Cubic interaction vertices for massive/massless continuous-spin fields and arbitrary spin fields

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ABSTRACT: We use light-cone gauge formalism to study interacting massive and massless continuous-spin fields and finite component arbitrary spin fields propagating in the flat space. Cubic interaction vertices for such fields are considered. We obtain parity invariant cubic vertices for coupling of one continuous-spin field to two arbitrary spin fields and cubic vertices for coupling of two continuous-spin fields to one arbitrary spin field. Parity invariant cubic vertices for self-interacting massive/massless continuous-spin fields are also obtained. We find the complete list of parity invariant cubic vertices for continuous-spin fields and arbitrary spin fields.

KEYWORDS: Field Theories in Higher Dimensions, Space-Time Symmetries, Higher Spin Symmetry

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1 Introduction

Continuous-spin field propagating in flat space is associated with continuous-spin representation of the Poincaré algebra (for review, see refs. [1]–[3]). From the point of view of field theory, the continuous-spin field provides interesting example of relativistic dynamical system which involves infinite number of coupled finite component fields. In this respect the continuous-spin field theory has some features in common with string theory and higher-spin theory. For example, we note that the continuous-spin field is decomposed into an infinite chain of coupled scalar, vector, and totally symmetric tensor fields which consists of every spin just once. We recall then a similar infinite chain of scalar, vector and totally symmetric fields appears in the theory of higher-spin gauge field in AdS space [4, 5]. We note also the intriguing discussions about possible interrelations between continuous-spin field theory and the string theory in refs. [6, 7].¹ In view of just mentioned and other interesting features, the continuous-spin field theory has attracted some interest recently (see, e.g., refs. [9]–[17]).

In ref. [18], we developed the light-cone gauge formulation of massless and massive continuous-spin fields propagating in the flat space $R^{d-1,1}$ with arbitrary $d \geq 4$.² Also, in ref. [18], we applied our formulation to study parity invariant cubic vertices for coupling of massless continuous-spin fields to massive arbitrary spin fields and obtained complete list of such cubic vertices.³ Cubic vertices involving continuous-spin fields have one interesting and intriguing feature in common with interaction vertices of string theory and higher-spin theory. It turns out that the interaction vertices for coupling of continuous-spin fields to arbitrary spin fields involve infinite number of derivatives. It seems therefore highly likely that the continuous-spin field theory is the interesting and promising direction to go.

This paper is a continuation of the investigation of cubic interaction vertices for continuous-spin fields and arbitrary spin field begun in ref. [18]. In ref. [18], we studied cubic vertices which involve massless continuous-spin fields and massive/massless arbitrary spin fields, while, in this paper, we study cubic vertices which involve massive/massless continuous-spin fields and massive/massless arbitrary spin fields. We recall that, in general, a continuous-spin field is labelled by mass parameter, which we denote by m , $m^2 \leq 0$, and continuous-spin parameter, which we denote by κ , $\kappa > 0$. To indicate such continuous-spin field we use the shortcut $(m, \kappa)_{CSF}$. A finite-component arbitrary spin field is labeled by mass parameter, denoted by m , $m^2 \geq 0$, and spin value denoted by integer s , $s \geq 0$. Such finite component field will be denoted as (m, s) . We now note that cubic vertices

¹Note that, in ref. [8], it was observed that the continuous-spin massless fields do not appear in perturbative string theory spectrum. We believe that continuous-spin fields could be appeared in the full-fledged string theory.

²Light-cone gauge free continuous-spin massless fields in $R^{3,1}$ and $R^{4,1}$ were discussed in ref. [3]. For $d \geq 4$, the discussion of UIR of the Poincaré algebra $iso(d-1, 1)$ may be found in ref. [1]. We think that our continuous-spin massive field is associated with the tachyonic UIR of the Poincaré algebra which is discussed in section 3 in ref. [1].

³We note that continuous-spin massless/massive fields are infinite component fields. In this paper, finite component massless/massive fields with arbitrary but fixed values of spin will be referred to as arbitrary spin fields.

involving massless continuous-spin fields and massive/massless arbitrary spin fields can be classified as

Cubic vertices with one massless continuous-spin field:

$$(0, s_1)-(0, s_2)-(0, \kappa_3)_{CSF} \quad - \quad (1.1)$$

$$(0, s_1)-(m_2, s_2)-(0, \kappa_3)_{CSF} \quad m_2^2 > 0, \quad (1.2)$$

$$(m_1, s_1)-(m_2, s_2)-(0, \kappa_3)_{CSF} \quad m_1 = m, \quad m_2 = m, \quad m^2 > 0, \quad (1.3)$$

$$(m_1, s_1)-(m_2, s_2)-(0, \kappa_3)_{CSF} \quad m_1 \neq m_2, \quad m_1^2 > 0, \quad m_2^2 > 0; \quad (1.4)$$

Cubic vertices with two massless continuous-spin field:

$$(0, \kappa_1)_{CSF}-(0, \kappa_2)_{CSF}-(0, s_3) \quad - \quad (1.5)$$

$$(0, \kappa_1)_{CSF}-(0, \kappa_2)_{CSF}-(m_3, s_3), \quad m_3^2 > 0, \quad (1.6)$$

Cubic vertices with three massless continuous-spin field:

$$(0, \kappa_1)_{CSF}-(0, \kappa_2)_{CSF}-(0, \kappa_3)_{CSF} \quad ? \quad (1.7)$$

Cubic vertices for fields in (1.1)–(1.6) were studied in ref. [18]. Namely, in ref. [18], we built the complete list of parity invariant cubic vertices for fields in (1.3), (1.4), (1.6).⁴ Also, in ref. [18], we demonstrated that, there are no parity invariant vertices for fields in (1.1), (1.5). This is to say that, in the framework of $so(d-2)$ covariant light-cone gauge formalism, there are no parity invariant cubic vertices for coupling of continuous-spin massless fields to finite component arbitrary spin massless fields. It remains to investigate whether cubic vertices describing self interacting continuous-spin massless field (1.7) do exist.⁵

In this paper, we study cubic vertices which involve, among other fields, at least one massive continuous-spin field. Such cubic vertices we separate in the following three groups.

Cubic vertices with one continuous-spin field:

$$(0, s_1)-(0, s_2)-(m_3, \kappa_3)_{CSF}, \quad m_3^2 < 0; \quad (1.8)$$

$$(0, s_1)-(m_2, s_2)-(m_3, \kappa_3)_{CSF}, \quad m_2^2 > 0, \quad m_3^2 < 0, \quad (1.9)$$

$$(m_1, s_1)-(m_2, s_2)-(m_3, \kappa_3)_{CSF}, \quad m_1^2 > 0, \quad m_2^2 > 0, \quad m_3^2 < 0; \quad (1.10)$$

Cubic vertices with two continuous-spin fields:

$$(m_1, \kappa_1)_{CSF}-(0, \kappa_2)_{CSF}-(0, s_3), \quad m_1^2 < 0, \quad (1.11)$$

$$(m_1, \kappa_1)_{CSF}-(m_2, \kappa_2)_{CSF}-(0, s_3), \quad m_1 = m, \quad m_2 = m, \quad m^2 < 0, \quad (1.12)$$

$$(m_1, \kappa_1)_{CSF}-(m_2, \kappa_2)_{CSF}-(0, s_3), \quad m_1^2 < 0, \quad m_2^2 < 0, \quad m_1 \neq m_2, \quad (1.13)$$

$$(0, \kappa_1)_{CSF}-(m_2, \kappa_2)_{CSF}-(m_3, s_3), \quad m_2^2 < 0, \quad m_3^2 > 0, \quad (1.14)$$

$$(m_1, \kappa_1)_{CSF}-(m_2, \kappa_2)_{CSF}-(m_3, s_3), \quad m_1^2 < 0, \quad m_2^2 < 0, \quad m_3^2 > 0; \quad (1.15)$$

⁴Cubic vertices for fields in (1.2) were lost while preparing paper in ref. [18]. We present our result for the cubic vertices (1.2) in appendix E in this paper.

⁵Cubic vertex for three massless continuous-spin fields (1.7) is a problem for the future research.

Cubic vertices with three continuous-spin fields:

$$(0, \kappa_1)_{CSF}-(0, \kappa_2)_{CSF}-(m_3, \kappa_3)_{CSF}, \quad m_3^2 < 0, \quad (1.16)$$

$$(m_1, \kappa_1)_{CSF}-(m_2, \kappa_2)_{CSF}-(0, \kappa_3)_{CSF}, \quad m_1 = m, \quad m_2 = m, \quad m^2 < 0, \quad (1.17)$$

$$(m_1, \kappa_1)_{CSF}-(m_2, \kappa_2)_{CSF}-(0, \kappa_3)_{CSF}, \quad m_1^2 < 0, \quad m_2^2 < 0, \quad m_1^2 \neq m_2^2, \quad (1.18)$$

$$(m_1, \kappa_1)_{CSF}-(m_2, \kappa_2)_{CSF}-(m_3, \kappa_3)_{CSF}, \quad m_1^2 < 0, \quad m_2^2 < 0, \quad m_3^2 < 0. \quad (1.19)$$

In this paper, we build the complete list of parity invariant cubic vertices for fields in (1.8)–(1.19).⁶ With exception of vertices (1.7), results in this paper together with the ones in ref. [18] provide the exhaustive solution to the problem of description of all parity invariant cubic vertices for massive/massless continuous spin fields and massive/massless arbitrary spin fields in (1.1)–(1.19).

We now note the general features of our vertices. **i)** As in string field theory, our vertices involve infinite number of derivatives. Also, as in light-cone gauge string field theory, our vertices are nonlocal w.r.t. the spatial light-cone coordinate x^- and expandable in derivatives w.r.t. the transverse spatial transverse coordinates x^i . **ii)** Our vertices for massive fields are singular when the masses tend to zero. **iii)** In contrast to the massless continuous-spin fields, the massive continuous-spin fields admit coupling to arbitrary spin massless fields.

This paper is organized as follows. In section 2, we briefly review the manifestly $so(d-2)$ covariant light-cone gauge formulation of free continuous-spin massless and massive fields in $R^{d-1,1}$ developed in ref. [18]. For the reader convenience, we also recall the textbook light-cone gauge formulation of finite-component arbitrary spin massless and massive fields. In section 3, we review the setup for studying cubic vertices which was developed in ref. [18]. We present the complete system of equations required to determine cubic vertices uniquely. For the case of parity invariant cubic vertices, we present the particular form of the complete system of equations which is convenient for our study.

In section 4, we present our result for parity invariant cubic vertices describing coupling of one continuous-spin massive field to two massive/massless arbitrary spin fields, while section 5 is devoted to study of cubic vertices for coupling of two massive/massless continuous-spin field to one massive/massless arbitrary spin field. Section 6 is devoted to cubic vertices for self-interacting massive/massless continuous-spin fields. Namely, we consider vertices for coupling of one massless continuous-spin field to two massive continuous-spin fields and vertices for coupling of one massive continuous-spin field to two massless continuous-spin fields. Cubic vertices for self-interacting massive continuous-spin field are also studied.

In section 7, we summarize our conclusions and suggest directions for future research.

In appendix A, we describe notation we use in this paper. In appendices B, C, D we outline some details of the derivation of cubic vertices. Namely, in appendix B, we discuss

⁶Lorentz covariant cubic vertex for coupling of massless continuous-spin field to two massive scalar fields (case (1.4) with $s_1 = s_2 = 0$) was studied in ref. [19], while Lorentz covariant cubic vertex for coupling of massive continuous-spin field to two massive scalar fields (case (1.10) with $s_1 = s_2 = 0$) was studied in ref. [20].

vertices for coupling of one continuous-spin field to two arbitrary spin fields, while, in appendix C, we discuss vertices for coupling of two continuous-spin fields to one arbitrary spin fields. In appendix D, we outline details of the derivation of vertices for self-interacting massive/massless continuous-spin fields. In appendix E, we study cubic vertex for one massless continuous-spin field, one arbitrary spin massless field, and one arbitrary spin massive field.

2 Light-cone gauge formulation of free continuous-spin fields and arbitrary spin massive and massless fields

Detailed discussion of light-cone formulation to free continuous-spin fields and arbitrary spin massive/massless fields may be found in section 2 in ref. [18]. Here, in order to make our presentation self-contained, we briefly review the most important ingredients of the light-cone formulation.

Poincaré algebra in light-cone frame. We use the method in ref. [21] which reduces the problem of finding a new dynamical system to a problem of finding a new solution for commutators of a basic symmetry algebra. For the case under consideration, basic symmetries are associated with the Poincaré algebra.⁷ Therefore we start with the presentation of a realization of the Poincaré algebra symmetries on a space of light-cone gauge fields. The commutation relations of the Poincaré algebra $iso(d-1, 1)$ are given by⁸

$$[P^\mu, J^{\nu\rho}] = \eta^{\mu\nu} P^\rho - \eta^{\mu\rho} P^\nu, \quad [J^{\mu\nu}, J^{\rho\sigma}] = \eta^{\nu\rho} J^{\mu\sigma} + 3 \text{ terms}, \quad (2.1)$$

where P^μ are the translation generators, while $J^{\mu\nu}$ are generators of the $so(d-1, 1)$ Lorentz algebra. The P^μ taken to be hermitian, while the $J^{\mu\nu}$ are considered to be anti-hermitian. Flat metric $\eta^{\mu\nu}$ is taken to be mostly positive.

For the discussion of light-cone formulation, we use, in place of the Lorentz basis coordinates x^μ , the light-cone basis coordinates $x^\pm, x^i$, where the $so(d-2)$ algebra vector indices i, j take values $i, j = 1, \dots, d-2$, while the coordinates $x^\pm$ are defined as $x^\pm \equiv (x^{d-1} \pm x^0)/\sqrt{2}$. The coordinate x^+ is considered as an evolution parameter. We note then that the $so(d-1, 1)$ Lorentz algebra vector X^μ is decomposed as X^+, X^-, X^i , while non vanishing elements of the flat metric are given by $\eta_{+-} = \eta_{-+} = 1$, $\eta_{ij} = \delta_{ij}$. We note also that, in light-cone approach, generators of the Poincaré algebra are separated into kinematical and dynamical generators defined as

$$P^+, P^i, J^{+i}, J^{+-}, J^{ij}, \quad \text{kinematical generators}; \quad (2.2)$$

$$P^-, J^{-i}, \quad \text{dynamical generators}. \quad (2.3)$$

In the field theory, the kinematical generators are quadratic in fields when $x^+ = 0$.⁹ The dynamical generators involve quadratic and, in general, higher order terms in fields. Before

⁷Application of light-cone formalism for studying vertices in string theory and arbitrary spin field theories may be found in refs. [22, 23] and [24]–[31], while discussion of vertices of 11d SUGRA is given in refs. [32, 33].

⁸Indices $\mu, \nu, \rho, \sigma = 0, 1, \dots, d-1$ are vector indices of the Lorentz algebra $so(d-1, 1)$.

⁹The generators P^i, P^+, J^{ij} do not depend on x^+ , while the generators J^{+i}, J^{+-} take the form $G = G_1 + x^+ G_2$, where G_1 is quadratic in fields, while G_2 involves quadratic and higher order terms in fields.

discussion of the field theoretical realization of the kinematical and dynamical generators we discuss light-cone gauge description of the continuous-spin field, and arbitrary spin massive and massless fields.

Continuous-spin field, arbitrary spin massive field, and arbitrary spin massless field. In light-cone gauge approach, continuous-spin field, spin- s massive field, and spin- s massless field, are described by the following set of scalar, vector, and tensor fields of the $so(d-2)$ algebra:

$$\sum_{n=0}^{\infty} \oplus \phi^{i_1 \dots i_n}, \quad \text{for continuous-spin field;} \quad (2.4)$$

$$\sum_{n=0}^s \oplus \phi^{i_1 \dots i_n}, \quad \text{for spin-}s \text{ massive field;} \quad (2.5)$$

$$\phi^{i_1 \dots i_s}, \quad \text{for spin-}s \text{ massless field.} \quad (2.6)$$

In (2.4), (2.6), fields $\phi^{i_1 \dots i_n}$ ($\phi^{i_1 \dots i_s}$) with $n=0$ ($s=0$) and $n=1$ ($s=1$) are the respective scalar and vector fields of the $so(d-2)$ algebra, while fields with $n \geq 2$ ($s \geq 2$) are totally symmetric traceless tensor fields of the $so(d-2)$ algebra, $\phi^{iii_3 \dots i_n} = 0$ ($\phi^{iii_3 \dots i_s} = 0$). Traceless constraint for massive fields (2.5) will be described below.

To simplify our presentation, we use oscillators α^i, ζ, v to collect fields (2.4)–(2.6) into respective ket-vectors defined as

$$|\phi(p, \alpha)\rangle = \sum_{n=0}^{\infty} \frac{v^n}{n! \sqrt{n!}} \alpha^{i_1} \dots \alpha^{i_n} \phi^{i_1 \dots i_n}(p) |0\rangle, \quad \text{for continuous-spin field;} \quad (2.7)$$

$$|\phi_s(p, \alpha)\rangle = \sum_{n=0}^s \frac{\zeta^{s-n}}{n! \sqrt{(s-n)!}} \alpha^{i_1} \dots \alpha^{i_n} \phi^{i_1 \dots i_n}(p) |0\rangle, \quad \text{for spin-}s \text{ massive field;} \quad (2.8)$$

$$|\phi_s(p, \alpha)\rangle = \frac{1}{s!} \alpha^{i_1} \dots \alpha^{i_s} \phi^{i_1 \dots i_s}(p) |0\rangle, \quad \text{for spin-}s \text{ massless field;} \quad (2.9)$$

where the argument p stands for the momenta p^i, β . Argument α of ket-vectors in (2.4) and (2.5) stands for the respective set of oscillators α^i, v and α^i, ζ , while the argument α in (2.6) stands for α^i . Ket-vectors (2.7)–(2.9) satisfy the following algebraic constraints

$$(N_\alpha - N_v)|\phi\rangle = 0, \quad \bar{\alpha}^2|\phi\rangle = 0, \quad \text{for continuous-spin field;} \quad (2.10)$$

$$(N_\alpha + N_\zeta - s)|\phi_s\rangle = 0, \quad (\bar{\alpha}^2 + \bar{\zeta}^2)|\phi_s\rangle = 0, \quad \text{for spin-}s \text{ massive field;} \quad (2.11)$$

$$(N_\alpha - s)|\phi_s\rangle = 0, \quad \bar{\alpha}^2|\phi_s\rangle = 0, \quad \text{for spin-}s \text{ massless field.} \quad (2.12)$$

From the first constraints in (2.11), (2.12), we learn that ket-vector $|\phi\rangle$ (2.8) is degree- s homogeneous polynomials in the α^i, ζ , while the ket-vector (2.9) is degree- s homogeneous polynomial in the α^i . The second constraints in (2.10), (2.12) amount to tracelessness constraints for the respective tensor fields in (2.4), (2.6). The second constraint in (2.11) is required for the massive fields (2.5) to be associated with irreps of the $so(d-1)$ algebra. Sometimes we prefer to use an infinite chain of massive/massless fields which consists of

every spin just once. Such chain of massive/massless fields is described by the ket-vector

$$|\phi(p, \alpha)\rangle = \sum_{s=0}^{\infty} |\phi_s(p, \alpha)\rangle, \quad (2.13)$$

where, in (2.13), the ket-vector $|\phi_s(p, \alpha)\rangle$ stands for the ket-vector of spin- s massive/massless field given in (2.8)/(2.9).

Field-theoretical realization of Poincaré algebra. To get a field theoretical realization of the Poincaré algebra we need the realization of generators (2.2), (2.3) in terms of differential operators acting on ket-vector $|\phi\rangle$ (2.7)–(2.9). The realization in term of differential operators is given by,¹⁰

Kinematical generators:

$$P^i = p^i, \quad P^+ = \beta, \quad J^{+i} = \partial_{p^i} \beta, \quad J^{+-} = \partial_{\beta} \beta, \quad (2.14)$$

$$J^{ij} = p^i \partial_{p^j} - p^j \partial_{p^i} + M^{ij}, \quad M^{ij} = \alpha^i \bar{\alpha}^j - \alpha^j \bar{\alpha}^i; \quad (2.15)$$

Dynamical generators:

$$P^- = -\frac{p^i p^i + m^2}{2\beta}, \quad J^{-i} = -\partial_{\beta} p^i + \partial_{p^i} P^- + \frac{1}{\beta} (M^{ij} p^j + M^i), \quad (2.16)$$

$$\beta \equiv p^+, \quad \partial_{\beta} \equiv \partial/\partial\beta, \quad \partial_{p^i} \equiv \partial/\partial p^i, \quad (2.17)$$

where, in (2.17), we explain our notation. Operator M^{ij} (2.15) stands for a spin operator of the $so(d-2)$ algebra. Commutation relations for the operators M^{ij} , M^i take the form

$$[M^{ij}, M^{kl}] = \delta^{jk} M^{il} + 3 \text{ terms}, \quad [M^i, M^{jk}] = \delta^{ij} M^k - \delta^{ik} M^j, \quad (2.18)$$

$$[M^i, M^j] = -m^2 M^{ij}. \quad (2.19)$$

Realization of the operator M^i on space of various ket-vectors is given by

Continuous-spin field (massless, $m^2 = 0$, and massive, $m^2 < 0$):

$$M^i = g \bar{\alpha}^i + A^i \bar{g}, \quad A^i = \alpha^i - \alpha^2 \frac{1}{2N_{\alpha} + d - 2} \bar{\alpha}^i, \quad (2.20)$$

$$g = g_v \bar{v}, \quad \bar{g} = -v g_v, \quad g_v = \left(\frac{F_v}{(N_v + 1)(2N_v + d - 2)} \right)^{1/2}, \quad (2.21)$$

$$F_v = \kappa^2 - N_v(N_v + d - 3)m^2; \quad (2.22)$$

Arbitrary spin massive field, $m^2 > 0$:

$$M^i = m(\zeta \bar{\alpha}^i - \alpha^i \bar{\zeta}); \quad (2.23)$$

Arbitrary spin massless field, $m = 0$:

$$M^i = 0. \quad (2.24)$$

¹⁰In this paper, without loss of generality, the generators of the Poincaré algebra are analysed for $x^+ = 0$.

For massive/massless continuous-spin fields, eigenvalues of the 2nd- and 4th-order Casimir operators of the Poincaré algebra are given by $C_2 = m^2$, $C_4 = \kappa^2$, while for arbitrary spin- s massive/massless fields one has the relations $C_2 = m^2$, $C_4 = m^2 s(s+d-3)$. Note that, in terms of the operators M^i , M^{ij} (2.18), (2.19), the 4th-order Casimir operator of the Poincaré algebra takes the form $C_4 = -M^i M^i - \frac{1}{2} m^2 M^{ij} M^{ij}$.

Now we are ready to present a field theoretical realization of the Poincaré algebra generators. This is to say that, at quadratic order in fields, a field theoretical realization of the generators (2.2), (2.3) takes the form

$$G_{[2]} = \int \beta d^{d-1} p \langle \phi(p) | G | \phi(p) \rangle, \quad d^{d-1} p \equiv d\beta d^{d-2} p, \quad (2.25)$$

where G stands for the differential operators presented in (2.14)–(2.16), while $G_{[2]}$ stands for the field theoretical generators. Equal time commutator of $G_{[2]}$ (2.25) with $|\phi\rangle$ takes the form $[[\phi], G_{[2]}] = G|\phi\rangle$. Also we recall that the light-cone gauge action takes the form

$$S = \int dx^+ d^{d-1} p \langle \phi(p) | i \beta \partial^- | \phi(p) \rangle + \int dx^+ P^-, \quad (2.26)$$

where $\partial^- = \partial/\partial x^+$ and P^- is the Hamiltonian. For free fields, the Hamiltonian is obtained from relations (2.16), (2.25).

3 Cubic interaction vertices and light-cone gauge dynamical principle

Detailed discussion of the setup we use for studying the light-cone gauge cubic vertices may be found in section 3 in ref. [18]. Here, to make our presentation self-contained, we just briefly review our result for complete system of equations required to determine the cubic vertices uniquely.

In theories of interacting fields, the dynamical generators $G^{\text{dyn}} = P^-, J^{-i}$ (2.3) can be expanded as

$$G^{\text{dyn}} = \sum_{n=2}^{\infty} G_{[n]}^{\text{dyn}}, \quad (3.1)$$

where $G_{[n]}^{\text{dyn}}$ stands for a functional that has n powers of ket-vector $|\phi\rangle$. At quadratic order in fields, contribution to G^{dyn} is governed by $G_{[2]}$ (2.25), while, at cubic order in fields, contributions to G^{dyn} are described by $G_{[3]}^{\text{dyn}}$. Our aim in this section is to describe the complete system of equations which allows us to determine $G_{[3]}^{\text{dyn}} = P_{[3]}^-, J_{[3]}^{-i}$ uniquely.

We start with the presentation of the expressions for dynamical generators $P_{[3]}^-, J_{[3]}^{-i}$ given by

$$P_{[3]}^- = \int d\Gamma_3 \langle \Phi_{[3]} | | p_{[3]}^- \rangle, \quad (3.2)$$

$$J_{[3]}^{-i} = \int d\Gamma_3 \langle \Phi_{[3]} | j_{[3]}^{-i} \rangle - \frac{1}{3} \left(\sum_{a=1,2,3} \partial_{p_a^i} \langle \Phi_{[3]} | \right) | p_{[3]}^- \rangle, \quad (3.3)$$

where we use the notation

$$\langle \Phi_{[3]} | \equiv \prod_{a=1,2,3} \langle \phi(p_a, \alpha_a) |, \quad \langle \phi(p_a, \alpha_a) | \equiv |\phi(p_a, \alpha_a)\rangle^\dagger, \quad (3.4)$$

$$d\Gamma_3 \equiv (2\pi)^{d-1} \delta^{d-1} \left(\sum_{a=1,2,3} p_a \right) \prod_{a=1,2,3} \frac{d^{d-1} p_a}{(2\pi)^{(d-1)/2}}, \quad d^{d-1} p_a = d^{d-2} p_a d\beta_a. \quad (3.5)$$

The ket-vectors $|p_{[3]}^-\rangle$ and $|j_{[3]}^{-i}\rangle$ appearing in (3.2), (3.3) can be presented as

$$|p_{[3]}^-\rangle = p_{[3]}^-(\mathbb{P}, \beta_a, \alpha_a)|0\rangle, \quad |j_{[3]}^{-i}\rangle = j_{[3]}^{-i}(\mathbb{P}, \beta_a, \alpha_a)|0\rangle, \quad (3.6)$$

where index $a = 1, 2, 3$ labels three fields entering dynamical generators (3.2), (3.3). Quantities $p_{[3]}^-$ and $j_{[3]}^{-i}$ in (3.6) are referred to as densities. Often the density $p_{[3]}^-$ is referred to as cubic interaction vertex. The quantities β_a (3.6) are three light-cone momenta (2.17), while the quantity α_a (3.6) is a shortcut for the oscillators entering ket-vectors (2.7)–(2.9): for continuous-spin field, the α_a stands for oscillators α_a^i , v_a , while, for massive and massless fields, the α_a stands for oscillators α_a^i , ζ_a and α_a^i respectively. A quantity $\mathbb{P}$ in (3.6) stands for a momentum $\mathbb{P}^i$ defined by the relations

$$\mathbb{P}^i \equiv \frac{1}{3} \sum_{a=1,2,3} \check{\beta}_a p_a^i, \quad \check{\beta}_a \equiv \beta_{a+1} - \beta_{a+2}, \quad \beta_a \equiv \beta_{a+3}. \quad (3.7)$$

Complete system of equations for cubic vertex. The complete system of equations for the cubic vertex $p_{[3]}^-$ and the density $j_{[3]}^{-i}$ discussed in ref. [18] takes the form

$$\mathbf{J}^{+-} |p_{[3]}^-\rangle = 0, \quad \text{kinematical } J^{+-} \text{ - symmetry}; \quad (3.8)$$

$$\mathbf{J}^{ij} |p_{[3]}^-\rangle = 0, \quad \text{kinematical } J^{ij} \text{ - symmetries}; \quad (3.9)$$

$$|j_{[3]}^{-i}\rangle = -(\mathbf{P}^-)^{-1} \mathbf{J}^{-i\dagger} |p_{[3]}^-\rangle, \quad \text{dynamical } P^-, J^{-i} \text{ symmetries}; \quad (3.10)$$

Light-cone gauge dynamical principle:

$$|p_{[3]}^-\rangle \text{ and } |j_{[3]}^{-i}\rangle \quad \text{are expandable in } \mathbb{P}^i; \quad (3.11)$$

$$|p_{[3]}^-\rangle \neq \mathbf{P}^- |V\rangle, \quad |V\rangle \quad \text{is expandable in } \mathbb{P}^i; \quad (3.12)$$

$$|p_{[3]}^-\rangle, |j_{[3]}^{-i}\rangle, |V\rangle \quad \text{are finite for } \mathbf{P}^- = 0, \quad (3.13)$$

where operators $\mathbf{J}^{+-}$, $\mathbf{J}^{ij}$, $\mathbf{P}^-$, $\mathbf{J}^{-i\dagger}$ appearing in (3.8)–(3.13) are given

$$\mathbf{J}^{+-} = \mathbb{P}^j \partial_{\mathbb{P}^j} + \sum_{a=1,2,3} \beta_a \partial_{\beta_a}, \quad (3.14)$$

$$\mathbf{J}^{ij} = \mathbb{P}^i \partial_{\mathbb{P}^j} - \mathbb{P}^j \partial_{\mathbb{P}^i} + \mathbf{M}^{ij}, \quad \mathbf{M}^{ij} \equiv \sum_{a=1,2,3} M_a^{ij}, \quad (3.15)$$

$$\mathbf{P}^- = \frac{\mathbb{P}^i \mathbb{P}^i}{2\beta} - \sum_{a=1,2,3} \frac{m_a^2}{2\beta_a}, \quad \beta \equiv \beta_1 \beta_2 \beta_3, \quad (3.16)$$

$$\mathbf{J}^{-i\dagger} = -\frac{\mathbb{P}^i}{\beta} \mathbb{N}_\beta + \frac{1}{\beta} \mathbb{M}^{ij} \mathbb{P}^j + \sum_{a=1,2,3} \frac{\check{\beta}_a}{6\beta_a} m_a^2 \partial_{\mathbb{P}^i} + \frac{1}{\beta_a} M_a^{i\dagger}, \quad (3.17)$$

$$\mathbb{N}_\beta \equiv \frac{1}{3} \sum_{a=1,2,3} \check{\beta}_a \beta_a \partial_{\beta_a}, \quad \mathbb{M}^{ij} \equiv \frac{1}{3} \sum_{a=1,2,3} \check{\beta}_a M_a^{ij}, \quad \check{\beta}_a \equiv \beta_{a+1} - \beta_{a+2}. \quad (3.18)$$

3.1 Equations for parity invariant cubic interaction vertices

Cubic vertices depend, among other things, on the following variables

$$\mathbb{P}^i, \quad \beta_a, \quad \alpha_a^i, \quad a = 1, 2, 3. \quad (3.19)$$

Note also that, if cubic Hamiltonian $P_{[3]}^-$ (3.2) involves continuous-spin field $\langle \phi(p_a, \alpha_a) |$, then cubic vertex, besides variables in (3.19), depends on the oscillators v_a (3.2), while if cubic Hamiltonian $P_{[3]}^-$ (3.2) involves massive field $\langle \phi(p_a, \alpha_a) |$, then cubic vertex, besides variables in (3.19), depends on the oscillator ζ_a . Restrictions imposed by J^{ij} -symmetries (3.9) imply that the vertex $p_{[3]}^-$ depends on invariants of the $so(d-2)$ algebra. The oscillators v_a, ζ_a , and momenta β_a are the invariants of the $so(d-2)$ algebra. Besides these invariants, in the problem under consideration, the remaining invariants of the $so(d-2)$ algebra can be constructed by using the $\mathbb{P}^i, \alpha_a^i$, the delta-Kroneker δ^{ij} , and the Levi-Civita symbol $\epsilon^{i_1 \dots i_{d-2}}$. We ignore invariants that involve one Levi-Civita symbol.¹¹ This is to say that, in this paper, vertices that depend on the invariants given by

$$\mathbb{P}^i \mathbb{P}^i, \quad \alpha_a^i \mathbb{P}^i, \quad \alpha_a^i \alpha_b^i, \quad v_a, \quad \zeta_a, \quad \beta_a, \quad (3.20)$$

are referred to as parity invariant vertices. If $P_{[3]}^-$ (3.2) involves continuous-spin field $\langle \phi(p_a, \alpha_a) |$, or massless arbitrary spin field $\langle \phi(p_a, \alpha_a) |$, then by virtue of the second constraints in (2.10), (2.12) the invariant $\alpha_a^i \alpha_a^i$ does not contribute to the $P_{[3]}^-$, while, if $P_{[3]}^-$ (3.2) involves massive field $\langle \phi(p_a, \alpha_a) |$, then by virtue of the second constraint in (2.10), the contribution of $\alpha_a^i \alpha_a^i$, can be replaced by the contribution of $(-\zeta_a^2)$. Also note that, by using field redefinitions, one can remove all terms in the vertex $p_{[3]}^-$ that are proportional to $\mathbb{P}^i \mathbb{P}^i$ (see appendix B in ref. [27]). This implies that, in the vertex $p_{[3]}^-$, we can drop down a dependence on the invariant $\mathbb{P}^i \mathbb{P}^i$. Taking the above-said into account, we note that cubic vertices in the list (1.8)–(1.19) take the following respective forms:

Cubic vertices with one continuous-spin field:

$$(0, s_1)-(0, s_2)-(m_3, \kappa_3)_{CSF}, \quad p_{[3]}^- = p_{[3]}^-(\beta_a, B_a, \alpha_{aa+1}, v_3); \quad (3.21)$$

$$(0, s_1)-(m_2, s_2)-(m_3, \kappa_3)_{CSF}, \quad p_{[3]}^- = p_{[3]}^-(\beta_a, B_a, \alpha_{aa+1}, \zeta_2, v_3); \quad (3.22)$$

$$(m_1, s_1)-(m_2, s_2)-(m_3, \kappa_3)_{CSF}, \quad p_{[3]}^- = p_{[3]}^-(\beta_a, B_a, \alpha_{aa+1}, \zeta_1, \zeta_2, v_3); \quad (3.23)$$

Cubic vertices with two continuous-spin fields:

$$(m_1, \kappa_1)_{CSF}-(0, \kappa_2)_{CSF}-(0, s_3), \quad (3.24)$$

$$(m, \kappa_1)_{CSF}-(m, \kappa_2)_{CSF}-(0, s_3), \quad (3.25)$$

$$(m_1, \kappa_1)_{CSF}-(m_2, \kappa_2)_{CSF}-(0, s_3), \quad p_{[3]}^- = p_{[3]}^-(\beta_a, B_a, \alpha_{aa+1}, v_1, v_2); \quad (3.26)$$

$$(0, \kappa_1)_{CSF}-(m_2, \kappa_2)_{CSF}-(m_3, s_3), \quad (3.27)$$

$$(m_1, \kappa_1)_{CSF}-(m_2, \kappa_2)_{CSF}-(m_3, s_3), \quad p_{[3]}^- = p_{[3]}^-(\beta_a, B_a, \alpha_{aa+1}, v_1, v_2, \zeta_3); \quad (3.28)$$

¹¹For finite-component fields, recent interesting discussion of parity-odd vertices may be found in ref. [34]. Generalization of our discussion to the case of light-cone gauge parity-odd vertices is straightforward. We do not discuss parity-odd vertices because their inclusion would make our discussion unwieldy. Note that full actions in light-cone gauge bosonic string field theories are formulated entirely in terms of the parity invariant vertices. We believe therefore that full actions in light-cone gauge bosonic continuous-spin field theories can also be formulated entirely in terms of the parity invariant vertices. For massless fields in 5d flat space, the pattern of the derivation of the light-cone gauge parity-odd cubic vertices may be found in section 8.1 in ref. [27].

Cubic vertices with three continuous-spin fields:

$$(0, \kappa_1)_{CSF^+} (0, \kappa_2)_{CSF^+} (m_3, \kappa_3)_{CSF^+}, \quad (3.29)$$

$$(m, \kappa_1)_{CSF^+} (m, \kappa_2)_{CSF^+} (0, \kappa_3)_{CSF^+}, \quad (3.30)$$

$$(m_1, \kappa_1)_{CSF^+} (m_2, \kappa_2)_{CSF^+} (0, \kappa_3)_{CSF^+}, \quad (3.31)$$

$$(m_1, \kappa_1)_{CSF^+} (m_2, \kappa_2)_{CSF^+} (m_3, \kappa_3)_{CSF^+}, \quad p_{[3]}^- = p_{[3]}^-(\beta_a, B_a, \alpha_{aa+1}, v_a); \quad (3.32)$$

where we introduce the notation

$$B_a \equiv \frac{\alpha_a^i \mathbb{P}^i}{\beta_a}, \quad \alpha_{ab} \equiv \alpha_a^i \alpha_b^i. \quad (3.33)$$

In other words, in place of the invariant $\alpha_a^i \mathbb{P}^i$ (3.20), we prefer to use the invariant B_a (3.33).

Using representations for cubic vertices in (3.21)–(3.32), we now present more convenient form of equations for the cubic vertices. Namely, using $\mathbf{J}^{-i}$ (3.17), we find the following relation

$$\mathbf{J}^{-i\dagger} |p_{[3]}^-\rangle = \mathbf{P}^- \sum_{a=1,2,3} \frac{2\check{\beta}_a}{3\beta_a} \alpha_a^i \partial_{B_a} |p_{[3]}^-\rangle + \mathbb{P}^i G_\beta |p_{[3]}^-\rangle + \sum_{a=1,2,3} \frac{\alpha_a^i}{\beta_a} G_{a,\mathbb{P}^2} |p_{[3]}^-\rangle, \quad (3.34)$$

where explicit form of operators $G_{a,\mathbb{P}^2}$, G_β may be found in appendices B, C, D. Using explicit form of operators $G_{a,\mathbb{P}^2}$, G_β , we then conclude that equations (3.10), (3.13), (3.34) lead to the equations

$$G_a |p_{[3]}^-\rangle = 0, \quad a = 1, 2, 3; \quad (3.35)$$

$$G_\beta |p_{[3]}^-\rangle = 0. \quad (3.36)$$

In turn, equations (3.34)–(3.36) and the ones in (3.10) imply the following expressions for density $|j_{[3]}^{-i}\rangle$ corresponding to the respective vertices with one continuous-spin field (3.21)–(3.23), two continuous-spin field (3.24), (3.28) and three continuous-spin field (3.30)

$$|j_{[3]}^{-i}\rangle = - \sum_{a=1,2,3} \frac{2\check{\beta}_a}{3\beta_a} \alpha_a^i \partial_{B_a} |p_{[3]}^-\rangle - \frac{2\beta}{\beta_3^3} \frac{g_{v_3} \partial_{v_3}}{2N_3 + d - 2} \partial_{B_3}^2 |p_{[3]}^-\rangle, \quad (3.37)$$

$$|j_{[3]}^{-i}\rangle = - \sum_{a=1,2,3} \frac{2\check{\beta}_a}{3\beta_a} \alpha_a^i \partial_{B_a} |p_{[3]}^-\rangle - \sum_{a=1,2} \frac{2\beta}{\beta_a^3} \frac{g_{v_a} \partial_{v_a}}{2N_a + d - 2} \partial_{B_a}^2 |p_{[3]}^-\rangle, \quad (3.38)$$

$$|j_{[3]}^{-i}\rangle = - \sum_{a=1,2,3} \frac{2\check{\beta}_a}{3\beta_a} \alpha_a^i \partial_{B_a} |p_{[3]}^-\rangle - \sum_{a=1,2,3} \frac{2\beta}{\beta_a^3} \frac{g_{v_a} \partial_{v_a}}{2N_a + d - 2} \partial_{B_a}^2 |p_{[3]}^-\rangle, \quad (3.39)$$

where operator N_a is defined in (A.7). We note also that, in terms of vertices presented in (3.21)–(3.32), equations given in (3.8) takes the form

$$\sum_{a=1,2,3} \beta_a \partial_{\beta_a} p_{[3]}^- = 0. \quad (3.40)$$

Summarizing our discussion in this section, we note that, the cubic vertices describing interactions of fields in (3.21)–(3.32) should satisfy equations (3.35), (3.36), (3.40). The

density $|j_{[3]}^{-i}\rangle$ corresponding to three groups of cubic vertices in (3.21)–(3.32) are given by the respective three relations in (3.37)–(3.39). Equations (3.35), (3.36), (3.40) constitute the complete system of equations which to the best of our present understanding determine parity invariant vertices uniquely. Note that, if we consider the light-cone gauge formulation of the Einstein and Yang-Mills theories, then we can verify that eqs. (3.35), (3.36), (3.40) admit to fix the cubic interaction vertices unambiguously (see ref. [27]). Therefore it seems reasonable to use eqs. (3.35), (3.36), (3.40) for studying the cubic vertices of the continuous-spin field theory. We now consider solutions of equations (3.35), (3.36), (3.40) for vertices (3.21)–(3.32) in turn.

4 Parity invariant cubic vertices for one continuous-spin massive field and two massless fields

In this section, we discuss parity invariant cubic vertices which involve one massive continuous-spin field and two finite component massless/massive fields. According to our classification, such vertices can be separated into three particular cases given in (1.8)–(1.10). Let us discuss these particular cases in turn.

4.1 One continuous-spin massive field and two massless fields

We now discuss parity invariant cubic vertices for one continuous-spin massive field and two arbitrary spin massless fields. This is to say that, using the shortcut $(m, \kappa)_{CSF}$ for a continuous-spin massive field and the shortcut $(0, s)$ for a spin- s massless field, we study cubic vertices for the following three fields:

$$(0, s_1)-(0, s_2)-(m_3, \kappa_3)_{CSF} \quad m_3^2 < 0, \quad \text{two massless fields and one continuous-spin massive field.} \quad (4.1)$$

Relation (4.1) tells us that the spin- s_1 and spin- s_2 massless fields carry the respective external line indices $a = 1$ and $a = 2$, while the continuous-spin massive field corresponds to $a = 3$.

For fields (4.1), we find the following general solution to cubic vertex $p_{[3]}^-$ (see appendix B)

$$p_{[3]}^- = U_v U_\Gamma U_\beta U_B U_W U_B^{-1} V_A^{(7)}, \quad A = +, -, \quad (4.2)$$

$$p_{[3]}^- = p_{[3]}^-(\beta_a, B_a, \alpha_{aa+1}, v_3), \quad (4.3)$$

$$V_A^{(7)} = V_A^{(7)}(B_3, \alpha_{aa+1}). \quad (4.4)$$

In (4.2), we introduce two vertices $V_A^{(7)}$ labelled by the superscript $A = +, -$. In (4.3) and (4.4), the arguments of the generic vertex $p_{[3]}^-$ and the vertices $V_A^{(7)}$ are shown explicitly. The definition of the arguments B_a and α_{ab} may be found in (A.3). Various quantities U appearing in (4.2) are differential operators w.r.t. the B_a and α_{aa+1} . These quantities will

be presented below. For two vertices $V_A^{(7)}$ (4.4), we find the following solution:

$$V_{\pm}^{(7)} = F\left(\alpha_3, \beta_3, \gamma_3; \frac{1 \pm z_3}{2}\right) V_{\pm}, \quad V_{\pm} = V_{\pm}(\alpha_{12}, \alpha_{23}, \alpha_{31}), \quad (4.5)$$

$$\alpha_3 = \nu_3 + \frac{1}{2} + \sigma_3, \quad \beta_3 = \nu_3 + \frac{1}{2} - \sigma_3, \quad \sigma_3 = \left(\frac{(d-3)^2}{4} + \frac{\kappa_3^2}{m_3^2}\right)^{1/2}, \quad (4.6)$$

$$\gamma_3 = \nu_3 + 1, \quad z_3 = \frac{2B_3}{\kappa_3}, \quad (4.7)$$

where, in (4.5) and below, the $F(\alpha, \beta, \gamma; x)$ stands for the hypergeometric function. For the hypergeometric function, we use notation and convention in chapter 15 in ref. [35]. In (4.5), in place of the variable B_3 , we use new variable z_3 (4.7). Operator ν_3 is defined below.

We see that the generic vertex $p_{[3]}^-$ (4.3) depends on the ten variables, while, the vertices V_A (4.5) depend only on the three variables. By definition, the vertices V_A (4.5) are expandable in the three variables α_{12} , α_{23} , α_{31} . The general solution (4.2) for the vertex $p_{[3]}^-$ is expressed in terms of the operators U , ν_3 acting on the vertices $V_{\pm}$ (4.5). To complete the description of the vertex $p_{[3]}^-$ we now provide expressions for the operators U , ν_3 . These operators are given by

$$U_v = v_3^{N_3}, \quad N_3 = N_{B_3} + N_{\alpha_{31}} + N_{\alpha_{23}}, \quad (4.8)$$

$$U_{\Gamma} = \left(\left(-\frac{\kappa_3^2}{m_3^2} \right)^{N_3} \frac{2^{N_3} \Gamma(N_3 + \frac{d-2}{2})}{\Gamma(N_3 - \lambda_{3+}) \Gamma(N_3 - \lambda_{3-}) \Gamma(N_3 + 1)} \right)^{1/2}, \quad \lambda_{3\pm} = \frac{d-3}{2} \pm \sigma_3, \quad (4.9)$$

$$U_{\beta} = \exp\left(-\frac{\check{\beta}_3}{2\beta_3} \kappa_3 \partial_{B_3}\right), \quad (4.10)$$

$$U_{\partial\alpha} = \exp\left(-\frac{2B_1 B_2}{m_3^2} \partial_{\alpha_{12}} + \frac{2B_1}{m_3^2} \left(B_3 + \frac{1}{2} \kappa_3\right) \partial_{\alpha_{31}} + \frac{2B_2}{m_3^2} \left(B_3 - \frac{1}{2} \kappa_3\right) \partial_{\alpha_{23}}\right), \quad (4.11)$$

$$U_B = \left(-\frac{\kappa_3^2}{m_3^2} + \frac{4}{m_3^2} B_3^2\right)^{-(\nu_3+1)/2}, \quad (4.12)$$

$$U_W = \sum_{n=0}^{\infty} \frac{\Gamma(\nu_3+n)}{4^n n! \Gamma(\nu_3+2n)} W_3^n, \quad W_3 = 2\alpha_{12} \partial_{\alpha_{31}} \partial_{\alpha_{23}}, \quad (4.13)$$

$$\nu_3 = N_{\alpha_{23}} + N_{\alpha_{31}} + \frac{d-4}{2}, \quad (4.14)$$

where Γ (4.9), (4.13) stands for the Gamma-function. Quantities $\check{\beta}_a$, N_{B_a} , $N_{\alpha_{ab}}$, N_a appearing in (4.8)–(4.14) are defined in (A.3)–(A.7).

Expressions above-presented in (4.2)–(4.14) provide the complete generating form description of cubic interaction vertices for two chains of totally symmetric massless fields (2.13) with one continuous-spin massive field. Now our aim is to describe cubic vertices for one continuous-spin massive field and two totally symmetric massless fields with arbitrary but fixed spin- s_1 and spin- s_2 values. Using the first algebraic constraint for massless spin- s field in (2.12) it is easy to see that vertices we are interested in must satisfy the algebraic constraints

$$(N_{\alpha_a} - s_a) |p_{[3]}^-\rangle = 0, \quad a = 1, 2. \quad (4.15)$$

Two constraints given in (4.15) tell us simply that the cubic vertex $p_{[3]}^-$ should be degree- s_1 and degree- s_2 homogeneous polynomial in the oscillators α_1^i and α_2^i respectively. It is easy to check, that, in terms of the vertices $V_{\pm}$ (4.5), algebraic constraints (4.15) take the form

$$(N_{\alpha_{12}} + N_{\alpha_{31}} - s_1)V_A = 0, \quad (N_{\alpha_{12}} + N_{\alpha_{23}} - s_2)V_A = 0. \quad (4.16)$$

Vertices V_A satisfy one and the same equations (4.16). Therefore to simplify our presentation we drop the superscript A and use a vertex V in place of the vertices V_A , i.e., we use $V = V_A$. Doing so, we note that the general solution to constraints (4.16) can be presented as

$$V = V(s_1, s_2; n), \quad V(s_1, s_2; n) = \alpha_{12}^n \alpha_{23}^{s_2-n} \alpha_{31}^{s_1-n}. \quad (4.17)$$

The integer n appearing in (4.17) is the freedom of our solution for the vertex V . In other words the integer n labels all possible cubic vertices that can be constructed for three fields in (4.1). In order for vertices (4.17) to be sensible, the integer n should satisfy the restrictions

$$n \geq 0, \quad s_1 - n \geq 0, \quad s_2 - n \geq 0, \quad (4.18)$$

which amount to the requirement that the powers of all variables $\alpha_{12}, \alpha_{23}, \alpha_{31}$ in (4.17) be non-negative. From (4.18), we see that allowed values of n are given by

$$n = 0, 1, \dots, s_{\min}, \quad s_{\min} \equiv \min(s_1, s_2). \quad (4.19)$$

Expressions for cubic interaction vertices given in (4.2)–(4.14), (4.17) and allowed values for n presented in (4.19) provide the complete description and classification of cubic interaction vertices that can be constructed for two spin- s_1 and spin- s_2 massless fields and one continuous-spin massive field. From (4.19), we find that, given spin values s_1 and s_2 , a number of cubic vertices V_+ (or V_-) which can be built for the fields in (4.1) is given by

$$N = s_{\min} + 1. \quad (4.20)$$

Cubic vertex for continuous-spin massive and massless scalar fields. For illustration purposes we consider cubic vertex for one continuous-spin massive field and two massless scalar fields. For two scalar fields, we have spin values $s_1 = 0, s_2 = 0$, i.e., $s_{\min} = 0$. Plugging $s_{\min} = 0$ in (4.19), we get $n = 0$. This implies that there is only one vertex V_+ and one vertex V_- . Plugging $n = 0$ in (4.17), we get $V_{\pm} = 1$. In turn, plugging $V_{\pm} = 1$ in (4.5), we get the vertices $V_{\pm}^{(7)}$:

$$V_{\pm}^{(7)} = F\left(\lambda_{3+}, \lambda_{3-}, \frac{d-2}{2}; \frac{1}{2} \pm \frac{B_3}{\kappa_3}\right), \quad \lambda_{3\pm} = \frac{d-3}{2} \pm \left(\frac{(d-3)^2}{4} + \frac{\kappa_3^2}{m_3^2}\right)^{1/2}. \quad (4.21)$$

Finally, plugging $V_{\pm}^{(7)}$ (4.21) into (4.2) and using expressions for operators U (4.8)–(4.14), we get the full expressions for two cubic interaction vertices $p_{[3]}^-$,

$$p_{[3]}^- = UF \left(\lambda_{3+}, \lambda_{3-}, \frac{d-2}{2}; \frac{v_3 B_3}{\kappa_3} - \frac{\beta_1}{\beta_3} \right), \quad (4.22)$$

$$p_{[3]}^- = UF \left(\lambda_{3+}, \lambda_{3-}, \frac{d-2}{2}; -\frac{v_3 B_3}{\kappa_3} - \frac{\beta_2}{\beta_3} \right), \quad (4.23)$$

$$U = \left(\left(-\frac{\kappa_3^2}{m_3^2} \right)^{N_{B_3}} \frac{2^{N_{B_3}} \Gamma(N_{B_3} + \frac{d-2}{2})}{\Gamma(N_{B_3} - \lambda_{3+}) \Gamma(N_{B_3} - \lambda_{3-}) \Gamma(N_{B_3} + 1)} \right)^{1/2}, \quad N_{B_3} = B_3 \partial_{B_3}, \quad (4.24)$$

where $\lambda_{3\pm}$ are given in (4.21).

4.2 One continuous-spin massive field, one massless field, and one massive field

We discuss parity invariant cubic vertices for one continuous-spin massive field, one arbitrary spin massless field, and one arbitrary spin massive field. This is to say that, using the shortcut $(m, \kappa)_{CSF}$ for a continuous-spin massive field and the shortcut (m, s) for a mass- m and spin- s massive field, we study cubic vertices for the following three fields:

$$(0, s_1)-(m_2, s_2)-(m_3, \kappa_3)_{CSF} \quad m_2^2 > 0, \quad m_3^2 < 0, \\ \text{one massless field, one massive field, and one continuous-spin massive field.} \quad (4.25)$$

Relation (4.25) tells us that spin- s_1 massless and spin- s_2 massive fields carry the respective external line indices $a = 1$ and $a = 2$, while the continuous-spin massive field corresponds to $a = 3$.

For fields (4.25), we find the following general solution to cubic vertex $p_{[3]}^-$ (see appendix B):

$$p_{[3]}^- = U_v U_\Gamma U_\beta U_\zeta U_{\partial B} U_{\partial \alpha} U_B U_W U_B^{-1} V_A^{(9)}, \quad A = +, -, \quad (4.26)$$

$$p_{[3]}^- = p_{[3]}^-(\beta_a, B_a, \alpha_{aa+1}, \zeta_2, v_3), \quad (4.27)$$

$$V_A^{(9)} = V_A^{(9)}(B_2, B_3, \alpha_{aa+1}). \quad (4.28)$$

In (4.26), we introduce two vertices $V_A^{(9)}$ labelled by the superscripts $A = +, -$. In (4.27) and (4.28), the arguments of the generic vertex $p_{[3]}^-$ and the vertices $V_A^{(9)}$ are shown explicitly. The definition of the arguments B_a and α_{ab} may be found in (A.3). Various quantities U appearing in (4.26) are differential operators w.r.t. the B_a and α_{aa+1} . These quantities will be presented below. For two vertices $V_{\pm}^{(9)}$ (4.28), we find the following solution:

$$V_{\pm}^{(9)} = F \left(\alpha_3, \beta_3, \gamma_3; \frac{1 \pm z_3}{2} \right) V_{\pm}, \quad V_{\pm} = V_{\pm}(B_2, \alpha_{12}, \alpha_{23}, \alpha_{31}), \quad (4.29)$$

$$\alpha_3 = \nu_3 + \frac{1}{2} + \sigma_3, \quad \beta_3 = \nu_3 + \frac{1}{2} - \sigma_3, \quad \sigma_3 = \left(\frac{(d-3)^2}{4} + \frac{\kappa_3^2}{m_3^2} \right)^{1/2}, \quad (4.30)$$

$$\gamma_3 = \nu_3 + 1, \quad z_3 = \frac{2m_3^2 B_3}{\kappa_3(m_3^2 - m_2^2)}, \quad (4.31)$$

where the $F(\alpha, \beta, \gamma; x)$ stands for the hypergeometric function. In (4.29), in place of the variable B_3 , we use new variable z_3 (4.31). A quantity ν_3 is defined below.

From (4.27), we see that the generic vertex $p_{[3]}^-$ depends on the eleven variables, while, from (4.29), we learn that the vertices V_A depend only on the four variables. Note also that, by definition, the vertices V_A (4.29) are expandable in the variables $B_2, \alpha_{12}, \alpha_{23}, \alpha_{31}$. From (4.26), we see that the general solution for the vertex $p_{[3]}^-$ is expressed in terms of the operators U, ν_3 and the vertices V_A (4.29). In order to complete the description of the vertex $p_{[3]}^-$ we should provide expressions for the operators U, ν_3 . These operators are given by

$$U_v = v_3^{N_3}, \quad N_3 = N_{B_3} + N_{\alpha_{31}} + N_{\alpha_{23}}, \quad (4.32)$$

$$U_\Gamma = \left(\left(-\frac{\kappa_3^2}{m_3^2} \right)^{N_3} \frac{2^{N_3} \Gamma(N_3 + \frac{d-2}{2})}{\Gamma(N_3 - \lambda_{3+}) \Gamma(N_3 - \lambda_{3-}) \Gamma(N_3 + 1)} \right)^{1/2}, \quad \lambda_{3\pm} = \frac{d-3}{2} \pm \sigma_3, \quad (4.33)$$

$$U_\beta = \exp \left(-\frac{\check{\beta}_2}{2\beta_2} \zeta_2 m_2 \partial_{B_2} - \frac{\check{\beta}_3}{2\beta_3} \kappa_3 \partial_{B_3} \right), \quad (4.34)$$

$$U_\zeta = \exp \left(-\frac{\zeta_2}{m_2} B_1 \partial_{\alpha_{12}} + \frac{\zeta_2}{m_2} \left(B_3 - \frac{1}{2} \kappa_3 \right) \partial_{\alpha_{23}} - \frac{m_3^2}{2m_2} \zeta_2 \partial_{B_2} \right), \quad (4.35)$$

$$U_{\partial B} = \exp \left(\frac{m_2^2}{2m_3^2} \kappa_3 \partial_{B_3} \right) \quad (4.36)$$

$$U_{\partial\alpha} = \exp \left(\frac{\kappa_3}{m_3^2} B_1 \partial_{\alpha_{31}} - \frac{\kappa_3}{m_3^2} B_2 \partial_{\alpha_{23}} \right. \\ \left. + \frac{2}{m_2^2 - m_3^2} B_1 B_2 \partial_{\alpha_{12}} + \frac{2(m_2^2 + m_3^2)}{(m_2^2 - m_3^2)^2} B_2 B_3 \partial_{\alpha_{23}} - \frac{2}{m_2^2 - m_3^2} B_3 B_1 \partial_{\alpha_{31}} \right), \quad (4.37)$$

$$U_B = \left(-\frac{\kappa_3^2}{m_3^2} + \frac{4m_3^2}{(m_2^2 - m_3^2)^2} B_3^2 \right)^{-(\nu_3+1)/2}, \quad (4.38)$$

$$U_W = \sum_{n=0}^{\infty} \frac{\Gamma(\nu_3 + n)}{4^n n! \Gamma(\nu_3 + 2n)} W_3^n, \quad W_3 = 2\alpha_{12} \partial_{\alpha_{31}} \partial_{\alpha_{23}} - \frac{4m_2^2}{(m_2^2 - m_3^2)^2} B_2^2 \partial_{\alpha_{23}}^2, \quad (4.39)$$

$$\nu_3 = N_{\alpha_{23}} + N_{\alpha_{31}} + \frac{d-4}{2}, \quad (4.40)$$

where quantities $\check{\beta}_a, N_{B_a}, N_{\alpha_{ab}}, N_a$ appearing in (4.32)–(4.40) are defined in (A.3)–(A.7).

Expressions above-presented in (4.26)–(4.40) provide the complete generating form description of cubic vertices for coupling of one continuous-spin massive field to arbitrary spin massless and massive fields. Namely, these vertices describe an coupling of one continuous-spin massive field to two chains of massless and massive fields (2.13). Now our aim is to describe cubic vertices for coupling of one continuous-spin massive field to massless and massive fields having the respective arbitrary but fixed spin- s_1 and spin- s_2 values. Using the first algebraic constraints in (2.11), (2.12) it is easy to see that vertices we are interested in must satisfy the algebraic constraints

$$(N_{\alpha_1} - s_1) |p_{[3]}^- \rangle = 0, \quad (N_{\alpha_2} + N_{\zeta_2} - s_2) |p_{[3]}^- \rangle = 0. \quad (4.41)$$

First constraint in (4.41) tells us that the $p_{[3]}^-$ should be degree- s_1 homogeneous polynomial in the α_1^i , while, from the second constraint in (4.41), we learn that the $p_{[3]}^-$ should be degree-

s_2 homogeneous polynomial in the α_2^i, ζ_2 . In terms of the $V_\pm$ (4.29), constraints (4.41) take the form

$$(N_{\alpha_{12}} + N_{\alpha_{31}} - s_1)V_\pm = 0, \quad (N_{B_2} + N_{\alpha_{12}} + N_{\alpha_{23}} - s_2)V_\pm = 0. \quad (4.42)$$

Vertices $V_\pm$ satisfy one and the same equations (4.42). Therefore to simplify our presentation we drop the superscripts $\pm$ and use a vertex V in place of the vertices $V_\pm$, i.e., we use $V = V_\pm$. Doing so, we note that the general solution to constraints (4.42) can be presented

$$V = V(s_1, s_2; n, k), \quad V(s_1, s_2; n, k) = B_2^{s_2-s_1+k-n} \alpha_{12}^{s_1-k} \alpha_{23}^n \alpha_{31}^k. \quad (4.43)$$

The integers n, k appearing in (4.43) are the freedom of our solution for the vertex V . In other words, the integers n, k label all possible cubic vertices that can be constructed for three fields in (4.25). In order for vertices (4.43) to be sensible, the integers n, k should satisfy the restrictions

$$0 \leq k \leq s_1, \quad 0 \leq n \leq s_2 - s_1 + k, \quad (4.44)$$

which amount to the requirement that the powers of all variables $B_2, \alpha_{12}, \alpha_{23}, \alpha_{31}$ in (4.43) be non-negative.

Expressions for cubic interaction vertices given in (4.26)–(4.40), (4.43) and values of n, k which satisfy restrictions (4.44) provide the complete description and classification of cubic interaction vertices that can be constructed for one spin- s_1 massless field, one spin- s_2 massive field and one continuous-spin massive field.

4.3 One continuous-spin massive field and two massive fields

In this section, we discuss parity invariant cubic vertices for one continuous-spin massive field and two arbitrary spin massive fields. This is to say that, using the shortcut $(m, \kappa)_{CSF}$ for a continuous-spin massive field and the shortcut (m, s) for a spin- s massive field, we study cubic vertices for the following three fields:

$$(m_1, s_1)-(m_2, s_2)-(m_3, \kappa_3)_{CSF} \quad \text{two massive fields and one continuous-spin massive field} \\ m_1^2 > 0, \quad m_2^2 > 0, \quad m_3^2 < 0. \quad (4.45)$$

Relation (4.45) tells us that the spin- s_1 and spin- s_2 massive fields carry the respective external line indices $a = 1, 2$, while the continuous-spin massive field corresponds to $a = 3$.

For fields (4.45), we find the following general solution to cubic vertex $p_{[3]}^-$ (see appendix B)

$$p_{[3]}^- = U_v U_\Gamma U_\beta U_\zeta U_{\partial B} U_{\partial \alpha} U_B U_W U_B^{-1} V_A^{(7)}, \quad A = +, -, \quad (4.46)$$

$$p_{[3]}^- = p_{[3]}^-(\beta_a, B_a, \alpha_{aa+1}, \zeta_1, \zeta_2, v_3), \quad (4.47)$$

$$V_A^{(7)} = V_A^{(7)}(B_a, \alpha_{aa+1}). \quad (4.48)$$

In (4.46), we introduce two vertices $V_A^{(7)}$ labelled by the superscripts $A = \pm$. In (4.47) and (4.48), the arguments of the generic vertex $p_{[3]}^-$ and the vertices $V_A^{(7)}$ are shown explicitly. The definition of the arguments B_a and α_{ab} may be found in (A.3). Various quantities

U appearing in (4.46) are differential operators w.r.t. the B_a and α_{aa+1} . These operators are given below. For two vertices $V_A^{(7)}$ (4.48), we find the following solution:

$$V_{\pm}^{(7)} = F\left(\alpha_3, \beta_3, \gamma_3; \frac{1 \pm z_3}{2}\right) V_{\pm}, \quad V_{\pm} = V_{\pm}(B_1, B_2, \alpha_{12}, \alpha_{23}, \alpha_{31}), \quad (4.49)$$

$$\alpha_3 = \nu_3 + \frac{1}{2} + \sigma_3, \quad \beta_3 = \nu_3 + \frac{1}{2} - \sigma_3, \quad \sigma_3 = \left(\frac{(d-3)^2}{4} + \frac{\kappa_3^2}{m_3^2}\right)^{1/2}, \quad (4.50)$$

$$\gamma_3 = \nu_3 + 1, \quad z_3 = -\frac{2m_3^2}{\kappa_3\sqrt{D}}B_3, \quad (4.51)$$

where the $F(\alpha, \beta, \gamma; x)$ stands for the hypergeometric function. In (4.49), in place of the variable B_3 , we use new variable z_3 (4.51). Quantities D , ν_3 are defined below.

From (4.47), we see that the generic vertex $p_{[3]}^-$ depends on the twelve variables, while, from (4.49), we learn that the vertices $V_{\pm}$ depend only on the five variables. Note also that, by definition, the vertices $V_{\pm}$ (4.49) are expandable in the five variables $B_1, B_2, \alpha_{12}, \alpha_{23}, \alpha_{31}$. From (4.46), we see that the general solution for the vertex $p_{[3]}^-$ is expressed in terms of the operators U , ν_3 and the vertices $V_{\pm}$ (4.49). In order to complete the description of the vertex $p_{[3]}^-$ we should provide expressions for the operators U , ν_3 . These operators are given by

$$U_v = v_3^{N_3}, \quad N_3 = N_{B_3} + N_{\alpha_{31}} + N_{\alpha_{23}}, \quad (4.52)$$

$$U_{\Gamma} = \left(\left(-\frac{\kappa_3^2}{m_3^2}\right)^{N_3} \frac{2^{N_3} \Gamma(N_3 + \frac{d-2}{2})}{\Gamma(N_3 - \lambda_{3+}) \Gamma(N_3 - \lambda_{3-}) \Gamma(N_3 + 1)}\right)^{1/2}, \quad \lambda_{3\pm} = \frac{d-3}{2} \pm \sigma_3, \quad (4.53)$$

$$U_{\beta} = \exp\left(-\frac{\check{\beta}_1}{2\beta_1} \zeta_1 m_1 \partial_{B_1} - \frac{\check{\beta}_2}{2\beta_2} \zeta_2 m_2 \partial_{B_2} - \frac{\check{\beta}_3}{2\beta_3} \kappa_3 \partial_{B_3}\right), \quad (4.54)$$

$$U_{\zeta} = \exp\left(-\frac{\zeta_1}{2m_1} (m_2^2 - m_3^2) \partial_{B_1} - \frac{\zeta_2}{2m_2} (m_3^2 - m_1^2) \partial_{B_2} - \frac{\zeta_1}{m_1} \left(B_3 + \frac{1}{2} \kappa_3\right) \partial_{\alpha_{31}} \right. \\ \left. + \left(\frac{\zeta_1}{m_1} B_2 - \frac{\zeta_2}{m_2} B_1 + \frac{m_3^2 - m_1^2 - m_2^2}{2m_1 m_2} \zeta_1 \zeta_2\right) \partial_{\alpha_{12}} + \frac{\zeta_2}{m_2} \left(B_3 - \frac{1}{2} \kappa_3\right) \partial_{\alpha_{23}}\right), \quad (4.55)$$

$$U_{\partial B} = \exp\left(\frac{m_2^2 - m_1^2}{2m_3^2} \kappa_3 \partial_{B_3}\right), \quad (4.56)$$

$$U_{\partial \alpha} = \exp\left(\frac{\kappa_3}{m_3^2} B_1 \partial_{\alpha_{31}} - \frac{\kappa_3}{m_3^2} B_2 \partial_{\alpha_{23}} + \sum_{a=1,2,3} \frac{2h_{aa+1}}{D} B_a B_{a+1} \partial_{\alpha_{aa+1}}\right), \quad (4.57)$$

$$U_B = \left(-\frac{\kappa_3^2}{m_3^2} + \frac{4m_3^2}{D} B_3^2\right)^{-(\nu_3+1)/2}, \quad (4.58)$$

$$U_W = \sum_{n=0}^{\infty} \frac{\Gamma(\nu_3 + n)}{4^n n! \Gamma(\nu_3 + 2n)} W_3^n, \quad (4.59)$$

$$\nu_3 = N_{\alpha_{23}} + N_{\alpha_{31}} + \frac{d-4}{2}, \quad W_3 = -\frac{4m_1^2}{D} B_1^2 \partial_{\alpha_{31}}^2 - \frac{4m_2^2}{D} B_2^2 \partial_{\alpha_{23}}^2 + 2\alpha_{12} \partial_{\alpha_{31}} \partial_{\alpha_{23}}, \quad (4.60)$$

$$h_{aa+1} = m_a^2 + m_{a+1}^2 - m_{a+2}^2, \quad (4.61)$$

$$D = m_1^4 + m_2^4 + m_3^4 - 2(m_1^2 m_2^2 + m_2^2 m_3^2 + m_3^2 m_1^2), \quad (4.62)$$

where quantities $\check{\beta}_a$, N_{B_a} , $N_{\alpha_{ab}}$, N_a appearing in (4.52)–(4.60) are defined in (A.3)–(A.7).

Expressions above-presented in (4.46)–(4.62) provide the complete generating form description of cubic interaction vertices for coupling of one continuous-spin massive field to two chains of totally symmetric massive fields (2.13). Now our aim is to describe cubic vertices for coupling of one continuous-spin massive field to two totally symmetric massive fields having arbitrary but fixed spin- s_1 and spin- s_2 values. Using the first algebraic constraint in (2.11) it is easy to see that vertices we are interested in must satisfy the algebraic constraints

$$(N_{\alpha_a} + N_{\zeta_a} - s_a)|p_{[3]}^-\rangle = 0, \quad a = 1, 2. \quad (4.63)$$

Two constraints given in (4.63) tell us simply that the cubic vertex $p_{[3]}^-$ should be degree- s_1 and degree- s_2 homogeneous polynomial in the oscillators α_1^i, ζ_1 and α_2^i, ζ_2 respectively. It is easy to check, that, in terms of the vertices $V_{\pm}$ (4.49), algebraic constraints (4.63) take the form

$$(N_{B_1} + N_{\alpha_{12}} + N_{\alpha_{31}} - s_1)V_A = 0, \quad (N_{B_2} + N_{\alpha_{12}} + N_{\alpha_{23}} - s_2)V_A = 0. \quad (4.64)$$

Vertices V_A satisfy one and the same equations (4.64). Therefore to simplify our presentation we drop the superscript A and use a vertex V in place of the vertices V_A , i.e., we use $V = V_A$. Doing so, we note that the general solution to constraints (4.64) can be presented as

$$V = V(s_1, s_2; n_1, n_2, n_3), \quad V(s_1, s_2; n_1, n_2, n_3) = B_1^{s_1 - n_2 - n_3} B_2^{s_2 - n_1 - n_3} \alpha_{12}^{n_3} \alpha_{23}^{n_1} \alpha_{31}^{n_2}. \quad (4.65)$$

The integers n_1, n_2, n_3 appearing in (4.65) are the freedom of our solution for the vertex V . In other words, the three integers n_a label all possible cubic vertices that can be constructed for three fields in (4.45). In order for vertices (4.65) to be sensible, the integers n_a should satisfy the restrictions

$$n_2 + n_3 \leq s_1, \quad n_1 + n_3 \leq s_2, \quad n_a \geq 0, \quad a = 1, 2, 3, \quad (4.66)$$

which amount to the requirement that the powers of B_1, B_2, α_{aa+1} in (4.65) be non-negative. Expressions for cubic vertices given in (4.46)–(4.60), (4.65) and restrictions on values of n_a presented in (4.66) provide the complete description and classification of cubic interaction vertices that can be constructed for two spin- s_1 and spin- s_2 massive fields and one continuous-spin massive field.

5 Parity invariant cubic vertices for two continuous-spin massive/massless fields and one arbitrary spin massive/massless field

In this section, we discuss parity invariant cubic vertices which involve two continuous-spin massive/massless fields and one finite component massive/massless field. According to our classification, such vertices can be separated into five particular cases given in (1.11)–(1.15). Let us discuss these particular cases in turn.

5.1 One continuous-spin massive field, one continuous-spin massless field and one arbitrary spin massless field

We start with discussion of vertices involving one continuous-spin massive field, one continuous-spin massless spin and one arbitrary spin massless field. This is to say that, using the shortcuts $(m, \kappa)_{CSF}$ and $(0, \kappa)_{CSF}$ for the respective continuous-spin massive and massless fields and the shortcut $(0, s)$ for a spin- s massless field, we study cubic vertices for the following three fields:

$$(m_1, \kappa_1)_{CSF}-(0, \kappa_2)_{CSF}-(0, s_3), \quad m_1^2 < 0, \quad (5.1)$$

one continuous-spin massive field, one continuous-spin massless field, and one massless field

Relation (5.1) tells us that massive and massless continuous-spin fields carry the respective external line indices $a = 1$ and $a = 2$, while the massless spin- s_3 field corresponds to $a = 3$.

For fields (5.1), we find the following general solution to cubic vertex $p_{[3]}^-$ (see appendix C)

$$p_{[3]}^- = U_v U_\Gamma U_\beta U_{\partial B} U_{\partial \alpha} U_B U_{JB} U_W U_B^{-1} V_{AB}^{(8)}, \quad A, B = +, -, \quad (5.2)$$

$$p_{[3]}^- = p_{[3]}^-(\beta_a, B_a, \alpha_{aa+1}, v_1, v_2), \quad (5.3)$$

$$V_{AB}^{(8)} = V_{AB}^{(8)}(B_1, B_2, \alpha_{aa+1}). \quad (5.4)$$

In (5.2), we introduce four vertices $V_{AB}^{(8)}$ labelled by the superscripts A, B. In (5.3) and (5.4), the arguments of the generic vertex $p_{[3]}^-$ and the vertices $V_{AB}^{(8)}$ are shown explicitly. The definition of the arguments B_a and α_{ab} may be found in (A.3). Various operators U appearing in (5.2) are defined below. For the four vertices $V_{AB}^{(8)}$ (5.4), we find the following solution:

$$V_{AB}^{(8)} = F_A J_B V_{AB}, \quad V_{AB} = V_{AB}(\alpha_{12}, \alpha_{23}, \alpha_{31}), \quad A, B = +, -, \quad (5.5)$$

$$F_\pm = F\left(\alpha_1, \beta_1, \gamma_1; \frac{1 \pm z_1}{2}\right), \quad J_+ = I_{\nu_2}(\sqrt{z_2}), \quad J_- = K_{\nu_2}(\sqrt{z_2}), \quad (5.6)$$

$$\alpha_1 = \nu_1 + \frac{1}{2} + \sigma_1, \quad \beta_1 = \nu_1 + \frac{1}{2} - \sigma_1, \quad \sigma_1 = \left(\frac{(d-3)^2}{4} + \frac{\kappa_1^2}{m_1^2}\right)^{1/2}, \quad (5.7)$$

$$\gamma_1 = \nu_1 + 1, \quad z_1 = \frac{2B_1}{\kappa_1}, \quad z_2 = \frac{4\kappa_2}{m_1^2} B_2, \quad (5.8)$$

where $F(\alpha, \beta, \gamma; x)$ stands for the hypergeometric function, while I_ν and K_ν are the modified Bessel functions. In (5.6), in place of the variables B_1 and B_2 we use the respective variables z_1 and z_2 (5.8). Operators ν_1, ν_2 are defined below.

From (5.3), we see that the generic vertex $p_{[3]}^-$ depends on the eleven variables, while, from (5.5), we learn that the vertices V_{AB} depend only on the three variables. Note also that, by definition, the vertices V_{AB} (5.5) are expandable in the three variables $\alpha_{12}, \alpha_{23}, \alpha_{31}$. We now present the explicit expressions for the operators U, ν_1, ν_2 . These operators

are given by

$$U_v = v_1^{N_1} v_2^{N_2}, \quad N_1 = N_{B_1} + N_{\alpha_{12}} + N_{\alpha_{31}}, \quad N_2 = N_{B_2} + N_{\alpha_{12}} + N_{\alpha_{23}}, \quad (5.9)$$

$$U_\Gamma = \left(\left(-\frac{\kappa_1^2}{m_1^2} \right)^{N_1} \frac{2^{N_1+N_2} \Gamma(N_1 + \frac{d-2}{2}) \Gamma(N_2 + \frac{d-2}{2})}{\Gamma(N_1 - \lambda_{1+}) \Gamma(N_1 - \lambda_{1-}) \Gamma(N_1 + 1) \Gamma(N_2 + 1)} \right)^{1/2}, \quad (5.10)$$

$$\lambda_{1\pm} = \frac{d-3}{2} \pm \sigma_1, \quad (5.11)$$

$$U_\beta = \exp \left(-\frac{\check{\beta}_1}{2\beta_1} \kappa_1 \partial_{B_1} - \frac{\check{\beta}_2}{2\beta_2} \kappa_2 \partial_{B_2} \right), \quad (5.12)$$

$$U_{\partial B} = \exp \left(\frac{\kappa_2}{2} \partial_{B_2} \right), \quad (5.13)$$

$$U_{\partial\alpha} = \exp \left(-\frac{\kappa_1 \kappa_2}{m_1^2} \partial_{\alpha_{12}} + \frac{\kappa_1}{m_1^2} B_2 \partial_{\alpha_{12}} - \frac{\kappa_1}{m_1^2} B_3 \partial_{\alpha_{31}} + \frac{2\kappa_2}{m_1^2} B_1 \partial_{\alpha_{12}} \right. \\ \left. + \frac{2B_1 B_2}{m_1^2} \partial_{\alpha_{12}} - \frac{2B_2 B_3}{m_1^2} \partial_{\alpha_{23}} + \frac{2B_3 B_1}{m_1^2} \partial_{\alpha_{31}} \right), \quad (5.14)$$

$$U_B = \left(-\frac{\kappa_1^2}{m_1^2} + \frac{4}{m_1^2} B_1^2 \right)^{-(\nu_1+1)/2}, \quad U_{JB} = \left(\frac{4\kappa_2}{m_1^2} B_2 \right)^{-\nu_2/2}, \quad (5.15)$$

$$U_W = U_{\nu_1, W_1} U_{\nu_2, W_{23}}, \quad U_{\nu, W} \equiv \sum_{n=0}^{\infty} \frac{\Gamma(\nu+n)}{4^n n! \Gamma(\nu+2n)} W^n, \quad (5.16)$$

$$W_1 = 2\alpha_{23} \partial_{\alpha_{12}} \partial_{\alpha_{31}} - \partial_{\alpha_{12}}^2, \quad W_{23} = 2\alpha_{31} \partial_{\alpha_{12}} \partial_{\alpha_{23}}, \quad (5.17)$$

$$\nu_1 = N_{\alpha_{12}} + N_{\alpha_{31}} + \frac{d-4}{2}, \quad \nu_2 = N_{\alpha_{12}} + N_{\alpha_{23}} + \frac{d-4}{2}, \quad (5.18)$$

where quantities $\check{\beta}_a$, N_{B_a} , $N_{\alpha_{ab}}$, N_a appearing in (5.9)–(5.18) are defined in (A.3)–(A.7).

Expressions (5.2)–(5.18) provide the complete generating form description of cubic vertices for coupling of continuous-spin massless and massive fields to arbitrary spin massless field. Namely, these vertices describe coupling of two continuous-spin fields to chain of massless fields (2.13). Now our aim is to describe cubic vertices for coupling of continuous-spin massless and massive fields to massless field with arbitrary but fixed spin- s_3 value. Using the first algebraic constraint in (2.12), it is easy to see that vertices we are interested in must satisfy the algebraic constraint

$$(N_{\alpha_3} - s_3) |p_{[3]}^-\rangle = 0, \quad (5.19)$$

which implies that the cubic vertex $p_{[3]}^-$ should be degree- s_3 homogeneous polynomial in the α_3^i . It is easy to check, that, in terms of the vertices V_{AB} (5.5), algebraic constraint (5.19) takes the form

$$(N_{\alpha_{23}} + N_{\alpha_{31}} - s_3) V = 0, \quad V = V_{AB}, \quad (5.20)$$

where to simplify the notation we drop the superscripts AB in V_{AB} . Obviously, general solution to constraints (5.20) can be presented as

$$V = V(s_3; n, k), \quad V(s_3; n, k) = \alpha_{12}^n \alpha_{23}^k \alpha_{31}^{s_3-k}, \quad (5.21)$$

$$n \geq 0, \quad k \geq 0, \quad s_3 - k \geq 0. \quad (5.22)$$

The integers n, k appearing in (5.21) are the freedom of our solution for the vertex V , i.e., the integers n, k label all possible cubic vertices that can be constructed for three fields in (5.1). In order for vertices (5.21) to be sensible, the integers should satisfy the restrictions in (5.22), which amount to the requirement that the powers of all variables $\alpha_{12}, \alpha_{23}, \alpha_{31}$ in (5.21) be non-negative. From (5.22), we see that allowed values of n, k are given by

$$k = 0, 1, \dots, s_3, \quad n = 0, 1, \dots, \infty. \quad (5.23)$$

Expressions for cubic vertices given in (5.2)–(5.18), (5.21) and allowed values for n and k presented in (5.23) provide the complete description and classification of cubic interaction vertices that can be constructed for one continuous-spin massive field, one continuous-spin massless field, and one spin- s_3 massless field.

5.2 Two continuous-spin massive fields with the same mass values and one massless field

In this section, we discuss parity invariant cubic vertices for two continuous-spin massive fields having the same masses and one arbitrary spin massless field. This is to say that, using the shortcut $(m, \kappa)_{CSF}$ for a continuous-spin massive field and the shortcut $(0, s)$ for a spin- s massless field, we study cubic vertices for the following three fields:

$$(m_1, \kappa_1)_{CSF}-(m_2, \kappa_2)_{CSF}-(0, s_3), \quad m_1 = m, \quad m_2 = m, \quad m^2 < 0, \quad (5.24)$$

two continuous-spin massive fields with the same masses and one massless field.

Relation (5.24) tells us that the massive continuous-spin fields carry the respective external line indices $a = 1, 2$, while the massless spin- s_3 field corresponds to $a = 3$.

For fields (5.24), we find the following general solution to cubic vertex $p_{[3]}^-$ (see appendix C)

$$p_{[3]}^- = U_v U_\Gamma U_\beta U_{\partial B} U_{\partial \alpha} U_e U_W V_{AB}^{(7)}, \quad A, B = 1, 2, \dots, 6, \quad (5.25)$$

$$p_{[3]}^- = p_{[3]}^-(\beta_a, B_a, \alpha_{aa+1}, v_1, v_2), \quad (5.26)$$

$$V_{AB}^{(7)} = V_{AB}^{(7)}(B_a, Z, \alpha_{12}), \quad (5.27)$$

$$Z = \left(B_1 - \frac{1}{2}\kappa_1\right)\alpha_{23} + \left(B_2 + \frac{1}{2}\kappa_2\right)\alpha_{31}. \quad (5.28)$$

In (5.25), we introduce vertices $V_{AB}^{(7)}$ labelled by the superscripts A, B . In (5.26) and (5.27), the arguments of the generic vertex $p_{[3]}^-$ and the vertices $V_{AB}^{(7)}$ are shown explicitly. The definition of the arguments B_a and α_{ab} may be found in (A.3). Operators U appearing in (5.25) are defined below. For the vertices $V_{AB}^{(7)}$ (5.27), we find the following solution:

$$V_{AB}^{(6)} = E_{1A} E_{2B} V_{AB}, \quad V_{AB} = V_{AB}(B_3, Z, \alpha_{12}), \quad A, B = 1, 2, \dots, 6, \quad (5.29)$$

$$E_{a1} = B_a^\rho, \quad E_{a2} = B_a^\rho \ln B_a, \quad \text{for } \kappa_a^2 = -\frac{(d-3)^2}{4}m^2, \quad (5.30)$$

$$E_{a3} = B_a^{\rho_{a+}}, \quad E_{a4} = B_a^{\rho_{a-}}, \quad \text{for } \kappa_a^2 < -\frac{(d-3)^2}{4}m^2, \quad (5.31)$$

$$E_{a5} = B_a^{\rho_{a+}} + B_a^{\rho_{a-}}, \quad E_{a6} = i(B_a^{\rho_{a+}} - B_a^{\rho_{a-}}), \quad \text{for } \kappa_a^2 > -\frac{(d-3)^2}{4}m^2, \quad (5.32)$$

$$\rho = -\omega_Z, \quad \rho_{a\pm} = -\omega_Z \pm \left(\frac{(d-3)^2}{4} + \frac{\kappa_a^2}{m^2}\right)^{1/2}, \quad \omega_Z = N_{\alpha_{12}} + N_Z + \frac{d-3}{2}, \quad (5.33)$$

where $a = 1, 2$, $N_Z = Z\partial_Z$. The generic vertex $p_{[3]}^-$ (5.26) depends on eleven variables, while, the vertices V_{AB} (5.29) depend only on three variables. By definition, the vertices V_{AB} (5.29) are expandable in the three variables B_3 , Z , α_{12} . In order to complete the description of the vertex $p_{[3]}^-$ we now present the explicit form of the operators U . These operators are given by

$$U_v = v_1^{N_1} v_2^{N_2}, \quad N_1 = N_{B_1} + N_{\alpha_{12}} + N_{\alpha_{31}}, \quad N_2 = N_{B_2} + N_{\alpha_{12}} + N_{\alpha_{23}}, \quad (5.34)$$

$$U_\Gamma = \prod_{a=1,2} \left(\left(-\frac{\kappa_a^2}{m^2} \right)^{N_a} \frac{2^{N_a} \Gamma(N_a + \frac{d-2}{2})}{\Gamma(N_a - \lambda_{a+}) \Gamma(N_a - \lambda_{a-}) \Gamma(N_a + 1)} \right)^{1/2}, \quad (5.35)$$

$$\lambda_{a\pm} = \frac{d-3}{2} \pm \sigma_a, \quad (5.36)$$

$$U_\beta = \exp \left(-\frac{\check{\beta}_1}{2\beta_1} \kappa_1 \partial_{B_1} - \frac{\check{\beta}_2}{2\beta_2} \kappa_2 \partial_{B_2} \right), \quad (5.37)$$

$$U_{\partial B} = \exp \left(-\frac{\kappa_1}{2} \partial_{B_1} + \frac{\kappa_2}{2} \partial_{B_2} \right), \quad (5.38)$$

$$U_{\partial\alpha} = \exp \left(-\frac{\kappa_1 \kappa_2}{m^2} \partial_{\alpha_{12}} - \frac{\kappa_2}{m^2} B_1 \partial_{\alpha_{12}} + \frac{\kappa_1}{m^2} B_2 \partial_{\alpha_{12}} - \frac{\kappa_1}{m^2} B_2 B_3 \partial_Z + \frac{\kappa_2}{m^2} B_3 B_1 \partial_Z \right. \\ \left. + \frac{1}{2m^2} B_1 B_2 \partial_{\alpha_{12}} - \frac{1}{m^2} B_1 B_2 B_3 \partial_Z \right), \quad (5.39)$$

$$U_e = \left(-\frac{m^2}{\kappa_1^2} \right)^{\omega_1/2} \left(-\frac{m^2}{\kappa_2^2} \right)^{\omega_2/2}, \quad (5.40)$$

$$U_W = U_{\omega_1, W_1} U_{\omega_2, W_{21}}, \quad U_{\omega, W} = \sum_{n=0}^{\infty} \frac{\Gamma(\omega+n)}{4^n n! \Gamma(\omega+2n)} W^n, \quad (5.41)$$

$$W_1 = \frac{1}{m^2} B_2^2 B_3^2 \partial_Z^2 - \partial_{\alpha_{12}}^2 - 2B_2 \partial_{\alpha_{12}} \partial_{B_1}, \quad W_{21} = \frac{1}{m^2} B_1^2 B_3^2 \partial_Z^2 - 2B_1 \partial_{\alpha_{12}} \partial_{B_2}, \quad (5.42)$$

$$\omega_1 = N_{B_1} + N_{\alpha_{12}} + N_Z + \frac{d-3}{2}, \quad \omega_2 = N_{B_2} + N_{\alpha_{12}} + N_Z + \frac{d-3}{2}, \quad N_Z = Z\partial_Z, \quad (5.43)$$

where quantities $\check{\beta}_a$, N_{B_a} , $N_{\alpha_{ab}}$, N_a appearing in (5.34)–(5.43) are defined in (A.3)–(A.7).

Expressions above-presented in (5.25)–(5.43) provide the complete generating form description of cubic vertices for coupling of two continuous-spin massive fields to chain of massless fields (2.13). Now our aim is to describe cubic vertices for coupling of two continuous-spin massive fields to massless field having arbitrary but fixed spin- s_3 value. Using the first algebraic constraint in (2.12), it is easy to see that vertices we are interested in must satisfy the algebraic constraint

$$(N_{\alpha_3} - s_3) |p_{[3]}^- \rangle = 0, \quad (5.44)$$

which implies that the vertex $p_{[3]}^-$ should be degree- s_3 homogeneous polynomial in the oscillators α_3^i . In terms of the vertices V_{AB} (5.29), algebraic constraint (5.44) takes the form

$$(N_{B_3} + N_Z - s_3) V = 0, \quad V \equiv V_{AB}, \quad (5.45)$$

where to simplify the notation we drop the superscripts AB in V_{AB} . General solution to constraint (5.45) can be presented as

$$V = V(s_3; n, k), \quad V(s_3; n, k) = B_3^{s_3-k} Z^k \alpha_{12}^n, \quad (5.46)$$

$$0 \leq k \leq s_3, \quad n \geq 0. \quad (5.47)$$

The integers n and k appearing in (5.46) are the freedom of our solution for the vertex V . In other words, these integers label all possible cubic vertices that can be constructed for three fields in (5.24). In order for vertices (5.46) to be sensible, the integers n, k should satisfy the restrictions in (5.47) which amount to the requirement that the powers of all variables B_3, Z and α_{12} in (5.46) be non-negative. From (5.47), we see that allowed values of n, k are given by

$$k = 0, 1, \dots, s_3, \quad n = 0, 1, \dots, \infty. \quad (5.48)$$

Expressions for cubic interaction vertices given in (5.25)–(5.43), (5.46) and values for n, k presented in (5.48) provide the complete description and classification of cubic vertices that can be constructed for two continuous-spin massive fields and one spin- s_3 massless field (5.24).

5.3 Two continuous-spin massive field with different masses and one massless field

In this section, we discuss parity invariant cubic vertices for two continuous-spin massive fields having different masses and one arbitrary spin massless field. This is to say that, using the shortcut $(m, \kappa)_{CSF}$ for a continuous-spin massive field and the shortcut $(0, s)$ for a spin- s massless field, we study cubic vertices for the following three fields:

$$(m_1, \kappa_1)_{CSF} - (m_2, \kappa_2)_{CSF} - (0, s_3), \quad m_1^2 < 0, \quad m_2^2 < 0, \quad m_1 \neq m_2, \\ \text{two continuous-spin massive fields with different masses and one massless field.} \quad (5.49)$$

Relation (5.49) tells us that two continuous-spin massive fields carry the external line indices $a = 1, 2$, while the spin- s_3 massless field corresponds to $a = 3$.

For fields (5.49), we find the following general solution to cubic vertex $p_{[3]}^-$ (see appendix C)

$$p_{[3]}^- = U_v U_\Gamma U_\beta U_{\partial B} U_{\partial \alpha} U_B U_W U_B^{-1} V_{AB}^{(8)}, \quad A, B = +, -, \quad (5.50)$$

$$p_{[3]}^- = p_{[3]}^-(\beta_a, B_a, \alpha_{aa+1}, v_1, v_2), \quad (5.51)$$

$$V_{AB}^{(8)} = V_{AB}^{(8)}(B_1, B_2, \alpha_{aa+1}). \quad (5.52)$$

In (5.50), we introduce four vertices $V_{AB}^{(8)}$ labelled by the superscripts $A, B = \pm$. In (5.51) and (5.52), the arguments of the generic vertex $p_{[3]}^-$ and the vertices $V_{AB}^{(8)}$ are shown explicitly. The definition of the arguments B_a and α_{ab} may be found in (A.3). Various quantities U appearing in (5.50) are differential operators w.r.t. the B_a and α_{aa+1} . These quantities

will be presented below. For four vertices $V_{AB}^{(8)}$ (5.52), we find the following solution:

$$V_{AB}^{(6)} = F_{1A} F_{2B} V, \quad V_{AB} = V_{AB}(\alpha_{12}, \alpha_{23}, \alpha_{31}), \quad A, B = +, -, \quad (5.53)$$

$$F_{a\pm} = F\left(\alpha_a, \beta_a, \gamma_a; \frac{1 \pm z_a}{2}\right), \quad \alpha_a = \nu_a + \frac{1}{2} + \sigma_a, \quad \beta_a = \nu_a + \frac{1}{2} - \sigma_a, \quad (5.54)$$

$$\gamma_a = \nu_a + 1, \quad z_a = \frac{2m_a^2 B_a}{\kappa_a(m_1^2 - m_2^2)}, \quad \sigma_a = \left(\frac{(d-3)^2}{4} + \frac{\kappa_a^2}{m_a^2}\right)^{1/2}, \quad a = 1, 2, \quad (5.55)$$

where the $F(\alpha, \beta, \gamma; x)$ stands for the hypergeometric function. In (5.53), in place of the variables B_1, B_2 , we use new variables z_1, z_2 defined in (5.55). Operators ν_a are given below.

The generic vertex $p_{[3]}^-$ (5.51) depends on the eleven variables, while, the vertices V_{AB} (5.53) depend only on the three variables. Note also that, by definition, the vertices V_{AB} (5.53) are expandable in the three variables $\alpha_{12}, \alpha_{23}, \alpha_{31}$. In order to complete the description of the vertex $p_{[3]}^-$ we now provide expressions for the operators U, ν_1, ν_2 . These operators are given by

$$U_v = v_1^{N_1} v_2^{N_2}, \quad N_1 = N_{B_1} + N_{\alpha_{12}} + N_{\alpha_{31}}, \quad N_2 = N_{B_2} + N_{\alpha_{12}} + N_{\alpha_{23}}, \quad (5.56)$$

$$U_\Gamma = \prod_{a=1,2} \left(\left(-\frac{\kappa_a^2}{m_a^2} \right)^{N_a} \frac{2^{N_a} \Gamma(N_a + \frac{d-2}{2})}{\Gamma(N_a - \lambda_{a+}) \Gamma(N_a - \lambda_{a-}) \Gamma(N_a + 1)} \right)^{1/2}, \quad \lambda_{a\pm} = \frac{d-3}{2} \pm \sigma_a, \quad (5.57)$$

$$U_\beta = \exp \left(-\frac{\check{\beta}_1}{2\beta_1} \kappa_1 \partial_{B_1} - \frac{\check{\beta}_2}{2\beta_2} \kappa_2 \partial_{B_2} \right), \quad (5.58)$$

$$U_{\partial B} = \exp \left(-\frac{m_2^2}{2m_1^2} \kappa_1 \partial_{B_1} + \frac{m_1^2}{2m_2^2} \kappa_2 \partial_{B_2} \right), \quad (5.59)$$

$$U_{\partial\alpha} = \exp \left(-\frac{\kappa_1 \kappa_2 (m_1^2 + m_2^2)}{2m_1^2 m_2^2} \partial_{\alpha_{12}} + \frac{\kappa_1}{m_1^2} (B_2 \partial_{\alpha_{12}} - B_3 \partial_{\alpha_{31}}) + \frac{\kappa_2}{m_2^2} (B_3 \partial_{\alpha_{23}} - B_1 \partial_{\alpha_{12}}) \right. \\ \left. + \frac{2(m_1^2 + m_2^2)}{(m_1^2 - m_2^2)^2} B_1 B_2 \partial_{\alpha_{12}} - \frac{2}{m_1^2 - m_2^2} B_2 B_3 \partial_{\alpha_{23}} + \frac{2}{m_1^2 - m_2^2} B_3 B_1 \partial_{\alpha_{31}} \right), \quad (5.60)$$

$$U_B = \prod_{a=1,2} \left(-\frac{\kappa_a^2}{m_a^2} + \frac{4m_a^2}{(m_1^2 - m_2^2)^2} B_a^2 \right)^{-(\nu_a+1)/2}, \quad (5.61)$$

$$U_W = U_{\nu_1, W_1} U_{\nu_2, W_{21}}, \quad U_{\nu, W} = \sum_{n=0}^{\infty} \frac{\Gamma(\nu+n)}{4^n n! \Gamma(\nu+2n)} W^n, \quad (5.62)$$

$$W_1 = 2\alpha_{23} \partial_{\alpha_{12}} \partial_{\alpha_{31}} - \partial_{\alpha_{12}}^2, \quad W_{21} = 2\alpha_{31} \partial_{\alpha_{12}} \partial_{\alpha_{23}}, \quad (5.63)$$

$$\nu_1 = N_{\alpha_{12}} + N_{\alpha_{31}} + \frac{d-4}{2}, \quad \nu_2 = N_{\alpha_{12}} + N_{\alpha_{23}} + \frac{d-4}{2}, \quad (5.64)$$

where quantities $\check{\beta}_a, N_{B_a}, N_{\alpha_{ab}}, N_a$ appearing in (5.56)–(5.64) are defined in (A.3)–(A.7).

Expressions (5.50)–(5.64) provide the complete generating form description of cubic vertices for coupling of two continuous-spin massive fields having different masses to chain of massless fields (2.13). Now our aim is to describe cubic vertices for coupling of two continuous-spin massive fields having different masses to massless field having arbitrary

but fixed spin- s_3 value. Using the first algebraic constraint in (2.12), it is easy to see that vertices we are interested in must satisfy the algebraic constraint

$$(N_{\alpha_3} - s_3)|p_{[3]}^-\rangle = 0, \quad (5.65)$$

which implies that the cubic vertex $p_{[3]}^-$ should be degree- s_3 homogeneous polynomial in the oscillators α_3^i . In terms of the vertices V_{AB} (5.53), algebraic constraint (5.65) takes the form

$$(N_{\alpha_{23}} + N_{\alpha_{31}} - s_3)V = 0, \quad V = V_{AB}. \quad (5.66)$$

General solution to constraint (5.66) can be presented as

$$V = V(s_3; n, k), \quad V(s_3; n, k) = \alpha_{12}^n \alpha_{23}^k \alpha_{31}^{s_3-k}, \quad (5.67)$$

$$0 \leq k \leq s_3, \quad n \geq 0. \quad (5.68)$$

The integers n, k appearing in (5.67) are the freedom of our solution for the vertex V . In other words, these integers label all possible cubic vertices that can be constructed for three fields in (5.49). In order for vertices (5.67) to be sensible, the integers n, k should satisfy the restrictions in (5.68) which amount to the requirement that the powers of all variables $\alpha_{12}, \alpha_{23}, \alpha_{31}$ in (5.67) be non-negative. From (5.68), we see that allowed values of n, k are given by

$$k = 0, 1, \dots, s_3, \quad n = 0, 1, \dots, \infty. \quad (5.69)$$

Relations (5.50)–(5.64), (5.67), and (5.69) provide the complete description and classification of cubic vertices that can be constructed for fields in (5.49).

5.4 One continuous-spin massless field, one continuous-spin massive field, and one arbitrary spin massive field

In this section, we discuss parity invariant cubic vertices for one continuous-spin massless field, one continuous-spin massive field, and one arbitrary spin massive field. This is to say that, using the shortcut $(m, \kappa)_{CSF}$ for a continuous-spin mass- m field and the shortcut (m, s) for the mass- m and spin- s field, we study cubic vertices for the following three fields:

$$(0, \kappa_1)_{CSF} - (m_2, \kappa_2)_{CSF} - (m_3, s_3), \quad m_2^2 < 0, \quad m_3^2 > 0, \quad (5.70)$$

one continuous-spin massless field, one continuous-spin massive field, and one massive field.

Relation (5.70) tells us that two continuous-spin massless and massive fields carry the external line indices $a = 1, 2$, while the spin- s_3 massive field corresponds to $a = 3$.

For fields (5.70), we find the following general solution to cubic vertex $p_{[3]}^-$ (see appendix C)

$$p_{[3]}^- = U_v U_\Gamma U_\beta U_\zeta U_{\partial B} U_{\partial \alpha} U_{JB} U_B U_W U_B^{-1} V_{AB}^{(8)}, \quad A, B = +, -, \quad (5.71)$$

$$p_{[3]}^- = p_{[3]}^-(\beta_a, B_a, \alpha_{aa+1}, v_1, v_2, \zeta_3), \quad (5.72)$$

$$V_{AB}^{(8)} = V_{AB}^{(8)}(B_a, \alpha_{aa+1}). \quad (5.73)$$

In (5.71), we introduce four vertices $V_{AB}^{(8)}$ labelled by the superscripts $AB = \pm$. In (5.72) and (5.73), the arguments of the generic vertex $p_{[3]}^-$ and the vertices $V_{AB}^{(8)}$ are shown explicitly. The definition of the arguments B_a and α_{ab} may be found in (A.3). Various quantities U appearing in (5.71) are differential operators w.r.t. the B_a and α_{aa+1} . These quantities will be presented below. For four vertices $V_{AB}^{(8)}$ (5.73), we find the following solution:

$$V_{AB}^{(8)} = J_{1A} F_{2B} V, \quad V_{AB} = V_{AB}(B_3, \alpha_{12}, \alpha_{23}, \alpha_{31}), \quad (5.74)$$

$$J_{1+} = I_{\nu_1}(\sqrt{z_1}), \quad J_{1-} = K_{\nu_1}(\sqrt{z_1}), \quad z_1 = -\frac{4\kappa_1 B_1}{m_2^2 - m_3^2}, \quad (5.75)$$

$$F_{2\pm} = F\left(\alpha_2, \beta_2, \gamma_2; \frac{1 \pm z_2}{2}\right), \quad \alpha_2 = \nu_2 + \frac{1}{2} + \sigma_2, \quad \beta_2 = \nu_2 + \frac{1}{2} - \sigma_2, \quad (5.76)$$

$$\gamma_2 = \nu_2 + 1, \quad z_2 = \frac{2m_2^2 B_2}{\kappa_2(m_2^2 - m_3^2)}, \quad \sigma_2 = \left(\frac{(d-3)^2}{4} + \frac{\kappa_2^2}{m_2^2}\right)^{1/2}, \quad (5.77)$$

where $F(\alpha, \beta, \gamma; x)$ is the hypergeometric function, while I_ν , K_ν are the modified Bessel functions. In (5.74), in place of B_1 , B_2 , we use z_1 (5.75) and z_2 (5.77). Operators ν_a are given below.

The generic vertex $p_{[3]}^-$ (5.72) depends on the twelve variables, while, the vertices V_{AB} (5.74) depend only on the four variables. Note also that, by definition, the vertices V_{AB} (5.74) are expandable in the four variables B_3 , α_{12} , α_{23} , α_{31} . In order to complete the description of the vertex $p_{[3]}^-$ we now provide expressions for the operators U , ν_1 , ν_2 . These operators are given by

$$U_\nu = \nu_1^{N_1} \nu_2^{N_2}, \quad N_1 = N_{B_1} + N_{\alpha_{12}} + N_{\alpha_{31}}, \quad N_2 = N_{B_2} + N_{\alpha_{12}} + N_{\alpha_{23}}, \quad (5.78)$$

$$U_\Gamma = \left(\frac{2^{N_1} \Gamma(N_1 + \frac{d-2}{2})}{\Gamma(N_1 + 1)} \left(-\frac{\kappa_2^2}{m_2^2} \right)^{N_2} \frac{2^{N_2} \Gamma(N_2 + \frac{d-2}{2})}{\Gamma(N_2 - \lambda_{2+}) \Gamma(N_2 - \lambda_{2-}) \Gamma(N_2 + 1)} \right)^{1/2},$$

$$\lambda_{2\pm} = \frac{d-3}{2} \pm \sigma_2, \quad (5.79)$$

$$U_\beta = \exp \left(-\frac{\check{\beta}_1}{2\beta_1} \kappa_1 \partial_{B_1} - \frac{\check{\beta}_2}{2\beta_2} \kappa_2 \partial_{B_2} - \frac{\check{\beta}_3}{2\beta_3} m_3 \zeta_3 \partial_{B_3} \right), \quad (5.80)$$

$$U_\zeta = \exp \left(\frac{\zeta_3}{m_3} \left(B_1 - \frac{1}{2} \kappa_1 \right) \partial_{\alpha_{31}} - \frac{\zeta_3}{m_3} \left(B_2 + \frac{1}{2} \kappa_2 \right) \partial_{\alpha_{23}} + \frac{m_2^2}{2m_3} \zeta_3 \partial_{B_3} \right), \quad (5.81)$$

$$U_{\partial B} = \exp \left(-\frac{\kappa_1(m_2^2 + m_3^2)}{2(m_2^2 - m_3^2)} \kappa_1 \partial_{B_1} - \frac{m_3^2}{2m_2^2} \kappa_2 \partial_{B_2} \right), \quad (5.82)$$

$$U_{\partial\alpha} = \exp \left(-\frac{\kappa_2}{q_1} \partial_{\alpha_{12}} - \frac{\kappa_2 B_1}{m_2^2} \partial_{\alpha_{12}} - \frac{2m_2^2}{\kappa_1 q_1^2} B_2 \partial_{\alpha_{12}} + \frac{2m_3^2}{\kappa_1 q_1^2} B_3 \partial_{\alpha_{31}} + \frac{\kappa_2 B_3}{m_2^2} \partial_{\alpha_{23}} \right. \\ \left. - \frac{2B_1 B_3}{\kappa_1 q_1} \partial_{\alpha_{31}} + \frac{2B_1 B_2}{\kappa_1 q_1} \partial_{\alpha_{12}} + \frac{2(m_2^2 + m_3^2) B_2 B_3}{\kappa_1^2 q_1^2} \partial_{\alpha_{23}} \right), \quad q_1 = \frac{m_2^2 - m_3^2}{\kappa_1}, \quad (5.83)$$

$$U_{JB} = \left(-\frac{4\kappa_1 B_1}{m_2^2 - m_3^2} \right)^{-\nu_1/2}, \quad U_B = \left(-\frac{\kappa_2^2}{m_2^2} + \frac{4m_2^2}{(m_2^2 - m_3^2)^2} B_2^2 \right)^{-(\nu_2+1)/2}, \quad (5.84)$$

$$U_W = U_{\nu_1, W_1} U_{\nu_2, W_{23}}, \quad U_{\nu, W} = \sum_{n=0}^{\infty} \frac{\Gamma(\nu+n)}{4^n n! \Gamma(\nu+2n)} W^n, \quad (5.85)$$

$$W_1 = 2\alpha_{23}\partial_{\alpha_{12}}\partial_{\alpha_{31}} - \partial_{\alpha_{12}}^2 - \frac{4m_3^2}{(m_2^2 - m_3^2)^2} B_3^2 \partial_{\alpha_{31}}^2, \quad (5.86)$$

$$W_{23} = 2\alpha_{31}\partial_{\alpha_{12}}\partial_{\alpha_{23}} - \frac{4m_3^2}{(m_2^2 - m_3^2)^2} B_3^2 \partial_{\alpha_{23}}^2, \quad (5.87)$$

$$\nu_1 = N_{\alpha_{12}} + N_{\alpha_{31}} + \frac{d-4}{2}, \quad \nu_2 = N_{\alpha_{12}} + N_{\alpha_{23}} + \frac{d-4}{2}, \quad (5.88)$$

where quantities $\tilde{\beta}_a$, N_{B_a} , $N_{\alpha_{ab}}$, N_a appearing in (5.78)–(5.88) are defined in (A.3)–(A.7).

Expressions above-presented in (5.71)–(5.88) provide the complete generating form description of cubic vertices for coupling of two continuous-spin fields to chain of massive fields (2.13). Now our aim is to describe cubic vertices for coupling of two continuous-spin fields to massive field having arbitrary but fixed spin- s_3 value. Using the first algebraic constraint (2.11), it is easy to see that vertices we are interested in must satisfy the algebraic constraint

$$(N_{\alpha_3} + N_{\zeta_3} - s_3)|p_{[3]}^-\rangle = 0, \quad (5.89)$$

which implies that the vertex $p_{[3]}^-$ should be degree- s_3 homogeneous polynomial in the oscillators α_3^i , ζ_3 . In terms of the vertices V_{AB} (5.74), algebraic constraint (5.89) takes the form

$$(N_{B_3} + N_{\alpha_{23}} + N_{\alpha_{31}} - s_3)V = 0, \quad V = V_{AB}. \quad (5.90)$$

General solution to constraint (5.90) can be presented as

$$V = V(s_3; n_1, n_2, n_3), \quad V(s_3; n_1, n_2, n_3) = B_3^{s_3 - n_1 - n_2} \alpha_{12}^{n_3} \alpha_{23}^{n_1} \alpha_{31}^{n_2}, \quad (5.91)$$

$$n_1 + n_2 \leq s_3, \quad n_a \geq 0, \quad \text{for } a = 1, 2, 3. \quad (5.92)$$

The integers n_1 , n_2 , n_3 appearing in (5.91) are the freedom of our solution for the vertex V . These integers label all possible cubic vertices that can be constructed for three fields in (5.70). In order for vertices (5.91) to be sensible, the integers n_a should satisfy the restrictions in (5.92) which amount to the requirement that the powers of all variables B_3 , α_{12} , α_{23} , α_{31} in (5.91) be non-negative. Relations given in (5.71)–(5.88), (5.91), and (5.92) provide the complete description and classification of cubic interaction vertices that can be constructed for fields in (5.70).

5.5 Two continuous-spin massive fields and one massive field

In this section, we discuss parity invariant cubic vertices for two continuous-spin massive fields and one arbitrary spin massive field. This is to say that, using the shortcut $(m, \kappa)_{CSF}$ for a continuous-spin massive field and the shortcut (m, s) for mass- m and spin- s massive field, we study cubic vertices for the following three fields:

$$(m_1, \kappa_1)_{CSF} - (m_2, \kappa_2)_{CSF} - (m_3, s_3), \quad m_1^2 < 0, \quad m_2^2 < 0, \quad m_3^2 > 0, \quad (5.93)$$

two continuous-spin massive fields and one massive field.

Relation (5.93) tells us that the two continuous-spin massive fields carry the external line indices $a = 1, 2$, while the spin- s_3 massive field corresponds to $a = 3$.

For fields (5.93), we find the following general solution to cubic vertex $p_{[3]}^-$ (see appendix C)

$$p_{[3]}^- = U_v U_\Gamma U_\beta U_\zeta U_{\partial B} U_{\partial \alpha} U_B U_W U_B^{-1} V_{AB}^{(8)}, \quad A, B = +, -, \quad (5.94)$$

$$p_{[3]}^- = p_{[3]}^-(\beta_a, B_a, \alpha_{aa+1}, v_1, v_2, \zeta_3), \quad (5.95)$$

$$V_{AB}^{(8)} = V_{AB}^{(8)}(B_a, \alpha_{aa+1}). \quad (5.96)$$

In (5.94), we introduce four vertices $V_{AB}^{(8)}$ labelled by the superscripts A, B. In (5.95) and (5.96), the arguments of the generic vertex $p_{[3]}^-$ and the vertices $V_{AB}^{(8)}$ are shown explicitly. The definition of the arguments B_a and α_{ab} may be found in (A.3). Various quantities U appearing in (5.94) are differential operators w.r.t. the B_a and α_{aa+1} . These quantities will be presented below. For four vertices $V_{AB}^{(8)}$ (5.96), we find the following solution:

$$V_{AB}^{(6)} = F_{1A} F_{2B} V_{AB}, \quad V_{AB} = V_{AB}(B_3, \alpha_{12}, \alpha_{23}, \alpha_{31}), \quad A, B = +, -, \quad (5.97)$$

$$F_{a\pm} = F\left(\alpha_a, \beta_a, \gamma_a; \frac{1 \pm z_a}{2}\right), \quad \alpha_a = \nu_a + \frac{1}{2} + \sigma_a, \quad \beta_a = \nu_a + \frac{1}{2} - \sigma_a, \quad (5.98)$$

$$\gamma_a = \nu_a + 1, \quad z_a = -\frac{2m_a^2}{\kappa_a \sqrt{D}} B_a, \quad \sigma_a = \left(\frac{(d-3)^2}{4} + \frac{\kappa_a^2}{m_a^2}\right)^{1/2}, \quad a = 1, 2, \quad (5.99)$$

where the $F(\alpha, \beta, \gamma; x)$ stands for the hypergeometric function. In (5.97), in place of the variables B_a , we use new variables z_a defined in (5.99). A quantity D and operators ν_a are defined below.

The generic vertex $p_{[3]}^-$ (5.95) depends on the twelve variables, while, the vertices V_{AB} (5.97) depend only on the four variables. Note also that, by definition, the vertices V_{AB} (5.97) are expandable in the four variables $B_3, \alpha_{12}, \alpha_{23}, \alpha_{31}$. To complete the description of the vertex $p_{[3]}^-$ we provide expressions for the operators U, ν_a . These operators are given by

$$U_v = v_1^{N_1} v_2^{N_2}, \quad N_1 = N_{B_1} + N_{\alpha_{12}} + N_{\alpha_{31}}, \quad N_2 = N_{B_2} + N_{\alpha_{12}} + N_{\alpha_{23}}, \quad (5.100)$$

$$U_\Gamma = \prod_{a=1,2} \left(\left(-\frac{\kappa_a^2}{m_a^2} \right)^{N_a} \frac{2^{N_a} \Gamma(N_a + \frac{d-2}{2})}{\Gamma(N_a - \lambda_{a+}) \Gamma(N_a - \lambda_{a-}) \Gamma(N_a + 1)} \right)^{1/2}, \quad \lambda_{a\pm} = \frac{d-3}{2} \pm \sigma_a, \quad (5.101)$$

$$U_\beta = \exp \left(-\frac{\check{\beta}_1}{2\beta_1} \kappa_1 \partial_{B_1} - \frac{\check{\beta}_2}{2\beta_2} \kappa_2 \partial_{B_2} - \frac{\check{\beta}_3}{2\beta_3} m_3 \zeta_3 \partial_{B_3} \right), \quad (5.102)$$

$$U_\zeta = \exp \left(\frac{\zeta_3}{m_3} \left(B_1 - \frac{1}{2} \kappa_1 \right) \partial_{\alpha_{31}} - \frac{\zeta_3}{m_3} \left(B_2 + \frac{1}{2} \kappa_2 \right) \partial_{\alpha_{23}} - \frac{\zeta_3}{2m_3} (m_1^2 - m_2^2) \partial_{B_3} \right), \quad (5.103)$$

$$U_{\partial B} = \exp \left(-\frac{m_2^2}{2m_1^2} \kappa_1 \partial_{B_1} + \frac{m_1^2}{2m_2^2} \kappa_2 \partial_{B_2} \right), \quad (5.104)$$

$$U_{\partial \alpha} = \exp \left(-\frac{\kappa_1 \kappa_2 h_{12}}{2m_1^2 m_2^2} \partial_{\alpha_{12}} + \frac{\kappa_1}{m_1^2} (B_2 \partial_{\alpha_{12}} - B_3 \partial_{\alpha_{31}}) + \frac{\kappa_2}{m_2^2} (B_3 \partial_{\alpha_{23}} - B_1 \partial_{\alpha_{12}}) \right. \\ \left. + \frac{2}{D} h_{12} B_1 B_2 \partial_{\alpha_{12}} + \frac{2}{D} h_{23} B_2 B_3 \partial_{\alpha_{23}} + \frac{2}{D} h_{31} B_3 B_1 \partial_{\alpha_{31}} \right), \quad (5.105)$$

$$U_B = \prod_{a=1,2} \left(-\frac{\kappa_a^2}{m_a^2} + \frac{4m_a^2}{D} B_a^2 \right)^{-(\nu_a+1)/2}, \quad (5.106)$$

$$U_W = U_{\nu_1, W_1} U_{\nu_2, W_{21}}, \quad U_{\nu, W} = \sum_{n=0}^{\infty} \frac{\Gamma(\nu+n)}{4^n n! \Gamma(\nu+2n)} W^n, \quad (5.107)$$

$$W_1 = 2\alpha_{23} \partial_{\alpha_{12}} \partial_{\alpha_{31}} - \partial_{\alpha_{12}}^2, \quad W_{21} = 2\alpha_{31} \partial_{\alpha_{12}} \partial_{\alpha_{23}}, \quad (5.108)$$

$$\nu_1 = N_{\alpha_{12}} + N_{\alpha_{31}} + \frac{d-4}{2}, \quad \nu_2 = N_{\alpha_{12}} + N_{\alpha_{23}} + \frac{d-4}{2}, \quad (5.109)$$

$$h_{aa+1} = m_a^2 + m_{a+1}^2 - m_{a+2}^2, \quad (5.110)$$

$$D = m_1^4 + m_2^4 + m_3^4 - 2(m_1^2 m_2^2 + m_2^2 m_3^2 + m_3^2 m_1^2), \quad (5.111)$$

where quantities $\tilde{\beta}_a$, N_{B_a} , $N_{\alpha_{ab}}$, N_a appearing in (5.100)–(5.109) are defined in (A.3)–(A.7).

Expressions above-presented in (5.94)–(5.111) provide the complete generating form description of cubic vertices for coupling of two continuous-spin massive fields to chain of massive fields (2.13). Now our aim is to describe cubic vertices for coupling of two continuous-spin massive fields to one massive field having arbitrary but fixed spin- s_3 value. Using the first algebraic constraint (2.11), is easy to see that vertices we are interested in must satisfy the algebraic constraint

$$(N_{\alpha_3} + N_{\zeta_3} - s_3) |p_{[3]}^-\rangle = 0, \quad (5.112)$$

which implies that the cubic vertex $p_{[3]}^-$ should be degree- s_3 homogeneous polynomial in the oscillators α_3^i , ζ_3 . In terms of the vertices V_{AB} (5.97), algebraic constraint (5.112) takes the form

$$(N_{B_3} + N_{\alpha_{23}} + N_{\alpha_{31}} - s_3) V = 0, \quad V \equiv V_{AB}, \quad (5.113)$$

where to simplify our notation we drop the superscripts A, B and use a vertex V in place of the vertex V_{AB} . General solution to constraint (5.113) can be presented as

$$V = V(s_3; n_1, n_2, n_3), \quad V(s_3; n_1, n_2, n_3) = B_3^{s_3-n_1-n_2} \alpha_{12}^{n_3} \alpha_{23}^{n_1} \alpha_{31}^{n_2}, \quad (5.114)$$

$$n_1 + n_2 \leq s_3, \quad n_a \geq 0, \quad \text{for } a = 1, 2, 3. \quad (5.115)$$

The integers n_1, n_2, n_3 in (5.114) are the freedom of our solution for the vertex V . In other words, the integers n_a label all possible cubic vertices that can be constructed for three fields in (5.93). In order for vertices (5.114) to be sensible, the integers n_a should satisfy the restrictions (5.115) which amount to the requirement that the powers of all variables B_3 , α_{12} , α_{23} , α_{31} in (5.114) be non-negative. Relations in (5.94)–(5.109), (5.114), and (5.115) provide the complete description and classification of cubic interaction vertices that can be constructed for fields in (5.93).

6 Parity invariant cubic vertices for three continuous-spin fields

In this section, we discuss parity invariant cubic vertices which involve three continuous-spin fields. According to our classification, such vertices can be separated into four particular cases given in (1.16)–(1.19). Let us discuss these particular cases in turn.

6.1 Two continuous-spin massless fields and one continuous-spin massive field

We start with discussion of parity invariant cubic vertices for two continuous-spin massless fields and one continuous-spin massive field. This is to say that, using the shortcut $(m, \kappa)_{CSF}$ for a mass- m continuous-spin field, we study cubic vertices for the following three fields:

$$(0, \kappa_1)_{CSF} - (0, \kappa_2)_{CSF} - (m_3, \kappa_3)_{CSF}, \quad m_3^2 < 0, \quad \text{two continuous-spin massless fields and one continuous-spin massive field.} \quad (6.1)$$

Relation (6.1) tells us that the massless continuous-spin fields carry the external line indices $a = 1, 2$, while the massive continuous-spin field corresponds to $a = 3$.

For fields (6.1), we find the following general solution to cubic vertex $p_{[3]}^-$ (see appendix D)

$$p_{[3]}^- = U_v U_\Gamma U_\beta U_{\partial B} U_{\partial \alpha} U_{JB} U_B U_W U_B^{-1} V_{ABC}^{(7)}, \quad A, B, C = +, -, \quad (6.2)$$

$$p_{[3]}^- = p_{[3]}^-(\beta_a, B_a, \alpha_{aa+1}, v_a), \quad (6.3)$$

$$V_{ABC}^{(7)} = V_{ABC}^{(7)}(B_a, \alpha_{aa+1}). \quad (6.4)$$

In (6.2), we introduce eight vertices $V_{ABC}^{(7)}$ labelled by the superscripts A, B, C. In (6.3) and (6.4), the arguments of the generic vertex $p_{[3]}^-$ and the vertices $V_{ABC}^{(7)}$ are shown explicitly. The definition of the arguments B_a and α_{ab} may be found in (A.3). Various quantities U appearing in (6.2) are differential operators w.r.t. the B_a and α_{aa+1} . These quantities will be presented below. For eight vertices $V_{ABC}^{(7)}$ (6.4), we find the following solution:

$$V_{ABC}^{(6)} = J_{1A} J_{2B} F_{3C} V_{ABC}, \quad V_{ABC} = V_{ABC}(\alpha_{12}, \alpha_{23}, \alpha_{31}), \quad (6.5)$$

$$J_{a+} = I_{\nu_a}(\sqrt{z_a}), \quad J_{a-} = K_{\nu_a}(\sqrt{z_a}), \quad z_1 = \frac{4\kappa_1 B_1}{m_3^2}, \quad z_2 = -\frac{4\kappa_2 B_2}{m_3^2}, \quad a = 1, 2, \quad (6.6)$$

$$F_{3\pm} = F\left(\alpha_3, \beta_3, \gamma_3; \frac{1 \pm z_3}{2}\right), \quad \alpha_3 = \nu_3 + \frac{1}{2} + \sigma_3, \quad \beta_3 = \nu_3 + \frac{1}{2} - \sigma_3, \quad (6.7)$$

$$\gamma_3 = \nu_3 + 1, \quad z_3 = \frac{2B_3}{\kappa_3}, \quad \sigma_3 = \left(\frac{(d-3)^2}{4} + \frac{\kappa_3^2}{m_3^2}\right)^{1/2}, \quad (6.8)$$

where, $I_\nu(x)$ and $K_\nu(x)$ (6.6) are the modified Bessel functions, while the $F(\alpha, \beta, \gamma; x)$ (6.7) is the hypergeometric function. In (6.5), in place of the variables B_1, B_2, B_3 , we use new variables z_1, z_2, z_3 defined in (6.6) and (6.8). Operators ν_a are defined below.

The generic vertex $p_{[3]}^-$ (6.3) depends on the twelve variables, while, the vertices V_{ABC} (6.5) depend only on the three variables. Note also that, by definition, the vertices V_{ABC} (6.5) are expandable in the three variables $\alpha_{12}, \alpha_{23}, \alpha_{31}$. To complete the description of the vertex $p_{[3]}^-$ we provide expressions for the operators U, ν_a . These operators are

given by

$$U_v = v_1^{N_1} v_2^{N_2} v_3^{N_3}, \quad N_a = N_{B_a} + N_{\alpha_{aa+1}} + N_{\alpha_{a+2a}}, \quad (6.9)$$

$$U_\Gamma = \left(\left(-\frac{\kappa_3^2}{m_3^2} \right)^{N_3} \frac{2^{N_3} \Gamma(N_3 + \frac{d-2}{2})}{\Gamma(N_3 - \lambda_{3+}) \Gamma(N_3 - \lambda_{3-}) \Gamma(N_3 + 1)} \prod_{a=1,2} \frac{2^{N_a} \Gamma(N_a + \frac{d-2}{2})}{\Gamma(N_a + 1)} \right)^{1/2}, \quad (6.10)$$

$$\lambda_{3\pm} = \frac{d-3}{2} \pm \sigma_3, \quad (6.11)$$

$$U_\beta = \exp \left(- \sum_{a=1,2,3} \frac{\check{\beta}_a}{2\beta_a} \kappa_a \partial_{B_a} \right), \quad (6.12)$$

$$U_{\partial B} = \exp \left(\frac{\kappa_1}{2} \partial_{B_1} - \frac{\kappa_2}{2} \partial_{B_2} \right), \quad (6.13)$$

$$U_{\partial\alpha} = \exp \left(\frac{1}{m_3^2} (2B_1 B_3 + \kappa_3 B_1 + 2\kappa_1 B_3 - \kappa_1 \kappa_3) \partial_{\alpha_{31}} \right. \\ \left. + \frac{1}{m_3^2} (2B_2 B_3 - \kappa_3 B_2 - 2\kappa_2 B_3 - \kappa_2 \kappa_3) \partial_{\alpha_{23}} - \frac{2}{m_3^2} (B_1 B_2 - \kappa_1 \kappa_2) \partial_{\alpha_{12}} \right), \quad (6.14)$$

$$U_{JB} = \left(\frac{4\kappa_1}{m_3^2} B_1 \right)^{-\nu_1/2} \left(-\frac{4\kappa_2}{m_3^2} B_2 \right)^{-\nu_2/2}, \quad U_B = \left(-\frac{\kappa_3^2}{m_3^2} + \frac{4}{m_3^2} B_3^2 \right)^{-(\nu_3+1)/2}, \quad (6.15)$$

$$U_W = U_{\nu_1, W_1} U_{\nu_2, W_{23}} U_{\nu_3, W_{312}}, \quad U_{\nu, W} \equiv \sum_{n=0}^{\infty} \frac{\Gamma(\nu+n)}{4^n n! \Gamma(\nu+2n)} W^n, \quad (6.16)$$

$$W_1 = 2\alpha_{23} \partial_{\alpha_{12}} \partial_{\alpha_{31}} - \partial_{\alpha_{12}}^2 - \partial_{\alpha_{31}}^2, \quad W_{23} = 2\alpha_{31} \partial_{\alpha_{12}} \partial_{\alpha_{23}} - \partial_{\alpha_{23}}^2, \quad (6.17)$$

$$W_{312} = 2\alpha_{12} \partial_{\alpha_{23}} \partial_{\alpha_{31}}, \quad \nu_a = N_{\alpha_{aa+1}} + N_{\alpha_{a+2a}} + \frac{d-4}{2}, \quad (6.18)$$

where quantities $\check{\beta}_a$, N_{B_a} , $N_{\alpha_{ab}}$, N_a appearing in (6.9)–(6.18) are defined in (A.3)–(A.7).

As we have already said, the vertices V_{ABC} (6.5) should be expandable in the three variables α_{12} , α_{23} , α_{31} . Using the simplified notation $V = V_{ABC}$, we then note that a general representative of the vertex V can be chosen to be

$$V = V(n_1, n_2, n_3), \quad V(n_1, n_2, n_3) = \alpha_{12}^{n_3} \alpha_{23}^{n_1} \alpha_{31}^{n_2}, \quad (6.19)$$

$$n_a \geq 0, \quad a = 1, 2, 3. \quad (6.20)$$

The three integers n_a in (6.19) are the freedom of our solution for the vertex V . In other words, the integers n_a label all possible cubic vertices that can be constructed for three fields in (6.1). In order for vertices (6.19) to be sensible, the integers n_a should satisfy the restrictions (6.20) which amount to the requirement that the powers of all variables α_{12} , α_{23} , α_{31} in (6.19) be non-negative. Relations given in (6.2)–(6.18), (6.19), and (6.20) provide the complete description and classification of cubic interaction vertices that can be constructed for fields in (6.1).

6.2 One continuous-spin massless field and two continuous-spin massive fields with equal masses

In this section, we discuss parity invariant cubic vertices for two continuous-spin massive fields having equal masses and one continuous-spin massless field. This is to say that, using the shortcut $(m, \kappa)_{CSF}$ for a continuous-spin field, we study cubic vertices for the following three fields:

$$(m_1, \kappa_1)_{CSF} - (m_2, \kappa_2)_{CSF} - (0, \kappa_3)_{CSF}, \quad m_1 = m, \quad m_2 = m, \quad m^2 < 0, \quad (6.21)$$

two continuous-spin massive fields with equal masses and one continuous-spin massless field.

Relation (6.21) tells us that the massive continuous-spin fields having equal masses carry the external line indices $a = 1, 2$, while the continuous-spin massless field corresponds to $a = 3$.

For fields (6.21), we find the following general solution to cubic vertex $p_{[3]}^-$ (see appendix D)

$$p_{[3]}^- = U_v U_\Gamma U_\beta U_{\partial B} U_{\partial \alpha} U_e U_W V_{ABC}^{(6)}, \quad A, B = 1, \dots, 6, \quad C = +, -, \quad (6.22)$$

$$p_{[3]}^- = p_{[3]}^-(\beta_a, B_a, \alpha_{aa+1}, v_a), \quad (6.23)$$

$$V_{ABC}^{(6)} = V_{ABC}^{(6)}(B_a, \alpha_{aa+1}). \quad (6.24)$$

In (6.22), we introduce vertices $V_{ABC}^{(6)}$ labelled by the superscripts A, B, C . In (6.23) and (6.24), the arguments of the generic vertex $p_{[3]}^-$ and the vertices $V_{ABC}^{(6)}$ are shown explicitly. The definition of the arguments B_a and α_{ab} may be found in (A.3). Various quantities U appearing in (6.22) are differential operators w.r.t. the B_a and α_{aa+1} . These quantities will be presented below. For the vertices $V_{ABC}^{(6)}$ (6.24), we find the following solution:

$$V_{ABC}^{(6)} = E_{1A} E_{2B} T_{3C} V_{ABC}, \quad V_{ABC} = V_{ABC}(\alpha_{12}, \alpha_{23}, \alpha_{31}), \quad A, B = 1, \dots, 6, \quad C = \pm, \quad (6.25)$$

$$E_{a1} = B_a^{\rho_a}, \quad E_{a2} = B_a^{\rho_a} \ln B_a, \quad \text{for } \kappa_a^2 = -\frac{(d-3)^2}{4} m^2, \quad (6.26)$$

$$E_{a3} = B_a^{\rho_{a+}}, \quad E_{a4} = B_a^{\rho_{a-}}, \quad \text{for } \kappa_a^2 < -\frac{(d-3)^2}{4} m^2, \quad (6.27)$$

$$E_{a5} = B_a^{\rho_{a+}} + B_a^{\rho_{a-}}, \quad E_{a6} = i(B_a^{\rho_{a+}} - B_a^{\rho_{a-}}), \quad \text{for } \kappa_a^2 > -\frac{(d-3)^2}{4} m^2, \quad (6.28)$$

$$\rho_a = -\nu_a - \frac{1}{2}, \quad \rho_{a\pm} = -\nu_a - \frac{1}{2} \pm \sigma_a, \quad \sigma_a = \left(\frac{(d-3)^2}{4} + \frac{\kappa_a^2}{m^2} \right)^{1/2}, \quad (6.29)$$

$$T_{3+} = \cos(\Omega_3 B_3), \quad T_{3-} = \frac{\sin(\Omega_3 B_3)}{\Omega_3}, \quad (6.30)$$

where $a = 1, 2$, while operators ν_a and Ω_3 are defined below. Note that, for κ_a^2 in (6.28), the σ_a is purely imaginary.

The generic vertex $p_{[3]}^-$ (6.23) depends on the twelve variables, while, the vertices V_{ABC} (6.25) depend only on the three variables. Note also that, by definition, the vertices V_{ABC} (6.25) are expandable in the three variables $\alpha_{12}, \alpha_{23}, \alpha_{31}$. To complete the description of the vertex $p_{[3]}^-$ (6.22) we should provide expressions for the operators U, ν_a ,

Ω_3 . These operators are given by

$$U_v = v_1^{N_1} v_2^{N_2} v_3^{N_3}, \quad N_a = N_{B_a} + N_{\alpha_{aa+1}} + N_{\alpha_{a+2a}}, \quad (6.31)$$

$$U_\Gamma = \left(\frac{2^{N_3} \Gamma(N_3 + \frac{d-2}{2})}{\Gamma(N_3+1)} \prod_{a=1,2} \left(-\frac{\kappa_a^2}{m^2} \right)^{N_a} \frac{2^{N_a} \Gamma(N_a + \frac{d-2}{2})}{\Gamma(N_a - \lambda_{a+}) \Gamma(N_a - \lambda_{a-}) \Gamma(N_a+1)} \right)^{1/2}, \quad (6.32)$$

$$\lambda_{a\pm} = \frac{d-3}{2} \pm \sigma_a, \quad (6.33)$$

$$U_\beta = \exp \left(- \sum_{a=1,2,3} \frac{\check{\beta}_a}{2\beta_a} \kappa_a \partial_{B_a} \right), \quad (6.34)$$

$$U_{\partial B} = \exp \left(-\frac{\kappa_1}{2} \partial_{B_1} + \frac{\kappa_2}{2} \partial_{B_2} \right), \quad (6.35)$$

$$U_{\partial\alpha} = \exp \left(\left(B_3^2 + \frac{1}{4} \kappa_3^2 \right) \frac{(B_2 \partial_{\alpha_{23}} - B_1 \partial_{\alpha_{31}})}{2\kappa_3 m^2} - \frac{B_3 B_1}{2m^2} \partial_{\alpha_{31}} - \frac{B_2 B_3}{2m^2} \partial_{\alpha_{23}} + \frac{B_1 B_2}{2m^2} \partial_{\alpha_{12}} \right. \\ \left. - \frac{\kappa_1 B_3}{m^2} \partial_{\alpha_{31}} + \frac{\kappa_2 B_3}{m^2} \partial_{\alpha_{23}} - \frac{\kappa_2 B_1}{m^2} \partial_{\alpha_{12}} + \frac{\kappa_1 B_2}{m^2} \partial_{\alpha_{12}} - \frac{\kappa_1 \kappa_2}{m^2} \partial_{\alpha_{12}} - \frac{\kappa_2 \kappa_3}{m^2} \partial_{\alpha_{23}} - \frac{\kappa_3 \kappa_1}{m^2} \partial_{\alpha_{31}} \right) \quad (6.36)$$

$$U_e = \left(-\frac{m^2}{\kappa_1^2} \right)^{\omega_1/2} \left(-\frac{m^2}{\kappa_2^2} \right)^{\omega_2/2} \quad (6.37)$$

$$U_W = U_{\omega_1, W_1} U_{\omega_2, W_{23}}, \quad U_{\omega, W} \equiv \sum_{n=0}^{\infty} \frac{\Gamma(\omega+n)}{4^n n! \Gamma(\omega+2n)} W^n, \quad (6.38)$$

$$W_1 = -2B_2 \partial_{B_1} \partial_{\alpha_{12}} + 2\kappa_3 \partial_{B_1} \partial_{\alpha_{31}} - \partial_{\alpha_{12}}^2 + 2\alpha_{23} \partial_{\alpha_{12}} \partial_{\alpha_{31}}, \quad (6.39)$$

$$W_{23} = -2B_1 \partial_{B_2} \partial_{\alpha_{12}} - 2\kappa_3 \partial_{B_2} \partial_{\alpha_{23}} + 2\alpha_{31} \partial_{\alpha_{12}} \partial_{\alpha_{23}}, \quad (6.40)$$

$$W_{321} = \frac{1}{\kappa_3} \{ \nu_3, B_2 \partial_{\alpha_{23}} - B_1 \partial_{\alpha_{31}} \} + 2\alpha_{12} \partial_{\alpha_{31}} \partial_{\alpha_{23}}, \quad (6.41)$$

$$\omega_a = N_{B_a} + \nu_a + \frac{1}{2}, \quad \nu_a = N_{\alpha_{aa+1}} + N_{\alpha_{a+2a}} + \frac{d-4}{2}, \quad (6.42)$$

$$\Omega_3 = \left(-\frac{1+W_{321}}{m^2} \right)^{1/2}, \quad (6.43)$$

where quantities $\check{\beta}_a$, N_{B_a} , $N_{\alpha_{ab}}$, N_a appearing in (6.31)–(6.43) are defined in (A.3)–(A.7).

Using simplified notation for the vertices $V = V_{ABC}$, we note that the general expression for the V can be presented as

$$V = V(n_1, n_2, n_3), \quad V(n_1, n_2, n_3) = \alpha_{12}^{n_3} \alpha_{23}^{n_1} \alpha_{31}^{n_2}, \quad (6.44)$$

$$n_a \geq 0, \quad a = 1, 2, 3. \quad (6.45)$$

The three integers n_a in (6.44) are the freedom of our solution for the vertex V . In other words these integers label all possible cubic vertices that can be constructed for three fields in (6.21). In order for vertices (6.44) to be sensible, the n_a should satisfy the restrictions (6.45) which amount to the requirement that the powers of all variables α_{12} ,

α_{23}, α_{31} in (6.44) be non-negative. Relations given in (6.22)–(6.43), (6.44), and (6.45) provide the complete description and classification of cubic interaction vertices that can be constructed for fields in (6.21).

6.3 One continuous-spin massless field and two continuous-spin massive fields with nonequal masses

In this section, we discuss parity invariant cubic vertices for two continuous-spin massive fields having different masses and one continuous-spin massless field. This is to say that, using the shortcut $(m, \kappa)_{CSF}$ for a mass- m continuous-spin field, we study cubic vertices for the following three fields:

$$(m_1, \kappa_1)_{CSF} - (m_2, \kappa_2)_{CSF} - (0, \kappa_3)_{CSF}, \quad m_1^2 < 0, \quad m_2^2 < 0, \quad m_1^2 \neq m_2^2, \quad (6.46)$$

two continuous-spin massive fields with nonequal masses and one continuous-spin massless field.

Relation (6.46) tells us that the massive continuous-spin fields carry the external line indices $a = 1, 2$, while the continuous-spin massless field corresponds to $a = 3$.

For fields (6.46), we find the following general solution to cubic vertex $p_{[3]}^-$ (see appendix D)

$$p_{[3]}^- = U_\nu U_\Gamma U_\beta U_{\partial B} U_{\partial \alpha} U_B U_{JB} U_W U_B^{-1} V_{ABC}^{(7)}, \quad A, B, C = +, -, \quad (6.47)$$

$$p_{[3]}^- = p_{[3]}^-(\beta_a, B_a, \alpha_{aa+1}, \nu_a), \quad (6.48)$$

$$V_{ABC}^{(7)} = V_{ABC}^{(7)}(B_a, \alpha_{aa+1}). \quad (6.49)$$

In (6.47), we introduce eight vertices $V_{ABC}^{(7)}$ labelled by the superscripts $A, B, C = \pm$. In (6.48) and (6.49), the arguments of the generic vertex $p_{[3]}^-$ and the vertices $V_{ABC}^{(7)}$ are shown explicitly. The definition of the arguments B_a and α_{ab} may be found in (A.3). Various quantities U appearing in (6.47) are differential operators w.r.t. the B_a and α_{aa+1} . These quantities will be presented below. For eight vertices $V_{ABC}^{(7)}$ (6.49), we find the following solution:

$$V_{ABC}^{(6)} = F_{1A} F_{2B} J_{3C} V_{ABC}, \quad V_{ABC} = V_{ABC}(\alpha_{12}, \alpha_{23}, \alpha_{31}), \quad A, B, C = +, -, \quad (6.50)$$

$$F_{a\pm} = F\left(\alpha_a, \beta_a, \gamma_a; \frac{1 \pm z_a}{2}\right), \quad \alpha_a = \nu_a + \frac{1}{2} + \sigma_a, \quad \beta_a = \nu_a + \frac{1}{2} - \sigma_a, \quad (6.51)$$

$$\gamma_a = \nu_a + 1, \quad z_a = \frac{2m_a^2 B_a}{\kappa_a(m_1^2 - m_2^2)}, \quad \sigma_a = \left(\frac{(d-3)^2}{4} + \frac{\kappa_a^2}{m_a^2}\right)^{1/2}, \quad a = 1, 2, \quad (6.52)$$

$$J_{3+} = I_{\nu_3}(\sqrt{z_3}), \quad J_{3-} = K_{\nu_3}(\sqrt{z_3}), \quad z_3 = -\frac{4\kappa_3 B_3}{m_1^2 - m_2^2}, \quad (6.53)$$

where $F(\alpha, \beta, \gamma; x)$ stands for the hypergeometric function, while $I_\nu(x)$ and $K_\nu(x)$ are the modified Bessel functions. In (6.50), in place of the variables B_a , we use new variables z_a defined in (6.52), (6.53). Operators ν_a are defined below.

From (6.48), we see that the generic vertex $p_{[3]}^-$ depends on the twelve variables, while, from (6.50), we learn that the vertices V_{ABC} depend only on the three variables. Note also that, by definition, the vertices V_{ABC} (6.50) are expandable in the three variables α_{12} , α_{23} , α_{31} . To complete our description of the vertex $p_{[3]}^-$ (6.47) we should provide expressions for the operators U , ν_a . These operators are given by

$$U_v = v_1^{N_1} v_2^{N_2} v_3^{N_3}, \quad N_a = N_{B_a} + N_{\alpha_{aa+1}} + N_{\alpha_{a+2a}}, \quad (6.54)$$

$$U_\Gamma = \left(\frac{2^{N_3} \Gamma(N_3 + \frac{d-2}{2})}{\Gamma(N_3+1)} \prod_{a=1,2} \left(-\frac{\kappa_a^2}{m_a^2} \right)^{N_a} \frac{2^{N_a} \Gamma(N_a + \frac{d-2}{2})}{\Gamma(N_a - \lambda_{a+}) \Gamma(N_a - \lambda_{a-}) \Gamma(N_a+1)} \right)^{1/2}, \quad (6.55)$$

$$\lambda_{a\pm} = \frac{d-3}{2} \pm \sigma_a, \quad (6.56)$$

$$U_\beta = \exp \left(- \sum_{a=1,2,3} \frac{\check{\beta}_a}{2\beta_a} \kappa_a \partial_{B_a} \right), \quad (6.57)$$

$$U_{\partial B} = \exp \left(-\frac{m_1^2}{2m_2^2} \kappa_1 \partial_{B_1} + \frac{m_1^2}{2m_2^2} \kappa_2 \partial_{B_2} - \frac{m_1^2 + m_2^2}{2(m_1^2 - m_2^2)} \kappa_3 \partial_{B_3} \right), \quad (6.58)$$

$$U_{\partial\alpha} = \exp \left(\left(\frac{2(m_1^2 + m_2^2)}{(m_1^2 - m_2^2)^2} B_1 B_2 - \frac{\kappa_2 B_1}{m_2^2} + \frac{\kappa_1 B_2}{m_1^2} - \frac{\kappa_1 \kappa_2 (m_1^2 + m_2^2)}{2m_1^2 m_2^2} \right) \partial_{\alpha_{12}} \right. \\ \left. + \left(\frac{2B_1 B_3}{\kappa_3 q_3} - \frac{\kappa_1 B_3}{m_1^2} - \frac{2m_1^2}{\kappa_3 q_3^2} B_1 - \frac{\kappa_1}{q_3} \right) \partial_{\alpha_{31}} - \left(\frac{2B_2 B_3}{\kappa_3 q_3} - \frac{\kappa_2 B_3}{m_2^2} - \frac{2m_2^2}{\kappa_3 q_3^2} B_2 - \frac{\kappa_2}{q_3} \right) \partial_{\alpha_{23}} \right), \quad (6.59)$$

$$q_3 = (m_1^2 - m_2^2) / \kappa_3, \quad (6.60)$$

$$U_B = \prod_{a=1,2} \left(-\frac{\kappa_a^2}{m_a^2} + \frac{4m_a^2}{(m_1^2 - m_2^2)^2} B_a^2 \right)^{-(\nu_a+1)/2}, \quad U_{JB} = \left(-\frac{4\kappa_3}{m_1^2 - m_2^2} B_3 \right)^{-\nu_3/2}, \quad (6.61)$$

$$U_W = U_{\nu_1, W_1} U_{\nu_2, W_{23}} U_{\nu_3, W_{312}}, \quad U_{\nu, W} \equiv \sum_{n=0}^{\infty} \frac{\Gamma(\nu+n)}{4^n n! \Gamma(\nu+2n)} W^n, \quad (6.62)$$

$$W_1 = 2\alpha_{23} \partial_{\alpha_{12}} \partial_{\alpha_{31}} - \partial_{\alpha_{12}}^2 - \partial_{\alpha_{31}}^2, \quad W_{23} = 2\alpha_{31} \partial_{\alpha_{12}} \partial_{\alpha_{23}} - \partial_{\alpha_{23}}^2, \quad (6.63)$$

$$W_{312} = 2\alpha_{12} \partial_{\alpha_{23}} \partial_{\alpha_{31}}, \quad \nu_a = N_{\alpha_{aa+1}} + N_{\alpha_{a+2a}} + \frac{d-4}{2}, \quad (6.64)$$

where quantities $\check{\beta}_a$, N_{B_a} , $N_{\alpha_{ab}}$, N_a appearing in (6.54)–(6.64) are defined in (A.3)–(A.7).

Using the notation for the vertices $V = V_{ABC}$, we note that the general V can be presented as

$$V = V(n_1, n_2, n_3), \quad V(n_1, n_2, n_3) = \alpha_{12}^{n_3} \alpha_{23}^{n_1} \alpha_{31}^{n_2}, \quad (6.65)$$

$$n_a \geq 0, \quad a = 1, 2, 3. \quad (6.66)$$

The three integers n_a in (6.65) are the freedom of our solution for the vertex V . These integers label all possible cubic vertices that can be constructed for three fields in (6.46). In order for vertices (6.65) to be sensible, the integers n_a should satisfy the restrictions (6.66) which amount to the requirement that the powers of all variables α_{12} , α_{23} , α_{31} in (6.65)

be non-negative. Relations given in (6.47)–(6.64), (6.65), and (6.66) provide the complete description and classification of cubic interaction vertices that can be constructed for fields in (6.46).

6.4 Three continuous-spin massive fields

Finally, we discuss parity invariant cubic vertices for three continuous-spin massive fields. Using shortcut $(m, \kappa)_{CSF}$ for a continuous-spin massive field, we study cubic vertices for the following three fields:

$$(m_1, \kappa_1)_{CSF} - (m_2, \kappa_2)_{CSF} - (m_3, \kappa_3)_{CSF}, \quad m_1^2 < 0, \quad m_2^2 < 0, \quad m_3^2 < 0, \\ \text{three continuous-spin massive fields.} \quad (6.67)$$

Relation (6.67) tells us that the mass- m_a continuous-spin massive field carry the external line index a , where $a = 1, 2, 3$.

For fields (6.67), we find the following general solution to cubic vertex $p_{[3]}^-$ (see appendix D)

$$p_{[3]}^- = U_v U_\Gamma U_\beta U_{\partial B} U_{\partial \alpha} U_B U_W U_B^{-1} V_{ABC}^{(7)}, \quad A, B, C = +, -, \quad (6.68)$$

$$p_{[3]}^- = p_{[3]}^-(\beta_a, B_a, \alpha_{aa+1}, v_a), \quad (6.69)$$

$$V_{ABC}^{(7)} = V_{ABC}^{(7)}(B_a, \alpha_{aa+1}). \quad (6.70)$$

In (6.68), we introduce eight vertices $V_{ABC}^{(7)}$ labelled by the superscripts A, B, C . In (6.69) and (6.70), the arguments of the generic vertex $p_{[3]}^-$ and the vertices $V_{ABC}^{(7)}$ are shown explicitly. The definition of the arguments B_a and α_{ab} may be found in (A.3). Various quantities U appearing in (6.68) are differential operators w.r.t. the B_a and α_{aa+1} . These operators are given below. For eight vertices $V_{ABC}^{(7)}$ (6.70), we find the following solution:

$$V_{ABC}^{(7)} = F_{1A} F_{2B} F_{3C} V_{ABC}, \quad V_{ABC} = V_{ABC}(\alpha_{12}, \alpha_{23}, \alpha_{31}), \quad A, B, C = +, -, \quad (6.71)$$

$$F_{a\pm} = F\left(\alpha_a, \beta_a, \gamma_a; \frac{1 \pm z_a}{2}\right), \quad \alpha_a = \nu_a + \frac{1}{2} + \sigma_a, \quad \beta_a = \nu_a + \frac{1}{2} - \sigma_a, \quad (6.72)$$

$$\gamma_a = \nu_a + 1, \quad z_a = -\frac{2m_a^2 B_a}{\kappa_a \sqrt{D}}, \quad \sigma_a = \left(\frac{(d-3)^2}{4} + \frac{\kappa_a^2}{m_a^2}\right)^{1/2}, \quad a = 1, 2, 3, \quad (6.73)$$

where the $F(\alpha, \beta, \gamma; x)$ is the hypergeometric function. In (6.71), in place of the variables B_a , we use new variables z_a defined in (6.73). Quantities ν_a and D are defined below in (6.82), (6.84).

The generic vertex $p_{[3]}^-$ (6.69) depends on the twelve variables, while, the vertices V_{ABC} (6.71) depend only on the three variables. Note also that, by definition, the vertices V_{ABC} (6.71) are expandable in the three variables $\alpha_{12}, \alpha_{23}, \alpha_{31}$. To complete our description of the vertex $p_{[3]}^-$ (6.69) we should provide expressions for the operators U , ν_a .

These operators are given by

$$U_v = v_1^{N_1} v_2^{N_2} v_3^{N_3}, \quad N_a = N_{B_a} + N_{\alpha_{a+1}} + N_{\alpha_{a+2a}}, \quad (6.74)$$

$$U_\Gamma = \prod_{a=1,2,3} \left(\left(-\frac{\kappa_a^2}{m_a^2} \right)^{N_a} \frac{2^{N_a} \Gamma(N_a + \frac{d-2}{2})}{\Gamma(N_a - \lambda_{a+}) \Gamma(N_a - \lambda_{a-}) \Gamma(N_a + 1)} \right)^{1/2}, \quad \lambda_{a\pm} = \frac{d-3}{2} \pm \sigma_a, \quad (6.75)$$

$$U_\beta = \exp \left(- \sum_{a=1,2,3} \frac{\check{\beta}_a}{2\beta_a} \kappa_a \partial_{B_a} \right), \quad (6.76)$$

$$U_{\partial B} = \exp \left(- \sum_{a=1,2,3} \frac{m_a^2 - m_{a+2}^2}{2m_a^2} \kappa_a \partial_{B_a} \right), \quad (6.77)$$

$$U_{\partial\alpha} = \exp \left(\sum_{a=1,2,3} \left(\frac{\kappa_{a+1} B_{a+2}}{m_{a+1}^2} - \frac{\kappa_{a+2} B_{a+1}}{m_{a+2}^2} - \frac{h_{a+1a+2} \kappa_{a+1} \kappa_{a+2}}{2m_{a+1}^2 m_{a+2}^2} \right) \partial_{\alpha_{a+1a+2}} \right) \\ \times \exp \left(\sum_{a=1,2,3} \frac{2h_{a+1a+2}}{D} B_{a+1} B_{a+2} \partial_{\alpha_{a+1a+2}} \right), \quad (6.78)$$

$$U_B = \prod_{a=1,2,3} \left(-\frac{\kappa_a^2}{m_a^2} + \frac{4m_a^2}{D} B_a^2 \right)^{-(\nu_a+1)/2}, \quad (6.79)$$

$$U_W = U_{\nu_1, W_1} U_{\nu_2, W_{23}} U_{\nu_3, W_{312}}, \quad U_{\nu, W} \equiv \sum_{n=0}^{\infty} \frac{\Gamma(\nu+n)}{4^n n! \Gamma(\nu+2n)} W^n, \quad (6.80)$$

$$W_1 = 2\alpha_{23} \partial_{\alpha_{12}} \partial_{\alpha_{31}} - \partial_{\alpha_{12}}^2 - \partial_{\alpha_{31}}^2, \quad W_{23} = 2\alpha_{31} \partial_{\alpha_{12}} \partial_{\alpha_{23}} - \partial_{\alpha_{23}}^2, \quad (6.81)$$

$$W_{312} = 2\alpha_{12} \partial_{\alpha_{23}} \partial_{\alpha_{31}}, \quad \nu_a = N_{\alpha_{aa+1}} + N_{\alpha_{a+2a}} + \frac{d-4}{2}, \quad (6.82)$$

$$h_{aa+1} = m_a^2 + m_{a+1}^2 - m_{a+2}^2, \quad (6.83)$$

$$D = m_1^4 + m_2^4 + m_3^4 - 2(m_1^2 m_2^2 + m_2^2 m_3^2 + m_3^2 m_1^2), \quad (6.84)$$

where $\check{\beta}_a$, N_{B_a} , $N_{\alpha_{ab}}$, N_a are defined in (A.3)–(A.7).

Using the notation for the vertices $V = V_{ABC}$, we note that the general representative of the vertex V can be presented as

$$V = V(n_1, n_2, n_3), \quad V(n_1, n_2, n_3) = \alpha_{12}^{n_3} \alpha_{23}^{n_1} \alpha_{31}^{n_2}, \quad (6.85)$$

$$n_a \geq 0, \quad a = 1, 2, 3. \quad (6.86)$$

The three integers n_a in (6.85) are the freedom of our solution for the vertex V . These integers label all possible cubic vertices that can be constructed for three fields in (6.67). In order for vertices (6.85) to be sensible, the integers n_a should satisfy the restrictions (6.86) which amount to the requirement that the powers of all variables α_{12} , α_{23} , α_{31} in (6.85) be non-negative. Relations given in (6.68)–(6.84), (6.85), and (6.86) provide the complete description and classification of cubic interaction vertices that can be constructed for fields in (6.67).

7 Conclusions

In this paper, we used the light-cone gauge approach to construct the parity invariant cubic vertices for continuous-spin massive/massless fields and arbitrary spin massive/massless fields. We investigated three types of the parity invariant cubic vertices: a) vertices describing coupling of one continuous-spin massive/massless field to two arbitrary spin massive/massless fields; b) vertices describing coupling of two continuous-spin massive/massless fields to one arbitrary spin massive/massless field; c) vertices for self-interacting massive/massless continuous-spin fields. We obtained the complete list of such cubic vertices. With exception of cubic vertices for massless self-interacting continuous-spin field, results in this paper together with the ones in ref. [18] provide the exhaustive solution to the problem of description of all parity invariant cubic vertices for coupling of massive/massless continuous-spin fields to arbitrary spin massive/massless fields as well as vertices for self-interacting massive/massless continuous-spin fields. Results in this paper might have the following applications and generalizations.

- i) Sometimes light-cone gauge formulation turns out to be good starting point for deriving Lorentz covariant formulations. It is the parity invariant light-cone gauge vertices that turn out to be convenient for deriving of Lorentz covariant and BRST gauge invariant formulations [36]. For example, all light-cone gauge vertices for massive/massless fields in ref. [27] have straightforwardly been cast into BRST gauge invariant form in ref. [37].¹² Various BRST formulations of free continuous-spin fields were discussed in refs. [43]–[45]. We expect therefore that our results in this paper may be good starting point for deriving Lorentz covariant and BRST gauge invariant formulations of vertices for interacting continuous-spin fields. Various applications of BRST approach for studying interacting finite-component fields may be found, e.g., in refs. [46–49].
- ii) From the perspective of investigation of interrelations between theory of continuous-spin fields and string theory it is important to extend our study to the case of mixed-symmetry fields. During last time, various interesting descriptions of mixed-symmetry fields were obtained in refs. [50]–[53]. For example, light-cone gauge description in ref. [50] and methods in this paper provide opportunity for investigation of interacting mixed-symmetry continuous-spin fields. Also, for finite-component mixed-symmetry fields, we mention interesting formulations developed in refs. [54–64]. Generalization of these formulations to the case continuous-spin fields could also be of some interest.
- iii) In this paper, we investigated cubic vertices for light-cone gauge continuous-spin fields propagating in flat space. Extension of our investigation to the case of continuous-spin fields in AdS space [14, 15] could be of some interest. In this respect we note that light-cone gauge formulation of free continuous-spin AdS fields was recently obtained in ref. [50], while for finite component light-cone gauge AdS fields, the systematic method for building cubic vertices was developed in ref. [65]. We think therefore

¹²For Lorentz covariant formulations of massless higher-spin fields in flat space, see also refs. [38–42].

that results in refs. [50, 65] provide opportunity for investigation of cubic vertices for continuous-spin AdS fields. Also we note that, for finite-component interacting AdS fields, many interesting formulations and results were obtained in refs. [66]–[72–80], which, upon a generalization, might be useful for investigation of interaction vertices of continuous-spin AdS fields. Also we note frame-like formulations of continuous-spin AdS fields obtained in refs. [51, 81] which seem to be convenient for studying interacting AdS fields.

- iv) Generalization of our results to the case of interacting supersymmetric continuous-spin field theories could be of great interest. As is known, for finite component fields, supersymmetry imposes additional constraints and leads to more simple interacting vertices. We expect therefore that, for continuous-spin field, supersymmetry might simplify a structure of interactions vertices. Supermultiplets for continuous-spin representations are considered in refs. [3, 16]. For finite-component fields, recent study of higher-spin supersymmetric theories can be found in refs. [82–85]. Use of twistor-like variables for discussion of supersymmetric theories turns out to be helpful. Recent various interesting applications of twistor-like variables may be found in refs. [86–90]. Finally, we note that investigation of various algebraic aspects of continuous-spin field theory along the line in refs. [91–93] could also be very interesting.

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A Notation and conventions

Unless otherwise specified, the vector indices of the $so(d-2)$ algebra i, j, k, l run over $1, \dots, d-2$. We refer to creation operators α^i, v, ζ and the respective annihilation operators $\bar{\alpha}^i, \bar{v}, \bar{\zeta}$ as oscillators. Our conventions for the commutation relations, the vacuum $|0\rangle$, and hermitian conjugation rules are as follows

$$[\bar{\alpha}^i, \alpha^j] = \delta^{ij}, \quad [\bar{v}, v] = 1, \quad [\bar{\zeta}, \zeta] = 1, \quad \bar{\alpha}^i|0\rangle = 0, \quad \bar{v}|0\rangle = 0, \quad \bar{\zeta}|0\rangle = 0, \quad (\text{A.1})$$

$$\alpha^{i\dagger} = \bar{\alpha}^i, \quad v^\dagger = \bar{v}, \quad \zeta^\dagger = \bar{\zeta}. \quad (\text{A.2})$$

Throughout this paper we use the following definitions for momentum $\mathbb{P}^i$ and quantities B_a, α_{ab}

$$\mathbb{P}^i \equiv \frac{1}{3} \sum_{a=1,2,3} \check{\beta}_a p_a^i, \quad \check{\beta}_a \equiv \beta_{a+1} - \beta_{a+2}, \quad B_a \equiv \frac{\alpha_a^i \mathbb{P}^i}{\beta_a}, \quad \alpha_{ab} \equiv \alpha_a^i \alpha_b^i, \quad (\text{A.3})$$

where $\beta_a \equiv \beta_{a+3}$. Our notation for the scalar product of the oscillators and various quantities constructed out of the B_a , α_{ab} and derivatives of the B_a , α_{ab} are as follows

$$\alpha^2 \equiv \alpha^i \alpha^i, \quad \bar{\alpha}^2 \equiv \bar{\alpha}^i \bar{\alpha}^i, \quad N_\alpha \equiv \alpha^i \bar{\alpha}^i, \quad N_\zeta \equiv \zeta \bar{\zeta}, \quad N_v \equiv v \bar{v}, \quad (\text{A.4})$$

$$N_{B_a} = B_a \partial_{B_a}, \quad \partial_{B_a} = \partial / \partial B_a, \quad (\text{A.5})$$

$$N_{\alpha_{ab}} = \alpha_{ab} \partial_{\alpha_{ab}}, \quad \partial_{\alpha_{ab}} = \partial / \partial \alpha_{ab}, \quad (\text{A.6})$$

$$N_a = N_{B_a} + N_{\alpha_{aa+1}} + N_{\alpha_{a+2a}}, \quad (\text{A.7})$$

$$\nu_a = N_{\alpha_{aa+1}} + N_{\alpha_{a+2a}} + \frac{d-4}{2}, \quad (\text{A.8})$$

$$g_{v_a} = \left(\frac{F_a}{(N_a+1)(2N_a+d-2)} \right)^{1/2}, \quad F_a = \kappa_a^2 - N_a(N_a+d-3)m_a^2, \quad (\text{A.9})$$

$$U_{v_a} = v_a^{N_a}, \quad (\text{A.10})$$

$$U_{\Gamma_a} = \left(\frac{2^{N_a} \Gamma(N_a + \frac{d-2}{2})}{\Gamma(N_a+1)} \right)^{1/2}, \quad \text{for massless continuous-spin field} \quad (\text{A.11})$$

$$U_{\Gamma_a} = \left(\left(-\frac{\kappa_a^2}{m_a^2} \right)^{N_a} \frac{2^{N_a} \Gamma(N_a + \frac{d-2}{2})}{\Gamma(N_a - \lambda_{a+}) \Gamma(N_a - \lambda_{a-}) \Gamma(N_a+1)} \right)^{1/2},$$

$$\lambda_{a\pm} = \frac{d-3}{2} \pm \left(\frac{(d-3)^2}{4} + \frac{\kappa_a^2}{m_a^2} \right)^{1/2}, \quad \text{for massive continuous-spin field} \quad (\text{A.12})$$

$$U_{B_a} = \left(\frac{1}{e_a} - \frac{4\kappa_a^2 e_a}{D} B_a^2 \right)^{-(\nu_a+1)/2}, \quad (\text{A.13})$$

$$U_{\nu, W} = \sum_{n=0}^{\infty} \frac{\Gamma(\nu+n)}{4^n n! \Gamma(\nu+2n)} W^n, \quad (\text{A.14})$$

$$U_D = \exp \left(\frac{2}{D} h_{12} B_1 B_2 \partial_{\alpha_{12}} + \frac{2}{D} h_{23} B_2 B_3 \partial_{\alpha_{23}} + \frac{2}{D} h_{31} B_3 B_1 \partial_{\alpha_{31}} \right). \quad (\text{A.15})$$

Our notation for the quantities constructed out of the masses m_a and continuous-spin parameters κ_a are as follows

$$r_a = \frac{1}{2} m_{a+1}^2 + \frac{1}{2} m_{a+2}^2 - \frac{1}{4} m_a^2, \quad q_a = \frac{m_{a+1}^2 - m_{a+2}^2}{\kappa_a}, \quad (\text{A.16})$$

$$e_a = -\frac{m_a^2}{\kappa_a^2}, \quad u_a = 1 + \frac{(d-2)(d-4)m_a^2}{4\kappa_a^2}, \quad (\text{A.17})$$

$$D = m_1^4 + m_2^4 + m_3^4 - 2(m_1^2 m_2^2 + m_2^2 m_3^2 + m_3^2 m_1^2), \quad (\text{A.18})$$

$$h_{aa+1} = m_a^2 + m_{a+1}^2 - m_{a+2}^2. \quad (\text{A.19})$$

B Derivation of vertices $p_{[3]}^-$ (4.2), (4.26), (4.46)

We consider three vertices in (3.21)–(3.23). For these vertices, we outline the derivation of the respective solutions in (4.2), (4.26), (4.46). We split our derivation in several steps.

Realization G_a , G_β on $p_{[3]}^-$ (3.23) for arbitrary masses. For $p_{[3]}^-$ (3.23), we now find G_β and $G_{a,\mathbb{P}^2}$ in (3.34). We use M_a^i , $a = 1, 2$ for massive field (2.23) and M_3^i for continuous-spin field (2.20). Plugging such M_a^i into $\mathbf{J}^{-i\dagger}$ (3.17), we cast $\mathbf{J}^{-i\dagger}|p_{[3]}^-\rangle$ into the form given in (3.34) with the following G_β and $G_{a,\mathbb{P}^2}$:

$$G_{1,\mathbb{P}^2} = G_1, \quad G_{2,\mathbb{P}^2} = G_2, \quad G_{3,\mathbb{P}^2} = G_3 + \mathbf{P}^- \frac{2\beta}{\beta_3^3} \alpha_3^i \frac{g_{v_3} \partial_{v_3}}{2N_3 + d - 2} \partial_{B_3}^2, \quad (\text{B.1})$$

$$G_1 = \left(B_3 - \frac{\beta_1}{\beta_3} g_{v_3} \partial_{v_3} \right) \partial_{\alpha_{31}} - \left(B_2 + \frac{\beta_1}{\beta_2} m_2 \zeta_2 \right) \partial_{\alpha_{12}} + \frac{1}{2} \left(\frac{\check{\beta}_1}{\beta_1} m_1^2 + m_2^2 - m_3^2 \right) \partial_{B_1} + m_1 \partial_{\zeta_1}, \quad (\text{B.2})$$

$$G_2 = \left(B_1 - \frac{\beta_2}{\beta_1} \zeta_1 m_1 \right) \partial_{\alpha_{12}} - \left(B_3 + \frac{\beta_2}{\beta_3} g_{v_3} \partial_{v_3} \right) \partial_{\alpha_{23}} + \frac{1}{2} \left(\frac{\check{\beta}_2}{\beta_2} m_2^2 + m_3^2 - m_1^2 \right) \partial_{B_2} + m_2 \partial_{\zeta_2}, \quad (\text{B.3})$$

$$\begin{aligned} G_3 = & \left(B_2 - \frac{\beta_3}{\beta_2} m_2 \zeta_2 \right) \partial_{\alpha_{23}} - \left(B_1 + \frac{\beta_3}{\beta_1} m_1 \zeta_1 \right) \partial_{\alpha_{31}} + \frac{1}{2} \left(\frac{\check{\beta}_3}{\beta_3} m_3^2 + m_1^2 - m_2^2 \right) \partial_{B_3} + v_3 g_{v_3} \\ & + \frac{g_{v_3} \partial_{v_3}}{2N_3 + d - 2} \left(\frac{2\beta_1}{\beta_3} B_1 \partial_{B_3} \partial_{\alpha_{31}} + \frac{2\beta_2}{\beta_3} B_2 \partial_{B_3} \partial_{\alpha_{23}} \right. \\ & \left. + 2\alpha_{12} \partial_{\alpha_{31}} \partial_{\alpha_{23}} + \alpha_{11} \partial_{\alpha_{31}}^2 + \alpha_{22} \partial_{\alpha_{23}}^2 + \frac{\beta}{\beta_3^2} \sum_{b=1}^3 \frac{m_b^2}{\beta_b} \partial_{B_3}^2 \right), \end{aligned} \quad (\text{B.4})$$

$$G_\beta = -\frac{1}{\beta} \mathbb{N}_\beta - \frac{1}{\beta_1^2} m_1 \zeta_1 \partial_{B_1} - \frac{1}{\beta_2^2} m_2 \zeta_2 \partial_{B_2} - \frac{1}{\beta_3^2} g_{v_3} \partial_{v_3} \partial_{B_3}, \quad (\text{B.5})$$

where g_{v_a} are given in (A.9). Using G_a , G_β (B.2)–(B.5), we now consider equations (3.35), (3.36).

Vertex $V^{(3)}$. Multiplying G_3 (B.4) on the left by $(2N_3 + d - 2)/\kappa_3$, we use the transformations¹³

$$p_{[3]}^- = U_{v_3} V^{(1)}, \quad V^{(1)} = U_{\Gamma_3} V^{(2)}, \quad V^{(2)} = U_\beta V^{(3)}, \quad (\text{B.6})$$

$$U_\beta = \exp \left(-\frac{\check{\beta}_1}{2\beta_1} m_1 \zeta_1 \partial_{B_1} - \frac{\check{\beta}_2}{2\beta_2} m_2 \zeta_2 \partial_{B_2} - \frac{\check{\beta}_3}{2\beta_3} \kappa_3 \partial_{B_3} \right), \quad (\text{B.7})$$

¹³To investigate equations $G_a p_{[3]}^- = 0$ it is convenient to use equivalence class for the G_a . Namely, the G_a and $(2N_a + d - 2)G_a$ are considered to be equivalent. For detailed discussion of transformations governed by operators U_{v_3} , U_{Γ_3} , U_β (B.6), see **Steps 2,3,4** in appendix C in ref. [18].

where U_{ν_3} , U_{Γ_3} are given in (A.10), (A.12). Realization of G_a (B.2)–(B.4) on $V^{(3)}$ takes the form

$$G_1 = \left(B_3 + \frac{1}{2}\kappa_3\right) \partial_{\alpha_{31}} - \left(B_2 - \frac{1}{2}m_2\zeta_2\right) \partial_{\alpha_{12}} + \frac{1}{2}(m_2^2 - m_3^2) \partial_{B_1} + m_1 \partial_{\zeta_1}, \quad (\text{B.8})$$

$$G_2 = \left(B_1 + \frac{1}{2}\zeta_1 m_1\right) \partial_{\alpha_{12}} - \left(B_3 - \frac{1}{2}\kappa_3\right) \partial_{\alpha_{23}} + \frac{1}{2}(m_3^2 - m_1^2) \partial_{B_2} + m_2 \partial_{\zeta_2}, \quad (\text{B.9})$$

$$\begin{aligned} G_3 = & \frac{2N_3 + d - 2}{\kappa_3} \left(\left(B_2 + \frac{m_2}{2}\zeta_2\right) \partial_{\alpha_{23}} - \left(B_1 - \frac{m_1}{2}\zeta_1\right) \partial_{\alpha_{31}} + \frac{m_1^2 - m_2^2}{2} \partial_{B_3} \right) \\ & + \frac{F_3}{\kappa_3^2} - \left(B_1 + \frac{3}{2}m_1\zeta_1\right) \partial_{B_3} \partial_{\alpha_{31}} - \left(B_2 - \frac{3}{2}m_2\zeta_2\right) \partial_{B_3} \partial_{\alpha_{23}} \\ & + 2\alpha_{12} \partial_{\alpha_{31}} \partial_{\alpha_{23}} + \alpha_{11} \partial_{\alpha_{31}}^2 + \alpha_{22} \partial_{\alpha_{23}}^2 + \left(\frac{1}{4}m_3^2 - \frac{1}{2}m_1^2 - \frac{1}{2}m_2^2\right) \partial_{B_3}^2, \end{aligned} \quad (\text{B.10})$$

$$G_\beta = -\frac{1}{\beta} \mathbb{N}_\beta. \quad (\text{B.11})$$

Dependence of $V^{(3)}$ on β_a . We fix dependence of $V^{(3)}$ on β_a . To this end we use eqs. (3.36), (3.40) and relation (B.11) to note the following equations for the vertex $V^{(3)}$:

$$\mathbb{N}_\beta V^{(3)} = 0, \quad \sum_{a=1,2,3} \beta_a \partial_{\beta_a} V^{(3)} = 0. \quad (\text{B.12})$$

Equations (B.12) imply that $V^{(3)}$ is independent of β_1 , β_2 , β_3 . Thus, the vertex $V^{(3)}$ is given by

$$V^{(3)} = V^{(3)}(B_a, \alpha_{aa+1}, \zeta_1, \zeta_2). \quad (\text{B.13})$$

We now use results above-obtained to study three vertices (3.21)–(3.23) in turn.

$$\text{Case} \quad m_1 = 0, \quad m_2 = 0, \quad m_3^2 < 0. \quad (\text{B.14})$$

Step 1. This case corresponds to the vertex $p_{[3]}^-$ (3.21). Setting $m_1 = 0$, $m_2 = 0$ and ignoring ζ_1 , ζ_2 in (B.1)–(B.13), we use vertex $V^{(3)}$ given by

$$V^{(3)} = V^{(3)}(B_a, \alpha_{aa+1}), \quad (\text{B.15})$$

where the realization of operators G_a (B.8)–(B.10) on the vertex V_3 (B.15) takes the form¹⁴

$$G_1 = \left(B_3 + \frac{1}{2}\kappa_3\right) \partial_{\alpha_{31}} - B_2 \partial_{\alpha_{12}} - \frac{1}{2}m_3^2 \partial_{B_1}, \quad (\text{B.16})$$

$$G_2 = B_1 \partial_{\alpha_{12}} - \left(B_3 - \frac{1}{2}\kappa_3\right) \partial_{\alpha_{23}} + \frac{1}{2}m_3^2 \partial_{B_2}, \quad (\text{B.17})$$

$$\begin{aligned} G_3 = & -r_3 \partial_{B_3}^2 + e_3 (N_{B_3} + \nu_3) (N_{B_3} + \nu_3 + 1) + u_3 \\ & + \frac{2}{\kappa_3} (N_{B_3} + \nu_3 + 1) (B_2 \partial_{\alpha_{23}} - B_1 \partial_{\alpha_{31}}) - B_1 \partial_{B_3} \partial_{\alpha_{31}} - B_2 \partial_{B_3} \partial_{\alpha_{23}} + 2\alpha_{12} \partial_{\alpha_{31}} \partial_{\alpha_{23}}, \end{aligned} \quad (\text{B.18})$$

where, here and below, we use quantities (A.16)–(A.19) taken for masses given in (B.14).

¹⁴For massless fields corresponding to $a = 1, 2$, α_{11} -, α_{22} -terms in (B.4) can be ignored in (B.18) by virtue of the second constraint in (2.12).

Step 2. We use the transformation

$$V^{(3)} = U_{\partial\alpha} V^{(4)}, \quad U_{\partial\alpha} = \exp\left(-\frac{2B_1 B_2}{m_3^2} \partial_{\alpha_{12}} + \frac{2B_1}{m_3^2} \left(B_3 + \frac{1}{2} \kappa_3\right) \partial_{\alpha_{31}} + \frac{2B_2}{m_3^2} \left(B_3 - \frac{1}{2} \kappa_3\right) \partial_{\alpha_{23}}\right). \quad (\text{B.19})$$

Realization of G_a (B.16)–(B.18) on $V^{(4)}$ takes the form

$$G_1 = -\frac{1}{2} m_3^2 \partial_{B_1}, \quad G_2 = \frac{1}{2} m_3^2 \partial_{B_2}, \quad (\text{B.20})$$

$$G_3 = -\frac{D}{4\kappa_3^2 e_3} \partial_{B_3}^2 + e_3 (N_{B_3} + \nu_3) (N_{B_3} + \nu_3 + 1) + u_3 + W_3, \quad W_3 = 2\alpha_{12} \partial_{\alpha_{31}} \partial_{\alpha_{23}}. \quad (\text{B.21})$$

From (B.20) and equations $G_1 V^{(4)} = 0$, $G_2 V^{(4)} = 0$, we see that $V^{(4)}$ is independent of B_1, B_2 ,

$$V^{(4)} = V^{(4)}(B_3, \alpha_{aa+1}). \quad (\text{B.22})$$

Step 3. We make the transformations

$$V^{(4)} = U_{B_3} V^{(5)}, \quad V^{(5)} = U_{\nu_3, W_3} V^{(6)}, \quad V^{(6)} = U_{B_3}^{-1} V^{(7)}, \quad (\text{B.23})$$

where U_{B_3} , U_{ν_3, W_3} are given in (A.13), (A.14). Realizations of G_3 (B.21) on $V^{(5)}$, $V^{(6)}$, $V^{(7)}$ take the forms

$$G_3 = \left(B_3^2 - \frac{D}{4\kappa_3^2 e_3^2}\right) \partial_{B_3}^2 + \left(1 - \frac{4\kappa_3^2 e_3^2}{D} B_3^2\right)^{-1} (\nu_3^2 - 1 + W_3) + \frac{u_3}{e_3}, \quad \text{for } V^{(5)}; \quad (\text{B.24})$$

$$G_3 = \left(B_3^2 - \frac{D}{4\kappa_3^2 e_3^2}\right) \partial_{B_3}^2 + \left(1 - \frac{4\kappa_3^2 e_3^2}{D} B_3^2\right)^{-1} (\nu_3^2 - 1) + \frac{u_3}{e_3}, \quad \text{for } V^{(6)}; \quad (\text{B.25})$$

$$G_3 = -\frac{D}{4\kappa_3^2 e_3} \partial_{B_3}^2 + e_3 (N_{B_3} + \nu_3) (N_{B_3} + \nu_3 + 1) + u_3, \quad \text{for } V^{(7)}. \quad (\text{B.26})$$

For ν_3 (A.8) and W_3 (B.21), we note the relation which admits us to get G_3 in (B.25),

$$(\nu_3^2 + W_3) U_{\nu_3, W_3} = U_{\nu_3, W_3} \nu_3^2. \quad (\text{B.27})$$

Equation $G_3 V^{(7)} = 0$ with G_3 as in (B.26) is the second-order differential equation w.r.t. B_3 . Two independent solutions of this equation are given in (4.5).

$$\text{Case} \quad m_1 = 0, \quad m_2^2 > 0, \quad m_3^2 < 0. \quad (\text{B.28})$$

Step 1. This case corresponds to the vertex $p_{[3]}^-$ (3.22). Setting $m_1 = 0$ and ignoring ζ_1 in (B.1)–(B.13), we use the vertex $V^{(3)}$ given by

$$V^{(3)} = V^{(3)}(B_a, \alpha_{aa+1}, \zeta_2), \quad (\text{B.29})$$

where the realization of operators G_a (B.8)–(B.10) on the vertex $V^{(3)}$ (B.29) takes the form¹⁵

$$G_1 = \left(B_3 + \frac{1}{2}\kappa_3\right) \partial_{\alpha_{31}} - \left(B_2 - \frac{1}{2}m_2\zeta_2\right) \partial_{\alpha_{12}} + \frac{1}{2}(m_2^2 - m_3^2)\partial_{B_1}, \quad (\text{B.30})$$

$$G_2 = B_1\partial_{\alpha_{12}} - \left(B_3 - \frac{1}{2}\kappa_3\right) \partial_{\alpha_{23}} + \frac{1}{2}m_3^2\partial_{B_2} + m_2\partial_{\zeta_2}, \quad (\text{B.31})$$

$$\begin{aligned} G_3 = & \frac{2N_3 + d - 2}{\kappa_3} \left(\left(B_2 + \frac{1}{2}m_2\zeta_2\right) \partial_{\alpha_{23}} - B_1\partial_{\alpha_{31}} - \frac{1}{2}m_2^2\partial_{B_3} \right) \\ & + \frac{F_3}{\kappa_3^2} - B_1\partial_{B_3}\partial_{\alpha_{31}} - \left(B_2 - \frac{3}{2}m_2\zeta_2\right) \partial_{B_3}\partial_{\alpha_{23}} \\ & + 2\alpha_{12}\partial_{\alpha_{31}}\partial_{\alpha_{23}} + \alpha_{22}\partial_{\alpha_{23}}^2 + \left(\frac{1}{4}m_3^2 - \frac{1}{2}m_2^2\right) \partial_{B_3}^2. \end{aligned} \quad (\text{B.32})$$

Step 2. We use the transformation

$$V^{(3)} = U_\zeta V^{(4)}, \quad U_\zeta = \exp \left(-\frac{\zeta_2}{m_2} B_1 \partial_{\alpha_{12}} + \frac{\zeta_2}{m_2} \left(B_3 - \frac{1}{2}\kappa_3 \right) \partial_{\alpha_{23}} - \frac{m_3^2}{2m_2} \zeta_2 \partial_{B_2} \right). \quad (\text{B.33})$$

Realization of G_a (B.30)–(B.32) on $V^{(4)}$ (B.33) takes the form¹⁶

$$G_1 = \left(B_3 + \frac{1}{2}\kappa_3\right) \partial_{\alpha_{31}} - B_2\partial_{\alpha_{12}} + \frac{1}{2}(m_2^2 - m_3^2)\partial_{B_1}, \quad G_2 = m_2\partial_{\zeta_2}, \quad (\text{B.34})$$

$$\begin{aligned} G_3 = & -r_3\partial_{B_3}^2 + q_3(N_{B_3} + \nu_3 + 1)\partial_{B_3} + e_3(N_{B_3} + \nu_3)(N_{B_3} + \nu_3 + 1) + u_3 \\ & + \frac{2}{\kappa_3}(N_{B_3} + \nu_3 + 1)(B_2\partial_{\alpha_{23}} - B_1\partial_{\alpha_{31}}) - B_1\partial_{B_3}\partial_{\alpha_{31}} - B_2\partial_{B_3}\partial_{\alpha_{23}} + 2\alpha_{12}\partial_{\alpha_{31}}\partial_{\alpha_{23}}, \end{aligned} \quad (\text{B.35})$$

where, here and below, we use quantities (A.16)–(A.19) taken for masses given in (B.28). From G_2 in (B.34) and the equation $G_2 V^{(4)} = 0$, we see that the $V^{(4)}$ is independent of ζ_2 ,

$$V^{(4)} = V^{(4)}(B_a, \alpha_{aa+1}). \quad (\text{B.36})$$

Step 3. We use the transformations

$$V^{(4)} = U_{1\partial\alpha} V^{(5)}, \quad U_{1\partial\alpha} = \exp \left(\frac{2B_1 B_2}{m_2^2 - m_3^2} \partial_{\alpha_{12}} - \frac{2B_1}{m_2^2 - m_3^2} \left(B_3 + \frac{1}{2}\kappa_3 \right) \partial_{\alpha_{31}} \right). \quad (\text{B.37})$$

Realization G_1, G_3 (B.34), (B.35) on $V^{(5)}$ (B.37) takes the form

$$G_1 = \frac{1}{2}(m_2^2 - m_3^2)\partial_{B_1}, \quad (\text{B.38})$$

$$\begin{aligned} G_3 = & -r_3\partial_{B_3}^2 + q_3(N_{B_3} + \nu_3 + 1)\partial_{B_3} + e_3(N_{B_3} + \nu_3)(N_{B_3} + \nu_3 + 1) + u_3 \\ & + \frac{2}{\kappa_3}(N_{B_3} + \nu_3 + 1)B_2\partial_{\alpha_{23}} - B_2\partial_{\alpha_{23}}\partial_{B_3} + 2\alpha_{12}\partial_{\alpha_{31}}\partial_{\alpha_{23}}. \end{aligned} \quad (\text{B.39})$$

From (B.38) and equation $G_1 V^{(4)} = 0$, we see that the $V^{(5)}$ is independent of B_1 ,

$$V^{(5)} = V^{(5)}(B_2, B_3, \alpha_{aa+1}). \quad (\text{B.40})$$

¹⁵For massless field corresponding to $a = 1$, α_{11} -term in (B.4) can be ignored in (B.32) by virtue of the second constraint in (2.12).

¹⁶For massive field corresponding to $a = 2$, operator G_3 (B.35) involves, besides α_{22} -term (B.4), ζ_3^2 -term, i.e., we get $(\alpha_{11} + \zeta_2^2)$ -term which can be ignored by virtue of the second constraint in (2.11).

Step 4. We use the transformation

$$V^{(5)} = U_{\partial B} U_{23\partial\alpha} V^{(6)}, \quad U_{\partial B} = \exp\left(\frac{q_3}{2e_3} \partial_{B_3}\right), \quad U_{23\partial\alpha} = \exp\left(\left(\frac{B_2}{\kappa_3 e_3} + \frac{2h_{23}}{D} B_2 B_3\right) \partial_{\alpha_{23}}\right) \quad (\text{B.41})$$

and note the relation

$$U_{1\partial\alpha} U_{\partial B} U_{23\partial\alpha} = U_{\partial B} U_{\partial\alpha}, \quad U_{\partial\alpha} = \exp\left(\frac{1}{\kappa_3 e_3} B_2 \partial_{\alpha_{23}} - \frac{1}{\kappa_3 e_3} B_1 \partial_{\alpha_{31}}\right) U_D, \quad (\text{B.42})$$

where U_D is defined in (A.15). Realization of G_3 (B.39) on $V^{(6)}$ takes the form

$$G_3 = -\frac{D}{4\kappa_3^2 e_3} \partial_{B_3}^2 + e_3(N_{B_3} + \nu_3)(N_{B_3} + \nu_3 + 1) + u_3 + W_3, \quad (\text{B.43})$$

$$W_3 = 2\alpha_{12} \partial_{\alpha_{31}} \partial_{\alpha_{23}} - \frac{4m_2^2}{(m_2^2 - m_3^2)^2} B_2^2 \partial_{\alpha_{23}}^2. \quad (\text{B.44})$$

Step 5. We make the transformations

$$V^{(6)} = U_{B_3} V^{(7)}, \quad V^{(7)} = U_{\nu_3, W_3} V^{(8)}, \quad V^{(8)} = U_{B_3}^{-1} V^{(9)}, \quad (\text{B.45})$$

where U_{B_3} , U_{ν_3, W_3} are given in (A.13), (A.14). Realizations of G_3 (B.43) on $V^{(7)}$, $V^{(8)}$, $V^{(9)}$ take the forms

$$G_3 = \left(B_3^2 - \frac{D}{4\kappa_3^2 e_3^2}\right) \partial_{B_3}^2 + \left(1 - \frac{4\kappa_3^2 e_3^2}{D} B_3^2\right)^{-1} (\nu_3^2 - 1 + W_3) + \frac{u_3}{e_3}, \quad \text{for } V^{(7)}, \quad (\text{B.46})$$

$$G_3 = \left(B_3^2 - \frac{D}{4\kappa_3^2 e_3^2}\right) \partial_{B_3}^2 + \left(1 - \frac{4\kappa_3^2 e_3^2}{D} B_3^2\right)^{-1} (\nu_3^2 - 1) + \frac{u_3}{e_3}, \quad \text{for } V^{(8)}, \quad (\text{B.47})$$

$$G_3 = -\frac{D}{4\kappa_3^2 e_3} \partial_{B_3}^2 + e_3(N_{B_3} + \nu_3)(N_{B_3} + \nu_3 + 1) + u_3, \quad \text{for } V^{(9)}. \quad (\text{B.48})$$

For ν_3 (A.8) and W_3 (B.44), we note the relation which admits us to get G_3 in (B.47),

$$(\nu_3^2 + W_3) U_{\nu_3, W_3} = U_{\nu_3, W_3} \nu_3^2. \quad (\text{B.49})$$

Equation $G_3 V^{(9)} = 0$ with G_3 as in (B.48) is the second-order differential equation w.r.t. B_3 . Two independent solutions of this equation are given in (4.29).

$$\text{Case} \quad m_1^2 > 0, \quad m_2^2 > 0, \quad m_3^2 < 0. \quad (\text{B.50})$$

Step 1. This case corresponds to the vertex $p_{[3]}^-$ (3.23). Vertex $V^{(3)}$ is given in (B.13), where the realization of G_a on $V^{(3)}$ is given in (B.8)–(B.10). We make the transformation

$$\begin{aligned} V^{(3)} = U_\zeta V^{(4)}, \quad U_\zeta = \exp & \left(-\frac{\zeta_1}{2m_1} (m_2^2 - m_3^2) \partial_{B_1} - \frac{\zeta_2}{2m_2} (m_3^2 - m_1^2) \partial_{B_2} \right. \\ & + \left(\frac{\zeta_1}{m_1} B_2 - \frac{\zeta_2}{m_2} B_1 + \frac{m_3^2 - m_1^2 - m_2^2}{2m_1 m_2} \zeta_1 \zeta_2 \right) \partial_{\alpha_{12}} \\ & \left. - \frac{\zeta_1}{m_1} \left(B_3 + \frac{1}{2} \kappa_3 \right) \partial_{\alpha_{31}} + \frac{\zeta_2}{m_2} \left(B_3 - \frac{1}{2} \kappa_3 \right) \partial_{\alpha_{23}} \right). \end{aligned} \quad (\text{B.51})$$

Realization of G_a (B.8)–(B.10) on $V^{(4)}$ (B.51) takes the form

$$G_1 = m_1 \partial_{\zeta_1}, \quad G_2 = m_2 \partial_{\zeta_2}, \quad (\text{B.52})$$

$$G_3 = -r_3 \partial_{B_3}^2 + q_3 (N_{B_3} + \nu_3 + 1) \partial_{B_3} + e_3 (N_{B_3} + \nu_3) (N_{B_3} + \nu_3 + 1) + u_3 \\ + \frac{2}{\kappa_3} (N_{B_3} + \nu_3 + 1) (B_2 \partial_{\alpha_{23}} - B_1 \partial_{\alpha_{31}}) - (B_1 \partial_{\alpha_{31}} + B_2 \partial_{\alpha_{23}}) \partial_{B_3} + 2\alpha_{12} \partial_{\alpha_{31}} \partial_{\alpha_{23}}, \quad (\text{B.53})$$

where, here and below, we use quantities defined in (A.16)–(A.19). From (B.52) and the equations $G_1 V^{(4)} = 0$, $G_2 V^{(4)} = 0$, we see that the $V^{(4)}$ is independent of ζ_1 , ζ_2 ,

$$V^{(4)} = V^{(4)}(B_a, \alpha_{aa+1}). \quad (\text{B.54})$$

Step 2. We make the transformation

$$V^{(4)} = U_{\partial B} U_{\partial \alpha} V^{(5)}, \quad U_{\partial B} = \exp\left(\frac{q_3}{2e_3} \partial_{B_3}\right), \quad U_{\partial \alpha} = \exp\left(\frac{B_2 \partial_{\alpha_{23}} - B_1 \partial_{\alpha_{31}}}{\kappa_3 e_3}\right) U_D, \quad (\text{B.55})$$

where U_D is defined in (A.15). Realization of G_3 (B.53) on $V^{(5)}$ (B.55) takes the form

$$G_3 = -\frac{D}{4\kappa_3^2 e_3} \partial_{B_3}^2 + e_3 (N_{B_3} + \nu_3) (N_{B_3} + \nu_3 + 1) + u_3 + W_3, \\ W_3 = -\frac{4m_1^2}{D} B_1^2 \partial_{\alpha_{31}}^2 - \frac{4m_2^2}{D} B_2^2 \partial_{\alpha_{23}}^2 + 2\alpha_{12} \partial_{\alpha_{31}} \partial_{\alpha_{23}}. \quad (\text{B.56})$$

Step 3. We make the transformations

$$V^{(4)} = U_{B_3} V^{(5)}, \quad V^{(5)} = U_{\nu_3, W_3} V^{(6)}, \quad V^{(6)} = U_{B_3}^{-1} V^{(7)}, \quad (\text{B.57})$$

where U_{B_3} , U_{ν_3, W_3} are given in (A.13), (A.14). Realization of G_3 (B.56) on $V^{(7)}$ takes the form

$$G_3 = -\frac{D}{4\kappa_3^2 e_3} \partial_{B_3}^2 + e_3 (N_{B_3} + \nu_3) (N_{B_3} + \nu_3 + 1) + u_3. \quad (\text{B.58})$$

For ν_3 (A.8) and W_3 (B.56), we note the helpful relation,

$$(\nu_3^2 + W_3) U_{\nu_3, W_3} = U_{\nu_3, W_3} \nu_3^2. \quad (\text{B.59})$$

Equation $G_3 V^{(9)} = 0$ with G_3 as in (B.58) is the second-order differential equation w.r.t. B_3 . Two independent solutions of this equation are given in (4.49).

C Derivation of $p_{[3]}^-$ (5.2), (5.25), (5.50), (5.71), (5.94)

We consider five vertices in (3.24), (3.28). For these vertices, we outline the derivation of the respective solutions in (5.2), (5.25), (5.50), (5.71), (5.94). We split our derivation in several steps.

Realization of G_a , G_β on $p_{[3]}^-$ (3.28) for arbitrary masses. First, for $p_{[3]}^-$ (3.28), we find G_β and $G_{a,\mathbb{P}^2}$ in (3.34). We use M_3^i for massive field (2.23) and M_a^i $a = 1, 2$ for continuous-spin field (2.20). Plugging such M_a^i into $\mathbf{J}^{-i\dagger}$ (3.17), we cast $\mathbf{J}^{-i\dagger}|p_{[3]}^- \rangle$ into the form given in (3.34) with the following G_β and $G_{a,\mathbb{P}^2}$:

$$G_{a,\mathbb{P}^2} = G_a + \mathbf{P}^- \frac{2\beta}{\beta_a^3} \alpha_a^i \frac{g_{v_a} \partial_{v_a}}{2N_a + d - 2} \partial_{B_a}^2, \quad a = 1, 2; \quad G_{3,\mathbb{P}^2} = G_3, \quad (\text{C.1})$$

$$\begin{aligned} G_1 = & \left(B_3 - \frac{\beta_1}{\beta_3} m_3 \zeta_3 \right) \partial_{\alpha_{31}} - \left(B_2 \partial_{\alpha_{12}} + \frac{\beta_1}{\beta_2} g_{v_2} \partial_{v_2} \right) \partial_{\alpha_{12}} + \frac{1}{2} \left(\frac{\check{\beta}_1}{\beta_1} m_1^2 + m_2^2 - m_3^2 \right) \partial_{B_1} + v_1 g_{v_1} \\ & + \frac{g_{v_1} \partial_{v_1}}{2N_1 + d - 2} \left(\frac{2\beta_2}{\beta_1} B_2 \partial_{B_1} \partial_{\alpha_{12}} + \frac{2\beta_3}{\beta_1} B_3 \partial_{B_1} \partial_{\alpha_{31}} + 2\alpha_{23} \partial_{\alpha_{12}} \partial_{\alpha_{31}} \right. \\ & \left. + \alpha_{33} \partial_{\alpha_{31}}^2 + \frac{\beta}{\beta_1^2} \sum_{b=1}^3 \frac{m_b^2}{\beta_b} \partial_{B_1}^2 \right), \end{aligned} \quad (\text{C.2})$$

$$\begin{aligned} G_2 = & \left(B_1 - \frac{\beta_2}{\beta_1} g_{v_1} \partial_{v_1} \right) \partial_{\alpha_{12}} - \left(B_3 + \frac{\beta_2}{\beta_3} m_3 \zeta_3 \right) \partial_{\alpha_{23}} + \frac{1}{2} \left(\frac{\check{\beta}_2}{\beta_2} m_2^2 + m_3^2 - m_1^2 \right) \partial_{B_2} + v_2 g_{v_2} \\ & + \frac{g_{v_2} \partial_{v_2}}{2N_2 + d - 2} \left(\frac{2\beta_3}{\beta_2} B_3 \partial_{B_2} \partial_{\alpha_{23}} + \frac{2\beta_1}{\beta_2} B_1 \partial_{B_2} \partial_{\alpha_{12}} + 2\alpha_{31} \partial_{\alpha_{12}} \partial_{\alpha_{23}} \right. \\ & \left. + \alpha_{33} \partial_{\alpha_{23}}^2 + \frac{\beta}{\beta_2^2} \sum_{b=1}^3 \frac{m_b^2}{\beta_b} \partial_{B_2}^2 \right), \end{aligned} \quad (\text{C.3})$$

$$G_3 = \left(B_2 - \frac{\beta_3}{\beta_2} g_{v_2} \partial_{v_2} \right) \partial_{\alpha_{23}} - \left(B_1 + \frac{\beta_3}{\beta_1} g_{v_1} \partial_{v_1} \right) \partial_{\alpha_{31}} + \frac{1}{2} \left(\frac{\check{\beta}_3}{\beta_3} m_3^2 + m_1^2 - m_2^2 \right) \partial_{B_3} + m_3 \partial_{\zeta_3}, \quad (\text{C.4})$$

$$G_\beta = -\frac{1}{\beta} \mathbb{N} - \frac{1}{\beta_1^2} g_{v_1} \partial_{v_1} \partial_{B_1} - \frac{1}{\beta_2^2} g_{v_2} \partial_{v_2} \partial_{B_2} - \frac{1}{\beta_3^2} m_3 \zeta_3 \partial_{B_3}, \quad (\text{C.5})$$

where g_{v_a} are given in (A.9). Using G_a , G_β (C.2)–(C.5), we now consider equations (3.35), (3.36).

Vertex $V^{(3)}$. We multiply G_a (C.2), (C.3), on the left by $(2N_a + d - 2)/\kappa_a$, $a = 1, 2$ and use the transformations¹⁷

$$p_{[3]}^- = U_{v_1} U_{v_2} V^{(1)}, \quad V^{(1)} = U_{\Gamma_1} U_{\Gamma_2} V^{(2)}, \quad V^{(2)} = U_\beta V^{(3)}, \quad (\text{C.6})$$

$$U_\beta = \exp \left(-\frac{\check{\beta}_1}{2\beta_1} \kappa_1 \partial_{B_1} - \frac{\check{\beta}_2}{2\beta_2} \kappa_2 \partial_{B_2} - \frac{\check{\beta}_3}{2\beta_3} m_3 \zeta_3 \partial_{B_3} \right), \quad (\text{C.7})$$

¹⁷For detailed discussion of transformations governed by operators $U_{v_1} U_{v_2}$, $U_{\Gamma_1} U_{\Gamma_2}$, U_β (C.6) see **Steps 2,3,4** in appendix D in ref. [18].

where U_{v_a} , U_{Γ_a} are defined in (A.10), (A.12). Realization of G_a , G_β (C.2)–(C.5) on $V^{(3)}$ takes the form

$$\begin{aligned}
 G_1 = & \frac{2N_1 + d - 2}{\kappa_1} \left(\left(B_3 + \frac{1}{2} m_3 \zeta_3 \right) \partial_{\alpha_{31}} - \left(B_2 - \frac{1}{2} \kappa_2 \right) \partial_{\alpha_{12}} + \frac{1}{2} (m_2^2 - m_3^2) \partial_{B_1} \right) \\
 & + \frac{F_1}{\kappa_1^2} - \left(B_2 + \frac{3}{2} \kappa_2 \right) \partial_{B_1} \partial_{\alpha_{12}} - \left(B_3 - \frac{3}{2} m_3 \zeta_3 \right) \partial_{B_1} \partial_{\alpha_{31}} + 2\alpha_{23} \partial_{\alpha_{12}} \partial_{\alpha_{31}} \\
 & + \alpha_{33} \partial_{\alpha_{31}}^2 + \frac{1}{4} (m_1^2 - 2m_2^2 - 2m_3^2) \partial_{B_1}^2, \tag{C.8}
 \end{aligned}$$

$$\begin{aligned}
 G_2 = & \frac{\kappa_2}{2N_2 + d - 2} \left(\left(B_1 + \frac{1}{2} \kappa_1 \right) \partial_{\alpha_{12}} - \left(B_3 - \frac{1}{2} m_3 \zeta_3 \right) \partial_{\alpha_{23}} + \frac{1}{2} (m_3^2 - m_1^2) \partial_{B_2} \right) \\
 & + \frac{F_2}{\kappa_2^2} - \left(B_3 + \frac{3}{2} m_3 \zeta_3 \right) \partial_{B_2} \partial_{\alpha_{23}} - \left(B_1 - \frac{3}{2} \kappa_1 \right) \partial_{B_2} \partial_{\alpha_{12}} + 2\alpha_{31} \partial_{\alpha_{12}} \partial_{\alpha_{23}} \\
 & + \alpha_{33} \partial_{\alpha_{23}}^2 + \frac{1}{4} (m_2^2 - 2m_3^2 - 2m_1^2) \partial_{B_2}^2, \tag{C.9}
 \end{aligned}$$

$$G_3 = \left(B_2 + \frac{1}{2} \kappa_2 \right) \partial_{\alpha_{23}} - \left(B_1 - \frac{1}{2} \kappa_1 \right) \partial_{\alpha_{31}} + \frac{1}{2} (m_1^2 - m_2^2) \partial_{B_3} + m_3 \partial_{\zeta_3}, \tag{C.10}$$

$$G_\beta = -\frac{1}{\beta} \mathbb{N}, \tag{C.11}$$

where F_a are given in (A.9).

Dependence of $V^{(3)}$ on β_a . We fix dependence of $V^{(3)}$ on β_a . To this end we use eqs. (3.36), (3.40) and relation (C.11) to note the following equations for the vertex $V^{(3)}$:

$$\mathbb{N}_\beta V^{(3)} = 0, \quad \sum_{a=1,2,3} \beta_a \partial_{\beta_a} V^{(3)} = 0. \tag{C.12}$$

Equations (C.12) imply that $V^{(3)}$ is independent of β_1 , β_2 , β_3 . Thus, the vertex $V^{(3)}$ is given by

$$V^{(3)} = V^{(3)}(B_a, \alpha_{aa+1}, \zeta_3). \tag{C.13}$$

We now use results above-obtained for the study of five vertices in (3.24), (3.28) in turn.

$$\text{Case} \quad m_1^2 < 0, \quad m_2 = 0, \quad m_3 = 0. \tag{C.14}$$

Step 1. This case corresponds to the first vertex $p_{[3]}^-$ (3.24). In (C.6), we use operator U_{Γ_2} given in (A.11). Then, in all remaining relations in (C.1)–(C.13), we set $m_2 = 0$, $m_3 = 0$ and, ignoring ζ_3 in (C.1)–(C.13), we proceed with vertex $V^{(3)}$ given by

$$V^{(3)} = V^{(3)}(B_a, \alpha_{aa+1}), \tag{C.15}$$

where the realization of operators G_a (C.8)–(C.10) on the vertex $V^{(3)}$ (C.15) takes the form¹⁸

$$G_1 = \frac{2N_1 + d - 2}{\kappa_1} \left(B_3 \partial_{\alpha_{31}} - \left(B_2 - \frac{1}{2} \kappa_2 \right) \partial_{\alpha_{12}} \right) + \frac{F_1}{\kappa_1^2} - \left(B_2 + \frac{3}{2} \kappa_2 \right) \partial_{B_1} \partial_{\alpha_{12}} \\ - B_3 \partial_{B_1} \partial_{\alpha_{31}} + 2\alpha_{23} \partial_{\alpha_{12}} \partial_{\alpha_{31}} + \alpha_{33} \partial_{\alpha_{31}}^2 + \frac{1}{4} m_1^2 \partial_{B_1}^2, \quad (\text{C.16})$$

$$G_2 = \frac{2N_2 + d - 2}{\kappa_2} \left(\left(B_1 + \frac{1}{2} \kappa_1 \right) \partial_{\alpha_{12}} - B_3 \partial_{\alpha_{23}} - \frac{1}{2} m_1^2 \partial_{B_2} \right) + \frac{F_2}{\kappa_2^2} - B_3 \partial_{B_2} \partial_{\alpha_{23}} \\ - \left(B_1 - \frac{3}{2} \kappa_1 \right) \partial_{B_2} \partial_{\alpha_{12}} + 2\alpha_{31} \partial_{\alpha_{12}} \partial_{\alpha_{23}} + \alpha_{33} \partial_{\alpha_{23}}^2 - \frac{1}{2} m_1^2 \partial_{B_2}^2, \quad (\text{C.17})$$

$$G_3 = \left(B_2 + \frac{1}{2} \kappa_2 \right) \partial_{\alpha_{23}} - \left(B_1 - \frac{1}{2} \kappa_1 \right) \partial_{\alpha_{31}} + \frac{1}{2} m_1^2 \partial_{B_3}. \quad (\text{C.18})$$

Step 2. We make the transformation

$$V^{(3)} = U_{1\partial\alpha} V^{(4)}, \quad U_{1\partial\alpha} = \exp \left(\frac{2B_3}{m_1^2} \left(B_1 - \frac{1}{2} \kappa_1 \right) \partial_{\alpha_{31}} - \frac{2B_3}{m_1^2} \left(B_2 + \frac{1}{2} \kappa_2 \right) \partial_{\alpha_{23}} \right). \quad (\text{C.19})$$

Realization of G_a (C.16)–(C.18) on $V^{(4)}$ takes the form

$$G_1 = -r_1 \partial_{B_1}^2 + e_1 (N_{B_1} + \nu_1) (N_{B_1} + \nu_1 + 1) + u_1 \\ - \frac{2}{\kappa_1} (N_{B_1} + \nu_1 + 1) \left(B_2 - \frac{1}{2} \kappa_2 \right) \partial_{\alpha_{12}} - \left(B_2 + \frac{3}{2} \kappa_2 \right) \partial_{B_1} \partial_{\alpha_{12}} + 2\alpha_{23} \partial_{\alpha_{12}} \partial_{\alpha_{31}}, \quad (\text{C.20})$$

$$G_2 = -r_2 \partial_{B_2}^2 + q_2 (N_{B_2} + \nu_2 + 1) \partial_{B_2} + u_2 \\ + \frac{2}{\kappa_2} (N_{B_2} + \nu_2 + 1) \left(B_1 + \frac{1}{2} \kappa_1 \right) \partial_{\alpha_{12}} - \left(B_1 - \frac{3}{2} \kappa_1 \right) \partial_{B_2} \partial_{\alpha_{12}} + 2\alpha_{31} \partial_{\alpha_{12}} \partial_{\alpha_{23}}, \quad (\text{C.21})$$

$$G_3 = \frac{1}{2} m_1^2 \partial_{B_3}, \quad (\text{C.22})$$

where, here and below, we use quantities (A.16)–(A.19) taken for masses given in (C.14). From (C.22) and equation $G_3 V^{(4)} = 0$, we see that $V^{(4)}$ is independent of B_3 ,

$$V^{(4)} = V^{(4)}(B_1, B_2, \alpha_{aa+1}). \quad (\text{C.23})$$

Step 3. We make the transformation

$$V^{(4)} = U_{\partial B} U_{2\partial\alpha} U_{3\partial\alpha} V^{(5)}, \quad U_{\partial B} = \exp \left(\frac{\kappa_2}{2} \partial_{B_2} \right) \\ U_{2\partial\alpha} = \exp \left(\frac{\kappa_2}{\kappa_1 e_1} \partial_{\alpha_{12}} - \frac{B_2}{\kappa_1 e_1} \partial_{\alpha_{12}} - \frac{2B_1 B_2}{\kappa_1^2 e_1} \partial_{\alpha_{12}} \right), \quad U_{3\partial\alpha} = \exp \left(-\frac{2\kappa_2}{\kappa_1^2 e_1} B_1 \partial_{\alpha_{12}} \right). \quad (\text{C.24})$$

Realization of G_1, G_2 (C.20), (C.21) on $V^{(5)}$ (C.24) takes the form

$$G_1 = -\frac{D}{4\kappa_1^2 e_1} \partial_{B_1}^2 + e_1 (N_{B_1} + \nu_1) (N_{B_1} + \nu_1 + 1) + u_1 + \frac{4}{q_2} B_2 \partial_{\alpha_{12}}^2 + 2\alpha_{23} \partial_{\alpha_{12}} \partial_{\alpha_{31}}, \\ G_2 = q_2 (N_{B_2} + \nu_2 + 1) \partial_{B_2} + u_2 + \left(\frac{4\kappa_1^2 e_1}{D} B_1^2 - \frac{1}{e_1} \right) \partial_{\alpha_{12}}^2 + 2\alpha_{31} \partial_{\alpha_{12}} \partial_{\alpha_{23}}. \quad (\text{C.25})$$

¹⁸For massless field corresponding to $a = 3$, α_{33} -terms in (C.2), (C.3) can be ignored in (C.16), (C.17) by virtue of the second constraint in (2.12).

Also, using $U_{\partial\alpha}$ given in (5.14), we note the relation

$$U_{1\partial\alpha}U_{\partial B}U_{2\partial\alpha}U_{3\partial\alpha} = U_{\partial B}U_{\partial\alpha}. \quad (\text{C.26})$$

Step 4. We make the transformations

$$V^{(5)} = U_{B_1} \left(-\frac{4}{q_2} B_2 \right)^{-\nu_2/2} V^{(6)}, \quad V^{(6)} = U_{\nu_1, W_1} U_{\nu_2, W_{23}} V^{(7)}, \quad V^{(7)} = U_{B_1}^{-1} V^{(8)}, \quad (\text{C.27})$$

where U_{B_1} , $U_{\nu, W}$ are given in (A.13), (A.14) and we use the notation

$$\begin{aligned} W_1 &= 2\alpha_{23}\partial_{\alpha_{12}}\partial_{\alpha_{31}} - \partial_{\alpha_{12}}^2, \\ W_2 &= 2\alpha_{31}\partial_{\alpha_{12}}\partial_{\alpha_{23}} - \partial_{\alpha_{12}}^2, \quad W_{23} = 2\alpha_{31}\partial_{\alpha_{12}}\partial_{\alpha_{23}}. \end{aligned} \quad (\text{C.28})$$

Realizations of G_1 , G_2 (C.25) on $V^{(6)}$, $V^{(7)}$, $V^{(8)}$ (C.27) take the forms

$$\begin{aligned} G_1 &= \left(B_1^2 - \frac{D}{4\kappa_1^2 e_1^2} \right) \partial_{B_1}^2 + \left(1 - \frac{4\kappa_1^2 e_1^2}{D} B_1^2 \right)^{-1} (\nu_1^2 - 1 + W_1) + \frac{u_1}{e_1}, \\ G_2 &= u_2 + q_2 (N_{B_2} + 1) \partial_{B_2} - \frac{q_2}{4B_2} (\nu_2^2 + W_2), \end{aligned} \quad \text{for } V^{(6)}; \quad (\text{C.29})$$

$$\begin{aligned} G_1 &= \left(B_1^2 - \frac{D}{4\kappa_1^2 e_1^2} \right) \partial_{B_1}^2 + \left(1 - \frac{4\kappa_1^2 e_1^2}{D} B_1^2 \right)^{-1} (\nu_1^2 - 1) + \frac{u_1}{e_1}, \\ G_2 &= u_2 + q_2 (N_{B_2} + 1) \partial_{B_2} - \frac{q_2}{4B_2} \nu_2^2, \end{aligned} \quad \text{for } V^{(7)}; \quad (\text{C.30})$$

$$\begin{aligned} G_1 &= -\frac{D}{4\kappa_1^2 e_1} \partial_{B_1}^2 + e_1 (N_{B_1} + \nu_1) (N_{B_1} + \nu_1 + 1) + u_1, \\ G_2 &= u_2 + q_2 (N_{B_2} + 1) \partial_{B_2} - \frac{q_2}{4B_2} \nu_2^2, \end{aligned} \quad \text{for } V^{(8)}. \quad (\text{C.31})$$

For ν_a (A.8) and W -operators (C.28), we note the relations which admit us to get G_1 , G_2 in (C.30),

$$\begin{aligned} (\nu_1^2 + W_1) U_{\nu_1, W_1} &= U_{\nu_1, W_1} \nu_1^2, & \nu_1^2 U_{\nu_2, W_{23}} &= U_{\nu_2, W_{23}} \nu_1^2, \\ (\nu_2^2 + W_2) U_{\nu_1, W_1} &= U_{\nu_1, W_1} (\nu_2^2 + W_{23}), & (\nu_2^2 + W_{23}) U_{\nu_2, W_{23}} &= U_{\nu_2, W_{23}} \nu_2^2. \end{aligned} \quad (\text{C.32})$$

Equations $G_1 V^{(8)} = 0$ and $G_2 V^{(8)} = 0$ with G_1 and G_2 as in (C.31) constitute a system of two decoupled second-order differential equations w.r.t. B_1 and B_2 . Four independent solutions of these equations are given in (5.5).

$$\text{Case} \quad m_1 = m, \quad m_2 = m, \quad m_3 = 0, \quad m^2 < 0. \quad (\text{C.33})$$

Step 1. This case corresponds to the second vertex $p_{[3]}^-$ (3.24). In all relations in (C.1)–(C.13), we set $m_1 = m$, $m_2 = m$, $m_3 = 0$ and, ignoring ζ_3 in (C.1)–(C.13), we use vertex $V^{(3)}$ given by

$$V^{(3)} = V^{(3)}(B_a, \alpha_{aa+1}), \quad (\text{C.34})$$

where the realization of operators G_a (C.8)–(C.10) on the vertex $V^{(3)}$ (C.34) takes the form¹⁹

$$G_1 = \frac{2N_1 + d - 2}{\kappa_1} \left(B_3 \partial_{\alpha_{31}} - \left(B_2 - \frac{1}{2} \kappa_2 \right) \partial_{\alpha_{12}} + \frac{1}{2} m^2 \partial_{B_1} \right) + \frac{F_1}{\kappa_1^2} - \left(B_2 + \frac{3}{2} \kappa_2 \right) \partial_{B_1} \partial_{\alpha_{12}} - B_3 \partial_{B_1} \partial_{\alpha_{31}} + 2\alpha_{23} \partial_{\alpha_{12}} \partial_{\alpha_{31}} - \frac{1}{4} m^2 \partial_{B_1}^2, \quad (\text{C.35})$$

$$G_2 = \frac{2N_2 + d - 2}{\kappa_2} \left(\left(B_1 + \frac{1}{2} \kappa_1 \right) \partial_{\alpha_{12}} - B_3 \partial_{\alpha_{23}} - \frac{1}{2} m^2 \partial_{B_2} \right) + \frac{F_2}{\kappa_2^2} - B_3 \partial_{B_2} \partial_{\alpha_{23}} - \left(B_1 - \frac{3}{2} \kappa_1 \right) \partial_{B_2} \partial_{\alpha_{12}} + 2\alpha_{31} \partial_{\alpha_{12}} \partial_{\alpha_{23}} - \frac{1}{4} m^2 \partial_{B_2}^2, \quad (\text{C.36})$$

$$G_3 = \left(B_2 + \frac{1}{2} \kappa_2 \right) \partial_{\alpha_{23}} - \left(B_1 - \frac{1}{2} \kappa_1 \right) \partial_{\alpha_{31}}. \quad (\text{C.37})$$

Step 2. From (C.37) and equation $G_3 V^{(4)} = 0$, we find

$$V^{(3)} = V^{(4)}(B_a, \alpha_{12}, Z), \quad Z = \left(B_1 - \frac{1}{2} \kappa_1 \right) \alpha_{23} + \left(B_2 + \frac{1}{2} \kappa_2 \right) \alpha_{31}. \quad (\text{C.38})$$

Realization of G_1, G_2 (C.35), (C.36) on $V^{(4)}$ (C.38) takes the form

$$G_1 = -r_1 \partial_{B_1}^2 + q_1 (N_{B_1} + \nu_Z + 1) \partial_{B_1} + e_1 (N_{B_1} + \nu_Z) (N_{B_1} + \nu_Z + 1) + u_1 + \frac{2}{\kappa_1} (N_{B_1} + \nu_Z + 1) \left(B_3 \left(B_2 + \frac{1}{2} \kappa_2 \right) \partial_Z - \left(B_2 - \frac{1}{2} \kappa_2 \right) \partial_{\alpha_{12}} \right) - \left(B_2 + \frac{3}{2} \kappa_2 \right) \partial_{B_1} \partial_{\alpha_{12}} - B_3 \left(B_2 + \frac{1}{2} \kappa_2 \right) \partial_{B_1} \partial_Z, \\ G_2 = -r_2 \partial_{B_2}^2 + q_2 (N_{B_2} + \nu_Z + 1) \partial_{B_2} + e_2 (N_{B_2} + \nu_Z) (N_{B_2} + \nu_Z + 1) + u_2 + \frac{2}{\kappa_2} (N_{B_2} + \nu_Z + 1) \left(\left(B_1 + \frac{1}{2} \kappa_1 \right) \partial_{\alpha_{12}} - B_3 \left(B_1 - \frac{1}{2} \kappa_1 \right) \partial_Z \right) - \left(B_1 - \frac{3}{2} \kappa_1 \right) \partial_{B_2} \partial_{\alpha_{12}} - B_3 \left(B_1 - \frac{1}{2} \kappa_1 \right) \partial_{B_2} \partial_Z, \quad (\text{C.39})$$

$$\nu_Z = N_{\alpha_{12}} + N_Z + \frac{d-4}{2}, \quad N_Z = Z \partial_Z, \quad (\text{C.40})$$

where, here and below, we use quantities (A.16)–(A.19) taken for masses given in (C.33).

Step 3. We make the transformation

$$V^{(4)} = U_{\partial B} U_{\partial \alpha} V^{(5)}, \quad U_{\partial B} = \exp \left(\frac{q_1}{2e_1} \partial_{B_1} + \frac{q_2}{2e_2} \partial_{B_2} \right), \\ U_{\partial \alpha} = \exp \left(-\frac{\kappa_1 \kappa_2}{m^2} \partial_{\alpha_{12}} - \frac{\kappa_2}{m^2} B_1 \partial_{\alpha_{12}} + \frac{\kappa_1}{m^2} B_2 \partial_{\alpha_{12}} + \frac{1}{2m^2} B_1 B_2 \partial_{\alpha_{12}} - \frac{\kappa_1}{m^2} B_2 B_3 \partial_Z + \frac{\kappa_2}{m^2} B_3 B_1 \partial_Z - \frac{1}{m^2} B_1 B_2 B_3 \partial_Z \right). \quad (\text{C.41})$$

¹⁹For massless field corresponding to $a = 3$, α_{33} -terms in (C.2), (C.3) can be ignored in (C.35), (C.36) by virtue of the second constraint in (2.12).

Realization of G_1, G_2 (C.39) on $V^{(5)}$ (C.41) takes the form

$$\begin{aligned} G_1 &= e_1(N_{B_1} + \nu_Z)(N_{B_1} + \nu_Z + 1) + u_1 - \frac{1}{e_2} \partial_{\alpha_{12}}^2 - 2B_2 \partial_{B_1} \partial_{\alpha_{12}} + \frac{1}{m^2} B_2^2 B_3^2 \partial_Z^2, \\ G_2 &= e_2(N_{B_2} + \nu_Z)(N_{B_2} + \nu_Z + 1) + u_2 - \frac{1}{e_1} \partial_{\alpha_{12}}^2 - 2B_1 \partial_{B_2} \partial_{\alpha_{12}} + \frac{1}{m^2} B_1^2 B_3^2 \partial_Z^2. \end{aligned} \quad (\text{C.42})$$

Step 4. We make the transformation

$$V^{(5)} = U_e V^{(6)}, \quad U_e = e_1^{\omega_1/2} e_2^{\omega_2/2} \quad (\text{C.43})$$

$$\omega_1 = N_{B_1} + \nu_Z + \frac{1}{2}, \quad \omega_2 = N_{B_1} + \nu_Z + \frac{1}{2}, \quad \nu_Z = N_{\alpha_{12}} + N_Z + \frac{d-4}{2}, \quad (\text{C.44})$$

where $N_Z = Z \partial_Z$, while e_a are given in (A.17). Realization of G_1, G_2 (C.42) on $V^{(6)}$ (C.43) takes the form

$$G_a = (N_{B_a} + \nu_Z)(N_{B_a} + \nu_Z + 1) + \frac{u_a}{e_a} + W_a, \quad a = 1, 2, \quad (\text{C.45})$$

$$\begin{aligned} W_1 &= \frac{1}{m^2} B_2^2 B_3^2 \partial_Z^2 - \partial_{\alpha_{12}}^2 - 2B_2 \partial_{\alpha_{12}} \partial_{B_1}, \\ W_2 &= \frac{1}{m^2} B_1^2 B_3^2 \partial_Z^2 - \partial_{\alpha_{12}}^2 - 2B_1 \partial_{\alpha_{12}} \partial_{B_2}, \quad W_{21} = \frac{1}{m^2} B_1^2 B_3^2 \partial_Z^2 - 2B_1 \partial_{\alpha_{12}} \partial_{B_2}. \end{aligned} \quad (\text{C.46})$$

Step 5. We make the transformations

$$V^{(6)} = U_{\omega_1, W_1} U_{\omega_2, W_{21}} V^{(7)}, \quad (\text{C.47})$$

where we use notation in (C.44), (C.46), (A.14). Realization of G_a (C.45) on $V^{(7)}$ (C.47) takes the form

$$G_a = (N_{B_a} + \nu_Z)(N_{B_a} + \nu_Z + 1) + \frac{u_a}{e_a}, \quad a = 1, 2. \quad (\text{C.48})$$

For ω_a (C.44) and W -operators (C.46), we note the relations which admit us to get G_a in (C.48),

$$\begin{aligned} (\omega_1^2 + W_1) U_{\omega_1, W_1} &= U_{\omega_1, W_1} \omega_1^2, & \omega_1^2 U_{\omega_2, W_{21}} &= U_{\omega_2, W_{21}} \omega_1^2, \\ (\omega_2^2 + W_2) U_{\omega_1, W_1} &= U_{\omega_1, W_1} (\omega_2^2 + W_{21}), & (\omega_2^2 + W_{21}) U_{\omega_2, W_{21}} &= U_{\omega_2, W_{21}} \omega_2^2. \end{aligned} \quad (\text{C.49})$$

Equations $G_1 V^{(7)} = 0$ and $G_2 V^{(7)} = 0$ with G_1 and G_2 as in (C.48) constitute a system of two decoupled second-order differential equations w.r.t. B_1 and B_2 . All independent solutions of these equations are given in (5.29).

$$\text{Case} \quad m_1^2 < 0, \quad m_2^2 < 0, \quad m_3 = 0, \quad m_1 \neq m_2. \quad (\text{C.50})$$

Step 1. This case corresponds to the third vertex $p_{[3]}^-$ (3.24). In relations (C.1)–(C.13), we set $m_3 = 0$ and, ignoring ζ_3 in (C.1)–(C.13), we use vertex $V^{(3)}$ given by

$$V^{(3)} = V^{(3)}(B_a, \alpha_{aa+1}), \quad (\text{C.51})$$

where the realization of operators G_a (C.8)–(C.10) on the vertex $V^{(3)}$ (C.51) takes the form²⁰

$$G_1 = \frac{2N_1 + d - 2}{\kappa_1} \left(B_3 \partial_{\alpha_{31}} - (B_2 - \frac{1}{2} \kappa_2) \partial_{\alpha_{12}} + \frac{m_2^2}{2} \partial_{B_1} \right) + \frac{F_1}{\kappa_1^2} - \left(B_2 + \frac{3}{2} \kappa_2 \right) \partial_{B_1} \partial_{\alpha_{12}} \\ - B_3 \partial_{B_1} \partial_{\alpha_{31}} + 2\alpha_{23} \partial_{\alpha_{12}} \partial_{\alpha_{31}} + \alpha_{33} \partial_{\alpha_{31}}^2 + \frac{1}{4} (m_1^2 - 2m_2^2) \partial_{B_1}^2, \quad (\text{C.52})$$

$$G_2 = \frac{2N_2 + d - 2}{\kappa_2} \left(\left(B_1 + \frac{1}{2} \kappa_1 \right) \partial_{\alpha_{12}} - B_3 \partial_{\alpha_{23}} - \frac{m_1^2}{2} \partial_{B_2} \right) + \frac{F_2}{\kappa_2^2} - B_3 \partial_{B_2} \partial_{\alpha_{23}} \\ - \left(B_1 - \frac{3}{2} \kappa_1 \right) \partial_{B_2} \partial_{\alpha_{12}} + 2\alpha_{31} \partial_{\alpha_{12}} \partial_{\alpha_{23}} + \alpha_{33} \partial_{\alpha_{23}}^2 + \frac{1}{4} (m_2^2 - 2m_1^2) \partial_{B_2}^2, \quad (\text{C.53})$$

$$G_3 = \left(B_2 + \frac{1}{2} \kappa_2 \right) \partial_{\alpha_{23}} - \left(B_1 - \frac{1}{2} \kappa_1 \right) \partial_{\alpha_{31}} + \frac{1}{2} (m_1^2 - m_2^2) \partial_{B_3}. \quad (\text{C.54})$$

Step 2. We make the transformation

$$V^{(3)} = UV^{(4)}, \quad U_{1\partial\alpha} = \exp \left(\frac{2B_3}{m_1^2 - m_2^2} \left(B_1 - \frac{1}{2} \kappa_1 \right) \partial_{\alpha_{31}} - \frac{2B_3}{m_1^2 - m_2^2} \left(B_2 + \frac{1}{2} \kappa_2 \right) \partial_{\alpha_{23}} \right). \quad (\text{C.55})$$

Realization of G_1, G_2, G_3 (C.52)–(C.54) on $V^{(4)}$ (C.55) takes the form

$$G_1 = -r_1 \partial_{B_1}^2 + q_1 (N_{B_1} + \nu_1 + 1) \partial_{B_1} + e_1 (N_{B_1} + \nu_1) (N_{B_1} + \nu_1 + 1) + u_1 \\ - \frac{2}{\kappa_1} (N_{B_1} + \nu_1 + 1) \left(B_2 - \frac{1}{2} \kappa_2 \right) \partial_{\alpha_{12}} - \left(B_2 + \frac{3}{2} \kappa_2 \right) \partial_{B_1} \partial_{\alpha_{12}} + 2\alpha_{23} \partial_{\alpha_{12}} \partial_{\alpha_{31}}, \\ G_2 = -r_2 \partial_{B_2}^2 + q_2 (N_{B_2} + \nu_2 + 1) \partial_{B_2} + e_2 (N_{B_2} + \nu_2) (N_{B_2} + \nu_2 + 2) + u_2 \\ + \frac{2}{\kappa_2} (N_{B_2} + \nu_2 + 1) \left(B_1 + \frac{1}{2} \kappa_1 \right) \partial_{\alpha_{12}} - \left(B_1 - \frac{3}{2} \kappa_1 \right) \partial_{B_2} \partial_{\alpha_{12}} + 2\alpha_{31} \partial_{\alpha_{12}} \partial_{\alpha_{23}}, \quad (\text{C.56})$$

$$G_3 = \frac{1}{2} (m_1^2 - m_2^2) \partial_{B_3}, \quad (\text{C.57})$$

where, here and below, we use quantities (A.16)–(A.19) taken for masses given in (C.50). From (C.57) and equation $G_3 V^{(4)} = 0$, we see that $V^{(4)}$ is independent of B_3 ,

$$V^{(4)} = V^{(4)}(B_1, B_2, \alpha_{aa+1}). \quad (\text{C.58})$$

Step 3. We make the transformation

$$V^{(4)} = U_{\partial B} U_{2\partial\alpha} U_{3\partial\alpha} V^{(5)}, \quad U_{\partial B} = \exp \left(\frac{q_1}{2e_1} \partial_{B_1} + \frac{q_2}{2e_2} \partial_{B_2} \right), \\ U_{2\partial\alpha} = \exp \left(\frac{B_1}{\kappa_2 e_2} \partial_{\alpha_{12}} - \frac{B_2}{\kappa_1 e_1} \partial_{\alpha_{12}} - \frac{h_{12}}{2\kappa_1 \kappa_2 e_1 e_2} \partial_{\alpha_{12}} \right), \quad U_{3\partial\alpha} = \exp \left(\frac{2h_{12}}{D} B_1 B_2 \partial_{\alpha_{12}} \right). \quad (\text{C.59})$$

²⁰For massless field corresponding to $a = 3$, α_{33} -terms in (C.2), (C.3) can be ignored in (C.52), (C.53) by virtue of the second constraint in (2.12).

Realization of G_1, G_2 (C.56) on $V^{(5)}$ (C.59) takes the form

$$\begin{aligned} G_1 &= -\frac{D}{4\kappa_1^2 e_1} \partial_{B_1}^2 + e_1(N_{B_1} + \nu_1)(N_{B_1} + \nu_1 + 1) + u_1 \\ &\quad + \left(\frac{4\kappa_2^2 e_2}{D} B_2^2 - \frac{1}{e_2} \right) \partial_{\alpha_{12}}^2 + 2\alpha_{23} \partial_{\alpha_{12}} \partial_{\alpha_{31}}, \\ G_2 &= -\frac{D}{4\kappa_2^2 e_2} \partial_{B_2}^2 + e_2(N_{B_2} + \nu_2)(N_{B_2} + \nu_2 + 1) + u_2 \\ &\quad + \left(\frac{4\kappa_1^2 e_1}{D} B_1^2 - \frac{1}{e_1} \right) \partial_{\alpha_{12}}^2 + 2\alpha_{31} \partial_{\alpha_{12}} \partial_{\alpha_{23}}. \end{aligned} \quad (\text{C.60})$$

Also, using $U_{\partial\alpha}$ in (5.60), we note the relation

$$U_{1\partial\alpha} U_{\partial B} U_{2\partial\alpha} U_{3\partial\alpha} = U_{\partial B} U_{\partial\alpha}. \quad (\text{C.61})$$

Step 4. We make the transformation

$$V^{(5)} = U_{B_1} U_{B_2} V^{(6)}, \quad V^{(6)} = U_{\nu_1, W_1} U_{\nu_2, W_{21}} V^{(7)}, \quad V^{(7)} = U_{B_1}^{-1} U_{B_2}^{-1} V^{(8)}, \quad (\text{C.62})$$

where $U_{B_a}, U_{\nu, W}$ are defined in (A.13), (A.14) and we use the notation

$$\begin{aligned} W_1 &= 2\alpha_{23} \partial_{\alpha_{12}} \partial_{\alpha_{31}} - \partial_{\alpha_{12}}^2, \\ W_2 &= 2\alpha_{31} \partial_{\alpha_{12}} \partial_{\alpha_{23}} - \partial_{\alpha_{12}}^2, \quad W_{21} = 2\alpha_{31} \partial_{\alpha_{12}} \partial_{\alpha_{23}}. \end{aligned} \quad (\text{C.63})$$

Realizations of G_1, G_2 (C.60) on $V^{(6)}, V^{(7)}, V^{(8)}$ (C.62) take the forms

$$G_a = \left(B_a^2 - \frac{D}{4\kappa_a^2 e_a^2} \right) \partial_{B_a}^2 + \frac{u_a}{e_a} + \left(1 - \frac{4\kappa_a^2 e_a^2}{D} B_a^2 \right)^{-1} (\nu_a^2 - 1 + W_a), \quad \text{for } V^{(6)}; \quad (\text{C.64})$$

$$G_a = \left(B_a^2 - \frac{D}{4\kappa_a^2 e_a^2} \right) \partial_{B_a}^2 + \left(1 - \frac{4\kappa_a^2 e_a^2}{D} B_a^2 \right)^{-1} (\nu_a^2 - 1) + \frac{u_a}{e_a}, \quad \text{for } V^{(7)}; \quad (\text{C.65})$$

$$G_a = -\frac{D}{4\kappa_a^2 e_a} \partial_{B_a}^2 + e_a(N_{B_a} + \nu_a)(N_{B_a} + \nu_a + 1) + u_a, \quad \text{for } V^{(8)}; \quad (\text{C.66})$$

where $a = 1, 2$. For ν_a (A.8) and W -operators (C.63), we note the relations which admit us to get G_a in (C.65),

$$\begin{aligned} (\nu_1^2 + W_1) U_{\nu_1, W_1} &= U_{\nu_1, W_1} \nu_1^2, & \nu_1^2 U_{\nu_2, W_{21}} &= U_{\nu_2, W_{21}} \nu_1^2, \\ (\nu_2^2 + W_2) U_{\nu_1, W_1} &= U_{\nu_1, W_1} (\nu_2^2 + W_{21}), & (\nu_2^2 + W_{21}) U_{\nu_2, W_{21}} &= U_{\nu_2, W_{21}} \nu_2^2. \end{aligned} \quad (\text{C.67})$$

Equations $G_1 V^{(8)} = 0$ and $G_2 V^{(8)} = 0$ with G_1 and G_2 as in (C.66) constitute a system of two decoupled second-order differential equations w.r.t. B_1 and B_2 . Four independent solutions of these equations are given in (5.53).

$$\text{Case} \quad m_1 = 0, \quad m_2^2 < 0, \quad m_3^2 > 0. \quad (\text{C.68})$$

Step 1. This case corresponds to the first vertex $p_{[3]}^-$ in (3.28). In (C.6), we use operator U_{Γ_1} (A.11). Then, in the remaining relations (C.1)–(C.13), we set $m_1 = 0$ and use vertex $V^{(3)}$ given by

$$V^{(3)} = V^{(3)}(B_a, \alpha_{aa+1}, \zeta_3), \quad (\text{C.69})$$

where the realization of operators G_a (C.8)–(C.10) on the vertex $V^{(3)}$ (C.69) takes the form

$$\begin{aligned}
 G_1 &= \frac{2N_1 + d - 2}{\kappa_1} \left(\left(B_3 + \frac{1}{2} m_3 \zeta_3 \right) \partial_{\alpha_{31}} - \left(B_2 - \frac{1}{2} \kappa_2 \right) \partial_{\alpha_{12}} + \frac{1}{2} (m_2^2 - m_3^2) \partial_{B_1} \right) \\
 &\quad + \frac{F_1}{\kappa_1^2} - \left(B_2 + \frac{3}{2} \kappa_2 \right) \partial_{B_1} \partial_{\alpha_{12}} - \left(B_3 - \frac{3}{2} m_3 \zeta_3 \right) \partial_{B_1} \partial_{\alpha_{31}} + 2\alpha_{23} \partial_{\alpha_{12}} \partial_{\alpha_{31}} \\
 &\quad + \alpha_{33} \partial_{\alpha_{31}}^2 - \frac{1}{2} (m_2^2 + m_3^2) \partial_{B_1}^2, \\
 G_2 &= \frac{\kappa_2}{2N_2 + d - 2} \left(\left(B_1 + \frac{1}{2} \kappa_1 \right) \partial_{\alpha_{12}} - \left(B_3 - \frac{1}{2} m_3 \zeta_3 \right) \partial_{\alpha_{23}} + \frac{1}{2} m_3^2 \partial_{B_2} \right) \\
 &\quad + \frac{F_2}{\kappa_2^2} - \left(B_3 + \frac{3}{2} m_3 \zeta_3 \right) \partial_{B_2} \partial_{\alpha_{23}} - \left(B_1 - \frac{3}{2} \kappa_1 \right) \partial_{B_2} \partial_{\alpha_{12}} + 2\alpha_{31} \partial_{\alpha_{12}} \partial_{\alpha_{23}} \\
 &\quad + \alpha_{33} \partial_{\alpha_{23}}^2 + \frac{1}{4} (m_2^2 - 2m_3^2) \partial_{B_2}^2, \\
 G_3 &= \left(B_2 + \frac{1}{2} \kappa_2 \right) \partial_{\alpha_{23}} - \left(B_1 - \frac{1}{2} \kappa_1 \right) \partial_{\alpha_{31}} - \frac{1}{2} m_2^2 \partial_{B_3} + m_3 \partial_{\zeta_3}.
 \end{aligned} \tag{C.70}$$

$$G_3 = \left(B_2 + \frac{1}{2} \kappa_2 \right) \partial_{\alpha_{23}} - \left(B_1 - \frac{1}{2} \kappa_1 \right) \partial_{\alpha_{31}} - \frac{1}{2} m_2^2 \partial_{B_3} + m_3 \partial_{\zeta_3}. \tag{C.71}$$

Step 2. We make the transformations

$$V^{(3)} = U_\zeta V^{(4)}, \quad U_\zeta = \exp \left(\frac{\zeta_3}{m_3} \left(B_1 - \frac{1}{2} \kappa_1 \right) \partial_{\alpha_{31}} - \frac{\zeta_3}{m_3} \left(B_2 + \frac{1}{2} \kappa_2 \right) \partial_{\alpha_{23}} + \frac{m_2^2}{2m_3} \zeta_3 \partial_{B_3} \right). \tag{C.72}$$

Realization of G_1, G_2, G_3 (C.70), (C.71) on $V^{(4)}$ (C.72) takes the form²¹

$$\begin{aligned}
 G_1 &= -r_1 \partial_{B_1}^2 + q_1 (N_{B_1} + \nu_1 + 1) \partial_{B_1} + u_1 \\
 &\quad + \frac{2}{\kappa_1} (N_{B_1} + \nu_1 + 1) \left(B_3 \partial_{\alpha_{31}} - \left(B_2 - \frac{1}{2} \kappa_2 \right) \partial_{\alpha_{12}} \right) \\
 &\quad - \left(B_2 + \frac{3}{2} \kappa_2 \right) \partial_{B_1} \partial_{\alpha_{12}} - B_3 \partial_{B_1} \partial_{\alpha_{31}} + 2\alpha_{23} \partial_{\alpha_{12}} \partial_{\alpha_{31}}, \\
 G_2 &= -r_2 \partial_{B_2}^2 + q_2 (N_{B_2} + \nu_2 + 1) \partial_{B_2} + e_2 (N_{B_2} + \nu_2) (N_{B_2} + \nu_2 + 1) + u_2 \\
 &\quad + \frac{2}{\kappa_2} (N_{B_2} + \nu_2 + 1) \left(\left(B_1 + \frac{1}{2} \kappa_1 \right) \partial_{\alpha_{12}} - B_3 \partial_{\alpha_{23}} \right) \\
 &\quad - B_3 \partial_{B_2} \partial_{\alpha_{23}} - \left(B_1 - \frac{3}{2} \kappa_1 \right) \partial_{B_2} \partial_{\alpha_{12}} + 2\alpha_{31} \partial_{\alpha_{12}} \partial_{\alpha_{23}}, \\
 G_3 &= m_3 \partial_{\zeta_3},
 \end{aligned} \tag{C.73}$$

$$G_3 = m_3 \partial_{\zeta_3}, \tag{C.74}$$

where, here and below, we use quantities (A.16)–(A.19) taken for masses given in (C.68). From (C.74) and equation $G_3 V^{(4)} = 0$, we see that $V^{(4)}$ is independent of ζ_3 ,

$$V^{(4)} = V^{(4)}(B_1, B_2, \alpha_{aa+1}). \tag{C.75}$$

²¹For massive field corresponding to $a = 3$, operators G_1, G_2 (C.73) involve, besides α_{33} -term (C.70), ζ_3^2 -term, i.e., we get $(\alpha_{33} + \zeta_3^2)$ -term which can be ignored by virtue of the second constraint in (2.11).

Step 3. We make the transformation

$$V^{(4)} = U_{\partial B} U_{\partial \alpha} V^{(5)}, \quad U_{\partial B} = \exp \left(-\frac{r_1}{q_1} \partial_{B_1} + \frac{q_2}{2e_2} \partial_{B_2} \right), \quad (\text{C.76})$$

$$U_{\partial \alpha} = \exp \left(-\frac{\kappa_2}{q_1} \partial_{\alpha_{12}} + \frac{B_1}{\kappa_2 e_2} \partial_{\alpha_{12}} - \frac{2m_2^2}{\kappa_1 q_1^2} B_2 \partial_{\alpha_{12}} + \frac{2m_3^2}{\kappa_1 q_1^2} B_3 \partial_{\alpha_{31}} - \frac{B_3}{\kappa_2 e_2} \partial_{\alpha_{23}} \right. \\ \left. - \frac{2B_1 B_3}{\kappa_1 q_1} \partial_{\alpha_{31}} + \frac{2B_1 B_2}{\kappa_1 q_1} \partial_{\alpha_{12}} + \frac{2h_{23} B_2 B_3}{\kappa_1^2 q_1^2} \partial_{\alpha_{23}} \right). \quad (\text{C.77})$$

Realization of G_1, G_2 (C.73) on $V^{(5)}$ (C.76) takes the form

$$G_1 = q_1 (N_{B_1} + \nu_1 + 1) \partial_{B_1} + u_1 \\ + \left(\frac{4\kappa_2^2 e_2}{D} B_2^2 - \frac{1}{e_2} \right) \partial_{\alpha_{12}}^2 - \frac{4m_3^2}{D} B_3^2 \partial_{\alpha_{31}}^2 + 2\alpha_{23} \partial_{\alpha_{12}} \partial_{\alpha_{31}}, \\ G_2 = -\frac{D}{4\kappa_2^2 e_2} \partial_{B_2}^2 + e_2 (N_{B_2} + \nu_2) (N_{B_2} + \nu_2 + 1) + u_2 \\ - \frac{4m_3^2}{D} B_3^2 \partial_{\alpha_{23}}^2 + \frac{4B_1}{q_1} \partial_{\alpha_{12}}^2 + 2\alpha_{31} \partial_{\alpha_{12}} \partial_{\alpha_{23}}. \quad (\text{C.78})$$

Step 4. We make the transformations

$$V^{(5)} = \left(-\frac{4}{q_1} B_1 \right)^{-\nu_1/2} U_{B_2} V^{(6)}, \quad V^{(6)} = U_{\nu_1, W_1} U_{\nu_2, W_{23}} V^{(7)}, \quad V^{(7)} = U_{B_2}^{-1} V^{(8)}, \quad (\text{C.79})$$

where $U_{B_2}, U_{\nu, W}$ are defined in (A.13), (A.14) and we use the notation

$$W_1 = 2\alpha_{23} \partial_{\alpha_{12}} \partial_{\alpha_{31}} - \partial_{\alpha_{12}}^2 - \frac{4m_3^2}{D} B_3^2 \partial_{\alpha_{31}}^2, \\ W_2 = 2\alpha_{31} \partial_{\alpha_{12}} \partial_{\alpha_{23}} - \partial_{\alpha_{12}}^2 - \frac{4m_3^2}{D} B_3^2 \partial_{\alpha_{23}}^2, \quad W_{23} = 2\alpha_{31} \partial_{\alpha_{12}} \partial_{\alpha_{23}} - \frac{4m_3^2}{D} B_3^2 \partial_{\alpha_{23}}^2. \quad (\text{C.80})$$

Realizations of G_1, G_2 (C.78) on $V^{(6)}, V^{(7)}, V^{(8)}$ (C.79) take the forms

$$G_1 = u_1 + q_1 (N_{B_1} + \nu_1 + 1) \partial_{B_1} - \frac{q_1}{4B_1} (\nu_1^2 + W_1), \\ G_2 = \left(B_2^2 - \frac{D}{4\kappa_2^2 e_2^2} \right) \partial_{B_2}^2 + \frac{u_2}{e_2} + \left(1 - \frac{4\kappa_2^2 e_2^2}{D} B_2^2 \right)^{-1} (\nu_2^2 - 1 + W_2), \quad \text{for } V^{(6)}; \quad (\text{C.81})$$

$$G_1 = u_1 + q_1 (N_{B_1} + \nu_1 + 1) \partial_{B_1} - \frac{q_1}{4B_1} \nu_1^2, \\ G_2 = \left(B_2^2 - \frac{D}{4\kappa_2^2 e_2^2} \right) \partial_{B_2}^2 + \left(1 - \frac{4\kappa_2^2 e_2^2}{D} B_2^2 \right)^{-1} (\nu_2^2 - 1) + \frac{u_2}{e_2}, \quad \text{for } V^{(7)}; \quad (\text{C.82})$$

$$G_1 = u_1 + q_1 (N_{B_1} + \nu_1 + 1) \partial_{B_1} - \frac{q_1}{4B_1} \nu_1^2, \\ G_2 = -\frac{D}{4\kappa_2^2 e_2} \partial_{B_2}^2 + e_2 (N_{B_2} + \nu_2) (N_{B_2} + \nu_2 + 1) + u_2, \quad \text{for } V^{(8)}. \quad (\text{C.83})$$

For ν_a (A.8) and W -operators (C.80), we note the relations which admit us to get G_1, G_2 in (C.82),

$$\begin{aligned} (\nu_1^2 + W_1)U_{\nu_1, W_1} &= U_{\nu_1, W_1}\nu_1^2, & \nu_1^2 U_{\nu_2, W_{23}} &= U_{\nu_2, W_{23}}\nu_1^2, \\ (\nu_2^2 + W_2)U_{\nu_1, W_1} &= U_{\nu_1, W_1}(\nu_2^2 + W_{23}), & (\nu_2^2 + W_{23})U_{\nu_2, W_{23}} &= U_{\nu_2, W_{23}}\nu_2^2. \end{aligned} \quad (\text{C.84})$$

Equations $G_1 V^{(8)} = 0$ and $G_2 V^{(8)} = 0$ with G_1 and G_2 as in (C.83) constitute a system of two decoupled second-order differential equations w.r.t. B_1 and B_2 . Four independent solutions of these equations are given in (5.74).

$$\text{Case} \quad m_1^2 < 0, \quad m_2^2 < 0, \quad m_3^2 > 0. \quad (\text{C.85})$$

Step 1. This case corresponds to the second vertex $p_{[3]}^-$ (3.28). For this case, vertex $V^{(3)}$ is given in (C.13), where the realization of operators G_a (C.4)–(C.6) on the vertex $V^{(3)}$ (C.13) is given in (C.8)–(C.10). We make the transformation

$$\begin{aligned} V^{(3)} &= U_\zeta V^{(4)}, \\ U_\zeta &= \exp \left(\frac{\zeta_3}{m_3} \left(B_1 - \frac{1}{2} \kappa_1 \right) \partial_{\alpha_{31}} - \frac{\zeta_3}{m_3} \left(B_2 + \frac{1}{2} \kappa_2 \right) \partial_{\alpha_{23}} - \frac{\zeta_3}{2m_3} (m_1^2 - m_2^2) \partial_{B_3} \right). \end{aligned} \quad (\text{C.86})$$

Realization of G_a (C.8)–(C.10) on $V^{(4)}$ (C.86) takes the form²²

$$\begin{aligned} G_1 &= -r_1 \partial_{B_1}^2 + q_1 (N_{B_1} + \nu_1 + 1) \partial_{B_1} + e_1 (N_{B_1} + \nu_1) (N_{B_1} + \nu_1 + 1) + u_1 \\ &\quad + \frac{2}{\kappa_1} (N_{B_1} + \nu_1 + 1) \left(B_3 \partial_{\alpha_{31}} - \left(B_2 - \frac{1}{2} \kappa_2 \right) \partial_{\alpha_{12}} \right) \\ &\quad - \left(B_2 + \frac{3}{2} \kappa_2 \right) \partial_{B_1} \partial_{\alpha_{12}} - B_3 \partial_{B_1} \partial_{\alpha_{31}} + 2\alpha_{23} \partial_{\alpha_{12}} \partial_{\alpha_{31}}, \\ G_2 &= -r_2 \partial_{B_2}^2 + q_2 (N_{B_2} + \nu_2 + 1) \partial_{B_2} + e_2 (N_{B_2} + \nu_2) (N_{B_2} + \nu_2 + 1) + u_2 \\ &\quad + \frac{2}{\kappa_2} (N_{B_2} + \nu_2 + 1) \left(\left(B_1 + \frac{1}{2} \kappa_1 \right) \partial_{\alpha_{12}} - B_3 \partial_{\alpha_{23}} \right) \\ &\quad - B_3 \partial_{B_2} \partial_{\alpha_{23}} - \left(B_1 - \frac{3}{2} \kappa_1 \right) \partial_{B_2} \partial_{\alpha_{12}} + 2\alpha_{31} \partial_{\alpha_{12}} \partial_{\alpha_{23}}, \end{aligned} \quad (\text{C.87})$$

$$G_3 = m_3 \partial_{\zeta_3}, \quad (\text{C.88})$$

where we use the quantities (A.16)–(A.19) taken for masses given in (C.85). From (C.88) and equation $G_3 V^{(4)} = 0$, we see that $V^{(4)}$ is independent of ζ_3 ,

$$V^{(4)} = V^{(4)}(B_a, \alpha_{aa+1}). \quad (\text{C.89})$$

²²For massive field corresponding to $a = 3$, operators G_1, G_2 (C.87) involve, besides α_{33} -term (B.4), ζ_3^2 -term, i.e., we get $(\alpha_{33} + \zeta_3^2)$ -term which can be ignored by virtue of the second constraint in (2.11).

Step 2. We make the transformation

$$V^{(4)} = U_{\partial B} U_{\partial \alpha} V^{(5)},$$

$$U_{\partial B} = \exp \left(\frac{q_1}{2e_1} \partial_{B_1} + \frac{q_2}{2e_2} \partial_{B_2} \right), \quad (C.90)$$

$$U_{\partial \alpha} = \exp \left(\frac{B_1}{\kappa_2 e_2} \partial_{\alpha_{12}} - \frac{B_2}{\kappa_1 e_1} \partial_{\alpha_{12}} + \frac{B_3}{\kappa_1 e_1} \partial_{\alpha_{31}} - \frac{B_3}{\kappa_2 e_2} \partial_{\alpha_{23}} - \frac{h_{12}}{2\kappa_1 \kappa_2 e_1 e_2} \partial_{\alpha_{12}} \right. \\ \left. + \frac{2}{D} h_{12} B_1 B_2 \partial_{\alpha_{12}} + \frac{2}{D} h_{23} B_2 B_3 \partial_{\alpha_{23}} + \frac{2}{D} h_{31} B_3 B_1 \partial_{\alpha_{31}} \right). \quad (C.91)$$

Realization of G_1, G_2 (C.87) on $V^{(5)}$ (C.90) takes the form

$$G_1 = -\frac{D}{4\kappa_1^2 e_1} \partial_{B_1}^2 + e_1 (N_{B_1} + \nu_1) (N_{B_1} + \nu_1 + 1) + u_1 \\ + \left(\frac{4\kappa_2^2 e_2}{D} B_2^2 - \frac{1}{e_2} \right) \partial_{\alpha_{12}}^2 - \frac{4m_3^2}{D} B_3^2 \partial_{\alpha_{31}}^2 + 2\alpha_{23} \partial_{\alpha_{12}} \partial_{\alpha_{31}}, \\ G_2 = -\frac{D}{4\kappa_2^2 e_2} \partial_{B_2}^2 + e_2 (N_{B_2} + \nu_2) (N_{B_2} + \nu_2 + 1) + u_2 \\ + \left(\frac{4\kappa_1^2 e_1}{D} B_1^2 - \frac{1}{e_1} \right) \partial_{\alpha_{12}}^2 - \frac{4m_3^2}{D} B_3^2 \partial_{\alpha_{23}}^2 + 2\alpha_{31} \partial_{\alpha_{12}} \partial_{\alpha_{23}}. \quad (C.92)$$

Step 3. We make the transformations

$$V^{(5)} = U_{B_1} U_{B_2} V^{(6)}, \quad V^{(6)} = U_{\nu_1, W_1} U_{\nu_2, W_{23}} V^{(7)}, \quad V^{(7)} = U_{B_1}^{-1} U_{B_2}^{-1} V^{(8)}, \quad (C.93)$$

where $U_{B_a}, U_{\nu, W}$ are given in (A.13), (A.14) and we use the notation

$$W_1 = 2\alpha_{23} \partial_{\alpha_{12}} \partial_{\alpha_{31}} - \partial_{\alpha_{12}}^2 - \frac{4m_3^2}{D} B_3^2 \partial_{\alpha_{31}}^2, \\ W_2 = 2\alpha_{31} \partial_{\alpha_{12}} \partial_{\alpha_{23}} - \partial_{\alpha_{12}}^2 - \frac{4m_3^2}{D} B_3^2 \partial_{\alpha_{23}}^2, \quad W_{23} = 2\alpha_{31} \partial_{\alpha_{12}} \partial_{\alpha_{23}} - \frac{4m_3^2}{D} B_3^2 \partial_{\alpha_{23}}^2. \quad (C.94)$$

Realizations of G_1, G_2 (C.92) on $V^{(6)}, V^{(7)}, V^{(8)}$ (C.93) take the forms

$$G_a = \left(B_a^2 - \frac{D}{4\kappa_a^2 e_a^2} \right) \partial_{B_a}^2 + \left(1 - \frac{4\kappa_a^2 e_a^2}{D} B_a^2 \right)^{-1} (\nu_a^2 - 1 + W_a) + \frac{u_a}{e_a}, \quad \text{for } V^{(6)}, \quad (C.95)$$

$$G_a = \left(B_a^2 - \frac{D}{4\kappa_a^2 e_a^2} \right) \partial_{B_a}^2 + \left(1 - \frac{4\kappa_a^2 e_a^2}{D} B_a^2 \right)^{-1} (\nu_a^2 - 1) + \frac{u_a}{e_a}, \quad \text{for } V^{(7)}, \quad (C.96)$$

$$G_a = -\frac{D}{4\kappa_a^2 e_a} \partial_{B_a}^2 + e_a (N_{B_a} + \nu_a) (N_{B_a} + \nu_a + 1) + u_a, \quad \text{for } V^{(8)}, \quad (C.97)$$

where, in (C.95)–(C.97), $a = 1, 2$. For ν_a (A.8) and W -operators (C.94), we note the relations which admit us to get G_a in (C.96),

$$(\nu_1^2 + W_1) U_{\nu_1, W_1} = U_{\nu_1, W_1} \nu_1^2, \quad \nu_1^2 U_{\nu_2, W_{23}} = U_{\nu_2, W_{23}} \nu_1^2, \\ (\nu_2^2 + W_2) U_{\nu_1, W_1} = U_{\nu_1, W_1} (\nu_2^2 + W_{23}), \quad (\nu_2^2 + W_{23}) U_{\nu_2, W_{23}} = U_{\nu_2, W_{23}} \nu_2^2. \quad (C.98)$$

Equations $G_1 V^{(8)} = 0$ and $G_2 V^{(8)} = 0$ with G_1 and G_2 as in (C.97) constitute a system of two decoupled second-order differential equations w.r.t. B_1 and B_2 . Four independent solutions of these equations are given in (5.97).

D Derivation of vertices $p_{[3]}^-$ (6.2), (6.22), (6.47), (6.68)

We consider four vertices in (3.29). For these vertices, we outline the derivation of the respective solutions in (6.2), (6.22), (6.47), (6.68). We split our derivation in several steps.

Realization of G_a , G_β on $p_{[3]}^-$ (3.29) for arbitrary masses. First, for $p_{[3]}^-$ (3.29) with arbitrary m_1, m_2, m_3 , we find G_β and $G_{a,\mathbb{P}^2}$ in (3.34). We use M_a^i , $a = 1, 2, 3$ for continuous-spin field (2.20). Plugging such M_a^i into $\mathbf{J}^{-i\dagger}$ (3.17), we cast $\mathbf{J}^{-i\dagger}|p_{[3]}^- \rangle$ into the form given in (3.34) with the following G_β and $G_{a,\mathbb{P}^2}$:

$$G_{a,\mathbb{P}^2} = G_a + \mathbf{P}^- \frac{2\beta}{\beta_a^3} \alpha_a^i \frac{g_{v_a} \partial_{v_a}}{2N_a + d - 2} \partial_{B_a}^2, \quad a = 1, 2, 3 \quad (\text{D.1})$$

$$\begin{aligned} G_a = & \left(B_{a+2} - \frac{\beta_a}{\beta_{a+2}} g_{v_{a+2}} \partial_{v_{a+2}} \right) \partial_{\alpha_{a+2a}} - \left(B_{a+1} + \frac{\beta_a}{\beta_{a+1}} g_{v_{a+1}} \partial_{v_{a+1}} \right) \partial_{\alpha_{aa+1}} \\ & + \frac{1}{2} \left(\frac{\check{\beta}_a}{\beta_a} m_a^2 + m_{a+1}^2 - m_{a+2}^2 \right) \partial_{B_a} + v_a g_{v_a} \\ & + \frac{g_{v_a} \partial_{v_a}}{2N_a + d - 2} \left(\frac{2\beta_{a+1}}{\beta_a} B_{a+1} \partial_{B_a} \partial_{\alpha_{aa+1}} + \frac{2\beta_{a+2}}{\beta_a} B_{a+2} \partial_{B_a} \partial_{\alpha_{a+2a}} \right. \\ & \left. + 2\alpha_{a+1a+2} \partial_{\alpha_{aa+1}} \partial_{\alpha_{a+2a}} + \frac{\beta}{\beta_a^2} \sum_{b=1,2,3} \frac{m_b^2}{\beta_b} \partial_{B_a}^2 \right), \end{aligned} \quad (\text{D.2})$$

$$G_\beta = -\frac{1}{\beta} \mathbb{N}_\beta - \sum_{a=1,2,3} \frac{1}{\beta_a^2} g_{v_a} \partial_{v_a} \partial_{B_a}, \quad (\text{D.3})$$

where g_{v_a} are given in (A.9). Using G_a, G_β (D.2), (D.3), we now consider equations (3.35), (3.36).

Vertex $V^{(3)}$. We multiply G_a (D.2) on the left by $(2N_a + d - 2)/\kappa_a$ and make the transformation²³

$$p_{[3]}^- = U_{v_1} U_{v_2} U_{v_3} V^{(1)}, \quad V^{(1)} = U_{\Gamma_1} U_{\Gamma_2} U_{\Gamma_3} V^{(2)}, \quad V^{(2)} = U_\beta V^{(3)}, \quad (\text{D.4})$$

$$U_\beta = \exp \left(-\frac{\check{\beta}_1}{2\beta_1} \kappa_1 \partial_{B_1} - \frac{\check{\beta}_2}{2\beta_2} \kappa_2 \partial_{B_2} - \frac{\check{\beta}_3}{2\beta_3} \kappa_3 \partial_{B_3} \right), \quad (\text{D.5})$$

where U_{v_a}, U_{Γ_a} are defined in (A.10), (A.12). Using notation in (A.16), (A.17), we find that realization of G_a, G_β (D.2), (D.3) on $V^{(3)}$ (D.4) takes the form

$$\begin{aligned} G_a = & -r_a \partial_{B_a}^2 + q_a (N_{B_a} + \nu_a + 1) \partial_{B_a} + e_a (N_{B_a} + \nu_a) (N_{B_a} + \nu_a + 1) + u_a \\ & + \frac{2}{\kappa_a} (N_a + \nu_a + 1) \left(\left(B_{a+2} + \frac{1}{2} \kappa_{a+2} \right) \partial_{\alpha_{a+2a}} - \left(B_{a+1} - \frac{1}{2} \kappa_{a+1} \right) \partial_{\alpha_{aa+1}} \right) \\ & - \left(B_{a+1} + \frac{3}{2} \kappa_{a+1} \right) \partial_{B_a} \partial_{\alpha_{aa+1}} - \left(B_{a+2} - \frac{3}{2} \kappa_{a+2} \right) \partial_{B_a} \partial_{\alpha_{a+2a}} + 2\alpha_{a+1a+2} \partial_{\alpha_{aa+1}} \partial_{\alpha_{a+2a}}, \end{aligned} \quad (\text{D.6})$$

$$G_\beta = -\frac{1}{\beta} \mathbb{N}_\beta. \quad (\text{D.7})$$

²³To investigate equations $G_a p_{[3]}^- = 0$ it is convenient to use equivalence class for the G_a . Namely, the G_a and $(2N_a + d - 2)G_a$ are considered to be equivalent.

Dependence of $V^{(3)}$ on β_a . We fix dependence of $V^{(3)}$ on β_a . To this end we use eqs. (3.36), (3.40) and relation (D.7) to note the following equations for the vertex $V^{(3)}$:

$$\mathbb{N}_\beta V^{(3)} = 0, \quad \sum_{a=1,2,3} \beta_a \partial_{\beta_a} V^{(3)} = 0. \quad (\text{D.8})$$

Equations (D.8) imply that $V^{(3)}$ is independent of $\beta_1, \beta_2, \beta_3$. Thus, the vertex $V^{(3)}$ is given by

$$V^{(3)} = V^{(3)}(B_a, \alpha_{aa+1}). \quad (\text{D.9})$$

We now use results above-obtained for the study of four vertices in (3.29) in turn.

$$\text{Case} \quad m_1 = 0, \quad m_2 = 0, \quad m_3^2 < 0. \quad (\text{D.10})$$

Step 1. This case corresponds to the first vertex $p_{[3]}^-$ (3.29). In (D.4), we use operators $U_{\Gamma_1}, U_{\Gamma_2}$ given in (A.11). Then, in all remaining relations in (D.1)–(D.9), we set $m_1 = 0, m_2 = 0$ and use vertex $V^{(3)}$ given in (D.9), where the realization of operators G_a (D.6) on the vertex $V^{(3)}$ (D.9) takes the form

$$\begin{aligned} G_a = & -r_a \partial_{B_a}^2 + q_a (N_{B_a} + \nu_a + 1) \partial_{B_a} + u_a \\ & + \frac{2}{\kappa_a} (N_{B_a} + \nu_a + 1) \left(\left(B_{a+2} + \frac{1}{2} \kappa_{a+2} \right) \partial_{\alpha_{a+2a}} - \left(B_{a+1} - \frac{1}{2} \kappa_{a+1} \right) \partial_{\alpha_{aa+1}} \right) \\ & - \left(B_{a+1} + \frac{3}{2} \kappa_{a+1} \right) \partial_{B_a} \partial_{\alpha_{aa+1}} - \left(B_{a+2} - \frac{3}{2} \kappa_{a+2} \right) \partial_{B_a} \partial_{\alpha_{a+2a}} + 2\alpha_{a+1a+2} \partial_{\alpha_{aa+1}} \partial_{\alpha_{a+2a}}, \end{aligned} \quad (\text{D.11})$$

$$\begin{aligned} G_3 = & -r_3 \partial_{B_3}^2 + e_3 (N_{B_3} + \nu_3) (N_{B_3} + \nu_3 + 1) + u_3 \\ & + \frac{2}{\kappa_3} (N_{B_3} + \nu_3 + 1) \left(\left(B_2 + \frac{1}{2} \kappa_2 \right) \partial_{\alpha_{23}} - \left(B_1 - \frac{1}{2} \kappa_1 \right) \partial_{\alpha_{31}} \right) \\ & - \left(B_1 + \frac{3}{2} \kappa_1 \right) \partial_{B_3} \partial_{\alpha_{31}} - \left(B_2 - \frac{3}{2} \kappa_2 \right) \partial_{B_3} \partial_{\alpha_{23}} + 2\alpha_{12} \partial_{\alpha_{31}} \partial_{\alpha_{23}}, \end{aligned} \quad (\text{D.12})$$

where $a = 1, 2$, and we use quantities (A.16), (A.17) taken for masses given in (D.10).

Step 2. We make the transformation

$$V^{(3)} = U_{\partial B} U_{\partial \alpha} V^{(4)}, \quad U_{\partial B} = \exp \left(-\frac{r_1}{q_1} \partial_{B_1} - \frac{r_2}{q_2} \partial_{B_2} \right), \quad (\text{D.13})$$

$$\begin{aligned} U_{\partial \alpha} = & \exp \left((2B_1 B_3 + \kappa_3 B_1 - \kappa_1 \kappa_3) \frac{\partial_{\alpha_{31}}}{m_3^2} + (2B_2 B_3 - \kappa_3 B_2 - \kappa_2 \kappa_3) \frac{\partial_{\alpha_{23}}}{m_3^2} \right. \\ & \left. - \frac{2B_1 B_2}{m_3^2} \partial_{\alpha_{12}} + \frac{2B_3}{m_3^2} (\kappa_1 \partial_{\alpha_{31}} - \kappa_2 \partial_{\alpha_{23}}) + \frac{2\kappa_1 \kappa_2}{m_3^2} \partial_{\alpha_{12}} \right). \end{aligned} \quad (\text{D.14})$$

Realization of G_a , G_3 (D.11), (D.12) on $V^{(4)}$ (D.13) takes the form

$$\begin{aligned}
 G_1 &= q_1(N_{B_1} + \nu_1 + 1)\partial_{B_1} + u_1 \\
 &\quad + \left(\frac{4\kappa_3^2 e_3}{D}B_3^2 - \frac{1}{e_3}\right)\partial_{\alpha_{31}}^2 + \frac{4}{q_2}B_2\partial_{\alpha_{12}}^2 + 2\alpha_{23}\partial_{\alpha_{12}}\partial_{\alpha_{31}}, \\
 G_2 &= q_2(N_{B_2} + \nu_2 + 1)\partial_{B_2} + u_2 \\
 &\quad + \left(\frac{4\kappa_3^2 e_3}{D}B_3^2 - \frac{1}{e_3}\right)\partial_{\alpha_{23}}^2 + \frac{4}{q_1}B_1\partial_{\alpha_{12}}^2 + 2\alpha_{31}\partial_{\alpha_{23}}\partial_{\alpha_{12}}, \\
 G_3 &= -\frac{D}{4\kappa_3^2 e_3}\partial_{B_3}^2 + e_3(N_{B_3} + \nu_3)(N_{B_3} + \nu_3 + 1) + u_3 \\
 &\quad + \frac{4}{q_1}B_1\partial_{\alpha_{31}}^2 + \frac{4}{q_2}B_2\partial_{\alpha_{23}}^2 + 2\alpha_{12}\partial_{\alpha_{31}}\partial_{\alpha_{23}}. \tag{D.15}
 \end{aligned}$$

Step 3. We make the transformation

$$\begin{aligned}
 V^{(4)} &= \left(-\frac{4}{q_1}B_1\right)^{-\nu_1/2} \left(-\frac{4}{q_2}B_2\right)^{-\nu_2/2} U_{B_3} V^{(5)}, \quad V^{(5)} = U_{\nu_1, W_1} U_{\nu_2, W_{23}} U_{\nu_3, W_{312}} V^{(6)}, \\
 V^{(6)} &= U_{B_3}^{-1} V^{(7)}, \tag{D.16}
 \end{aligned}$$

where U_{B_3} , $U_{\nu, W}$ are given in (A.13), (A.14) and we use the notation

$$\begin{aligned}
 W_a &= 2\alpha_{a+1a+2}\partial_{\alpha_{aa+1}}\partial_{\alpha_{a+2a}} - \partial_{\alpha_{aa+1}}^2 - \partial_{\alpha_{a+2a}}^2, & a = 1, 2, 3, \\
 W_{23} &= 2\alpha_{31}\partial_{\alpha_{12}}\partial_{\alpha_{23}} - \partial_{\alpha_{23}}^2, & W_{32} = 2\alpha_{12}\partial_{\alpha_{23}}\partial_{\alpha_{31}} - \partial_{\alpha_{23}}^2, & W_{312} = 2\alpha_{12}\partial_{\alpha_{23}}\partial_{\alpha_{31}}. \tag{D.17}
 \end{aligned}$$

Realizations of G_1 , G_2 , G_3 (D.15) on $V^{(5)}$, $V^{(6)}$, $V^{(7)}$ (D.16) take the following forms

$$\begin{aligned}
 G_a &= u_a + q_a(N_{B_a} + 1)\partial_{B_a} - \frac{q_a}{4B_a}(\nu_a^2 + W_a), & a = 1, 2, \\
 G_3 &= \left(B_3^2 - \frac{D}{4\kappa_3^2 e_3^2}\right)\partial_{B_3}^2 + \left(1 - \frac{4\kappa_3^2 e_3^2}{D}B_3^2\right)^{-1}(\nu_3^2 - 1 + W_3) + \frac{u_3}{e_3}, & \text{for } V^{(5)}; \tag{D.18}
 \end{aligned}$$

$$\begin{aligned}
 G_a &= u_a + q_a(N_{B_a} + 1)\partial_{B_a} - \frac{q_a}{4B_a}\nu_a^2, & a = 1, 2, \\
 G_3 &= \left(B_3^2 - \frac{D}{4\kappa_3^2 e_3^2}\right)\partial_{B_3}^2 + \left(1 - \frac{4\kappa_3^2 e_3^2}{D}B_3^2\right)^{-1}(\nu_3^2 - 1) + \frac{u_3}{e_3}, & \text{for } V^{(6)}; \tag{D.19}
 \end{aligned}$$

$$\begin{aligned}
 G_a &= u_a + q_a(N_{B_a} + 1)\partial_{B_a} - \frac{q_a}{4B_a}\nu_a^2, & a = 1, 2, \\
 G_3 &= -\frac{D}{4\kappa_3^2 e_3}\partial_{B_3}^2 + e_3(N_{B_3} + \nu_3)(N_{B_3} + \nu_3 + 1) + u_3, & \text{for } V^{(7)}. \tag{D.20}
 \end{aligned}$$

For ν_a (A.8) and W -operators (D.17), we note the relations which admit us to get $G_{1,2,3}$ in (D.19),

$$\begin{aligned} (\nu_1^2 + W_1)U_{\nu_1, W_1} &= U_{\nu_1, W_1}\nu_1^2, & \nu_1^2 U_{\nu_2, W_{23}} &= U_{\nu_2, W_{23}}\nu_1^2, \\ \nu_1^2 U_{\nu_3, W_{312}} &= U_{\nu_3, W_{312}}\nu_1^2, \end{aligned} \quad (D.21)$$

$$\begin{aligned} (\nu_2^2 + W_2)U_{\nu_1, W_1} &= U_{\nu_1, W_1}(\nu_2^2 + W_{23}), & (\nu_2^2 + W_{23})U_{\nu_2, W_{23}} &= U_{\nu_2, W_{23}}\nu_2^2, \\ \nu_2^2 U_{\nu_3, W_{312}} &= U_{\nu_3, W_{312}}\nu_2^2, \end{aligned} \quad (D.22)$$

$$\begin{aligned} (\nu_3^2 + W_3)U_{\nu_1, W_1} &= U_{\nu_1, W_1}(\nu_3^2 + W_{32}), & (\nu_3^2 + W_{32})U_{\nu_2, W_{23}} &= U_{\nu_2, W_{23}}(\nu_3^2 + W_{312}), \\ (\nu_3^2 + W_{312})U_{\nu_3, W_{312}} &= U_{\nu_3, W_{312}}\nu_3^2. \end{aligned} \quad (D.23)$$

Equations $G_a V^{(7)} = 0$ with G_1, G_2, G_3 as in (D.20) constitute a system of three decoupled second-order differential equations w.r.t. B_1, B_2, B_3 . Eight independent solutions of these equations are given in (6.5).

$$\text{Case} \quad m_1 = m, \quad m_2 = m, \quad m_3 = 0, \quad m^2 < 0. \quad (D.24)$$

Step 1. This case corresponds to the second vertex $p_{[3]}^-$ (3.29). Now, in (D.4), we use operator U_{Γ_3} given in (A.11). Then, in all remaining relations in (D.1)–(D.9) we set $m_1 = m, m_2 = m, m_3 = 0$ and use the vertex $V^{(3)}$ given in (D.9), where the realization of operators G_a (D.6) on the vertex $V^{(3)}$ (D.9) takes the form

$$\begin{aligned} G_a &= -r_a \partial_{B_a}^2 + q_a (N_{B_a} + \nu_a + 1) \partial_{B_a} + e_a (N_{B_a} + \nu_a) (N_{B_a} + \nu_a + 1) + u_a \\ &\quad + \frac{2}{\kappa_a} (N_{B_a} + \nu_a + 1) \left(\left(B_{a+2} + \frac{1}{2} \kappa_{a+2} \right) \partial_{\alpha_{a+2a}} - \left(B_{a+1} + \frac{1}{2} \kappa_{a+1} \right) \partial_{\alpha_{aa+1}} \right) \\ &\quad - \left(B_{a+1} + \frac{3}{2} \kappa_{a+1} \right) \partial_{B_a} \partial_{\alpha_{aa+1}} - \left(B_{a+2} - \frac{3}{2} \kappa_{a+2} \right) \partial_{B_a} \partial_{\alpha_{a+2a}} + 2\alpha_{a+1a+2} \partial_{\alpha_{aa+1}} \partial_{\alpha_{a+2a}}, \end{aligned} \quad (D.25)$$

$$\begin{aligned} G_3 &= -r_3 \partial_{B_3}^2 + u_3 + \frac{2}{\kappa_3} (N_{B_3} + \nu_3 + 1) \left(\left(B_2 + \frac{1}{2} \kappa_2 \right) \partial_{\alpha_{23}} - \left(B_1 + \frac{1}{2} \kappa_1 \right) \partial_{\alpha_{31}} \right) \\ &\quad - \left(B_1 + \frac{3}{2} \kappa_1 \right) \partial_{B_3} \partial_{\alpha_{31}} - \left(B_2 - \frac{3}{2} \kappa_2 \right) \partial_{B_3} \partial_{\alpha_{23}} + 2\alpha_{12} \partial_{\alpha_{31}} \partial_{\alpha_{23}}, \end{aligned} \quad (D.26)$$

where $a = 1, 2$, and we use quantities (A.16), (A.17) taken for masses given in (D.24).

Step 2. We make the transformation

$$V^{(3)} = U_{\partial B} U_{\partial \alpha} V^{(4)}, \quad U_{\partial B} = \exp \left(\frac{q_1}{2e_1} \partial_{B_1} + \frac{q_2}{2e_2} \partial_{B_2} \right), \quad (D.27)$$

$$\begin{aligned} U_{\partial \alpha} &= \exp \left(\frac{B_3^2}{2\kappa_3 r_3} (B_2 \partial_{\alpha_{23}} - B_1 \partial_{\alpha_{31}}) - \frac{B_3}{2r_3} (B_1 \partial_{\alpha_{31}} + B_2 \partial_{\alpha_{23}}) \right. \\ &\quad + \frac{B_3}{\kappa_1 e_1} \partial_{\alpha_{31}} - \frac{B_3}{\kappa_2 e_2} \partial_{\alpha_{23}} - \frac{B_2}{\kappa_1 e_1} \partial_{\alpha_{12}} + \frac{B_1}{\kappa_2 e_2} \partial_{\alpha_{12}} - \frac{\kappa_1 \kappa_2}{m^2} \partial_{\alpha_{12}} - \frac{\kappa_2 \kappa_3}{m^2} \partial_{\alpha_{23}} - \frac{\kappa_3 \kappa_1}{m^2} \partial_{\alpha_{31}} \\ &\quad \left. - \frac{1}{2\kappa_1^2 e_1} B_1 B_2 \partial_{\alpha_{12}} + \frac{\kappa_3}{8r_3} (B_2 \partial_{\alpha_{23}} - B_1 \partial_{\alpha_{31}}) \right). \end{aligned} \quad (D.28)$$

Realization of G_a (D.25), (D.26) on $V^{(4)}$ (D.27) takes the form

$$\begin{aligned}
 G_1 &= e_1(N_{B_1} + \nu_1)(N_{B_1} + \nu_1 + 1) + u_1 \\
 &\quad - 2B_2\partial_{B_1}\partial_{\alpha_{12}} + 2\kappa_3\partial_{B_1}\partial_{\alpha_{31}} - \frac{1}{e_2}\partial_{\alpha_{12}}^2 + 2\alpha_{23}\partial_{\alpha_{12}}\partial_{\alpha_{31}}, \\
 G_2 &= e_2(N_{B_2} + \nu_2)(N_{B_2} + \nu_2 + 1) + u_2 \\
 &\quad - 2B_1\partial_{B_2}\partial_{\alpha_{12}} - 2\kappa_3\partial_{B_2}\partial_{\alpha_{23}} - \frac{1}{e_1}\partial_{\alpha_{12}}^2 + 2\alpha_{31}\partial_{\alpha_{12}}\partial_{\alpha_{23}}, \\
 G_3 &= -r_3\partial_{B_3}^2 + u_3 + \frac{1}{\kappa_3}\{\nu_3, B_2\partial_{\alpha_{23}} - B_1\partial_{\alpha_{31}}\} - \frac{1}{e_1}\partial_{\alpha_{31}}^2 - \frac{1}{e_2}\partial_{\alpha_{23}}^2 + 2\alpha_{12}\partial_{\alpha_{31}}\partial_{\alpha_{23}}. \quad (D.29)
 \end{aligned}$$

Step 3. We make the transformation

$$V^{(4)} = e_1^{\omega_1/2} e_2^{\omega_2/2} V^{(5)}, \quad V^{(5)} = U_{\omega_1, W_1} U_{\omega_2, W_{23}} V^{(6)}, \quad (D.30)$$

where e_a and $U_{\nu, W}$ are given in (A.17), (A.14) and we use the notation

$$\omega_a = N_{B_a} + \nu_a + \frac{1}{2}, \quad a = 1, 2, \quad (D.31)$$

$$W_1 = -2B_2\partial_{B_1}\partial_{\alpha_{12}} + 2\kappa_3\partial_{B_1}\partial_{\alpha_{31}} - \partial_{\alpha_{12}}^2 + 2\alpha_{23}\partial_{\alpha_{12}}\partial_{\alpha_{31}}, \quad (D.32)$$

$$W_2 = -2B_1\partial_{B_2}\partial_{\alpha_{12}} - 2\kappa_3\partial_{B_2}\partial_{\alpha_{23}} - \partial_{\alpha_{12}}^2 + 2\alpha_{31}\partial_{\alpha_{12}}\partial_{\alpha_{23}}, \quad (D.33)$$

$$W_3 = \frac{1}{\kappa_3}\{\nu_3, B_2\partial_{\alpha_{23}} - B_1\partial_{\alpha_{31}}\} - \partial_{\alpha_{31}}^2 - \partial_{\alpha_{23}}^2 + 2\alpha_{12}\partial_{\alpha_{31}}\partial_{\alpha_{23}}, \quad (D.34)$$

$$W_{23} = -2B_1\partial_{B_2}\partial_{\alpha_{12}} - 2\kappa_3\partial_{B_2}\partial_{\alpha_{23}} + 2\alpha_{31}\partial_{\alpha_{12}}\partial_{\alpha_{23}}, \quad (D.35)$$

$$W_{321} = \frac{1}{\kappa_3}\{\nu_3, B_2\partial_{\alpha_{23}} - B_1\partial_{\alpha_{31}}\} + 2\alpha_{12}\partial_{\alpha_{31}}\partial_{\alpha_{23}}. \quad (D.36)$$

Realizations of G_1, G_2, G_3 (D.29) on $V^{(5)}, V^{(6)}$ (D.30) take the forms

$$\begin{aligned}
 G_a &= (N_{B_a} + \nu_a)(N_{B_a} + \nu_a + 1) + \frac{u_a}{e_a} + W_a, & a = 1, 2, \\
 G_3 &= u_3 - r_3\partial_{B_3}^2 + W_3, & \text{for } V^{(5)}; \quad (D.37)
 \end{aligned}$$

$$\begin{aligned}
 G_a &= (N_{B_a} + \nu_a)(N_{B_a} + \nu_a + 1) + \frac{u_a}{e_a}, & a = 1, 2, \\
 G_3 &= u_3 - r_3\partial_{B_3}^2 + W_{321}, & \text{for } V^{(6)}. \quad (D.38)
 \end{aligned}$$

For ω_a and W -operators in (D.31)–(D.36), we note the relations which admit us to get $G_{1,2,3}$ in (D.38),

$$\begin{aligned}
 (\omega_1^2 + W_1)U_{\omega_1, W_1} &= U_{\omega_1, W_1}\omega_1^2, & \omega_1^2 U_{\omega_2, W_{23}} &= U_{\omega_2, W_{23}}\omega_1^2, \\
 (\omega_2^2 + W_2)U_{\omega_1, W_1} &= U_{\omega_1, W_1}(\omega_2^2 + W_{23}), & (\omega_2^2 + W_{23})U_{\omega_2, W_{23}} &= U_{\omega_2, W_{23}}\omega_2^2, \\
 W_3 U_{\omega_1, W_1} &= U_{\omega_1, W_1} W_{32}, & W_{32} U_{\omega_2, W_{23}} &= U_{\omega_2, W_{23}} W_{321}. \quad (D.39)
 \end{aligned}$$

Equations $G_a V^{(6)} = 0$ with G_1, G_2, G_3 as in (D.38) are three decoupled second-order differential equations w.r.t. B_1, B_2, B_3 . All independent solutions of these equations are given in (6.25).

$$\text{Case} \quad m_1^2 < 0, \quad m_2^2 < 0, \quad m_3 = 0, \quad m_1 \neq m_2. \quad (D.40)$$

Step 1. This case corresponds to the third vertex $p_{[3]}^-$ (3.29). In (D.4), we use operator U_{Γ_3} given in (A.11). Then, in all remaining relations in (D.1)–(D.9), we set $m_3 = 0$ and use vertex $V^{(3)}$ given in (D.9), where the realization of operators G_a (D.6) on the vertex $V^{(3)}$ (D.9) takes the form

$$\begin{aligned}
 G_a &= -r_a \partial_{B_a}^2 + q_a (N_{B_a} + \nu_a + 1) \partial_{B_a} + e_a (N_{B_a} + \nu_a) (N_{B_a} + \nu_a + 1) + u_a \\
 &\quad + \frac{2}{\kappa_a} (N_{B_a} + \nu_a + 1) \left(\left(B_{a+2} + \frac{1}{2} \kappa_{a+2} \right) \partial_{\alpha_{a+2a}} - \left(B_{a+1} - \frac{1}{2} \kappa_{a+1} \right) \partial_{\alpha_{aa+1}} \right) \\
 &\quad - \left(B_{a+1} + \frac{3}{2} \kappa_{a+1} \right) \partial_{B_a} \partial_{\alpha_{aa+1}} - \left(B_{a+2} - \frac{3}{2} \kappa_{a+2} \right) \partial_{B_a} \partial_{\alpha_{a+2a}} + 2\alpha_{a+1a+2} \partial_{\alpha_{aa+1}} \partial_{\alpha_{a+2a}}, \\
 G_3 &= -r_3 \partial_{B_3}^2 + q_3 (N_{B_3} + \nu_3 + 1) \partial_{B_3} + u_3 \\
 &\quad + \frac{2}{\kappa_3} (N_{B_3} + \nu_3 + 1) \left(\left(B_2 + \frac{1}{2} \kappa_2 \right) \partial_{\alpha_{23}} - \left(B_1 - \frac{1}{2} \kappa_1 \right) \partial_{\alpha_{31}} \right) \\
 &\quad - \left(B_1 + \frac{3}{2} \kappa_1 \right) \partial_{B_3} \partial_{\alpha_{31}} - \left(B_2 - \frac{3}{2} \kappa_2 \right) \partial_{B_3} \partial_{\alpha_{23}} + 2\alpha_{12} \partial_{\alpha_{31}} \partial_{\alpha_{23}}, \tag{D.41}
 \end{aligned}$$

where, in (D.41), $a = 1, 2$. In (D.41) and below, we use quantities (A.16)–(A.19) taken for masses given in (D.40).

Step 2. We make the transformation

$$\begin{aligned}
 V^{(3)} &= U_{\partial B} U_{\partial \alpha} V^{(4)}, \quad U_{\partial B} = \exp \left(\frac{q_1}{2e_1} \partial_{B_1} + \frac{q_2}{2e_2} \partial_{B_2} - \frac{r_3}{q_3} \partial_{B_3} \right), \tag{D.42} \\
 U_{\partial \alpha} &= \exp \left(\frac{2B_1 B_3 \partial_{\alpha_{31}} - 2B_2 B_3 \partial_{\alpha_{23}}}{\kappa_3 q_3} + \frac{B_3 \partial_{\alpha_{31}} - B_2 \partial_{\alpha_{12}}}{\kappa_1 e_1} + \frac{B_1 \partial_{\alpha_{12}} - B_3 \partial_{\alpha_{23}}}{\kappa_2 e_2} \right. \\
 &\quad - \frac{h_{12}}{2\kappa_1 \kappa_2 e_1 e_2} \partial_{\alpha_{12}} - \frac{\kappa_1}{q_3} \partial_{\alpha_{31}} + \frac{\kappa_2}{q_3} \partial_{\alpha_{23}} \\
 &\quad \left. + \frac{2h_{12}}{\kappa_3^2 q_3^2} B_1 B_2 \partial_{\alpha_{12}} - \frac{2m_1^2}{\kappa_3 q_3^2} B_1 \partial_{\alpha_{31}} + \frac{2m_2^2}{\kappa_3 q_3^2} B_2 \partial_{\alpha_{23}} \right). \tag{D.43}
 \end{aligned}$$

Realization of G_1, G_2, G_3 (D.41) on $V^{(4)}$ (D.42) takes the form

$$\begin{aligned}
 G_1 &= -\frac{D}{4\kappa_1^2 e_1} \partial_{B_1}^2 + e_1 (N_{B_1} + \nu_1) (N_{B_1} + \nu_1 + 1) + u_1 \\
 &\quad + \left(\frac{4\kappa_2^2 e_2}{\kappa_3^2 b_3^2} B_2^2 - \frac{1}{e_2} \right) \partial_{\alpha_{12}}^2 + \frac{4}{q_3} B_3 \partial_{\alpha_{31}}^2 + 2\alpha_{23} \partial_{\alpha_{12}} \partial_{\alpha_{31}}, \\
 G_2 &= -\frac{D}{4\kappa_2^2 e_2} \partial_{B_2}^2 + e_2 (N_{B_2} + \nu_2) (N_{B_2} + \nu_2 + 1) + u_2 \\
 &\quad + \left(\frac{4\kappa_1^2 e_1}{\kappa_3^2 b_3^2} B_1^2 - \frac{1}{e_1} \right) \partial_{\alpha_{12}}^2 + \frac{4}{q_3} B_3 \partial_{\alpha_{23}}^2 + 2\alpha_{31} \partial_{\alpha_{12}} \partial_{\alpha_{23}}, \\
 G_3 &= q_3 (N_{B_3} + \nu_3 + 1) \partial_{B_3} + u_3 \\
 &\quad + \left(\frac{4\kappa_1^2 e_1}{\kappa_3^2 q_3^2} B_1^2 - \frac{1}{e_1} \right) \partial_{\alpha_{31}}^2 + \left(\frac{4\kappa_2^2 e_2}{\kappa_3^2 q_3^2} B_2^2 - \frac{1}{e_2} \right) \partial_{\alpha_{23}}^2 + 2\alpha_{12} \partial_{\alpha_{31}} \partial_{\alpha_{23}}. \tag{D.44}
 \end{aligned}$$

Step 3. We make the transformations

$$\begin{aligned} V^{(4)} &= \left(-\frac{4}{q_3} B_3\right)^{-\nu_3/2} U_{B_1} U_{B_2} V^{(5)}, & V^{(5)} &= U_{\nu_1, W_1} U_{\nu_2, W_{23}} U_{\nu_3, W_{312}} V^{(6)}, \\ V^{(6)} &= U_{B_1}^{-1} U_{B_2}^{-1} V^{(7)}, \end{aligned} \quad (\text{D.45})$$

where U_{B_a} , $U_{\nu, W}$ are given in (A.13), (A.14) and we use the notation

$$\begin{aligned} W_a &= 2\alpha_{a+1a+2} \partial_{\alpha_{aa+1}} \partial_{\alpha_{a+2a}} - \partial_{\alpha_{aa+1}}^2 - \partial_{\alpha_{a+2a}}^2, & a &= 1, 2, 3, \\ W_{23} &= 2\alpha_{31} \partial_{\alpha_{12}} \partial_{\alpha_{23}} - \partial_{\alpha_{23}}^2, & W_{32} &= 2\alpha_{12} \partial_{\alpha_{23}} \partial_{\alpha_{31}} - \partial_{\alpha_{23}}^2, & W_{312} &= 2\alpha_{12} \partial_{\alpha_{23}} \partial_{\alpha_{31}}. \end{aligned} \quad (\text{D.46})$$

Realizations of G_1 , G_2 , G_3 (D.44) on $V^{(5)}$, $V^{(6)}$, $V^{(7)}$ (D.45) take the forms

$$\begin{aligned} G_a &= \left(B_a^2 - \frac{D}{4\kappa_a^2 e_a^2}\right) \partial_{B_a}^2 + \left(1 - \frac{4\kappa_a^2 e_a^2}{D} B_a^2\right)^{-1} (\nu_a^2 - 1 + W_a) + \frac{u_a}{e_a}, \\ G_3 &= u_3 + q_3 (N_{B_3} + 1) \partial_{B_3} - \frac{q_3}{4B_3} (\nu_3^2 + W_3), \end{aligned} \quad \text{for } V^{(5)}; \quad (\text{D.47})$$

$$\begin{aligned} G_a &= \left(B_a^2 - \frac{D}{4\kappa_a^2 e_a^2}\right) \partial_{B_a}^2 + \left(1 - \frac{4\kappa_a^2 e_a^2}{D} B_a^2\right)^{-1} (\nu_a^2 - 1) + \frac{u_a}{e_a}, \\ G_3 &= u_3 + q_3 (N_{B_3} + 1) \partial_{B_3} - \frac{q_3}{4B_3} \nu_3^2, \end{aligned} \quad \text{for } V^{(6)}; \quad (\text{D.48})$$

$$\begin{aligned} G_a &= -\frac{D}{4\kappa_a^2 e_a} \partial_{B_a}^2 + e_a (N_{B_a} + \nu_a) (N_{B_a} + \nu_a + 1) + u_a, \\ G_3 &= u_3 + q_3 (N_{B_3} + 1) \partial_{B_3} - \frac{q_3}{4B_a} \nu_3^2, \end{aligned} \quad \text{for } V^{(7)}; \quad (\text{D.49})$$

where, in (D.47)–(D.49), $a = 1, 2$. For ν_a (A.8) and W -operators (D.46), we note the relations (D.21)–(D.23) which admit us to get G_1 , G_2 , G_3 in (D.48). Equations $G_a V^{(7)} = 0$ with G_1 , G_2 , G_3 as in (D.49) constitute a system of three decoupled second-order differential equations w.r.t. B_1 , B_2 , B_3 . Eight independent solutions of these equations are given in (6.50).

$$\text{Case} \quad m_1^2 < 0, \quad m_2^2 < 0, \quad m_3^2 < 0. \quad (\text{D.50})$$

Step 1. This case corresponds to the fourth vertex $p_{[3]}^-$ in (3.32). Vertex $V^{(3)}$ for this case is given in (D.9), where the realization of operators G_a (D.2) on the vertex $V^{(3)}$ is given in (D.6). We make the transformations

$$V^{(3)} = U_{\partial B} U_{\partial \alpha} V^{(4)}, \quad U_{\partial B} = \exp \left(\sum_{a=1,2,3} \frac{q_a}{2e_a} \partial_{B_a} \right), \quad (\text{D.51})$$

$$U_{\partial \alpha} = \exp \left(\sum_{a=1,2,3} \frac{1}{\kappa_a e_a} (B_{a+2} \partial_{\alpha_{a+2a}} - B_{a+1} \partial_{\alpha_{aa+1}}) - \frac{h_{a+1a+2}}{2\kappa_{a+1} \kappa_{a+2} e_{a+1} e_{a+2}} \partial_{\alpha_{a+1a+2}} \right) U_D, \quad (\text{D.52})$$

where U_D is defined in (A.15). Realization of G_a (D.6) on $V^{(4)}$ (D.51) takes the form

$$\begin{aligned}
 G_a = & -\frac{D}{4\kappa_a^2 e_a} \partial_{B_a}^2 + e_a (N_{B_a} + \nu_a) (N_{B_a} + \nu_a + 1) + u_a \\
 & + \left(\frac{4\kappa_{a+1}^2 e_{a+1}}{D} B_{a+1}^2 - \frac{1}{e_{a+1}} \right) \partial_{\alpha_{aa+1}}^2 + \left(\frac{4\kappa_{a+2}^2 e_{a+2}}{D} B_{a+2}^2 - \frac{1}{e_{a+2}} \right) \partial_{\alpha_{a+2a}}^2 \\
 & + 2\alpha_{a+1a+2} \partial_{\alpha_{aa+1}} \partial_{\alpha_{a+2a}}, \quad a = 1, 2, 3,
 \end{aligned} \tag{D.53}$$

where, here and below, we use quantities defined in (A.16)–(A.19).

Step 2. We make the transformations

$$V^{(4)} = U_{B_1} U_{B_2} U_{B_3} V^{(5)}, \quad V^{(5)} = U_{\nu_1, W_1} U_{\nu_2, W_{23}} U_{\nu_3, W_{312}} V^{(6)}, \quad V^{(6)} = U_{B_1}^{-1} U_{B_2}^{-1} U_{B_3}^{-1} V^{(7)}, \tag{D.54}$$

where U_{B_a} , $U_{\nu, W}$ are given in (A.13), (A.14) and we use the notation

$$\begin{aligned}
 W_a &= 2\alpha_{a+1a+2} \partial_{\alpha_{aa+1}} \partial_{\alpha_{a+2a}} - \partial_{\alpha_{aa+1}}^2 - \partial_{\alpha_{a+2a}}^2, \quad a = 1, 2, 3, \\
 W_{23} &= 2\alpha_{31} \partial_{\alpha_{12}} \partial_{\alpha_{23}} - \partial_{\alpha_{23}}^2, \quad W_{32} = 2\alpha_{12} \partial_{\alpha_{23}} \partial_{\alpha_{31}} - \partial_{\alpha_{23}}^2, \quad W_{312} = 2\alpha_{12} \partial_{\alpha_{23}} \partial_{\alpha_{31}}.
 \end{aligned} \tag{D.55}$$

Realizations of G_a (D.53) on $V^{(5)}$, $V^{(6)}$, $V^{(7)}$ (D.54) take the forms

$$G_a = \left(B_a^2 - \frac{D}{4\kappa_a^2 e_a^2} \right) \partial_{B_a}^2 + \left(1 - \frac{4\kappa_a^2 e_a^2}{D} B_a^2 \right)^{-1} (\nu_a^2 - 1 + W_a) + \frac{u_a}{e_a}, \quad \text{for } V^{(5)}; \tag{D.56}$$

$$G_a = \left(B_a^2 - \frac{D}{4\kappa_a^2 e_a^2} \right) \partial_{B_a}^2 + \left(1 - \frac{4\kappa_a^2 e_a^2}{D} B_a^2 \right)^{-1} (\nu_a^2 - 1) + \frac{u_a}{e_a}, \quad \text{for } V^{(6)}; \tag{D.57}$$

$$G_a = -\frac{D}{4\kappa_a^2 e_a} \partial_{B_a}^2 + e_a (N_{B_a} + \nu_a) (N_{B_a} + \nu_a + 1) + u_a, \quad \text{for } V^{(7)}; \tag{D.58}$$

where, in (D.56)–(D.58), $a = 1, 2, 3$. For ν_a (A.8) and W -operators (D.55), we note the relations (D.21)–(D.23) which admit us to get G_a in (D.57). Equations $G_a V^{(7)} = 0$ with $G_{1,2,3}$ as in (D.58) constitute a system of three decoupled second-order differential equations w.r.t. B_1 , B_2 , B_3 . Eight independent solutions of these equations are given in (6.71).

E One continuous-spin massless field, one arbitrary spin massless field, and one arbitrary spin massive field

We use the shortcut $(0, \kappa)_{CSF}$ for a continuous-spin massless field, while the shortcuts $(0, s)$ and (m, s) are used for the respective spin- s massless and spin- s massive field. We now consider cubic vertices for the following three fields:

$$\begin{aligned}
 & (0, s_1) - (m_2, s_2) - (0, \kappa_3)_{CSF} \quad m_2^2 > 0, \\
 & \text{one massless field, one massive field, and one continuous-spin massless field.}
 \end{aligned} \tag{E.1}$$

Relation (E.1) tells us that spin- s_1 massless and spin- s_2 massive fields carry the respective external line indices $a = 1$ and $a = 2$, while the continuous-spin massless field corresponds to $a = 3$.

For fields (E.1), we find the following general solution to cubic vertex $p_{[3]}^-$

$$p_{[3]}^- = U_v U_\Gamma U_\beta U_\zeta U_{\partial B} U_{\partial \alpha} U_{JB} U_W V_\pm^{(6)}, \quad (\text{E.2})$$

$$p_{[3]}^- = p_{[3]}^-(\beta_a, B_a, \alpha_{aa+1}, \zeta_2, v_3), \quad (\text{E.3})$$

$$V_\pm^{(6)} = V_\pm^{(6)}(B_2, B_3, \alpha_{aa+1}). \quad (\text{E.4})$$

In (E.2), we introduce two vertices $V_\pm^{(6)}$ labelled by the superscripts $\pm$. In (E.3) and (E.4), the arguments of the generic vertex $p_{[3]}^-$ and the vertices $V_\pm^{(6)}$ are shown explicitly. The definition of the arguments B_a and α_{ab} may be found in (A.3). Various quantities U appearing in (E.2) are differential operators w.r.t. the B_a and α_{aa+1} . These quantities will be presented below. For two vertices $V_\pm^{(6)}$ (E.4), we find the following solution:

$$V_+^{(6)} = I_{\nu_3}(\sqrt{z_3})V_+, \quad V_-^{(6)} = K_{\nu_3}(\sqrt{z_3})V_-, \quad V_\pm = V_\pm(B_2, \alpha_{12}, \alpha_{23}, \alpha_{31}), \quad (\text{E.5})$$

$$z_3 = \frac{4\kappa_3 B_3}{m_2^2}, \quad (\text{E.6})$$

where, in (E.5), the I_ν and K_ν are the modified Bessel functions. In (E.5), in place of the variable B_3 , we use new variable z_3 (E.6). Operator ν_3 is defined below.

Generic vertex $p_{[3]}^-$ (E.3) depends on the eleven variables, while, the vertices $V_\pm$ (E.5) depend only on the four variables. By definition, the vertices $V_\pm$ (E.5) are expandable in the variables $B_2, \alpha_{12}, \alpha_{23}, \alpha_{31}$. The general solution (E.2) for the vertex $p_{[3]}^-$ is expressed in terms of the operators U, ν_3 acting on the vertices $V_\pm$ (E.5). To complete the description of the vertex $p_{[3]}^-$ we now provide expressions for the operators U, ν_3 given by

$$U_v = v_3^{N_3}, \quad N_3 = N_{B_3} + N_{\alpha_{31}} + N_{\alpha_{23}}, \quad (\text{E.7})$$

$$U_\Gamma = \left(\frac{2^{N_3} \Gamma(N_3 + \frac{d-2}{2})}{\Gamma(N_3 + 1)} \right)^{1/2}, \quad (\text{E.8})$$

$$U_\beta = \exp \left(-\frac{\check{\beta}_2}{2\beta_2} \zeta_2 m_2 \partial_{B_2} - \frac{\check{\beta}_3}{2\beta_3} \kappa_3 \partial_{B_3} \right), \quad (\text{E.9})$$

$$U_\zeta = \exp \left(-\frac{\zeta_2}{m_2} B_1 \partial_{\alpha_{12}} + \frac{\zeta_2}{m_2} \left(B_3 - \frac{1}{2} \kappa_3 \right) \partial_{\alpha_{23}} \right), \quad (\text{E.10})$$

$$U_{\partial B} = \exp \left(\frac{1}{2} \kappa_3 \partial_{B_3} \right) \quad (\text{E.11})$$

$$U_{\partial \alpha} = \exp \left(\frac{2\kappa_3}{m_2^2} B_2 \partial_{\alpha_{23}} + \frac{2B_1 B_2}{m_2^2} \partial_{\alpha_{12}} + \frac{B_2 B_3}{m_2^2} \partial_{\alpha_{23}} - \frac{2B_3 B_1}{m_2^2} \partial_{\alpha_{31}} \right), \quad (\text{E.12})$$

$$U_{JB} = \left(\frac{4\kappa_3 B_3}{m_2^2} \right)^{-\nu_3/2}, \quad (\text{E.13})$$

$$U_W = \sum_{n=0}^{\infty} \frac{\Gamma(\nu + n)}{4^n n! \Gamma(\nu + 2n)} W^n, \quad (\text{E.14})$$

$$W_3 = -\frac{4B_2^2}{m_2^2} \partial_{\alpha_{23}}^2 + 2\alpha_{12} \partial_{\alpha_{31}} \partial_{\alpha_{23}}, \quad \nu_3 = N_{\alpha_{23}} + N_{\alpha_{31}} + \frac{d-4}{2}, \quad (\text{E.15})$$

where quantities $\check{\beta}_a, N_{B_a}, N_{\alpha_{ab}}, N_a$ appearing in (E.7)–(E.15) are defined in (A.3)–(A.7).

Expressions (E.2)–(E.15) provide the complete generating form description of cubic vertices for coupling of one continuous-spin massless field to two chains of massless and massive fields (2.13). Now we describe cubic vertices for coupling of one continuous-spin massless field to massless and massive fields having the respective arbitrary but fixed spin- s_1 and spin- s_2 values. Using the first algebraic constraints in (2.11), (2.12), it is easy to see that vertices we are interested in must satisfy the algebraic constraints

$$(N_{\alpha_1} - s_1)|p_{[3]}^-\rangle = 0, \quad (N_{\alpha_2} + N_{\zeta_2} - s_2)|p_{[3]}^-\rangle = 0. \quad (\text{E.16})$$

In terms of the vertices $V_{\pm}$ (E.5), constraints (E.16) take the form

$$(N_{\alpha_{12}} + N_{\alpha_{31}} - s_1)V = 0, \quad (N_{B_2} + N_{\alpha_{12}} + N_{\alpha_{23}} - s_2)V = 0, \quad V = V_{\pm}, \quad (\text{E.17})$$

where to simplify our presentation we drop the superscripts $\pm$ and use a vertex V in place of the vertices $V_{\pm}$, i.e., we use $V = V_{\pm}$. Doing so, we note that the general solution to constraints (E.17) can be presented as

$$V = V(s_1, s_2; n, k), \quad V(s_1, s_2; n, k) = B_2^{s_2-s_1+k-n} \alpha_{12}^{s_1-k} \alpha_{23}^n \alpha_{31}^k, \quad (\text{E.18})$$

$$0 \leq k \leq s_1, \quad 0 \leq n \leq s_2 - s_1 + k. \quad (\text{E.19})$$

The integers n, k (E.18) are the freedom of our solution for the vertex V . In order for vertices (E.18) to be sensible, the integers n, k should satisfy restrictions (E.19) which amount to the requirement that the powers of all variables $B_2, \alpha_{12}, \alpha_{23}, \alpha_{31}$ in (E.18) be non-negative. Expressions for cubic interaction vertices given in (E.2)–(E.15), (E.18) and values of n, k given in (E.19) provide the complete description and classification of parity invariant cubic interaction vertices that can be constructed for one spin- s_1 massless field, one spin- s_2 massive field and one continuous-spin massless field.

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