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Self-consistent large-N analytical solutions of inhomogeneous condensates in quantum $\mathbb{C}P^{N-1}$ model

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Abstract: We give, for the first time, self-consistent large-N analytical solutions of inhomogeneous condensates in the quantum $\mathbb{C}P^{N-1}$ model in the large-N limit. We find a map from a set of gap equations of the $\mathbb{C}P^{N-1}$ model to those of the Gross-Neveu (GN) model (or the gap equation and the Bogoliubov-de Gennes equation), which enables us to find the self-consistent solutions. We find that the Higgs field of the $\mathbb{C}P^{N-1}$ model is given as a zero mode of solutions of the GN model, and consequently only topologically non-trivial solutions of the GN model yield nontrivial solutions of the $\mathbb{C}P^{N-1}$ model. A stable single soliton is constructed from an anti-kink of the GN model and has a broken (Higgs) phase inside its core, in which $\mathbb{C}P^{N-1}$ modes are localized, with a symmetric (confining) phase outside. We further find a stable periodic soliton lattice constructed from a real kink crystal in the GN model, while the Ablowitz-Kaup-Newell-Segur hierarchy yields multiple solitons at arbitrary separations.

Keywords: 1/N Expansion, Field Theories in Lower Dimensions, Sigma Models

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1 Introduction

Nonlinear sigma models such as the $\mathbb{C}P^{N-1}$ model in 1+1 dimensions [1–4] are known to share a number of phenomena common with 3+1 dimensional QCD, e.g. asymptotic freedom, dynamical mass generation, confinement, and instantons [5–12]. The mass gap can be best shown in the large-N analysis in which one solves the gap equations self-consistently, to be consistent with the Coleman-Mermin-Wagner (CMW) theorem forbidding a gapless excitations in 1+1 dimensions [13, 14]. The $\mathbb{C}P^{N-1}$ model, or the $\mathbb{C}P^1$ model equivalent to the $O(3)$ sigma model, appears in a wide range of physics from particle physics to condensed matter physics. The relation between the 1+1 dimensional Heisenberg antiferromagnetic spin chain and the $O(3)$ sigma model has been shown in refs. [15, 16]. Recently, the quantum phase transition, so-called deconfined criticality is proposed in the antiferromagnetic system [17–19]. The sigma model with topological term is known to describe the integer quantum Hall effect [20]. The supersymmetric $\mathbb{C}P^{N-1}$ model was also investigated [21, 22] for which the all order calculation in coupling constant is possible for Gell-Mann-Low function [12], and dynamical mass gap was proved by the mirror symmetry [23]. The analogy between 3+1 dimensional Yang-Mills theory and 1+1 dimensional sigma model, pointed out in ref. [5], has been recently revealed in a rather nontrivial way; a non-Abelian vortex string in a U(N) gauge theory with N scalar fields in the fundamental representation carries $\mathbb{C}P^{N-1}$ moduli [24–27] (see refs. [28–31] as a review), yielding a nontrivial relation between the $\mathbb{C}P^{N-1}$ model on the string worldsheet and the bulk gauge theory [32, 33]. The $\mathbb{C}P^{N-1}$ model defined on an interval [34–36] or on a ring [37, 38] was also studied. The $\mathbb{C}P^{N-1}$ model or the $O(3)$ sigma model at finite temperature and/or density was also investigated in which Berezinskii- Kosterlitz-Thouless transition at nonzero density was examined [39, 40]. One of recent developments is a resurgent structure of the $\mathbb{C}P^{N-1}$ model [41–45], in which a molecule of fractional instantons [46–49] called a bion, plays a crucial role. In spite of tremendous studies of the $\mathbb{C}P^{N-1}$ model, there was no study on inhomogeneous configurations (such as solitons) at quantum level,except for a numerical study of the $\mathbb{C}P^{N-1}$ model on an interval [36].

The situation is rather different for an interacting fermionic theory: the Gross-Neveu (GN) [50] or Nambu-Jona-Lasino model [51, 52], exhibiting dynamical symmetry breaking of discrete or continuous chiral symmetry, thereby sharing an important property with QCD [53–55]. This model is equivalent at the large-N limit or in the mean field approximation to a set of the Bogoliubov-de Gennes (BdG) equations and the gap equation, appearing in condensed matter systems such as conducting polymers [56–59], superconductors, superfluids and ultracold atomic gases [60–62]. Self-consistent analytical solutions such as a real kink [53, 56], a twisted (complex) kink [54], a real kink-anti-kink (polaron) [53, 63, 64], a real kink-anti-kink-kink [55, 65, 66] and more general real solutions [67] have been known. Recently, a theoretical progress has been achieved for inhomogeneous condensates in the 1+1 dimensional (chiral) GN model, e.g., the exact self-consistent and inhomogeneous condensates such as a real kink crystal [68, 69] (Larkin-Ovchinnikov(LO) state [70]), a chiral spiral (Fulde-Ferrell(FF) state [71]), and a twisted kink crystal [72, 73] (FF-LO state) have been found by mapping the equations to the nonlinear Schrödinger equation, and such states have been shown to be ground states in a certain region of the phase diagram for finite temperature and density [74]. More generally, multiple twisted kinks with arbitrary phase and positions [75, 76] can be further constructed systematically due to the integrable structure behind the model known as the Ablowitz-Kaup-Newell-Segur (AKNS) hierarchy for the nonlinear Schrödinger equation [77–79]. Recent developments include time-dependent soliton scatterings [80, 81], multi-component condensates [82–84], a ring geometry [85], and an interval with a Casimir force [86].

In the present work, we reveal an unexpected relation between these two completely different theories, the $\mathbb{C}P^{N-1}$ and GN models developed independently. By finding a map from a set of gap equations of the $\mathbb{C}P^{N-1}$ model to those of the GN model, we find self-consistent analytical solutions of stable inhomogeneous condensates in the quantum $\mathbb{C}P^{N-1}$ model, that is, a single soliton, a soliton lattice and multiple solitons at arbitrary separations.

2 Model and method

We consider the $\mathbb{C}P^{N-1}$ model on an infinite space:

$$S = \int dtdx \left[(D_\mu n_i)^*(D^\mu n_i) - \lambda(n_i^* n_i - r)\right], \tag{2.1}$$

where n^i $(i = 1, \cdots, N)$ are complex scalar fields, $D_\mu = \partial_\mu - iA_\mu$, and $\lambda(x)$ is a Lagrange multiplier. The "radius" r is known to have connection with a coupling constant $g_{\rm YM}$ in the Yang-Mills theory; $r = 4\pi/g_{\rm YM}^2$ if we realize this model on a non-Abelian vortex in U(N) gauge theory. Here we note that the model does not have kinetic term for A_μ and thus we focus on the case of $A_\mu = 0$ throughout this paper. We separate n_i fields into a classical field $n_1 = \sigma$ (real) and $n_i = \tau_i$ $(2, \cdots, N)$. Integrating out the τ_i fields, we obtain the effective action for σ as

$$S_{\text{eff}} = \int dtdx \left[(N-1)\text{Tr}\ln(-\partial_\mu \partial^\mu + \lambda) + \partial_\mu \sigma \partial^\mu \sigma - \lambda(\sigma^2 - r) \right]. \tag{2.2}$$

In the following we consider and the leading contribution of $1/N$ expansion and thus we replace $N-1$ to N.[1] One can formally write down the total energy functional as

$$E = N \sum_n \omega_n + \int dx \left[(\partial_x \sigma)^2 + \lambda(\sigma^2 - r) \right]. \tag{2.3}$$

The corresponding gap equations obtained from the static condition with respect to λ and σ are [36]

$$\frac{N}{2} \sum_n \frac{f_n^2}{\omega_n} + \sigma^2 - r = 0, \tag{2.4}$$

$$\partial_x^2 \sigma - \lambda \sigma = 0, \tag{2.5}$$

respectively, where $f_n(x)$ and ω_n are orthonormal eigenstates and eigenvalues of the following equation

$$(-\partial_x^2 + \lambda) f_n(x) = \omega_n^2 f_n(x). \tag{2.6}$$

We need to solve eqs. (2.4)–(2.6) in a self-consistent manner. We here note from eqs. (2.5) and (2.6) that σ is proportional to a zero mode f_0.

It is well known that assuming a uniform state in infinite system, one finds the confining (unbroken) phase with a constant λ to be a unique solution, to be consistent with the CMW theorem. For the case of a ring, in addition to it, there is a Higgs (broken) phase with a constant σ for a smaller ring [37, 38].

One of the main results of this paper is a map from those equations to the gap equation and eigenvalue equation for the GN model. In order to reduce the number of equations, we introduce the new field Δ such as

$$\Delta^2 + \partial_x \Delta = \lambda(x). \tag{2.7}$$

By using this function, we find a solution to eq. (2.5):

$$\sigma = A \exp\left[\int^x dy \Delta(y) \right], \tag{2.8}$$

where A is the integral constant. The energy in eq. (2.3) can be rewritten as

$$E_{\text{tot}} = N \sum_n \omega_n - r \int_{-\infty}^{\infty} dx (\Delta^2 + \partial_x \Delta) + \sigma \partial_x \sigma |_{-\infty}^{\infty}. \tag{2.9}$$

[1]We note that the large N limit is considered to obtain the self-consistent equations and the rest does not rely on the large N. Furthermore, the mean field approximation (for finite N) also yields the same self-consistent equations. Thus the results in the following are expected to be qualitatively correct even in the case of finite N.

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The rather nontrivial step is to rewrite eq. (2.6) as [See appendix A]

$$\begin{pmatrix} 0 & \partial_x + \Delta \\ -\partial_x + \Delta & 0 \end{pmatrix} \begin{pmatrix} f_n \\ g_n \end{pmatrix} = \omega_n \begin{pmatrix} f_n \\ g_n \end{pmatrix}, \tag{2.10}$$

where g_n's are auxiliary fields and the elimination of g_n yields eq. (2.6). We note that eq. (2.10) together with eq. (2.7) describes a supersymmetric quantum mechanics, in which the potential λ is given by the superpotential Δ [87]. Eq. (2.10) is the positive energy part of the BdG or Andreev equation which corresponds to the Hartree-Fock equation of the GN model with N flavors [See appendix B]

$$L_{\rm GN} = \bar{\psi} i \partial\!\!\!/ \psi + \frac{g^2}{2}(\bar{\psi}\psi)^2. \tag{2.11}$$

The corresponding Hartree-Fock equation becomes $H\psi = E\psi$, with $H = -i\gamma^5\partial_x + \gamma^0\Delta$, where $\gamma^5 = -\sigma_2$ and $\gamma^0 = \sigma_1$ with the Pauli matrices σ_i. Here Δ (real) satisfies $\langle\bar{\psi}\psi\rangle = -\Delta/g$, which is called a gap equation. It is known that the $\mathbb{Z}_2$ symmetry is spontaneously broken in the GN model, yielding two discrete vacua.

With a help of $g_n = (-\partial_x + \Delta) f_n/\omega_n$, one can show that g_n automatically gives a orthonormal set if f_n gives a orthonormal set. Eq. (2.10) has the particle-hole symmetry which enables us to obtain the set $\{-\omega_n, \tilde{f}_n, \tilde{g}_n\}$ from the set $\{\omega_n, f_n, g_n\}$ by $\tilde{f}_n = f_n$ and $\tilde{g}_n = -g_n$. By taking the derivative of eq. (2.4) with respect to x and by substituting eqs. (2.8) and $\omega_n g_n = (-\partial_x + \Delta) f_n$ into that, we obtain

$$\Delta = \frac{N}{2r}\sum_n f_n g_n = -\frac{N}{2r}\sum_n \tilde{f}_n \tilde{g}_n, \tag{2.12}$$

which has the same form with the gap equation for the GN model. Here we note that corresponding fermionic coupling $Ng^2 = N/2r$ is proportional to the 't Hooft coupling in an underlying U(N) gauge theory $Ng^2_{\rm YM}$. Since we solve the differentiated one instead of eq. (2.4) itself, we need to fix the integration constant A for σ by substituting eq. (2.8) into eq. (2.4). For the BdG equation (2.10) and gap equation (2.12), various exact self-consistent solutions are already known. From eq. (2.10) one can immediately find the zero mode solution

$$f_0(x) \propto \exp\left[\int^x dy \Delta(y)\right], \tag{2.13}$$

where the corresponding auxiliary field is $g_0(x) = 0$. The zero mode solutions f_0 in the $\mathbb{C}P^{N-1}$ and GN models are identical. As denoted below eq. (2.6), the Higgs field $\sigma(x)$ in the $\mathbb{C}P^{N-1}$ model is proportional to the zero mode, thereby exists only when corresponding Δ in the GN model is topologically nontrivial with allowing a normalizable zero mode [88].

3 Self-consistent analytical solutions

In the GN model, a constant gap $\Delta = m$ is a solution which can be called the Bardeen-Cooper-Schrieffer (BCS) phase, whereas that for $m = 0$ is called a normal phase. We show that the BCS and normal phases in the GN model correspond to the confining and Higgs

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phases in the $\mathbb{C}P^{N-1}$ model, respectively. For the constant solution, $\omega_n = \sqrt{(\pi n/L)^2 + m^2}$ and the degenerated eigenfunctions are $f_n^{(1)} = \sqrt{2}\sin \pi nx/L, f_n^{(2)} = \sqrt{2}\cos \pi nx/L$. For both the cases, $g_n^{(i)}(x) = (-\partial_x + m)f_n/\omega_n$ $(i = 1, 2)$. Here we consider the periodic boundary condition in domain $[-L/2, L/2]$. The infinite system can be obtained by taking the proper limit of $L \to \infty$. The substitution $\Delta = m$ and corresponding eigenstates into eq. (2.12) yields

$$m = \frac{N}{r}\sum_n \frac{m}{\omega_n}, \tag{3.1}$$

while eq. (2.4) becomes

$$\sigma^2 = r - N\sum_n \frac{1}{\omega_n}. \tag{3.2}$$

We find that the condition (3.1) for $m \neq 0$ and (3.2) for $\sigma = 0$ are equivalent

$$1 = \frac{N}{r}\sum_n \frac{1}{\omega_n}, \tag{3.3}$$

which gives the well known renormalization condition of the coupling constant $g^2 = 4\pi/r$. This results in two possibilities $\{\lambda = m^2, \sigma = 0\}$ (confining phase) and $\{\lambda = 0, \sigma = \text{const}\}$ (Higgs phase), but only the former satisfies the gap equation (2.4) and the latter is not allowed in the infinite system [34, 35, 37, 38].

The solution $\Delta = -m\tanh mx$ is known as a topological kink solution interpolating two discrete vacua of the GN model, which has a zero mode localized near the kink. In the case of kink solution, the eigenvalue is the same with the constant solution $\omega_n = \sqrt{(\pi n/L)^2 + m^2}$ while the degenerated eigenfunctions are $f_n^{(i)} = (\partial_x - m\tanh mx)g_n^{(i)}/\omega_n$ with $g_n^{(1)} = \sqrt{2}\sin \pi nx/L$, $g_n^{(2)} = \sqrt{2}\cos \pi nx/L$. We also have a normalizable zero mode $f_0(x) \propto 1/\cosh mx$, $g_0(x) = 0$. Thus eq. (2.12) yields

$$-m\tanh mx = \frac{N}{r}\sum_n \frac{-m\tanh mx}{\omega_n}, \tag{3.4}$$

which indeed gives the same condition with eq. (3.1). On the other hand, eq. (2.4) implies

$$\sigma^2 = r - N\sum_n \frac{1}{\omega_n} + \frac{m^2}{\cosh^2 mx} N\sum_n \frac{1}{\omega_n^3}. \tag{3.5}$$

In the case of $m \neq 0$, eq. (3.4) yields eq. (3.3) and we reach at

$$\sigma = \frac{m}{\cosh mx}\sqrt{N\sum_n \frac{1}{\omega_n^3}}, \tag{3.6}$$

which has a bright solitonic profile. Again, it is indeed proportional to the zero mode solution. In this case, the mass gap function becomes

$$\lambda(x) = m^2(1 - 2\cosh^{-2} mx), \tag{3.7}$$

which has a gray soliton configuration and is called the Pöschl-Teller potential [87]. Since all the eigenenergies of this solution are non-negative, the solution is stable. In figure 1, we

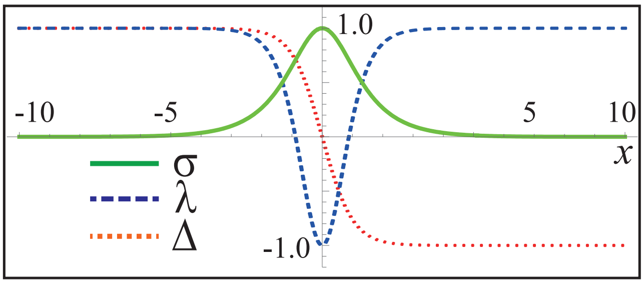

Figure 1. The configuration of σ (solid line) and λ (dashed line) for $\Delta = -m \tanh mx$ (dotted line). Here we normalize as $\sigma(0) = 1$ and $m = 1$.

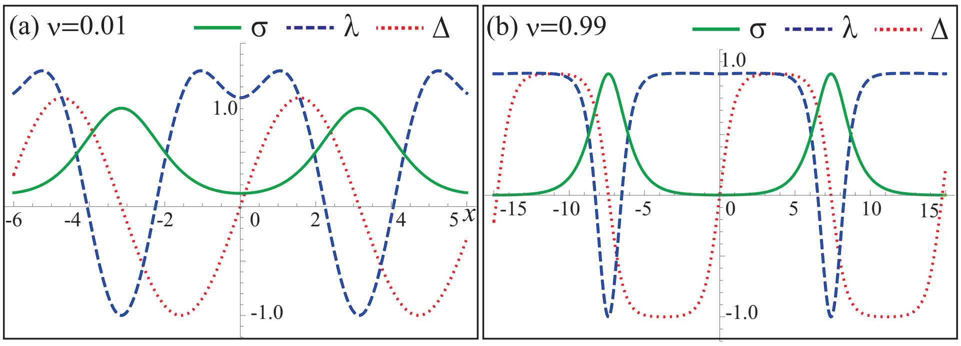

Figure 2. The bright soliton lattice configuration of σ (solid line) and λ (dashed line) for $\nu = 10^{-2}$ (left figure) and $\nu = 1 - 10^{-2}$ (right figure). The auxiliary field Δ (dotted line) are also plotted. Here we set, $m = 1$ and normalize the peak of σ to be 1.

plot the configuration of $\sigma(x)$ and the mass gap function $\lambda(x)$. The energy of the soliton can be calculated by the energy E_s for the soliton configuration in eqs. (3.6) and (3.7) subtracted by E_0 for the confining phase ($\sigma_0 = 0$ and $\lambda_0 = m^2$), for both of which the third term in eq. (2.9) vanishes from the equation of motion (2.5) and ω_n's are the same. We thus obtain

$$E_s - E_0 = \int_{-\infty}^{\infty} dxr(\lambda_0 - \lambda_s) = 4rm. \tag{3.8}$$

Since σ has a localized profile function, soliton core is in the Higgs (broken) phase where the $\mathbb{C}P^{N-1}$ modes are localized, while the bulk is in the confining (symmetric) phase, in contrast to a uniform system allowing only the confining phase in infinite system to be consistent with the CMW theorem. It is known that the correlation function behaves at large distance as $x^{-1/N}$ in 1+1 dimension [89], which inhibits the long-range order for finite N. Here we have obtained the Higgs phase localized with length $\sim 1/m$, thus the robustness of our solitonic solution is expected if N is sufficiently large as $\ln(1/m) \ll N$.[2]

The above solutions can be obtained from a soliton lattice obtained from a real kink crystal in the GN model:

$$\Delta(x) = m\mathrm{sn}(mx, \nu), \tag{3.9}$$

[2]It is also the case of the $\mathbb{C}P^{N-1}$ model on a ring: the Higgs phase is allowed for a smaller ring [37, 38].

where sn, cn, and dn (appearing later) are the Jacobi functions and ν is elliptic parameter. Here the periodicity of the above solution is given by $\ell = 4K(\nu)/m$, where $K(\nu)$ is a complete elliptic integral of the first kind. This solution together with eq. (2.8) gives a soliton lattice:

$$\sigma = A\left[\frac{-\sqrt{\nu}\,\mathrm{cn}(x,\nu)+\mathrm{dn}(x,\nu)}{1-\sqrt{\nu}}\right]^{\pm\frac{1}{\sqrt{\nu}}}. \tag{3.10}$$

In figure 2, we plot the mass gap function λ and σ for $\nu = 10^{-2}$ and $\nu = 1-10^{-2}$. The auxiliary field Δ are also plotted. The Higgs field σ in this solution has a bright soliton lattice profile. By taking $\nu = 1$ limit for $\Delta = m\mathrm{sn}(mx + K(\nu), \nu)$, λ becomes constant and $\sigma = 0$ in the whole system. This limit corresponds to the constant solution discussed above. On the other hand, $\Delta = m\mathrm{sn}(mx+2K(\nu), \nu)$ reduces to $\lambda = m[1-2/\cosh^2 mx]$ and $\sigma(x) \propto m/\cosh mx$. This corresponds to the kink solution [See appendix A]. Our periodic soliton solutions can be put on a ring, while the previous studies on the $\mathbb{C}P^{N-1}$ model on a ring dealt with only constant configurations [37, 38].

4 Higher order self-consistent analytical solutions

In the GN model, the integrable structure enables us to systematically construct all possible exact self-consistent solutions [78, 79]. The above solutions belong to the lowest order ($n = 1$) of the AKNS hierarchy (denoted by AKNS$_n$ for $n = 1, 2, \cdots$) for the nonlinear Schrödinger equation [78, 79] [See appendix B]. The configuration of a kink-anti-kink (polaron) in the GN model [63, 64] (in AKNS$_2$) does not yield a nontrivial solution in the $\mathbb{C}P^{N-1}$ model, while the three kink solution (in AKNS$_3$) [55, 65, 66]

$$\Delta = k\tanh[kx - k\delta + R] - \frac{\omega_b e^R[\sinh(m_+x - k\delta + 2R) + \sinh(m_-x + k\delta)]}{\cosh(m_+x - k\delta + 2R) + e^{2R}\cosh(m_-x + k\delta)},$$

does. Here $\omega_b = \sqrt{m^2 - k^2}$, $R = (1/2)\ln(m_+/m_-)$, and $m_\pm = m \pm k$. In figure 3, we plot the configurations of σ, λ, and Δ for various parameter choices. The symmetric case $\delta = 0$ (a) looks like a double copy of a single soliton in figure 1. For larger δ the middle kink is closer to the right anti-kink than the left anti-kink in Δ as (b), and then the amplitude of the Higgs field σ localized in the right soliton of λ decreases with increasing δ. On the other hand, the parameter k controls the soliton-soliton distance [(a), (c), and (d)]. The two solitons merge for larger k and eventually becomes one soliton in $k \to 1$. This is possible because the three kink solution belongs to the same topological sector with the single kink solution in the GN model. In general, AKNS$_{2k+1}$ ($k = 1, 2, \cdots$) yields solutions of k solitons with arbitrary positions exhibiting the similar behaviors.

5 Summary

We have found the map from the GN model to the $\mathbb{C}P^{N-1}$ model, which enables us to construct, for the first time, the exact self-consistent stable inhomogeneous solutions of the $\mathbb{C}P^{N-1}$ model; a single soliton, a soliton lattice and multiple solitons with arbitrary separations. The Higgs (broken) phase appears inside the soliton cores where the Higgs field σ has bright solitonic profiles and the $\mathbb{C}P^{N-1}$ moduli are confined.

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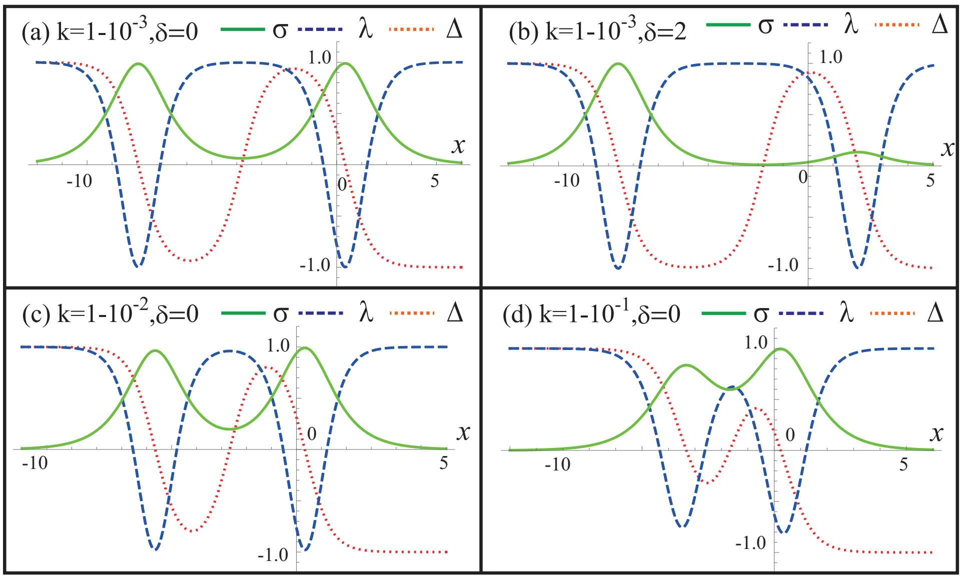

Figure 3. The two bright soliton configuration of σ (solid line) and λ (dashed line). The auxiliary field Δ (dotted line) are also plotted. Here we set, $m = 1$. In figure (a) and (b) we plot the case of $\delta = 0$ and $\delta = 2$, respectively, with $k = 1 - 10^{-3}$. In figure (c) and (d) we plot the case of $k = 1 - 10^{-2}$ and $k = 1 - 10^{-1}$, respectively, with $\delta = 0$. In the figure, we normalize σ such that the hight of the highest peak is 1.

It is an open question whether there is a map to the chiral GN model with continuous chiral symmetry, which allows a variety of complex solutions. In the (chiral) GN model, the inhomogeneous phase is stabilized at the low temperature and high density [74], or in the presence of a chiral chemical potential, equivalent to the constant Zeeman magnetic field on the superconductivity [60]. Such analogies in the $\mathbb{C}P^{N-1}$ model may imply a possibility of a crystaline phase. While our periodic soliton lattice can be put on a ring, an extension to an interval [34–36] is also possible to calculate a Casimir force [90], since the exact solutions in the GN model on an interval have been found recently [86]. Another relation between the $\mathbb{C}P^{N-1}$ model and the GN model in $2+1$ dimensions has recently been found in ref. [91] in which the large-N free energy densities for the both theories are found to be remarkably similar. Though it would be important to see whether the similar structure also appears in the $1+1$ dimensions, we leave it as a future problem. The connection between our formalism and the bosonization scheme in $1+1$ dimensions should be also important. The former gives the coincidence of the self-consistent equations in $\mathbb{C}P^{N-1}$ model and the GN model, whereas the latter yields the sine-Gordon model as the bosonized model of the GN model [89]. We also leave it as a future problem. Physical consequences of our solitons on a non-Abelian vortex in supersymmetric gauge theories [28–31] or dense QCD [92] will be an important problem to be explored.

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A Alternative mapping

In this appendix, we show an alternative map from the Gross-Neveu model to the $\mathbb{C}P^{N-1}$ model. In our formalism, f_n's are chosen as upper components of BdG equation ($u_n = f_n,\ v_n = g_n$) in

$$\begin{pmatrix} 0 & \partial_x + \Delta \\ -\partial_x + \Delta & 0 \end{pmatrix} \begin{pmatrix} u_n \\ v_n \end{pmatrix} = \omega_n \begin{pmatrix} u_n \\ v_n \end{pmatrix}, \tag{A.1}$$

with

$$\lambda = \Delta^2 + \partial_x \Delta,\ \sigma \propto \exp\left(\int^x dy \Delta \right). \tag{A.2}$$

For the same Δ, one can also define

$$\tilde{\lambda} = \Delta^2 - \partial_x \Delta,\ \tilde{\sigma} \propto \exp\left(- \int^x dy \Delta \right). \tag{A.3}$$

These functions satisfy

$$\partial_x^2 \tilde{\sigma} - \tilde{\lambda} \sigma = 0, \tag{A.4}$$

$$(-\partial_x^2 + \tilde{\lambda}) v_n = \omega_n^2 v_n, \tag{A.5}$$

$$(N/2r) \sum_n u_n v_n = \Delta. \tag{A.6}$$

This implies that the lower component can also be mapped to the $\mathbb{C}P^{N-1}$ model ($v_n = f_n$, $u_n = g_n$) with the Higgs field $\tilde{\sigma}$ and the mass gap function $\tilde{\lambda}$. Thus the single Δ corresponds to two solutions in the $\mathbb{C}P^{N-1}$ model (for $\Delta = m$, those are identical).

For instance, in the case of the kink solution, we obtain

$$\Delta = m \tanh mx, \tag{A.7}$$

$$\lambda = m^2,\ \sigma = 0, \tag{A.8}$$

$$\tilde{\lambda} = m^2(1 - 2\mathrm{sech}^2 mx), \qquad \tilde{\sigma} = A\,\mathrm{sech}\, mx, \tag{A.9}$$

whereas in the case of the anti-kink solution, we obtain

$$\Delta = \ -m \tanh mx, \tag{A.10}$$

$$\lambda = m^2(1 - 2\mathrm{sech}^2 mx),\ \sigma = A\,\mathrm{sech}\, mx, \tag{A.11}$$

$$\tilde{\lambda} = m^2,\ \tilde{\sigma} = 0. \tag{A.12}$$

Thus both the solutions correspond to the same solution in the $\mathbb{C}P^{N-1}$ model.

B Chiral Gross-Neveu model, Bogoliubov-de Gennes equation, and AKNS hierarchy

In this appendix, we briefly summarize the self-consistent treatment of Gross-Neveu model studied in refs. [78, 79]. The Lagrangian of the chiral Gross-Neveu model with N flavor is given by

$$L = \bar{\psi} i \partial\!\!\!/ \psi + \frac{g^2}{2} \left[(\bar{\psi}\psi)^2 + (\bar{\psi} i \gamma^5 \psi)^2 \right], \tag{B.1}$$

where $g > 0$. By introducing the auxiliary fields $\Delta_1 = -g^2 \langle \bar{\psi}\psi \rangle$ and $\Delta_2 = -g^2 \langle \bar{\psi} i \gamma_5 \psi \rangle$, and by taking the large N approximation (or mean field approximation) one can obtain the following effective Lagrangian

$$L_{\text{eff}} = \bar{\psi} i \partial\!\!\!/ \psi + \left(\Delta_1 \bar{\psi}\psi + \Delta_2 \bar{\psi} i \gamma^5 \psi \right) - \frac{1}{2g^2} \left(\Delta_1^2 + \Delta_2^2 \right). \tag{B.2}$$

Thus we obtain the following total energy

$$E_{\text{tot}} = \int dx \psi^\dagger H \psi + \frac{1}{2g^2} \int dx (\Delta_1^2 + \Delta_2^2), \tag{B.3}$$

with the Bogoliubov-de Gennes (BdG) Hamiltonian

$$H = -i\gamma^0 \gamma^1 \frac{d}{dx} - \gamma^0 \left(\Delta_1 + i \gamma^5 \Delta_2 \right). \tag{B.4}$$

The consistency condition of the auxiliary field Δ_1 and Δ_2 are called the gap equations

$$\langle \bar{\psi}\psi \rangle = -\frac{1}{g^2} \Delta_1, \ \langle \bar{\psi} i \gamma^5 \psi \rangle = -\frac{1}{g^2} \Delta_2, \tag{B.5}$$

which must be solved in a consistent manner with the BdG equation $H\psi = E\psi$. Here the left hand sides of the gap equations can be, respectively, rewritten as $N\langle \bar{\psi}_1 \psi_1 \rangle$ and $N\langle \bar{\psi}_1 i\gamma^5 \psi_1 \rangle$, since the N flavors gives the same contributions, e.g., $\langle \bar{\psi}_1 \psi_1 \rangle = \langle \bar{\psi}_2 \psi_2 \rangle = \cdots = \langle \bar{\psi}_N \psi_N \rangle$. Thus we can rewrite the gap equations as

$$\Delta_1 = -g^2 N \langle \bar{\psi}_1 \psi_1 \rangle, \ \Delta_2 = -g^2 N \langle \bar{\psi}_1 i \gamma^5 \psi_1 \rangle. \tag{B.6}$$

In the following, we use the chiral representation $\gamma_0 = \sigma_1$, $\gamma_1 = -i\sigma_2$, and $\gamma_5 = \sigma_3$.

For the BdG Hamiltonian, the Gor'kov resolvent $R(x;E) = 1/\langle x|(H-E)|x\rangle$ satisfies the Dikii-Eilenberger equation

$$\partial_x R(x;E)\sigma_3 = \left[Q(E,\Delta), R(x;E)\sigma_3 \right], \tag{B.7}$$

$$Q(E,\Delta) = \begin{pmatrix} iE & -i\Delta \\ i\Delta^* & -iE \end{pmatrix}, \tag{B.8}$$

where $\Delta = \Delta_1 - i\Delta_2$. We note that the BdG equation can be written as $\partial_x \psi = Q\psi$. The Gor'kov resolvent must satisfies the conditions $\det R = -\frac{1}{4}$, $\mathrm{Tr} R\sigma_3 = 0$, and $R^\dagger = R$.

The Dikii-Eilenberger equation and the BdG equation can be rewritten as

$$\partial_t Q - \partial_x R\sigma_3 + [Q, R\sigma_3] = 0, \quad \partial_x \psi = Q\psi, \quad \partial_t \psi = R\sigma_3 \psi, \tag{B.9}$$

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with the constraint $\partial_t Q = 0$. The first equation is the integrable condition (zero curvature condition) of this system; $\partial_x\partial_t\psi = \partial_t\partial_x\psi$. Since we find the connection between BdG system to the AKNS system, by using the machinery of the integrable system, one can systematically expand the resolvent $R\sigma_3$ which yields AKNS_n as

$$R\sigma_3 = i\sum_{j=1}^{n+2} c_j V^{(j)},\ V^{(n)} = \sum_{j=0}^{n-1}(2E)^{n-1-k}M^{(j)}, \tag{B.10}$$

where c_j's are positive constants. Here $M^{(i)}_{i,j}$ components of the matrices $M^{(i)}$ satisfy $M^{(i)}_{11} = -M^{(i)}_{22}$, $M^{(i)}_{12} = (M^{(i)}_{21})^*$, and first few components are given by

$$M^{(0)}_{11} = -\frac{i}{2}, M^{(0)}_{12} = 0, \tag{B.11}$$

$$M^{(1)}_{11} = 0, M^{(1)}_{12} = i\Delta, \tag{B.12}$$

$$M^{(2)}_{11} = -i|\Delta|^2, M^{(2)}_{12} = \partial_x\Delta, \tag{B.13}$$

$$M^{(3)}_{11} = -2i\Im(\Delta^*\partial_x\Delta), M^{(3)}_{12} = \partial_x^2\Delta - 2|\Delta|^2, \tag{B.14}$$

$$\begin{aligned} M^{(4)}_{11} &= 2i\Re(\Delta^*\partial_x^2\Delta) - 2i|\partial_x\Delta|^2 - 3|\Delta|^4, \\ M^{(4)}_{12} &= -\partial_x^3\Delta + 6|\Delta|^2\partial_x\Delta. \end{aligned} \tag{B.15}$$

The higher components are calculable with a help of the following formula

$$\frac{i}{2}\left[\sigma_3, M^{(n+1)}\right] = \partial_x M^{(n)} + \left[M^{(1)}, M^{(n)}\right]. \tag{B.16}$$

We can also obtain the nonlinear Schrödinger equations for this system as $\sum_{j=1}^{n+1} c_j M^{(j)}_{12} = 0$. The AKNS_0, AKNS_1, AKNS_2 for instance, yield

$$-\frac{i}{2}\partial_x\Delta + c_1\Delta = 0, \tag{B.17}$$

$$-\frac{1}{4}(\partial_x^2\Delta - 2|\Delta|^2\Delta) - c_1\frac{1}{2}\partial_x\Delta + c_2\Delta = 0, \tag{B.18}$$

$$\frac{i}{8}(\partial_x^3\Delta - 8|\Delta|^2\partial_x^2\Delta) - c_1\frac{1}{4}(\partial_x^2\Delta - 2|\Delta|^2\Delta) - c_2\frac{1}{2}\partial_x\Delta + c_3\Delta = 0. \tag{B.19}$$

The fermionic solutions are also calculable as

$$\psi_1^2 = CV_{12}\sqrt{\frac{iV_{11}-\omega}{iV_{11}+\omega}}\exp\left[i\omega\int_0^x dx\left(\frac{U_{12}}{V_{12}} + \frac{U_{21}}{V_{21}}\right)\right], \tag{B.20}$$

$$\psi_1^2 = -CV_{21}\sqrt{\frac{iV_{11}+\omega}{iV_{11}-\omega}}\exp\left[i\omega\int_0^x dx\left(\frac{U_{12}}{V_{12}} + \frac{U_{21}}{V_{21}}\right)\right], \tag{B.21}$$

where $\psi = (\psi_1, \psi_2)^T$ and C is the normalization constant. The square-root of those function must be taken such as $v/u = iV_{21}/(iV_{11} - \omega)$.

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