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Grassmannian and flag sigma models on interval: phase structure and L-dependence

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ABSTRACT: We discuss the two-dimensional Grassmannian $SU(N)/S(U(N-2) \times U(2))$ and the flag $SU(N)/S(U(N-2) \times U(1) \times U(1))$ sigma models on a finite interval and construct analytical solutions of gap equations in the large- N limit. We show that the flag model admits a homogeneous solution for “mixed” Dirichlet-Neumann (DN) boundary conditions only for sufficiently large length L and undergoes a phase transition from the phase of partly broken gauge symmetry $U(1)$ to the symmetric phase $U(1) \times U(1)$ for large L . On the other hand, the Grassmannian model has a detached phase with one massive and one massless non-zero condensates that completely break $U(2)$ gauge symmetry. This phase lives on a region of L bounded from above and has to use the Robin boundary conditions. We also examine the L -dependence of the total energy and detect the linear growth inherent to confining string in all phases.

KEYWORDS: Sigma Models, Effective Field Theories, Spontaneous Symmetry Breaking, $1/N$ Expansion

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1 Introduction

As it can be seen from a lot of works on QCD-like theories a phase structure, geometry and vacuum have nontrivial influence on each other. A review of non-perturbative Casimir effects in theories possessing mass gap, confinement and chiral symmetry breaking phenomena was recently done in [1]. It was natural to probe these problems for the well-studied $\mathbb{CP}^{N-1}$ non-linear sigma model (NLSM). This toy model has asymptotic freedom, dynamical mass generation [2–4] and may undergo the Higgs-Coulomb phase transition on the interval in the large- N limit for special boundary conditions [5–8].

In this paper, we continue the study of homogeneous solutions for non-supersymmetric 2d NLSMs on the Grassmannian $\text{Gr}(N, 2) = \text{SU}(N)/S(\text{U}(N-2) \times \text{U}(2))$ ([9]) and flag $F(N, 2, 1) = \text{SU}(N)/S(\text{U}(N-2) \times \text{U}(1) \times \text{U}(1))$ ([10]), that was started in [11]. Since these manifolds have positive Ricci curvature, the corresponding field theories are asymptotically free. Also there is dynamical mass generation [12, 13]. The phase structure of the compactified 2d $\text{SU}(3)/S(\text{U}(1) \times \text{U}(1))$ flag sigma model with respect to θ -terms was considered in [14]. Study of instanton-like solutions in the Grassmannian sigma model on $\mathbb{R} \times S^1$ was carried out in [15, 16]. Being a direct generalization of the $\mathbb{CP}^{N-1}$ NLSM, $F(N, 2, 1)$ and $\text{Gr}(N, 2)$ models are effective theories describing orientational moduli on the worldsheet of the non-Abelian strings [17, 18]. Therefore the corresponding theories on a finite interval reveal composite strings stretched between two domain walls. The flag sigma models also appear in the low-energy description of anti-ferromagnetic spin chains [19–21]. The $1/N$ -expansion for the model with $\text{SU}(N)/S(\text{U}(N-2) \times \text{U}(1) \times \text{U}(1))$ target space was constructed in [22].

The scope of the paper includes the study of the phase structure of $F(N, 2, 1)$ and $\text{Gr}(N, 2)$ models depending on the length of the interval under various boundary conditions

in the large- N limit. In section 2 we construct the Lagrangian for the simplest flag sigma model $F(N, 2, 1)$. In general, there are two different coupling constants. Therefore two different dimensional parameters, Λ_1 and Λ_2 (e.g. $\Lambda_1 \geq \Lambda_2$), are generated. The theory passes into $\text{Gr}(N, 2)$ if the coupling constants coincide. Then we get the effective action by integrating out all but two fields. The gap equations for the theory on the interval are constructed in section 3. Section 4 is the main part of this paper. Here we investigate the phase structure of the models. For $F(N, 2, 1)$ model the region of $L > \Lambda_2^{-1}$ corresponds to unbroken $U(1)_A \times U(1)_B$ gauge symmetry. For $\Lambda_2^{-1} > L > \Lambda_1^{-1}$ we are in phase with $U(1)_A$ symmetry. The quantity Λ_1^{-1} plays the role of ultraviolet cutoff (or string thickness) since for $L < \Lambda_1^{-1}$ there is no solution of the gap equations, and the gauge symmetry can not be broken completely. On the other hand, when the coupling constants coincide and $F(N, 2, 1)$ turns to $\text{Gr}(N, 2)$, for special choice of boundaries we have a solution only for $\Lambda_1^{-1} > L > 0$. This region corresponds to the phase with completely broken $U(2)$ gauge symmetry. In section 5, we examine the L -dependence of the total energy and find out that the linear term corresponding to the string tension appears in all phases. Also we analyze the Lüscher term that appears only in phases with broken gauge symmetry and signals about massless degrees of freedom on the string worldsheet [2, 23]. The calculations of the L -dependence in some phases are given in the appendix A.

2 Effective action

Following [12] we start from the Lagrangian for two $\mathbb{CP}^{N-1}$ models with some coupling constants and interaction terms

$$L = \frac{1}{g_1^2} |D_{1\mu} \psi_1|^2 + \frac{1}{g_2^2} |D_{2\mu} \psi_2|^2 - i \frac{1}{\sqrt{g_1 g_2}} C_\mu \psi_2^\dagger \partial^\mu \psi_1 - i \frac{1}{\sqrt{g_1 g_2}} C_\mu^* \psi_1^\dagger \partial^\mu \psi_2 + \frac{1}{4} \left(\frac{g_1}{g_2} + \frac{g_2}{g_1} \right) C_\mu^* C^\mu \quad (2.1)$$

where ψ_1 and ψ_2 are two complex N -vectors, such that $\psi_i^\dagger \psi_j = \delta_{ij}$ ($i = 1, 2$), and coupled to the Abelian gauge fields A_μ and B_μ via covariant derivatives $D_{1\mu} = \partial_\mu - iA_\mu$ and $D_{2\mu} = \partial_\mu - iB_\mu$. If one starts from different coupling constants for each complex space this leads to the flag manifold. The Grassmannian manifold corresponds to the case with equal coupling constants. Note that the presence of two different coupling constants leads to the generation of two different dimensional parameters.

It is convenient to rewrite the Lagrangian in terms of the $N \times 2$ matrix $Z_{i\alpha} = (\psi_{1\alpha}, \psi_{2\alpha})$ ($\alpha = 1, \dots, N$)

$$L = \text{Tr}((D_\mu Z)^\dagger (D^\mu Z) - \lambda(Z^\dagger Z - R)) \quad (2.2)$$

where we have introduced matrix $R = \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix}$ with $r_1 = N/g_1^2$ and $r_2 = N/g_2^2$ and matrix of Lagrange multipliers $\lambda = \begin{pmatrix} \lambda_1 & \lambda_3 \\ \lambda_3^* & \lambda_2 \end{pmatrix}$ imposing the constraints $(\psi_{i\alpha})^\dagger \psi_{j\alpha} = \delta_{ij} r_j$. Covariant derivative is $D_\mu Z = \partial_\mu Z - Z \tilde{A}_\mu$ with $\tilde{A}_\mu = -i \begin{pmatrix} A_\mu & \frac{1}{2} C_\mu \\ \frac{1}{2} C_\mu^* & B_\mu \end{pmatrix}$. At classical level the gauge

Figure 1. Quiver diagram for $F(N, 2, 1)$.

fields can be eliminated by their equations of motions. Thus, for convenience they are equaled to zero in the effective action below.

Taking into account that λ transforms in the adjoint representation of the $U(2)$ gauge group one can see that in the case of $r_1 = r_2$ the Lagrangian (2.2) is invariant under the local $U(2)$ transformation. Vice versa, when $r_1 \neq r_2$ the local symmetry is explicitly broken down to $U(1)_A \times U(1)_B$. As well known, a linear model with $G_{\text{global}} \times H_{\text{local}}$ is gauge equivalent to the non-linear sigma model corresponding to coset space G/H (see [24]). So it gives us $\text{Gr}(N, 2)$ and $F(N, 2, 1)$ manifolds.

Let us turn for a moment to the standard quiver GLSM formulation (figure 1). There are two dynamical bifundamental fields u and Z . However, using the action of $U(1) \times U(2)$, we can bring the vector u to the canonical form $(1, 0)^T$ leaving only one dynamical field Z as in the construction above. Nonzero vev of u makes the matrix R to have different eigenvalues breaking $U(2)$ symmetry. Note that it could happen only for the finite L due to the Coleman theorem.

Let us get back to the eq. (2.2). In order to perform Gaussian integration in the partition function, it is convenient to represent the Lagrangian in terms of the $2N \times 2N$ matrix

$$L = (\psi_1^\dagger, \psi_2^\dagger)(M^T \otimes I) \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} + r_1 \lambda_1 + r_2 \lambda_2 \quad (2.3)$$

where $M = \begin{pmatrix} -\partial^2 + \lambda_1 & \lambda_3 \\ \lambda_3^* & -\partial^2 + \lambda_2 \end{pmatrix}$ and $I_{N \times N}$ is unit matrix. Integrating out all but two fields σ_1 and σ_2 (the first components of vectors ψ_1 and ψ_2 respectively), that will get non-zero vacuum expectation values (VEVs), we come to the effective action

$$S_{\text{eff}} = 2N \text{Tr} \log \begin{pmatrix} -\partial^2 + \lambda_1 & \lambda_3 \\ \lambda_3^* & -\partial^2 + \lambda_2 \end{pmatrix} + \int d^2x \left[(\partial_\mu \sigma_1)^2 + \lambda_1 (|\sigma_1|^2 - r_1) + (\partial_\mu \sigma_2)^2 + \lambda_2 (|\sigma_2|^2 - r_2) + \lambda_3 \sigma_1^* \sigma_2 + \lambda_3^* \sigma_1 \sigma_2^* \right]. \quad (2.4)$$

Here we have changed $N \rightarrow 2N + 1$ for convenience. Note that the main difference between this action and the one in $\mathbb{CP}^{N-1}$ model with two condensates (see [25]) is the presence of the off-diagonal elements λ_3, λ_3^* that mix two sectors ψ_1 and ψ_2 . As it will be seen below, this fact leads to crucial consequences for the phase structure.

In order to calculate partition function in large- N limit we should find saddle-points of the action. We will consider only time independent solutions since time-translation symmetry is unbroken.

3 Gap equations

From eq. (2.4) it follows that the total energy of the system is the sum of the energy of the quantum fluctuations and the energy of the condensates σ_1 and σ_2 ([26])

$$E = 2N \sum_n \omega_n + \int_0^L dx \left[(\partial_\mu \sigma_1)^2 + \lambda_1 (|\sigma_1|^2 - r_1) + (\partial_\mu \sigma_2)^2 + \lambda_2 (|\sigma_2|^2 - r_2) + \lambda_3 \sigma_1^* \sigma_2 + \lambda_3^* \sigma_1 \sigma_2^* \right] \quad (3.1)$$

where ω_n^2 are the eigenvalues of the problem

$$\begin{pmatrix} -\partial_x^2 + \lambda_1 & \lambda_3 \\ \lambda_3^* & -\partial_x^2 + \lambda_2 \end{pmatrix} \begin{pmatrix} f_{1,n} \\ f_{2,n} \end{pmatrix} = \omega_n^2 \begin{pmatrix} f_{1,n} \\ f_{2,n} \end{pmatrix}. \quad (3.2)$$

Varying the total energy with respect to the Lagrange multipliers we get the following saddle-point equations

$$N \sum_n \frac{1}{\omega_n} \begin{pmatrix} \frac{|f_{1,n}|^2}{\theta_1} & \frac{f_{1,n} f_{2,n}^*}{\theta_2} \\ \frac{f_{1,n}^* f_{2,n}}{\theta_1} & \frac{|f_{2,n}|^2}{\theta_2} \end{pmatrix} + \begin{pmatrix} |\sigma_1|^2 & \sigma_1^* \sigma_2 \\ \sigma_1 \sigma_2^* & |\sigma_2|^2 \end{pmatrix} - \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix} = 0 \quad (3.3)$$

where we have used the normalization

$$\theta_1 = \int_0^L dx |f_{1,n}(x)|^2, \quad \theta_2 = \int_0^L dx |f_{2,n}(x)|^2, \quad \theta_1 + \theta_2 = 2. \quad (3.4)$$

The off-diagonal elements are complex conjugate to each other, thus $\theta_1 = \theta_2 = 1$. Variation with respect to σ_i gives us the equations of motion

$$\begin{pmatrix} -\partial_x^2 + \lambda_1 & \lambda_3 \\ \lambda_3^* & -\partial_x^2 + \lambda_2 \end{pmatrix} \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \quad (3.5)$$

We limit ourselves to considering only real homogeneous condensates, so the saddle-point equations turn out to be as follow

$$N \sum_n \frac{1}{\omega_n} \begin{pmatrix} f_{1,n}^2 & f_{1,n} f_{2,n} \\ f_{1,n} f_{2,n} & f_{2,n}^2 \end{pmatrix} + \begin{pmatrix} \sigma_1^2 & \sigma_1 \sigma_2 \\ \sigma_1 \sigma_2 & \sigma_2^2 \end{pmatrix} - \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix} = 0 \quad (3.6)$$

and

$$\begin{pmatrix} \lambda_1 & \lambda_3 \\ \lambda_3 & \lambda_2 \end{pmatrix} \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \quad (3.7)$$

As will be shown in the next section the qualitative behavior of the solutions of these equations strongly depends on the parameters r_1/r_2 , λ_3 and boundary conditions.

Figure 2. Schematic depiction of the phase structure. The system (4.1)–(4.3) has the solution only for $L > L_1$.

4 Phase structure

First of all, note that for the models with $L \rightarrow \infty$ there is a unique confinement phase with the dimensional parameters $m_i \equiv \Lambda_i = \Lambda_{uv} \exp(-2\pi/N g_i^2)$ ($i = 1, 2$) and zero VEVs of ψ_i . As was discussed in [13] for unbounded theory the solution exists only for $\lambda_3 = 0$. For the models on the finite interval with $\lambda_3 = 0$ from (3.6) and (3.7) it follows

$$N \sum_n \frac{f_{1,n}^2}{\omega_n} + \sigma_1^2 - r_1 = 0, \quad \lambda_1 \sigma_1 = 0 \quad (4.1)$$

$$N \sum_n \frac{f_{2,n}^2}{\omega_n} + \sigma_2^2 - r_2 = 0, \quad \lambda_2 \sigma_2 = 0 \quad (4.2)$$

$$\sigma_1 \sigma_2 = 0 \quad (4.3)$$

where $r_i = \frac{2N}{\pi} \log(\Lambda_{uv}/\Lambda_i)$. For the last equation we used the condition $\sum_n \frac{f_{1,n} f_{2,n}}{\omega_n} = 0$ that is satisfied for the diagonal operator in (3.2). Let us impose “mixed” DN-ND boundary conditions:

$$\psi_{i\alpha}(0) = 0, \quad \partial_x \psi_{i\alpha}(L) = 0, \quad \text{if } \alpha = 2, \dots, N+1; \quad (4.4)$$

$$\partial_x \psi_{i\alpha}(0) = 0, \quad \psi_{i\alpha}(L) = 0, \quad \text{if } \alpha = N+2, \dots, 2N+1; \quad (4.5)$$

$$\partial_x \sigma_i(0) = \partial_x \sigma_i(L). \quad (4.6)$$

In this way the first terms in (4.1) and (4.2) have the form

$$\left(\frac{N}{2} \sum_n \frac{f_{i,n}^2}{\omega_n} \right)_{DN} + \left(\frac{N}{2} \sum_n \frac{f_{i,n}^2}{\omega_n} \right)_{ND}. \quad (4.7)$$

It was shown by Milekhin in [8] that the equations (4.1) and (4.2) with boundaries (4.4)–(4.6) have the solutions for all L with Higgs-Coulomb phase transitions in $L_1 \approx 1/\Lambda_1$ and $L_2 \approx 1/\Lambda_2$ respectively that is depicted schematically in figure 2. However, now, due to the additional condition $\sigma_1 \sigma_2 = 0$, we have the solution only on the region $(L_1, +\infty)$. For $L \in (L_1, L_2)$ there is the massless non-zero field σ_2 that breaks $U(1)_A \times U(1)_B$ gauge symmetry to $U(1)_A$. For $L > L_2$ we are in the symmetric (Coulomb) phase. Note that for $r_1 = r_2$ (i.e. Grassmannian manifold) $L_1 = L_2$ and there is only the Coulomb phase.

Now let us examine the case of $\lambda_3 = \text{const} \neq 0$. The first thing we can say about this phase is that it does not exist for $L \rightarrow \infty$, as was mentioned at the beginning of

this section. To study the models on the finite interval, it is convenient to diagonalize the operator in (3.2)

$$\begin{pmatrix} -\partial_x^2 + \frac{\lambda_1 + \lambda_2 + \sqrt{(\lambda_1 - \lambda_2)^2 + 4\lambda_3^2}}{2} & 0 \\ 0 & -\partial_x^2 + \frac{\lambda_1 + \lambda_2 - \sqrt{(\lambda_1 - \lambda_2)^2 + 4\lambda_3^2}}{2} \end{pmatrix} \begin{pmatrix} \tilde{f}_{1,n} \\ \tilde{f}_{2,n} \end{pmatrix} = \omega_n^2 \begin{pmatrix} \tilde{f}_{1,n} \\ \tilde{f}_{2,n} \end{pmatrix}. \quad (4.8)$$

In the case at hand, the “quantum” part of the total energy has the form

$$2N \left(\sum_n \omega_{1,n} + \sum_k \omega_{2,k} \right) \quad (4.9)$$

where $\omega_{1,n}^2$ and $\omega_{2,k}^2$ are eigenvalues of the problems

$$\left(-\partial_x^2 + \frac{\lambda_1 + \lambda_2 + \sqrt{(\lambda_1 - \lambda_2)^2 + 4\lambda_3^2}}{2} \right) \tilde{f}_{1,n} = \omega_{1,n}^2 \tilde{f}_{1,n} \quad (4.10)$$

$$\left(-\partial_x^2 + \frac{\lambda_1 + \lambda_2 - \sqrt{(\lambda_1 - \lambda_2)^2 + 4\lambda_3^2}}{2} \right) \tilde{f}_{2,k} = \omega_{2,k}^2 \tilde{f}_{2,k}. \quad (4.11)$$

After variation with respect to $\lambda_{1,2,3}, \sigma_{1,2}$ and some algebra we have the following saddle-point equations

$$N \sum_n \frac{\tilde{f}_{1,n}^2}{\omega_{1,n}} + N \sum_k \frac{\tilde{f}_{2,k}^2}{\omega_{2,k}} + \sigma_1^2 + \sigma_2^2 - r_1 - r_2 = 0 \quad (4.12)$$

$$\frac{(\lambda_1 - \lambda_2)}{\sqrt{(\lambda_1 - \lambda_2)^2 + 4\lambda_3^2}} \left(N \sum_n \frac{\tilde{f}_{1,n}^2}{2\omega_{1,n}} - N \sum_k \frac{\tilde{f}_{2,k}^2}{2\omega_{2,k}} \right) + \sigma_1^2 - \sigma_2^2 + r_1 - r_2 = 0 \quad (4.13)$$

$$\frac{2\lambda_3}{\sqrt{(\lambda_1 - \lambda_2)^2 + 4\lambda_3^2}} \left(N \sum_n \frac{\tilde{f}_{1,n}^2}{\omega_{1,n}} - N \sum_k \frac{\tilde{f}_{2,k}^2}{\omega_{2,k}} \right) + 2\sigma_1\sigma_2 = 0 \quad (4.14)$$

$$\left(\lambda_1 - \frac{\lambda_3^2}{\lambda_2} \right) \sigma_1 = 0 \quad (4.15)$$

$$\sigma_2 = -\frac{\lambda_3}{\lambda_2} \sigma_1. \quad (4.16)$$

The last two equations illustrate two possibilities: σ_i are zero or not. Let us first consider non-zero $\sigma_i \neq 0$ that completely break initial gauge symmetry. In this case from eq. (4.15) we get $\lambda_1\lambda_2 = \lambda_3^2$. Then, using (4.13) and (4.14) we get $r_1 = r_2$. This means that the non-zero λ_3 is compatible only with the Grassmannian model. Finally, using the notations $\sigma \equiv \sigma_1 = -\frac{\lambda_3}{\lambda_2} \sigma_2$ and $r \equiv r_1$ we come to the system of equations

$$N \sum_n \frac{\tilde{f}_{1,n}^2}{\omega_{1,n}} - r = 0; \quad (-\partial_x^2 + \lambda_1 + \lambda_2) \tilde{f}_{1,n} = \omega_{1,n}^2 \tilde{f}_{1,n} \quad (4.17)$$

$$N \sum_k \frac{\tilde{f}_{2,k}^2}{\omega_{2,k}} + \frac{\lambda_1 + \lambda_2}{\lambda_2} \sigma^2 - r = 0; \quad -\partial_x^2 \tilde{f}_{2,k} = \omega_{2,k}^2 \tilde{f}_{2,k} \quad (4.18)$$

$$\lambda_1\lambda_2 = \lambda_3^2. \quad (4.19)$$

We now find out what kind of boundaries we could impose on the fields. According to [8], the eq. (4.17) has the solution on the region $L \in (\Lambda_1^{-1}, +\infty)$ for DN-ND boundaries and on $(0, +\infty)$ for DD-NN boundaries on the $\tilde{f}_{1,n}$ modes. On the other hand, eq. (4.18) with non-zero σ has the solution on $(0, \Lambda_1^{-1})$ for DN-ND and has no solution for DD-NN boundaries on the $\tilde{f}_{2,n}$ modes ($\Lambda_1 = \Lambda_2$ and is taken from the unbounded theory). In this way we are forced to choose the DD-NN boundaries on $\tilde{f}_{1,n}$ and DN-ND on $\tilde{f}_{2,n}$. It corresponds to the solution on the region $L \in (0, \Lambda_1^{-1})$. If $\lambda_1 = \lambda_2 = \lambda_3$ the boundary conditions have the following form in terms of the fields $\psi_{i\alpha}$

$$\psi_{i\alpha}(0) = 0, \quad \partial_x(\psi_{1\alpha} - \psi_{2\alpha})(L) = (\psi_{1\alpha} + \psi_{2\alpha})(L) = 0, \quad \text{if } \alpha = 2, \dots, N+1; \quad (4.20)$$

$$\partial_x \psi_{i\alpha}(0) = 0, \quad \partial_x(\psi_{1\alpha} + \psi_{2\alpha})(L) = (\psi_{1\alpha} - \psi_{2\alpha})(L) = 0, \quad \text{if } \alpha = N+2, \dots, 2N+1; \quad (4.21)$$

$$\partial_x \sigma_i(0) = \partial_x \sigma_i(L). \quad (4.22)$$

Let us consider the “classical” part of the energy density

$$\lambda_1 \sigma_1^2 + \lambda_2 \sigma_2^2 + 2\lambda_2 \sigma_1 \sigma_2 - (\lambda_1 + \lambda_2)r. \quad (4.23)$$

The quadratic form in this expression can be diagonalized, for example, as follows

$$\lambda_1 \sigma_1^2 + \lambda_2 \sigma_2^2 + 2\lambda_2 \sigma_1 \sigma_2 = \lambda_2 \left(\sigma_2 + \frac{\lambda_3}{\lambda_2} \sigma_1 \right)^2 + \left(\lambda_1 - \frac{\lambda_3^2}{\lambda_2} \right) \sigma_1^2. \quad (4.24)$$

Using the eqs. (4.16) and (4.19) and also Sylvester’s law of inertia we can conclude that there is one massive and one massless field. It can be note that the detection of the massless degree of freedom allows us to expect the appearance of corresponding power corrections in the total energy (see the next section 5).

Let us now investigate the case $\sigma_1 = \sigma_2 = 0$. As before, from eqs. (4.13) and (4.14) we get the condition $r_1 = r_2 = r$, and the system of equations is

$$N \sum_n \frac{\tilde{f}_{1,n}^2}{\omega_{1,n}} - r = 0, \quad \left(-\partial_x^2 + \frac{\lambda_1 + \lambda_2 + \sqrt{(\lambda_1 - \lambda_2)^2 + 4\lambda_3^2}}{2} \right) \tilde{f}_{1,n} = \omega_{1,n}^2 \tilde{f}_{1,n} \quad (4.25)$$

$$N \sum_k \frac{\tilde{f}_{2,k}^2}{\omega_{2,k}} - r = 0, \quad \left(-\partial_x^2 + \frac{\lambda_1 + \lambda_2 - \sqrt{(\lambda_1 - \lambda_2)^2 + 4\lambda_3^2}}{2} \right) \tilde{f}_{2,k} = \omega_{2,k}^2 \tilde{f}_{2,k}. \quad (4.26)$$

This implies an equality

$$N \sum_n \frac{\tilde{f}_{1,n}^2}{\omega_{1,n}} = N \sum_k \frac{\tilde{f}_{2,k}^2}{\omega_{2,k}} \quad (4.27)$$

that is not valid for the translation-invariant λ_1 and λ_2 . Thus, the phase with zero σ_i and non-zero λ_3 is absent.

We have studied $\text{Gr}(N, 2)$ and $F(N, 2, 1)$ NLSM on the interval and find that for $\lambda_3 = 0$ the length of the interval L is bounded from below, vice versa, for $\lambda_3 \neq 0$ L is bounded from above. It is now very simple to build a model that lives on a region of L that bounded from

below and above. Indeed, let us consider NLSM on $F(N, 3, 1)$ manifold with the following effective action

$$\begin{aligned}
 S_{\text{eff}} = & (N-1) \text{Tr} \ln \begin{pmatrix} -\partial^2 + \lambda_1 & \lambda_{12} & \lambda_{13} \\ \lambda_{12} & -\partial^2 + \lambda_2 & \lambda_{23} \\ \lambda_{13} & \lambda_{23} & -\partial^2 + \lambda_3 \end{pmatrix} \\
 & + \int d^2x [(\partial_\mu \sigma_1)^2 + \lambda_1(|\sigma_1|^2 - r_1) + (\partial_\mu \sigma_2)^2 + \lambda_2(|\sigma_2|^2 - r_2) \\
 & + (\partial_\mu \sigma_3)^2 + \lambda_3(|\sigma_3|^2 - r_3) + 2\lambda_{12}\sigma_1\sigma_2 + 2\lambda_{13}\sigma_1\sigma_3 + 2\lambda_{23}\sigma_2\sigma_3]. \quad (4.28)
 \end{aligned}$$

and suppose that $\lambda_{12} = \lambda_{13} = 0$, $\lambda_{23} \neq 0$ and $r_3 = r_2 > r_1$ in gap equations. One can easily make sure that it corresponds to a phase that lives on a region of $L \in (\Lambda_1^{-1}, \Lambda_2^{-1})$, where $\Lambda_i = \Lambda_{uv} \exp(-2\pi/Ng_i^2)$ ($i = 1, 2, 3$) and $g_2 = g_3$.

5 L-dependence

We now investigate L-dependence of the total energy for the different phases. Firstly consider the flag model (see figure 2). Performing calculations parallel to the work [5] (see appendix A) we get the following expressions

$$E(L) = \frac{NLm_1^2}{\pi} \sum_{n=1}^{+\infty} (-1)^{n+1} \frac{K_1(2Lm_1n)}{Lm_1n} + \frac{NLm_1^2}{4\pi} + (m_1 \leftrightarrow m_2), \quad \text{if } L > L_2, \quad m_i L \gg 1 \quad (5.1)$$

$$E(L) = \frac{NLm_1^2}{\pi} \sum_{n=1}^{+\infty} (-1)^{n+1} \frac{K_1(2Lm_1n)}{Lm_1n} + \frac{NLm_1^2}{4\pi} + \frac{N\pi}{96L}, \quad \text{if } L_2 > L > L_1, \quad m_i L \gg 1 \quad (5.2)$$

where K_1 is modified Bessel function, $m_1 = \Lambda_1$ and $m_2 = \Lambda_2$. The first terms in both equations correspond to the Casimir energy for a massive complex scalar field with “mixed” DN boundary conditions and are negligible if $m_i L \gg 1$. The Lüscher term in the Higgs phase $\frac{N\pi}{96L}$ is merely the massless limit of the first term. One can associate its appearance precisely with the presence of a massless degree of freedom σ_2 in this phase. Note that the boundary terms $m_i/2$ in the Casimir energy and the $1/L$ corrections of masses are absent for these boundary conditions.

For the Grassmannian model in the phase with $\lambda_3 \neq 0$ and completely broken $U(2)$ gauge symmetry the total energy is

$$E(L) = \frac{NLm^2}{4\pi} - \frac{N\pi}{8L} + \frac{N\pi}{96L}, \quad \text{if } L < L_1, \quad mL \gg 1 \quad (5.3)$$

where $m^2 = \Lambda_1^2 + \Lambda_2^2$. As was deduced in the previous section (see eq. (4.24)) there is one massless field, thus the term $\frac{N\pi}{96L}$ appears.

6 Conclusion

We have studied the homogeneous solutions for 2d NLSMs of the Grassmannian $\text{Gr}(N, 2)$ and the flag $F(N, 2, 1)$ manifolds on the interval and found out the strong interdependence between the phase structure and the boundary conditions. The flag model with the “mixed” DN boundaries may undergo the phase transition from the phase with $U(1) \times U(1)$ gauge symmetry to the phase with $U(1)$. However, for $L < \Lambda_1^{-1}$ there is no solution for these boundaries. On the other hand, for the Grassmannian model the Robin boundaries (4.20) are compatible only with the phase with completely broken $U(2)$ gauge symmetry and $L < \Lambda_1^{-1}$.

The generalizations to other boundary conditions can be made:

- (1) For the systems $F(N, 2, 1)$ and $\text{Gr}(N, 2)$ on the interval the minimal length L appears if any of the following boundary conditions are imposed on the fields ψ_1 and ψ_2 : “mixed” DN-ND, periodic and “unmixed” DD, NN. Indeed, the $\mathbb{CP}^{N-1}$ model with listed boundaries gets non-zero VEV for a sufficiently small L (see [7, 8] and [26, 27] respectively). Then, eq. (4.3) tells us that the saddle-point equations have no solution in this region. The generalization for the maximal length seems to be more subtle and is the subject of further study.
- (2) The systems $F(N, 2, 1)$ and $\text{Gr}(N, 2)$ have no completely deconfined phase, that can be deduced from (1) and eq. (4.19).¹

Also, it seems interesting to consider a model with general flag manifold. We expect it to have more complicated phase structure. For example, 2d sigma model with the complete flag $SU(N)/U(1)^{N-1}$ target space was recently considered in [29, 30] and it was argued that the theory could be gapless in the infrared limit. Another point worth researching is the construction of inhomogeneous solutions for the Grassmannian and flag NLSMs.

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A Derivation of L-dependence

Let us hold a derivation of eq. (5.2) for total energy L -dependence in the phase with partially broken symmetry. It corresponds

$$\lambda_3 = 0, \quad m_2 = 0, \quad \sigma_2 \neq 0, \quad m_1 \neq 0, \quad \sigma_1 = 0$$

¹After the first version of this paper, the paper [28] appeared in arXiv. Among other things, it was argued that $\mathbb{CP}^{N-1}$ with twisted b.c. $n_k(x) = e^{i2\pi k/N} n_k(x+L)$, $k = 1, \dots, N$, have a unique confinement phase with dynamical generation of the mass gap. It allows one to generalize the statement (2) to this type of boundary conditions.

therefore the gap eq. (4.1) with “mixed” DN b.c. is

$$N \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} \frac{dk}{2\pi L} \frac{1}{k^2 + \left(\frac{\pi(n-1/2)}{L}\right)^2 + \lambda_1} - r_1 = 0 \quad (\text{A.1})$$

One can calculate the sum using the following formulas

$$\sum_{\mathbb{Z}} \frac{1}{\left(\frac{\pi n}{L}\right)^2 + \omega^2} = \frac{2L}{\omega} \left(\frac{1}{2} + \frac{1}{\exp(2L\omega) - 1} \right) \quad (\text{A.2})$$

and

$$2 \sum_{\mathbb{N}} \frac{1}{\left(\frac{\pi(n-1/2)}{L}\right)^2 + \omega^2} = \sum_{\mathbb{Z}} \frac{1}{\left(\frac{\pi n}{2L}\right)^2 + \omega^2} - \sum_{\mathbb{Z}} \frac{1}{\left(\frac{\pi n}{L}\right)^2 + \omega^2}. \quad (\text{A.3})$$

Thus we have

$$r_1 - \frac{N}{2\pi L} \int_0^{+\infty} dk \left[\frac{L}{\sqrt{k^2 + m_1^2}} + \frac{4L}{\sqrt{k^2 + m_1^2}} \frac{1}{\exp(4L\sqrt{k^2 + m_1^2} - 1)} - \frac{2L}{\sqrt{k^2 + m_1^2}} \frac{1}{\exp(2L\sqrt{k^2 + m_1^2} - 1)} \right] = 0. \quad (\text{A.4})$$

Let us introduce a quantity

$$Q(m_1 L) = \int_0^{\infty} \frac{2dk}{\sqrt{k^2 + (m_1 L)^2}} \frac{1}{\exp(2\sqrt{k^2 + (m_1 L)^2} - 1)} \quad (\text{A.5})$$

and use the equation $\Lambda_1 = \Lambda_{uv} \exp(-2\pi/N g_1^2)$ then come to the following equalities

$$2\pi r_1 - N \log(\Lambda_{uv} L) = -N \log(\Lambda_1 L) = -N \log(m_1 L) + 2NQ(2m_1 L) - NQ(m_1 L). \quad (\text{A.6})$$

If $m_1 L \gg 1$

$$Q(m_1 L) = \frac{\sqrt{\pi/m_1} e^{-2m_1 L}}{\sqrt{L}} \quad (\text{A.7})$$

Therefore $m_1 = \Lambda_1$ and $1/L$ corrections are absent.

After Pauli-Villars regularization effective action is

$$S_{\text{eff}}^{\text{reg}} = 2N \sum_{i=0}^2 c_i \text{Tr} \log(-\partial^2 + m_1^2 + \tilde{m}_i^2) - \int d^2 x r_1 m_1^2 + \dots \quad (\text{A.8})$$

where “...” means m_1 -independent terms, and

$$\tilde{m}_0 = 0, \quad c_0 = 1, \quad c_1 = \frac{\tilde{m}_2^2}{\tilde{m}_1^2 - \tilde{m}_2^2}, \quad c_2 = \frac{-\tilde{m}_1^2}{\tilde{m}_1^2 - \tilde{m}_2^2}. \quad (\text{A.9})$$

Varying it with respect to m_1^2 , we get

$$\int dx_0 r_1 = N \sum_{i=0}^2 c_i \text{Tr} \frac{1}{-\partial^2 + m_1^2 + \tilde{m}_i^2}. \quad (\text{A.10})$$

If $m_1 L \gg 1$ then the approximation (A.7) is valid, and

$$r_1 = \frac{1}{2\pi L} \left(\frac{L}{2} \log \left(\frac{m_1^2 + \tilde{m}_2^2}{m_1^2} \right) + \frac{L\tilde{m}_2^2}{2(\tilde{m}_1^2 - \tilde{m}_2^2)} \log \left(\frac{m_1^2 + \tilde{m}_2^2}{m_1^2 + \tilde{m}_1^2} \right) \right). \quad (\text{A.11})$$

Suppose that $\tilde{m}_1^2 = aM^2$ and $\tilde{m}_2^2 = M^2$ and take

$$a \rightarrow 1, \quad M \rightarrow \infty \quad (\text{A.12})$$

then regularised value for r_1 is

$$r_1^{\text{reg}} = -\frac{N}{4\pi} \quad (\text{A.13})$$

The quantity $\text{Tr} \log(-\partial^2 + m_1^2)$ is the Casimir energy for a massive scalar field. For mixed DN-ND boundary conditions, it is

$$\frac{Lm_1^2}{\pi} \sum_{n=1}^{+\infty} (-1)^{n+1} \frac{K_1(2Lm_1 n)}{Lm_1 n}. \quad (\text{A.14})$$

The quantity $\text{Tr} \log(-\partial^2 + m_2^2)$, where $m_2 = 0$, is the Casimir energy for a massless scalar field

$$\frac{N\pi}{96L}. \quad (\text{A.15})$$

Putting it all together we get

$$E(L) = \frac{NLm_1^2}{\pi} \sum_{n=1}^{+\infty} (-1)^{n+1} \frac{K_1(2Lm_1 n)}{Lm_1 n} + \frac{NLm_1^2}{4\pi} + \frac{N\pi}{96L}, \quad \text{if } L_2 > L > L_1, \quad m_1 L \gg 1. \quad (\text{A.16})$$

Similarly one can get eq. (5.3).

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