Numerical study of the $\mathcal{N} = 2$ Landau-Ginzburg model with two superfields

Okuto Morikawa

*Department of Physics, Kyushu University,
744 Motoooka, Nishi-ku, Fukuoka, 819-0395, Japan*

E-mail: o-morikawa@phys.kyushu-u.ac.jp

ABSTRACT: In the low energy limit, the two-dimensional massless $\mathcal{N} = 2$ Wess-Zumino (WZ) model with a quasi-homogeneous superpotential is believed to become a superconformal field theory. This conjecture of the Landau-Ginzburg (LG) description has been studied numerically in the case of the A_2 , A_3 , and E_6 minimal models. In this paper, by using a supersymmetric-invariant non-perturbative formulation, we simulate the WZ model with two superfields corresponding to the D_3 , D_4 , and E_7 models. Then, we numerically determine the central charge, and obtain the results that are consistent with the conjectured correspondence. We hope that this numerical approach, when further developed, will be useful to investigate superstring theory via the LG/Calabi-Yau correspondence.

KEYWORDS: Conformal Field Models in String Theory, Extended Supersymmetry, Field Theories in Lower Dimensions, Lattice Quantum Field Theory

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1 Introduction

At an extremely low-energy scale, the two-dimensional massless $\mathcal{N} = 2$ Wess-Zumino (WZ) model [1] with a quasi-homogeneous superpotential is believed to become an $\mathcal{N} = 2$ superconformal field theory (SCFT). This conjecture of the Landau-Ginzburg (LG) description has been studied from various aspects [2–23]. However, we have no complete proof of this conjecture. It is difficult to prove because the coupling constant becomes strong at the low-energy region and the perturbative theory possesses infrared (IR) divergences; the LG description is a truly non-perturbative phenomenon. An interesting approach to this issue may be a numerical and non-perturbative technique on the basis of the lattice field theory.

By using such numerical approaches, the A_2 and A_3 minimal models were simulated in refs. [24–26], where superpotentials in the corresponding WZ model are given as the cubic and quartic ones containing a single superfield; see table 1 [15]. These studies are based on either a lattice formulation in ref. [27], or a supersymmetry-preserving formulation with a momentum cutoff in ref. [28]; both non-perturbative formulations make essential use of the existence of the Nicolai map [29–32].¹ In above numerical studies, their results of the scaling dimension and the central charge are consistent with the expected values in the A_2 and A_3 minimal models within numerical errors. Therefore, we have now numerical evidences for the LG/SCFT correspondence in the case of the A_2 , A_3 , and E_6 ($\cong A_2 \otimes A_3$) minimal models.

In this paper, on the basis of the momentum cutoff regularization [28] and the analysis in ref. [26], we simulate the two-dimensional $\mathcal{N} = 2$ WZ model corresponding to D - and E -type theories. The method in ref. [26] is generalized to the WZ model with multiple superfields and more complicated superpotentials. Then, from an IR behavior of the

¹For some related works, see refs. [26, 28] and references therein. It appears that the two-dimensional massless $\mathcal{N} = 2$ WZ model is numerically studied in ref. [33].

Algebra	Superpotential W	Central charge c
A_n	$\Phi^{n+1}, n \geq 1$	$3 - 6/(n+1)$
D_n	$\Phi^{n-1} + \Phi\Phi'^2, n \geq 3$	$3 - 6/2(n-1)$
E_6	$\Phi^3 + \Phi'^4$	$3 - 6/12$
E_7	$\Phi^3 + \Phi\Phi'^3$	$3 - 6/18$
E_8	$\Phi^3 + \Phi'^5$	$3 - 6/30$

Table 1. ADE classification.

energy-momentum tensor (EMT), we numerically determine the central charge of the D_3 , D_4 , and E_7 models; we obtain the results that are consistent with the conjectured correspondence. We also measure the “effective central charge” [25, 26], which is analogous to the Zamolodchikov’s c -function [34, 35]. Although the theoretical background of the formulation is not completely obvious so far, our computational results support the validity of the formulation even if we consider multi-superfield theories. We hope to apply this approach to some models which is neither a minimal model nor a product of minimal models (Gepner model [10, 13]), and then develop a numerical method to investigate superstring theory via the LG/Calabi-Yau correspondence [16, 36–38].

2 Nicolai mapping for the multi-superfield WZ model

Our numerical simulation is based on the formulation in ref. [28]. The detailed discussions for the formulation are given in ref. [26]. These preceding studies treat the two-dimensional $\mathcal{N} = 2$ WZ model with a single superfield. In this section, we summarize basic formulas of the formulation for the WZ model with multiple superfields.

Suppose that the system is defined in a two-dimensional Euclidean physical box $L_0 \times L_1$. In what follows, we work in the momentum space with an ultraviolet (UV) cutoff,

$$p_\mu = \frac{2\pi}{L_\mu} n_\mu \quad \left(n_\mu = 0, \pm 1, \pm 2, \dots, \pm \frac{L_\mu}{2a} \right), \quad (2.1)$$

where the Greek index μ runs over 0 and 1, and repeated indices are not summed over; L_μ/a is taken as even integers, and a is a unit of dimensionful quantities. A limit $a \rightarrow 0$ removes the UV cutoff, being similar to the *continuum limit*. In fact, when we take L_μ/a as *odd* integers, the unit a itself is the lattice spacing in the dimensional reduction of the four-dimensional lattice formulation [39] based on the SLAC derivative [40, 41].

It is well recognized that the regularization based on the SLAC derivative violates the locality. The four-dimensional SLAC derivative is plagued by the pathology that the locality of the theory is not automatically restored in the continuum limit [42–45]. In the two- or three-dimensional case, on the other hand, one can argue the restoration of the locality in the continuum limit within perturbation theory for *massive* WZ models [28]. For *massless* models, it is not clear whether the restoration is automatically accomplished because perturbation techniques are hindered by the IR divergences. We believe that the numerical results in refs. [25, 26] and ours below support the validity of the formulation.

For simplicity, we set $a = 1$. We basically use the complex coordinates for the momentum, $p_z = (p_0 - ip_1)/2$ and $p_{\bar{z}} = (p_0 + ip_1)/2$. In general, the two-dimensional $\mathcal{N} = 2$ WZ model contains N_Φ superfields, $\{\Phi_I\}_{I=1,\dots,N_\Phi}$. A supermultiplet Φ_I consists of a complex scalar A_I , left- and right-handed spinors $(\psi_I, \bar{\psi}_I)$, and a complex auxiliary field F_I . Then, the action of the two-dimensional $\mathcal{N} = 2$ WZ model with a quasi-homogeneous superpotential $W(\{A\})$ is given by

$$S = \frac{1}{L_0 L_1} \sum_p \sum_I \left[4p_z A_I^*(-p) p_{\bar{z}} A_I(p) - F_I^*(-p) F_I(p) - F_I(-p) \frac{\partial W(\{A\})}{\partial A_I}(p) - F_I^*(-p) \frac{\partial W(\{A\})^*}{\partial A_I^*}(p) + (\bar{\psi}_1, \psi_2)_I(-p) \sum_J \left(\frac{2\delta_{IJ} p_z}{\frac{\partial^2 W(\{A\})}{\partial A_I \partial A_J} *} \frac{\frac{\partial^2 W(\{A\})^*}{\partial A_I^* \partial A_J^*} *}{2\delta_{IJ} p_{\bar{z}}} \right) \begin{pmatrix} \psi_1 \\ \bar{\psi}_2 \end{pmatrix}_J(p) \right], \quad (2.2)$$

where $*$ denotes the convolution

$$(\varphi_1 * \varphi_2)(p) \equiv \frac{1}{L_0 L_1} \sum_q \varphi_1(q) \varphi_2(p - q). \quad (2.3)$$

The field products in $\partial W(\{A\})/\partial A_I$ and $\partial W(\{A\})/\partial A_I \partial A_J$ are understood as this convolution. Integrating over the auxiliary fields $\{F\}$, we obtain the action in terms of the physical component fields,

$$S = S_B + \frac{1}{L_0 L_1} \sum_p \sum_{I,J} (\bar{\psi}_1, \psi_2)_I(-p) \left(\frac{2i\delta_{IJ} p_z}{\frac{\partial^2 W(\{A\})}{\partial A_I \partial A_J} *} \frac{\frac{\partial^2 W(\{A\})^*}{\partial A_I^* \partial A_J^*} *}{2i\delta_{IJ} p_{\bar{z}}} \right) \begin{pmatrix} \psi_1 \\ \bar{\psi}_2 \end{pmatrix}_J(p), \quad (2.4)$$

where S_B is the bosonic part of the total action

$$S_B = \frac{1}{L_0 L_1} \sum_p \sum_I N_I^*(-p) N_I(p), \quad N_I(p) = 2p_z A_I(p) + \frac{\partial W(\{A\})^*}{\partial A_I^*}(p). \quad (2.5)$$

The new variables $\{N\}$ (2.5) specify the so-called Nicolai map [29–32], the change of variables from $\{A\}$ to $\{N\}$. This mapping simplifies the path-integral weight drastically; the partition function where the fermion fields are integrated is given by

$$\mathcal{Z} = \int \prod_{|p_\mu| \leq \pi} \prod_I [dN_I(p) dN_I^*(p)] e^{-S_B} \sum_k \text{sign det} \frac{\partial(\{N\}, \{N^*\})}{\partial(\{A\}, \{A^*\})} \Big|_{\{A\}=\{A\}_k}, \quad (2.6)$$

where $\{A\}_k$ ($k = 1, 2, \dots$) is a set of solutions of the equation

$$2ip_z A_I(p) + \frac{\partial W(\{A\})^*}{\partial A_I^*}(p) - N_I(p) = 0. \quad (2.7)$$

The weight $\exp(-S_B)$ is a Gaussian function of the variables $\{N\}$. All we should do to obtain configurations of $\{N\}$ and $\{A\}$ is the following: we generate complex random numbers $\{N(p)\}$ from the Gaussian distribution for each momentum, and then, solve numerically the algebraic equation (2.7) with respect to $\{A\}$. Note that, however, our numerical root-finding analysis may suffer from the systematic error associated with some undiscovered solutions to eq. (2.7). Ref. [26] addresses the difficulty of the algorithm.

3 Central charge from the EMT correlator

In a two-dimensional SCFT, the central charge c appears in the two-point function of EMT; in terms of the Fourier mode $T(p) \equiv T_{zz}(p)$, we have [26]²

$$\langle T(p)T(-p) \rangle = L_0 L_1 \frac{\pi c p_z^3}{12 p_{\bar{z}}}. \quad (3.1)$$

The two-dimensional $\mathcal{N} = 2$ WZ model, which itself is not superconformal invariant and hence does not behave as eq. (3.1), is believed to give an LG description of SCFT. Thus, the IR behavior of the WZ model would be governed by relations as eq. (3.1) in SCFT. The central charge can be computed from the fit function (3.1) in the IR region.

Let us write down explicit expressions of EMT and its correlator in the WZ model (2.2). Since our formulation preserves some spacetime symmetries exactly, we can straightforwardly construct Noether currents associated with the symmetries, for example, the supercurrent and EMT. To remove the ambiguity of EMT, we require the traceless condition $T_{z\bar{z}} = T_{\bar{z}z} = 0$ in the free-field limit [26] (see also refs. [51, 52]). Following the corresponding computation in ref. [26], EMT is given in the momentum space by

$$T(p) = \frac{\pi}{L_0 L_1} \sum_q \sum_I \left[4(p-q)_z q_z A_I^*(p-q) A_I(q) - i q_z \psi_{2I}(p-q) \bar{\psi}_{2I}(q) + i(p-q)_z \psi_{2I}(p-q) \bar{\psi}_{2I}(q) \right]. \quad (3.2)$$

Like as the single-supermultiplet case, it can turn out that this expression of EMT is the super-transformation of the supercurrent; for the definition of the super-transformation, see appendix A in ref. [26]. Thus we can obtain a less noisy form of the EMT correlator

$$\langle T(p)T(-p) \rangle = -\frac{1}{16} 2ip_z \langle S_z^-(p) S_z^+(-p) + S_z^+(p) S_z^-(-p) \rangle, \quad (3.3)$$

where $S_z^\pm$ is the supercurrent defined by

$$S_z^+(p) = \frac{4\pi}{L_0 L_1} \sum_q \sum_I i(p-q)_z A_I(p-q) \bar{\psi}_{2I}(q), \quad (3.4)$$

$$S_z^-(p) = -\frac{4\pi}{L_0 L_1} \sum_q \sum_I i(p-q)_z A_I^*(p-q) \psi_{2I}(q). \quad (3.5)$$

Since the formulation exactly preserves the supersymmetry, this relation between EMT and the supercurrent holds.

4 Numerical setup and sampling configurations

In what follows, we consider the $N_\Phi = 2$ WZ model corresponding to the D_3 , D_4 , and E_7 minimal models. The superpotential is defined by

$$W(\{\Phi\}) = \frac{\lambda_1}{n} \Phi_1^n + \frac{\lambda_2}{2} \Phi_1 \Phi_2^2 \quad \text{for } D_n, \quad (4.1)$$

$$W(\{\Phi\}) = \frac{\lambda_1}{3} \Phi_1^3 + \frac{\lambda_2}{3} \Phi_1 \Phi_2^3 \quad \text{for } E_7. \quad (4.2)$$

²In the spacetime with the complex coordinates $z \equiv x_0 + ix_1$ and $\bar{z} \equiv x_0 - ix_1$, the EMT correlator is given by $\langle T(z)T(0) \rangle = c/2z^4$. This convention is identical to that of refs. [26, 49, 50].

L	8	16	24
$(+++)_3$	640	640	638
$(++++-)_3$	0	0	2
$(+++++-)_3$	0	0	0
Δ	3	3	3
δ	0.0016(30)	0.0015(16)	-0.0007(11)
core·hour [h]	1.60	63.78	649.73

Table 2. Classification of configurations for D_3 . The symbol $(+\dots+-\dots-)_{{n_+}-{n_-}}$ with n_+ plus and n_- minus symbols implies that, for a configuration $\{N(p)\}$, we find $(n_+ + n_-)$ solutions, $\{A(p)\}_k$ ($k = 1, \dots, n_+ + n_-$); the n_+ solutions take $\text{sign det} \frac{\partial(N, N^*)_i}{\partial(A, A^*)_j} = +1$ and the n_- solutions take -1 . The number of obtained configurations for each setup is shown. Δ (4.3) and δ (4.4) are numerically computed.

L	32	40	44
$(+++)_3$	639	633	632
$(++++-)_3$	1	7	7
$(+++++-)_3$	0	0	1
Δ	3	3	3
δ	0.00019(85)	-0.00078(69)	-0.00092(63)
core·hour [h]	3440.33	13426.08	25623.00

Table 3. Classification of configurations for D_3 (continued).

We set the couplings λ_1 and λ_2 to 0.3 in the unit of $a = 1$. The box size $L = L_0 = L_1$ is taken as even integers 8, 16, 24, 32, 40, and 44 for D_3 ; 8, 16, 24, 32, 36, 40 and 42 for D_4 ; 8, 16, and 24 for E_7 . Other numerical setups are identical to those in ref. [26], and hence see ref. [26] for more detailed information of our program.

The classification of obtained configurations are tabulated in tables 2–6. We also listed the Witten index [26, 46, 47],

$$\Delta \equiv \left\langle \sum_k \text{sign det} \frac{\partial(\{N\}, \{N^*\})}{\partial(\{A\}, \{A^*\})} \bigg|_{\{A\}=\{A\}_k} \right\rangle = n \quad \text{for } D_n, E_n, \quad (4.3)$$

and the one-point supersymmetry Ward-Takahashi identity [26, 48]

$$\delta \equiv \frac{\langle S_B \rangle}{(L+1)^2} - 1 = 0. \quad (4.4)$$

Δ and δ should be reproduced numerically, and indicate the quality of our configurations. The computational time is given in core·hour on the Intel Xeon E5 2.0 GHz for D_3 with $L = 8, \dots, 44$ and D_4 with $L = 8, \dots, 32, 40$ and the Intel Xeon Gold 3.0 GHz for D_4 with $L = 36, 42$ and E_7 with $L = 8, \dots, 24$.

L	8	16	24	32
$(++++)_4$	640	638	629	626
$(+++++-)_4$	0	2	9	13
$(+++++--)_4$	0	0	2	1
$(+++++---)_4$	0	0	0	0
$(++++-)_3$	0	0	0	0
$(+++++)_5$	0	0	0	0
$(+++++-----)_4$	0	0	0	0
Δ	4	4	4	4
δ	-0.0028(31)	0.0018(17)	-0.0020(11)	-0.00095(84)
core-hour [h]	1.73	72.28	847.83	5220.60

Table 4. Classification of configurations for D_4 . Same as table 2, but for D_4 .

L	36	40	42
$(++++)_4$	604	606	603
$(+++++-)_4$	23	24	35
$(+++++--)_4$	12	8	9
$(+++++---)_4$	1	0	0
$(++++-)_3$	0	1	0
$(+++++)_5$	0	1	0
$(+++++-----)_4$	0	0	1
Δ	4	4.000(2)	4
δ	-0.00008(73)	-0.00025(87)	0.00019(64)
core-hour [h]	4328.58	22264.12	12272.93

Table 5. Classification of configurations for D_4 (continued).

5 Numerical determination of the central charge

Let us show the main result of this paper, the numerical determination of the central charge for the D_3 , D_4 , and E_7 models.

We plot the correlation function $\langle T(p)T(-p) \rangle$ (3.3) in figures 1–3 for the maximal box size with the fitting curve (3.1); the central charge c is obtained from the fit in the IR region $\frac{2\pi}{L} \leq |p| < \frac{4\pi}{L}$. The left panel in each figure is devoted to the real part of the two-point function and the right one is the imaginary part. The horizontal axis indicates the momentum p_0 , and the momentum p_1 is fixed to the positive minimal value $p_1 = 2\pi/L$. In table 7, we tabulate the numerical results of the central charge for all box sizes in the D_3 , D_4 , and E_7 models.

L	8	16	24
$(++++++)_7$	639	628	582
$(++++++)_7$	1	10	20
$(++++++)_6$	0	2	27
$(++++++)_6$	0	0	5
$(++++++)_8$	0	0	3
$(++++++)_6$	0	0	2
$(++++++)_7$	0	0	1
Δ	7	6.997(2)	6.952(9)
δ	-0.0013(32)	-0.0004(16)	0.0009(17)
core·hour [h]	1.30	60.28	750.92

Table 6. Classification of configurations for E_7 . Same as table 2, but for E_7 .

The central charge for the maximal box size in table 7 reads

$$c = 1.595(31)(41) \quad \text{for } D_3, \quad (5.1)$$

$$c = 2.172(48)(39) \quad \text{for } D_4, \quad (5.2)$$

$$c = 2.638(47)(59) \quad \text{for } E_7. \quad (5.3)$$

This is the main result in this paper. Here, a number in the second parentheses indicates the systematic error associated with the finite-volume effect. In eqs. (5.1) and (5.3), we estimate this as follows: we pick out the largest three volumes for each minimal model; from the central values at two smaller ones, we extrapolate to the larger L regime as a linear function of the inverse volume $1/L^2$, and then, obtain an extrapolated value at the maximal volume (see figure 4); the systematic error is identified with the deviation between this and the central value in eqs. (5.1) and (5.3).³ In eq. (5.2), since we have more than two would-be convergent results at large L , the systematic error is estimated by the maximum deviation of the central values at the three largest volumes as in ref. [26]. Eqs. (5.1)–(5.3) are consistent with the expected values, 1.5 for D_3 within $\sim 1.3\sigma$, 2 for D_4 within 2σ , and $2.666\dots$ for E_7 within the numerical errors; the standard deviations are evaluated by the sum of the statistical and systematic errors.

As mentioned in refs. [25, 26], it is interesting to plot the “effective central charge,” which is analogous to the Zamolodchikov’s c -function [34, 35]. This is obtained from the fit in a variety of momentum regions from IR to UV; we take the fitted momentum regions as $\frac{2\pi}{L}n \leq |p| < \frac{2\pi}{L}(n+1)$ for $n \in \mathbb{Z}_+$. Then figure 5 shows that the “effective central charge” connects the IR central charge to an UV value $c \approx 6$. This number is consistent with the central charge $c = 3N_\Phi$ in the expected free $\mathcal{N} = 2$ SCFT. Recall that N_Φ is the number of supermultiplets in the free $\mathcal{N} = 2$ WZ model, and we have set $N_\Phi = 2$ in this paper.

³The fit function $\propto 1/L^2$ is a possible choice, but there would be no theoretical evidence to support this choice. Because the behavior of the $L \rightarrow \infty$ limit appears not quite smooth as in figure 4, we do not attempt to extrapolate to the infinite volume limit.

Figure 1. $\langle T(p)T(-p) \rangle$ for D_3 , $L = 44$, and $p_1 = \pi/22$. The fitting curve (3.1) is depicted at once.

Figure 2. $\langle T(p)T(-p) \rangle$ for D_4 , $L = 42$, and $p_1 = \pi/21$. The fitting curve (3.1) is depicted at once.

Figure 3. $\langle T(p)T(-p) \rangle$ for E_7 , $L = 24$, and $p_1 = \pi/12$. The fitting curve (3.1) is depicted at once.

Algebra	L	$\chi^2/\text{d.o.f.}$	c	Expected value
D_3	8	6.056	2.786(34)	1.5
	16	5.496	2.141(31)	
	24	3.122	1.867(28)	
	32	2.682	1.711(29)	
	40	0.476	1.591(32)	
	44	3.598	1.595(31)	
D_4	8	3.216	2.907(36)	2
	16	3.738	2.509(34)	
	24	1.946	2.466(42)	
	32	2.832	2.202(40)	
	36	1.109	2.211(70)	
	40	2.276	2.175(48)	
	42	1.177	2.172(48)	
E_7	8	2.220	2.964(36)	2.666...
	16	1.800	2.639(35)	
	24	1.364	2.638(47)	

Table 7. The central charge obtained from the fit of the EMT correlator. The fitted momentum range is $\frac{2\pi}{L} \leq |p| < \frac{4\pi}{L}$.

Figure 4. Systematic error estimation for the central charge.

Figure 5. “Effective central charge,” which changes as the function of $|p| = \frac{2\pi}{L}n$ with fitted momentum regions, $\frac{2\pi}{L}n \leq |p| < \frac{2\pi}{L}(n+1)$, for $n \in \mathbb{Z}_+$.

6 Conclusion

In this paper, we numerically studied the two-dimensional $\mathcal{N} = 2$ WZ model corresponding to the D_3 , D_4 , and E_7 minimal models. Utilizing the supersymmetry-preserving formulation with a momentum cutoff [28], we numerically determined the central charge from the IR behavior of the WZ model. Although the theoretical background of our computational approach is not clear so far, our results for the theories with two superfields are consistent with the conjectured correspondence between the LG model and the minimal series of SCFT. In the paper and the preceding studies [24–26], we have the numerical evidences of typical minimal models: the A_2 , A_3 , D_3 , D_4 , E_6 , and E_7 models in table 1; the A_4 or E_8 ($\cong A_2 \otimes A_4$) minimal model is left to be simulated.

To investigate superstring theory by using our numerical approach, we may start from the numerical simulation of the A_4 minimal model, or simpler theories with several supermultiplets which is not a Gepner model.

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