ϵ -regular basis for non-polylogarithmic multiloop integrals and total cross section of the process $e^+e^- \rightarrow 2(Q\bar{Q})$

R.N. Lee^a and A.I. Onishchenko^{b,c}

^a*Theory department, Budker Institute of Nuclear Physics,
11 Academician Lavrentiev prospect, 630090 Novosibirsk, Russia*

^b*Bogoliubov Laboratory of Theoretical Physics, Joint Institute for Nuclear Research,
Joliot-Curie 6, 141980 Dubna, Moscow region, Russia*

^c*Skobeltsyn Institute of Nuclear Physics, Moscow State University,
1(2), Leninskie gory, GSP-1, 119991 Moscow, Russia*

E-mail: r.n.lee@inp.nsk.su, onish@theor.jinr.ru

ABSTRACT: We argue that in many physical calculations where the “elliptic” sectors are involved, one can express the results via iterated integrals with almost all weights being rational. Our method is based on the existence of ϵ -regular basis, which is akin to the ϵ -finite basis defined in ref. [1]. As a demonstration of our technique, we calculate the photon contribution to the total cross section of the production of two $Q\bar{Q}$ pairs in the electron-positron collisions.

KEYWORDS: NLO Computations, QCD Phenomenology

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1 Introduction

Multiloop calculations have been an extremely rapidly developing subject in quantum field theory for the last thirty years. The two important milestones in the beginning of this long journey have been the invention of the integration-by-part reduction [2, 3] and differential equations technique [4, 5]. These achievements have paved the way for the computer-assisted calculations which resulted in a burst-like growth of many physically relevant calculations. While doing these calculations, we have learned that the results can often be expressed in terms of iterated integrals with rational weights, viz., the multiple polylogarithms (MPLs), [6, 7]. This class of functions is thoroughly investigated in many aspects, including the identities among them and algorithms for their effective numerical evaluation.

However, for some, at first relatively rare, cases it appeared that other functions may also be involved, the (arguably) simplest example being the phase-space integral for three massive particles. Later it was realized that the possibility to express the results in terms of MPLs corresponds¹ to the existence of ϵ -form of the differential system [8, 9]. This is the form, where the dependence on dimensional regularization parameter ϵ in the right-hand side of the differential system is pulled out as the overall factor ϵ . The transformation to this form resembles passing to the interaction picture in quantum mechanics, with $\epsilon = 2 - d/2$ being the small parameter, and the iterated integrals arising naturally from the perturbative expansion of a path-ordered exponent (evolution operator). In ref. [10] a criterion of reducibility has been given. The non-polylogarithmic loop integrals correspond to the case where one can not express the “unperturbed” evolution operator in terms of rational or algebraic functions. In particular, for 3- and 4- equal-mass particles phase space integrals, this operator involves the complete elliptic integrals [11, 12]. There is probably not much we can do about this fact, but the price we have to pay multiplies when we

¹With some reservations related to the possibility to find a rationalizing variable change.

go to higher orders in ϵ . Namely, each successive integration involves the kernel which contains functions entering the unperturbed evolution operator. The situation only gets more complicated when these results enter the right-hand side of the differential equations for the master integrals in higher sectors. This is, of course, a poor man's approach, which, for the above-mentioned examples, is superseded by two alternative approaches, one resulting in the iterative integrals over modular forms [13, 14], the other giving up the iterative structure of the integrals in favor of algebraic weights [15, 16]² (see also refs. [17, 18]). While we readily acknowledge a nice geometric picture behind each of these two approaches, one might nevertheless wonder if there is a narrower, simpler class of functions sufficient for the physical applications. A hint for the positive answer to this question can be found already in the paper by G. Racah [19] written in 1934! In this paper the total cross section of the e^+e^- -pair production by a photon with energy ω in the field of a nucleus with charge $Z|e|$ has been calculated. The result was obtained by the direct integration of the spectrum and has the following form³

$$\sigma(\gamma Z \rightarrow e^+e^-Z) = \frac{\alpha(Z\alpha)^2}{m^2} \left\{ \frac{692 + 468\xi + 76\xi^2 + 108\xi^3}{27(1+\xi)^3} K(\xi^2) - \frac{692 + 360\xi + 692\xi^2}{27(1+\xi)^3} E(\xi^2) - \frac{4(1-\xi)^2}{(1+\xi)^2} \int_0^\xi \frac{K(\eta^2)d\eta}{1-\eta} + \frac{16(1-\xi)^2}{(1+\xi)^2} \int_0^\xi \frac{d\zeta}{1-\zeta^2} \int_0^\zeta \frac{K(\eta^2)d\eta}{1-\eta} \right\}, \quad (1.1)$$

where K and E are the complete elliptic integrals of the first and second kinds, respectively, and $\xi = \frac{\omega-2m}{\omega+2m}$ (m is the electron mass). Note the modern look of this result which at the Racah time could have been underestimated. The appearance of the elliptic integrals is not surprising: they come due to the cut sunrise integral with two massive lines and one HQET line. However the rational integration weights are very remarkable. On general ground we would have expected the proliferation of transcendental weights in each successive integration. Let us also note that the two-fold integral in eq. (1.1) can be easily converted to a one-fold integral by changing the order of integration and taking the integral over ζ in terms of logarithms. Moreover, it is easy to understand that a similar k -fold iterated integral with all but the innermost weight being rational can be reduced to one-fold integral which integrand is expressed via complete elliptic integral function K and multiple polylogarithms. We note that the latter form is also known for the kite graph, thanks to ref. [17].

In the present paper we demonstrate that, in some sense, the appearance of the iterated integrals with most of the weights being rational is a general pattern. Our method is based on the existence of a special basis of master integrals which we call ϵ -regular basis. It has many features common to ϵ -finite basis of ref. [1]. Speaking loosely, our ϵ -regular basis is that which remains a well-defined basis at $\epsilon = 0$.

As an immediate application of our method, we calculate the photon contribution to the total Born cross section of the process $e^+e^- \rightarrow 2(Q\bar{Q})$ with the full account of the quark mass.

²Here by “iterated integral” we mean the integral of the form $\int_{x_0 < x_1 < \dots < x_n < x} dx_1 f_1(x_1) \dots dx_n f_n(x_n)$, where the dependence on the kinematic parameter is only via the upper limit x .

³A typographical error in the numerical coefficient of the last term has been corrected in ref. [20] by Racah himself. Note also that we use modern convention for the arguments of elliptic integrals, so that, e.g., $K(z) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-z\sin^2\theta}}$.

2 ϵ -regular basis

Let us elaborate on the notion of ϵ -regular basis of master integrals. While our definition is akin to that of ϵ -finite basis in ref. [1], it somehow differs in a few details.

Suppose that we have a family $\mathcal{F} = \{j(k|\mathbf{n})|k = 1, \dots, K \text{ and } \mathbf{n} \in \mathbb{Z}^{N_k}\}$ of L -loop integrals which may belong to one or more prototypes (big topologies):

$$j(k|n_1 \dots n_{N_k}) = \int \frac{d^d l_1 \dots d^d l_L}{D_{k1}^{n_1} \dots D_{kN_k}^{n_{N_k}}}. \quad (2.1)$$

Here $d = 4 - 2\epsilon$ and $k = 1, \dots, K$ enumerates different prototypes of our family. We define the *basis* of $\mathcal{F}$ as a finite set of functions $j_1, \dots, j_M$ such that any integral of $\mathcal{F}$ can be uniquely represented as their linear combination with coefficients being the rational functions of kinematic invariants and space-time dimension over rational numbers. In what follows we will always understand the linear combinations and linear (in)dependence in this way. In particular, from the uniqueness we conclude that $j_1, \dots, j_M$ are all linearly independent. This is a usual definition of the basis of master integrals except that we allow also for the linear combinations of integrals as its elements (as “master integrals”). Then it is easy to understand that there exists such a basis $F_1, \dots, F_M$ that the following conditions are fulfilled:

1. The ϵ -expansion of each F_m starts from ϵ^0 , i.e., $F_m = F_m^{(0)} + O(\epsilon)$.
2. The leading terms $F_1^{(0)}, \dots, F_M^{(0)}$ are linearly independent (in the above sense).

We will call any basis with these properties the ϵ -regular basis. From the second condition it follows, in particular, that $F_m^{(0)} \neq 0$ for any $m = 1, \dots, M$.

Let us first explain why such a basis necessarily exists. Suppose that we have arbitrary basis $F_1, \dots, F_M$, then we can multiply each element by a suitable power of ϵ so that the redefined basis satisfies the first condition. Now assume $F_1, \dots, F_M$ satisfies the first condition, but there is a vanishing linear combination $F_{m_0}^{(0)} - \sum_{m < m_0} c_m F_m^{(0)} = 0$. Then we redefine the m_0 -th element of the basis by replacing

$$F_{m_0} \rightarrow \frac{1}{\epsilon} \left[F_{m_0} - \sum_{m < m_0} c_m F_m \right]. \quad (2.2)$$

It is easy to understand that proceeding in this way we will finally obtain the ϵ -regular basis.⁴

The rationale behind our definition is the following. Suppose that we want to calculate some quantity P whose ϵ -expansion starts from $O(\epsilon^0)$ and which is a linear combination of the integrals of $\mathcal{F}$ (possibly, with coefficients singular at $\epsilon = 0$). Then this quantity can be

⁴Here we silently rely on some properties of the multiloop integrals. In particular, we assume that consecutive terms of ϵ expansion generate infinite or, at least, large enough set of mutually transcendental functions to form the suitable basis. For generic set, the ϵ -regular basis does not necessarily exist. Consider, e.g., the linear span of $F_1(x) = \exp(\epsilon x)$, $F_2(x) = \exp(-\epsilon x)$. Obviously, the described process will not terminate for these two functions.

reduced to linear combination of elements $F_1, \dots, F_M$ of the ϵ -regular basis with coefficients which are necessarily regular at $\epsilon = 0$. The proof by contradiction is almost trivial: suppose there are singular coefficients with the maximal pole order $n > 0$. Then the coefficient of ϵ^{-n} is a non-trivial linear combination of the leading terms $F_1^{(0)}, \dots, F_M^{(0)}$, which, by second condition, can not be zero. Therefore, the expansion of P should start from ϵ^{-n} , which contradicts our assumption. Therefore, we can claim that any physical quantity well-defined at $\epsilon = 0$ and expressed, in dimensional regularization, as a linear combination of multiloop integrals of $\mathcal{F}$, is a linear combination of the leading terms $\mathcal{F}_1^{(0)}, \dots, \mathcal{F}_M^{(0)}$ of ϵ -regular basis. There is another important property of the ϵ -regular basis. Consider the differential system $d\mathbf{F} = M_F \mathbf{F}$. Then we can claim that M_F has a finite $\epsilon \rightarrow 0$ limit. The proof goes along the same lines as above.

Let us remark about the similarity of the introduced ϵ -regular basis and ϵ -finite basis of ref. [1]. They are both engineered to secure the finite coefficients in front of the master integrals in the physical quantities. However, our definition shows that, once this physical quantity is finite, also the elements of the basis can be chosen finite. While this basis is obviously relevant for calculations of explicitly finite quantities, like Born cross sections calculated in the present paper, it is expected to be also applicable to renormalized quantities. This is because the counter-terms are usually expressed via the multiloop integrals with lower loops multiplied by rational coefficients, thus, the condition of being expressible as linear combination of the multiloop integrals is fulfilled.

Let us now assume that the differential system for some master integrals $\mathbf{J} = (J_1, \dots, J_M)^\top$ is in ϵ -form:

$$d\mathbf{J} = \epsilon A(\mathbf{x}) \mathbf{J}, \quad (2.3)$$

where $\mathbf{x}$ denotes the kinematic parameters, and the matrix 1-form A satisfies $dA = A \wedge A = 0$. Then there is an ϵ -regular basis $\mathbf{F} = (F_1, \dots, F_M)^\top$ satisfying

$$d\mathbf{F} = B(\epsilon, \mathbf{x}) \mathbf{F} = \sum_{k=0}^K \epsilon^k B_k(\mathbf{x}) \mathbf{F}, \quad (2.4)$$

where $K < \infty$ and B_0 is strictly lower triangular, i.e., lower triangular with zero diagonal. In order to prove this, let us formulate two “moves” which will eventually render $\mathbf{J}$ into $\mathbf{F}$. Preliminary step is to multiply all $\mathbf{J}$ by a common factor ϵ^n , where n is chosen so that the modified basis $\tilde{\mathbf{J}} = \epsilon^n \mathbf{J}$ is regular at $\epsilon = 0$ and for some m we had $\tilde{J}_m^{(0)} \neq 0$. These requirements uniquely fix n . In what follows we will denote our current basis by $\mathbf{J}$.

From now on we repetitively do the following. On each repetition we assume that $B(\epsilon, \mathbf{x}) = \sum_{k=0}^{K < \infty} \epsilon^k B_k(\mathbf{x})$ and that $B_0(\mathbf{x})$ is strictly lower triangular and check that these assumptions also hold in the end of the cycle. Assume that the first $m-1$ integrals satisfy both conditions, but first m integrals don't. It means that either integral J_m starts from higher order of ϵ , or that its leading term $J_m^{(0)}$ is linearly dependent on $J_1^{(0)}, \dots, J_{m-1}^{(0)}$, i.e., that $J_m^{(0)} = c_1 J_1^{(0)} + \dots + c_{m-1} J_{m-1}^{(0)}$ with some rational coefficients $c_1, \dots, c_{m-1}$.

1. Let J_m start from higher order of ϵ . We replace J_m with $\tilde{J}_m = \epsilon^{-1} J_m$. After this replacement the matrix $B(\epsilon, \mathbf{x})$ in the right-hand side of the differential system is

altered. Namely, the m -th row is divided by ϵ and the m -th column is multiplied by ϵ , while $B_{m,m}$ is unchanged. Let us first prove that $B(\epsilon, \mathbf{x})$ is regular at $\epsilon = 0$. Obviously, the poles might appear only on m -th row in positions $1, \dots, m-1$. However, that would mean that some linear combination of $J_1, \dots, J_{m-1}$ is vanishing at $\epsilon \rightarrow 0$, which by assumption is not the case. So, there are no poles in the new matrix B . Some elements on m -th row, starting from position $m+1$ might now be of order ϵ^0 . Therefore, the new matrix B_0 is not strictly lower triangular one. However, this is fixed by re-numerating the integrals. Note that m -th column of the matrix B is suppressed at least as ϵ . Let the index of last nonzero element of the m -th row of new B_0 be $\tilde{m}$ ($\tilde{m} > m$). Then we re-numerate integrals $J_{m+1}, \dots, J_{\tilde{m}}$ to be the new $\tilde{J}_m, \dots, \tilde{J}_{\tilde{m}-1}$ and $\tilde{J}_m$ to be the new $\tilde{\tilde{J}}_{\tilde{m}}$. So, the overall change is

$$\mathbf{J} = \begin{pmatrix} \vdots \\ J_m \\ J_{m+1} \\ \vdots \\ J_{\tilde{m}} \\ \vdots \end{pmatrix} \longrightarrow \begin{pmatrix} \vdots \\ J_{m+1} \\ \vdots \\ J_{\tilde{m}} \\ \epsilon^{-1} J_m \\ \vdots \end{pmatrix}. \quad (2.5)$$

After this change the new matrix B is regular at $\epsilon = 0$ and its leading term B_0 is strictly lower triangular.

2. Let $J_m^{(0)} = c_1 J_1^{(0)} + \dots + c_{m-1} J_{m-1}^{(0)}$. Then we pass from J_m to $\tilde{J}_m = J_m - c_1 J_1 - \dots - c_{m-1} J_{m-1}$. This new integral is suppressed at least as ϵ^1 , so on the next iteration we will hit case 1. Thus, the change is

$$\mathbf{J} = \begin{pmatrix} \vdots \\ J_m \\ \vdots \end{pmatrix} \longrightarrow \begin{pmatrix} \vdots \\ J_m - \sum_{k=1}^{m-1} c_k J_k \\ \vdots \end{pmatrix}. \quad (2.6)$$

It is clear that the new matrix B is regular at $\epsilon = 0$ and B_0 is strictly lower triangular.

Naturally, when we have processed the case $m = M$, the algorithm terminates, and the current set of master integrals is obviously an ϵ -regular basis $\mathbf{F}$. Suppose that our initial matrix A was lower block-triangular with diagonal blocks corresponding to sectors. Then the above algorithm tries to conserve the block structure as much as possible.

From eq. (2.4) we see that the leading coefficients satisfy the equation

$$d\mathbf{F}^{(0)} = B_0(\mathbf{x})\mathbf{F}^{(0)}, \quad (2.7)$$

which can be solved in quadratures as $\mathbf{F}^{(0)} = \int B_0(\mathbf{x})\mathbf{F}^{(0)}$, and, thanks to the strict lower-triangularity of the matrix B_0 , this does not lead to circular definition.

Note that the form (2.7) of the differential system for finite master integrals with strictly lower-triangular matrix $B_0(\mathbf{x})$ was already advocated in ref. [21]. Thus, our notion

of ϵ -regular basis allows us to algorithmically obtain the desired form for the systems reducible to ϵ -form.

Suppose now that some sectors can not be reduced to ϵ -form by a chosen class of transformations (involving either rational or algebraic functions). Then it is obvious that it is not possible to obtain finite basis satisfying (2.4) with elements of B_k being rational (or even algebraic) functions and with B_0 being strictly lower triangular. Otherwise, this would mean that we can obtain coefficients of ϵ expansion in terms of iterated integrals with weights equal to the elements of B_k . However, we can still apply the algorithm described above to the form where as many as possible blocks are reduced to ϵ -form. That is, instead of ϵ -form, eq. (2.3), we start from the equation of the form

$$d\mathbf{J} = \sum_{k=0}^K \epsilon^k A_k(\mathbf{x}) \mathbf{J}, \quad (2.8)$$

where A_0 has a few non-zero columns corresponding to the integrals of irreducible sector(s).⁵ Then, proceeding in a similar way as before, we will still be able to obtain the same form (2.4), where however the matrix B_0 contains now some non-zero diagonal blocks corresponding to irreducible sectors. The solution of the homogeneous system for these blocks is then nontrivial. However, once this solution is known, the leading terms of the integrals in the higher sectors are obtained as iterated integrals with rational weights.

3 Pedagogical example: derivation of Racah result for $\sigma_B(\gamma Z \rightarrow e^+ e^- Z)$

Let us first demonstrate how the Racah result is obtained via the modern differential equations technique. The total cross section of the process is expressed as a sum of diagrams depicted in figure 1.

Contributions of both diagrams can be expressed via integrals of the following form

$$j_{n_1 \dots n_7} = \int \frac{d^d l d^d q}{(2\pi)^{2d}} \theta(q^0 - l^0) \theta(l^0 + k^0) \prod_{k=1}^3 \Im \frac{1}{(D_k - i0)^{n_k}} \prod_{k=4}^7 \frac{1}{(D_k - i0)^{n_k}},$$

$$D_1 = 1 - (k + l)^2, \quad D_2 = 1 - (l - q)^2, \quad D_3 = -2q \cdot v = -2q_0,$$

$$D_4 = 1 - l^2, \quad D_5 = 1 - (k + l - q)^2, \quad D_6 = -q^2, \quad D_7 = -2l \cdot v = -2l_0, \quad (3.1)$$

where D_7 denotes an irreducible numerator. Here and below we put the electron mass to unit for simplicity.

The IBP reduction⁶ reveals 4 master integrals:

$$j_{11110000}, j_{11200000}, j_{11110010}, j_{11110100}. \quad (3.2)$$

⁵One might wonder if this form is general enough for any differential system appearing in multiloop calculation to be reducible to it. While we believe that the answer to this question is positive, and even after restricting the sum by $K = 1$, we admit that this question requires some further considerations. In the present paper we assume the existence of this form. Note however that our algorithm can tolerate even putting K to infinity in eqs. (2.4) and (2.8).

⁶We use LiteRed for the IBP reduction, [22].

Figure 1. Cut diagrams for the calculation of the total cross section of e^+e^- pair production by a photon in the field of nucleus. Cut thin line denotes the cut propagators $-2\pi i\delta(p^2-m^2)(\hat{p}+m)$ of the electron, cut double line denotes the cut propagator $-2\pi i\delta(2q \cdot v) = -2\pi i\delta(2q_0)$ of a heavy particle, interaction vertex with the heavy particle is $-iv^\mu$ ($v = (1, \mathbf{0})$ is a four-velocity of the nucleus).

It is easy to understand that all these integrals are finite at $\epsilon = 0$. However, if we express the cross section via these master integrals, the coefficients will have poles:

$$\sigma = \frac{(16\pi)^3 \alpha (Z\alpha)^2}{\omega} \left\{ \left(-\frac{1}{4\epsilon} + \frac{7}{4} + O(\epsilon) \right) j_{1110010} + \left(\frac{1}{2\epsilon} - 2 + O(\epsilon) \right) j_{1110100} \right. \\ \left. \times \left(\frac{3}{16\epsilon} - \frac{69\omega^2 + 58}{72\omega^2} + O(\epsilon) \right) j_{1110000} + \frac{\omega^2 - 4}{8\omega} \left(\frac{1}{\epsilon} - \frac{34\omega^2 + 4}{9\omega^2} + O(\epsilon) \right) j_{1120000} \right\}, \quad (3.3)$$

where $\omega = k_0$ and we have truncated the coefficients at ϵ^0 to save space. Since we calculate the finite quantity, the ϵ^{-1} terms should cancel, which gives the condition

$$\frac{1}{2} j_{1110100}^{(0)} - \frac{1}{4} j_{1110010}^{(0)} + \frac{\omega^2 - 4}{8\omega} j_{1120000}^{(0)} + \frac{3}{16} j_{1110000}^{(0)} = 0, \quad (3.4)$$

where $j_k^{(0)}$ denotes the leading $\propto \epsilon^0$ term of j_k .

Before we proceed further, let us underline, that in usual approach we would have concluded here, that we need to know all integrals up to ϵ^1 . Meanwhile, the first two integrals can not be expressed via polylogarithms. Their leading terms $j_{1110000}^{(0)}$ and $j_{1120000}^{(0)}$ read

$$j_{1110000}^{(0)} = \frac{(\xi^2 + 1) E(\xi^2) + (\xi^2 - 1) K(\xi^2)}{24\pi^3(1 - \xi)^3}, \quad (3.5)$$

$$j_{1120000}^{(0)} = -\frac{(\xi + 1) (E(\xi^2) + (\xi - 1) K(\xi^2))}{64\pi^3(\xi - 1)^2}. \quad (3.6)$$

Therefore, the next terms $j_{1110000}^{(1)}$ and $j_{1120000}^{(1)}$ are likely to involve modular forms in the integration weights. From this point of view, in order to arrive to the Racah result in the standard approach one would need some non-trivial functional relations between iterated integrals of modular forms.

In the approach advocated here, we first pass to ϵ -regular basis. In particular, thanks to the relation (3.4), we can pass from $j_{1110100}$ to the linear combination

$$\tilde{j} = \frac{1}{2\epsilon} \left[j_{1110100} - \frac{1}{2} j_{1110010} + \frac{\omega^2 - 4}{4\omega} j_{1120000} + \frac{3}{8} j_{1110000} \right]. \quad (3.7)$$

For the purpose of simpler presentation, we prefer to pass to a slightly different combination

$$\frac{1-\epsilon}{2\epsilon}j_{1110100} - \frac{1-3\epsilon}{4\epsilon}j_{1110010} + \frac{\omega^2-4}{8\omega\epsilon}j_{1120000} + \frac{3-4\epsilon}{16\epsilon}j_{1110000} = j_{1110110}, \quad (3.8)$$

whose coefficients differ from those in $\tilde{j}$ only by ϵ^0 terms, but which can be represented as one specific integral $j_{1110110}$. In terms of new master integrals the cross section has the form

$$\sigma = \frac{(16\pi)^3 \alpha(Z\alpha)^2}{\omega} \left\{ j_{1110110} + \frac{1}{4}j_{1110010} - \frac{(\omega^2-4)(7\omega^2+4)}{72\omega^3}j_{1120000} - \frac{21\omega^2+116}{144\omega^2}j_{1110000} \right\}, \quad (3.9)$$

where we have omitted $O(\epsilon^1)$ terms in the coefficients. Note the absence of poles in the coefficients. The differential system for the column

$$\mathbf{j} = (j_{1110000}, j_{1120000}, j_{1110010}, j_{1110110})^\top \quad (3.10)$$

has the form $\partial_\xi \mathbf{j} = M \mathbf{j}$ with

$$M = \begin{pmatrix} 0 & -\frac{8}{(1-\xi)^2} & 0 & 0 \\ \frac{(1-2\epsilon)(3-4\epsilon)}{8\xi} & \frac{(1-2\epsilon)(\xi^2+4\xi+1)}{\xi(1-\xi^2)} & 0 & 0 \\ 0 & -\frac{2}{(1+\xi)^2} & -\frac{2(1-2\epsilon)}{1-\xi^2} & 0 \\ 0 & -\frac{1}{2(\xi+1)^2} & -\frac{1-3\epsilon}{1-\xi^2} & -\frac{2}{1-\xi^2} \end{pmatrix} \quad (3.11)$$

This system has a regular limit $\epsilon \rightarrow 0$, therefore, from now on we can safely put $\epsilon = 0$. As we have seen in the previous section, we can find the transformation which reduces the differential system to strictly lower-triangular form everywhere except the irreducible block corresponding to the “elliptic” sector. Indeed, passing to new master integrals $\mathbf{F} = (F_1, F_2, F_3, F_4)^\top$ related to $\mathbf{j}$ via

$$\mathbf{j} = T \mathbf{F}, \quad (3.12)$$

$$T|_{\epsilon=0} = \begin{pmatrix} -\frac{16\xi}{3(1-\xi^2)} & \frac{16(1+\xi^2)}{3(1-\xi^2)} & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & \frac{1-\xi}{1+\xi} & -\frac{1-\xi}{1+\xi} & 0 \\ 0 & 0 & 0 & \frac{1-\xi}{1+\xi} \end{pmatrix}, \quad (3.13)$$

we obtain the differential system

$$\partial_\xi \mathbf{F} = M_F \mathbf{F}, \quad (3.14)$$

$$M_F|_{\epsilon=0} = \begin{pmatrix} 0 & \frac{1-\xi}{\xi(1+\xi)} & 0 & 0 \\ \frac{1}{1-\xi^2} & \frac{4}{1-\xi^2} & 0 & 0 \\ \frac{1}{1-\xi^2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{1-\xi^2} & 0 \end{pmatrix} \quad (3.15)$$

Using eqs. (3.5), (3.6), and (3.12) we obtain

$$\begin{aligned} F_1^{(0)}(\xi) &= \frac{1+\xi}{128\pi^3} K(\xi^2), \\ F_2^{(0)}(\xi) &= \frac{1+\xi}{128\pi^3(1-\xi)^2} [E(\xi^2) - (1-\xi)K(\xi^2)] \end{aligned} \quad (3.16)$$

The results for $F_{3,4}^{(0)}$ can be obtained by direct integration of eq. (3.14):

$$F_3^{(0)}(\xi) = \int_0^\xi d\eta \frac{F_1^{(0)}(\eta)}{1-\eta^2} = \frac{1}{128\pi^3} \int_0^\xi \frac{K(\eta^2)d\eta}{1-\eta}, \quad (3.17)$$

$$F_4^{(0)}(\xi) = \int_0^\xi d\zeta \frac{F_3^{(0)}(\zeta)}{1-\zeta^2} = \frac{1}{128\pi^3} \int_0^\xi \frac{d\zeta}{1-\zeta^2} \int_0^\zeta \frac{K(\eta^2)d\eta}{1-\eta}. \quad (3.18)$$

The integration constants are fixed by the condition of vanishing of $F_{3,4}^{(0)}$ at the threshold. In terms of the integrals F_1, F_2, F_3, F_4 the cross section has the form

$$\begin{aligned} \sigma = \frac{(16\pi)^3 \alpha(Z\alpha)^2}{m^2} & \left\{ \frac{\xi(25\xi^2 - 8\xi + 25) F_1^{(0)}}{27(1+\xi)^4} - \frac{(1-\xi)^2(173\xi^2 + 90\xi + 173) F_2^{(0)}}{216(1+\xi)^4} \right. \\ & \left. - \frac{(1-\xi)^2 F_3^{(0)}}{8(1+\xi)^2} + \frac{(1-\xi)^2 F_4^{(0)}}{2(1+\xi)^2} \right\} \end{aligned} \quad (3.19)$$

Substituting results for F_{1-4} into eq. (3.19), we obtain the Racah result (1.1).

4 Total Born cross section of $e^+e^- \rightarrow 2(Q\bar{Q})$

Let us now apply the same technique to the calculation of total Born cross section for the production of two heavy quark anti-quark pairs in e^+e^- annihilation.⁷ The latter can be conveniently expressed via $4m$ discontinuity of photon polarization operator as⁸

$$\sigma = \frac{e^2}{6s^2} \text{Disc}_{4m} \Pi_\mu^\mu \quad (4.1)$$

where e is electron charge and $\Pi^{\mu\nu}$ is the photon polarization operator. The corresponding diagrams are shown in figure 2. To calculate the $4m$ discontinuity of photon polarization operator we first use IBP relations to reduce the diagrams in figure 2 to master integrals and consider all $4m$ cuts of the latter. There are 14 distinct cut master integrals depicted in figure 3.

The master integrals in the lowest sector are related to the phase space of four massive particles. They satisfy a homogeneous differential system whose solution at $d=2$ in terms

⁷Here we restrict ourselves by c or b -quarks pairs, when it is sufficient to consider only photon exchange contribution.

⁸By $4m$ discontinuity we mean the contribution of cuts with four massive lines going on-shell.

Figure 2. Photon polarization diagrams with non-zero $4m$ cuts contributing to total cross-section of e^+e^- annihilation into two heavy quark anti-quark pairs.

Figure 3. The cut master integrals. The solid (dashed) lines denote massive (massless) propagators, thin vertical line corresponds to the cut.

of the complete elliptic integrals was found in ref. [12]. Using the threshold asymptotics to fix the boundary conditions we have

$$j_1|_{d=2} = f(s) = \frac{16\pi}{s} [\mathbf{K}(1-k_-) \mathbf{K}(k_+) - \mathbf{K}(k_-) \mathbf{K}(1-k_+)] , \quad (4.2)$$

where

$$k_{\pm} = \frac{1}{2} \left[1 \pm \left(1 - \frac{8}{s} \right) \sqrt{1 - \frac{16}{s}} + \frac{16}{s} \sqrt{1 - \frac{4}{s}} \right] . \quad (4.3)$$

At $d = 4$, the master integrals $j_1|_{d=4}$, $j_2|_{d=4}$, $j_3|_{d=4}$ are expressed as linear combinations of $f(s)$, $f'(s)$ and $f''(s)$. The explicit form of these expressions is not essential at the moment. Therefore, if we write $j_k|_{d=4-2\epsilon} = \sum_n \epsilon^n j_k^{(n)}$ ($k = 1, 2, 3$), we do know the leading coeffi-

cients $j_k^{(0)}$, but, to the best of our knowledge, not more. Meanwhile, if we express the total cross section via j_{1-14} , the coefficients in front of $j_{1,2,3,4}$ will have second order poles in ϵ :

$$\sigma = \frac{\alpha^2 \alpha_s^2 e_Q^2}{24\pi^2 s^4 (s-4)\epsilon^2} (17s^4 + 59s^3 - 146s^2 - 872s + 576) \times \left\{ \frac{1}{8}(8-s)j_1 + \frac{1}{3}(6-s)j_2 + \frac{1}{24}s(s-16)j_3 + \frac{1}{6}(4-s)j_4 \right\} + \mathcal{O}(\epsilon^{-1}). \quad (4.4)$$

Therefore, within the standard approach we would have to calculate $j_k^{(n)}$ with $n = 1, 2$. This looks like a very challenging task. However, using the methods described in section 2, we pass to ϵ -regular basis $\mathbf{F} = (F_1, \dots, F_{14})^\top$,

$$\mathbf{j} = T\mathbf{F}, \quad (4.5)$$

where the matrix T can be found in the supplementary material (file Tj2F.m). In terms of these new functions, the cross section reads

$$\sigma = \frac{\alpha^2 \alpha_s^2 e_Q^2}{27\pi^2 s^2} \left\{ \frac{3s^2 + 430s + 128}{96s} F_1^{(0)}(s) + \frac{85s^3 + 990s^2 + 16680s - 92608}{144(s-4)s} F_2^{(0)}(s) - \frac{1019s^2 - 1286s - 12120}{36s} F_3^{(0)}(s) - \frac{(s-6)(49s^2 - 336s + 464)\beta}{12(s-4)^2} F_4^{(0)}(s) - \frac{(s-1)(s+2)}{s} F_5^{(0)}(s) - \frac{2(s^2 + 2s - 18)\beta}{s-4} F_6^{(0)}(s) + \frac{5s^2 - 12s - 50}{5s} F_7^{(0)}(s) + \frac{5s^2 - 2s - 60}{s-4} \beta F_8^{(0)}(s) + \frac{s^2 + s + 4}{s} F_9^{(0)}(s) + 2 \frac{s^2 - 2}{s} F_{10}^{(0)}(s) - \frac{2(s+2)(s-2)^2 \beta}{(s-4)s} F_{12}^{(0)}(s) \right\}, \quad (4.6)$$

where e_Q is the heavy quark charge and $\beta = \sqrt{1 - 4/s}$. Note that only leading terms $F_k^{(0)}$ enter this expression, as it should be. The system of differential equations for the column $\mathbf{F}^{(0)} = (F_1^{(0)}, \dots, F_{14}^{(0)})^\top$ has the form $\partial_s \mathbf{F}^{(0)} = M_F \mathbf{F}^{(0)}$, where

$$M_F = \begin{pmatrix} \frac{3}{2(s-16)} & -\frac{1}{s} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{3}{2(s-16)} & -\frac{1}{s} & -\frac{2}{s} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{3}{16s\beta^2} & \frac{s-16}{8s^2\beta^2} & -\frac{s+8}{2s^2\beta^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{3\beta}{2(s-16)} & -\frac{1}{s\beta} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{s} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{s\beta^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{s\beta} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2s\beta} & 0 & 0 & -\frac{1}{s} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{s\beta} & 0 & -\frac{1}{s\beta} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{s} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{s^2\beta^2} & -\frac{3}{4s\beta} & -\frac{4}{s^2\beta^2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{4s\beta} & 0 & \frac{1}{s\beta} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{s\beta} & \frac{1}{s\beta} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{s} & 0 & 0 & \frac{1}{s\beta} & 0 & 0 \end{pmatrix} \quad (4.7)$$

Figure 4. The normalized cross-section $\tilde{\sigma} = \sigma / \left[\left(\frac{\alpha}{\pi} \right)^2 \left(\frac{\alpha_s}{4\pi} \right)^2 e_Q^2 \right]$ as a function of $\sqrt{s}$.

Apart from the 3×3 upper left block, the matrix M_F is strictly lower triangular. Therefore, the integrals $F_{4,\dots,14}$ can be expressed via iterated integrals of F_{1-3} . If we pass from s to β via $s = \frac{4}{1-\beta^2}$, it is easy to see that the weights in these iterated integrals are rational in β . Using the above matrix, it is trivial to write $F_4^{(0)}, \dots, F_{14}^{(0)}$ as iterated integrals of $F_1^{(0)}$ and $F_2^{(0)}$. It worth noting that all appearing iterated integrals can be transformed into one-fold integrals of MPLs and complete elliptic integrals. Explicit form of $F_1^{(0)}, \dots, F_{14}^{(0)}$ is presented in the appendix.⁹

To check the obtained result we have performed the same calculation of total cross-section by squaring tree level matrix element and integrating it numerically over the final particles phase space with the use of massive Monte Carlo algorithm **RAMBO** [23]. The comparison between analytical and numerical¹⁰ results for the normalized cross-section¹¹ $\tilde{\sigma}(s) = \sigma(s) / \left[\left(\frac{\alpha}{\pi} \right)^2 \left(\frac{\alpha_s}{4\pi} \right)^2 e_Q^2 \right]$ can be found in table 1. The plot of normalized cross-section $\tilde{\sigma}$ is shown in figure 4.

5 Conclusion

In the present paper we have elaborated a method to treat the multiloop integrals in the case when they can not be expressed via polylogarithmic functions. Our method is based on

⁹Note that the integrals $F_{11}^{(0)}$, $F_{13}^{(0)}$, and $F_{14}^{(0)}$ do not enter the cross section and presented only for completeness.

¹⁰We used non-adaptive Monte Carlo with 5 million sampling points.

¹¹The correct relation of σ and $\tilde{\sigma}$ with proper account of m_Q reads $\sigma(s) = \frac{1}{m_Q^2} \left(\frac{\alpha}{\pi} \right)^2 \left(\frac{\alpha_s}{4\pi} \right)^2 e_Q^2 \tilde{\sigma}(s/m_Q^2)$.

$\sqrt{s}$	$\tilde{\sigma}_{\text{exact}}$	$\tilde{\sigma}_{\text{MC}}$	$\sqrt{s}$	$\tilde{\sigma}_{\text{exact}}$	$\tilde{\sigma}_{\text{MC}}$
5	0.0544	0.0544(1)	60	2.4767	2.4789(70)
10	2.4303	2.4309(6)	70	2.1785	2.1739(70)
15	3.7172	3.7165(20)	80	1.9338	1.9338(70)
20	4.0277	4.0277(30)	100	1.5614	1.5642(80)
30	3.7620	3.7605(40)	200	0.7304	0.74(1)
40	3.2806	3.2810(50)	500	0.2281	0.22(1)
50	2.8415	2.8374(60)	1000	0.0870	0.09(1)

Table 1. Comparison between analytical $\tilde{\sigma}_{\text{exact}}$ and numerical $\tilde{\sigma}_{\text{MC}}$ results for the normalized cross-section $\tilde{\sigma} = \sigma / \left[\left(\frac{\alpha}{\pi} \right)^2 \left(\frac{\alpha_s}{4\pi} \right)^2 e_Q^2 \right]$ at different values of $\sqrt{s}$.

the notion of ϵ -regular basis of master integrals defined as a set of master integrals which are finite and linearly independent at $\epsilon = 0$. This basis exists in any multiloop setup. The advantage of using this basis is that it allows one to calculate any finite physical quantity circumventing the necessity to expand the master integrals in ϵ . This is especially advantageous for the non-polylogarithmic cases, when expansion in ϵ is typically quite complicated. Our approach results in the iterated integrals with almost all weights being rational functions. As an illustration of the advocated technique, we have calculated the photon contribution to the total Born cross section of the process $e^+e^- \rightarrow 2(Q\bar{Q})$. We have expressed this cross section via iterated integrals with only the right-most integration weight being transcendental function. Alternatively, our result can be presented as a one-fold integral of the expression depending on dilogarithms, complete elliptic integrals, and elementary functions. We anticipate our method to be applicable to other problems where the non-polylogarithmic integrals are involved.

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A $F_1^{(0)}, \dots, F_{14}^{(0)}$ via iterated integrals

Let us present here the explicit formulas for the leading terms $F_1^{(0)}, \dots, F_{14}^{(0)}$. The integrals $F_1^{(0)}$, $F_2^{(0)}$, and $F_3^{(0)}$ are expressed via $j_1|_{d=2}$, eq. (4.2). We have

$$F_1^{(0)}(s) = (s - 16)f(s) = \frac{16\pi(s - 16)}{s} [K(1 - k_-)K(k_+) - K(k_-)K(1 - k_+)], \quad (\text{A.1})$$

$$F_2^{(0)}(s) = \frac{3s}{2(s - 16)} F_1^{(0)}(s) - s \frac{d}{ds} F_1^{(0)}(s), \quad (\text{A.2})$$

$$F_3^{(0)}(s) = \frac{12s}{(s - 16)^2} F_1^{(0)}(s) + \frac{s(s - 64)}{4(s - 16)} \frac{d}{ds} F_1^{(0)}(s) + \frac{s^2}{2} \frac{d^2}{ds^2} F_1^{(0)}(s), \quad (\text{A.3})$$

$$F_4^{(0)}(s) = \frac{3}{2}I_1 \left[\frac{\beta}{s-16} \right] - I_2 \left[\frac{1}{s\beta} \right], \quad (\text{A.4})$$

$$F_5^{(0)}(s) = I_1 \left[\frac{1}{s} \right], \quad (\text{A.5})$$

$$F_6^{(0)}(s) = -\frac{3}{2}I_1 \left[\frac{1}{s\beta^2}, \frac{\beta}{s-16} \right] + I_2 \left[\frac{1}{s\beta^2}, \frac{1}{s\beta} \right], \quad (\text{A.6})$$

$$F_7^{(0)}(s) = \frac{3}{2}I_1 \left[\frac{1}{s\beta}, \frac{\beta}{s-16} \right] - I_2 \left[\frac{1}{s\beta}, \frac{1}{s\beta} \right], \quad (\text{A.7})$$

$$F_8^{(0)}(s) = \frac{1}{2}I_1 \left[\frac{1}{s\beta} \right] - \frac{3}{2}I_1 \left[\frac{1}{s}, \frac{\beta}{s-16} \right] + I_2 \left[\frac{1}{s}, \frac{1}{s\beta} \right], \quad (\text{A.8})$$

$$F_9^{(0)}(s) = -\frac{1}{2}I_1 \left[\frac{1}{s\beta}, \frac{1}{s\beta} \right] - \frac{3}{2}I_1 \left[\frac{1}{s\beta}, \frac{1}{s\beta^2}, \frac{\beta}{s-16} \right] + I_2 \left[\frac{1}{s\beta}, \frac{1}{s\beta^2}, \frac{1}{s\beta} \right] \\ + \frac{3}{2}I_1 \left[\frac{1}{s\beta}, \frac{1}{s}, \frac{\beta}{s-16} \right] - I_2 \left[\frac{1}{s\beta}, \frac{1}{s}, \frac{1}{s\beta} \right], \quad (\text{A.9})$$

$$F_{10}^{(0)}(s) = \frac{3}{2}I_1 \left[\frac{1}{s}, \frac{1}{s\beta}, \frac{\beta}{s-16} \right] - I_2 \left[\frac{1}{s}, \frac{1}{s\beta}, \frac{1}{s\beta} \right], \quad (\text{A.10})$$

$$F_{11}^{(0)}(s) = -\frac{1}{2}I_1 \left[\frac{1}{s\beta^2}, \frac{1}{s\beta} \right] + \frac{1}{2}I_1 \left[\frac{1}{s}, \frac{1}{s\beta} \right] - \frac{3}{4}I_1 \left[\frac{1}{s\beta^2}, \frac{1}{s\beta^2}, \frac{\beta}{s-16} \right] \\ + \frac{1}{4}I_2 \left[\frac{1}{s\beta^2}, \frac{1}{s\beta^2}, \frac{1}{s\beta} \right] + \frac{3}{2}I_1 \left[\frac{1}{s\beta^2}, \frac{1}{s}, \frac{\beta}{s-16} \right] - I_2 \left[\frac{1}{s\beta^2}, \frac{1}{s}, \frac{1}{s\beta} \right] \\ + \frac{3}{8}I_1 \left[\frac{1}{s}, \frac{1}{s\beta^2}, \frac{\beta}{s-16} \right] - \frac{1}{4}I_2 \left[\frac{1}{s}, \frac{1}{s\beta^2}, \frac{1}{s\beta} \right] - \frac{3}{2}I_1 \left[\frac{1}{s}, \frac{1}{s}, \frac{\beta}{s-16} \right] \\ + I_2 \left[\frac{1}{s}, \frac{1}{s}, \frac{1}{s\beta} \right] - \frac{9}{8}I_1 \left[\frac{1}{s\beta}, \frac{1}{s\beta}, \frac{\beta}{s-16} \right] + \frac{3}{4}I_2 \left[\frac{1}{s\beta}, \frac{1}{s\beta}, \frac{1}{s\beta} \right], \quad (\text{A.11})$$

$$F_{12}^{(0)}(s) = -\frac{1}{4}I_1 \left[\frac{1}{s\beta}, \frac{1}{s} \right] + \frac{3}{2}I_1 \left[\frac{1}{s\beta}, \frac{1}{s\beta}, \frac{\beta}{s-16} \right] - I_2 \left[\frac{1}{s\beta}, \frac{1}{s\beta}, \frac{1}{s\beta} \right], \quad (\text{A.12})$$

$$F_{13}^{(0)}(s) = -\frac{1}{2}I_1 \left[\frac{1}{s\beta}, \frac{1}{s\beta^2}, \frac{1}{s\beta} \right] + \frac{1}{2}I_1 \left[\frac{1}{s\beta}, \frac{1}{s}, \frac{1}{s\beta} \right] - \frac{1}{4}I_1 \left[\frac{1}{s\beta}, \frac{1}{s\beta}, \frac{1}{s} \right] \\ - \frac{3}{8}I_1 \left[\frac{1}{s\beta}, \frac{1}{s\beta^2}, \frac{1}{s\beta^2}, \frac{\beta}{s-16} \right] + \frac{1}{4}I_2 \left[\frac{1}{s\beta}, \frac{1}{s\beta^2}, \frac{1}{s\beta^2}, \frac{1}{s\beta} \right] \\ + \frac{3}{2}I_1 \left[\frac{1}{s\beta}, \frac{1}{s\beta^2}, \frac{1}{s}, \frac{\beta}{s-16} \right] - I_2 \left[\frac{1}{s\beta}, \frac{1}{s\beta^2}, \frac{1}{s}, \frac{1}{s\beta} \right] \\ + \frac{3}{8}I_1 \left[\frac{1}{s\beta}, \frac{1}{s}, \frac{1}{s\beta^2}, \frac{\beta}{s-16} \right] - \frac{1}{4}I_2 \left[\frac{1}{s\beta}, \frac{1}{s}, \frac{1}{s\beta^2}, \frac{1}{s\beta} \right] \\ - \frac{3}{2}I_1 \left[\frac{1}{s\beta}, \frac{1}{s}, \frac{1}{s}, \frac{\beta}{s-16} \right] + I_2 \left[\frac{1}{s\beta}, \frac{1}{s}, \frac{1}{s}, \frac{1}{s\beta} \right] \\ + \frac{3}{8}I_1 \left[\frac{1}{s\beta}, \frac{1}{s\beta}, \frac{1}{s\beta}, \frac{\beta}{s-16} \right] - \frac{1}{4}I_2 \left[\frac{1}{s\beta}, \frac{1}{s\beta}, \frac{1}{s\beta}, \frac{1}{s\beta} \right], \quad (\text{A.13})$$

$$\begin{aligned}
 F_{14}^{(0)}(s) = & -\frac{1}{2}I_1 \left[\frac{1}{s}, \frac{1}{s\beta}, \frac{1}{s\beta} \right] - \frac{1}{4}I_1 \left[\frac{1}{s\beta}, \frac{1}{s\beta}, \frac{1}{s} \right] - \frac{3}{2}I_1 \left[\frac{1}{s}, \frac{1}{s\beta}, \frac{1}{s\beta^2}, \frac{\beta}{s-16} \right] \\
 & + I_2 \left[\frac{1}{s}, \frac{1}{s\beta}, \frac{1}{s\beta^2}, \frac{1}{s\beta} \right] + \frac{3}{2}I_1 \left[\frac{1}{s}, \frac{1}{s\beta}, \frac{1}{s}, \frac{\beta}{s-16} \right] - I_2 \left[\frac{1}{s}, \frac{1}{s\beta}, \frac{1}{s}, \frac{1}{s\beta} \right] \\
 & + \frac{3}{2}I_1 \left[\frac{1}{s\beta}, \frac{1}{s\beta}, \frac{1}{s\beta}, \frac{\beta}{s-16} \right] - I_2 \left[\frac{1}{s\beta}, \frac{1}{s\beta}, \frac{1}{s\beta}, \frac{1}{s\beta} \right].
 \end{aligned} \tag{A.14}$$

where we introduced the following notation for iterated integrals:

$$I[w_1(s), \dots, w_n(s)] = \int_{16}^s ds_1 w_1(s_1) \int_{16}^{s_1} ds_2 w_2(s_2) \dots \int_{16}^{s_{n-1}} ds_n w_n(s_n), \tag{A.15}$$

$$I_{1,2}[w_1(s), \dots, w_n(s)] = I[w_1(s), \dots, w_n(s) F_{1,2}^{(0)}(s)]. \tag{A.16}$$

It is remarkable, that all weights, apart from the right-most ones, are restricted to the three-letter alphabet $\left\{ \frac{1}{s}, \frac{1}{s\beta} = \frac{1}{\sqrt{s(s-4)}}, \frac{1}{s\beta^2} = \frac{1}{s-4} \right\}$.

Note that the iterated integrals above can be easily turned into one-fold integrals of multiple polylogarithms and complete elliptic integrals. Indeed,

$$I[w_1(s), \dots, w_n(s)] = \int_{16}^s ds_n w_n(s_n) \left[\int_{s_n}^s ds_{n-1} w_{n-1}(s_{n-1}) \dots \int_{s_2}^s ds_1 w_1(s_1) \right], \tag{A.17}$$

and, since the weights $w_1, \dots, w_{n-1}$ are rational, the bracketed quantity is expressed via polylogarithms. This may be convenient for numerical purposes. We present the corresponding expressions for the integrals $F_{4-10}^{(0)}$ and $F_{12}^{(0)}$ entering the cross section:

$$F_4^{(0)}(s) = \frac{\beta s F_1^{(0)}(s)}{s-4} - \int_{16}^s ds_1 \frac{4\beta_1(s_1+2)F_1^{(0)}(s_1)}{(s_1-16)(s_1-4)^2}, \tag{A.18}$$

$$F_5^{(0)}(s) = \int_{16}^s ds_1 \frac{F_1^{(0)}(s_1)}{s_1}, \tag{A.19}$$

$$F_6^{(0)}(s) = \int_{16}^s ds_1 \frac{\beta_1 F_1^{(0)}(s_1)}{(s_1-4)^2} \left\{ -s_1 - \frac{4(s_1+2)}{s_1-16} \left[2\ln \frac{\beta_1}{\beta} + \ln \frac{s_1}{s} \right] \right\}, \tag{A.20}$$

$$F_7^{(0)}(s) = \int_{16}^s ds_1 \frac{F_1^{(0)}(s_1)}{(s_1-4)^2} \left\{ s_1 - 4 + \frac{4\beta_1(s_1+2)}{s_1-16} \left[\ln \frac{1+\beta_1}{1-\beta_1} - \ln \frac{1+\beta}{1-\beta} \right] \right\}, \tag{A.21}$$

$$F_8^{(0)}(s) = \int_{16}^s ds_1 \frac{\beta_1 F_1^{(0)}(s_1)}{2(s_1-4)^2} \left\{ -s_1 + 4 - \frac{8\beta_1(s_1+2)\ln \frac{s_1}{s}}{s_1-16} \right\}, \tag{A.22}$$

$$\begin{aligned}
 F_9^{(0)}(s) = & \int_{16}^s ds_1 \frac{\beta_1 F_1^{(0)}(s_1)}{2(s_1-4)^2} \left\{ (s_1+4) \left[\ln \frac{1+\beta_1}{1-\beta_1} - \ln \frac{1+\beta}{1-\beta} \right] \right. \\
 & \left. - \frac{16(s_1+2)}{s_1-16} \left[\ln \frac{1+\beta}{1-\beta} \ln \frac{\beta_1}{\beta} + \text{Li}_2(-\beta_1) - \text{Li}_2(\beta_1) - \text{Li}_2(-\beta) + \text{Li}_2(\beta) \right] \right\},
 \end{aligned} \tag{A.23}$$

$$F_{10}^{(0)}(s) = \int_{16}^s ds_1 \frac{F_1^{(0)}(s_1)}{(s_1-4)^2} \left\{ -(s_1-4) \ln \frac{s_1}{s} - \frac{\beta_1(s_1+2)}{s_1-16} \left[\frac{1}{2} \ln \frac{1+\beta_1}{1-\beta_1} \ln s_1 + \frac{1}{2} \ln \frac{1+\beta}{1-\beta} \ln s \right. \right. \\ \left. \left. - \ln \frac{1+\beta_1}{1-\beta_1} \ln s - \text{Li}_2\left(\frac{1-\beta_1}{2}\right) + \text{Li}_2\left(\frac{1+\beta_1}{2}\right) + \text{Li}_2\left(\frac{1-\beta}{2}\right) - \text{Li}_2\left(\frac{1+\beta}{2}\right) \right] \right\}, \quad (\text{A.24})$$

$$F_{12}^{(0)}(s) = \int_{16}^s ds_1 \frac{F_1^{(0)}(s_1)}{(s_1-4)^2} \left\{ \frac{(3s_1+4)(s_1-4)}{4s_1} \left[\ln \frac{1+\beta}{1-\beta} - \ln \frac{1+\beta_1}{1-\beta_1} \right] \right. \\ \left. - \frac{2\beta_1(s_1+2)}{s_1-16} \left[\ln \frac{1+\beta}{1-\beta} - \ln \frac{1+\beta_1}{1-\beta_1} \right]^2 \right\}. \quad (\text{A.25})$$

Here $\beta_1 = \sqrt{1-4/s_1}$ and $F_1^{(0)}(s)$ is defined in eq. (A.1) via complete elliptic integral K.

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