

## Subleading soft theorem for multiple soft gravitons

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ABSTRACT: We derive the subleading soft graviton theorem in a generic quantum theory of gravity for arbitrary number of soft external gravitons and arbitrary number of finite energy external states carrying arbitrary mass and spin. Our results are valid to all orders in perturbation theory when the number of non-compact space-time dimensions is six or more, but only for tree amplitudes for five or less non-compact space-time dimensions due to enhanced contribution to loop amplitudes from the infrared region.

KEYWORDS: Scattering Amplitudes, Superstrings and Heterotic Strings

ARXIV EPRINT: [1707.06803](https://arxiv.org/abs/1707.06803)

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## 1 Introduction

Soft graviton theorem expresses the scattering amplitude of finite energy external states and low energy gravitons in terms of the amplitude without the low energy gravitons [1–4]. They have been investigated intensively during the last few years [5–30] due to their connection to asymptotic symmetries [31–41]. They have also been investigated in string theory [42–56]. In particular in specific quantum field theories and string theories, amplitudes with several finite energy external states and one soft graviton have been analyzed to subsubleading order, leading to the subsubleading soft graviton theorem in these theories. A general proof of the soft graviton theorem in a generic quantum theory of gravity was given in [55–57] for one external soft graviton and arbitrary number of other finite energy external states carrying arbitrary mass and spin.

For specific theories, soft graviton amplitudes with two soft gravitons have also been investigated in [26, 30, 58–63]. Our goal in this paper will be to derive, in a generic quantum theory of gravity, the form of the soft graviton theorem to the first subleading order in soft momentum for arbitrary number of soft gravitons and for arbitrary number of finite energy external states carrying arbitrary mass and spin. The limit we consider is when all the soft momenta become small at the same rate. As discussed in section 2.3, in order to avoid enhanced contribution to loop diagrams from the infrared region, we shall restrict our analysis to the case where the number of non-compact space-time dimensions  $D$  is six or more. For  $D \leq 5$  our analysis will be valid for tree amplitudes. We expect that even in  $D=5$ , where the amplitudes are infrared finite, the enhanced infrared contributions of the type described in section 2.3 will cancel in the sum over graphs and our result will be valid also for  $D = 5$  to all loop orders. However, we have not proved this yet.

Our final result for an amplitude with  $N$  external finite energy particles carrying polarizations and momenta  $(\epsilon_i, p_i)$  for  $i = 1, \dots, N$ , and  $M$  soft gravitons carrying polarizations and momenta  $(\epsilon_r, k_r)$  for  $r = 1, \dots, M$ , takes the form

$$\begin{aligned}
 A = & \left\{ \prod_{i=1}^N \epsilon_{i, \alpha_i}(p_i) \right\} \left[ \left\{ \prod_{r=1}^M S_r^{(0)} \right\} \Gamma^{\alpha_1 \dots \alpha_N} + \sum_{s=1}^M \left\{ \prod_{\substack{r=1 \\ r \neq s}}^M S_r^{(0)} \right\} [S_s^{(1)} \Gamma]^{\alpha_1 \dots \alpha_N} \right. \\
 & \left. + \sum_{\substack{r, u=1 \\ r < u}}^M \left\{ \prod_{\substack{s=1 \\ s \neq r, u}}^M S_s^{(0)} \right\} \left\{ \sum_{j=1}^N \{p_j \cdot (k_r + k_u)\}^{-1} \mathcal{M}(p_j; \epsilon_r, k_r, \epsilon_u, k_u) \right\} \Gamma^{\alpha_1 \dots \alpha_N} \right], \quad (1.1)
 \end{aligned}$$

where

$$S_r^{(0)} = \sum_{\ell=1}^N (p_\ell \cdot k_r)^{-1} \epsilon_{r, \mu\nu} p_\ell^\mu p_\ell^\nu, \quad (1.2)$$

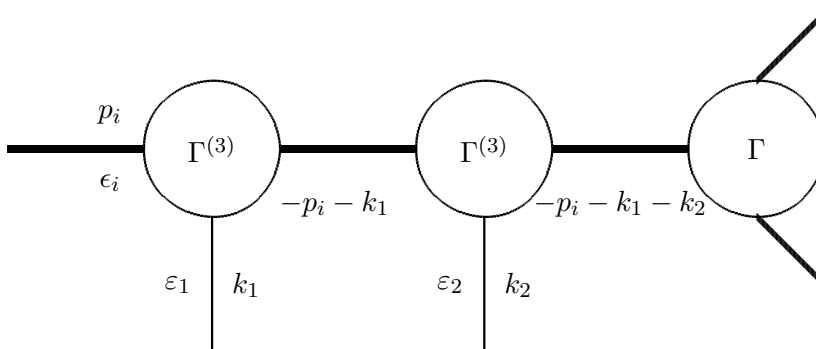
$$\begin{aligned}
 [S_s^{(1)} \Gamma]^{\alpha_1 \dots \alpha_N} = & \sum_{j=1}^N (p_j \cdot k_s)^{-1} \epsilon_{s, b\mu} k_{sa} p_j^\mu \\
 & \times \left[ p_j^b \frac{\partial \Gamma^{\alpha_1 \dots \alpha_N}}{\partial p_{ja}} - p_j^a \frac{\partial \Gamma^{\alpha_1 \dots \alpha_N}}{\partial p_{jb}} + (J^{ab})_{\beta_j}^{\alpha_j} \Gamma^{\alpha_1 \dots \alpha_{j-1} \beta_j \alpha_{j+1} \dots \alpha_N} \right], \quad (1.3)
 \end{aligned}$$

$$\begin{aligned}
 \mathcal{M}(p_i; \epsilon_1, k_1, \epsilon_2, k_2) = & (p_i \cdot k_1)^{-1} (p_i \cdot k_2)^{-1} \left\{ -k_1 \cdot k_2 p_i \cdot \epsilon_1 \cdot p_i \ p_i \cdot \epsilon_2 \cdot p_i \right. \\
 & \left. + 2p_i \cdot k_2 p_i \cdot \epsilon_1 \cdot p_i p_i \cdot \epsilon_2 \cdot k_1 + 2p_i \cdot k_1 p_i \cdot \epsilon_2 \cdot p_i p_i \cdot \epsilon_1 \cdot k_2 - 2p_i \cdot k_1 p_i \cdot k_2 p_i \cdot \epsilon_1 \cdot \epsilon_2 \cdot p_i \right\} \\
 & + (k_1 \cdot k_2)^{-1} \left\{ -(k_2 \cdot \epsilon_1 \cdot \epsilon_2 \cdot p_i)(k_2 \cdot p_i) - (k_1 \cdot \epsilon_2 \cdot \epsilon_1 \cdot p_i)(k_1 \cdot p_i) \right. \\
 & + (k_2 \cdot \epsilon_1 \cdot \epsilon_2 \cdot p_i)(k_1 \cdot p_i) + (k_1 \cdot \epsilon_2 \cdot \epsilon_1 \cdot p_i)(k_2 \cdot p_i) - \epsilon_1^{\gamma\delta} \epsilon_2 \cdot \gamma \delta (k_1 \cdot p_i)(k_2 \cdot p_i) \\
 & \left. - 2(p_i \cdot \epsilon_1 \cdot k_2)(p_i \cdot \epsilon_2 \cdot k_1) + (p_i \cdot \epsilon_2 \cdot p_i)(k_2 \cdot \epsilon_1 \cdot k_2) + (p_i \cdot \epsilon_1 \cdot p_i)(k_1 \cdot \epsilon_2 \cdot k_1) \right\}, \quad (1.4)
 \end{aligned}$$

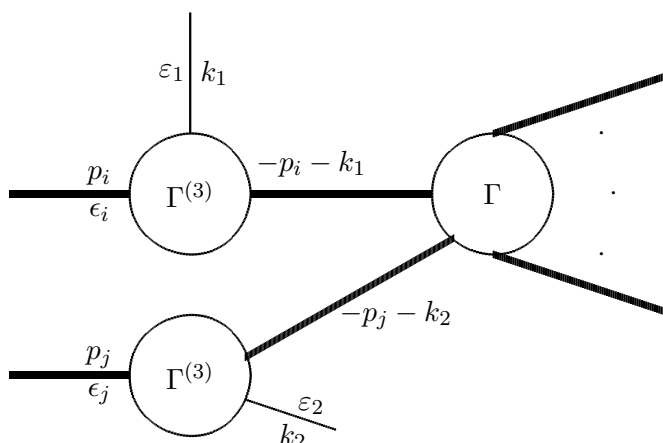
and  $\Gamma^{\alpha_1 \dots \alpha_N}$  is defined such that

$$\Gamma(\epsilon_1, p_1, \dots, \epsilon_N, p_N) \equiv \left\{ \prod_{i=1}^N \epsilon_{i, \alpha_i} \right\} \Gamma^{\alpha_1 \dots \alpha_N}, \quad (1.5)$$

gives the amplitude without the soft gravitons, including the momentum conserving delta function. The indices  $\alpha, \beta, \gamma, \delta$  run over all the fields of the theory and  $J^{ab}$  is the (reducible) representation of the spin angular momentum generator on the fields. The indices  $a, b$  as well as  $\mu, \nu, \rho$  are space-time coordinate/momentum labels. We shall use Einstein summation convention for the indices  $\alpha, \beta, \dots$  carried by the fields and also for the space-time coordinate labels  $a, b, \dots$  and  $\mu, \nu, \dots$ , but not for the indices  $r, s, \dots$  labelling the external soft gravitons and  $i, j, \dots$  labelling the external finite energy particles. For the signature of the space-time metric we shall use mostly + sign convention.



**Figure 1.** A leading contribution to the amplitude with two soft gravitons.



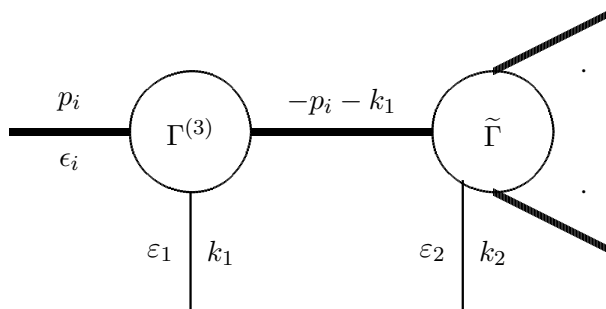
**Figure 2.** A leading contribution to the amplitude with two soft gravitons.

The rest of the paper is organized as follows. In section 2 we prove the subleading soft graviton theorem for two external soft gravitons and arbitrary number of external states of arbitrary mass and spin. In section 3 we carry out various consistency checks of this formula. These include test of gauge invariance and also comparison with existing results. In particular we find that neither the first nor the second line of (1.1) is gauge invariant by itself but their sum is gauge invariant. We generalize the result to the case of multiple soft gravitons in section 4.

Derivation of double soft theorem from asymptotic symmetries has been pursued in [64].

## 2 Amplitudes with two soft gravitons

In this section we shall analyze an amplitude with arbitrary number of finite energy external states and two soft gravitons in the limit when the momenta carried by the soft gravitons become soft at the same rate. The relevant diagrams are shown in figures 1–5. We use the convention that all external momenta are ingoing, thick lines represent finite energy propagators and thin lines represent soft propagators.  $\varepsilon_r, k_r$  for  $r = 1, 2$  represent the



**Figure 3.** A subleading contribution to the amplitude with two soft gravitons. The subamplitude  $\tilde{\Gamma}$  excludes all diagrams where the soft particle carrying momentum  $k_2$  gets attached to one of external lines of  $\tilde{\Gamma}$ .

polarizations and momenta carried by the soft gravitons subject to the constraint

$$\eta^{\mu\nu} \varepsilon_{r,\mu\nu} = 0, \quad k_r^\mu \varepsilon_{r,\mu\nu} = 0. \tag{2.1}$$

$\Gamma^{(3)}$  and  $\Gamma^{(4)}$  denote one particle irreducible (1PI) three and four point functions and  $\Gamma$  denotes full amputated Green’s function. In figure 3,  $\tilde{\Gamma}$  denotes sum of all amputated Feynman diagrams in which the soft graviton is not attached to an external leg via a 1PI three point function. The internal thick lines of the diagrams represent full quantum corrected propagators carrying finite momentum. For figures 1 and 3 we also have to consider diagrams where the two soft gravitons are exchanged.

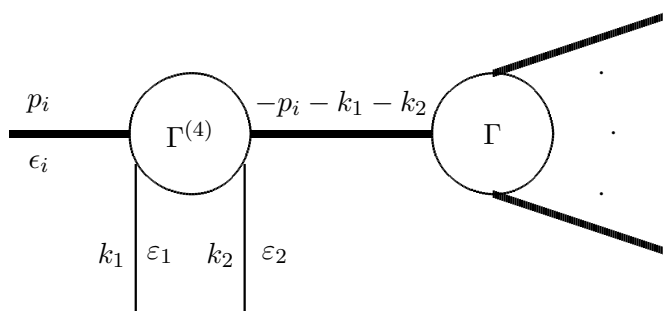
Among these diagrams the contributions from figure 1 and figure 2 have two nearly on-shell propagators giving two powers of soft momentum in the denominators. For example in figure 1 the line carrying momentum  $p_i + k_1$  is proportional to

$$\{(p_i + k_1)^2 + M_i^2\}^{-1} = (2p_i \cdot k_1)^{-1}, \tag{2.2}$$

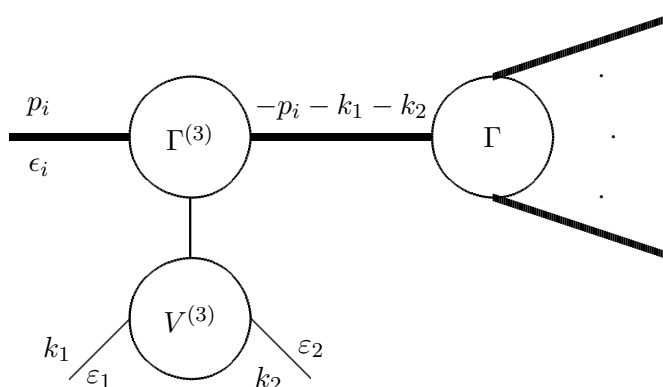
using the on-shell condition  $k_1^2 = 0$ ,  $p_i^2 + M_i^2 = 0$  if the mass of the internal state is the same as the mass of the  $i$ -th external state. Therefore the contribution from these diagrams begins at the leading order. The rest of the diagrams have only one nearly on-shell propagator and therefore their contribution begins at the subleading order. The contribution from figure 5 is somewhat deceptive — it appears to have one nearly on-shell propagator carrying finite energy giving one power of soft momentum in the denominator and a soft internal propagator giving two powers of soft momentum in the denominator. However the three graviton vertex has two powers of soft momentum in the numerator. Therefore the contribution from this diagram begins with one inverse power of soft momentum and is subleading.

### 2.1 Expressions for the vertices and propagators

Our strategy for deriving the vertices will be the same as that in [55–57]. We begin with the 1PI effective action of the theory and use Lorentz covariant gauge fixing conditions such that the propagators computed from this gauge fixed action do not have double poles. We now find the coupling of the soft graviton to the rest of the fields by covariantizing this action. As in [56, 57] we shall assume that all the fields carry tangent space indices so that



**Figure 4.** A subleading contribution to the amplitude with two soft gravitons.



**Figure 5.** A subleading contribution to the amplitude with two soft gravitons.

covariantization corresponds to replacing ordinary derivatives by covariant derivatives and then converting the tensor indices arising from derivatives to tangent space indices by contraction with inverse vielbeins. For simplicity we shall choose a gauge in which the metric always has determinant  $-1$  so that we do not need to worry about the multiplicative factor of  $\sqrt{-\det g}$  while covariantizing the action. This is done by parametrizing the metric as

$$g_{\mu\nu} = (e^{2S\eta})_{\mu\nu} = \eta_{\mu\nu} + 2S_{\mu\nu} + 2S_{\mu\rho}S^\rho_\nu + \dots, \quad S_{\mu\nu} = S_{\nu\mu}, \quad S_\mu^\mu = 0, \quad (2.3)$$

where all indices are raised and lowered by the flat metric  $\eta$ . We also introduce the vielbein fields

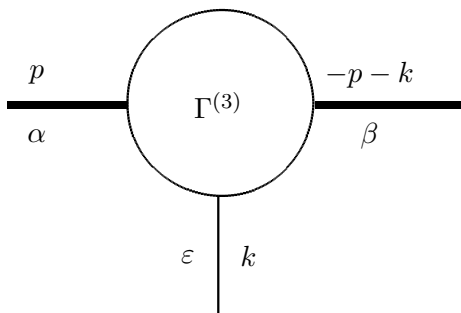
$$e_\mu^a = (e^{S\eta})_\mu^a = \delta_\mu^a + S_\mu^a + \frac{1}{2}S_\mu^b S_b^a + \dots, \quad E_a^\mu = (e^{-S\eta})_a^\mu = \delta_a^\mu - S_a^\mu + \frac{1}{2}S_a^b S_b^\mu + \dots. \quad (2.4)$$

Covariantization of the action now involves the following step. Let  $\{\phi_\alpha\}$  denote the set of all the fields of the theory. We replace a chain of ordinary derivatives  $\partial_{a_1} \dots \partial_{a_n}$  acting on a field  $\phi_\alpha$  by

$$E_{a_1}^{\mu_1} \dots E_{a_n}^{\mu_n} D_{\mu_1} \dots D_{\mu_n} \quad (2.5)$$

where

$$D_\mu \phi_\alpha = \partial_\mu \phi_\alpha + \frac{1}{2} \omega_\mu^{ab} (J_{ab})_\alpha^\beta \phi_\beta, \quad (2.6)$$



**Figure 6.** A 1PI vertex involving two finite energy particles and one soft particle.

with  $(J_{ab})_{\alpha}^{\beta}$  representing the action of spin angular momentum generator on all the fields, normalized so that acting on a covariant vector field  $\phi_c$ , we have

$$(J^{ab})_c^d = \delta_c^a \eta^{bd} - \delta_c^b \eta^{ad}. \quad (2.7)$$

For our analysis we shall only need the expression for  $\omega_{\mu}^{ab}$  to first order in  $S_{\mu\nu}$ . This is given by<sup>1</sup>

$$\omega_{\mu}^{ab} = \partial^b S_{\mu}^a - \partial^a S_{\mu}^b. \quad (2.8)$$

For each pair of covariant derivatives acting on the field  $\phi_{\alpha}$ , we also have a contribution from the Christoffel symbol

$$D_{\mu} D_{\nu} \phi_{\alpha} = \dots - \left\{ \begin{matrix} \rho \\ \mu \nu \end{matrix} \right\} D_{\rho} \phi_{\alpha}, \quad (2.9)$$

where

$$\left\{ \begin{matrix} \rho \\ \mu \nu \end{matrix} \right\} = \partial_{\mu} S_{\nu}^{\rho} + \partial_{\nu} S_{\mu}^{\rho} - \partial^{\rho} S_{\mu\nu} + \text{terms involving quadratic and higher powers of } S, \quad (2.10)$$

and  $\dots$  terms represent the usual derivatives and spin connection term. Since we shall compute subleading soft graviton amplitudes we shall only keep terms up to first order in the derivatives of soft gravitons. Also for amplitudes with two soft gravitons we only need to keep up to terms with two powers of soft graviton field  $S_{\mu\nu}$ . As we shall see, for specific vertices we can make further truncation of the action.

Let us now derive the form of the three point vertex involving one soft graviton and two finite energy fields, as shown in figure 6. For this we first express the quadratic part of the 1PI action as

$$\frac{1}{2} \int \frac{d^D q_1}{(2\pi)^D} \frac{d^D q_2}{(2\pi)^D} (2\pi)^D \delta^{(D)}(q_1 + q_2) \phi_{\alpha}(q_1) \mathcal{K}^{\alpha\beta}(q_2) \phi_{\beta}(q_2), \quad (2.11)$$

where we take

$$\mathcal{K}^{\alpha\beta}(q) = \mathcal{K}^{\beta\alpha}(-q). \quad (2.12)$$

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<sup>1</sup>Terms involving higher powers of  $S$  will give rise to vertices that have two or more soft gravitons, and a power of soft momentum. Such vertices will not contribute to the amplitude to subleading order in soft momentum.

For grassmann odd fields there will be an extra minus sign in this equation, but it does not affect the final results. If the soft graviton carries polarization  $\varepsilon$  and momentum  $k$ , then the coupling of single soft graviton to the fields  $\phi_\alpha$ , obtained by covariantizing (2.11), takes the form [57]

$$\begin{aligned}
 S^{(3)} &= \frac{1}{2} \int \frac{d^D q_1}{(2\pi)^D} \frac{d^D q_2}{(2\pi)^D} (2\pi)^D \delta^{(D)}(q_1 + q_2 + k) \\
 &\times \Phi_\alpha(q_1) \left[ -\varepsilon_{\mu\nu} q_2^\nu \frac{\partial}{\partial q_{2\mu}} \mathcal{K}^{\alpha\beta}(q_2) + \frac{1}{2} (k_b \varepsilon_{a\mu} - k_a \varepsilon_{b\mu}) \frac{\partial}{\partial q_{2\mu}} \mathcal{K}^{\alpha\gamma}(q_2) (J^{ab})_\gamma^\beta \right. \\
 &\left. - \frac{1}{2} \frac{\partial^2 \mathcal{K}^{\alpha\beta}(q_2)}{\partial q_{2\mu} \partial q_{2\nu}} q_{2\rho} (k_\mu \varepsilon_\nu^\rho + k_\nu \varepsilon_\mu^\rho - k^\rho \varepsilon_{\mu\nu}) \right] \Phi_\beta(q_2). \tag{2.13}
 \end{aligned}$$

In this equation the first term inside the square bracket represents the effect of multiplication by  $E_a^\mu = \delta_a^\mu - S_a^\mu$  in (2.5). The second term is the effect of the spin connection (2.8) appearing in the definition of the covariant derivative in (2.6) and the third term is the effect of the Christoffel symbol appearing in (2.9). From this we can derive an expression for the soft graviton vertex shown in figure 6 to order  $k$ :

$$\begin{aligned}
 \Gamma^{(3)\alpha\beta}(\varepsilon, k; p, -p - k) &= \frac{i}{2} \left[ -\varepsilon_{\mu\nu} (p + k)^\nu \frac{\partial}{\partial p_\mu} \mathcal{K}^{\alpha\beta}(-p - k) - \varepsilon_{\mu\nu} p^\nu \frac{\partial}{\partial p_\mu} \mathcal{K}^{\beta\alpha}(p) \right. \\
 &+ \frac{1}{2} (k_a \varepsilon_{b\mu} - k_b \varepsilon_{a\mu}) \frac{\partial}{\partial p_\mu} \mathcal{K}^{\alpha\gamma}(-p - k) (J^{ab})_\gamma^\beta - \frac{1}{2} (k_a \varepsilon_{b\mu} - k_b \varepsilon_{a\mu}) \frac{\partial}{\partial p_\mu} \mathcal{K}^{\beta\gamma}(p) (J^{ab})_\gamma^\alpha \\
 &- \frac{1}{2} \frac{\partial^2 \mathcal{K}^{\alpha\beta}(-p - k)}{\partial p_\mu \partial p_\nu} (-p_\rho - k_\rho) (k_\mu \varepsilon_\nu^\rho + k_\nu \varepsilon_\mu^\rho - k^\rho \varepsilon_{\mu\nu}) \\
 &\left. - \frac{1}{2} \frac{\partial^2 \mathcal{K}^{\beta\alpha}(p)}{\partial p_\mu \partial p_\nu} p_\rho (k_\mu \varepsilon_\nu^\rho + k_\nu \varepsilon_\mu^\rho - k^\rho \varepsilon_{\mu\nu}) \right]. \tag{2.14}
 \end{aligned}$$

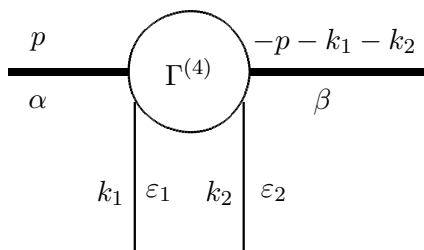
Using (2.12), (2.1), and expanding each term in Taylor series in the soft momentum  $k$ , we arrive at the following expression for the vertex  $\Gamma^{(3)}$  in figure 6 to order  $k$ :

$$\begin{aligned}
 \Gamma^{(3)}(\varepsilon, k; p, p - k) &= i \left[ -\varepsilon_{\mu\nu} p^\nu \frac{\partial \mathcal{K}(-p)}{\partial p_\mu} - \frac{1}{2} \varepsilon_{\mu\nu} p^\nu k_\rho \frac{\partial^2 \mathcal{K}(-p)}{\partial p_\mu \partial p_\rho} + \frac{1}{2} k_a \varepsilon_{b\mu} \frac{\partial \mathcal{K}(-p)}{\partial p_\mu} J^{ab} - \frac{1}{2} k_a \varepsilon_{b\mu} (J^{ab})^T \frac{\partial \mathcal{K}(-p)}{\partial p_\mu} \right], \tag{2.15}
 \end{aligned}$$

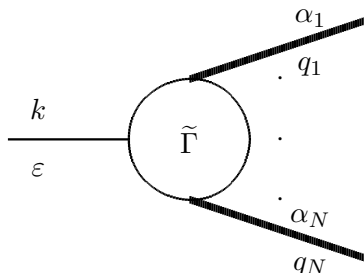
where we have used a matrix notation and  $(J^{ab})^T$  denotes the transpose of  $J^{ab}$ , i.e.  $((J^{ab})^T)_\gamma^\alpha = (J^{ab})_\gamma^\alpha$ .

Next we consider the four point vertex containing two soft gravitons and two finite energy particles as shown in figure 7. Since this vertex appears in figure 4 which begins contributing at the subleading order, we need to evaluate this to leading power in the soft momentum. Therefore we can ignore the spin connection and Christoffel symbol terms in the expression for the covariant derivatives appearing in (2.5), and only focus on the contribution from the  $E_a^\mu$  terms. Since we have two soft gravitons, we need to keep terms





**Figure 7.** A 1PI vertex involving two finite energy particles and two soft particles.



**Figure 8.** An amputated amplitude with one external soft particle and many external finite energy particles. We exclude from this any diagram where the soft particle gets attached to one of the external lines.

quadratic in the soft graviton field  $S_{\mu\nu}$ . These can come from two sources — either one power of  $S$  from two  $E_a^\mu$ 's or two powers of  $S$  from a single  $E_a^\mu$ . The resulting action is given by

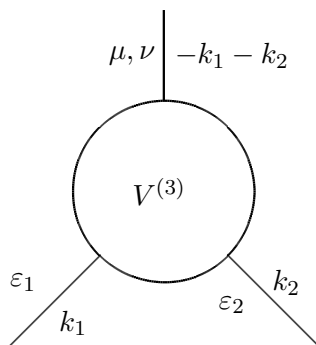
$$\frac{1}{2} \int \frac{d^D q_1}{(2\pi)^D} \frac{d^D q_2}{(2\pi)^D} \frac{d^D \ell_1}{(2\pi)^D} \frac{d^D \ell_2}{(2\pi)^D} (2\pi)^D \delta^{(D)}(q_1 + q_2 + \ell_1 + \ell_2) \Phi_\alpha(q_1) \Phi_\beta(q_2) \times \left[ \frac{1}{2} S_{\mu\nu}(\ell_1) S_{\rho\sigma}(\ell_2) q_2^\nu q_2^\sigma \frac{\partial^2 \mathcal{K}^{\alpha\beta}(q_2)}{\partial q_{2\mu} \partial q_{2\rho}} + \frac{1}{2} S_\mu{}^b S_{b\nu} q_2^\nu \frac{\partial \mathcal{K}^{\alpha\beta}(q_2)}{\partial q_{2\mu}} \right]. \quad (2.16)$$

Using this and the symmetry (2.12), we get the following form of the vertex shown in figure 7 to leading order in soft momenta, written in the matrix notation:

$$\Gamma^{(4)}(\varepsilon_1, k_1, \varepsilon_2, k_2; p, -p - k_1 - k_2) = i \left[ \varepsilon_{1,\mu\nu} \varepsilon_{2,\rho\sigma} p^\nu p^\sigma \frac{\partial^2 \mathcal{K}(-p)}{\partial p_\mu \partial p_\rho} + \frac{1}{2} \left( \varepsilon_{1,\mu}{}^b \varepsilon_{2,b\nu} + \varepsilon_{2,\mu}{}^b \varepsilon_{1,b\nu} \right) p^\nu \frac{\partial \mathcal{K}(-p)}{\partial p_\mu} \right]. \quad (2.17)$$

Next let us consider the contribution from the amplitude in figure 8 for off-shell external momenta  $q_1, \dots, q_N$ . This can be obtained by covariantizing the truncated Green's function  $\Gamma^{\alpha_1 \dots \alpha_N}(q_1, \dots, q_N)$  without the soft graviton. Since this amplitude appears inside figure 3 which begins contributing at the subleading order, we only need the leading contribution from this amplitude. This is easily computed using the covariantization procedure, giving the result [56]

$$\tilde{\Gamma}^{\alpha_1 \dots \alpha_N}(\varepsilon, k; q_1, \dots, q_N) = - \sum_{i=1}^N \varepsilon_{\mu\nu} q_i^\mu \frac{\partial}{\partial q_{i\nu}} \Gamma^{\alpha_1 \dots \alpha_N}(q_1, \dots, q_N), \quad (2.18)$$



**Figure 9.** A 1PI vertex involving three soft gravitons.

reflecting the effect of having to multiply every factor of momentum (derivative with respect to space-time coordinates) by inverse vielbeins as in (2.5).

The next vertex to be evaluated is the three point vertex of three soft gravitons as shown in figure 9, involving external on-shell soft gravitons carrying momenta  $k_1, k_2$  and polarizations  $\varepsilon_1, \varepsilon_2$  respectively and internal soft graviton carrying momenta  $-k_1 - k_2$  and polarization labelled by the pair of indices  $(\mu, \nu)$ . This vertex appears in figure 5 which begins contributing at the subleading order. Therefore we need to evaluate this vertex to leading order in soft momenta — given by the Einstein-Hilbert action. This is best done by regarding the external soft gravitons as background field  $S_{\mu\nu}$  so that the vertex can be regarded as the one point function of the internal graviton in the presence of soft graviton background. This is proportional to  $R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$  computed from the soft graviton metric. Evaluating this to quadratic order in  $S_{\rho\sigma}$  we can read out the vertex. Using standard results on the expansion of connection and curvature in powers of fluctuations in the metric (see e.g. [65, 66]) we find that the vertex takes the form:

$$\begin{aligned}
 &V_{\mu\nu}^{(3)}(\varepsilon_1, k_1, \varepsilon_2, k_2) \\
 &= \frac{i}{2} \varepsilon_{1,ab} \varepsilon_{2,cd} \left[ \left\{ \eta_{\mu\nu} \eta^{ac} \eta^{bd} k_1^\rho k_{2\rho} - 2\eta^{ad} \eta^c{}_\nu k_2^b k_{2\mu} - 2\eta^{cb} \eta^a{}_\nu k_1^d k_{1\mu} + 2\eta^{ad} \eta^c{}_\nu k_{1\mu} k_2^b \right. \right. \\
 &\quad \left. \left. + 2\eta^{cb} \eta^a{}_\mu k_1^d k_{2\nu} - 2\eta^{ac} \eta^{bd} k_{1\mu} k_{2\nu} - 4\eta^a{}_\nu \eta^c{}_\mu k_1^d k_2^b + 2\eta^c{}_\mu \eta^d{}_\nu k_2^b k_2^a + 2\eta^a{}_\mu \eta^b{}_\nu k_1^d k_1^c \right\} \right. \\
 &\quad \left. + \{ \mu \leftrightarrow \nu \} \right]. \tag{2.19}
 \end{aligned}$$

We now turn to the computation of the propagators. In the normalization in which the three point vertex of figure 9 is given by (2.19), the soft graviton propagator in the de Donder gauge takes the form:

$$G_{\mu\nu,\rho\sigma}(k) = -\frac{1}{2} \left( \eta_{\mu\rho} \eta_{\nu\sigma} + \eta_{\nu\rho} \eta_{\mu\sigma} - \frac{2}{d-2} \eta_{\mu\nu} \eta_{\rho\sigma} \right) \frac{i}{k^2}, \tag{2.20}$$

where  $\mu, \nu$  are the indices carried by one of the gravitons and  $\rho, \sigma$  are the indices carried by the other graviton.

The final ingredient is the propagator for an internal finite energy line carrying momentum  $q$ . This is given by  $i\mathcal{K}_{\alpha\beta}^{-1}(q)$ . We define

$$\Xi_{\alpha\beta}^j(q) = i\mathcal{K}_{\alpha\beta}^{-1}(q)(q^2 + M_j^2), \quad (2.21)$$

where  $M_j$  is the mass of the  $j$ -th external state. Then the propagator can be expressed as

$$\Delta(q) = (q^2 + M_j^2)^{-1} \Xi^j(q), \quad (2.22)$$

where we have adopted the matrix notation dropping the indices  $\alpha, \beta$ .

Now from (2.21) we have

$$\mathcal{K}(q) \Xi^i(q) = i(q^2 + M_i^2). \quad (2.23)$$

Taking derivatives of this with respect to momenta we arrive at the following relations:

$$\begin{aligned} \frac{\partial \mathcal{K}(-p)}{\partial p_\mu} \Xi^i(-p) &= -\mathcal{K}(-p) \frac{\partial \Xi^i(-p)}{\partial p_\mu} + 2i p^\mu, \\ \frac{\partial^2 \mathcal{K}(-p)}{\partial p_\mu \partial p_\nu} \Xi^i(-p) &= -\frac{\partial \mathcal{K}(-p)}{\partial p_\mu} \frac{\partial \Xi^i(-p)}{\partial p_\nu} - \frac{\partial \mathcal{K}(-p)}{\partial p_\nu} \frac{\partial \Xi^i(-p)}{\partial p_\mu} - \mathcal{K}(-p) \frac{\partial^2 \Xi^i(-p)}{\partial p_\mu \partial p_\nu} + 2i \eta^{\mu\nu}, \end{aligned} \quad (2.24)$$

Finally rotational invariance of  $\mathcal{K}$  implies the following relations:

$$\begin{aligned} (J^{ab})^T \mathcal{K}(-p) &= -\mathcal{K}(-p) J^{ab} + p^a \frac{\partial \mathcal{K}(-p)}{\partial p_b} - p^b \frac{\partial \mathcal{K}(-p)}{\partial p_a}, \\ J^{ab} \Xi^i(-p) &= -\Xi^i(-p) (J^{ab})^T - p^a \frac{\partial \Xi^i(-p)}{\partial p_b} + p^b \frac{\partial \Xi^i(-p)}{\partial p_a}. \end{aligned} \quad (2.25)$$

## 2.2 Evaluation of the diagrams

We begin with the evaluation of figure 1. Even though we can use the form (2.22) for the internal propagator for any  $j$ , (2.22) being independent of  $j$  due to (2.21), we shall use the form (2.22) with  $j = i$  when the soft gravitons attach to the  $i$ -th external line. In this case the propagator carrying momentum  $-p_i - k$  for some soft momentum  $k$  takes the form

$$\Delta(-p_i - k) = \{(p_i + k)^2 + M_i^2\}^{-1} \Xi^i(-p_i - k) = (2p_i \cdot k + k^2)^{-1} \Xi^i(-p_i - k). \quad (2.26)$$

We now define

$$\Gamma_{(i)}^{\alpha_i}(p_i) = \left\{ \prod_{\substack{j=1 \\ j \neq i}}^N \epsilon_{j, \alpha_j} \right\} \Gamma^{\alpha_1 \dots \alpha_N}(p_1, \dots, p_N), \quad (2.27)$$

with the understanding that  $\Gamma_{(i)}^{\alpha_i}(p_i)$  also implicitly depends on the  $p_j$ 's and  $\epsilon_j$ 's for  $j \neq i$ . Using this we can express the contribution from figure 1 as

$$\begin{aligned} A_1 &\equiv \sum_{i=1}^N (2p_i \cdot k_1)^{-1} (2p_i \cdot (k_1 + k_2) + 2k_1 \cdot k_2)^{-1} \epsilon_i^T \Gamma^{(3)}(\varepsilon_1, k_1; p_i, -p_i - k_1) \Xi^i(-p_i - k_1) \\ &\quad \Gamma^{(3)}(\varepsilon_2, k_2; p_i + k_1, -p_i - k_1 - k_2) \Xi^i(-p_i - k_1 - k_2) \Gamma_{(i)}(p_i + k_1 + k_2), \end{aligned} \quad (2.28)$$

where we have summed over soft graviton insertion on different external legs. We now use the expression (2.15) for  $\Gamma^{(3)}$  and manipulate this expression as follows:

1. Take all the  $J^{ab}$  factors to the extreme right using (2.25) and their derivatives with respect to  $p^\mu$ .
2. Expand  $\mathcal{K}$ ,  $\Xi^i$  and  $\Gamma_{(i)}$  in Taylor series expansion in  $k_1$ ,  $k_2$ , and keep up to the first subleading terms in soft momenta.
3. Use the relations (2.24) to move all momentum derivatives to the extreme right to the extent possible.
4. Finally use the on-shell condition

$$\epsilon_i^T \mathcal{K}(-p) = 0, \quad (2.29)$$

to set all terms in which the left-most  $\mathcal{K}$  does not have a derivative acting on it to zero.

While these steps are sufficient to arrive at the final result given in (2.33), for the analysis of section 4 we shall need some of the results that appear in the intermediate stages. For example, Taylor series expansion in  $k$ , together with the use of (2.24), (2.25) leads to the result

$$\Gamma^{(3)}(\varepsilon, k; p, -p-k) \Xi^i(-p-k) = \left[ 2 \varepsilon^{\mu\nu} p_\mu p_\nu + i \varepsilon_{\mu\nu} p^\nu \mathcal{K}(-p) \frac{\partial \Xi^i(-p)}{\partial p_\mu} + 2 \varepsilon_{b\mu} k_a p^\mu (J^{ab})^T + \mathcal{K}(-p) \mathcal{Q}(p, k) \right] \quad (2.30)$$

to subleading order. Here

$$\mathcal{Q}(p, k) \equiv \frac{i}{2} k \cdot p \varepsilon_{b\mu} \frac{\partial^2 \Xi(-p)}{\partial p_\mu \partial p_b} + i \varepsilon_{b\mu} k_a \frac{\partial \Xi(-p)}{\partial p_\mu} (J^{ab})^T, \quad (2.31)$$

denotes a term that receives contribution from subleading order in soft momentum. We shall see that its contribution to the amplitude vanishes due to (2.29). Using (2.30) we can express the amplitude (2.28) as

$$\begin{aligned} A_1 = & \sum_{i=1}^N (2p_i \cdot k_1)^{-1} (2p_i \cdot (k_1 + k_2) + 2k_1 \cdot k_2)^{-1} \\ & \epsilon_i^T \left[ 2 \varepsilon_1^{\mu\nu} p_{i\mu} p_{i\nu} + i \varepsilon_{1,\mu\nu} p_i^\nu \mathcal{K}(-p_i) \frac{\partial \Xi^i(-p_i)}{\partial p_{i\mu}} + 2 \varepsilon_{1,b\mu} k_{1a} p_i^\mu (J^{ab})^T + \mathcal{K}(-p_i) \mathcal{Q}(p_i, k_1) \right] \\ & \left[ 2 \varepsilon_2^{\rho\sigma} (p_{i\rho} + k_{1\rho})(p_{i\sigma} + k_{1\sigma}) + i \varepsilon_{2,\rho\sigma} (p_i^\sigma + k_1^\sigma) \mathcal{K}(-p_i - k_1) \frac{\partial \Xi^i(-p_i - k_1)}{\partial p_{i\rho}} \right. \\ & \left. + 2 \varepsilon_{2,d\rho} k_{2c} (p_i^\rho + k_1^\rho) (J^{cd})^T + \mathcal{K}(-p_i) \mathcal{Q}(p_i, k_2) \right] \Gamma_{(i)}(p_i + k_1 + k_2), \quad (2.32) \end{aligned}$$

to first subleading order. Expanding the terms inside the second square bracket and  $\Gamma_{(i)}$  in a Taylor series expansion in  $k_1$  and  $k_2$ , and using (2.29), (2.25), we get, up to subleading order,

$$\begin{aligned}
 A_1 = & \sum_{i=1}^N (p_i \cdot k_1)^{-1} \{p_i \cdot (k_1 + k_2) + k_1 \cdot k_2\}^{-1} \epsilon_i^T \left[ \epsilon_1^{\sigma\tau} p_{i\sigma} p_{i\tau} \varepsilon_{2\mu\nu} p_i^\nu p_i^\mu \Gamma_{(i)}(p_i) \right. \\
 & + 2\varepsilon_1^{\sigma\tau} p_{i\sigma} p_{i\tau} \varepsilon_{2,\mu\nu} k_1^\nu p_i^\mu \Gamma_{(i)}(p_i) + \varepsilon_1^{\sigma\tau} p_{i\sigma} p_{i\tau} k_{2a} \varepsilon_{2,b\mu} p_i^\mu (J^{ab})^T \Gamma_{(i)}(p_i) \\
 & + \varepsilon_{1,b\sigma} k_{1a} p_i^\sigma \varepsilon_{2,\mu\nu} p_i^\nu p_i^\mu (J^{ab})^T \Gamma_{(i)}(p_i) + \varepsilon_1^{\sigma\tau} p_{i\sigma} p_{i\tau} \varepsilon_{2,\mu\nu} p_i^\nu p_i^\mu (k_1 + k_2)_\rho \frac{\partial \Gamma_{(i)}(p_i)}{\partial p_{i\rho}} \\
 & \left. + \frac{1}{2} i (k_1 \cdot p_i) \varepsilon_{1,\mu\sigma} \varepsilon_{2,\rho\nu} p_i^\sigma p_i^\nu \frac{\partial \mathcal{K}(-p_i)}{\partial p_{i\mu}} \frac{\partial \Xi^i(-p_i)}{\partial p_{i\rho}} \Gamma_{(i)}(p_i) \right]. \quad (2.33)
 \end{aligned}$$

To this, we also need to add an expression in which we interchange  $(k_1, \varepsilon_1) \leftrightarrow (k_2, \varepsilon_2)$ . This gives the amplitude

$$\begin{aligned}
 A'_1 = & \sum_{i=1}^N (p_i \cdot k_2)^{-1} (p_i \cdot (k_2 + k_1) + 2k_2 \cdot k_1)^{-1} \epsilon_i^T \left[ \varepsilon_2^{\sigma\tau} p_{i\sigma} p_{i\tau} \varepsilon_{1,\mu\nu} p_i^\nu p_i^\mu \Gamma_{(i)}(p_i) \right. \\
 & + 2\varepsilon_2^{\sigma\tau} p_{i\sigma} p_{i\tau} \varepsilon_{1,\mu\nu} k_2^\nu p_i^\mu \Gamma_{(i)}(p_i) + \varepsilon_2^{\sigma\tau} p_{i\sigma} p_{i\tau} k_{1a} \varepsilon_{1,b\mu} p_i^\mu (J^{ab})^T \Gamma_{(i)}(p_i) \\
 & + \varepsilon_{2,b\sigma} k_{2a} p_i^\sigma \varepsilon_{1,\mu\nu} p_i^\nu p_i^\mu (J^{ab})^T \Gamma_{(i)}(p_i) + \varepsilon_2^{\sigma\tau} p_{i\sigma} p_{i\tau} \varepsilon_{1,\mu\nu} p_i^\nu p_i^\mu (k_2 + k_1)_\rho \frac{\partial \Gamma_{(i)}(p_i)}{\partial p_{i\rho}} \\
 & \left. + \frac{1}{2} i (k_2 \cdot p_i) \varepsilon_{2,\mu\sigma} \varepsilon_{1,\rho\nu} p_i^\sigma p_i^\nu \frac{\partial \mathcal{K}(-p_i)}{\partial p_{i\mu}} \frac{\partial \Xi^i(-p_i)}{\partial p_{i\rho}} \Gamma_{(i)}(p_i) \right]. \quad (2.34)
 \end{aligned}$$

The contribution from figure 2 can be evaluated by knowing the result for single soft graviton insertion since the two parts of the diagram on which the two soft gravitons are inserted can be evaluated independently. We shall express this as

$$\begin{aligned}
 & (2p_i \cdot k_1)^{-1} (2p_j \cdot k_2)^{-1} \{ \varepsilon_i^T \Gamma^{(3)}(\varepsilon_1, k_1; p_i, -p_i - k_1) \Xi^i(-p_i - k_1) \} \\
 & \otimes \{ \varepsilon_j^T \Gamma^{(3)}(\varepsilon_2, k_2; p_j, -p_j - k_2) \Xi^j(-p_j - k_2) \} \Gamma_{(i,j)}(p_i + k_1, p_j + k_2), \quad (2.35)
 \end{aligned}$$

where  $\Gamma_{(i,j)}^{\alpha_i \alpha_j}$  is defined in the same way as  $\Gamma_{(i)}$  except that we now strip off both the polarization tensors of the  $i$ -th and the  $j$ -th leg:

$$\Gamma_{(i,j)}^{\alpha_i \alpha_j}(p_i, p_j) \equiv \left\{ \prod_{\substack{\ell=1 \\ \ell \neq i, j}}^N \epsilon_{\ell, \alpha_\ell} \right\} \Gamma^{\alpha_1 \dots \alpha_N}(p_1, \dots, p_N). \quad (2.36)$$

It is understood that in (2.35) the terms inside the first curly bracket contracts with the first index  $\alpha_i$  of  $\Gamma_{(i,j)}$  and the terms inside the second bracket contracts with the second index  $\alpha_j$  of  $\Gamma_{(i,j)}$ . By manipulating the matrices acting on the  $i$ -th and the  $j$ -th leg independently in the same way as before, using the results

$$\epsilon_{i,\alpha} \Gamma_{(i,j)}^{\alpha\beta}(p_i, p_j) = \Gamma_{(j)}^\beta(p_j), \quad \epsilon_{j,\beta} \Gamma_{(i,j)}^{\alpha\beta}(p_i, p_j) = \Gamma_{(i)}^\alpha(p_i), \quad (2.37)$$

and summing over insertions on all external legs, we arrive at the following result for the amplitude up to first subleading order:

$$\begin{aligned}
 A_2 = & \sum_{\substack{i,j=1 \\ i \neq j}}^N (p_i \cdot k_1)^{-1} (p_j \cdot k_2)^{-1} \varepsilon_{1,\mu\nu} p_i^\mu p_i^\nu \varepsilon_{2,\rho\sigma} p_j^\rho p_j^\sigma \Gamma(\varepsilon_1, k_1, \varepsilon_2, k_2; \varepsilon_1, p_1, \dots, \varepsilon_N, p_N) \quad (2.38) \\
 & + \sum_{\substack{i,j=1 \\ i \neq j}}^N (p_i \cdot k_1)^{-1} (p_j \cdot k_2)^{-1} \varepsilon_{2,\rho\sigma} p_j^\rho p_j^\sigma \varepsilon_i^T \left[ \varepsilon_{1,\mu\nu} p_i^\mu p_i^\nu k_{1\tau} \frac{\partial \Gamma_{(i)}(p_i)}{\partial p_{i\tau}} + k_{1a} \varepsilon_{1,b\mu} p_i^\mu (J^{ab})^T \Gamma_{(i)}(p_i) \right] \\
 & + \sum_{\substack{i,j=1 \\ i \neq j}}^N (p_i \cdot k_2)^{-1} (p_j \cdot k_1)^{-1} \varepsilon_{1,\rho\sigma} p_j^\rho p_j^\sigma \varepsilon_i^T \left[ \varepsilon_{2,\mu\nu} p_i^\mu p_i^\nu k_{2\tau} \frac{\partial \Gamma_{(i)}(p_i)}{\partial p_{i\tau}} + k_{2a} \varepsilon_{2,b\mu} p_i^\mu (J^{ab})^T \Gamma_{(i)}(p_i) \right].
 \end{aligned}$$

Next we consider the contribution from figure 3. The contribution from this term has at most one pole in the soft momentum and therefore begins at subleading order. Therefore we only need the leading contribution from this diagram. For this we use the result (2.18) for the off-shell amplitude shown in figure 8. This gives the following expression for the contribution from figure 3:

$$A_3 = - \sum_{i=1}^N (2p_i \cdot k_1)^{-1} \varepsilon_i^T \Gamma^{(3)}(\varepsilon_1, k_1; p_i, -p_i - k_1) \Xi^i(-p_1 - k_1) \sum_{j=1}^N \varepsilon_j^T \varepsilon_2^{\mu\nu} p_{j\mu} \frac{\partial}{\partial p_j^\nu} \Gamma_{(i,j)}(p_i, p_j), \quad (2.39)$$

where again we have summed over the insertion of the first soft graviton on all external finite energy states. We can now manipulate this using the form of  $\Gamma^{(3)}$  given earlier. This leads to

$$A_3 = - \sum_{i=1}^N (p_i \cdot k_1)^{-1} \varepsilon_1^{\rho\sigma} p_{i\rho} p_{i\sigma} \sum_{j=1}^N \varepsilon_j^T \varepsilon_2^{\mu\nu} p_{j\mu} \frac{\partial}{\partial p_j^\nu} \Gamma_{(j)}. \quad (2.40)$$

The diagram obtained by interchanging  $(k_1, \varepsilon_1) \leftrightarrow (k_2, \varepsilon_2)$  gives

$$A'_3 = - \sum_{i=1}^N (p_i \cdot k_2)^{-1} \varepsilon_2^{\rho\sigma} p_{i\rho} p_{i\sigma} \sum_{j=1}^N \varepsilon_j^T \varepsilon_1^{\mu\nu} p_{j\mu} \frac{\partial}{\partial p_j^\nu} \Gamma_{(j)}. \quad (2.41)$$

Figure 4 also begins contributing at the subleading order. Therefore we only need its leading contribution, which is given by

$$A_4 = \sum_{i=1}^N \{2p_i \cdot (k_1 + k_2)\}^{-1} \varepsilon_i^T \Gamma^{(4)}(\varepsilon_1, k_1, \varepsilon_2, k_2; p_i, -p_i - k_1 - k_2) \Xi^i(-p_i - k_1 - k_2) \Gamma_{(i)}(p_i). \quad (2.42)$$

This can be evaluated using the expression (2.17) for the vertex  $\Gamma^{(4)}$  shown in figure 7 and manipulating the resulting expression in the same way as the previous diagrams. The result is

$$\begin{aligned}
 A_4 = & \sum_{i=1}^N \{p_i \cdot (k_1 + k_2)\}^{-1} \varepsilon_i^T \left[ -2\varepsilon_{1,\mu}{}^\nu \varepsilon_{2,\nu\rho} p_i^\rho p_i^\mu \right. \\
 & \left. - \frac{i}{2} \left( \varepsilon_{1,\mu\sigma} \varepsilon_{2,\rho\nu} p_i^\sigma p_i^\nu + \varepsilon_{1,\rho\sigma} \varepsilon_{2,\mu\nu} p_i^\sigma p_i^\nu \right) \frac{\partial \mathcal{K}(-p_i)}{\partial p_{i\mu}} \frac{\partial \Xi^i(-p_i)}{\partial p_{i\rho}} \right] \Gamma_{(i)}(p_i). \quad (2.43)
 \end{aligned}$$

Finally we turn to the computation of the diagram shown in figure 5. Its contribution is given by

$$A_5 = V^{(3)\mu\nu}(\varepsilon_1, k_1, \varepsilon_2, k_2) G_{\mu\nu, \rho\sigma}(k_1 + k_2) \sum_{i=1}^N \epsilon_i^T \Gamma^{(3)(\rho\sigma)}(k_1 + k_2; p_i, -p_i - k_1 - k_2) \Gamma_{(i)}(p_i), \quad (2.44)$$

where  $V^{(3)}$  and  $G_{\mu\nu, \rho\sigma}$  have been defined in (2.19) and (2.20) respectively, and  $\Gamma^{(3)(\rho\sigma)}$  is defined via the equation

$$\Gamma^{(3)}(\varepsilon, k; p, -p - k) = \varepsilon_{\rho\sigma} \Gamma^{(3)(\rho\sigma)}(k; p, -p - k). \quad (2.45)$$

Using the leading order expression for  $\Gamma^{(3)}$  given in (2.15), and the relations (2.24), (2.25), (2.29) this can be brought to the form

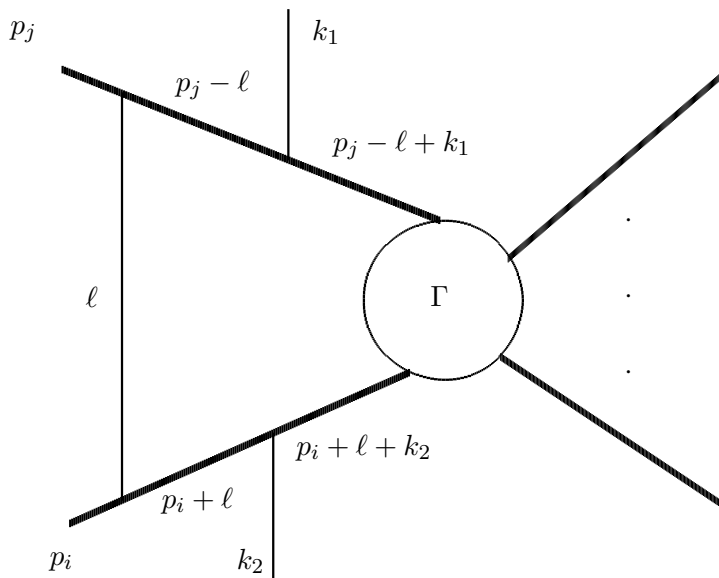
$$\begin{aligned} A_5 = & \sum_{i=1}^N \{p_i \cdot (k_1 + k_2)\}^{-1} (k_1 \cdot k_2)^{-1} \epsilon_i^T \left[ -(k_2 \cdot \varepsilon_1 \cdot \varepsilon_2 \cdot p_i)(k_2 \cdot p_i) - (k_1 \cdot \varepsilon_2 \cdot \varepsilon_1 \cdot p_i)(k_1 \cdot p_i) \right. \\ & + (k_2 \cdot \varepsilon_1 \cdot \varepsilon_2 \cdot p_i)(k_1 \cdot p_i) + (k_1 \cdot \varepsilon_2 \cdot \varepsilon_1 \cdot p_i)(k_2 \cdot p_i) - \varepsilon_1^{cd} \varepsilon_{2, cd} (k_1 \cdot p_i)(k_2 \cdot p_i) \\ & \left. - 2(p_i \cdot \varepsilon_1 \cdot k_2)(p_i \cdot \varepsilon_2 \cdot k_1) + (p_i \cdot \varepsilon_2 \cdot p_i)(k_2 \cdot \varepsilon_1 \cdot k_2) + (p_i \cdot \varepsilon_1 \cdot p_i)(k_1 \cdot \varepsilon_2 \cdot k_1) \right] \Gamma_{(i)}(p_i) \quad (2.46) \end{aligned}$$

The full amplitude is given by

$$\begin{aligned} A = & A_1 + A'_1 + A_2 + A_3 + A'_3 + A_4 + A_5 \\ = & \left\{ \sum_{i=1}^N (p_i \cdot k_1)^{-1} \varepsilon_{1, \mu\nu} p_i^\mu p_i^\nu \right\} \left\{ \sum_{j=1}^N (p_j \cdot k_2)^{-1} \varepsilon_{2, \rho\sigma} p_j^\rho p_j^\sigma \right\} \Gamma(\varepsilon_1, p_1, \dots, \varepsilon_N, p_N) \\ & + \left\{ \sum_{j=1}^N (p_j \cdot k_2)^{-1} \varepsilon_{2, \rho\sigma} p_j^\rho p_j^\sigma \right\} \sum_{i=1}^N (p_i \cdot k_1)^{-1} \varepsilon_{1, b\mu} k_{1a} p_i^\mu \epsilon_i^T \left[ p_i^b \frac{\partial \Gamma_{(i)}(p_i)}{\partial p_{ia}} - p_i^a \frac{\partial \Gamma_{(i)}(p_i)}{\partial p_{ib}} + (J^{ab})^T \Gamma_{(i)}(p_i) \right] \\ & + \left\{ \sum_{j=1}^N (p_j \cdot k_1)^{-1} \varepsilon_{1, \rho\sigma} p_j^\rho p_j^\sigma \right\} \sum_{i=1}^N (p_i \cdot k_2)^{-1} \varepsilon_{2, b\mu} k_{2a} p_i^\mu \epsilon_i^T \left[ p_i^b \frac{\partial \Gamma_{(i)}(p_i)}{\partial p_{ia}} - p_i^a \frac{\partial \Gamma_{(i)}(p_i)}{\partial p_{ib}} + (J^{ab})^T \Gamma_{(i)}(p_i) \right] \\ & + \left\{ \sum_{i=1}^N \{p_i \cdot (k_1 + k_2)\}^{-1} \mathcal{M}(p_i; \varepsilon_1, k_1, \varepsilon_2, k_2) \right\} \Gamma(\varepsilon_1, p_1, \dots, \varepsilon_N, p_N), \quad (2.47) \end{aligned}$$

where

$$\begin{aligned} \mathcal{M}(p_i; \varepsilon_1, k_1, \varepsilon_2, k_2) = & (p_i \cdot k_1)^{-1} (p_i \cdot k_2)^{-1} \left\{ -(k_1 \cdot k_2)(p_i \cdot \varepsilon_1 \cdot p_i)(p_i \cdot \varepsilon_2 \cdot p_i) \right. \\ & + 2(p_i \cdot k_2)(p_i \cdot \varepsilon_1 \cdot p_i)(p_i \cdot \varepsilon_2 \cdot k_1) + 2(p_i \cdot k_1)(p_i \cdot \varepsilon_2 \cdot p_i)(p_i \cdot \varepsilon_1 \cdot k_2) \\ & \left. - 2(p_i \cdot k_1)(p_i \cdot k_2)(p_i \cdot \varepsilon_1 \cdot \varepsilon_2 \cdot p_i) \right\} \\ & + (k_1 \cdot k_2)^{-1} \left\{ -(k_2 \cdot \varepsilon_1 \cdot \varepsilon_2 \cdot p_i)(k_2 \cdot p_i) - (k_1 \cdot \varepsilon_2 \cdot \varepsilon_1 \cdot p_i)(k_1 \cdot p_i) \right. \\ & + (k_2 \cdot \varepsilon_1 \cdot \varepsilon_2 \cdot p_i)(k_1 \cdot p_i) + (k_1 \cdot \varepsilon_2 \cdot \varepsilon_1 \cdot p_i)(k_2 \cdot p_i) - \varepsilon_1^{cd} \varepsilon_{2, cd} (k_1 \cdot p_i)(k_2 \cdot p_i) \\ & \left. - 2(p_i \cdot \varepsilon_1 \cdot k_2)(p_i \cdot \varepsilon_2 \cdot k_1) + (p_i \cdot \varepsilon_2 \cdot p_i)(k_2 \cdot \varepsilon_1 \cdot k_2) + (p_i \cdot \varepsilon_1 \cdot p_i)(k_1 \cdot \varepsilon_2 \cdot k_1) \right\}. \quad (2.48) \end{aligned}$$



**Figure 10.** A possible subleading contribution in five non-compact dimensions.

Here we have used the shorthand notation  $p_i \cdot \varepsilon_1 \cdot p_i \equiv \varepsilon_{1,\mu\nu} p_i^\mu p_i^\nu$  etc.  $\mathcal{M}$  receives contributions from the first two terms in (2.33) and (2.34) and also from (2.43) and (2.46).

### 2.3 Infrared issues

In our analysis we have assumed that possible soft factors in the denominator arise from propagators but not from the 1PI vertices. This holds when the number of non-compact space-time dimensions  $D$  is sufficiently high. However we shall now show that for  $D \leq 5$ , individual contributions violate this condition due to infrared effects in the loop. Let us consider for example the diagram shown in figure 10. In the 1PI effective field theory, this corresponds to a graph similar to one shown in figure 3, but with both soft gravitons connected to the vertex  $\tilde{\Gamma}$ . If there is no inverse power of soft momenta from  $\tilde{\Gamma}$  then this contribution is subsubleading and can be ignored. However let us consider the limit in which the loop momentum  $\ell$  in figure 10 becomes soft — of the same order as the external soft momenta. In this limit each of the propagators carrying momenta  $p_i + \ell$ ,  $p_i + \ell + k_2$ ,  $p_j - \ell$  and  $p_j - \ell + k_1$  gives one power of soft momentum in the denominator and the soft propagator carrying momentum  $\ell$  gives two powers of soft momentum in the denominator. On the other hand in  $D$  non-compact space-time dimensions the loop momentum integration measure goes as  $D$  powers of soft momentum. Therefore the net power of soft momentum that we get from this graph for soft  $\ell$  is  $D - 6$ , and in  $D = 5$  this integral can give a term containing one power of soft momentum in the denominator, giving a subleading contribution. Since we have not included these diagrams in our analysis we conclude that for loop amplitudes our result is valid for  $D \geq 6$ . It is easy to see by simple power counting that higher loop amplitudes do not lead to any additional enhancement from the infrared region of loop momenta.



Similar analysis can be carried out for multiple soft graviton amplitudes of the kind described in section 4. As we connect each external soft graviton to an internal nearly on-shell line carrying finite energy, the number of powers of soft momentum in the denominator goes up by one. However the required number of powers of soft momentum in the denominator of the subleading contribution also goes up by one. Therefore the result of section 4 continues to be valid for loop amplitudes for  $D \geq 6$ , irrespective of the number of external soft gravitons.

Even though this analysis shows that individual diagrams can give contributions beyond what we have included in our analysis for  $D \leq 5$ , we expect that for  $D = 5$  such contributions will cancel when we sum over all diagrams. This expectation arises out of standard results on factorization of soft loops [67, 68] that tells us that after summing over graphs, the contribution from the region of soft loop momentum takes the form of a product of an amplitude without soft loop and a soft factor that arises from graphs like figure 10 without the external soft lines. Since the graphs like figure 10 without soft external lines do not receive large contribution from the small  $\ell$  region, and are furthermore independent of the external soft momenta, their contribution may be absorbed into the definition of the amplitude without the soft gravitons. Therefore we conclude that the contribution from the loop momentum integration region for small  $\ell$  in graphs like figure 10 must cancel in the sum over graphs. Nevertheless since our general analysis relies on the analysis of individual contributions of different graphs of the type shown in figure 1–5, and since the coefficients of Taylor series expansion of these individual contributions as well as those not included in figure 1–5 (like figure 10) do receive large contribution from small loop momentum region, we cannot give a foolproof argument that our general result is not affected by infrared contributions of the type described above.

Note that similar infrared enhancement also occurs for amplitudes with single soft graviton, but by analyzing the tensor structure of these contributions it was argued in [57] that gauge invariance prevents corrections to the soft theorem from such effects to subleading order for  $D \geq 5$ . Similar argument has not been developed for multiple soft graviton amplitudes.

This problem of course does not arise for tree amplitudes where the vertices are always polynomial in momenta. Therefore for tree amplitudes our results hold in all dimensions.

### 3 Consistency checks

In this section we shall carry out various consistency checks on our result. First we shall check the internal consistency of our result. Then we shall compare our results with the previous results derived for specific theories.

#### 3.1 Internal consistency

The first internal consistency check of our result comes from the requirement of gauge invariance. This means that if we make the transformation

$$\varepsilon_{r,\mu\nu} \rightarrow k_{r\mu} \xi_{r\nu} + k_{r\nu} \xi_{r\mu}, \quad r = 1, 2, \tag{3.1}$$

for any vector  $\xi_r$  satisfying  $k_r \cdot \xi_r = 0$ , the result (2.47) does not change. Checking this involves tedious but straightforward algebra, and needs use of the equations

$$\sum_{i=1}^N p_{i\mu} \Gamma^{\alpha_1 \cdots \alpha_N} = 0, \quad (3.2)$$

and

$$\sum_{i=1}^N \left[ p_i^b \frac{\partial \Gamma^{\alpha_1 \cdots \alpha_N}}{\partial p_{ia}} - p_i^a \frac{\partial \Gamma^{\alpha_1 \cdots \alpha_N}}{\partial p_{ib}} + (J^{ab})_{\beta_i}^{\alpha_i} \Gamma^{\alpha_1 \cdots \alpha_{i-1} \beta_i \alpha_{i+1} \cdots \alpha_N} \right] = 0, \quad (3.3)$$

reflecting respectively translational and rotational invariance of the amplitude without the soft graviton. While making this analysis we also need to be careful to ensure that while passing  $p_{i\mu}$  through  $\partial/\partial p_{j\nu}$  in order to make use of (3.2), we have to take into account the extra terms proportional to  $\delta_{ij} \delta_\mu^\nu$ . For this reason the terms in the third and fourth lines of (2.47) are not gauge invariant by themselves — their gauge variation cancels against the variation of the term in the last line of (2.47). More specifically if we denote by  $\delta_r$  the gauge variation:

$$\delta_r : \quad \varepsilon_{r,\mu\nu} \rightarrow \varepsilon_{r,\mu\nu} + k_{r\mu} \xi_{r\nu} + k_{r\nu} \xi_{r\mu}, \quad (3.4)$$

for some vector  $\xi_r$  satisfying  $k_r \cdot \xi_r = 0$ , then under  $\delta_1$  the term in the third line of (2.47) remains unchanged, but the term in the fourth line changes by

$$-2 \sum_{i=1}^N (p_i \cdot k_2)^{-1} \varepsilon_{2,b\mu} p_i^b p_i^\mu k_2 \cdot \xi_1 \epsilon_i^T \Gamma_{(i)}. \quad (3.5)$$

On the other hand we get, after using momentum conservation equation  $\sum_{j=1}^N p_j \Gamma_{(i)} = 0$ ,

$$\sum_{i=1}^N \{p_i \cdot (k_1 + k_2)\}^{-1} \delta_1 \mathcal{M}(p_i; \varepsilon_1, k_1, \varepsilon_2, k_2) \epsilon_i^T \Gamma_{(i)} = 2 \sum_{i=1}^N (p_i \cdot k_2)^{-1} \varepsilon_{2,b\mu} p_i^b p_i^\mu k_2 \cdot \xi_1 \epsilon_i^T \Gamma_{(i)}. \quad (3.6)$$

Using this one can easily verify that the  $\delta_1$  variation of the fourth line and the last line of (2.47) cancel. A similar analysis shows that the  $\delta_2$  variation of the third and the last lines of (2.47) cancel, and that the fourth line of (2.47) is invariant under  $\delta_2$ .

The second consistency requirement arises from the fact that individual terms in (2.47) depend on the off-shell data on  $\Gamma^{\alpha_1 \cdots \alpha_N}$  while the actual result should be insensitive to such off-shell extension. For example if we add to  $\Gamma^{\alpha_1 \cdots \alpha_N}$  any term proportional to  $p_i^2 + M_i^2$ , it does not affect the on-shell amplitude without the soft gravitons since it vanishes on-shell. However  $\partial \Gamma^{\alpha_1 \cdots \alpha_N} / \partial p_{i\mu}$  receives a contribution proportional to  $p_i^\mu$  that does not vanish on-shell. We note however that in (2.47) the derivatives of  $\Gamma^{\alpha_1 \cdots \alpha_N}$  come in a very special combination that vanishes under addition of any term to  $\partial \Gamma / \partial p_{i\mu}$  proportional to  $p_i^\mu$ . Therefore (2.47) is not sensitive to such additional terms in  $\Gamma$ .

More generally we can add to  $\Gamma^{\alpha_1 \cdots \alpha_N}$  any term proportional to  $\mathcal{K}^{\alpha_i \beta}(-p_i) \mathcal{G}_\beta^{\alpha_1 \cdots \alpha_{i-1} \alpha_{i+1} \cdots \alpha_N}$  for any function  $\mathcal{G}$ , since its contribution to on-shell amplitudes without the soft gravitons vanishes due to (2.29). Using (2.29) and the rotational invariance of  $\mathcal{K}$  described in (2.25), is easy to see however that the addition of such terms to  $\Gamma$  does not affect (2.47).

### 3.2 Comparison with known results

In order to compare the amplitude with known results, it is convenient to rewrite the amplitude (2.47) as a sum of two terms  $\mathcal{A}_1 + \mathcal{A}_2$  by adding and subtracting a specific term given in the last two lines of (3.7):

$$\begin{aligned}
 \mathcal{A}_1 = & \left\{ \sum_{i=1}^N (p_i \cdot k_1)^{-1} \varepsilon_{1,\mu\nu} p_i^\mu p_i^\nu \right\} \left\{ \sum_{j=1}^N (p_j \cdot k_2)^{-1} \varepsilon_{2,\rho\sigma} p_j^\rho p_j^\sigma \right\} \Gamma(\epsilon_1, p_1, \dots, \epsilon_N, p_N) \\
 & + \left\{ \sum_{j=1}^N (p_j \cdot k_2)^{-1} \varepsilon_{2,\rho\sigma} p_j^\rho p_j^\sigma \right\} \sum_{i=1}^N (p_i \cdot k_1)^{-1} \varepsilon_{1,b\mu} k_{1a} p_i^\mu \epsilon_i^T \left[ p_i^b \frac{\partial \Gamma_{(i)}(p_i)}{\partial p_{ia}} - p_i^a \frac{\partial \Gamma_{(i)}(p_i)}{\partial p_{ib}} + (J^{ab})^T \Gamma_{(i)}(p_i) \right] \\
 & + \left\{ \sum_{j=1}^N (p_j \cdot k_1)^{-1} \varepsilon_{1,\rho\sigma} p_j^\rho p_j^\sigma \right\} \sum_{i=1}^N (p_i \cdot k_2)^{-1} \varepsilon_{2,b\mu} k_{2a} p_i^\mu \epsilon_i^T \left[ p_i^b \frac{\partial \Gamma_{(i)}(p_i)}{\partial p_{ia}} - p_i^a \frac{\partial \Gamma_{(i)}(p_i)}{\partial p_{ib}} + (J^{ab})^T \Gamma_{(i)}(p_i) \right] \\
 & + (k_1 \cdot k_2)^{-1} \sum_{i=1}^N (p_i \cdot k_1)^{-1} (p_i \cdot k_2)^{-1} \left\{ (k_1 \cdot \varepsilon_2 \cdot k_1)(p_i \cdot \varepsilon_1 \cdot p_i)(p_i \cdot k_2) \right. \\
 & \qquad \qquad \qquad \left. + (k_2 \cdot \varepsilon_1 \cdot k_2)(p_i \cdot \varepsilon_2 \cdot p_i)(p_i \cdot k_1) \right\} \epsilon_i^T \Gamma_{(i)}(p_i), \tag{3.7}
 \end{aligned}$$

$$\mathcal{A}_2 = \left\{ \sum_{i=1}^N \mathcal{N}(p_i; \varepsilon_1, k_1, \varepsilon_2, k_2) \right\} \Gamma(\epsilon_1, p_1, \dots, \epsilon_N, p_N), \tag{3.8}$$

where

$$\begin{aligned}
 \mathcal{N}(p_i; \varepsilon_1, k_1, \varepsilon_2, k_2) = & \{p_i \cdot (k_1 + k_2)\}^{-1} \mathcal{M}(p_i; \varepsilon_1, k_1, \varepsilon_2, k_2) \\
 & - (k_1 \cdot k_2)^{-1} (p_i \cdot k_1)^{-1} (p_i \cdot k_2)^{-1} \\
 & \times \left\{ (k_1 \cdot \varepsilon_2 \cdot k_1)(p_i \cdot \varepsilon_1 \cdot p_i)(p_i \cdot k_2) + (k_2 \cdot \varepsilon_1 \cdot k_2)(p_i \cdot \varepsilon_2 \cdot p_i)(p_i \cdot k_1) \right\}, \tag{3.9}
 \end{aligned}$$

$\mathcal{M}$  being given in (2.48). With this definition  $\mathcal{A}_1$  and  $\mathcal{A}_2$  can be shown to be separately gauge invariant.

Refs. [26, 30] computed the double soft limit for scattering of gravitons in Einstein gravity using CHY scattering equations [69–73]. Since our result is valid for general finite energy external states in any theory, it must also be valid for scattering of gravitons. Therefore we can compare the two results. The contribution in [26, 30] comes from two separate terms, the degenerate solutions and non-degenerate solutions. The contribution from the degenerate solutions agrees with our amplitude  $\mathcal{A}_2$  given in (3.8) up to a sign after using momentum conservation rules (3.2). The contribution from the non-degenerate solutions were evaluated in [30] to give only the first three lines of (3.7). However the analysis was carried out in a gauge in which  $k_1 \cdot \varepsilon_2 = 0$  and  $k_2 \cdot \varepsilon_1 = 0$ . For this choice of gauge the contribution from the last two lines of (3.7) vanishes. Therefore, up to the issue with signs mentioned above, there is agreement between our results and the results in pure gravity derived from CHY equations in [26, 30], with (3.7) giving the full gauge invariant version of the contribution from non-degenerate solutions of CHY equations. By carefully reanalyzing the double soft limit of the CHY formula for the scattering amplitudes we have been able to show that the result obtained from the CHY formula actually agrees with ours including the sign [74].

Ref. [59] computed the double soft limit of graviton scattering amplitude in four space-time dimensions using BCFW recursion relations [75]. This analysis was also carried out in

the gauge  $k_1 \cdot \varepsilon_2 = 0$  and  $k_2 \cdot \varepsilon_1 = 0$ . In this gauge the subleading contribution to  $\mathcal{A}_1$  comes only from the second and the third lines which, written in the spinor helicity notation, has the standard form involving derivatives with respect to the spinor helicity variables, called ‘non-contact terms’ in [59]. Therefore we focus on the  $\mathcal{A}_2$  term. Ref. [30] showed that the contribution from the degenerate solution to the CHY equations agrees with the ‘contact terms’ computed in [59] using BCFW recursion relations. Therefore our result for  $\mathcal{A}_2$  agrees with the contact terms of [59] up to the sign factor discussed earlier. We have also verified this independently by noting that in the gauge  $k_1 \cdot \varepsilon_2 = 0$  and  $k_2 \cdot \varepsilon_1 = 0$  many of the terms in  $\mathcal{A}_2$  vanish and the remaining terms take the form

$$\sum_{i=1}^N \{p_i \cdot (k_1 + k_2)\}^{-1} \left[ - (p_i \cdot k_1)^{-1} (p_i \cdot k_2)^{-1} (k_1 \cdot k_2) (p_i \cdot \varepsilon_1 \cdot p_i) (p_i \cdot \varepsilon_2 \cdot p_i) - 2(p_i \cdot \varepsilon_1 \cdot \varepsilon_2 \cdot p_i) - (k_1 \cdot k_2)^{-1} \varepsilon_1^{cd} \varepsilon_{2,cd} (k_1 \cdot p_i) (k_2 \cdot p_i) \right] \Gamma(\varepsilon_1, p_1, \dots, \varepsilon_N, p_N) . \quad (3.10)$$

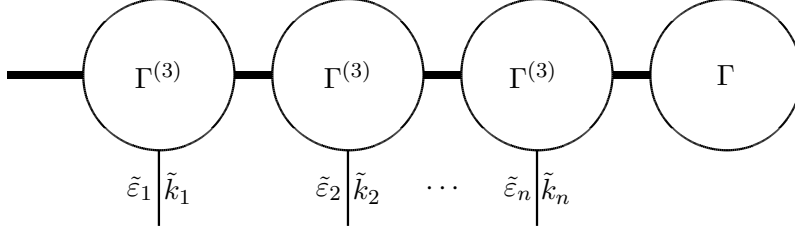
By expressing this in the spinor helicity notation we find that when the two soft gravitons carry the same helicity (3.10) vanishes. This is in agreement with the result of [59]. On the other hand when the two soft gravitons carry opposite helicities,  $\mathcal{A}_2$  gives a non-zero result that agrees with the ‘contact terms’ of [59] up to a sign. We have not tried to resolve this discrepancy in sign between our results and that of [59]. However given that we have now verified that the CHY result for contact terms actually comes with a sign opposite to that found in [26, 30] and agrees with our amplitude  $\mathcal{A}_2$  [74], it seems that the difference in sign between our results and the BCFW results may be due to some differences in convention, e.g. the difference in the choice of sign of the graviton polarization tensor.<sup>2</sup>

#### 4 Amplitudes with arbitrary number of soft gravitons

The method described in the earlier sections can now be generalized to derive the expression for the amplitude with multiple soft gravitons when the momenta carried by all the soft gravitons become small at the same rate. We shall first write down the result and then explain how we arrive at it. The subleading soft graviton amplitude with  $M$  soft gravitons carrying momenta  $k_1, \dots, k_M$  and polarizations  $\varepsilon_1, \dots, \varepsilon_M$  and  $N$  finite energy particles carrying momenta  $p_1, \dots, p_N$  and polarizations  $\varepsilon_1, \dots, \varepsilon_N$  is given by

$$\begin{aligned} A = & \prod_{r=1}^M \left\{ \sum_{i=1}^N (p_i \cdot k_r)^{-1} \varepsilon_{r,\mu\nu} p_i^\mu p_i^\nu \right\} \Gamma(\varepsilon_1, p_1, \dots, \varepsilon_N, p_N) \\ & + \sum_{s=1}^M \sum_{j=1}^N (p_j \cdot k_s)^{-1} \varepsilon_{s,b\mu} k_{sa} p_j^\mu \varepsilon_j^T \left[ p_j^b \frac{\partial \Gamma_{(j)}(p_j)}{\partial p_{ja}} - p_j^a \frac{\partial \Gamma_{(j)}(p_j)}{\partial p_{jb}} + (J^{ab})^T \Gamma_{(j)}(p_j) \right] \\ & \times \prod_{\substack{r=1 \\ r \neq s}}^M \left\{ \sum_{i=1}^N (p_i \cdot k_r)^{-1} \varepsilon_{r,\mu\nu} p_i^\mu p_i^\nu \right\} \\ & + \sum_{\substack{r,u=1 \\ r < u}}^M \left\{ \sum_{j=1}^N \{p_j \cdot (k_r + k_u)\}^{-1} \mathcal{M}(p_j; \varepsilon_r, k_r, \varepsilon_u, k_u) \varepsilon_j^T \Gamma_{(j)}(p_j) \right\} \prod_{\substack{s=1 \\ s \neq r,u}}^M \left\{ \sum_{i=1}^N (p_i \cdot k_s)^{-1} \varepsilon_{s,\mu\nu} p_i^\mu p_i^\nu \right\} , \end{aligned} \quad (4.1)$$

<sup>2</sup>We have used the convention that the graviton polarization tensors in four dimensions are given by squares of the gauge field polarization tensors without any extra sign.



**Figure 11.** A leading contribution to the amplitude with multiple soft gravitons.

where  $\mathcal{M}(p_j; \varepsilon_r, k_r, \varepsilon_u, k_u)$  has been defined in (2.48). Independently of the general argument given below, we have used Cadabra [76, 77] and Mathematica [78] to check (4.1) explicitly for amplitudes with three soft gravitons.

We begin by reviewing the derivation of the leading term given in the first line of (4.1). For this note that this term may be rearranged as

$$\sum_{\substack{A_1, \dots, A_N; A_i \subset \{1, \dots, M\} \\ A_i \cap A_j = \emptyset \text{ for } i \neq j; A_1 \cup A_2 \cup \dots \cup A_N = \{1, \dots, M\}}} \prod_{i=1}^N \left\{ \prod_{r \in A_i} \varepsilon_{r, \mu\nu} p_i^\mu p_i^\nu \right\} \left\{ \prod_{r \in A_i} (p_i \cdot k_r)^{-1} \right\} \Gamma(\varepsilon_1, p_1, \dots, \varepsilon_N, p_N). \quad (4.2)$$

Physically the  $i$ -th term in the product represents the contribution from the soft gravitons in the set  $A_i$  attached to the  $i$ -th finite energy external line. To see how we get this factor, let us denote the momenta of the soft gravitons attached from the outermost end to the innermost end of the  $i$ -th line in a given graph by  $\tilde{k}_1, \dots, \tilde{k}_n$ . The corresponding polarizations are denoted by  $\tilde{\varepsilon}_1, \dots, \tilde{\varepsilon}_n$ . This is shown in figure 11. The unordered set  $\{\tilde{k}_1, \dots, \tilde{k}_n\}$  coincides with the set  $\{k_s; s \in A_i\}$ . A similar statement holds for the polarizations. The leading contribution from the products of three point vertices and propagators associated with the  $i$ -th line of the graph may be computed using (2.30), (2.29) and is given by

$$\left\{ \prod_{r=1}^n \tilde{\varepsilon}_{r, \mu\nu} p_i^\mu p_i^\nu \right\} \{p_i \cdot \tilde{k}_1\}^{-1} \{p_i \cdot (\tilde{k}_1 + \tilde{k}_2)\}^{-1} \dots \{p_i \cdot (\tilde{k}_1 + \dots + \tilde{k}_n)\}^{-1}. \quad (4.3)$$

The total contribution obtained after summing over all permutations of the momenta  $\tilde{k}_1, \dots, \tilde{k}_n$  using (A.1) is given by

$$\begin{aligned} & \left\{ \prod_{r=1}^n \tilde{\varepsilon}_{r, \mu\nu} p_i^\mu p_i^\nu \right\} \sum_{\text{permutations of } \tilde{k}_1, \dots, \tilde{k}_n} \{p_i \cdot \tilde{k}_1\}^{-1} \{p_i \cdot (\tilde{k}_1 + \tilde{k}_2)\}^{-1} \dots \{p_i \cdot (\tilde{k}_1 + \dots + \tilde{k}_n)\}^{-1} \\ &= \left\{ \prod_{r=1}^n \tilde{\varepsilon}_{r, \mu\nu} p_i^\mu p_i^\nu \right\} \left\{ \prod_{s=1}^n (p_i \cdot \tilde{k}_s)^{-1} \right\} = \left\{ \prod_{r \in A_i} \varepsilon_{r, \mu\nu} p_i^\mu p_i^\nu \right\} \left\{ \prod_{s \in A_i} (p_i \cdot k_s)^{-1} \right\}. \end{aligned} \quad (4.4)$$

This reproduces (4.2).

We now turn to the analysis of the subleading terms. For this let us first analyze the contribution from the products of the propagators and vertices in figure 11 to subleading

order. Using (2.30) this may be expressed as

$$\begin{aligned}
 & \{2p_i \cdot \tilde{k}_1\}^{-1} \{2p_i \cdot (\tilde{k}_1 + \tilde{k}_2) + 2\tilde{k}_1 \cdot \tilde{k}_2\}^{-1} \cdots \left\{ 2p_i \cdot (\tilde{k}_1 + \cdots + \tilde{k}_n) + 2 \sum_{\substack{r,u=1 \\ r < u}}^n \tilde{k}_r \cdot \tilde{k}_u \right\}^{-1} \\
 & \epsilon_i^T \left[ 2\tilde{\varepsilon}_1^{\mu\nu} p_{i\mu} p_{i\nu} + i\tilde{\varepsilon}_{1,\mu\nu} p_i^\nu \mathcal{K}(-p_i) \frac{\partial \Xi^i(-p_i)}{\partial p_{i\mu}} + 2\tilde{\varepsilon}_{1,b\mu} \tilde{k}_{1a} p_i^\mu (J^{ab})^T + \mathcal{K}(-p_i) \mathcal{Q}(p_i, \tilde{k}_1) \right] \\
 & \left[ 2\tilde{\varepsilon}_2^{\mu\nu} \{p_{i\mu} + \tilde{k}_{1\mu}\} \{p_{i\nu} + \tilde{k}_{1\nu}\} + i\tilde{\varepsilon}_{2,\mu\nu} (p_i^\nu + \tilde{k}_1^\nu) \mathcal{K}(-p_i - \tilde{k}_1) \frac{\partial \Xi^i(-p_i - \tilde{k}_1)}{\partial p_{i\mu}} \right. \\
 & \left. + 2\tilde{\varepsilon}_{2,b\mu} \tilde{k}_{2a} p_i^\mu (J^{ab})^T + \mathcal{K}(-p_i) \mathcal{Q}(p_i, \tilde{k}_2) \right] \\
 & \cdots \left[ 2\tilde{\varepsilon}_n^{\mu\nu} \{p_{i\mu} + \tilde{k}_{1\mu} + \cdots + \tilde{k}_{(n-1)\mu}\} \{p_{i\nu} + \tilde{k}_{1\nu} + \cdots + \tilde{k}_{(n-1)\nu}\} \right. \\
 & \left. + i\tilde{\varepsilon}_{n,\mu\nu} (p_i^\nu + \tilde{k}_1^\nu + \cdots + \tilde{k}_{n-1}^\nu) \mathcal{K}(-p_i - \tilde{k}_1 - \tilde{k}_2 - \cdots - \tilde{k}_{n-1}) \frac{\partial \Xi^i(-p_i - \tilde{k}_1 - \tilde{k}_2 - \cdots - \tilde{k}_{n-1})}{\partial p_{i\mu}} \right. \\
 & \left. + 2\tilde{\varepsilon}_{n,b\mu} \tilde{k}_{na} p_i^\mu (J^{ab})^T + \mathcal{K}(-p_i) \mathcal{Q}(p_i, \tilde{k}_n) \right] \Gamma_{(i)}(p_i + \tilde{k}_1 + \cdots + \tilde{k}_n). \quad (4.5)
 \end{aligned}$$

First let us analyze the contribution from the  $\tilde{k}_r \cdot \tilde{k}_u$  terms in the denominator. Since this is subleading, we need to expand one of the denominators to first order in  $\tilde{k}_r \cdot \tilde{k}_u$ , set  $\tilde{k}_r \cdot \tilde{k}_u = 0$  in the rest of the denominators, and pick the leading contribution from all other factors. This leads to

$$- \left\{ \sum_{m=2}^n \sum_{\substack{r,u=1 \\ r < u}}^m \frac{\tilde{k}_r \cdot \tilde{k}_u}{p_i \cdot (\tilde{k}_1 + \cdots + \tilde{k}_m)} \right\} \left\{ \prod_{\ell=1}^n \frac{1}{p_i \cdot (\tilde{k}_1 + \cdots + \tilde{k}_\ell)} \right\} \left\{ \prod_{s=1}^n \tilde{\varepsilon}_{s,\mu\nu} p_i^\mu p_i^\nu \right\} \epsilon_i^T \Gamma_{(i)}(p_i). \quad (4.6)$$

After performing the sum over all permutations of  $\tilde{k}_1, \dots, \tilde{k}_n$  using (A.2) this gives

$$- \prod_{s=1}^n \left\{ (p_i \cdot \tilde{k}_s)^{-1} \tilde{\varepsilon}_{s,\mu\nu} p_i^\mu p_i^\nu \right\} \sum_{\substack{r,u=1 \\ r < u}}^n \tilde{k}_r \cdot \tilde{k}_u \{p_i \cdot (\tilde{k}_r + \tilde{k}_u)\}^{-1}. \quad (4.7)$$

Next we consider the terms involving the contraction of  $\tilde{\varepsilon}_u$  with  $\tilde{k}_r$  for  $r < u$ , coming from the first term inside each square bracket in (4.5). Since this term is subleading, once we pick one of these factors we must pick the leading terms from all the other factors. Again using (2.29) we can express the sum of all such contributions as

$$2 \sum_{\substack{r,u=1 \\ r < u}}^n \left\{ \prod_{\substack{s=1 \\ s \neq u}}^n \tilde{\varepsilon}_{s,\mu\nu} p_i^\mu p_i^\nu \right\} \tilde{\varepsilon}_{u,\mu\nu} p_i^\mu \tilde{k}_r^\nu \left\{ \prod_{m=1}^n \frac{1}{p_i \cdot (\tilde{k}_1 + \cdots + \tilde{k}_m)} \right\} \epsilon_i^T \Gamma_{(i)}(p_i). \quad (4.8)$$

After summing over all permutations of  $(\tilde{k}_1, \tilde{\varepsilon}_1), \dots, (\tilde{k}_n, \tilde{\varepsilon}_n)$  using (A.3) this gives

$$\begin{aligned}
 & 2 \left\{ \prod_{s=1}^n (p_i \cdot \tilde{k}_s)^{-1} \right\} \sum_{\substack{r,u=1 \\ r < u}}^n \{p_i \cdot (\tilde{k}_r + \tilde{k}_u)\}^{-1} \left\{ \prod_{\substack{s=1 \\ s \neq r,u}}^n \tilde{\varepsilon}_{s,\mu\nu} p_i^\mu p_i^\nu \right\} \\
 & \left\{ (p_i \cdot \tilde{k}_u) (p_i \cdot \tilde{\varepsilon}_r \cdot p_i) (p_i \cdot \tilde{\varepsilon}_u \cdot \tilde{k}_r) + (p_i \cdot \tilde{k}_r) (p_i \cdot \tilde{\varepsilon}_u \cdot p_i) (p_i \cdot \tilde{\varepsilon}_r \cdot \tilde{k}_u) \right\} \epsilon_i^T \Gamma_{(i)}(p_i). \quad (4.9)
 \end{aligned}$$

We now turn to the rest of the contribution from (4.5) in which we drop the  $\tilde{k}_r \cdot \tilde{k}_u$  factors in the denominator and also the terms involving contraction of  $\tilde{k}_r$  with  $\tilde{\varepsilon}_u$  in the first term inside each square bracket. Our first task will be to expand the factors of  $\mathcal{K}$  and  $\Xi^i$  in Taylor series expansion in powers of the soft momenta. It is easy to see however that to the first subleading order, the order  $\tilde{k}^\mu$  terms in the expansion of  $\Xi^i$  do not contribute to the amplitude. This is due to the fact that once we have picked a subleading term proportional to  $\tilde{k}_s^\rho \partial^2 \Xi^i / \partial p_i^\mu \partial p_i^\rho$ , we must replace the argument of  $\mathcal{K}$  by  $-p_i$  in the accompanying factor and in all other factors we must pick the leading term. In this case repeated use of (2.29) shows that the corresponding contribution vanishes. Therefore we can replace all factors of  $\partial \Xi^i(-p_i - \tilde{k}_1 - \dots) / \partial p_i^\mu$  by  $\partial \Xi^i(-p_i) / \partial p_i^\mu$ . Similar argument shows that all the  $\mathcal{K}(-p_i) \mathcal{Q}$  terms, and the terms involving contraction of  $\tilde{\varepsilon}_u$  with  $\tilde{k}_r$  in the second term inside each square bracket in (4.5), give vanishing contribution at the subleading order. This allows us to express the rest of the contribution from (4.5) as

$$\begin{aligned}
 & (2p_i \cdot \tilde{k}_1)^{-1} \{2p_i \cdot (\tilde{k}_1 + \tilde{k}_2)\}^{-1} \dots \{2p_i \cdot (\tilde{k}_1 + \dots + \tilde{k}_n)\}^{-1} \\
 & \epsilon_i^T \left[ 2\mathcal{E}_1 + 2\mathcal{L}_1 + 2\tilde{\varepsilon}_{1,b\mu} \tilde{k}_{1a} p_i^\mu (J^{ab})^T \right] \\
 & \left[ 2\mathcal{E}_2 + 2\mathcal{L}_2 + i \tilde{\varepsilon}_{2,\mu\nu} p_i^\nu \tilde{k}_{1\rho} \frac{\partial \mathcal{K}(-p_i)}{\partial p_{i\rho}} \frac{\partial \Xi^i(-p_i)}{\partial p_{i\mu}} + 2\tilde{\varepsilon}_{2,b\mu} \tilde{k}_{2a} p_i^\mu (J^{ab})^T \right] \dots \\
 & \left[ 2\mathcal{E}_n + 2\mathcal{L}_n + i \tilde{\varepsilon}_{n,\mu\nu} p_i^\nu (\tilde{k}_{1\rho} + \dots + \tilde{k}_{n-1,\rho}) \frac{\partial \mathcal{K}(-p_i)}{\partial p_{i\rho}} \frac{\partial \Xi^i(-p_i)}{\partial p_{i\mu}} + 2\tilde{\varepsilon}_{n,b\mu} \tilde{k}_{na} p_i^\mu (J^{ab})^T \right] \\
 & \Gamma_{(i)}(p_i + \tilde{k}_1 + \dots + \tilde{k}_n), \tag{4.10}
 \end{aligned}$$

where,

$$\mathcal{E}_s = \tilde{\varepsilon}_s^{\mu\nu} p_{i\mu} p_{i\nu}, \quad \mathcal{L}_s = \frac{i}{2} \tilde{\varepsilon}_s^{\mu\nu} p_{i\nu} \mathcal{K}(-p_i) \frac{\partial \Xi^i(-p_i)}{\partial p_{i\mu}}. \tag{4.11}$$

We now expand (4.10) in powers of soft momenta. Even though  $\mathcal{L}_s$  is leading order, its contribution to the amplitude vanishes by (2.29) unless there is some other matrix sitting between  $\epsilon_i^T$  and  $\mathcal{L}_s$ . The possible terms come from picking up either the term proportional to  $\partial \mathcal{K} / \partial p_{i\rho} \partial \Xi / \partial p_{i\mu}$  or  $(J^{ab})^T$  from one of the factors. Both these terms are subleading and therefore we can pick at most one such term in the product, with the other factors being given by  $\mathcal{E}_s + \mathcal{L}_s$ . Therefore if we expand (4.10) and pick the subleading factor from the  $r$ -th term in the product, then in the product of  $\mathcal{E}_s + \mathcal{L}_s$ , we can drop all factors of  $\mathcal{L}_s$  for  $s < r$  since they sit to the left of the subleading factor and will vanish due to (2.29). This gives the following expression for the subleading contribution to (4.10):

$$\begin{aligned}
 & (p_i \cdot \tilde{k}_1)^{-1} \{p_i \cdot (\tilde{k}_1 + \tilde{k}_2)\}^{-1} \dots \{p_i \cdot (\tilde{k}_1 + \dots + \tilde{k}_n)\}^{-1} \\
 & \epsilon_i^T \left[ \sum_{r=1}^n \left\{ \prod_{s=1}^{r-1} \mathcal{E}_s \right\} \left[ \frac{i}{2} \tilde{\varepsilon}_{r,\mu\nu} p_i^\nu (\tilde{k}_{1\rho} + \dots + \tilde{k}_{r-1,\rho}) \frac{\partial \mathcal{K}(-p_i)}{\partial p_{i\rho}} \frac{\partial \Xi^i(-p_i)}{\partial p_{i\mu}} + \tilde{\varepsilon}_{r,b\mu} \tilde{k}_{ra} p_i^\mu (J^{ab})^T \right] \right. \\
 & \left. \left\{ \prod_{s=r+1}^n (\mathcal{E}_s + \mathcal{L}_s) \right\} \Gamma_{(i)}(p_i) \right. \\
 & \left. + (p_i \cdot \tilde{k}_1)^{-1} \{p_i \cdot (\tilde{k}_1 + \tilde{k}_2)\}^{-1} \dots \{p_i \cdot (\tilde{k}_1 + \dots + \tilde{k}_n)\}^{-1} \left\{ \prod_{s=1}^n \mathcal{E}_s \right\} \sum_{r=1}^n \tilde{k}_{r\rho} \frac{\partial \Gamma_{(i)}(p_i)}{\partial p_{i\rho}} \right]. \tag{4.12}
 \end{aligned}$$

The last term comes from the Taylor series expansion of  $\Gamma_{(i)}$  in powers of soft momenta. In the product the  $(\mathcal{E}_s + \mathcal{L}_s)$ 's are ordered from left to right in the order of increasing  $s$ . We now manipulate the product  $\prod_{s=r+1}^n (\mathcal{E}_s + \mathcal{L}_s)$  as follows. If the subleading factor is the one proportional to  $\partial\mathcal{K}/\partial p_{i\rho} \partial\Xi/\partial p_{i\mu}$  then we leave the product of the factors  $(\mathcal{E}_s + \mathcal{L}_s)$  for  $s > r$  unchanged. However if the subleading factor is the one proportional to  $(J^{ab})^T$ , then we expand the product of the factors  $(\mathcal{E}_s + \mathcal{L}_s)$  for  $s > r$  as

$$(\mathcal{E}_{r+1} + \mathcal{L}_{r+1}) \cdots (\mathcal{E}_n + \mathcal{L}_n) = \mathcal{E}_{r+1} \cdots \mathcal{E}_n + \sum_{u=r+1}^n \mathcal{E}_{r+1} \cdots \mathcal{E}_{u-1} \mathcal{L}_u (\mathcal{E}_{u+1} + \mathcal{L}_{u+1}) \cdots (\mathcal{E}_n + \mathcal{L}_n). \quad (4.13)$$

Using this, and combining the contribution from the first term on the right hand side of (4.13) with the last term in (4.12), we can express (4.12) as

$$\begin{aligned} & (p_i \cdot \tilde{k}_1)^{-1} \{p_i \cdot (\tilde{k}_1 + \tilde{k}_2)\}^{-1} \cdots \{p_i \cdot (\tilde{k}_1 + \cdots + \tilde{k}_n)\}^{-1} \quad (4.14) \\ & \epsilon_i^T \left[ \sum_{r=1}^n \left\{ \prod_{\substack{s=1 \\ s \neq r}}^n \mathcal{E}_s \right\} \left\{ \tilde{\epsilon}_{r,b\mu} \tilde{k}_{ra} p_i^\mu (J^{ab})^T \Gamma_{(i)}(p_i) + \tilde{\epsilon}_r^{\mu\nu} p_{i\mu} p_{i\nu} \tilde{k}_{r\rho} \frac{\partial \Gamma_{(i)}}{\partial p_{i\rho}} \right\} \right. \\ & + \frac{i}{2} \sum_{r=1}^n \left\{ \prod_{s=1}^{r-1} \mathcal{E}_s \right\} \tilde{\epsilon}_{r,\mu\nu} p_i^\nu (\tilde{k}_{1\rho} + \cdots + \tilde{k}_{r-1,\rho}) \frac{\partial \mathcal{K}(-p_i)}{\partial p_{i\rho}} \frac{\partial \Xi^i(-p_i)}{\partial p_{i\mu}} \left\{ \prod_{s=r+1}^n (\mathcal{E}_s + \mathcal{L}_s) \right\} \Gamma_{(i)} \\ & \left. + \frac{i}{2} \sum_{r=1}^n \sum_{u=r+1}^n \left\{ \prod_{\substack{s=1 \\ s \neq r}}^{u-1} \mathcal{E}_s \right\} \tilde{\epsilon}_{r,b\mu} \tilde{k}_{ra} p_i^\mu (J^{ab})^T \tilde{\epsilon}_{u,\rho\sigma} p_i^\sigma \mathcal{K}(-p_i) \frac{\partial \Xi^i(-p_i)}{\partial p_{i\rho}} \left\{ \prod_{s=u+1}^n (\mathcal{E}_s + \mathcal{L}_s) \right\} \Gamma_{(i)} \right]. \end{aligned}$$

We now use (2.25) to move the  $\mathcal{K}(-p_i)$  factor in the last term to the left of  $(J^{ab})^T$  and use (2.29). This allows us to express (4.14) as

$$\begin{aligned} & (p_i \cdot \tilde{k}_1)^{-1} \{p_i \cdot (\tilde{k}_1 + \tilde{k}_2)\}^{-1} \cdots \{p_i \cdot (\tilde{k}_1 + \cdots + \tilde{k}_n)\}^{-1} \\ & \epsilon_i^T \left[ \sum_{r=1}^n \left\{ \prod_{\substack{s=1 \\ s \neq r}}^n \mathcal{E}_s \right\} \left\{ \tilde{\epsilon}_{r,b\mu} \tilde{k}_{ra} p_i^\mu (J^{ab})^T \Gamma_{(i)}(p_i) + \tilde{\epsilon}_r^{\mu\nu} p_{i\mu} p_{i\nu} \tilde{k}_{r\rho} \frac{\partial \Gamma_{(i)}}{\partial p_{i\rho}} \right\} \right. \\ & + \frac{i}{2} \sum_{r=1}^n \left\{ \prod_{s=1}^{r-1} \mathcal{E}_s \right\} \tilde{\epsilon}_{r,\mu\nu} p_i^\nu (\tilde{k}_{1\rho} + \cdots + \tilde{k}_{r-1,\rho}) \frac{\partial \mathcal{K}(-p_i)}{\partial p_{i\rho}} \frac{\partial \Xi^i(-p_i)}{\partial p_{i\mu}} \left\{ \prod_{s=r+1}^n (\mathcal{E}_s + \mathcal{L}_s) \right\} \Gamma_{(i)} \\ & + \frac{i}{2} \sum_{\substack{r,u=1 \\ r < u}}^n \left\{ \prod_{\substack{s=1 \\ s \neq r}}^{u-1} \mathcal{E}_s \right\} \tilde{\epsilon}_{r,b\mu} \tilde{k}_{ra} p_i^\mu \tilde{\epsilon}_{u,\rho\sigma} p_i^\sigma \left( p_i^a \frac{\partial \mathcal{K}(-p_i)}{\partial p_{ib}} - p_i^b \frac{\partial \mathcal{K}(-p_i)}{\partial p_{ia}} \right) \frac{\partial \Xi^i(-p_i)}{\partial p_{i\rho}} \\ & \left. \left\{ \prod_{s=u+1}^n (\mathcal{E}_s + \mathcal{L}_s) \right\} \Gamma_{(i)} \right]. \quad (4.15) \end{aligned}$$

It is easy to see that terms proportional to  $p_i^b \partial \mathcal{K} / \partial p_{ia}$  in the fourth line of (4.15) cancels



the terms in the third line of (4.15). Therefore we are left with

$$\begin{aligned}
 & (p_i \cdot \tilde{k}_1)^{-1} \{p_i \cdot (\tilde{k}_1 + \tilde{k}_2)\}^{-1} \cdots \{p_i \cdot (\tilde{k}_1 + \cdots + \tilde{k}_n)\}^{-1} \\
 & \epsilon_i^T \left[ \sum_{r=1}^n \left\{ \prod_{\substack{s=1 \\ s \neq r}}^n \mathcal{E}_s \right\} \left\{ \tilde{\epsilon}_{r,b\mu} \tilde{k}_{ra} p_i^\mu (J^{ab})^T \Gamma_{(i)}(p_i) + \tilde{\epsilon}_r^{\mu\nu} p_{i\mu} p_{i\nu} \tilde{k}_{r\rho} \frac{\partial \Gamma_{(i)}}{\partial p_{i\rho}} \right\} \right. \\
 & \left. + \frac{i}{2} \sum_{\substack{r,u=1 \\ r < u}}^n \left\{ \prod_{\substack{s=1 \\ s \neq r}}^{u-1} \mathcal{E}_s \right\} p_i \cdot \tilde{k}_r \tilde{\epsilon}_{r,b\mu} p_i^\mu \tilde{\epsilon}_{u,\rho\sigma} p_i^\sigma \frac{\partial \mathcal{K}(-p_i)}{\partial p_{ib}} \frac{\partial \Xi^i(-p_i)}{\partial p_{i\rho}} \left\{ \prod_{s=u+1}^n (\mathcal{E}_s + \mathcal{L}_s) \right\} \Gamma_{(i)} \right].
 \end{aligned} \tag{4.16}$$

First consider the term in the second line of (4.16). We sum over all permutations of  $(\tilde{\epsilon}_1, \tilde{k}_1), \dots, (\tilde{\epsilon}_n, \tilde{k}_n)$ . After the sum over  $r$  is performed, this expression is already invariant under the permutations of the soft gravitons inserted on the  $i$ -th line. Therefore we simply have to sum the expression in the first line over all permutations using (A.1), producing the result:

$$\left\{ \prod_{s=1}^n (p_i \cdot \tilde{k}_s)^{-1} \right\} \epsilon_i^T \left[ \sum_{r=1}^n \prod_{\substack{s=1 \\ s \neq r}}^n \{ \tilde{\epsilon}_s^{\mu\nu} p_{i\mu} p_{i\nu} \} \left\{ \tilde{\epsilon}_{r,b\mu} \tilde{k}_{ra} p_i^\mu (J^{ab})^T \Gamma_{(i)}(p_i) + \tilde{\epsilon}_r^{\mu\nu} p_{i\mu} p_{i\nu} \tilde{k}_{r\rho} \frac{\partial \Gamma_{(i)}}{\partial p_{i\rho}} \right\} \right]. \tag{4.17}$$

Since this is already subleading, we have to pick the leading contribution from all the other external legs, producing factors of  $\prod_{s \in A_j} \{ (p_j \cdot k_s)^{-1} \epsilon_{s,\mu\nu} p_j^\mu p_j^\nu \}$  after summing over permutations of the soft gravitons. Finally we sum over all ways of distributing the soft gravitons on the external lines. The net contribution from these terms is given by

$$\sum_{r=1}^M \sum_{i=1}^N (p_i \cdot k_r)^{-1} \epsilon_{r,b\rho} k_{ra} p_i^\rho \epsilon_i^T \left[ p_i^b \frac{\partial \Gamma_{(i)}(p_i)}{\partial p_{ia}} + (J^{ab})^T \Gamma_{(i)}(p_i) \right] \prod_{\substack{s=1 \\ s \neq r}}^M \left\{ \sum_{j=1}^N (p_j \cdot k_s)^{-1} \epsilon_{s,\mu\nu} p_j^\mu p_j^\nu \right\}. \tag{4.18}$$

We now combine this with the contribution from the sum of graphs where one soft graviton attaches to the amplitude via the vertex  $\tilde{\Gamma}$  shown in figure 8 and the other soft gravitons attach to the external lines. Using (2.18) we get the contribution from these graphs to be

$$- \sum_{r=1}^M \left[ \prod_{\substack{s=1 \\ s \neq r}}^M \left\{ \sum_{j=1}^N (p_j \cdot k_s)^{-1} \epsilon_{s,\mu\nu} p_j^\mu p_j^\nu \right\} \right] \sum_{i=1}^N \epsilon_{r,ab} p_i^a \epsilon_i^T \frac{\partial \Gamma_{(i)}(p_i)}{\partial p_{ib}}. \tag{4.19}$$

The sum of (4.18) and (4.19) reproduces the terms in the second and third line of (4.1).

Let us now turn to the contribution from the last line of (4.16). We express  $p_i \cdot \tilde{k}_r$  factor as

$$p_i \cdot (\tilde{k}_1 + \cdots + \tilde{k}_r) - p_i \cdot (\tilde{k}_1 + \cdots + \tilde{k}_{r-1}) \tag{4.20}$$

so that each term in (4.20) cancels one of the denominator factors in the first line of (4.16). Now we are supposed to sum over all permutations of the soft gravitons carrying the labels  $1, \dots, n$ . However instead of summing over all permutations of  $\tilde{k}_1, \dots, \tilde{k}_n$  in one step, let us first fix the positions of all soft gravitons except the one carrying momentum  $\tilde{k}_r$ , and sum over all insertions of the soft graviton carrying momentum  $\tilde{k}_r$  to the left of the one

carrying momentum  $\tilde{k}_u$ . Using (4.20) at each step, it is easy to see that the contributions from the terms cancel pairwise. For example for three soft gravitons, with 1 fixed to the left of 3, and the position of 2 summed over on all positions to the left of 3, we have

$$\begin{aligned} & \{p_i \cdot \tilde{k}_1\}^{-1} \{p_i \cdot (\tilde{k}_1 + \tilde{k}_2)\}^{-1} \{p_i \cdot (\tilde{k}_1 + \tilde{k}_2 + \tilde{k}_3)\}^{-1} \{p_i \cdot (\tilde{k}_1 + \tilde{k}_2) - p_i \cdot \tilde{k}_1\} \\ & + \{p_i \cdot \tilde{k}_2\}^{-1} \{p_i \cdot (\tilde{k}_1 + \tilde{k}_2)\}^{-1} \{p_i \cdot (\tilde{k}_1 + \tilde{k}_2 + \tilde{k}_3)\}^{-1} \{p_i \cdot \tilde{k}_2\} \\ & = \{p_i \cdot \tilde{k}_1\}^{-1} \{p_i \cdot (\tilde{k}_1 + \tilde{k}_2 + \tilde{k}_3)\}^{-1}. \end{aligned} \quad (4.21)$$

As a result of this pairwise cancellation, at the end we are left with only one term arising from the insertion of  $\tilde{k}_r$  just to the left of  $\tilde{k}_u$ . In order to express the result in a convenient form we relabel the gravitons attached to the  $i$ -th line from left to right, other than the one carrying momentum  $\tilde{k}_r$ , as

$$(\hat{\varepsilon}_1, \hat{k}_1), \dots, (\hat{\varepsilon}_{u-2}, \hat{k}_{u-2}), (\tilde{\varepsilon}_u, \tilde{k}_u), (\hat{\varepsilon}_{u+1}, \hat{k}_{u+1}), \dots, (\hat{\varepsilon}_n, \hat{k}_n). \quad (4.22)$$

and sum over all insertions of the graviton carrying the quantum numbers  $(\tilde{\varepsilon}_r, \tilde{k}_r)$  to the left of  $(\tilde{\varepsilon}_u, \tilde{k}_u)$ . Then for fixed  $r, s$  the result is given by

$$\begin{aligned} & (p_i \cdot \hat{k}_1)^{-1} \{p_i \cdot (\hat{k}_1 + \hat{k}_2)\}^{-1} \dots \{p_i \cdot (\hat{k}_1 + \dots + \hat{k}_{u-2})\}^{-1} \\ & \{p_i \cdot (\hat{k}_1 + \dots + \hat{k}_{u-2} + \tilde{k}_r + \tilde{k}_u)\}^{-1} \dots \{p_i \cdot (\hat{k}_1 + \dots + \hat{k}_{u-2} + \tilde{k}_r + \tilde{k}_u + \hat{k}_{u+1} + \dots + \hat{k}_n)\}^{-1} \\ & \epsilon_i^T \left[ \frac{i}{2} \left\{ \prod_{s=1}^{u-2} \hat{\mathcal{E}}_s \right\} \tilde{\varepsilon}_{r,b\mu} p_i^\mu \tilde{\varepsilon}_{u,\rho\sigma} p_i^\sigma \frac{\partial \mathcal{K}(-p_i)}{\partial p_{ib}} \frac{\partial \Xi^i(-p_i)}{\partial p_{i\rho}} \left\{ \prod_{s=u+1}^n (\hat{\mathcal{E}}_s + \hat{\mathcal{L}}_s) \right\} \right] \Gamma_{(i)}(p_i), \end{aligned} \quad (4.23)$$

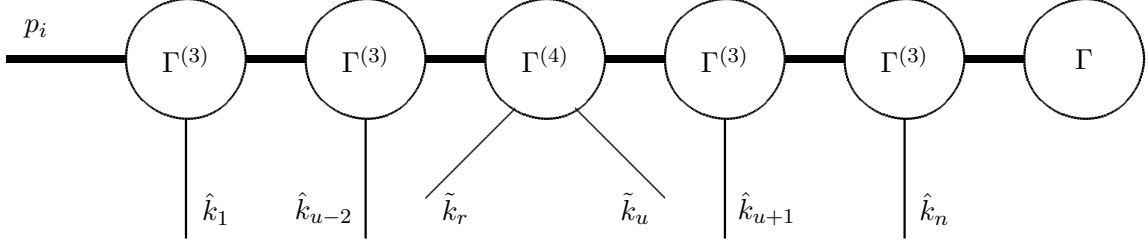
where

$$\hat{\mathcal{E}}_s = \varepsilon_s^{\mu\nu} p_{i\mu} p_{i\nu}, \quad \hat{\mathcal{L}}_s = \frac{i}{2} \varepsilon_s^{\mu\nu} p_{i\nu} \mathcal{K}(-p_i) \frac{\partial \Xi^i(-p_i)}{\partial p_{i\mu}}. \quad (4.24)$$

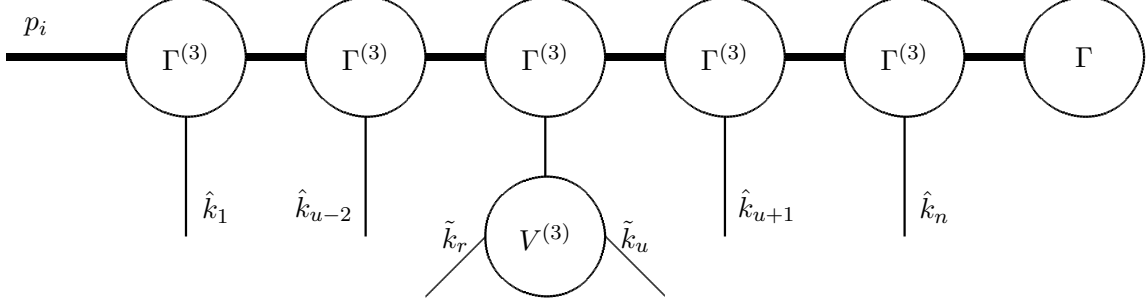
Next we add to this a term obtained by exchanging the positions of  $r$  and  $u$ . This is equivalent to exchanging the  $\rho$  and  $b$  indices in  $(\partial \mathcal{K} / \partial p_{ib})(\partial \Xi / \partial p_{i\rho})$  and gives

$$\begin{aligned} & (p_i \cdot \hat{k}_1)^{-1} \{p_i \cdot (\hat{k}_1 + \hat{k}_2)\}^{-1} \dots \{p_i \cdot (\hat{k}_1 + \dots + \hat{k}_{u-2})\}^{-1} \\ & \{p_i \cdot (\hat{k}_1 + \dots + \hat{k}_{u-2} + \tilde{k}_r + \tilde{k}_u)\}^{-1} \dots \{p_i \cdot (\hat{k}_1 + \dots + \hat{k}_{u-2} + \tilde{k}_r + \tilde{k}_u + \hat{k}_{u+1} + \dots + \hat{k}_n)\}^{-1} \\ & \epsilon_i^T \left[ \frac{i}{2} \left\{ \prod_{s=1}^{u-2} \hat{\mathcal{E}}_s \right\} \tilde{\varepsilon}_{r,b\mu} p_i^\mu \tilde{\varepsilon}_{u,\rho\sigma} p_i^\sigma \frac{\partial \mathcal{K}(-p_i)}{\partial p_{i\rho}} \frac{\partial \Xi^i(-p_i)}{\partial p_{ib}} \left\{ \prod_{s=u+1}^n (\hat{\mathcal{E}}_s + \hat{\mathcal{L}}_s) \right\} \right] \Gamma_{(i)}(p_i). \end{aligned} \quad (4.25)$$

Summing over the remaining permutations corresponds to treating the  $r$ -th and  $u$ -th graviton as one unit sitting together, and summing over all permutations of the  $n - 1$  objects generated this way. However it will be more convenient for us to first add to this the contribution from the diagrams shown in figure 12 and figure 13. Since these diagrams are subleading, we need to pick the leading contribution from all the vertices and propagators. For the product of any of the  $\Gamma^{(3)}$  vertices and the propagator to the right of this vertex, we can use (2.30) to generate factors of  $(\hat{\mathcal{E}}_s + \hat{\mathcal{L}}_s)$  in the numerator. Using (2.29) we can argue that in all such factors to the left of the vertex, where momentum  $\tilde{k}_r + \tilde{k}_u$  enters the



**Figure 12.** A subleading contribution to the amplitude with multiple soft gravitons.



**Figure 13.** Another subleading contribution to the amplitude with multiple soft gravitons.

finite energy line, we can drop the  $\widehat{\mathcal{L}}_s$  factors. Following analysis similar to the ones leading to (2.43) and (2.46), we arrive at the following results for figures 12 and 13 respectively,

$$\begin{aligned}
 & (p_i \cdot \hat{k}_1)^{-1} \{p_i \cdot (\hat{k}_1 + \hat{k}_2)\}^{-1} \cdots \{p_i \cdot (\hat{k}_1 + \cdots + \hat{k}_{u-2})\}^{-1} \\
 & \{p_i \cdot (\hat{k}_1 + \cdots + \hat{k}_{u-2} + \tilde{k}_r + \tilde{k}_u)\}^{-1} \cdots \{p_i \cdot (\hat{k}_1 + \cdots + \hat{k}_{u-2} + \tilde{k}_r + \tilde{k}_u + \hat{k}_{u+1} + \cdots + \hat{k}_n)\}^{-1} \\
 & \epsilon_i^T \left[ \left\{ \prod_{s=1}^{u-2} \widehat{\mathcal{E}}_s \right\} \left\{ -2\varepsilon_{r\mu}{}^\nu \varepsilon_{u,\nu\rho} p_i^\rho p_i^\mu - \frac{i}{2} \left( \varepsilon_{r,\mu\sigma} \varepsilon_{u,\rho\nu} p_i^\sigma p_i^\nu + \varepsilon_{r,\rho\sigma} \varepsilon_{u,\mu\nu} p_i^\sigma p_i^\nu \right) \frac{\partial \mathcal{K}(-p_i)}{\partial p_{i\mu}} \frac{\partial \Xi^i(-p_i)}{\partial p_{i\rho}} \right\} \right. \\
 & \left. \left\{ \prod_{s=u+1}^n (\widehat{\mathcal{E}}_s + \widehat{\mathcal{L}}_s) \right\} \right] \Gamma_{(i)}(p_i), \tag{4.26}
 \end{aligned}$$

and<sup>3</sup>

$$\begin{aligned}
 & (\tilde{k}_r \cdot \tilde{k}_u)^{-1} (p_i \cdot \hat{k}_1)^{-1} \{p_i \cdot (\hat{k}_1 + \hat{k}_2)\}^{-1} \cdots \{p_i \cdot (\hat{k}_1 + \cdots + \hat{k}_{u-2})\}^{-1} \\
 & \{p_i \cdot (\hat{k}_1 + \cdots + \hat{k}_{u-2} + \tilde{k}_r + \tilde{k}_u)\}^{-1} \cdots \{p_i \cdot (\hat{k}_1 + \cdots + \hat{k}_{u-2} + \tilde{k}_r + \tilde{k}_u + \hat{k}_{u+1} + \cdots + \hat{k}_n)\}^{-1} \\
 & \epsilon_i^T \left[ \left\{ \prod_{s=1}^{u-2} \widehat{\mathcal{E}}_s \right\} \left\{ -(\tilde{k}_u \cdot \tilde{\varepsilon}_r \cdot \tilde{\varepsilon}_u \cdot p_i)(\tilde{k}_u \cdot p_i) - (\tilde{k}_r \cdot \tilde{\varepsilon}_u \cdot \tilde{\varepsilon}_r \cdot p_i)(\tilde{k}_r \cdot p_i) \right. \right. \\
 & + (\tilde{k}_u \cdot \tilde{\varepsilon}_r \cdot \tilde{\varepsilon}_u \cdot p_i)(\tilde{k}_r \cdot p_i) + (\tilde{k}_r \cdot \tilde{\varepsilon}_u \cdot \tilde{\varepsilon}_r \cdot p_i)(\tilde{k}_u \cdot p_i) - \tilde{\varepsilon}_r^{cd} \varepsilon_{u,cd}(\tilde{k}_r \cdot p_i)(\tilde{k}_u \cdot p_i) \\
 & \left. \left. - 2(p_i \cdot \tilde{\varepsilon}_r \cdot \tilde{k}_u)(p_i \cdot \tilde{\varepsilon}_u \cdot \tilde{k}_r) + (p_i \cdot \tilde{\varepsilon}_u \cdot p_i)(\tilde{k}_u \cdot \tilde{\varepsilon}_r \cdot \tilde{k}_u) + (p_i \cdot \tilde{\varepsilon}_r \cdot p_i)(\tilde{k}_r \cdot \tilde{\varepsilon}_u \cdot \tilde{k}_r) \right\} \right. \\
 & \left. \left\{ \prod_{s=u+1}^n (\widehat{\mathcal{E}}_s + \widehat{\mathcal{L}}_s) \right\} \right] \Gamma_{(i)}(p_i). \tag{4.27}
 \end{aligned}$$

<sup>3</sup>We could have dropped the  $\widehat{\mathcal{L}}_s$  factors from (4.27) using (2.29), but will postpone this till the next step.

After adding these to (4.23), (4.25) the terms involving derivatives of  $\mathcal{K}$  and  $\Xi$  get canceled. Once these terms cancel, we can drop the terms proportional to  $\widehat{\mathcal{L}}_s$ . The result takes the form

$$\begin{aligned}
 & (p_i \cdot \hat{k}_1)^{-1} \{p_i \cdot (\hat{k}_1 + \hat{k}_2)\}^{-1} \cdots \{p_i \cdot (\hat{k}_1 + \cdots + \hat{k}_{u-2})\}^{-1} \\
 & \{p_i \cdot (\hat{k}_1 + \cdots + \hat{k}_{u-2} + \tilde{k}_r + \tilde{k}_u)\}^{-1} \cdots \{p_i \cdot (\hat{k}_1 + \cdots + \hat{k}_{u-2} + \tilde{k}_r + \tilde{k}_u + \hat{k}_{u+1} + \cdots + \hat{k}_n)\}^{-1} \\
 & (\tilde{k}_r \cdot \tilde{k}_u)^{-1} \epsilon_i^T \left[ \left\{ \prod_{s=1}^{u-2} \widehat{\mathcal{E}}_s \right\} \left\{ \prod_{s=u+1}^n \widehat{\mathcal{E}}_s \right\} \left\{ -2 \tilde{k}_r \cdot \tilde{k}_u \varepsilon_{r\mu}^\nu \varepsilon_{u,\nu\rho} p_i^\rho p_i^\mu - (\tilde{k}_u \cdot \tilde{\varepsilon}_r \cdot \tilde{\varepsilon}_u \cdot p_i)(\tilde{k}_u \cdot p_i) \right. \right. \\
 & - (\tilde{k}_r \cdot \tilde{\varepsilon}_u \cdot \tilde{\varepsilon}_r \cdot p_i)(\tilde{k}_r \cdot p_i) + (\tilde{k}_u \cdot \tilde{\varepsilon}_r \cdot \tilde{\varepsilon}_u \cdot p_i)(\tilde{k}_r \cdot p_i) + (\tilde{k}_r \cdot \tilde{\varepsilon}_u \cdot \tilde{\varepsilon}_r \cdot p_i)(\tilde{k}_u \cdot p_i) \\
 & - \tilde{\varepsilon}_r^{cd} \tilde{\varepsilon}_{u,cd} (\tilde{k}_r \cdot p_i)(\tilde{k}_u \cdot p_i) - 2(p_i \cdot \tilde{\varepsilon}_r \cdot \tilde{k}_u)(p_i \cdot \tilde{\varepsilon}_u \cdot \tilde{k}_r) + (p_i \cdot \tilde{\varepsilon}_u \cdot p_i)(\tilde{k}_u \cdot \tilde{\varepsilon}_r \cdot \tilde{k}_u) \\
 & \left. \left. + (p_i \cdot \tilde{\varepsilon}_r \cdot p_i)(\tilde{k}_r \cdot \tilde{\varepsilon}_u \cdot \tilde{k}_r) \right\} \right] \Gamma_{(i)}(p_i). \tag{4.28}
 \end{aligned}$$

We can now sum over all permutations of the soft gravitons carrying momenta  $\hat{k}_1, \dots, \hat{k}_{u-2}, \hat{k}_{u+1}, \dots, \hat{k}_n$  and the relative position of the unit carrying momentum  $\tilde{k}_r + \tilde{k}_u$  among these. The only factors that differ for different permutations are the factors in the first two lines of (4.28). Sum over permutations using (A.1) converts these to

$$\left\{ \prod_{s=1}^{u-1} (p_i \cdot \hat{k}_s)^{-1} \right\} \left\{ \prod_{s=u+1}^n (p_i \cdot \hat{k}_s)^{-1} \right\} \{p_i \cdot (\tilde{k}_r + \tilde{k}_u)\}^{-1} = \{p_i \cdot (\tilde{k}_r + \tilde{k}_u)\}^{-1} \left\{ \prod_{\substack{s=1 \\ s \neq r, u}}^n (p_i \cdot \tilde{k}_s)^{-1} \right\}. \tag{4.29}$$

where we have used the fact that the unordered set  $\{\tilde{k}_r, \tilde{k}_u, \hat{k}_1, \dots, \hat{k}_{u-2}, \hat{k}_{u+1}, \dots, \hat{k}_n\}$  corresponds to the set  $\{\tilde{k}_1, \dots, \tilde{k}_n\}$ . Using a similar relation for the polarizations we can express the product of  $\widehat{\mathcal{E}}_s$  factors in (4.28) as  $\prod_{s \neq r, u} \mathcal{E}_s$ . We now sum over all possible choices of  $r, u$  from the set  $\{1, \dots, n\}$ , and add to this the contribution (4.7), (4.9). This gives

$$\begin{aligned}
 & \sum_{\substack{r, u=1 \\ r < u}}^n \left\{ \prod_{\substack{s=1 \\ s \neq r, u}}^n (\tilde{\varepsilon}_s^{\mu\nu} p_{i\mu} p_{i\nu}) \right\} \left\{ \prod_{\substack{s=1 \\ s \neq r, u}}^n (p_i \cdot \tilde{k}_s)^{-1} \right\} \{p_i \cdot (\tilde{k}_r + \tilde{k}_u)\}^{-1} \mathcal{M}(p_i; \tilde{\varepsilon}_r, \tilde{k}_r, \tilde{\varepsilon}_u, \tilde{k}_u) \epsilon_i^T \Gamma_{(i)}(p_i) \\
 & = \sum_{\substack{r, u \in A_i \\ r < u}} \left\{ \prod_{\substack{s \in A_i \\ s \neq r, u}} (\varepsilon_s^{\mu\nu} p_{i\mu} p_{i\nu}) \right\} \left\{ \prod_{\substack{s \in A_i \\ s \neq r, u}} (p_i \cdot k_s)^{-1} \right\} \{p_i \cdot (k_r + k_u)\}^{-1} \mathcal{M}(p_i; \varepsilon_r, k_r, \varepsilon_u, k_u) \epsilon_i^T \Gamma_{(i)}(p_i), \tag{4.30}
 \end{aligned}$$

where we have used the fact that the set  $\{\tilde{k}_1, \dots, \tilde{k}_n\}$  corresponds to the set  $\{k_a; a \in A_i\}$ , and that a similar relation exists also for the polarization tensors.

Summing over all insertions of all other soft gravitons on other legs we now get the result

$$\begin{aligned}
 & \sum_{\substack{A_1, \dots, A_N; A_i \subset \{1, \dots, M\} \\ A_i \cap A_j = \emptyset \text{ for } i \neq j; A_1 \cup A_2 \cup \dots \cup A_N = \{1, \dots, M\}}} \sum_{i=1}^N \left[ \prod_{\substack{j=1 \\ j \neq i}}^N \prod_{q \in A_j} \left\{ (p_j \cdot k_q)^{-1} \varepsilon_{q,\mu\nu} p_j^\mu p_j^\nu \right\} \right] \\
 & \sum_{\substack{r, u \in A_i \\ r < u}} \left[ \{p_i \cdot (k_r + k_u)\}^{-1} \prod_{\substack{s \in A_i \\ s \neq r, u}} \left\{ (p_i \cdot k_s)^{-1} \varepsilon_{s,\mu\nu} p_i^\mu p_i^\nu \right\} \right. \\
 & \left. \mathcal{M}(p_i; \varepsilon_r, k_r, \varepsilon_u, k_u) \Gamma(\epsilon_1, p_1, \dots, \epsilon_N, p_N) \right]. \tag{4.31}
 \end{aligned}$$

After rearrangement of the sums and products, this reproduces the terms on the last line of (4.1). This completes our proof that amplitudes with multiple soft gravitons are given by (4.1).

Finally let us briefly discuss the gauge invariance of (4.1). For this it will be useful to use the compact notation for the amplitude  $A$  as given in eq. (1.1). Let us suppose that we transform  $\varepsilon_p$  by the gauge transformation  $\delta_p$  defined in (3.4). Then the non-vanishing contribution to  $\delta_p A$  is given by

$$\begin{aligned} & \left\{ \prod_{i=1}^N \epsilon_{i,\alpha_i} \right\} \sum_{\substack{s=1 \\ s \neq p}}^M \left\{ \prod_{\substack{r=1 \\ r \neq s,p}}^M S_r^{(0)} \right\} \delta_p S_p^{(0)} [S_s^{(1)} \Gamma]^{\alpha_1 \dots \alpha_p} \\ & + \left\{ \prod_{i=1}^N \epsilon_{i,\alpha_i} \right\} \sum_{\substack{r,u=1 \\ r < u}}^M \left\{ \prod_{\substack{s=1 \\ s \neq r,u}}^M S_s^{(0)} \right\} \left\{ \sum_{j=1}^N \{p_j \cdot (k_r + k_u)\}^{-1} \delta_p \mathcal{M}(p_j; \varepsilon_r, k_r, \varepsilon_u, k_u) \right\} \Gamma^{\alpha_1 \dots \alpha_N}. \end{aligned} \tag{4.32}$$

The first line of (4.32) can be evaluated using (3.5), and yields the result

$$- 2 \left\{ \prod_{i=1}^N \epsilon_{i,\alpha_i} \right\} \sum_{\substack{s=1 \\ s \neq p}}^M \left\{ \prod_{\substack{r=1 \\ r \neq s,p}}^M S_r^{(0)} \right\} S_s^{(0)} k_s \cdot \xi_p \Gamma^{\alpha_1 \dots \alpha_N}. \tag{4.33}$$

The second line of (4.32) receives contribution from the choices  $r = p$  or  $u = p$ . Since  $\mathcal{M}(p_j; \varepsilon_r, k_r, \varepsilon_u, k_u)$  is symmetric under the exchange of  $r$  and  $u$ , we can take  $u = p$  and replace the  $r < u$  constraint in the sum by  $r \neq p$ . Therefore the second line of (4.32) takes the form

$$\left\{ \prod_{i=1}^N \epsilon_{i,\alpha_i} \right\} \sum_{\substack{r=1 \\ r \neq p}}^M \left\{ \prod_{\substack{s=1 \\ s \neq r,p}}^M S_s^{(0)} \right\} \left\{ \sum_{j=1}^N \{p_j \cdot (k_r + k_p)\}^{-1} \delta_p \mathcal{M}(p_j; \varepsilon_r, k_r, \varepsilon_p, k_p) \right\} \Gamma^{\alpha_1 \dots \alpha_N}. \tag{4.34}$$

Using (3.6) we can now express this as

$$2 \left\{ \prod_{i=1}^N \epsilon_{i,\alpha_i} \right\} \sum_{\substack{r=1 \\ r \neq p}}^M \left\{ \prod_{\substack{s=1 \\ s \neq r,p}}^M S_s^{(0)} \right\} S_r^{(0)} k_r \cdot \xi_p \Gamma^{\alpha_1 \dots \alpha_N}. \tag{4.35}$$

This precisely cancels (4.33), establishing gauge invariance of the amplitude.

## Acknowledgments

We would like to thank Anirban Basu, Dileep Jatkar, Alok Laddha, Bhuwanesh Rao Patil and Arnab Priya Saha for discussions. The work of A.S. was supported in part by the JC Bose fellowship of the Department of Science and Technology, India. The work of MV was also supported by the SPM fellowship of CSIR. We also thank the people and Government of India for their continuous support for theoretical physics.

## A Summation identities

In this appendix we list three summation identities that are used in the analysis in section 4.

$$\sum_{\text{all permutations of subscripts } 1, \dots, n} \prod_{m=1}^n (a_1 + a_2 + \dots + a_m)^{-1} = \prod_{m=1}^n (a_m)^{-1}. \quad (\text{A.1})$$

$$\begin{aligned} & \sum_{\text{all permutations of subscripts } 1, \dots, n} \sum_{m=2}^n \sum_{\substack{r, u=1 \\ r < u}}^m b_{ru} (a_1 + \dots + a_m)^{-1} \prod_{\ell=1}^n (a_1 + \dots + a_\ell)^{-1} \\ &= \prod_{m=1}^n (a_m)^{-1} \sum_{\substack{r, u=1 \\ r < u}}^n b_{ru} (a_r + a_u)^{-1} \quad \text{for } b_{rs} = b_{sr} \text{ for } 1 \leq r < s \leq n. \end{aligned} \quad (\text{A.2})$$

$$\begin{aligned} & \sum_{\text{all permutations of subscripts } 1, \dots, n} \sum_{\substack{r, u=1 \\ r < u}}^n c_{ur} \prod_{\ell=1}^n (a_1 + \dots + a_\ell)^{-1} \\ &= \prod_{m=1}^n (a_m)^{-1} \sum_{\substack{r, u=1 \\ r < u}}^n (a_r + a_u)^{-1} (a_u c_{ur} + a_r c_{ru}). \end{aligned} \quad (\text{A.3})$$

The proof of these identities may be given as follows. Let us first consider (A.1). The summand on the left hand side may be expressed as

$$\int_0^\infty ds_1 e^{-s_1 a_1} \int_0^\infty ds_2 e^{-s_2 (a_1 + a_2)} \dots \int_0^\infty ds_n e^{-s_n (a_1 + \dots + a_n)}. \quad (\text{A.4})$$

Defining new variables

$$t_1 = s_1 + s_2 + \dots + s_n, \quad t_2 = s_2 + \dots + s_n, \dots, \quad t_n = s_n, \quad (\text{A.5})$$

we may express (A.4) as

$$\int_R dt_1 dt_2 \dots dt_n e^{-t_1 a_1 - t_2 a_2 - \dots - t_n a_n} \quad (\text{A.6})$$

where the integration range  $R$  is

$$\infty > t_1 \geq t_2 \geq \dots \geq t_{n-1} \geq t_n \geq 0. \quad (\text{A.7})$$

Summing over all permutations of the subscripts  $1, \dots, n$  can now be implemented by summing over permutations of  $t_1, \dots, t_n$ . This has the effect of making the integration range unrestricted, with each  $t_i$  running from 0 to  $\infty$ . The corresponding integral generates the right hand side of (A.1).

The proof of (A.3) follows from a simple variation of this. For this note that the coefficient of the  $c_{ur}$  term on the left hand side for  $r < u$  is given by a sum over permutations with the same summand as in (A.1), but with the restriction that we sum over those

permutations in which  $r$  comes before  $u$ . Translated to (A.7) this means that after summing over permutations the restriction  $t_r > t_u$  is still maintained. Therefore the result is

$$\int_{t_r > t_u} dt_1 dt_2 \cdots dt_n e^{-t_1 a_1 - t_2 a_2 - \cdots - t_n a_n}. \quad (\text{A.8})$$

This integral can be easily evaluated to give

$$(a_1 \cdots a_n)^{-1} a_u (a_r + a_u)^{-1}. \quad (\text{A.9})$$

This is precisely the coefficient of  $c_{ur}$  on the right hand side of (A.3). Similarly in the computation of the coefficient of  $c_{ru}$  for  $r < u$  we only sum over those permutations for which  $u$  comes before  $r$ . This has the effect of changing the constraint  $t_r \geq t_u$  to  $t_r \leq t_u$  in (A.8) and reproduces correctly the coefficient on  $c_{ru}$  on the right hand side of (A.3).

Finally let us consider (A.2). We begin with a different sum

$$\sum_{\text{all permutations of subscripts } 1, \dots, n} \prod_{\ell=1}^n \left( a_1 + a_2 + \cdots + a_\ell - \sum_{\substack{r, u=1 \\ r < u}}^{\ell} b_{ru} \right)^{-1}, \quad (\text{A.10})$$

and note that the first subleading term in a Taylor series expansion of (A.10) in powers of  $b_{mn}$ 's give the left hand side of (A.2). We now manipulate this as before, arriving at the analog of (A.4):

$$\int_0^\infty ds_1 e^{-s_1 a_1} \int_0^\infty ds_2 e^{-s_2 (a_1 + a_2 - b_{12})} \cdots \int_0^\infty ds_n e^{-s_n (a_1 + \cdots + a_n - \sum_{\substack{r, u=1 \\ r < u}}^n b_{ru})}. \quad (\text{A.11})$$

The change of variables given in (A.5) converts this to

$$\begin{aligned} & \int_R dt_1 dt_2 \cdots dt_n e^{-t_1 a_1 - t_2 a_2 - \cdots - t_n a_n} \\ & \exp \left[ (t_2 - t_3) b_{12} + (t_3 - t_4) (b_{12} + b_{23} + b_{13}) + \cdots + (t_{n-1} - t_n) \sum_{\substack{r, u=1 \\ r < u}}^{n-1} b_{ru} + t_n \sum_{\substack{r, u=1 \\ r < u}}^n b_{ru} \right] \\ & = \int_R dt_1 dt_2 \cdots dt_n e^{-t_1 a_1 - t_2 a_2 - \cdots - t_n a_n} \exp \left[ \sum_{\substack{r, u=1 \\ r < u}}^n b_{ru} t_u \right]. \end{aligned} \quad (\text{A.12})$$

We now expand the last factor of (A.12) in a Taylor series expansion and pick the coefficient of the  $b_{ru}$  term. This has the effect of multiplying the integrand by  $t_u$  and restrict the sum over permutations to those for which  $r$  remains to the left of  $u$ . However as  $b_{ru}$  is symmetric in  $r, u$ , there is also another term related to this one under the exchange of the subscripts  $r$  and  $u$ . Therefore the integral is given by

$$\int_{t_r > t_u} dt_1 dt_2 \cdots dt_n e^{-t_1 a_1 - t_2 a_2 - \cdots - t_n a_n} t_u + (r \leftrightarrow u). \quad (\text{A.13})$$

Evaluation of this integral gives

$$(a_1 \cdots a_n)^{-1} \left\{ \frac{a_u}{(a_r + a_u)^2} + \frac{a_r}{(a_r + a_u)^2} \right\} = (a_1 \cdots a_n)^{-1} (a_r + a_u)^{-1}. \quad (\text{A.14})$$

This is precisely the coefficient of  $b_{ru}$  on the right hand side of (A.2).

We can also give recursive proof of all the identities without using the integral representations. Let us begin with the identity (A.1). Let us suppose that it holds for  $(n - 1)$  objects. We now organise the sum over permutations of all subscripts  $1, \dots, n$  in (A.1) by first fixing the last element to be some integer  $i$ , and summing over all permutations of the subscripts other than  $i$ . This gives, using (A.1) for  $(n - 1)$  objects,

$$(a_1 \cdots a_{i-1} a_{i+1} \cdots a_n)^{-1} (a_1 + \cdots + a_n)^{-1}. \quad (\text{A.15})$$

We now sum over all possible choices of  $i$ . This gives

$$\sum_{i=1}^n (a_1 \cdots a_{i-1} a_{i+1} \cdots a_n)^{-1} (a_1 + \cdots + a_n)^{-1}. \quad (\text{A.16})$$

This can be written as

$$(a_1 \cdots a_n)^{-1} (a_1 + \cdots + a_n)^{-1} \sum_{i=1}^n a_i = (a_1 \cdots a_n)^{-1}, \quad (\text{A.17})$$

reproducing the right hand side of (A.1).

A recursive proof of (A.3) can be given as follows. Let us again assume that the identity is valid for  $(n - 1)$  objects. Now for  $u > r$ , the coefficient of  $c_{ur}$  on the left hand side involves a sum over permutations of the subscripts  $1, \dots, n$ , with the same summand as in identity (A.1), but with the restriction that  $r$  always appears to the left of  $u$  in the permutation. We now organise the sum as follows. First we fix the last element and sum over permutations of the first  $(n - 1)$  elements. If the last element is  $i$  with  $i \neq u$ , then the result, using (A.2) for  $(n - 1)$  objects, is given by

$$\left\{ \prod_{\substack{m=1 \\ m \neq i}}^n (a_m)^{-1} \right\} a_u (a_u + a_r)^{-1} (a_1 + \cdots + a_n)^{-1}. \quad (\text{A.18})$$

Note that  $i$  cannot be  $r$  since that will violate the rule that the  $r$  always appears to the left of  $u$ . On the other hand if the last element is  $u$  then the sum over permutations over the first  $(n - 1)$  elements becomes unrestricted and we can apply (A.1) to get

$$\left\{ \prod_{\substack{m=1 \\ m \neq u}}^n (a_m)^{-1} \right\} (a_1 + \cdots + a_n)^{-1}. \quad (\text{A.19})$$

Therefore the total answer, obtained by summing over all possible choices of the last element (other than  $r$ ), is

$$\sum_{i \neq r, u} \left\{ \prod_{\substack{m=1 \\ m \neq i}}^n (a_m)^{-1} \right\} a_u (a_u + a_r)^{-1} (a_1 + \cdots + a_n)^{-1} + \left\{ \prod_{\substack{m=1 \\ m \neq u}}^n (a_m)^{-1} \right\} (a_1 + \cdots + a_n)^{-1}. \quad (\text{A.20})$$

Elementary algebra reduces this to

$$(a_1 \cdots a_n)^{-1} a_u (a_r + a_u)^{-1}, \quad (\text{A.21})$$



which is the coefficient of  $c_{ur}$  on the right hand side of (A.3). The analysis for the case  $r > u$  is identical, with the roles of  $r$  and  $u$  interchanged.

Finally we turn to the proof of (A.2). By collecting the coefficients of  $b_{ru}$  on both sides and using the symmetry of  $b_{ru}$ , we can write this identity as

$$\begin{aligned} & \sum_{\text{all permutations of subscripts } 1, \dots, n} \sum_{\substack{m=2 \\ m \geq r, u}}^n (a_1 + \dots + a_m)^{-1} \prod_{\ell=1}^n (a_1 + \dots + a_\ell)^{-1} \\ &= (a_r + a_u)^{-1} \prod_{m=1}^n (a_m)^{-1}. \end{aligned} \quad (\text{A.22})$$

As before, we shall proceed by assuming this to be valid for  $(n-1)$  objects and then prove this for  $n$  objects. Let us first consider the contribution from the  $m = n$  term in the sum on the left hand side of (A.22). The contribution of this term is given by

$$(a_1 + \dots + a_n)^{-2} \sum_{\text{all permutations of subscripts } 1, \dots, n} \prod_{\ell=1}^{n-1} (a_1 + \dots + a_\ell)^{-1}. \quad (\text{A.23})$$

We now perform the sum over all permutations by fixing the last element to be some fixed number  $i$ , sum over permutations of the rest for which we can use (A.1), and then sum over all choices of  $i$ . This gives

$$(a_1 + \dots + a_n)^{-2} \sum_{i=1}^n \left\{ \prod_{\substack{m=1 \\ m \neq i}}^n (a_m)^{-1} \right\} = (a_1 + \dots + a_n)^{-1} \left\{ \prod_{m=1}^n (a_m)^{-1} \right\}. \quad (\text{A.24})$$

Next we consider the contribution to the sum in the left hand side of (A.22) for  $m \leq (n-1)$ . This is given by

$$(a_1 + \dots + a_n)^{-1} \sum_{\text{all permutations of subscripts } 1, \dots, n} \sum_{\substack{m=2 \\ m \geq r, u}}^{n-1} (a_1 + \dots + a_m)^{-1} \prod_{\ell=1}^{n-1} (a_1 + \dots + a_\ell)^{-1}. \quad (\text{A.25})$$

We again perform the sum over permutations by fixing the last element to be some fixed number  $i$ , summing over permutation of the rest of the objects, and then summing over  $i$ . Note however that now  $i$  cannot be either  $r$  or  $u$  since then we cannot satisfy the constraint  $m \geq r, u$ . The sum over permutations can now be performed using (A.22) for  $n-1$  objects and gives

$$\begin{aligned} & (a_1 + \dots + a_n)^{-1} \sum_{\substack{i=1 \\ i \neq r, s}}^n \left\{ \prod_{\substack{m=1 \\ m \neq i}}^n (a_m)^{-1} \right\} (a_r + a_u)^{-1} \\ &= (a_1 + \dots + a_n)^{-1} (a_r + a_u)^{-1} \left\{ \prod_{m=1}^n (a_m)^{-1} \right\} (a_1 + \dots + a_n - a_r - a_u). \end{aligned} \quad (\text{A.26})$$

Adding this to (A.24) we get

$$(a_r + a_u)^{-1} \left\{ \prod_{m=1}^n (a_m)^{-1} \right\}, \quad (\text{A.27})$$

which is precisely the right hand side of (A.22).

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