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Large- N $\mathbb{CP}^{N-1}$ sigma model on a Euclidean torus: uniqueness and stability of the vacuum

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ABSTRACT: In this paper we examine analytically the large- N gap equation and its solution for the 2D $\mathbb{CP}^{N-1}$ sigma model defined on a Euclidean spacetime torus of arbitrary shape and size (L, β) , β being the inverse temperature. We find that the system has a unique homogeneous phase, with the $\mathbb{CP}^{N-1}$ fields n_i acquiring a dynamically generated mass $\langle \lambda \rangle \geq \Lambda^2$ (analogous to the mass gap of $SU(N)$ Yang-Mills theory in 4D), for any β and L . Several related topics in the recent literature are discussed. One concerns the possibility, which turns out to be excluded according to our analysis, of a “Higgs-like” — or deconfinement — phase at small L and at zero temperature. Another topics involves “soliton-like” (inhomogeneous) solutions of the generalized gap equation, which we do not find. A related question concerns a possible instability of the standard $\mathbb{CP}^{N-1}$ vacuum on $\mathbb{R}^2$, which is shown not to occur. In all cases, the difference in the conclusions can be traced to the existence of certain zeromodes and their proper treatment. The $\mathbb{CP}^{N-1}$ model with twisted boundary conditions is also analyzed. The θ dependence and different limits involving N , β and L are briefly discussed.

KEYWORDS: 1/ N Expansion, Confinement, Duality in Gauge Field Theories, Nonperturbative Effects

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1 Introduction

The two dimensional $\mathbb{CP}^{N-1}$ sigma model has received constant attention from theoretical physicists ever since the pioneering works by D’Adda et al. [1] and Witten [2]. See also [3]–[21].¹ The model is interesting as a toy model for nonperturbative dynamics of QCD, sharing the properties of asymptotic freedom and confinement with the latter. It can

¹It is not our aim here to make a review of the vast literature on two dimensional $\mathbb{CP}^{N-1}$ sigma model; the references cited in the text below are strictly relevant ones to the discussion. The list [3]–[21] is certainly a partial and incomplete set of papers on the two dimensional $\mathbb{CP}^{N-1}$ sigma model.

also be related to some physical phenomena in condensed matter physics such as quantum Hall effects [22–26]. Another context in which this model emerges as an effective action is the internal quantum fluctuations of the nonAbelian vortex, in a color-flavor locked $SU(N)_{\text{cf}}$ symmetric vacuum [27–31].

In spite of much effort dedicated to the study of the model, there seem still to be some disagreement, new unconfirmed proposals, and not fully justified remarks in the current literature. It is our purpose to address some of these issues through a careful examination of the gap equations in a system defined on a Euclidean torus of arbitrary size and form (finite spatial length L and inverse-temperature β) in the large- N expansion, and try to resolve the controversies as much as possible.

We find that the two dimensional $\mathbb{CP}^{N-1}$ sigma models defined with doubly periodic conditions, possesses a unique ground state for any L and β , which goes over, in the $L \rightarrow \infty$ and at zero temperature ($\beta \rightarrow \infty$) limit, smoothly to the well-known vacuum with mass generation, and with no global $SU(N)$ symmetry breaking. Our results turn out to differ from some claims found in the existent literature, and agree with some others. One concerns the possibility, which appears to be excluded according to our analysis, of a “Higgs-like” — or deconfinement — phase at small L and at zero temperature. Another concerns “soliton-like” (inhomogeneous) solutions of the generalized gap equation, which we do not find. As will be discussed in detail below (see in particular section 3), in all cases the difference in the conclusions can be traced to the subtle role played by certain zeromodes (or in some cases, negative modes), and to their proper treatment.

The action of the $\mathbb{CP}^{N-1}$ sigma model is:

$$S = \int dt dx \left[r (D_\mu n_i)^\dagger (D^\mu n_i) - \lambda (n_i^\dagger n_i - 1) \right], \quad r \equiv \frac{4\pi}{g^2}, \quad (1.1)$$

where n_i with $i = 1, \dots, N$ are complex scalar fields, and the covariant derivative is

$$D_\mu = \partial_\mu - i A_\mu. \quad (1.2)$$

The action is invariant under the $U(1)$ gauge transformation

$$n_i \rightarrow e^{i\alpha} n_i, \quad A_\mu \rightarrow A_\mu - \partial_\mu \alpha. \quad (1.3)$$

Classically the $U(1)$ gauge field A_μ can be integrated out, giving

$$A_\mu = \frac{i}{2} \left(n_i^\dagger \partial_\mu n_i - \partial_\mu n_i^\dagger n_i \right), \quad (1.4)$$

which upon insertion into the action leads to the characteristic quartic interaction term among the fields n_i . $\lambda(x, t)$ is a Lagrange multiplier field enforcing the classical constraint

$$n_i^\dagger n_i = 1. \quad (1.5)$$

$r \equiv \frac{4\pi}{g^2}$ is the inverse of the coupling constant squared:² after an appropriate rescaling of the n_i fields (see eq. (2.1)), it represents the “size” of the $\mathbb{CP}^{N-1}$ manifold.

²Many different definitions of the coupling constant are used in the literature on the $2D$ $\mathbb{CP}^{N-1}$ sigma model. As this can potentially be confusing, we list the relations among them in appendix A.

The $\mathbb{CP}^{N-1}$ model also admits the introduction of a θ term

$$\Delta S = \frac{i\theta}{2\pi} \int d^2x \epsilon_{\mu\nu} \partial_\mu A_\nu = \frac{i\theta}{2\pi} \int d^2x F_{01} \equiv -i\theta Q, \quad (1.6)$$

where Q is the topological charge. Q is classically quantized on $\mathbb{R}^2$ as

$$Q = \frac{1}{2\pi i} \int d^2x \partial_\mu \left(n_i^\dagger \epsilon_{\mu\nu} \partial_\nu n_i \right) = \frac{1}{2\pi} \oint dx_\mu \frac{\partial \sigma}{\partial x_\mu} \in \mathbb{Z}, \quad (1.7)$$

where it is assumed that asymptotically $n_i = e^{i\sigma} w_i$ with w_i fixed and with $w_i^\dagger w_i = 1$.

The paper is organized as follows. In section 2 we discuss the large- N solution of the gap equation for the model defined on a Euclidean torus. In section 3 we discuss two related issues raised in the recent literature, regarding the uniqueness and stability of the standard confining vacuum. In section 4 the system with twisted boundary conditions is discussed. In section 5 we make a brief remark on the dependence on the topological θ angle. We make concluding remarks in section 6. Appendices A, B C, and D offer a collection of brief discussions on the coupling constant convention, Pauli-Villars regularization, various mathematical identities, and on the spontaneous symmetry breaking of the global $SU(N)$.

2 $2D$ $\mathbb{CP}^{N-1}$ sigma model on Euclidean torus

2.1 Analytic derivation of the gap equation

Let us consider this model on a ring of size L at a finite temperature $T = \beta^{-1}$. The Euclidean partition function Z is

$$Z = \int \mathcal{D}n \mathcal{D}\lambda \mathcal{D}A e^{-S_E - i\theta Q},$$

$$S_E = \int_0^\beta dt \int_0^L dx \left[|D_t n_i|^2 + |D_x n_i|^2 + \lambda(x) (|n_i|^2 - r) + \mathcal{E}_{uv} \right], \quad (2.1)$$

where bare energy density $\mathcal{E}_{uv}$ is introduced. Here fields $n_i(x, t)$ have the periodicity

$$n_i(x, t + \beta) = n_i(x, t), \quad n_i(x + L, t) = n_i(x, t), \quad i = 1, 2, \dots, N. \quad (2.2)$$

In this section we set $\theta = 0$ (the θ dependence will be discussed in section 5). Assuming translational invariance, $A_\mu(x, t)$ and $\lambda(x, t)$ can be replaced by constants

$$\langle A_\mu(x, t) \rangle = a_\mu, \quad \langle \lambda(x, t) \rangle = \lambda. \quad (2.3)$$

In the large- N limit, one can omit contributions coming from their fluctuations. The possibility that the n_i fields acquire a nonvanishing classical VEV,

$$\langle n_i(x, t) \rangle = \sigma_i, \quad (2.4)$$

will be discussed in section 3. For the moment, we set $\sigma_i = 0$.

Integration over the constant λ (and a_μ)

$$Z = \int_{\mathbb{R}^2} d^2 a_\mu \int_{-i\infty+\epsilon}^{i\infty+\epsilon} d\lambda Z_\lambda, \quad \text{def. } Z_\lambda, \quad (2.5)$$

can be done in the large- N limit by the saddle-point method:

$$Z \simeq Z_\lambda|_{\lambda=\lambda_{\text{sp}}}, \quad \left. \frac{d \ln Z_\lambda}{d\lambda} \right|_{\lambda=\lambda_{\text{sp}}} = 0. \quad (2.6)$$

Here in order to make the integral finite, the integration path for λ needs to be taken along the imaginary axis, whereas those for a_μ are along the real axis as usual. The pseudo free energy is

$$F_\lambda = -T \ln Z_\lambda. \quad (2.7)$$

Sometimes the adjective “pseudo” will be used to stress the fact that F_λ is still a function of λ and it is not yet extremized. When it acquires its stationary value

$$F = -T \ln Z, \quad (2.8)$$

with Z given in (2.6), one may refer to it as the real free energy. When it is not specified it should be clear from the context to which one we are referring to.

The strategy for solving the theory in the large- N limit is to first perform the Gaussian integration over the n_i fields, and then to consider the saddle-point approximation (2.6). After integrating out $n_i(x, t)$, the partition function Z_λ becomes

$$\begin{aligned} -\ln Z_\lambda = N \sum_{n,m \in \mathbb{Z}} \sum_I c_I \ln \left(\left(\frac{2n\pi}{\beta} + a_t \right)^2 + \left(\frac{2m\pi}{L} + a_x \right)^2 + \lambda + \lambda_I \right) \\ + \beta L (-\lambda r + \mathcal{E}_{\text{uv}}), \end{aligned} \quad (2.9)$$

where λ is determined by the saddle-point equation

$$\begin{aligned} 0 &= -\frac{1}{\beta L} \frac{\partial \ln Z_\lambda}{\partial \lambda} \\ &= \frac{N}{\beta L} \sum_{n,m \in \mathbb{Z}} \sum_I c_I \left(\left(\frac{2n\pi}{\beta} + a_t \right)^2 + \left(\frac{2m\pi}{L} + a_x \right)^2 + \lambda + \lambda_I \right)^{-1} - r. \end{aligned} \quad (2.10)$$

Here we use Pauli-Villars regularization scheme where c_I, λ_I , with $I = 0, 1, 2, 3$, satisfy³

$$\begin{aligned} c_0 = 1, \quad \lambda_0 = 0, \quad \lambda_{I \neq 0} = b_I \Lambda_{\text{uv}}^2 > 0, \\ \text{with} \quad \sum_I c_I = 0, \quad \sum_I c_I \lambda_I = 0, \end{aligned} \quad (2.12)$$

which gives

$$\sum_{I \neq 0} c_I \ln \lambda_I = -\ln \Lambda_{\text{uv}}^2 + \sum_{I \neq 0} c_I \ln b_I, \quad \sum_{I \neq 0} c_I \lambda_I \ln \lambda_I = \Lambda_{\text{uv}}^2 \sum_{I \neq 0} c_I b_I \ln b_I. \quad (2.13)$$

Λ_{uv} is the UV cutoff which has to be sent to infinity. See appendix B for more details.

³A simple possible choice is

$$c_1 = 1, c_2 = c_3 = -1, \quad \lambda_1 = 2\Lambda_{\text{uv}}^2, \quad \lambda_2 = \lambda_3 = \Lambda_{\text{uv}}^2. \quad (2.11)$$

In order to compute eq. (2.9) we first use⁴

$$\begin{aligned}\sum_I c_I \ln(\lambda' + \lambda_I) &= \lim_{s \rightarrow 0} \sum_I c_I \frac{1 - (\lambda' + \lambda_I)^{-s}}{s} \\ &= - \lim_{s \rightarrow 0} \sum_I \frac{c_I}{s \Gamma(s)} \int_0^\infty dt t^{s-1} e^{-t(\lambda' + \lambda_I)},\end{aligned}\quad (2.15)$$

where $\Gamma(s)$ is the gamma function. The key formula is the following identity

$$\sum_{n \in \mathbb{Z}} e^{-t(\frac{2\pi n}{L} + a)^2} = \frac{L}{2\sqrt{\pi t}} \sum_{n' \in \mathbb{Z}} e^{-\frac{(n'L)^2}{4t} + in'La}.\quad (2.16)$$

which follows from the Poisson summation formula (see appendix C). By using these formulae, eq. (2.9) can be cast in the form

$$\begin{aligned}-\ln Z_\lambda - \beta L(-\lambda r + \mathcal{E}_{\text{uv}}) \\ &= - \lim_{s \rightarrow 0} \sum_{n, m, I} \frac{N c_I}{\Gamma(s+1)} \int_0^\infty dt t^{s-1} e^{-t\left(\left(\frac{2n\pi}{\beta} + a_t\right)^2 + \left(\frac{2m\pi}{L} + a_x\right)^2 + \lambda + \lambda_I\right)} \\ &= - \lim_{s \rightarrow 0} \sum_{n', m', I} \frac{N \beta L c_I}{4\pi \Gamma(s+1)} e^{in'\beta a_t + im'La_x} \int_0^\infty dt t^{s-2} e^{-\frac{(n'\beta)^2 + (m'L)^2}{4t} - t(\lambda + \lambda_I)}.\end{aligned}\quad (2.17)$$

We now note that, while the divergent part in the second line comes from an infinite sum over (n, m) , the divergence in the last line arises from the $n' = m' = 0$ term alone, which thus can be separated neatly from the rest. This term can be calculated and gives $(\frac{N\beta L}{4\pi})$ times)

$$\begin{aligned}\lim_{s \rightarrow 0} \sum_I \frac{c_I}{\Gamma(s+1)} \int_0^\infty dt t^{s-2} e^{-t(\lambda + \lambda_I)} &= \lim_{s \rightarrow 0} \sum_I \frac{c_I \Gamma(s-1)}{\Gamma(s+1)(\lambda + \lambda_I)^{s-1}} \\ &= \lim_{s \rightarrow 0} \sum_I \frac{c_I (\lambda + \lambda_I)^{1-s}}{s(s-1)} = \lim_{s \rightarrow 0} \sum_I \frac{c_I (\lambda + \lambda_I)}{s(s-1)} (1 - s \ln(\lambda + \lambda_I) + \mathcal{O}(s^2)) \\ &= \sum_I c_I (\lambda + \lambda_I) \ln(\lambda + \lambda_I) \\ &= \lambda \ln \frac{\lambda}{e} + \lambda \sum_{I \neq 0} c_I \ln \lambda_I + \sum_{I \neq 0} c_I \lambda_I \ln \lambda_I + \mathcal{O}(\Lambda_{\text{uv}}^{-2}).\end{aligned}\quad (2.18)$$

⁴The formula with $p \in \mathbb{R}_{>0}$

$$\frac{1}{p^s} = \frac{1}{\Gamma(s)} \int_0^\infty dt t^{s-1} e^{-tp}\quad (2.14)$$

is correct only for $\text{Re}(s) > 0$. Using Pauli-Villars regularization and without any analytic continuation, the above formula is extended with $\text{Re}(s) > -2$ as is used in eq. (2.17). See appendix B for details.

On the other hand, a generic $(n', m') \neq (0, 0)$ term is equal to $(\frac{N\beta L}{4\pi})$ times)

$$\begin{aligned} & \int_0^\infty dt \, t^{-2} e^{-\frac{(n'\beta)^2 + (m'L)^2}{4t} - t(\lambda + \lambda_I)} \\ &= 4 \sqrt{\frac{\lambda + \lambda_I}{(n'\beta)^2 + (m'L)^2}} K_1 \left(\sqrt{(\lambda + \lambda_I)((n'\beta)^2 + (m'L)^2)} \right), \end{aligned} \quad (2.19)$$

where $K_1(x)$ is the modified Bessel function of the second kind. Note that in the limit in which the Pauli-Villars regulator masses λ_I ($I \geq 1$) are sent to infinity, this expression is nonvanishing only for $I = 0$. Therefore we find after relabeling $(n', m') \rightarrow (n, m)$,

$$\begin{aligned} -\frac{\ln Z_\lambda}{\beta L} &= -\frac{N}{4\pi} \lambda \ln \frac{\lambda}{e\Lambda^2} + M \\ &\quad - \sum_{(n,m) \in \mathbb{Z}^2 \setminus \{(0,0)\}} \frac{N}{\pi} \cos(n\beta a_t) \cos(mLa_x) \\ &\quad \times \sqrt{\frac{\lambda}{(n\beta)^2 + (mL)^2}} K_1 \left(\sqrt{\lambda((n\beta)^2 + (mL)^2)} \right), \end{aligned} \quad (2.20)$$

where the renormalized constants Λ, M are defined as

$$r = \frac{N}{4\pi} \left(\ln \frac{\Lambda_{\text{uv}}^2}{\Lambda^2} - \sum_{I \neq 0} c_I \ln b_I \right), \quad \mathcal{E}_{\text{uv}} = M + \frac{N\Lambda_{\text{uv}}^2}{4\pi} \sum_{I \neq 0} c_I b_I \ln b_I. \quad (2.21)$$

Being in an Euclidean space, this equation is invariant under the exchange of $(\beta, a_t) \leftrightarrow (L, a_x)$, as it should be. We note that the maximum of Z_λ (i.e., the minimum of the free energy) with respect to (a_t, a_x) for any given λ is at

$$a_x = 0, \quad a_t = 0, \quad (2.22)$$

as the coefficients of $\cos(n\beta a_t) \cos(mLa_x)$ in $\ln Z_\lambda$ are positive definite. Thus we set $a_x = a_t = 0$, and find:

$$\begin{aligned} -\frac{\ln Z_\lambda}{\beta L} &= -\frac{N}{4\pi} \lambda \ln \frac{\lambda}{e\Lambda^2} + M \\ &\quad - \sum_{(n,m) \in \mathbb{Z}^2 \setminus \{(0,0)\}} \frac{N}{\pi} \sqrt{\frac{\lambda}{(n\beta)^2 + (mL)^2}} K_1 \left(\sqrt{\lambda((n\beta)^2 + (mL)^2)} \right). \end{aligned} \quad (2.23)$$

The variation with respect to λ , the gap equation (2.10), becomes

$$\frac{1}{4\pi} \ln \frac{\lambda}{\Lambda^2} = \frac{1}{2\pi} \sum_{n,m \in \mathbb{Z}^2 \setminus \{0,0\}} K_0 \left(\sqrt{\lambda((n\beta)^2 + (mL)^2)} \right). \quad (2.24)$$

This is an *exact* formula valid for any (β, L) , at large- N . For the model at the zero temperature, $\beta = \infty$, but with generic value of L , only the $n = 0$ term is present and eq. (2.24) reduces exactly to the one found in ref. [32].

Figure 1. Left panel: a graphic solution of eq. (2.24) for $L = 1$ and $\beta = 1, 2, 3, 5$, top to bottom, all measured in the unit of $1/\Lambda^2$. The left-hand side of eq. (2.24) is shown in red and the right-hand sides are shown in blue (top), orange, green, and magenta, respectively. Right panel: the pseudo free energy normalized by N times length L as a function of λ . $\beta = 1, 2, 3, 5$, from the bottom to the top. The real free energy is the maximum along the real axis of λ .

The left-hand side of eq. (2.24) is a monotonically increasing function of $\lambda \in (0, \infty)$, varying from $-\infty$ to ∞ , whereas the right-hand side is a positive-definite function, monotonically decreasing from ∞ to 0 (figure 1, the left panel). This equation therefore possesses a unique solution with nonvanishing λ such that

$$\lambda \geq \Lambda^2 = \lim_{T \rightarrow 0, L \rightarrow \infty} \lambda, \quad (2.25)$$

for any (β, L) and such that it attains its minimum value Λ^2 in the case of the standard $2D$ $\mathbb{CP}^{N-1}$ sigma model vacuum on $\mathbb{R}^2$ ($\beta = \infty, L = \infty$). The pseudo free energy is plotted in figure 1, the right panel. This function is manifestly concave, as $\frac{d^2(\ln Z_\lambda)}{d\lambda^2} > 0$, which means that F_λ (2.7), somewhat counter-intuitively, is *maximized* along the real axis of λ . See subsection 3.4 for a more general explanation of this fact

Configurations of λ with various values of temperatures are shown in figure 2, the left panel. Substituting the solutions into eq. (2.23) we obtain the real free energy (2.8) shown in figure 2, the right panel.

2.2 Small size and low temperatures

We now study the limit

$$L\Lambda \rightarrow 0, \quad \beta\sqrt{\lambda} > \beta\Lambda \rightarrow \infty, \quad (2.26)$$

(recall $\lambda \geq \Lambda^2$). Moreover we assume

$$L\sqrt{\lambda} \rightarrow 0, \quad (2.27)$$

which will be verified a posteriori. At low-temperatures $T \ll \Lambda$, the pseudo free energy is obtained from eq. (2.9) with $a_\mu = 0$ as

$$\begin{aligned} F_\lambda &\equiv -T \ln Z_\lambda \\ &= N \sum_I c_I \sum_{n \in \mathbb{Z}} \sqrt{\left(\frac{2\pi n}{L}\right)^2 + \lambda + \lambda_I} + L(-\lambda r + \mathcal{E}_{uv}) + \mathcal{O}\left(e^{-\beta\sqrt{\lambda}}\right). \end{aligned} \quad (2.28)$$

Figure 2. Left panel: VEV of λ as a function of L , with $\Lambda = 1$ and $T = 0, 0.5, 1, \dots, 3.5$, respectively, from the bottom to the top. Right panel: the free energy for the same values and $M = 0$ as functions of L . In the right panel the curves refer to the temperatures $T = 0, 0.5, 1, \dots, 3.5$, from the top to the bottom.

Using the expansion

$$\sqrt{z + \lambda} = \sqrt{z} - \sum_{p=1}^{\infty} \frac{(-\lambda)^p}{z^{p-\frac{1}{2}}} \frac{\Gamma(p - \frac{1}{2})}{2\sqrt{\pi}p!}, \quad \text{for } |\lambda| < z, \quad (2.29)$$

and the Riemann zeta function $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$, the $n \neq 0$ terms in F_λ can be expanded for λ small (compared to $(\frac{2\pi}{L})^2$) as

$$\begin{aligned} F_\lambda = & -\frac{\pi N}{3L} + LM + N\sqrt{\lambda} - \frac{NL}{2\pi} \ln \frac{4\pi e^{-\gamma}}{\Lambda L} \lambda \\ & - \frac{2\pi N}{L} \sum_{p=2}^{\infty} \frac{\Gamma(p - \frac{1}{2})}{\sqrt{\pi}p!} \left(-\frac{\lambda L^2}{4\pi^2} \right)^p \zeta(2p - 1) + \mathcal{O}(e^{-\beta\sqrt{\lambda}}). \end{aligned} \quad (2.30)$$

In the calculation of the constant and linear terms, the regularization turns out to be crucial and we have used

$$\begin{aligned} \sum_I c_I \sum_{n \in \mathbb{Z}} \sqrt{\left(\frac{2\pi n}{L} \right)^2 + \lambda_I} &= -\frac{\pi}{3L} - \frac{L}{4\pi} \sum_{I \neq 0} c_I \lambda_I \ln \lambda_I, \\ \sum_I c_I \sum_{n=1}^{\infty} \frac{1}{\sqrt{\left(\frac{2\pi n}{L} \right)^2 + \lambda_I}} &= -\frac{L}{4\pi} \sum_{I \neq 0} c_I \left(\ln \frac{\lambda_I L^2}{16\pi^2} - 2\gamma \right) : \end{aligned} \quad (2.31)$$

see appendix B.2 for more details.

The gap equation for small $\sqrt{\lambda}L$ is given by

$$0 = \frac{\partial F_\lambda}{\partial \lambda} = \frac{NL}{4\pi} \left(\frac{2\pi}{\sqrt{\lambda}L} - 2 \ln \frac{4\pi e^{-\gamma}}{\Lambda L} - \zeta(3) \frac{\lambda L^2}{4\pi^2} + \mathcal{O}((\lambda L^2)^2) \right), \quad (2.32)$$

and its solution is

$$\sqrt{\lambda} = \frac{2\pi}{L} \left(\frac{1}{2\alpha} - \frac{\zeta(3)}{16\alpha^4} + \mathcal{O}(\alpha^{-6}) \right), \quad \alpha \equiv \ln \frac{4\pi e^{-\gamma}}{\Lambda L}, \quad (2.33)$$

where α is a large parameter

$$1 \gg \frac{\Lambda L}{4\pi e^{-\gamma}} = e^{-\alpha} \quad (\gg e^{-\frac{\pi}{LT}}). \quad (2.34)$$

Therefore, the free energy for small $L\Lambda$ is obtained as an expansion in powers of $\frac{1}{\alpha}$:

$$F = -T \ln Z = -\frac{\pi N}{3L} \left(1 - \frac{3}{2\alpha} + \frac{3\zeta(3)}{32\alpha^4} + \mathcal{O}\left(\alpha^{-6}, e^{-\frac{\pi}{LT\alpha}}\right) \right) + LM. \quad (2.35)$$

In the first term we recognize the Lüscher term $-\frac{\pi N}{3L}$, with its thermal corrections, for the $2(N-1) \simeq 2N$ degrees of freedom which, due to asymptotic freedom, behave as if they were massless fields. This agrees with the result obtained by Shifman et al. [32] strictly at $T = 0$.

2.3 High temperature and large size

The Euclidean torus possesses a symmetry $L \leftrightarrow \beta = T^{-1}$. By using this we can relate the case of high temperatures with large size with the one discussed in the previous subsection. Here we assume the limits

$$\beta\Lambda \rightarrow 0, \quad L\sqrt{\lambda} > L\Lambda \rightarrow \infty, \quad (2.36)$$

which implies also $\beta\sqrt{\lambda} \rightarrow 0$. In these limits, a solution of the gap equation is obtained, from the dual of (2.33), as

$$\sqrt{\lambda} = 2\pi T \left(\frac{1}{2\tilde{\alpha}} - \frac{\zeta(3)}{16\tilde{\alpha}^4} + \mathcal{O}(\tilde{\alpha}^{-6}) \right), \quad \text{with } \tilde{\alpha} \equiv \ln \frac{4\pi e^{-\gamma} T}{\Lambda}, \quad (2.37)$$

and the free energy as

$$F = -T \ln Z = -\frac{\pi N}{3} T^2 L \left(1 - \frac{3}{2\tilde{\alpha}} + \frac{3\zeta(3)}{32\tilde{\alpha}^4} + \mathcal{O}\left(\tilde{\alpha}^{-6}, e^{-\frac{\pi LT}{\tilde{\alpha}}}\right) \right) + LM. \quad (2.38)$$

The pressure and entropy of this system are then calculated as

$$P \equiv -\frac{\partial F}{\partial L}, \quad S \equiv -\frac{\partial F}{\partial T}, \quad (2.39)$$

and are shown for various configurations in figure 3 for F given in (2.38). In the extreme limit $L \gg \Lambda^{-1}$, $T \gg \Lambda$ we have

$$P \simeq \frac{\pi N}{3} T^2 - M, \quad S \simeq \frac{2\pi N}{3} TL. \quad (2.40)$$

This is to be compared with the other known limits

$$P \simeq \begin{cases} -\frac{N\Lambda^2}{4\pi} - M & \text{for } L \gg \Lambda^{-1}, \quad T \ll \Lambda, \\ -\frac{\pi N}{3L^2} - M & \text{for } L \ll \Lambda^{-1}, \quad T \ll \Lambda, \end{cases} \quad S \simeq 0 \quad \text{for } T \ll \Lambda. \quad (2.41)$$

Figure 3. Pressure and entropy density for $\Lambda = 1, M = 0$ and $T = 0, 0.5, 1, \dots, 3.5$, from the bottom to the top.

3 Uniqueness and stability of the $\mathbb{CP}^{N-1}$ sigma model vacuum

3.1 Deconfinement (Higgs) phase at small L ?

It was argued in ref. [32] (see also [14]) that at small $L < L_{\text{crit}} \sim \frac{1}{\Lambda}$ (at zero temperature, $\beta = \infty$) the system undergoes a phase transition into a deconfinement (or Higgs) phase⁵ where

$$\langle \lambda(x, t) \rangle = \lambda = 0, \quad \langle n_i \rangle = \delta_{i1} \sigma \neq 0. \quad (3.1)$$

Consider the *classical* equation of motion for the component σ

$$\lambda \sigma = 0, \quad (3.2)$$

that arises from the classical action (2.1), with $\partial\sigma = 0$. This appears to imply that two branches of solutions are possible, either $\lambda = 0$ or $\sigma = 0$. In the work [32] they argued that the system has a solution (3.1) without a mass gap, which becomes energetically favorable at small compactification length,⁶ as compared to the standard $\mathbb{CP}^{N-1}$ vacuum,

$$\langle \lambda(x) \rangle = \Lambda^2, \quad \langle \sigma \rangle = 0, \quad (L = \beta = \infty). \quad (3.3)$$

To examine whether a vacuum of the type, (3.1), i.e., with a nonvanishing VEV for $\hat{\sigma}^2 \equiv \sum_i |n_i|^2$ but without the dynamical mass generation, $\langle \lambda \rangle = 0$, can arise quantum mechanically at small L , we rewrite the part of the partition function due to the constant

⁵Even though they may not be entirely adequate, we stick for definiteness to the terminology used in [32], calling the standard $\mathbb{CP}^{N-1}$ phase at zero temperature, $L = \infty$, (3.3), as “confinement” phase; while using the word “deconfinement (or Higgs) phase”, for an eventual phase (3.1) in which no mass generation for n_i occurs, but with a non vanishing VEV for $\hat{\sigma}^2 \equiv \sum_i |n_i|^2$ signaling a spontaneous breaking of the global $\text{SU}(N)$ symmetry.

⁶This type of phase transition was first proposed in [33] for the model on a finite interval with Dirichlet boundary condition. Such a possibility was subsequently excluded [34]. For models with mixed boundary conditions, however, see [35, 36].

modes as

$$\begin{aligned} Z_\sigma &= \int d\hat{\sigma}^2 \int \prod_{i=1}^N d\sigma_i d\bar{\sigma}_i \delta\left(\sum_i |\sigma_i|^2 - \hat{\sigma}^2\right) e^{-\beta L \lambda \hat{\sigma}^2} \\ &\simeq \int d\hat{\sigma}^2 \left(\frac{\pi e \hat{\sigma}^2}{N}\right)^N e^{-\beta L \lambda \hat{\sigma}^2} = \int d\hat{\sigma}^2 e^{-S_{\hat{\sigma}}}, \end{aligned} \quad (3.4)$$

where we introduced an effective $\mathbb{CP}^{N-1}$ radius $\hat{\sigma} \equiv \sqrt{\sum_i |\sigma_i|^2}$, and

$$S_{\hat{\sigma}} = N \ln \frac{N}{\pi e \hat{\sigma}^2} + \beta L \lambda \hat{\sigma}^2 \quad (3.5)$$

is the effective action for $\hat{\sigma}^2$. The correct saddle-point equation for $\hat{\sigma}^2$ is therefore

$$\hat{\sigma}^2 = \frac{N}{\beta L \lambda}, \quad (3.6)$$

which replaces eq. (3.2).

To compare this saddle-point equation with eq. (3.2), we rewrite eq. (3.6) as

$$\lambda \hat{\sigma} = \frac{N}{\beta L \hat{\sigma}}. \quad (3.7)$$

The right-hand side comes from the volume integration of the zero-modes. Since there are N copies of zero-modes, one cannot omit this volume factor. Therefore, for any finite (β, L) , there exists only one branch of solutions of the coupled equations (3.6) and (2.24) and not two separate ones $\lambda = 0$ or $\sigma = 0$.

One might object by saying that, by first taking the limit, e.g., $\beta \rightarrow \infty$, eq. (3.7) yields indeed $\lambda \hat{\sigma} = 0$. As this point potentially involves a subtle question of ordering of various limits, let us proceed with care. Consider the full partition function Z_λ :

$$Z_\lambda = Z_{\text{mass}} Z_\sigma = \int d\hat{\sigma} Z_{\text{mass}} e^{-S_{\hat{\sigma}}}, \quad (3.8)$$

where Z_{mass} is the part of the partition function found earlier by integration over massive modes (corresponding to the second term in eq. (2.23)). The saddle-point equation for λ is

$$0 = -\frac{\partial \ln Z_{\text{mass}}}{\partial \lambda} + \beta L \hat{\sigma}^2. \quad (3.9)$$

The first term in this expression does not contain contributions from the zero-modes. Note that at this point the system of equations solved are eq. (3.7) and eq. (3.9), valid in the large N limit and at arbitrary (β, L) .

Eq. (3.9) is precisely the form of the gap equation (i.e., the saddle-point equation for λ) in the “deconfinement” phase, used in ref. [32]. However, the correct saddle-point equation for $\hat{\sigma}$ which accompanies it, is eq. (3.7), and not eq. (3.2). Use of eq. (3.7) shows that eq. (3.9) is equivalent to

$$\begin{aligned} 0 &= -\frac{\partial \ln Z_{\text{mass}}}{\partial \lambda} + \frac{N}{\lambda} = -\frac{\partial}{\partial \lambda} \left(\ln Z_{\text{mass}} + N \ln \frac{\pi}{\beta L \lambda} \right) \\ &= -\frac{\partial \ln Z_\lambda}{\partial \lambda}, \end{aligned} \quad (3.10)$$

Figure 4. VEV of $\frac{\hat{\sigma}^2}{N}$ for $\frac{T}{\Lambda} = 0.5, 1, 1.5, \dots, 3.5$, respectively, from the bottom to the top.

which is exactly eq. (2.24), derived and studied in the previous section, and which has been shown to possess a unique nonvanishing solution for λ , with $\lambda \geq \Lambda^2$, for any values of (β, L) . What eq. (3.6) tells us is that the two gap equations in the “deconfinement” phase and in the “confinement” phase are actually one and the same equation; its unique saddle point describes a single phase of the system, with dynamical generation of mass for the n_i fields. A “deconfinement” phase with $\langle \lambda \rangle = 0$ never appears in our system.⁷ The crucial point is that the coupled saddle-point equations for $(\lambda, \hat{\sigma})$ have a smooth $\beta \rightarrow \infty$ limit.

Thus as far as we can see there is no problem of ordering in which different limits, $N, \beta, L \rightarrow \infty$, are taken, as long as the vacuum property of the system at $\theta = 0$ is concerned. As will be discussed in section 5, the θ dependence of the free energy, instead, depends crucially on the order in which different limits are taken.

The VEV of $\hat{\sigma}$ vanishes at zero temperature due to the mass gap as

$$\hat{\sigma}^2 \leq \frac{NT}{L\Lambda^2}, \quad \lim_{T \rightarrow 0} \hat{\sigma}^2 = 0, \quad \forall L. \quad (3.11)$$

The volume integration of the large number of the zero-modes prevents $\hat{\sigma}^2$ from acquiring a nonvanishing VEV at zero temperature. Conversely, at any nonzero temperature, $T \neq 0$, the mean square $\frac{\hat{\sigma}^2}{N}$ takes a nonvanishing expectation value (figure 4), but this is unrelated to any spontaneous symmetry breaking. The absence of spontaneous breaking of the global $SU(N)$ symmetry in our system is discussed more explicitly in appendix D.

3.2 Absence of soliton-like solutions

Another issue concerns the possible existence of inhomogeneous, soliton-like solutions of the gap equation. It was recently argued [38] that the standard $2D \mathbb{CP}^{N-1}$ sigma model vacuum, with dynamical mass generation

$$\langle \lambda(x) \rangle = \Lambda^2, \quad (3.12)$$

⁷This conclusion is in agreement with the one reached in refs. [7, 37], but differs from that in ref. [32]. This latter fact can be traced to the proper treatment of the zero-modes, eq. (3.4), eq. (3.5).

is actually unstable against decay into a lattice of solitons, i.e., inhomogeneous configurations (n, λ) found in [39]. These papers deal with the system at zero temperature ($\beta = \infty$) and infinitely extended space ($L = \infty$). The crucial issue, as in the previous section 3.1, concerns a proper treatment of the zero-modes of the n_i fields.

In order to study a possible solution with a classical component, $n_i = \delta_{iN}\sigma$, one first integrates the fluctuations of the quantum fields n_i , $i = 1, 2, \dots, N-1$, to get an effective action

$$S = (N-1)\text{Tr} \log (-\partial^2 + \lambda) + \int d^2x [(\partial\sigma)^2 + \lambda(\sigma^2 - r)], \quad (3.13)$$

which, after functional variation with respect to $\lambda(x)$, yields

$$\sigma^2 = r - N \left(\int_{k \neq 0} \frac{|f_k(x)|^2}{2\omega_k} + \sum_i \frac{|f_{0i}(x)|^2}{2\omega_{0i}} \right), \quad (3.14)$$

$$(-\partial^2 + \lambda(x)) f_k(x) = \omega_k^2 f_k(x). \quad (3.15)$$

The last term in eq. (3.14) is the contribution from the eventual bound states (f_{0i}, ω_{0i}) : it has been separated from that of the continuum. Note that in any potential $\lambda(x)$ which asymptotically approaches a value Λ^2 and $\lambda(x) < \Lambda^2$ in some region of x , there is at least one bound state of energy less than Λ^2 , according to a known theorem in one-dimensional quantum mechanics [40]. The variation with respect to the classical field $\sigma \equiv n_N$ gives

$$(-\partial^2 + \lambda(x)) \sigma(x) = 0. \quad (3.16)$$

The phase of the system corresponds to the solution of the coupled equations (3.14)–(3.16). These generalized gap equations have been discussed in detail in [34] in the context of the $\mathbb{CP}^{N-1}$ defined on an interval with Dirichlet boundary conditions. Note that problems may arise if the potential $\lambda(x)$ admits a bound state with zero energy, $\omega_0 = 0$: the equation (3.14) would become ill defined due to an IR divergence.

The standard ground-state of the $\mathbb{CP}^{N-1}$ model corresponds to the homogeneous solution

$$\sigma(x) \equiv 0, \quad \lambda(x) \equiv \Lambda^2, \quad r - N \sum_k \frac{|f_k(x)|^2}{2\omega_k} = 0. \quad (3.17)$$

A strict formula of the gap equation with Pauli-Villars regularization is discussed in the next subsection. Note that in this case the spectrum of n_i fields is purely a continuum.

In ref. [39], by using a map from the chiral Gross-Neveu model [41, 42] to the $\mathbb{CP}^{N-1}$ model, it is claimed that a soliton-like configuration

$$\lambda = \Lambda^2 \left(1 - \frac{2}{\cosh^2 \Lambda x} \right), \quad \sigma(x) = C \frac{\Lambda}{\cosh \Lambda x}, \quad (3.18)$$

where the center of the soliton x_0 is taken at 0, satisfies the generalized gap equations (3.14)–(3.16). (More general solutions have subsequently been studied in [43].) In Gorsky et al. [38] the energy density of this soliton solution was evaluated and found to be lower than the one of the standard homogeneous confining vacuum; hence the claim of the potential instability of the standard vacuum.

Such a claim, however, is problematic, as there is a zero-energy bound state for each n_i field in the potential (3.18). The contribution from the latter in eq. (3.14) seems to be mysteriously missing in their proof [38, 39].

Note that in the $\mathbb{CP}^{N-1}$ system with Dirichlet boundary conditions at $x = \pm \frac{L}{2}$ studied in refs. [34, 44, 45], the classical field $\sigma(x)$ solves eq. (3.16) and obeys the boundary condition

$$n\left(-\frac{L}{2}\right) = n\left(\frac{L}{2}\right) = \sqrt{r}. \quad (3.19)$$

In other words, the $\mathbb{CP}^{N-1}$ constraint

$$\sum_{i=1}^N n_i^\dagger n_i = r, \quad (3.20)$$

is saturated by the classical component σ at the boundaries. On the other hand, the quantum fluctuations are required to vanish at the boundaries

$$n_i\left(\pm \frac{L}{2}\right) = 0, \quad i = 1, 2, \dots, N-1. \quad (3.21)$$

The difference in the boundary conditions (3.19) and (3.21), explains why a zero-mode $f_0(x) \propto \sigma(x)$ does not appear in the sum over quantum fluctuations of other components n_i ($i \neq N$) in the gap equation for the finite-width $\mathbb{CP}^{N-1}$ system.⁸ The classical function $\sigma(x)$, although normalizable (it diverges logarithmically at the boundaries), does not belong to the domain of the self-adjoint operator.

In the background of the “presumed” soliton potential $\lambda(x)$ of eq. (3.18), the n_i fields have one bound-state of zero energy each,

$$f_0(x) \propto \frac{1}{\cosh \Lambda x}, \quad \omega_0 = 0, \quad (3.22)$$

which is normalizable, satisfying the boundary condition $f_0(x = \pm\infty) = 0$, and therefore *must* be taken into account. It is orthogonal to the continuum modes $f_k(x)$. The presence of the zero-mode contribution, the infrared divergent last term in eq. (3.14), means that the gap equation is not satisfied by $\lambda(x)$ of eq. (3.18).

Another way to show that eq. (3.18) does not constitute a solution, is to consider a variational search for the solutions of the coupled equations eqs. (3.14)–(3.16), with respect to $\lambda(x), \sigma(x)$. For $\lambda(x), \sigma(x)$ near the configuration eq. (3.18), there is a bound state $f_0(x)$ for each $n_i(x)$: it is a near zero-energy bound-state mode for n_i : there are no reasons to omit it from the sum in eq. (3.14). As one approaches the configuration eq. (3.18), the failure of the gap equation eq. (3.14) is exacerbated. The variational search for $\lambda(x), \sigma(x)$ will drive one farther and farther away from eq. (3.18).

⁸Of course, it is crucial — and it was verified — that there are indeed no other zero energy solutions of $(-\partial^2 + \lambda(x)) f_0(x) = 0$, satisfying the boundary condition (3.21).

We conclude that the soliton eq. (3.18) is not a solution of the generalized gap equation.⁹ This also shows that the vacuum cannot be a lattice of solitons of this type, as suggested in [38]. A more general reason for the uniqueness and stability of the vacuum is given in the next two subsections.

3.3 Uniqueness of the saddle point on the real axis

We now make an even stronger statement about the uniqueness of the solution of the gap equation. We prove below that, under some generic assumptions, there are no solutions of eqs. (3.14)–(3.16) other than the standard confining vacuum.

Let us consider an arbitrary real function $\lambda(x)$ (as well as $A_x(x)$) with periodicity $\lambda(x+L) = \lambda(x)$ and assume that all mass-squared eigenvalues of

$$(-D_x^2 + \lambda(x)) f_n(x) = \omega_n^2 f_n(x), \quad (3.24)$$

are positive definite: $\omega_n^2 > 0$. We expect that the gauge fields vanish at the saddle point and hence just set $A_t = 0$, for simplicity. After integrating the n_i field fluctuations, the pseudo free energy is given by¹⁰

$$\begin{aligned} F_\lambda &= T \sum_{(n,m) \in \mathbb{Z}^2} \sum_I c_I \ln \left((2\pi nT)^2 + \omega_m^2 + \lambda_I \right) + \int_0^L dx (-\lambda(x)r + \mathcal{E}_{uv}) \\ &= \sum_{m \in \mathbb{Z}} \sum_I c_I \rho(\omega_m^2 + \lambda_I) + \int_0^L dx (-\lambda(x)r + \mathcal{E}_{uv}), \end{aligned} \quad (3.25)$$

where the function $\rho(\lambda)$ is defined as

$$\rho(\lambda) = 2T \ln \left(2 \sinh \left(\frac{\sqrt{\lambda}}{2T} \right) \right), \quad \lim_{T \rightarrow 0} \rho(\lambda) = \sqrt{\lambda}. \quad (3.26)$$

Here the well-known formula (B.3) for a harmonic oscillator with Pauli-Villars regularization has been used. The gap equation for $\lambda(x)$ is given by

$$0 = \frac{\delta F_\lambda}{\delta \lambda(x)} \Big|_{\text{saddle pt.}} = \sum_{n \in \mathbb{Z}} \sum_I c_I \rho'(\omega_n^2 + \lambda_I) |f_n(x)|^2 \Big|_{\text{saddle pt.}} - r, \quad (3.27)$$

where use was made of the formula

$$\frac{\delta \omega_n^2}{\delta \lambda(x)} = |f_n(x)|^2, \quad \text{with} \quad \int_0^L dx \bar{f}_n(x) f_m(x) = \delta_{nm}, \quad (3.28)$$

⁹Another soliton-like configuration was proposed in [43] in a twisted version of the $\mathbb{CP}^{N-1}$ model. Their main result is in their eq. (3.5), where the function $\lambda(x)$ is given by the known Pöschl-Teller potential,

$$\lambda(x) = -\frac{2}{\cosh^2 x}. \quad (3.23)$$

This potential is actually the same as eq. (3.18) used in [39], just shifted by a negative constant. It means that the zero-mode wave function eq. (3.22) describes a *negative* mode in the potential eq. (3.23). It appears that the configurations considered there [43] thus suffer from instability.

¹⁰We recall that λ_I 's denote the Pauli-Villars regulator masses (squared), as in section 2.

derived from eq. (3.24). Furthermore we have¹¹

$$\frac{\delta f_n(x)}{\delta \lambda(y)} = \sum_{m(\neq n)} \frac{f_n(y) \bar{f}_m(y)}{\omega_n^2 - \omega_m^2} f_m(x). \quad (3.29)$$

Let us consider now a one-parameter family of functions interpolating between two candidate gap functions $\xi_1(x)$ and $\xi_2(x)$:

$$\lambda(x) = (1-s) \xi_1(x) + s \xi_2(x), \quad (3.30)$$

and define

$$\Delta_{nm} \equiv \int dx \frac{\partial \lambda(x)}{\partial s} f_n(x) \bar{f}_m(x), \quad (3.31)$$

from which we find

$$\frac{d\omega_n^2}{ds} = \Delta_{nn}, \quad \frac{d\Delta_{nn}}{ds} = 2 \sum_{m(\neq n)} \frac{|\Delta_{nm}|^2}{\omega_n^2 - \omega_m^2}. \quad (3.32)$$

Now, the pseudo free energy F_λ is a function of s , and we find

$$\frac{d^2 F_\lambda}{ds^2} = \frac{d}{ds} \sum_{I,n} c_I \rho'(\omega_n^2 + \lambda_I) \Delta_{nn} = - \sum_I c_I (\mathcal{A}_I + \mathcal{B}_I), \quad (3.33)$$

where $\mathcal{A}_I$ and $\mathcal{B}_I$ are defined by

$$\begin{aligned} \mathcal{A}_I &\equiv - \sum_{n \in \mathbb{Z}} \rho''(\omega_n^2 + \lambda_I) |\Delta_{nn}|^2, \\ \mathcal{B}_I &\equiv - \sum_{(n,m) \in \mathbb{Z}^2}^{n \neq m} \frac{\rho'(\omega_n^2 + \lambda_I) - \rho'(\omega_m^2 + \lambda_I)}{\omega_n^2 - \omega_m^2} |\Delta_{nm}|^2. \end{aligned} \quad (3.34)$$

We make now the following two natural assumptions:

$$\exists C_\lambda, \forall x : |\xi_1(x) - \xi_2(x)| \leq C_\lambda < \infty, \quad \sum_n \frac{1}{\omega_n^2} < \infty. \quad (3.35)$$

It can then be shown that the $I \neq 0$ terms vanish in the large regulator-mass limit

$$\lim_{\Lambda_I \rightarrow \infty} \mathcal{A}_I = 0, \quad \lim_{\lambda_I \rightarrow \infty} \mathcal{B}_I = 0, \quad \text{for } I \neq 0, \quad (3.36)$$

and $\mathcal{A}_0, \mathcal{B}_0$ have positive definite values (see appendix B.3 for the proof). Therefore, given two arbitrary different functions $\xi_1(x), \xi_2(x)$, the pseudo free energy F_λ is a *concave function* in s .¹²

$$\forall \xi_1(x), \forall \xi_2(x), \forall s : \quad \lim_{\lambda_{I \neq 0} \rightarrow \infty} \frac{d^2 F_\lambda}{ds^2} = -(\mathcal{A}_0 + \mathcal{B}_0) < 0. \quad (3.37)$$

¹¹For simplicity, here the eigenvalues are assumed to be non-degenerate, as $\omega_n^2 \neq \omega_{-n}^2$ due to a non-trivial A_x . The degenerate case is obtained by taking the limit $\omega_{-n}^2 \rightarrow \omega_n^2$. We also set diagonal components to zero, $u_n^n = 0$, with an infinitesimal unitary transformation $\delta f_n(x) = i \sum_m u_n^m f_m(x)$ which is irrelevant for eq. (3.32).

¹²For constant λ this result reduces to that found in section 2.1, see figure 1.

Let us now assume that ξ_1 and ξ_2 are two distinct solutions of the gap equation. In other words, we assume, for *reductio ad absurdum*, that the solution of the gap equation is not unique, and there are, e.g., two solutions, ξ_1 and ξ_2 . The function F then should satisfy

$$\left. \frac{dF_\lambda}{ds} \right|_{s=0} = \left. \frac{dF_\lambda}{ds} \right|_{s=1} = 0. \quad (3.38)$$

But such a function cannot exist, in view of eq. (3.37).

We conclude that the solution of the gap equation (3.27) is unique and therefore it must be the homogeneous vacuum discussed in section 2.

3.4 Maximization of the free energy

Formally the pseudo free energy F_λ must take a *local maximum* when evaluated on a solution of the gap equation as a function of λ . We checked this explicitly for formula (2.23) in figure 1. For an ordinary field, such as n_i or A_μ , this would signal the presence of a tachyonic instability but this is not the case for λ which does not have a canonical kinetic term.¹³ The integrand for the path integral can be regarded as a holomorphic function of λ . Therefore, by fixing both the initial point ($\lambda = -i\infty + \epsilon$) and the final point ($\lambda = i\infty + \epsilon$), the partition function is invariant under continuous transformations of the path while avoiding any of poles. Here we note that to make the partition function finite, the path integration with respect to λ must be along the imaginary axis for large $|\lambda|$. A real configuration of λ discussed here appears as an intersection between the real axis and such a path. In this sense, we can choose any real VEV of λ as long as the mass spectrum is positive definite. (Here one cannot choose λ so that the mass spectrum contain zero-modes or negative modes due to existence of poles as we noted). When we apply the saddle-point approximation to this path integral, however, we have to use a certain real VEV of λ as a saddle point $\lambda = \lambda_{\text{sp}}$ which satisfies the gap equation (2.10). (Here we assume that there is no *complex saddle point* which can contribute to the partition function.) Around this saddle point, it is natural to assume that Z_λ behaves as

$$\ln Z_\lambda = \text{const.} + a \beta (\lambda - \lambda_{\text{sp}})^2 + \dots, \quad (3.39)$$

and the pseudo free energy F_λ must take the *minimum* at the saddle point along the imaginary axis,

$$\left. \frac{\partial^2 F_\lambda}{\partial (\text{Im } \lambda)^2} \right|_{\lambda=\lambda_{\text{sp}}} = a \quad \Rightarrow \quad a \in \mathbb{R}_{>0}. \quad (3.40)$$

Thus this feature means that, formally, the free energy must take a local maximum at the saddle point along *the real axis*,

$$\left. \frac{\partial^2 F_\lambda}{\partial (\text{Re } \lambda)^2} \right|_{\lambda=\lambda_{\text{sp}}} = -a < 0. \quad (3.41)$$

Therefore, counter-intuitively, with a generic $\lambda \in \mathbb{R}_{>0}$, the pseudo free energy F_λ can (or must) take smaller values than the real free energy $F \equiv -T \ln Z$,

$$F_\lambda \leq F_{\lambda_{\text{sp}}} \simeq F. \quad (3.42)$$

¹³Discussions on the quantum mechanically generated “kinetic” term for the field λ can be found in [9].

A lesson which follows from these discussions is that even if one finds a field configuration with smaller free energy than that of the vacuum (such as the one in [38], or simply a generic configurations in figure 1, the right panel), this is not necessarily a signal of instability.

4 Twisted boundary conditions

In this section the analysis of section 2 will be repeated with a twisted boundary condition:

$$n_i(x, t + \beta) = n_i(x, t), \quad n_i(x + L, t) = e^{i\xi_i} n_i(x, t), \quad i = 1, 2, \dots, N. \quad (4.1)$$

By using the local U(1) symmetry of the model, one can set

$$\sum_i \xi_i = 0. \quad (4.2)$$

It is possible to define periodic fields, $\tilde{n}_i$, by

$$n_i(x, t) = e^{i\frac{\xi_i x}{L}} \tilde{n}_i(x, t), \quad (4.3)$$

where

$$\tilde{n}_i(x, t + \beta) = \tilde{n}_i(x, t), \quad \tilde{n}_i(x + L, t) = \tilde{n}_i(x, t), \quad (4.4)$$

but this introduces background gauge fields

$$D_\mu n_i = (\partial_\mu + iA_\mu) n_i = e^{i\frac{\xi_i x}{L}} \left(\partial_\mu + iA_\mu + i\delta_{\mu 2} \frac{\xi_i}{L} \right) \tilde{n}_i(x, t), \quad (4.5)$$

Dropping the tildes from now on, $\tilde{n}_i \rightarrow n_i$, we have

$$S_E = \int_0^\beta dt \int dx \left(|D_t n_i|^2 + |D_x n_i|^2 + \lambda(x)(|n_i|^2 - r) \right), \quad (4.6)$$

where

$$D_t n_i = (\partial_t + iA_t) n_i, \quad D_x n_i = \left(\partial_\mu + iA_x + i\frac{\xi_i}{L} \right) n_i. \quad (4.7)$$

Due to the periodicity of the system we can set the gauge fields $A_t = a_t$ and $A_x = a_x$ equal to constants, and also set $\lambda(x, t) = \lambda$ constant. By integrating out $n_i(x, t)$, the partition function Z_λ is given by

$$\begin{aligned} -\ln Z_\lambda = & \sum_i \sum_{n, m \in \mathbb{Z}} \sum_I c_I \ln \left(\left(\frac{2n\pi}{\beta} + a_t \right)^2 + \left(\frac{2m\pi}{L} + a_x + \frac{\xi_i}{L} \right)^2 + \lambda + \lambda_I \right) \\ & + \beta L (-\lambda r + \mathcal{E}_{uv}). \end{aligned} \quad (4.8)$$

For simplicity we consider only the $\mathbb{Z}_N$ symmetric form of the phases ξ_i :

$$\xi_i = \frac{2\pi i}{N}, \quad i = 1, 2, \dots, N. \quad (4.9)$$

A useful observation made in refs. [47, 48] is to combine the double sum $\sum_i \sum_m$ into a single sum $\sum_k$

$$k = Nm + i, \quad k \in \mathbb{Z}, \quad (4.10)$$

and the first line of eq. (4.8) formally remains the same, with the replacements

$$L \rightarrow NL, \quad \sum_i \rightarrow 1. \quad (4.11)$$

We thus have

$$\begin{aligned} -\ln Z_\lambda &= \sum_{n,k \in \mathbb{Z}} \sum_I c_I \ln \left(\left(\frac{2n\pi}{\beta} + a_t \right)^2 + \left(\frac{2k\pi}{NL} + a_x \right)^2 + \lambda + \lambda_I \right) \\ &\quad + \beta L (-\lambda r + \mathcal{E}_{uv}). \end{aligned} \quad (4.12)$$

The analysis of section 2 can now be repeated step by step. Eq. (2.17) is then replaced by

$$\begin{aligned} &-\ln Z_\lambda - \beta L (-\lambda r + \mathcal{E}_{uv}) \\ &= -\lim_{s \rightarrow 0} \sum_{n,k,I} \frac{c_I}{\Gamma(s+1)} \int_0^\infty dt t^{s-1} e^{-t \left(\left(\frac{2n\pi}{\beta} + a_t \right)^2 + \left(\frac{2k\pi}{NL} + a_x \right)^2 + \lambda + \lambda_I \right)} \\ &= -\lim_{s \rightarrow 0} \sum_{n',k',I} \frac{\beta NL c_I}{4\pi \Gamma(s+1)} e^{in'\beta a_t + ik'NL a_x} \int_0^\infty dt t^{s-2} e^{-\frac{(n'\beta)^2 + (k'NL)^2}{4t} - t(\lambda + \lambda_I)}. \end{aligned} \quad (4.13)$$

Note that, remarkably, the explicit factor of N which is absent in the second line here — as compared to the same line in eq. (2.17) — reappears in the third line in front of the expression after use of the crucial identity (2.16). In the second line, divergences come from the infinite sum over n and k , whereas in the last line, the divergent part arises from the $n' = k' = 0$ term only. The $n' = k' = 0$ part (which is independent of the twisting) is given, as before, by

$$\lambda \left(\ln \frac{\lambda}{e} + \sum_{I \neq 0} c_I \ln \lambda_I \right) + \sum_{i \neq 0} c_I \lambda_i \ln \lambda_i + \mathcal{O}(\Lambda_{uv}^{-2}), \quad (4.14)$$

whereas a generic $(n', k') \neq (0, 0)$ term is

$$\begin{aligned} &\int_0^\infty dt t^{-2} e^{-\frac{(n'\beta)^2 + (k'NL)^2}{4t} - t(\lambda + \lambda_I)} \\ &= 4 \sqrt{\frac{\lambda + \lambda_I}{(n'\beta)^2 + (k'NL)^2}} K_1 \left(\sqrt{(\lambda + \lambda_I) ((n'\beta)^2 + (k'NL)^2)} \right). \end{aligned} \quad (4.15)$$

Note that only the $I = 0$ term survives when the UV regulator masses $\lambda_{1,2,3}$ are sent to ∞ ($\Lambda_{uv} \rightarrow \infty$). Therefore, we find, after renaming (n', k') as (n, k) ,

$$\begin{aligned} -\frac{\ln Z_\lambda}{\beta L} &= -\frac{N}{4\pi} \lambda \ln \frac{\lambda}{e\Lambda^2} + M \\ &\quad - \sum_{(n,k) \in \mathbb{Z}^2 \setminus \{(0,0)\}} \frac{N}{\pi} \cos(n\beta a_t) \cos(kNL a_x) \\ &\quad \times \sqrt{\frac{\lambda}{(n\beta)^2 + (kNL)^2}} K_1 \left(\sqrt{\lambda((n\beta)^2 + (kNL)^2)} \right), \end{aligned} \quad (4.16)$$

where Λ, M are determined by eq. (2.21). We again have the maximum Z_λ with respect to (a_t, a_x) with a given λ at $(0, 0)$. Thus we can set $a_x = a_t = 0$ and

$$-\frac{\ln Z_\lambda}{\beta L} = -\frac{N}{4\pi} \lambda \ln \frac{\lambda}{e\Lambda^2} + M - \sum_{(n,k) \in \mathbb{Z}^2 \setminus \{(0,0)\}} \frac{N}{\pi} \sqrt{\frac{\lambda}{(n\beta)^2 + (kNL)^2}} K_1 \left(\sqrt{\lambda((n\beta)^2 + (kNL)^2)} \right). \quad (4.17)$$

The gap equation is

$$0 = -\frac{1}{\beta L} \frac{\partial \ln Z_\lambda}{\partial \lambda} = -\frac{N}{4\pi} \ln \frac{\lambda}{\Lambda^2} + \frac{N}{2\pi} \sum_{n,k \in \mathbb{Z} \setminus \{(0,0)\}} K_0 \left(\sqrt{\lambda((n\beta)^2 + (kNL)^2)} \right). \quad (4.18)$$

This equation has a unique solution for λ , with $\lambda \geq \Lambda^2$, as in the non-twisted system, (2.25).

To conclude, the effect of the twist is to suppress the contributions from $k > 0$: the effect on the generation of the mass gap is quantitative but not qualitative. The system with a small or finite L , behaves as one with a large width, $NL \rightarrow \infty$. This is consistent with the original idea of the Eguchi-Kawai [49] volume independence in 4D pure YM theory, and in this context, we agree with [48].

5 θ dependence of the free energy density

The dependence of the energy density on the θ term eq. (1.6) has been debated extensively in the literature. We do not have much new to add here, except for making a few remarks for completeness.

In the standard extended 2D spacetime, it was explained [1, 2, 50] why in the $\frac{1}{N}$ expansion, the θ dependence appears at the leading perturbative order $\mathcal{O}(\frac{1}{N})$ and is not exponentially suppressed, as expected from the instantons, by e^{-cN} . To leading order in $\mathcal{O}(\frac{1}{N})$ the θ dependence of the free energy density is

$$\mathcal{F}(\theta) = \frac{3\Lambda^2}{2\pi} \frac{\theta^2}{N} + \dots, \quad (5.1)$$

and this is consistent with the classical electric field energy,

$$\frac{E(\theta)}{L} = \frac{1}{2} \left(\frac{e\theta}{2\pi} \right)^2, \quad (5.2)$$

in the background electric field

$$\mathcal{E} = \frac{e\theta}{2\pi} \quad (5.3)$$

(or charges $\pm \frac{\theta}{2\pi}$ at $x = \mp\infty$). The coupling constant

$$e^2 = \frac{12\pi\Lambda^2}{N} \quad (5.4)$$

is inferred from the coefficient of the kinetic term $F_{\mu\nu}^2$ generated by quantum effects (one-loop diagram of the n_i fields).

Also, the nonanalyticity in θ , a double vacuum degeneracy and a spontaneous breaking of CP symmetry which occur at $\theta = \pi$, have been shown to hold in the large- N limit. Such a result has been confirmed recently by using the ideas of generalized symmetry and mixed anomalies [51].

All this applies in the case of infinite line $L = \infty$ and at zero temperature ($\beta = \infty$). As noted by many authors, there is a subtle issue of the different order in which the limits are taken, $\beta, L \rightarrow \infty$, versus $N \rightarrow \infty$. Clearly, the well-known large- N results reviewed above correspond to taking the thermodynamic limit first,

$$\lim_{N \rightarrow \infty} \left(\lim_{\beta, L \rightarrow \infty} Z \right). \quad (5.5)$$

The θ dependence on an Euclidean torus with equal lengths ($L = \beta$) has been studied in ref. [52], by summing over the topological sectors. Their result can be easily generalized to our situation since it just depends on the volume $V = \beta L$ and is thus

$$\mathcal{F}(\theta) = -\frac{1}{\beta L} \log \frac{\vartheta_3\left(\frac{\theta}{2}, q\right)}{\vartheta_3(0, q)}, \quad (5.6)$$

where

$$q \equiv e^{i\pi\tau}, \quad \tau = \frac{iN}{6\beta L\Lambda^2}, \quad |q| < 1, \quad (5.7)$$

and $\vartheta_3(z, q)$ is Jacobi's theta function:

$$\vartheta_3(z, q) = 1 + 2 \sum_{n=1}^{\infty} q^{n^2} \cos 2nz. \quad (5.8)$$

In the thermodynamic limit (5.5),

$$N \ll \beta L\Lambda^2, \quad q = e^{-\frac{\pi N}{6\beta L\Lambda^2}} \simeq 1, \quad (5.9)$$

one first uses the re-summation formula

$$\vartheta_3(z, e^{i\pi\tau}) = \frac{\sqrt{i}}{\sqrt{\tau}} e^{-\frac{iz^2}{\pi\tau}} \vartheta_3\left(\frac{z}{\tau}, e^{-\frac{i\pi}{\tau}}\right), \quad (5.10)$$

and then takes the limit

$$e^{-\frac{i\pi}{\tau}} = e^{-\frac{6\pi\beta L\Lambda^2}{N}} \rightarrow 0, \quad e^{-\frac{iz^2}{\pi\tau}} \simeq 1 - \frac{3\beta L\Lambda^2}{2\pi N} \theta^2 + \dots, \quad (5.11)$$

leading to the well-known large- N result eq. (5.1). Also a cusp at $\theta = \pi$ and a first-order transition between two degenerate vacua is predicted [52].

In the opposite limit in which

$$N \gg \beta L\Lambda^2, \quad q = e^{-\frac{\pi N}{6\beta L\Lambda^2}} \rightarrow 0, \quad (5.12)$$

the formula (5.6) yields straightforwardly an instanton-like behavior,

$$\mathcal{F}(\theta) \simeq \frac{1}{\beta L} e^{-\frac{\pi N}{6\beta L \Lambda^2}} (1 - \cos \theta) + \dots \quad (5.13)$$

As pointed out in ref. [52], most lattice studies of the θ dependence in the $\mathbb{CP}^{N-1}$ model are done so far in the regime of thermodynamical limit (5.9), and the instanton behavior (5.13) which should appear in the limit (5.12), has not yet been corroborated, to the best of our knowledge.

6 Concluding remarks

In this paper we examined the solution of the gap equation of the large- N $\mathbb{CP}^{N-1}$ sigma model on a Euclidean torus of arbitrary shape and size, L and $\beta = T^{-1}$. It was found that the system has a unique ground-state for any L, β , in which a mass gap for the n_i fields

$$\langle \lambda \rangle \geq \Lambda^2, \quad (6.1)$$

is generated. The system is in a phase analogous to the confinement phase of QCD. In particular, a “deconfinement” phase proposed to appear at small L in some literature is shown not to occur in our system. In particular a spontaneous symmetry breaking never occurs. Another interesting proposal, that the system possesses soliton-like inhomogeneous solutions of the gap equation, and that the standard homogeneous ground-state of the $2D$ $\mathbb{CP}^{N-1}$ sigma model is unstable against decay into a lattice of such soliton like solutions, has been proven not to be justified.

We have presented a detailed discussion on the possible origin of the discrepancies found among different literatures and with the present work. The principal cause for such differences is traced to the presence of certain zeromodes, whose proper treatment (according to us) leads to the said differences.

Concerning the “deconfinement” phase, a short comment on some remarks made in ref. [47] is worthwhile. In this paper, the phases $e^{i\xi_i}$ appearing in the general twisted boundary condition (4.1), are considered as further dynamical degrees of freedom of the system. Associated with the global symmetry of the $\mathbb{CP}^{N-1}$ system, the authors introduce the concept of $SU(N)$ “center symmetry”, which acts on the “line operators”:

$$e^{i \oint A_k}, \quad A_k = \frac{i}{2} \left(n_k^\dagger \partial_\mu n_k - \partial_\mu n_k^\dagger n_k \right) dx^\mu, \quad (\text{no sum over } k). \quad (6.2)$$

The periodicity conditions such as the strict one

$$n_i(x, t + \beta) = n_i(x, t), \quad n_i(x + L, t) = n_i(x, t), \quad i = 1, 2, \dots, N \quad (6.3)$$

and a twisted one,

$$n_i(x, t + \beta) = n_i(x, t), \quad n_i(x + L, t) = e^{i\mu_i} n_i(x, t), \quad \mu_i = \frac{2\pi i}{N}, \quad i = 1, 2, \dots, N. \quad (6.4)$$

considered in section 2 and section 4, respectively, correspond to the VEV of $e^{i\mu_i}$ which breaks the $SU(N)$ “center symmetry” (in the case of eq. (6.3)), and *vis a vis*, which is center symmetric (for eq. (6.4)). Relying on an analogy with the physics of the 4D $SU(N)$ Yang-Mills theory, the authors of [47] then state that the background eq. (6.3) corresponds to a “deconfinement center broken phase”, and the background eq. (6.4) an analogue of “adjoint Higgs” (confinement?) phase. As shown in the main text here, however, for both boundary conditions the dynamics of the system describes a confinement phase with dynamical generation of the mass gap, $\langle \lambda \rangle \geq \Lambda^2$. One has the impression that, in using the concepts such as center symmetry, line operators, breaking of center symmetry as a criterion of a Higgs phase, etc., the formal analogy with the 4D YM theory has been stretched too far, beyond what is justified by the actual analysis of the dynamics.

In conclusion, the large N $\mathbb{CP}^{N-1}$ model on a 2D torus has a unique phase that looks like a confinement phase in the thermodynamic limit (large L and zero temperature $\beta = \infty$). The mass gap $\lambda \geq \Lambda^2$ is generated for all (β, L) , and we find no phase transitions as β and L are varied. We critically commented on a number of papers in the literature, analyzing carefully the points which may have caused the discrepancies. In particular, soliton-like solutions such as (3.12) [39], which are analogues of those discovered in the chiral Gross-Neveu model [41, 42] and which could have caused instabilities of the vacua [38], are actually found to be absent in the $\mathbb{CP}^{N-1}$ model. Moreover, we have seen that a field configuration with smaller free energy than the vacuum, unless it is proven to be a correct solution of the gap equations, is not necessarily a signal of instability if the field λ is involved. The elegant map from the chiral GN model to the $\mathbb{CP}^{N-1}$ model used by Yoshii and Nitta [39, 43, 46], seems to fail subtly to produce the solutions of the gap equation for the latter from those in the former, due to the zero or negative modes which affect differently the two distinct physical systems.

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Ref.	[1]	[2, 50]	[28, 30, 31]	[52]	[47]	[48]
$\mathcal{C}$	$\frac{N}{2f}$	$\frac{N}{g^2}$	$\frac{4\pi}{g^2}$	$\frac{N}{2g^2}$	$\frac{2}{g^2}$	$\frac{N}{f}$

Table 1. Different definitions of the coupling constant used in the literature.

A The coupling constant

The action is

$$S = \int d^2x \left[\mathcal{C} (D_\mu n_i)^\dagger D_\mu n_i - \lambda (n_i^\dagger n_i - 1) + \frac{i\theta}{2\pi} \epsilon_{\mu\nu} \partial_\mu A_\nu \right], \quad (\text{A.1})$$

where the Lagrange multiplier $\lambda(x)$ is imposing the unit radius of $\mathbb{CP}^{N-1}$. One then proceeds to path integrate

$$W = \int \mathcal{D}n^* \mathcal{D}n \mathcal{D}A_\mu \mathcal{D}\lambda e^{-S} \quad (\text{A.2})$$

to get the quantized version of the theory. The coefficient $\mathcal{C}$ in front of the kinetic term for n_i is the inverse bare coupling constant (or the square thereof); its choice is arbitrary and corresponds to a different definition of the coupling constant adopted. Many different choices are used in the literature, some of which are summarized in table 1. By rescaling the n_i fields, so as to render the kinetic terms of n_i canonical,

$$n_i \rightarrow \frac{1}{\sqrt{\mathcal{C}}} n_i, \quad \lambda \rightarrow \mathcal{C} \lambda, \quad (\text{A.3})$$

the action becomes

$$S = \int d^2x \left[(D_\mu n_i)^* D_\mu n_i - \lambda (n_i^* n_i - \mathcal{C}) + \frac{i\theta}{2\pi} \epsilon_{\mu\nu} \partial_\mu A_\nu \right]. \quad (\text{A.4})$$

In the large- N approximation, after the functional integration over the n_i fields, the gap equation shows the dynamically generated mass of the n_i fields which on $\mathbb{R}^2$ is

$$\sqrt{\langle \lambda \rangle} \equiv \Lambda = \mu e^{-\frac{2\pi\mathcal{C}(\mu)}{N}}, \quad (\text{A.5})$$

which is μ independent.

There is no particular reason to prefer one choice of $\mathcal{C}$ with respect to another. The one we use here is inherited from the nonAbelian vortex literature [28]–[31]. Here the $\mathbb{CP}^{N-1}$ model emerges as a long-distance effective action describing the fluctuations of the orientational zero-modes of the nonAbelian vortex [27, 28], which emerges, upon symmetry breaking of a $4D$ gauge theory,

$$(\text{SU}(N) \times \text{U}(1))_{\text{c}} \times \text{SU}(N)_{\text{f}} \xrightarrow{v} \text{SU}(N)_{\text{cf}} \quad (\text{A.6})$$

in a color-flavor locked Higgs vacuum, as a soliton excitation. The factor $\frac{4\pi}{g^2}$ in front of the $\mathbb{CP}^{N-1}$ action appears in a BPS saturated system, g being the $4D$ gauge coupling constant

at the mass scale v of the symmetry breaking (A.6). From the point of view of the RG flow of the $2D \mathbb{CP}^{N-1}$ model at the mass scale $\mu \leq v$, $g(v)$ plays the role of the UV cut-off (bare) coupling constant. Remarkably, the mass generated by the $2D \mathbb{CP}^{N-1}$ dynamics in the far infrared,

$$\sqrt{\langle \lambda \rangle} \equiv \Lambda = \mu e^{-\frac{8\pi^2}{Ng(\mu)^2}}, \quad (\text{A.7})$$

coincides with the RG invariant mass of the $4D \text{SU}(N)$ gauge theory in the Coulomb phase ($v = 0$). In other words, the $2D$ dynamics of the n_i fields appears to “carry on” the RG flow of the $4D \text{SU}(N)$ gauge theory (which was frozen at v) below the mass scale v . This is a manifestation of the beautiful $2D$ - $4D$ duality, first conjectured by N. Dorey [53] and subsequently confirmed explicitly in refs. [27, 30].

B Pauli-Villars regularization

Let us consider the following Pauli-Villars regularization

$$f(\lambda) \Big|_{\text{reg}(\lambda)} \equiv \sum_I c_I f(\lambda + \lambda_I), \quad (\text{B.1})$$

where $\{c_i, \lambda_i\}$ is determined so that, with a given $p \in \mathbb{Z}_{\geq 0}$,

$$c_0 = 1, \lambda_0 = 0, \quad \text{and} \quad \lambda^k \Big|_{\text{reg}(\lambda)} = \sum_I c_I \lambda_I^k = 0, \quad \text{for } k = 0, 1, 2, \dots, p. \quad (\text{B.2})$$

For instance, with $p \geq 0$, we rediscover the well-known formula for the harmonic oscillator,

$$\begin{aligned} & \sum_{n \in \mathbb{Z}} \ln \left(\left(\frac{2\pi n}{\beta} \right)^2 + \omega^2 \right) \Big|_{\text{reg}(\omega^2)} \\ &= \ln \omega^2 + 2 \sum_{n=1}^{\infty} \left[\ln \left\{ 1 + \left(\frac{\beta\omega}{2\pi n} \right)^2 \right\} + \ln \left(\frac{2\pi n}{\beta} \right)^2 \right] \Big|_{\text{reg}(\omega^2)} \\ &= 2 \ln \left[\omega \prod_{n=1}^{\infty} \left\{ 1 + \left(\frac{\beta\omega}{2\pi n} \right)^2 \right\} \right] \Big|_{\text{reg}(\omega^2)} \\ &= 2 \ln \left(2 \sinh \frac{\beta\omega}{2} \right) \Big|_{\text{reg}(\omega^2)}. \end{aligned} \quad (\text{B.3})$$

Note that this identity is exact for arbitrary finite regulator masses.

B.1 $G_s(x, \epsilon)$ and identities

Let us define a function $G_s(x, \epsilon)$ as ($s \in \mathbb{C} \setminus \mathbb{Z}_{\geq 0}$, $x, \epsilon \in \mathbb{R}_{>0}$),

$$G_s(x, \epsilon) \equiv \frac{1}{\Gamma(-s)} \int_{\epsilon}^{\infty} \frac{dt}{t^{s+1}} e^{-xt}. \quad (\text{B.4})$$

For $\text{Re } s < 0$, taking the limit of $\epsilon \rightarrow 0$ gives just a familiar integration

$$\lim_{\epsilon \rightarrow 0} G_s(x, \epsilon) = x^s, \quad \Leftrightarrow \quad \frac{1}{\Gamma(u)} \int_0^{\infty} dt t^{u-1} e^{-xt} = \frac{1}{x^u} \quad \text{for } \text{Re}(u) > 0. \quad (\text{B.5})$$

For $\text{Re } s > 0$, this limit gives divergences and the following reduction is found

$$\begin{aligned} G_s(x, \epsilon) &= \frac{1}{(-s)\Gamma(-s)} \left[\frac{e^{-xt}}{t^s} \right]_{\epsilon}^{\infty} + \frac{x}{(-s)\Gamma(-s)} \int_{\epsilon}^{\infty} \frac{dt}{t^s} e^{-xt} \\ &= x G_{s-1}(x, \epsilon) - \frac{1}{\Gamma(-(s-1))} \frac{e^{-x\epsilon}}{\epsilon^s}. \end{aligned} \quad (\text{B.6})$$

Therefore, we find, for $\text{Re } s < 0, k \in \mathbb{Z}_{>0}$,

$$G_{s+k}(x, \epsilon) = x^k G_s(x, \epsilon) - \sum_{n=1}^k \frac{x^{k-n}}{\Gamma(-(s+n-1)) \epsilon^{s+n}} e^{-x\epsilon}, \quad (\text{B.7})$$

and all divergent parts at $\epsilon = 0$ are proportional to

$$\frac{1}{\epsilon^{s+k}} (x\epsilon)^n, \quad \text{for } n = 0, 1, 2, \dots, k-1. \quad (\text{B.8})$$

Therefore, using Pauli-Villars regularization, the region of s where the limit of $\epsilon \rightarrow 0$ converges is extended as

$$\begin{aligned} \forall s \in \mathbb{C} \setminus \mathbb{Z}_{\geq 0} | \text{Re } s < p+1 \\ \int_0^{\infty} \frac{dt}{t^{s+1}} e^{-\lambda t} \Big|_{\text{reg}(\lambda)} = \lim_{\epsilon \rightarrow 0} \Gamma(-s) G_s(\lambda, \epsilon) \Big|_{\text{reg}(\lambda)} = \Gamma(-s) \lambda^s \Big|_{\text{reg}(\lambda)}. \end{aligned} \quad (\text{B.9})$$

Some integer cases, $n = 0, 1, \dots, p$, are defined by taking the limit as

$$\begin{aligned} \int_0^{\infty} \frac{dt}{t^{n+1}} e^{-\lambda t} \Big|_{\text{reg}(\lambda)} &= \lim_{s \rightarrow 0} \lim_{\epsilon \rightarrow 0} \Gamma(-(n+s)) G_{n+s}(\lambda, \epsilon) \Big|_{\text{reg}(\lambda)} \\ &= \lim_{s \rightarrow 0} \frac{\pi \lambda^{n+s}}{\sin(-\pi(n+s)) \Gamma(s+n+1)} \Big|_{\text{reg}(\lambda)} \\ &= \lim_{s \rightarrow 0} (-1)^{n+1} \frac{\lambda^{n+s}}{s n!} \Big|_{\text{reg}(\lambda)} = (-1)^{n+1} \frac{\lambda^n}{n!} \lim_{s \rightarrow 0} \frac{e^{s \log \lambda} - 1}{s} \Big|_{\text{reg}(\lambda)} \\ &= \frac{(-1)^{n+1}}{n!} \lambda^n \log \lambda \Big|_{\text{reg}(\lambda)}. \end{aligned} \quad (\text{B.10})$$

Therefore, we find the useful identities

$$\begin{aligned} \int_0^{\infty} \frac{dt}{t} e^{-\lambda t} \Big|_{\text{reg}(\lambda)} &= -\ln \lambda \Big|_{\text{reg}(\lambda)} & \text{for } p \geq 0, \\ \int_0^{\infty} \frac{dt}{t^{\frac{3}{2}}} e^{-\lambda t} \Big|_{\text{reg}(\lambda)} &= -2\sqrt{\pi \lambda} \Big|_{\text{reg}(\lambda)} & \text{for } p \geq 0, \\ \int_0^{\infty} \frac{dt}{t^2} e^{-\lambda t} \Big|_{\text{reg}(\lambda)} &= \lambda \ln \lambda \Big|_{\text{reg}(\lambda)} & \text{for } p \geq 1. \end{aligned} \quad (\text{B.11})$$

Here the right hand sides of these identities cannot be defined without regularization.

B.2 Eq. (2.31): regularized infinite sums

Let us apply the Abel-Plana summation formula

$$\begin{aligned} \sum_{n=0}^{N_{\text{uv}}} f(n) &= \frac{1}{2}f(0) + \frac{1}{2}f(N_{\text{uv}}) + \int_0^{N_{\text{uv}}} dx f(x) \\ &\quad + i \int_0^\infty dy \frac{f(iy) - f(-iy)}{e^{2\pi y} - 1} \\ &\quad - i \int_0^\infty dy \frac{f(N_{\text{uv}} + iy) - f(N_{\text{uv}} - iy)}{e^{2\pi y} - 1}, \end{aligned} \quad (\text{B.12})$$

to the function $f(n) = \sqrt{n^2 + \nu^2}$ by assuming $1, \nu \ll N_{\text{uv}}$. The first line of the right-hand side gives

$$\begin{aligned} &\frac{\nu}{2} + \frac{\sqrt{N_{\text{uv}}^2 + \nu^2}}{2} + \frac{1}{2} \left\{ N_{\text{uv}} \sqrt{N_{\text{uv}}^2 + \nu^2} + \nu^2 \ln \frac{N_{\text{uv}} + \sqrt{N_{\text{uv}}^2 + \nu^2}}{\nu} \right\} \\ &= \frac{N_{\text{uv}}(N_{\text{uv}} + 1)}{2} + \frac{\nu^2}{2} \ln \frac{2N_{\text{uv}}}{\nu} + \frac{\nu(\nu + 2)}{4} + \mathcal{O}\left(\frac{1}{N_{\text{uv}}^2}\right). \end{aligned} \quad (\text{B.13})$$

Due to the existence of branch cuts on the imaginary axis, the second line gives

$$i \int_0^\infty dy \frac{f(iy) - f(-iy)}{e^{2\pi y} - 1} = -2 \int_\nu^\infty dy \frac{\sqrt{y^2 - \nu^2}}{e^{2\pi y} - 1} \equiv -\mathcal{K}(\nu), \quad (\text{B.14})$$

which vanishes in the large ν limit. Actually, using reparametrization of the integral variable $y = \frac{\nu(t+1)}{2}$, $\mathcal{K}(\nu)$ can be expressed in terms of the modified Bessel function of the second kind $K_1(x)$ as,

$$\begin{aligned} \mathcal{K}(\nu) &= \int_1^\infty \frac{dt}{2t} \left(t^2 + \frac{1}{t^2} - 2 \right) \frac{\nu^2}{e^{\pi\nu(t+1/t)} - 1} \\ &= \int_0^\infty \frac{dt}{2t} \left(\frac{1}{t^2} - 1 \right) \frac{\nu^2}{e^{\pi\nu(t+1/t)} - 1} = \sum_{n=1}^\infty \nu^2 \int_0^\infty \frac{dt}{2t} \left(\frac{1}{t^2} - 1 \right) e^{-\pi n \nu(t+1/t)} \\ &= \sum_{n=1}^\infty \frac{\nu}{\pi n} K_1(2\pi n \nu). \end{aligned} \quad (\text{B.15})$$

The last term gives a constant term in the large N_{uv} limit,

$$\begin{aligned} 2 \int_0^\infty dy \frac{\text{Im} \sqrt{(N_{\text{uv}} + iy)^2 + \nu^2}}{e^{2\pi y} - 1} &\xrightarrow{N_{\text{uv}} \rightarrow \infty} 2 \int_0^\infty dy \frac{y}{e^{2\pi y} - 1} = \mathcal{K}(0), \\ \mathcal{K}(0) &= 2 \sum_{n=1}^\infty \int_0^\infty dy y e^{-2\pi n y} = \frac{1}{2\pi^2} \sum_{n=1}^\infty \frac{1}{n^2} = \frac{1}{12}, \end{aligned} \quad (\text{B.16})$$

where we used

$$\text{Im} \sqrt{(N_{\text{uv}} + iy)^2 + \nu^2} = y \left(1 + \mathcal{O}\left(\frac{\nu^2}{N_{\text{uv}}^2 + y^2}\right) \right). \quad (\text{B.17})$$

In summary, we get the following result

$$\begin{aligned}\mathcal{H}(\nu^2) &\equiv \sum_{n=1}^{\infty} \left\{ \sqrt{n^2 + \nu^2} - n - \frac{\nu^2}{2n} \right\} \\ &= \frac{1}{12} - \frac{\nu}{2} + \frac{\nu^2}{2} \left(\ln \frac{2}{\nu} + \frac{1}{2} - \gamma \right) - \sum_{n=1}^{\infty} \frac{\nu}{\pi n} K_1(2\pi n \nu),\end{aligned}\quad (\text{B.18})$$

where γ is Euler's constant defined by

$$\gamma \equiv \lim_{N_{\text{uv}} \rightarrow \infty} \left(\sum_{n=1}^{N_{\text{uv}}} \frac{1}{n} - \log N_{\text{uv}} \right). \quad (\text{B.19})$$

Using the above result, the infinite sum $\sum_{n=1}^{\infty} n$ is regularized ($p \geq 1$) as

$$\begin{aligned}\sum_I c_I \sum_{n \in \mathbb{Z}} \sqrt{n^2 + \lambda_I} &= \sum_I c_I \sqrt{\lambda_I} + 2 \sum_I c_I \mathcal{H}(\lambda_I) \\ &= \sum_{I \neq 0} c_I \left\{ \frac{1}{6} + \frac{\lambda_I}{2} \left(\ln \frac{4}{\lambda_I} + 1 - 2\gamma \right) + \mathcal{O}(e^{-2\pi\sqrt{\lambda_I}}) \right\} \\ &\stackrel{\lambda_I \neq 0 \sim \infty}{=} -\frac{1}{6} - \frac{1}{2} \sum_{I \neq 0} c_I \lambda_I \ln \lambda_I,\end{aligned}\quad (\text{B.20})$$

where $2\zeta(-1) = -\frac{1}{6}$ appears as it should be. Similarly we find,

$$\begin{aligned}\sum_I c_I \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2 + \lambda_I}} &= \sum_I c_I \sum_{n=1}^{\infty} \left\{ \frac{1}{\sqrt{n^2 + \lambda_I}} - \frac{1}{n} \right\} = 2 \sum_{I \neq 0} c_I \frac{d\mathcal{H}(\lambda_I)}{d\lambda_I} \\ &= \sum_{I \neq 0} c_I \left\{ -\frac{1}{2\sqrt{\lambda_I}} + \frac{1}{2} \ln \frac{4}{\lambda_I} - \gamma + \mathcal{O}(e^{-2\pi\sqrt{\lambda_I}}) \right\} \\ &\stackrel{\lambda_I \sim \infty}{=} \gamma - \frac{1}{2} \sum_{I \neq 0} c_I \ln \frac{\lambda_I}{4}.\end{aligned}\quad (\text{B.21})$$

B.3 Proof of eq. (3.36)

With the function

$$\rho(\lambda) = 2T \ln \left(2 \sinh \left(\frac{\sqrt{\lambda}}{2T} \right) \right) \quad \text{for } \lambda > 0, \quad (\text{B.22})$$

one can show that function $\rho(\lambda)$ satisfies the following inequalities for arbitrary $\lambda > 0$ and the limit

$$\rho'(\lambda) > 0, \quad \rho''(\lambda) < 0, \quad \rho'''(\lambda) > 0, \quad (\lambda \rho''(\lambda))' > 0, \quad \lim_{\lambda \rightarrow \infty} \lambda \rho''(\lambda) = 0. \quad (\text{B.23})$$

Especially, we find for $\lambda > 0$,

$$-\frac{1}{\omega^2} \lambda \rho''(\lambda) > -\frac{1}{\omega^2 + \lambda} \lambda \rho''(\lambda) \geq -\rho''(\omega^2 + \lambda) > 0. \quad (\text{B.24})$$

Therefore, by assuming that $\sum_n \omega_n^{-2}$ is finite, the large regulator-mass limit gives

$$\lim_{\lambda_I \rightarrow \infty} \sum_n \rho''(\omega_n^2 + \lambda_I) = 0. \quad (\text{B.25})$$

Using the inequality $\rho'' < 0$, we find

$$\mathcal{A}_I > 0, \quad \mathcal{B}_I > 0. \quad (\text{B.26})$$

Furthermore, with a natural assumption

$$\exists C_\lambda \in \mathbb{R}_{>0}, \forall x \in [0, L]: \quad |\partial_s \lambda(x)| \leq C_\lambda, \quad (\text{B.27})$$

we find $|\Delta_{nm}|$ has an upper bound as

$$|\Delta_{nm}| = \left| \int dx \partial_s \lambda(x) |f_n(x)|^2 \right| \leq \int dx |\partial_s \lambda(x)| |f_n(x)|^2 < \int dx C_\lambda |f_n(x)|^2 = C_\lambda, \quad (\text{B.28})$$

from which an inequality for $\mathcal{A}_I$ is given

$$0 < \mathcal{A}_I = - \sum_{n \in \mathbb{Z}} \rho''(\omega_n^2 + \lambda_I) |\Delta_{nm}|^2 < -C_\lambda^2 \sum_{n \in \mathbb{Z}} \rho''(\omega_n^2 + \lambda_I). \quad (\text{B.29})$$

Thus we find the large regulator-mass limit of $\mathcal{A}_{I \neq 0}$,

$$\lim_{\lambda_I \rightarrow \infty} \mathcal{A}_I = 0, \quad \text{for } I \neq 0. \quad (\text{B.30})$$

The inequality $\rho'''(\lambda) > 0$ gives,

$$-\rho''(x) < -\frac{\rho'(x) - \rho'(y)}{x - y} < -\rho''(y), \quad \text{for } x > y, \quad (\text{B.31})$$

and by using the completeness condition $\sum_n f_n(x) \bar{f}_n(y) = \delta(x - y)$, we find

$$\begin{aligned} \sum_n |\Delta_{nm}|^2 &= \int dx dy \partial_s \lambda(x) f_m(x) \left(\sum_n \bar{f}_n(x) f_n(y) \right) \bar{f}_m(y) \partial_s \lambda(y) \\ &= \int dx (\partial_s \lambda(x))^2 |f_m(x)|^2 < C_\lambda^2. \end{aligned} \quad (\text{B.32})$$

Combining these inequalities we obtain a bound for $\mathcal{B}_I$ as

$$\begin{aligned} 0 < \mathcal{B}_I &= - \sum_{(n,m) \in \mathbb{Z}^2}^{n \neq m} \frac{\rho'(\omega_n^2 + \lambda_I) - \rho'(\omega_m^2 + \lambda_I)}{\omega_n^2 - \omega_m^2} |\Delta_{nm}|^2 \\ &< -2 \sum_{\omega_n^2 > \omega_m^2} \rho''(\omega_m^2 + \lambda_I) |\Delta_{nm}|^2 < -2 \sum_m \rho''(\omega_m^2 + \lambda_I) \sum_n |\Delta_{nm}|^2 \\ &< -2 \sum_m \rho''(\omega_m^2 + \lambda_I) \times C_\lambda^2. \end{aligned} \quad (\text{B.33})$$

Finally, we also obtain the limit for $\mathcal{B}_{I \neq 0}$ as

$$\lim_{\lambda_I \rightarrow \infty} \mathcal{B}_I = 0, \quad \text{for } I \neq 0. \quad (\text{B.34})$$

C Poisson summation formula

Combining the identity

$$\sum_{n \in \mathbb{Z}} e^{inx} = 2\pi \sum_{m \in \mathbb{Z}} \delta(x + 2\pi m), \quad (\text{C.1})$$

and a Fourier transformation

$$\tilde{f}(p) \equiv \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}} f(x) e^{-ipx}, \quad (\text{C.2})$$

the Poisson summation formula is proved:

$$\begin{aligned} \sum_{n \in \mathbb{Z}} \tilde{f}(nq) e^{in\theta} &= \sum_n \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}} f(x) e^{-i(xq-\theta)n} \\ &= \frac{1}{|q|} \sum_n \int_{-\infty}^{\infty} \frac{dy}{\sqrt{2\pi}} f\left(\frac{y+\theta}{q}\right) e^{-iyn} \\ &= \frac{\sqrt{2\pi}}{|q|} \sum_{n \in \mathbb{Z}} f\left(\frac{2\pi n + \theta}{q}\right). \end{aligned} \quad (\text{C.3})$$

Applying the Poisson summation formula to a Fourier transformation of the Gaussian function $f(x) = e^{-t(x/L)^2}$

$$\tilde{f}(p) = \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}} e^{-t(\frac{x}{L})^2 - ipx} = \frac{L}{\sqrt{2t}} e^{-\frac{(pL)^2}{4t}}, \quad \text{for } L, t \in \mathbb{R}_{>0}, \quad (\text{C.4})$$

we obtain

$$\sum_{n \in \mathbb{Z}} e^{-t(\frac{2\pi n}{L} + a)^2} = \frac{L}{2\sqrt{\pi t}} \sum_{n \in \mathbb{Z}} e^{-\frac{(nL)^2}{4t} + inLa}. \quad (\text{C.5})$$

For instance, the property of the theta function $\theta(x)$ is shown as,

$$\theta(x) = \sum_{n \in \mathbb{Z}} e^{-n^2 \pi x} = \frac{1}{\sqrt{x}} \sum_{n \in \mathbb{Z}} e^{-\frac{n^2 \pi}{x}} = \frac{1}{\sqrt{x}} \theta(x^{-1}), \quad (\text{C.6})$$

with $x > 0$.

D Absence of spontaneous breaking of the global $\text{SU}(N)$ symmetry

Let us add the source term for n_i :

$$\Delta S_J = - \int d^2x (\vec{n} \cdot \vec{J}^\dagger + \text{h.c.}), \quad (\text{D.1})$$

then

$$\langle \vec{n}(x) \rangle_J \equiv \frac{\delta}{\delta \vec{J}(x)^\dagger} \ln Z[J]; \quad Z[J] = \int DA_\mu D\lambda Dn Dn^\dagger e^{-S - \Delta S_J}. \quad (\text{D.2})$$

Path integration over the $\vec{n}$ fields includes the constant modes $\vec{\sigma}$. The spontaneous symmetry breaking occurs if

$$\lim_{J \rightarrow 0} \langle \vec{n}(x) \rangle_J \neq 0. \quad (\text{D.3})$$

Let us take $J(x)$ to be constant $\vec{J}(x) = \vec{J}_c$, which is sufficient for our purpose. Also the translational-invariance assumption is made. Then one can easily calculate and obtain the partition function and the (pseudo-) free energy as

$$\ln Z[J] \sim \ln Z_\lambda[J] = \ln Z_\lambda + \frac{|\vec{J}_c|^2}{\lambda} L\beta \quad (\text{D.4})$$

$$F[J] \sim F_\lambda[J] = F_\lambda - \frac{|\vec{J}_c|^2}{\lambda} L \quad (\text{D.5})$$

with Z_λ, F_λ given in eq. (2.20), eq. (2.23) or eq. (2.30). Therefore, the VEV of $\vec{n}(x)$ is given by

$$\langle \vec{n}(x) \rangle_J \sim \frac{1}{L\beta} \frac{\partial}{\partial \vec{J}_c^\dagger} \ln Z_\lambda[J] \Big|_{\text{s.p.}} = \frac{\vec{J}_c}{\lambda} \Big|_{\text{s.p.}} \quad (\text{D.6})$$

(“s.p.” stands for the saddle-point-value) which obviously vanishes as long as the mass gap remains nonzero in the limit $\vec{J}_c \rightarrow 0$.

Next let us reconsider the gap equation:

$$0 = \frac{\partial F_\lambda[J]}{\partial \lambda} = \frac{\partial F_\lambda}{\partial \lambda} + \frac{|\vec{J}_c|^2}{\lambda^2} L. \quad (\text{D.7})$$

Here one can again show that

$$\frac{\partial^2 F_\lambda[J]}{\partial \lambda^2} < 0, \quad \lim_{\lambda \rightarrow 0} \frac{\partial F_\lambda[J]}{\partial \lambda} = \infty, \quad \lim_{\lambda \rightarrow \infty} \frac{\partial F_\lambda[J]}{\partial \lambda} = -\infty, \quad (\text{D.8})$$

which leads to the uniqueness of the solution $\lambda = \lambda_{\text{sol}}$ of the gap equation, even in the presence of an arbitrary $\vec{J}_c$. λ_{sol} satisfies

$$\lambda_{\text{sol}} \geq \lambda_{\text{sol}} \Big|_{J=0} \geq \Lambda^2. \quad (\text{D.9})$$

Thus

$$\lim_{J \rightarrow 0} \langle \vec{n}(x) \rangle_J = 0. \quad (\text{D.10})$$

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