{2\omega CC}^{\eta\bar{\eta}}$ (5.77). As shown below, this leads to essentially different formulae for components of the respective vertices $\Upsilon^{\eta\bar{\eta}}(\omega, \omega, C, C)$.

The choice $\mu = \bar{\mu} = 0$ in (5.60) yields

$$S_2 = -\frac{i\eta\bar{\eta}}{8} C * C * \left[\Delta_{p_2} \gamma * \bar{\Delta}_{\bar{p}_2} \bar{\Delta}_{\bar{p}_1+2\bar{p}_2} \bar{\gamma} + \Delta_{p_2} \Delta_{p_1+2p_2} \gamma * \bar{\Delta}_{\bar{p}_2} \bar{\gamma} \right] \quad (5.83)$$

leading to simplification of $W_{2\omega CC}^{\eta\bar{\eta}}$ and complication of $W_{2CC\omega}^{\eta\bar{\eta}}$. To obtain a form of $W_2^{\eta\bar{\eta}}$ and, hence, $\Upsilon^{\eta\bar{\eta}}(\omega, \omega, C, C)$ symmetric with respect to reordering of C and ω one can take an averaged sum of $\mu = 0$, $\bar{\mu} = -1$

$$S_2^{\eta\bar{\eta}} = -\frac{i\eta\bar{\eta}}{16} C * C * \left[\Delta_{p_2} \Delta_{p_1+2p_2} \gamma * \bar{\Delta}_{\bar{p}_1+2\bar{p}_2} \bar{\gamma} + \Delta_{p_2} \gamma * \bar{\Delta}_{\bar{p}_2} \bar{\Delta}_{\bar{p}_1+2\bar{p}_2} \bar{\gamma} \right. \\ \left. + \Delta_{p_1+2p_2} \gamma * \bar{\Delta}_{\bar{p}_2} \bar{\Delta}_{\bar{p}_1+2\bar{p}_2} \bar{\gamma} + \Delta_{p_2} \Delta_{p_1+2p_2} \gamma * \bar{\Delta}_{\bar{p}_2} \bar{\gamma} \right] . \quad (5.84)$$

Note that all resulting vertices $\Upsilon^{\eta\bar{\eta}}$ will be local.

5.3.3 $\eta\bar{\eta}$ vertex $\Upsilon^{\eta\bar{\eta}}$

From (2.3) we have

$$\Upsilon^{\eta\bar{\eta}}(\omega, \omega, C, C) = -\left(d_x W_1 + W_1 * W_1 + d_x W_2 + \omega * W_2 + W_2 * \omega \right) \Big|_{\eta\bar{\eta}} . \quad (5.85)$$

Plugging the obtained expressions for W_1 and W_2 into r.h.s. of eq. (5.85) one can calculate $\eta\bar{\eta}$ vertices in this order of perturbation theory. For the reader's convenience, the expressions for $d_x C$ and $d_x \omega$ in terms of lower order corrections obtained in [3] are collected in appendix C.

The procedure goes as follows. Since r.h.s. of (5.85) is by construction $z, \bar{z}$ -independent one can apply any projector $h_q \bar{h}_{\bar{q}}$ (q and $\bar{q}$ are not necessarily complex conjugated).

As an example, consider vertex $\Upsilon_{C\omega C\omega}^{\eta\bar{\eta}}$. Eq. (5.85) yields

$$\Upsilon_{C\omega C\omega} = -\left(d_x W_{2C\omega C} + d_x W_{2C\omega C} + d_x W_{2C\omega C} + d_x W_{1C\omega} + d_x W_{1C\omega} \right) \Big|_{C\omega C\omega} \\ - W_{1C\omega} * W_{1C\omega} - W_{2C\omega C} * \omega , \quad (5.86)$$

where underlined labels refer to the fields d_x is acting on. For instance, using formula (5.82) for $W_{2C\omega}^{\eta\bar{\eta}}$ one has (discarding the conjugated terms for brevity)

$$d_x W_{2C\omega}^{\eta\bar{\eta}} =: -\frac{\eta\bar{\eta}}{16} C * (d_x C) * \omega * h_{p_2+2t} \Delta_{p_2+t} \Delta_{p_1+2p_2+2t} \gamma * \bar{\Delta}_{p_1+2p_2+2t} \bar{\Delta}_{p_1+p_2+t} \bar{\gamma}, \quad (5.87)$$

whence it follows by virtue of (C.2)

$$d_x W_{2C\omega}^{\eta\bar{\eta}} = \frac{\eta\bar{\eta}}{16} C * \omega * C * \omega * h_{p_2+t_1+2t_2} \Delta_{p_2+t_1+t_2} \Delta_{p_1+2p_2+2t_1+2t_2} \times \gamma * \bar{\Delta}_{\bar{p}_1+2\bar{p}_2+2\bar{t}_1+2\bar{t}_2} \bar{\Delta}_{\bar{p}_1+\bar{p}_2+\bar{t}_1+\bar{t}_2} \bar{\gamma}. \quad (5.88)$$

In obtaining this and similar expressions one should be careful in keeping track of the homotopy parameters. For example in (5.87) p_2 acts on $d_x C$ which after substitution of (C.2) contributes $\omega * C$ implying that one should replace $p_2 \rightarrow p_2 + t_1$ and rename $t \rightarrow t_2$ since there are now two ω 's.

Analogously one obtains from (5.80), (5.82) and (5.5) by virtue of (C.1)–(C.13)

$$d_x W_{2C\omega}^{\eta\bar{\eta}} = -\frac{\eta\bar{\eta}}{16} C * \omega * C * \omega * \quad (5.89)$$

$$\times \left[(h_{p_1+2p_2+2t_1+2t_2} - h_{p_2+t_2}) \Delta_{p_2+t_1+t_2} \Delta_{p_1+2p_2+t_1+2t_2} \gamma * \bar{\Delta}_{\bar{p}_1+2\bar{p}_2+2\bar{t}_1+2\bar{t}_2} \bar{\Delta}_{\bar{p}_1+\bar{p}_2+\bar{t}_1+\bar{t}_2} \bar{\gamma} \right. \\ \left. - \frac{1}{2} \Delta_{p_2+t_2} \Delta_{p_1+2p_2+t_1+2t_2} \gamma * \bar{\Delta}_{\bar{p}_1+2\bar{p}_2+2\bar{t}_1+2\bar{t}_2} \bar{\Delta}_{\bar{p}_1+2\bar{p}_2+\bar{t}_1+2\bar{t}_2} \bar{\gamma} \right. \\ \left. - \frac{1}{2} \Delta_{p_1+2p_2+t_1+2t_2} \Delta_{p_1+2p_2+2t_1+2t_2} \gamma * \bar{\Delta}_{\bar{p}_1+2\bar{p}_2+2\bar{t}_1+2\bar{t}_2} \bar{\Delta}_{\bar{p}_2+\bar{t}_2} \bar{\gamma} \right. \\ \left. + \frac{1}{2} h_{p_1+2p_2+t_1+2t_2} \Delta_{p_1+2p_2+2t_1+2t_2} \Delta_{p_2+t_2} \gamma * \bar{\Delta}_{\bar{p}_1+2\bar{p}_2+\bar{t}_1+2\bar{t}_2} \bar{\Delta}_{\bar{p}_1+2\bar{p}_2+2\bar{t}_1+2\bar{t}_2} \bar{\gamma} \right],$$

$$d_x W_{2C\omega}^{\eta\bar{\eta}} = -\frac{\eta\bar{\eta}}{16} C * \omega * C * \omega * \quad (5.90)$$

$$\times \left[h_{p_2+2t_2} \Delta_{p_2+t_2} \Delta_{p_1+2p_2+t_1+2t_2} \gamma * \bar{\Delta}_{\bar{p}_1+2\bar{p}_2+\bar{t}_1+2\bar{t}_2} \bar{\Delta}_{\bar{p}_1+\bar{p}_2+\bar{t}_1+\bar{t}_2} \bar{\gamma} \right],$$

$$d_x W_{1C\omega}^{\eta\bar{\eta}} = \frac{\eta\bar{\eta}}{16} C * \omega * C * \omega * \left[(h_{p_2+t_1+t_2} \Delta_{p_2+t_1+2t_2} \Delta_{p_2+t_2} \gamma + h_{p_2+t_1+2t_2} \Delta_{p_2+2t_2} \Delta_{p_2+t_2} \gamma) * \right. \\ \left. \times * \bar{\Delta}_{\bar{p}_1+\bar{p}_2+\bar{t}_1+\bar{t}_2} \bar{\Delta}_{\bar{p}_1+2\bar{p}_2+2\bar{t}_1+2\bar{t}_2} \bar{\gamma} \right], \quad (5.91)$$

$$d_x W_{1C\omega}^{\eta\bar{\eta}} = \frac{\eta\bar{\eta}}{16} C * \omega * C * \omega * \left[(h_{p_1+2p_2+2t_1+2t_2} - h_{p_2+2t_2}) \Delta_{p_2+t_1+2t_2} \Delta_{p_1+2p_2+t_1+2t_2} \gamma * \right. \\ \left. \times * \bar{\Delta}_{\bar{p}_1+\bar{p}_2+\bar{t}_1+\bar{t}_2} \bar{\Delta}_{\bar{p}_1+2\bar{p}_2+2\bar{t}_1+2\bar{t}_2} \bar{\gamma} \right]. \quad (5.92)$$

Using star-exchange formulae (3.10)–(3.13) one obtains from (5.5) and (5.80)

$$W_{1C\omega}^{\eta\bar{\eta}} * W_{1C\omega}^{\eta\bar{\eta}} = \frac{\eta\bar{\eta}}{16} C * \omega * C * \omega * \left[\Delta_{p_1+2p_2+t_1+2t_2} \Delta_{p_1+2p_2+2t_1+2t_2} \gamma * \bar{\Delta}_{\bar{p}_2+\bar{t}_2} \bar{\Delta}_{\bar{p}_2+2\bar{t}_2} \bar{\gamma} \right], \quad (5.93)$$

$$W_{2C\omega}^{\eta\bar{\eta}} * \omega = \frac{\eta\bar{\eta}}{16} C * \omega * C * \omega * \left[(h_{p_1+2p_2+2t_1+2t_2} - h_{p_2+2t_2}) \right. \\ \times \Delta_{p_2+t_1+2t_2} \Delta_{p_1+2p_2+t_1+2t_2} \gamma * \bar{\Delta}_{\bar{p}_1+2\bar{p}_2+2\bar{t}_1+2\bar{t}_2} \bar{\Delta}_{\bar{p}_1+\bar{p}_2+\bar{t}_1+2\bar{t}_2} \bar{\gamma} \\ \left. - \frac{1}{2} \Delta_{p_2+2t_2} \Delta_{p_1+2p_2+t_1+2t_2} \gamma * \bar{\Delta}_{\bar{p}_1+2\bar{p}_2+2\bar{t}_1+2\bar{t}_2} \bar{\Delta}_{\bar{p}_1+2\bar{p}_2+\bar{t}_1+2\bar{t}_2} \bar{\gamma} \right. \\ \left. - \frac{1}{2} \Delta_{p_1+2p_2+t_1+2t_2} \Delta_{p_1+2p_2+2t_1+2t_2} \gamma * \bar{\Delta}_{\bar{p}_1+2\bar{p}_2+2\bar{t}_1+2\bar{t}_2} \bar{\Delta}_{\bar{p}_2+2\bar{t}_2} \bar{\gamma} \right. \\ \left. + \frac{1}{2} h_{p_1+2p_2+t_1+2t_2} \Delta_{p_1+2p_2+2t_1+2t_2} \Delta_{p_2+2t_2} \gamma * \bar{\Delta}_{\bar{p}_1+2\bar{p}_2+\bar{t}_1+2\bar{t}_2} \bar{\Delta}_{\bar{p}_1+2\bar{p}_2+2\bar{t}_1+2\bar{t}_2} \bar{\gamma} \right]. \quad (5.94)$$

Substitution of (5.88)–(5.94) into (5.86) and application of the cohomology projector

$$h_{p_1+2p_2+2t_1+2t_2} \bar{h}_{\bar{p}_1+2\bar{p}_2+2\bar{t}_1+2\bar{t}_2} \quad (5.95)$$

yields $\Upsilon_{C\omega C\omega}^{\eta\bar{\eta}}$ (4.21).

Other vertices are extracted from eq. (5.85) analogously by virtue of eqs. (5.5), (5.77)–(5.82) taking into account eqs. (C.1)–(C.13). To bring them to the form (4.17)–(4.22) identity (5.53) can be useful.

So far the vertices were calculated using (5.9). Now in order to account for local field redefinition (5.11) one has to add (5.44) to them. To do so we calculate δS_2 using (5.55) and eventually δW_2 using (5.62) via contracting homotopy Δ_0

$$\delta S_2 = -\frac{1}{2} \Delta_0 (\bar{\eta} \delta B_2^\eta * \bar{\gamma} + \eta \delta B_2^{\bar{\eta}} * \gamma), \quad (5.96)$$

$$\delta W_2 = -\frac{i}{2} \Delta_0 (\omega * \delta S_2 + \delta S_2 * \omega) \Big|_{\eta\bar{\eta}}, \quad (5.97)$$

where δB_2 is explicitly given in (5.11). Substituting (5.97) into (5.44) we eventually find $\delta \Upsilon^{\eta\bar{\eta}}$ in the form (4.23)–(4.25).

6 β -shift as star-product re-ordering

The local framework for analysis of HS equations (2.1)–(2.5) elaborated in [2, 3, 9] and in this paper rests on the specific choice of contracting homotopy operators in solving for z -dependence of master fields. This choice is driven by PLT that places constraint on homotopy shifts as in (2.25) and (2.26) in the PLT-even and -odd sectors, respectively.

As shown in [9], PLT extends to extra derivative shift (2.27) with any $\beta < 1$, preserving shift conditions (2.25) and (2.26) upon overall $(1 - \beta)$ -rescaling in (3.15). Still, the PLT requirement alone is insufficient for locality: to obtain local vertices in the one-form sector one has to take the limit $\beta \rightarrow -\infty$. On the computational side, $\beta \rightarrow -\infty$ limit cuts off non-local contributions keeping the local ones.

6.1 Star-product re-ordering

To understand the $\beta \rightarrow -\infty$ limit better we note that β -extended contracting homotopy (3.1) is related to that with $\beta = 0$ via β -dependent star-product re-ordering. (That such a map exists for $\beta = 1$ was in fact shown in [36].) Indeed, consider the following re-ordering operator that maps symbol $f(z, y)$ of original ordering (2.7) to the β -deformed one

$$O_\beta f(z, y) = \int \frac{dudv}{(2\pi)^2} f(z + v, y + \beta u) \exp(iu_\alpha v^\alpha), \quad (6.1)$$

$$O_\beta^{-1} f(z, y) = O_{-\beta} f(z, y) = \int \frac{dudv}{(2\pi)^2} f(z + v, y - \beta u) \exp(iu_\alpha v^\alpha). \quad (6.2)$$

It can be shown then by direct computation that

$$O_\beta \Delta_{q, \beta=\Delta_q, 0} O_\beta \quad (6.3)$$

and

$$h_{q,\beta} = h_{q,0} O_\beta. \quad (6.4)$$

In other words, $\Delta_{q,\beta}$ and cohomology projector $h_{q,\beta}$ appear from the re-ordering similarity transform of $\Delta_{q,0}$ and $h_{q,0}$ correspondingly,

$$\Delta_{q,\beta} = O_\beta^{-1} \Delta_{q,0} O_\beta, \quad h_{q,\beta} = h_{q,0} O_\beta. \quad (6.5)$$

It is easy to find star product $\star_\beta$ that corresponds to new ordering (6.1) from

$$f \star_\beta g = O_\beta (O_\beta^{-1} f * O_\beta^{-1} g), \quad (6.6)$$

where $*$ is the original star product (2.7). An elementary calculation gives

$$f \star_\beta g = \int \frac{du du' dv dv'}{(2\pi)^4} f(z+u', y+u) g(z-(1-\beta)v-v', y+v+(1-\beta)v') \exp(iu_\alpha v^\alpha + iu'_\alpha v'^\alpha). \quad (6.7)$$

The new star product contains extra integration over u' and v' . There are two points $\beta = 0$ and $\beta = 2$ that reduce it down to the normal (2.7) and anti-normal orderings of $y \pm z$ operators correspondingly and a point in between, $\beta = 1$, corresponding to their Weyl ordering. For an arbitrary β one can obtain from (6.7) the following product rules

$$y \star_\beta = y + i \frac{\partial}{\partial y} - i(1-\beta) \frac{\partial}{\partial z}, \quad \star_\beta y = y - i \frac{\partial}{\partial y} - i(1-\beta) \frac{\partial}{\partial z}, \quad (6.8)$$

$$z \star_\beta = z - i \frac{\partial}{\partial z} + i(1-\beta) \frac{\partial}{\partial y}, \quad \star_\beta z = z + i(1-\beta) \frac{\partial}{\partial y} + i \frac{\partial}{\partial z}, \quad (6.9)$$

which show that y and z still commute and their commutators with star-product elements $f(z, y)$ remain undeformed for any β

$$[y_\alpha, f]_{\star_\beta} = 2i \frac{\partial}{\partial y^\alpha} f, \quad [z_\alpha, f]_{\star_\beta} = -2i \frac{\partial}{\partial z^\alpha} f. \quad (6.10)$$

Since star product defined in (6.7) is associative and space-time independent one can consider HS equations (2.1)–(2.5) in the β -ordering. This amounts to simply replacing original star product (2.7) by (6.7) and modifying the central element (2.9) by (cf, eqs. (3.44) of [36])

$$\gamma_\beta = O_\beta (\exp(iz_\alpha y^\alpha)) k \theta^\alpha \theta_\alpha = \frac{1}{(1-\beta)^2} \exp\left(\frac{i}{1-\beta} z_\alpha y^\alpha\right) k \theta^\alpha \theta_\alpha. \quad (6.11)$$

We observe that, within the β -reordered HS equations, local HS interactions in the one-form sector are recovered in terms of conventional contracting homotopy in the limit $\beta \rightarrow -\infty$. Indeed, from (6.1) it follows that the reordering procedure does not affect Y - or Z - independent functions and, hence, Z -independent HS fields $\omega(Y, K|x)$ and $C(Y, K|x)$. In particular, HS vacuum (2.12) is not going to change in the new ordering. Therefore the perturbative expansion remains the same. For example, vertex $\Upsilon_{C\omega\omega C}^{\eta\eta}(\beta)$ is given by

$$\Upsilon_{C\omega\omega C}^{\eta\eta}(\beta) = -h_{0,0}(W_{1\beta} \star_\beta W_{1\beta}) \Big|_{C\omega\omega C}, \quad (6.12)$$

where W_1^β is the one-form first-order correction calculated in the β -ordering using conventional contracting homotopy (2.18) with $q = 0$ and, as opposed to (5.35), we use conventional projector $h_{0,0}$. One can see that (6.12) is exactly equal to the one in (5.35). Indeed, using (6.1) and (6.4) we have

$$h_{0,0}(W_{1\beta} \star_\beta W_{1\beta}) = h_{0,0}O_\beta(W_1 * W_1) = h_{0,\beta}(W_1 * W_1) \quad (6.13)$$

reproducing this way (5.35).

Therefore, one may argue that $\beta \rightarrow -\infty$ contracting homotopy (3.1) manifests a way to implement specific re-orderings of operators that renders HS vertices local. The details of such localization is yet to be understood. Put it differently, instead of using β -shifted contracting homotopies (3.1) one could have used the β -independent ones at the price of using modified star product $\star_\beta$ (6.7).

Note, however, that the limiting points $\beta = 1$ and $\beta = -\infty$ lead to singularities in (6.11). In fact, $\beta = 1$ corresponds to the Weyl ordering prescription analyzed in [8, 36–38] (and references therein). The singularity in (6.11) expresses the fact that inner Klein operators become δ -functions in the Weyl ordering [39, 40]. The analysis of [8] indicated that it is hard to handle the resulting divergencies. On the other hand, recent results of [36] show that one can proceed at least in the lowest order. From the perspective of this paper this can be understood as a consequence of the fact that, in the lowest order, β -shifted contracting homotopies are equivalent to the conventional ones with $\beta = 0$. It would be interesting to see to which extent the results of [36] can be extended to higher orders. In our analysis we were not able to make use of the $\beta \rightarrow 1$ limit.

The local limit singularity at $\beta \rightarrow -\infty$ in the star product is less clear demanding further investigation. In this case, formula (6.1) as well as $\star_{-\infty}$ to be well defined require precise specification of the functional class used. Leaving detailed analysis of this issue for the future, at this stage we therefore prefer to analyze the problem in terms of the standard HS star product (2.7) but using $\Delta_{q,-\infty}$.

Practically, the use of the reordering operator (6.1) sometimes simplifies calculations that include analysis of several contracting homotopy operators. Particularly, it is useful for the analysis of S_2 contribution to (anti)holomorphic vertices as we show now.

6.2 S_2 contribution to $\Upsilon^{\eta\eta}$

It was argued already that contribution to (anti)holomorphic part of the vertices from S_2 can be made to vanish at $\beta \rightarrow -\infty$. Let us demonstrate this in some detail. This contribution comes from the second term on the r.h.s. of (4.6) with the part of W_2 taken from (5.16). It potentially contributes to the three vertex structures $\Upsilon_{\omega\omega CC}^{\eta\eta}$, $\Upsilon_{CC\omega\omega}^{\eta\eta}$ and $\Upsilon_{\omega CC\omega}^{\eta\eta}$. Let us consider $\Upsilon_{\omega\omega CC}^{\eta\eta}$ for example,

$$-h_{0,\beta}(\omega * W_2')|_{\omega\omega CC} = \frac{1}{4}h_{0,\beta}(\omega * \Delta_{0,\beta}(\omega * \Delta_{0,\beta}(S_1 * S_1 - i\eta B_2 * \gamma))). \quad (6.14)$$

Using star-exchange relation (3.16) one brings it to

$$-h_{0,\beta}(\omega * W_2')|_{\omega\omega CC} = \frac{1}{4}\omega * \omega * h_{(1-\beta)(t_1+t_2),\beta} \Delta_{(1-\beta)t_2,\beta} \Delta_{0,\beta}(S_1 * S_1 - i\eta B_2 * \gamma). \quad (6.15)$$

A convenient way to tackle three consecutive β -homotopies is to use the similarity reordering identities (6.5) which allows one to reduce them to the much simpler β -independent ones. This gives

$$\frac{1}{4}\omega * \omega * h_{(1-\beta)a} \Delta_{(1-\beta)b} \Delta_0 O_\beta(S_1 * S_1 - i\eta B_2 * \gamma), \quad a = t_1 + t_2, \quad b = t_2, \quad (6.16)$$

where O_β is given by (6.1). Using the scaling property of $h_a \Delta_b \Delta_c$ (3.20) to calculate the vertex at $\beta \rightarrow -\infty$ it suffices to analyze the following limit

$$\lim_{\beta \rightarrow -\infty} (1 - \beta)^2 O_\beta(S_1 * S_1 - i\eta B_2 * \gamma) \Big|_{z \rightarrow (1-\beta)z}. \quad (6.17)$$

The vertex is then reproduced by applying $\frac{1}{4}\omega * \omega * h_a \Delta_b \Delta_0$ to the resulting limiting expression. This way we first of all find using the explicit (B.3) and changing the integration variables that

$$(1 - \beta)^2 O_\beta(B_2 * \gamma) \Big|_{z \rightarrow (1-\beta)z} = -\frac{i\eta}{2} \theta^2 C^* C \int d_\Delta^3 \tau (p_1 + p_2)^\alpha (y + p_1)_\alpha \times \\ \times \exp i((\tau_1 + \tau_2) z_\alpha y^\alpha - (\tau_1 p_2 - \tau_2 p_1)^\alpha z_\alpha - (1 - \tau_1 - \tau_2)(p_1 + p_2)^\alpha (y + p_1)_\alpha) \quad (6.18)$$

is β -independent.

Contribution from $S_1 * S_1$ is found from (B.1) by performing star-product integration

$$S_1 * S_1 = -\frac{\eta^2}{2} \theta^2 C^* C \epsilon^{\alpha\beta} \partial_{1\alpha} \partial_{2\beta} \int_{[0,1]^2} d\tau_1 d\tau_2 \times \\ \times \exp i(\tau_\circ z_\alpha y^\alpha - \tau_1 \tau_2 (p_1 + p_2)^\alpha y_\alpha - (\tau_2(1 - \tau_1)p_2 - \tau_1(1 - \tau_2)p_1)^\alpha z_\alpha + \tau_1 \tau_2 p_1^\alpha p_{2\alpha}), \\ \tau_\circ = \tau_1(1 - \tau_2) + \tau_2(1 - \tau_1), \quad (6.19)$$

where $\partial_{1,2\alpha}$ denote differentiation with respect to $p_{1,2}^\alpha$. Applying to it (6.1) one finds

$$O_\beta(S_1 * S_1) = -\frac{\eta^2}{2} \theta^2 C^* C \epsilon^{\alpha\beta} \int_{[0,1]^2} d\tau_1 d\tau_2 \partial_{1\alpha} \partial_{2\beta} \frac{1}{(1 - \beta \tau_\circ)^2} \\ \times \exp i\left(\frac{1}{1 - \beta \tau_\circ} (\tau_\circ z_\alpha y^\alpha - \tau_1 \tau_2 (p_1 + p_2)^\alpha y_\alpha - (\tau_2(1 - \tau_1)p_2 - \tau_1(1 - \tau_2)p_1)^\alpha z_\alpha + \tau_1 \tau_2 p_1^\alpha p_{2\alpha})\right). \quad (6.20)$$

We now take the limit

$$\lim_{\beta \rightarrow -\infty} (1 - \beta)^2 O_\beta(S_1 * S_1) \Big|_{z \rightarrow (1-\beta)z} = -\frac{\eta^2}{2} \theta^2 C^* C \\ \times \int d_\Delta^3 \tau [(\tau_1 + \tau_2) z_\alpha y^\alpha - z_\alpha (\tau_1 p_2 - \tau_2 p_1)^\alpha - (1 - \tau_1 - \tau_2)(p_1 + p_2)^\alpha (y + p_1)_\alpha - 2i] \\ \times \exp i((\tau_1 + \tau_2) z_\alpha y^\alpha - (\tau_1 p_2 - \tau_2 p_1)^\alpha z_\alpha - (1 - \tau_1 - \tau_2)(p_1 + p_2)^\alpha (y + p_1)_\alpha) \quad (6.21)$$

and note that the exponential parts of (6.21) and (6.18) identically coincide. Pre-exponentials are different but when combined together according to (6.17) $B_2 * \gamma$ cancels

out with 1 in $(1 - \tau_1 - \tau_2)(p_1 + p_2)^\alpha (y + p_1)_\alpha$ part of the pre-exponential of (6.21) bringing us to

$$\begin{aligned} \lim_{\beta \rightarrow -\infty} (1 - \beta)^2 O_\beta(S_1 * S_1 - i\eta B_2 * \gamma) \Big|_{z \rightarrow (1-\beta)z} &= -\frac{\eta^2}{2} \theta^2 C^* C \\ &\times \int d_\Delta^3 \tau [(\tau_1 + \tau_2) z_\alpha y^\alpha - z_\alpha (\tau_1 p_2 - \tau_2 p_1)^\alpha + (\tau_1 + \tau_2)(p_1 + p_2)^\alpha (y + p_1)_\alpha - 2i] \\ &\times \exp i((\tau_1 + \tau_2) z_\alpha y^\alpha - (\tau_1 p_2 - \tau_2 p_1)^\alpha z_\alpha - (1 - \tau_1 - \tau_2)(p_1 + p_2)^\alpha (y + p_1)_\alpha) . \end{aligned} \quad (6.22)$$

The latter can be rewritten as

$$\begin{aligned} \frac{i\eta^2}{2} \theta^2 C^* C \int d_\Delta^3 \tau \left[\tau_1 \frac{\partial}{\partial \tau_1} + \tau_2 \frac{\partial}{\partial \tau_2} + 2 \right] \\ \times \exp i((\tau_1 + \tau_2) z_\alpha y^\alpha - (\tau_1 p_2 - \tau_2 p_1)^\alpha z_\alpha - (1 - \tau_1 - \tau_2)(p_1 + p_2)^\alpha (y + p_1)_\alpha) . \end{aligned} \quad (6.23)$$

Partial integration amounts to

$$\begin{aligned} -\frac{i\eta^2}{2} \theta^2 C^* C \int d^3 \tau \left(\tau_1 \frac{\partial}{\partial \tau_1} + \tau_2 \frac{\partial}{\partial \tau_2} \right) X \\ \times \exp i((\tau_1 + \tau_2) z_\alpha y^\alpha - (\tau_1 p_2 - \tau_2 p_1)^\alpha z_\alpha - (1 - \tau_1 - \tau_2)(p_1 + p_2)^\alpha (y + p_1)_\alpha) , \end{aligned} \quad (6.24)$$

where (recall the definition of measure d_Δ^3 in (3.19))

$$X = \delta \left(1 - \sum \tau_i \right) \theta(\tau_1) \theta(\tau_2) \theta(\tau_3) . \quad (6.25)$$

Noting that

$$\frac{\partial}{\partial \tau_1} \delta(\dots) = \frac{\partial}{\partial \tau_2} \delta(\dots) = \frac{\partial}{\partial \tau_3} \delta(\dots) \quad (6.26)$$

and $\tau_1 + \tau_2 = 1 - \tau_3$ we further rewrite (6.24) in the following form

$$\begin{aligned} -\frac{i\eta^2}{2} \theta^2 \int d^3 \tau \frac{\partial}{\partial \tau_3} \left(\delta \left(1 - \sum \tau_i \right) (1 - \tau_3) \right) \theta(\tau_1) \theta(\tau_2) \theta(\tau_3) \\ \times \exp i((\tau_1 + \tau_2) z_\alpha y^\alpha - (\tau_1 p_2 - \tau_2 p_1)^\alpha z_\alpha - (1 - \tau_1 - \tau_2)(p_1 + p_2)^\alpha (y + p_1)_\alpha) . \end{aligned} \quad (6.27)$$

Yet another partial integration along τ_3 yields the final result

$$\begin{aligned} \lim_{\beta \rightarrow -\infty} (1 - \beta)^2 O_\beta(S_1 * S_1 - i\eta B_2 * \gamma) \Big|_{z \rightarrow (1-\beta)z} &= \\ &= \frac{i\eta^2}{2} \theta^2 C^* C \int_{[0,1]^2} d\tau_1 d\tau_2 \delta(1 - \tau_1 - \tau_2) \exp i(z_\alpha y^\alpha - (\tau_1 p_2 - \tau_2 p_1)^\alpha z_\alpha) . \end{aligned} \quad (6.28)$$

Using (3.16) and denoting by $a = p_1 + 2p_2$ and $b = p_2$ one can rewrite (6.28) in the form

$$\lim_{\beta \rightarrow -\infty} \beta^2 O_\beta(\Delta_a \gamma * \Delta_b \gamma - \Delta_a \Delta_b \gamma * \gamma) \Big|_{z \rightarrow (1-\beta)z} = \phi(y; k) * \gamma , \quad (6.29)$$

where $\phi(y; k)$ is the z -independent function given by

$$\phi(y; k) = 2i \int_{[0,1]^2} d\tau_1 d\tau_2 \delta(1 - \tau_1 - \tau_2) e^{i(\tau_1 a + \tau_2 b)^\alpha y_\alpha} k . \quad (6.30)$$

Let us note that this cancellation mechanism can be analyzed without precise calculation of the limit $\beta \rightarrow -\infty$ rather using a language of the functional classes of [9], where it is shown that $S_1 * S_1 - i\eta B_2 * \gamma$ enjoys a remarkable structure relation. The corresponding contribution to the vertex appears from application of $\omega * \omega * h_{t_1+t_2} \Delta_{t_2} \Delta_0$ to (6.28) and thanks to (3.18) is obviously ultra-local.

We now note that the non-zero result (6.28) can be completely eliminated by the following local z -independent field redefinition

$$B_2 \rightarrow B_2 + \delta B_2(y) = B_2^{\min}, \quad (6.31)$$

$$\delta B_2 = \frac{\eta}{2} C^* C \int_{[0,1]^2} d\tau_1 d\tau_2 \delta(1 - \tau_1 - \tau_2) \exp(-i(\tau_2 p_1 - \tau_1 p_2)^\alpha y_\alpha) k, \quad (6.32)$$

which is nothing but (5.10) corresponding to the minimal derivative vertex $\Upsilon(\omega, C, C)$ found in [26]. Indeed, the effect of such z -independent field redefinition for B_2 amounts to the corresponding change in fields S_2 and W_2 according to (5.13) and (5.16). This results in the change of the vertex (6.15)

$$\delta \Upsilon_{\omega C C}^{\eta\eta} = \frac{1}{4} \omega * \omega * h_{(1-\beta)(t_1+t_2), \beta} \Delta_{(1-\beta)t_2, \beta} \Delta_{0, \beta} (-i\eta \delta B_2 * \gamma). \quad (6.33)$$

To analyze its limiting value at $\beta \rightarrow -\infty$ one again uses the scaling property (3.20) which leads to the following expression

$$(1 - \beta)^2 O_\beta(-i\eta \delta B_2 * \gamma) \Big|_{z \rightarrow (1-\beta)z} = -i\eta \delta B_2 * \gamma, \quad (6.34)$$

where the r.h.s. is due to z -independence of δB_2 . Up to an overall sign it equals (6.28) thus cancelling it out.

Similar consideration applies to the rest of contributions to $\Upsilon_{CC\omega\omega}^{\eta\eta}$ and $\Upsilon_{\omega CC\omega}^{\eta\eta}$ which are expressed as well via (6.17) followed by some $h_a \Delta_b \Delta_c$ action. Therefore, we conclude that $\beta \rightarrow -\infty$ contribution from S_2 to (anti)holomorphic vertices is ultra-local in accordance with the general consideration of [9]. Moreover it vanishes at all for the minimally coupled $B_2^{\min}$ (5.10).

7 Conclusion

In this paper we identified the homotopy procedure that leads to local HS vertices not only in the zero-form sector of the HS equations as in [3] but also in the one-form sector. Specifically we have evaluated all vertices that are bilinear both in the zero-form fields C and in the one-form fields ω in the sector of equations (1.2) for ω . Since ω describes dynamical HS fields (not, say, just AdS_4 background) the corresponding vertices describe HS interactions of massless fields of all spins up to the quintic order at the Lagrangian level. All these vertices are shown to be spin-local. This conclusion is in agreement with the conjecture of [2] that HS gauge theories are spin-local not only in the lowest order but in the higher orders as well.

The same time, the obtained vertices agree with the lower-order results obtained in [26]–[28] by less sophisticated means. In particular, in [28] it was shown that η^2 and

$\bar{\eta}^2$ vertices in the one-form sector vanish in the AdS_4 background. Though higher-order vertices obtained in this paper are non-zero in the η^2 and $\bar{\eta}^2$ sectors, their restrictions to AdS_4 background one-form fields indeed vanish that provides a highly nontrivial test of the obtained results. This implies in turn a very important property of HS equations that the resulting currents on the r.h.s. of the HS equations are local and the coupling constants in front of them only depend on $\eta\bar{\eta}$ having definite signs independent of the phase of η . In particular, the gravitational constant as a coefficient in front of the stress tensor is positive. Moreover η^2 and $\bar{\eta}^2$ vertices vanish for one-forms that are no more than bilinear in Y' 's, particularly, for any gravitational or electromagnetic background.

To obtain corrections to the r.h.s.'s of the Fronsdal equations one has to project all obtained vertices to the sector of frame-like fields with the equal numbers of y and $\bar{y}$ in the bosonic case and differing by 1 in the fermionic. (For more detail see e.g. [31] and [34, 35] for analogous analysis of HS interactions.) After expanding in powers of y and $\bar{y}$, one has to evaluate integrals over the homotopy parameters like σ and τ in the η^2 sector or $\bar{\sigma}$, $\bar{\tau}$ in the $\bar{\eta}^2$ -sector and τ , $\bar{\tau}$ in the $\eta\bar{\eta}$ sector, that enter through the contracting homotopies Δ , $\bar{\Delta}$ and cohomology projectors h , $\bar{h}$. This will result in nontrivial coefficients in front of different types of vertices in terms of fields. Being straightforward, derivation of their explicit form is beyond the scope of this paper.

The homotopy procedure used in our analysis is based on the limit $\beta \rightarrow -\infty$ with respect to the parameter β that enters the shifted contracting homotopy $\Delta_{0,\beta}$. Remarkably, the limit is well defined and the terms with infinite towers of higher derivatives vanish in this limit so that the resulting vertices become spin-local. This phenomenon, which is general in nature, is anticipated to work in higher orders as well. The technique appropriate for this analysis is developed further in [9] based on the previous analysis of locality in [8]. In particular, in [9] it is shown that the shifted homotopy technique is well defined for any $-\infty < \beta < 1$ and a sufficient condition for the limit $\beta \rightarrow -\infty$ to be well defined is found and shown to be fulfilled for the vertices analysed in this paper. At any rate, the limiting β -shifted contracting homotopy identified in this paper is unique for higher orders as well since it makes no sense to choose any other value of β except for the limiting one $\beta \rightarrow -\infty$. The reason why the lowest-order analysis of locality in [2, 3], which was based on the shifted homotopy with $\beta = 0$, was successful is that, as shown in this paper, the relevant lower-order terms turn out to be β -independent. This phenomenon does not take place at higher-orders where the limit $\beta \rightarrow -\infty$ becomes fully significant.

The obtained vertices naturally split into two classes of η^2 , $\bar{\eta}^2$ and $\eta\bar{\eta}$ vertices that originate from different contracting homotopies prescribed by PLT. This led to their different locality properties. While the η^2 , $\bar{\eta}^2$ vertices turn out to be spin ultra-local, in accordance with their current interaction structure, the $\eta\bar{\eta}$ ones are spin-local⁶ but z -dominated in the terminology of [2].

It should be stressed that there is of course a great freedom in performing spin-local field redefinitions that preserve the class of spin-local vertices. The class of these field redefini-

⁶This situation is similar to what happens at lower orders. Vertex $\Upsilon(\omega, \omega, C)$ stems from PLT-even sector and is spin-ultra-local, however $\Upsilon(\omega, C, C)$ originates from the PLT-odd one and is spin-local.

tions is defined the same way as spin-local vertices as being free from contractions between different factors of zero-forms C in either holomorphic or antiholomorphic sector (or both). Spin-local vertices form an equivalence class with respect to spin-local field redefinitions. So in this paper we found a particular representative of the class of spin-local vertices that may or may not be most useful for applications. For instance the $\eta\bar{\eta}$ vertices presented in section 4.3 do not respect the invariance of the whole setup under the fundamental anti-automorphism ρ of the HS theory that relates opposite orderings of the field product factors. As such, it is not most convenient for the analysis of the minimal HS model resulting from the truncation induced by ρ (for more detail see [31]). As explained in section 5.3.2, this can however be easily cured by a minor modification of the contracting homotopy scheme.

A related comment is that while all possible local field redefinitions lead to HS vertices with different number of derivatives it might be crucial for higher-order locality to single out those that give minimal derivative couplings. We have shown that the local field redefinition (5.11) that makes lower order vertex $\Upsilon(\omega, C, C)$ to contain minimal number of derivatives results in an extra ultra-local cancellations within $\Upsilon(\omega, \omega, C, C)$ structures. Particularly, in this case field S second order contribution completely vanishes leading among other things to the vanishing of (anti)holomorphic vertices on any gravitational backgrounds.

The obtained results illustrate high efficiency of the nonlinear HS equations of [1] as compared to other approaches available in the literature. (It is not clear whether at all it is possible to compute quintic vertices with arbitrary parity breaking parameter ϕ in $\eta = |\eta| \exp i\phi$ for infinite towers of spins by other means like, e.g., holographic reconstruction). Though in this paper it has been developed specifically for the $4d$ HS theory, our approach is applicable to the analysis of $3d$ theory of [41] (to large extent this analysis is contained in the η^2 sector of the $4d$ theory), HS theory in any dimension of [42] and, most important, to the multi-particle theory of [43], conjectured to be related to String Theory.

Of course, it remains to be checked how our prescription works for other higher-order vertices, namely those containing more than two zero-forms C . The simplest vertex of this type is $\Upsilon(\omega, C, C, C)$ in (1.2). In particular, it is this vertex that contains the scalar field self-interaction corresponding to the quartic Lagrangian spin-zero vertex. The fact that η^2 and $\bar{\eta}^2$ vertices appear in the ultra-local form severely constrains potential non-localities of the $\Upsilon(\omega, C, C, C)$ vertex. Indeed, had the $\eta\eta$ and $\bar{\eta}\bar{\eta}$ vertices been just spin-local rather than ultra-local the integrability condition for that quartic vertex would resulted in star products of spin-local pieces which are generally non-local. This however never happens to ultra-local terms.

It would be interesting to compare the vertex of this type resulting from HS equations with those elaborated in [21–23]. This is the most urgent problem on the agenda that we leave for a future publication. The technical tools for the analysis of this problem are elaborated in this paper and in [9].

While the mechanism that brings vertices to their local form in the limit $\beta \rightarrow -\infty$ is essentially the suppression of higher derivative terms it is not entirely clear why it works in a so delicately fine tuned way. We have shown here that the effect of parameter $-\infty < \beta < 1$ is equivalent to a star-product re-ordering. An interesting problem for the future is to understand to which extent the limiting $\beta \rightarrow -\infty$ star product is defined on its own right.

An interesting feature of the developed formalism is that it treats differently one-forms ω and zero-forms C . In the sector of higher spins this is just what is needed given that these are zero-forms C that contain infinite tails of higher derivatives of Fronsdal fields. However, the most general version of the $4d$ HS theory [1] (see also [31]) contains the sector of topological (Killing-like) fields, each carrying at most a finite number of degrees of freedom. In this sector the roles of one-forms and zero-forms are swapped: zero-forms C^{top} carry a finite number of derivatives of the topological fields while one-forms ω^{top} contain infinite towers of derivatives. This can affect the analysis of locality if the HS and topological sectors get interacting that can happen if some of the topological fields acquire nontrivial VEVs. In particular this happens in the $3d$ HS theory of [41] where the topological sector is related to the dynamical one.

An important related issue is the study of the HS black hole solutions found in [40, 44, 45]. Analysis of field fluctuations around the BH solutions demands accurate choice of the field variables and specification of the relevant classes of functions. (Otherwise it is hard to distinguish between dynamical and pure gauge degrees of freedom [46].) A related point is that topological fields play a role of chemical potentials in the analysis of invariant functionals associated with boundary charges [47]. This suggests that the effects of (non)locality in presence of topological fields can in particular play a role in the analysis of HS black holes.

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A Derivation of contracting homotopy formula

Applying (3.1) to (3.30) we obtain

$$\begin{aligned} \Delta_{0,\beta} f(z, y, \theta) &= \frac{1}{(2\pi)^2} \int d^2u d^2v \int_0^1 d\tau \int_0^1 dt t^{p-1} \exp i[v_\alpha u^\alpha + \tau(tz + (1-t)u)_\alpha (\beta v + y)^\alpha] \\ &\quad \times (z - u)^\alpha \frac{\partial}{\partial \theta^\alpha} \phi(\tau(tz + (1-t)u), (1-\tau)(\beta v + y), \tau\theta, \tau). \end{aligned} \quad (\text{A.1})$$

Introducing new integration variables,

$$\tau_1 = t\tau, \quad \tau = \tau_1 + \tau_2, \quad 1 - \tau = \tau_3 \quad (\text{A.2})$$

with the Jacobian

$$\det \left| \frac{\partial(\tau, t)}{\partial \tau_i} \right| = (\tau_1 + \tau_2)^{-1} \quad (\text{A.3})$$

we obtain

$$\begin{aligned} \Delta_{0,\beta} f(z,y,\theta) &= \frac{1}{(2\pi)^2} \int d^2u d^2v \int d^3\Delta \tau \frac{\tau_1^{p-1}}{(\tau_1 + \tau_2)^p} \delta\left(1 - \sum_{i=1}^3 \tau_i\right) \\ &\times \exp i[v_\beta u^\beta + (\tau_1 z + \tau_2 u)_\beta (\beta v + y)^\beta] (z - u)^\alpha \frac{\partial}{\partial \theta^\alpha} \phi(\tau_1 z + \tau_2 u, \tau_3(\beta v + y), (\tau_1 + \tau_2)\theta, \tau_1 + \tau_2). \end{aligned} \quad (\text{A.4})$$

Then, the shift of the integration variables,

$$u_\alpha \rightarrow u_\alpha + \frac{\tau_1 \beta}{1 - \tau_2 \beta} z_\alpha, \quad v_\alpha \rightarrow (1 - \tau_2 \beta)^{-1} (v_\alpha + \tau_2 y_\alpha) \quad (\text{A.5})$$

yields using definition (3.19)

$$\begin{aligned} \Delta_{0,\beta} f(z,y,\theta) &= \frac{1}{(2\pi)^2} \int d^2u d^2v \int d^3\Delta \tau (\tau_1)^{p-1} (1 - \beta \tau_2)^{-3} \exp i \left[v_\beta u^\beta + \frac{\tau_1}{(1 - \beta \tau_2)} z_\beta y^\beta \right] \\ &\times ((1 - (\tau_1 + \tau_2)\beta)z - (1 - \beta \tau_2)u)^\alpha \frac{\partial}{\partial \theta^\alpha} \phi\left(\frac{\tau_1}{(1 - \beta \tau_2)} z + \tau_2 u, \frac{\tau_3}{1 - \beta \tau_2} (y + \beta v), \theta, \tau_1 + \tau_2\right). \end{aligned} \quad (\text{A.6})$$

To reduce this expression to the desired form (3.30) we finally change variables to

$$\tau'_1 = \frac{\tau_1}{1 - \beta \tau_2}, \quad \tau'_3 = \frac{\tau_3}{1 - \beta \tau_2}, \quad \tau'_2 = \frac{(1 - \beta)\tau_2}{1 - \beta \tau_2}. \quad (\text{A.7})$$

This change of variables, which we call *simplicial map*, is remarkable in several respects. Firstly, all $\tau'_i \in [0, 1]$, taking into account that

$$\sum_{i=1}^3 \tau_i = 1. \quad (\text{A.8})$$

Secondly, it preserves the class of simplices of unit perimeter since

$$\sum_{i=1}^3 \tau'_i = 1 \quad (\text{A.9})$$

as a consequence of (A.8). The Jacobian is

$$\det \left| \frac{\partial \tau'_i}{\partial \tau_j} \right| = \frac{1 - \beta}{(1 - \beta \tau_2)^3}. \quad (\text{A.10})$$

Using also that

$$1 - \beta \tau_2 = \frac{1 - \beta}{1 - \beta(1 - \tau'_2)} \quad (\text{A.11})$$

we finally obtain (3.33) upon discarding primes.

B Useful formulae

B.1 Lower order fields

Here we collect explicit formulae for perturbative master fields. For S_1 from (5.2) and W_1 from (5.5) it is easy to obtain

$$S_1^\eta = -i\eta \int_0^1 d\tau C(0, \bar{y}) \theta^\alpha \frac{\partial}{\partial p^\alpha} \exp(i\tau z_\alpha y^\alpha + i\tau p^\alpha z_\alpha) k + h.c. \quad (\text{B.1})$$

and

$$\begin{aligned} W_1^\eta = & \frac{\eta}{2} \int d_\Delta^3 \tau C(0, \bar{y}) \bar{*} \omega(0, \bar{y}) \frac{1}{\tau_1} t^\alpha \frac{\partial}{\partial p^\alpha} \exp(i\tau_1 z_\alpha y^\alpha + i\tau_3 t^\alpha y_\alpha + i\tau_1(p+t)^\alpha z_\alpha + i(1-\tau_3)p^\alpha t_\alpha) k \\ & + \frac{\eta}{2} \int d_\Delta^3 \tau \omega(0, \bar{y}) \bar{*} C(0, \bar{y}) \frac{1}{\tau_1} t^\alpha \frac{\partial}{\partial p^\alpha} \exp(i\tau_1 z_\alpha y^\alpha - i\tau_3 t^\alpha y_\alpha + i\tau_1(p+t)^\alpha z_\alpha - i(1-\tau_3)p^\alpha t_\alpha) k + h.c. \end{aligned} \quad (\text{B.2})$$

For B_2^η one finds from (5.9)

$$\begin{aligned} B_2^\eta = & \frac{\eta}{2} \int d_\Delta^3 \tau C(0, \bar{y}) \bar{*} C(0, \bar{y}) \\ & \times \frac{\partial}{\partial \tau_1} \exp(i(1-\tau_2-\tau_3)z_\alpha y^\alpha + i\tau_1(p_1+p_2)^\alpha (z-p_1)_\alpha + i(\tau_2 p_2 - \tau_3 p_1)^\alpha y_\alpha) k. \end{aligned} \quad (\text{B.3})$$

In these expressions field derivatives p and t (4.8) act on the left. Equivalently, by explicit action with differential operators p and t expressions (B.1)–(B.3) can be rewritten as follows

$$S_1^\eta = \eta \int_0^1 d\tau \tau \theta^\alpha z_\alpha C(-\tau z, \bar{y}) \exp(i\tau z_\alpha y^\alpha) k + h.c., \quad (\text{B.4})$$

$$W_1^\eta = -i\frac{\eta}{2} \int \frac{d^2 u d^2 v}{(2\pi)^2} \int d_\Delta^3 \tau \exp(i\tau_1 z_\alpha y^\alpha + iu_\alpha v^\alpha) z^\alpha \frac{\partial}{\partial v^\alpha} \quad (\text{B.5})$$

$$\begin{aligned} & \times \left(C((1-\tau_3)u - \tau_1 z, \bar{y}) \bar{*} \omega(v - \tau_1 z - \tau_3 y, \bar{y}) \right. \\ & \quad \left. + \omega(v - \tau_1 z + \tau_3 y, \bar{y}) \bar{*} C((1-\tau_3)u - \tau_1 z, \bar{y}) \right) k + h.c., \end{aligned} \quad (\text{B.6})$$

$$\begin{aligned} B_2^\eta = & \frac{\eta}{2} \int \frac{d^2 u d^2 v}{(2\pi)^2} \int d_\Delta^3 \tau \exp i(\tau_1 z_\alpha y^\alpha + u_\alpha v^\alpha) \\ & \times \frac{\partial}{\partial \tau_1} \left(C_1(\tau_1 u - \tau_1 z + \tau_3 y, \bar{y}) \bar{*} C_2(v - \tau_1 z - \tau_2 y, \bar{y}) \right) k. \end{aligned} \quad (\text{B.7})$$

B.2 Projector formulas

In calculation of (5.17) and similar it is convenient to use the following formulae

$$h_{0,\beta}(\omega * \Delta_{0,\beta}(\theta^\alpha f_\alpha(z, y))) = -\omega(0, \bar{y}) \frac{(1-\beta)}{(2\pi)^2} \int d^2 u d^2 v \exp(iu_\alpha v^\alpha + iy^\alpha t_\alpha) F^+(f), \quad (\text{B.8})$$

$$h_{0,\beta}(\Delta_{0,\beta}(\theta^\alpha f_\alpha(z, y)) * \omega) = -\frac{(1-\beta)}{(2\pi)^2} \int d^2 u d^2 v \exp(iu_\alpha v^\alpha + iy^\alpha t_\alpha) F^-(f) \omega(0, \bar{y}), \quad (\text{B.9})$$

where

$$F^\pm(f) = \int_0^1 d\sigma t^\alpha f_\alpha(v - (1-\beta)\sigma t, y \pm t + \beta u). \quad (\text{B.10})$$

Particularly, for

$$\tilde{f}_\alpha(z, y) = \phi_\alpha \exp(i\tau_\circ z_\alpha y^\alpha + iA^\alpha y_\alpha + iB^\alpha z_\alpha), \quad (\text{B.11})$$

where ϕ_α , A_α and B_α are some constant spinors, performing Gaussian integration these amount to

$$h_{0,\beta}(\omega * \Delta_{0,\beta}(\theta^\alpha \tilde{f}_\alpha(z, y))) = \omega(0, \bar{y}) H^+ \quad h_{0,\beta}(\Delta_{0,\beta}(\theta^\alpha \tilde{f}_\alpha(z, y)) * \omega) = H^- \omega(0, \bar{y}), \quad (\text{B.12})$$

where

$$\begin{aligned} H^\pm = & - \int_0^1 d\tau \frac{1-\beta}{(1-\beta\tau_\circ)^2} t^\alpha \phi_\alpha \\ & \times \exp\left(\frac{i}{1-\beta\tau_\circ} [(A + (1-\beta)\tau_\circ\tau t)^\alpha (y \pm t)_\alpha - B^\alpha (\beta A + (1-\beta)\tau t)_\alpha] + iy^\alpha t_\alpha\right). \end{aligned} \quad (\text{B.13})$$

B.3 Some homotopy integrals

In analysis of β -dependent vertices one encounters the following integrals (cf. (5.30) and (5.36)) that we need to evaluate at $\beta \rightarrow -\infty$

$$\begin{aligned} I_1(\beta) = & - \int_{[0,1]^2} d\tau_1 d\tau'_1 \frac{\beta^3 \tau_1 (1-\tau'_1)^2 (1-\tau_1)}{(1-\beta\tau_\circ)^4} \\ & \times \exp i \left(-\frac{\beta}{\xi} \tau_1 (1-\tau'_1) A - \frac{\beta}{\xi} \tau'_1 (1-\tau_1) B + \frac{1}{\xi} C + D \right), \end{aligned} \quad (\text{B.14})$$

$$\begin{aligned} I_2(\beta) = & \int_{[0,1]^2} d\tau_1 d\tau'_1 \frac{\beta^2 (1-\tau'_1) (1-\tau_1)}{(1-\beta\tau_\circ)^4} \\ & \times \exp i \left(-\frac{\beta}{\xi} \tau_1 (1-\tau'_1) A - \frac{\beta}{\xi} \tau'_1 (1-\tau_1) B + \frac{1}{\xi} C + D \right), \end{aligned} \quad (\text{B.15})$$

where

$$\tau_\circ = \tau_1 \circ \tau'_1 := \tau_1 + \tau'_1 - 2\tau_1 \tau'_1, \quad \xi = 1 - \beta\tau_\circ. \quad (\text{B.16})$$

Using inequalities (5.28) it is easy to see that both integrals vanish for $\tau_\circ \in [\epsilon, 1]$ at $\beta \rightarrow -\infty$, where $0 < \epsilon < 1$ is some fixed β -independent number. Therefore it suffices to analyze the integrands in the vicinity $\tau_\circ \rightarrow 0$. To do so it is convenient to introduce

$$J_{n_1, n_2}^{m_1, m_2}(\tau) = \int_{[0,1]^2} d\tau_1 d\tau_2 \tau_1^{m_1} \tau_2^{m_2} (1-\tau_1)^{n_1} (1-\tau_2)^{n_2} \delta(\tau - \tau_1 \circ \tau_2), \quad (\text{B.17})$$

where $m_{1,2} \geq 0$ and $n_{1,2} \geq 0$. For small τ , $J(\tau)$ can be easily calculated with the coefficients expressed via beta-function

$$J_{n_1, n_2}^{m_1, m_2}(\tau) = \begin{cases} \frac{m_1! m_2!}{(m_1+m_2+1)!} \tau^{m_1+m_2+1} + o(\tau^{m_1+m_2+1}), & m_1 + m_2 < n_1 + n_2, \\ \frac{n_1! n_2!}{(n_1+n_2+1)!} \tau^{n_1+n_2+1} + o(\tau^{n_1+n_2+1}), & m_1 + m_2 > n_1 + n_2, \\ \frac{2m_1! m_2!}{(m_1+m_2+1)!} \tau^{m_1+m_2+1} + o(\tau^{m_1+m_2+1}), & m_1 + m_2 = n_1 + n_2. \end{cases} \quad (\text{B.18})$$

Plugging $\delta(\tau - \tau_1 \circ \tau'_1)$ into (B.14), (B.15) and using (B.18) it is easy to take $\beta \rightarrow -\infty$ to obtain (after introducing simplicial integration variables) the following result

$$I_1(-\infty) = \int d^3_\Delta \tau \tau_1 \exp i(\tau_1 A + \tau_2 B + \tau_3 C + D), \quad (\text{B.19})$$

$$I_2(-\infty) = \int d^3_\Delta \tau \tau_3 \exp i(\tau_1 A + \tau_2 B + \tau_3 C + D). \quad (\text{B.20})$$

Let us prove (B.19). Expanding (B.14) in power series we find the following integrals to be evaluated

$$\begin{aligned} & - \int d\tau_1 d\tau'_1 \frac{\beta^{3+m+n+k} \tau_1^{m+1} \tau_1'^n (1-\tau_1)^{n+1} (1-\tau'_1)^{m+2}}{(1-\beta\tau_o)^{4+m+n+k}} \\ & \xrightarrow{\beta \rightarrow -\infty} \int_0^\infty du \frac{u^{m+n+k+2}}{(1+u)^{m+n+k+4}} \int_0^1 d\lambda \lambda^{m+1} (1-\lambda)^n, \end{aligned}$$

where in taking the limit we have introduced $u = -\beta\tau_o$ and applied (B.18) in the form of integration over λ . Summing up series back into exponential and introducing simplicial variables

$$\tau_1 = \frac{u\lambda}{1+u}, \quad \tau_2 = \frac{u(1-\lambda)}{1+u}, \quad \tau_3 = \frac{1}{1+u}, \quad \sum_i \tau_i = 1 \quad (\text{B.21})$$

we obtain (B.19). Eq. (B.20) can be worked out analogously.

B.4 W to the second order

Here we give explicit formulae used in calculation of the (anti)holomorphic part of W_2 for finite β

$$W_2^{\eta\eta} = \Delta_{0,\beta} (\{W_1^\eta, S_1^\eta\}_*) + \dots, \quad (\text{B.22})$$

where ellipses denotes the terms DS_2 that do not contribute to the final result.

$$\begin{aligned} \Delta_{0,\beta} (W_1^\eta \omega_C * S_1^\eta) &= i \frac{\eta^2}{4} \int_0^1 d\rho \int d^3_\Delta \tau \int_0^1 d\sigma \times \\ & \times \frac{\partial}{\partial(p_{2\gamma})} \frac{t_\alpha \partial}{\partial(p_{1\alpha})} \frac{[(1-\beta\tau_1 \circ \sigma)z - \beta(\tau_1\sigma + \tau_3(1-\sigma)t + \tau_1\sigma(p_1 + p_2))]_\gamma}{\tau_1 \zeta^3} \times \\ & \times \exp \left(\frac{i}{\zeta} \left[\rho\tau_1 \circ \sigma z_\alpha y^\alpha + \sigma(\tau_1 - \tau_3)p_{2\alpha} t^\alpha + \tau_1 \sigma p_{2\alpha} p_1^\alpha + (-\rho\tau_1(1-\sigma)z + \tau_1 \sigma y)^\alpha p_{1\alpha} \right. \right. \\ & \left. \left. + (-\rho\tau_1(1-\sigma)z - \rho\sigma\tau_3 z + \tau_1 \sigma y + \tau_3(1-\sigma)y)^\alpha t_\alpha + (\rho\sigma(1-\tau_1)z + \tau_1 \sigma y)^\alpha p_{2\alpha} \right] \right. \\ & \left. + i(1-\tau_3)p_{1\alpha} t^\alpha - i\beta \frac{(1-\rho)\tau_1\tau_3(1-2\sigma)}{\zeta} p_{1\alpha} t^\alpha + i\beta \frac{(1-\rho)\tau_3\sigma}{\zeta} p_{2\alpha} t^\alpha \right) \omega CC, \quad (\text{B.23}) \end{aligned}$$

$$\begin{aligned} \Delta_{0,\beta} (W_1^\eta C_\omega * S_1^\eta) &= i \frac{\eta^2}{4} \int_0^1 d\rho \int d^3_\Delta \tau \int_0^1 d\sigma \times \\ & \times \frac{\partial}{\partial(p_{2\gamma})} \frac{\partial}{\partial(p_{1\alpha})} t_\alpha \frac{[(1-\beta\tau_1 \circ \sigma)z - \beta((\tau_1\sigma - \tau_3(1-\sigma)t + \tau_1\sigma(p_1 + p_2))]_\gamma}{\tau_1 \zeta^3} \times \\ & \times \exp \left(\frac{i}{\zeta} \left[\rho\tau_1 \circ \sigma z_\alpha y^\alpha + \sigma(\tau_1 + \tau_3)p_{2\alpha} t^\alpha + \tau_1 \sigma p_{2\alpha} p_1^\alpha + (-\rho\tau_1(1-\sigma)z + \tau_1 \sigma y)^\alpha p_{1\alpha} \right. \right. \end{aligned}$$

$$\begin{aligned}
 & +(-\rho\tau_1(1-\sigma)z + \rho\sigma\tau_3z + \tau_1\sigma y - \tau_3(1-\sigma)y)^\alpha t_\alpha + (\rho\sigma(1-\tau_1)z + \tau_1\sigma y)^\alpha p_{2\alpha} \Big] \\
 & -i(1-\tau_3)p_{1\alpha}t^\alpha + i\beta\frac{(1-\rho)\tau_1\tau_3(1-2\sigma)}{\zeta}p_{1\alpha}t^\alpha - i\beta\frac{(1-\rho)\tau_3\sigma}{\zeta}p_{2\alpha}t^\alpha \Big) C\omega C, \quad (B.24)
 \end{aligned}$$

$$\begin{aligned}
 \Delta_{0,\beta}(S_1^\eta * W_1^\eta{}_{\omega C}) &= i\frac{\eta^2}{4} \int_0^1 d\rho \int d\Delta^3 \tau \int_0^1 d\sigma \times \\
 & \times \frac{\partial}{\partial(p_{1\gamma})} \frac{t_\alpha \partial}{\partial(p_{2\alpha})} \frac{[(1-\beta\tau_1 \circ \sigma)z - \beta((\tau_1\sigma - \tau_3(1-\sigma)t - \tau_1\sigma(p_1 + p_2)))]_\gamma}{\tau_1\zeta^3} \times \\
 & \times \exp\left(\frac{i}{\zeta} \left[\rho\tau_1 \circ \sigma z_\alpha y^\alpha + \sigma(\tau_1 + \tau_3)t_\alpha p_1^\alpha + \tau_1\sigma p_{2\alpha} p_1^\alpha + (-\rho\sigma(1-\tau_1)z + \tau_1\sigma y)^\alpha p_{1\alpha} \right. \right. \\
 & \left. \left. + (\rho\tau_1(1-\sigma)z - \rho\sigma\tau_3z + \tau_1\sigma y - \tau_3(1-\sigma)y)^\alpha t_\alpha + (\rho\tau_1(1-\sigma)z + \tau_1\sigma y)^\alpha p_{2\alpha} \right] \right. \\
 & \left. + i(1-\tau_3)p_{2\alpha}t^\alpha - i\beta\frac{(1-\rho)\tau_1\tau_3(1-2\sigma)}{\zeta}p_{2\alpha}t^\alpha - i\beta\frac{(1-\rho)\tau_3\sigma}{\zeta}t_\alpha p_1^\alpha \right) C\omega C, \quad (B.25)
 \end{aligned}$$

$$\begin{aligned}
 \Delta_{0,\beta}(S_1^\eta * W_1^\eta{}_{C\omega}) &= i\frac{\eta^2}{4} \int_0^1 d\rho \int d\Delta^3 \tau \int_0^1 d\sigma \times \\
 & \times \frac{\partial}{\partial(p_{1\gamma})} \frac{\partial}{\partial(p_{2\alpha})} t_\alpha \frac{[(1-\beta\tau_1 \circ \sigma)z + i\beta((\tau_1\sigma + \tau_3(1-\sigma)\partial_\omega + \tau_1\sigma(\partial_1 + \partial_2)))]_\gamma}{\tau_1\zeta^3} \times \\
 & \times \exp\left(\frac{i}{\zeta} \left[\rho\tau_1 \circ \sigma z_\alpha y^\alpha + \sigma(\tau_1 - \tau_3)t_\alpha p_1^\alpha + \tau_1\sigma p_{2\alpha} p_1^\alpha + (-\rho\sigma(1-\tau_1)z + \tau_1\sigma y)^\alpha p_{1\alpha} \right. \right. \\
 & \left. \left. + (\rho\tau_1(1-\sigma)z + \rho\sigma\tau_3z + \tau_1\sigma y + \tau_3(1-\sigma)y)^\alpha t_\alpha + (\rho\tau_1(1-\sigma)z + \tau_1\sigma y)^\alpha p_{2\alpha} \right] \right. \\
 & \left. + i(1-\tau_3)t_\alpha p_2^\alpha - i\beta\frac{(1-\rho)\tau_1\tau_3(1-2\sigma)}{\zeta}t_\alpha p_2^\alpha + i\beta\frac{(1-\rho)\tau_3\sigma}{\zeta}t_\alpha p_1^\alpha \right) CC\omega, \quad (B.26)
 \end{aligned}$$

where $\circ$ is defined in (5.26), p_1 and p_2 (4.8) differentiate C' s as seen from left and

$$\zeta = 1 - \beta(1 - \rho)(\tau_1 \circ \sigma). \quad (B.27)$$

C Lower-order vertices

For the reader's convenience the formulae for lower-order vertices are presented in this appendix.

$$d_x \omega = -\omega * \omega + \Upsilon_{\omega\omega C}^\eta + \Upsilon_{C\omega\omega}^\eta + \Upsilon_{\omega C\omega}^\eta + \Upsilon_{\omega\omega C}^{\bar{\eta}} + \Upsilon_{C\omega\omega}^{\bar{\eta}} + \Upsilon_{\omega C\omega}^{\bar{\eta}} + \dots, \quad (C.1)$$

$$d_x C = -[\omega, C]_* + \Upsilon_{\omega CC}^\eta + \Upsilon_{CC\omega}^\eta + \Upsilon_{C\omega C}^\eta + \Upsilon_{\omega CC}^{\bar{\eta}} + \Upsilon_{CC\omega}^{\bar{\eta}} + \Upsilon_{C\omega C}^{\bar{\eta}} + \dots, \quad (C.2)$$

where [3]

$$\Upsilon_{\omega\omega C}^\eta = \frac{\eta}{4i} \omega * \omega * C * h_{p+t_1+t_2} \Delta_p \Delta_{p+t_2} \gamma, \quad (C.3)$$

$$\Upsilon_{C\omega\omega}^\eta = \frac{\eta}{4i} C * \omega * \omega * h_{p+t_1+t_2} \Delta_{p+t_1+2t_2} \Delta_{p+2t_1+2t_2} \gamma, \quad (C.4)$$

$$\Upsilon_{\omega C\omega}^\eta = -\frac{\eta}{4i} \omega * C * \omega * (h_{p+t_1+t_2} \Delta_{p+t_1+2t_2} \Delta_{p+t_2} \gamma - h_{p+t_1+2t_2} \Delta_{p+2t_2} \Delta_{p+t_2} \gamma), \quad (C.5)$$

$$\Upsilon_{\omega CC}^\eta = \frac{\eta}{4i} \omega * C * C * h_{p_2} \Delta_{p_1+2p_2} \Delta_{p_1+2p_2+t} \gamma, \quad (C.6)$$

$$\Upsilon_{CC\omega}^\eta = \frac{\eta}{4i} C * C * \omega * h_{p_2+2t} \Delta_{p_2+t} \Delta_{p_1+2p_2+2t} \gamma, \quad (\text{C.7})$$

$$\Upsilon_{C\omega C}^\eta = \frac{\eta}{4i} C * \omega * C * (h_{p_1+2p_2+2t} - h_{p_2}) \Delta_{p_2+t} \Delta_{p_1+2p_2+t} \gamma \quad (\text{C.8})$$

and

$$\Upsilon_{\omega\omega C}^{\bar{\eta}} = \frac{\bar{\eta}}{4i} \omega * \omega * C * \bar{h}_{\bar{p}+\bar{t}_1+\bar{t}_2} \bar{\Delta}_{\bar{p}} \bar{\Delta}_{\bar{p}+\bar{t}_2} \bar{\gamma}, \quad (\text{C.9})$$

$$\Upsilon_{C\omega\omega}^{\bar{\eta}} = \frac{\bar{\eta}}{4i} C * \omega * \omega * \bar{h}_{\bar{p}+\bar{t}_1+\bar{t}_2} \bar{\Delta}_{p+\bar{t}_1+2\bar{t}_2} \bar{\Delta}_{p+2\bar{t}_1+2\bar{t}_2} \bar{\gamma}, \quad (\text{C.10})$$

$$\Upsilon_{\omega C\omega}^{\bar{\eta}} = -\frac{\bar{\eta}}{4i} \omega * C * \omega * (\bar{h}_{\bar{p}+\bar{t}_1+\bar{t}_2} \bar{\Delta}_{\bar{p}+\bar{t}_1+2\bar{t}_2} \bar{\Delta}_{\bar{p}+\bar{t}_2} \bar{\gamma} - \bar{h}_{\bar{p}+\bar{t}_1+2\bar{t}_2} \bar{\Delta}_{\bar{p}+2\bar{t}_2} \bar{\Delta}_{\bar{p}+\bar{t}_2} \bar{\gamma}), \quad (\text{C.11})$$

$$\Upsilon_{\omega C C}^{\bar{\eta}} = \frac{\bar{\eta}}{4i} \omega * C * C * \bar{h}_{\bar{p}_2} \bar{\Delta}_{\bar{p}_1+2\bar{p}_2} \bar{\Delta}_{\bar{p}_1+2\bar{p}_2+\bar{t}} \bar{\gamma}, \quad (\text{C.12})$$

$$\Upsilon_{C C \omega}^{\bar{\eta}} = \frac{\bar{\eta}}{4i} C * C * \omega * \bar{h}_{\bar{p}_2+2\bar{t}} \bar{\Delta}_{\bar{p}_2+\bar{t}} \bar{\Delta}_{\bar{p}_1+2\bar{p}_2+2\bar{t}} \bar{\gamma}, \quad (\text{C.13})$$

$$\Upsilon_{C\omega C}^{\bar{\eta}} = \frac{\bar{\eta}}{4i} C * \omega * C * (\bar{h}_{\bar{p}_1+2\bar{p}_2+2\bar{t}} - \bar{h}_{\bar{p}_2}) \bar{\Delta}_{\bar{p}_2+\bar{t}} \bar{\Delta}_{\bar{p}_1+2\bar{p}_2+\bar{t}} \bar{\gamma}. \quad (\text{C.14})$$

$\Upsilon(\omega, C, C)$ vertices given here are derived for B field given in (5.9).

To see that $\Upsilon(\omega, \omega, C)$ are ultra-local (i.e., C is y -independent) one can make sure using eqs. (2.18), (5.45) and (4.8) that for a, b, c independent of p_1

$$\begin{aligned} C(y) * h_{p_1+c} \Delta_{p_1+b} \Delta_{p_1+a} \gamma &= 2 \int_0^1 d\Delta^3 \tau C(y) * (b-c)_\gamma (a-c)^\gamma \exp(-i(p_1 + \tau_1 c + \tau_2 a + \tau_3 b)_\alpha y^\alpha) k \\ &= 2 \int_0^1 d\Delta^3 \tau C(0) * (b-c)_\gamma (a-c)^\gamma \exp(-i(\tau_1 c + \tau_2 a + \tau_3 b)_\alpha y^\alpha) k. \end{aligned} \quad (\text{C.15})$$

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