Semiclassical spectrum for BMN string in $Sch_5 \times S^5$

Hao Ouyang

*Institute of High Energy Physics and Theoretical Physics Center for Science Facilities,
Chinese Academy of Sciences,
19B Yuquan Road, Beijing 100049, P. R. China
School of Physical Sciences,
University of Chinese Academy of Sciences,
19A Yuquan Road Beijing 100049, P. R. China*

E-mail: ouyangh@ihep.ac.cn

ABSTRACT: We investigate the algebraic curve for string in $Sch_5 \times S^5$. We compute the semiclassical spectrum for BMN string in $Sch_5 \times S^5$ from the algebraic curve. We compare our results with the anomalous dimensions in $sl(2)$ sector of the null dipole deformation of $\mathcal{N} = 4$ super Yang-Mills theory.

KEYWORDS: Gauge-gravity correspondence, Integrable Field Theories

ARXIV EPRINT: [1709.06844](https://arxiv.org/abs/1709.06844)

Contents

1	Introduction	1
2	Algebraic curve for strings in $Sch_5 \times S^5$	2
2.1	$Sch_5 \times S^5$ from TsT transformation	2
2.2	Flat connection	4
2.3	BMN string	6
3	Semi-classical quantization of the BMN string	7
3.1	Fluctuation energies of excitations	9
3.2	One-loop shift	11
4	Comparison between the string and the gauge theory results	12
5	Conclusion and discussion	14

---

1 Introduction

Spectrum of superstrings in $AdS_5 \times S^5$ is related to the spectrum of scaling dimensions in planar $\mathcal{N} = 4$ supersymmetric Yang-Mills theory via the AdS/CFT duality [1–3]. Integrability on both sides of the duality helps us dramatically finding and understanding the AdS/CFT spectrum (For a big review, see [4]). In the $\mathcal{N} = 4$ supersymmetric Yang-Mills theory, the planar anomalous dimension matrix of infinitely long composite operators corresponds to Hamiltonian of integrable spin chain [5]. This implies that the spectrum can be solved efficiently by the Bethe ansatz.

On the string side, classical integrability of superstrings in $AdS_5 \times S^5$ follows from the existence of an infinite number of conserved charges [6] generated by the monodromy matrix of the Lax connection. Algebraic curve for classical solution of superstring in $AdS_5 \times S^5$ [7–11] can be obtained from the Lax connection. It plays an important role in studying the semiclassical strings in $AdS_5 \times S^5$.

In recent years, much attention has been enjoyed by the integrable deformations of AdS/CFT. One intriguing example is the Schrödinger spacetime [12–14]. Schrödinger spacetime can be obtained from AdS background by an appropriate TsT (T-duality-shift-T-duality) transformation [15] or null Melvin twist and has been shown to be classically integrable [16–18]. String theories in Schrödinger spacetime is dual to null dipole deformed field theories [19] (see also [20–22]). It is interesting to study the spectrum on both sides of the Schrödinger/dipole CFT duality with the methods of integrability.

Integrability in null dipole deformed $\mathcal{N} = 4$ super Yang-Mills was discussed in detail in [23]. The dipole deformation can be described as a Jordan cell Drinfeld-Reshetikhin

twist [24, 25] in the spin chain picture. The traditional Bethe ansatz is inapplicable due to the absence of a vacuum state. One-loop spectrum of the nontrivial twisted $sl(2)$ sector was instead obtained from the Baxter equation. In the large J limit, the anomalous dimension of the ground state perfectly matches the classical energy of the BMN string at order J^{-1} .

The purpose of this paper is to study the Schrödinger/dipole CFT duality by comparing semiclassical spectrum around classical string solutions to anomalous dimension of operators in the $sl(2)$ sector at order J^{-2} in the large J limit. One reason to study the order J^{-2} terms is that in the well studied AdS_5/CFT_4 correspondence, the gauge theory and string results match at order J^{-2} in the BMN limit [26]. One can expect that the null dipole deformation preserves this matching. Another reason is that at order J^{-2} we should consider one-loop quantum string theory corrections to the string energy, while the previous test at order J^{-1} involve purely bosonic classical string energies. We compute fluctuation energies of the excitations and the one-loop shift of the ground state energy from algebraic curve. We show that semiclassical spectrum around the BMN string solution perfectly matches the spin chain prediction.

This paper is organized as follows. In section 2 we discuss the $Sch_5 \times S^5$ background and TsT transformation in detail. We discuss the algebraic curve for strings in this background and obtain the quasi-momenta for the BMN string. In section 3, we review the algebraic curve method for computing the fluctuation energies around classical string solutions. Then we compute the semiclassical spectrum for the BMN strings. In section 4, we compare string theory results obtained in section 3 with the 1-loop spectrum in the $sl(2)$ sector of the null dipole deformation of $\mathcal{N} = 4$ super Yang-Mills theory.

2 Algebraic curve for strings in $Sch_5 \times S^5$

2.1 $Sch_5 \times S^5$ from TsT transformation

Schrödinger spacetime can be constructed by applying a TsT transformations to the AdS background [12–14, 19]. In this paper we are interested in a particular case of $Sch_5 \times S^5$ obtained by acting a TsT transformation on $AdS_5 \times S^5$.¹ We begin with the $AdS_5 \times S^5$ solution of type IIB supergravity

$$ds^2 = R^2 \left(\frac{-2dx^+ dx^- + (dx^1)^2 + (dx^2)^2 + dz^2}{z^2} + (d\psi + A)^2 + ds_{\mathbb{CP}^2}^2 \right), \quad (2.1)$$

$$ds_{\mathbb{CP}^2}^2 = d\phi_1^2 + \frac{1}{4} \sin^2 \phi_1^2 (\cos \phi_1^2 (d\theta_1 + \cos \phi_2 d\theta_2)^2 + d\phi_2^2 + \sin^2 \phi_2 d\theta_2^2), \quad (2.2)$$

$$A = \frac{1}{2} \sin^2 \phi_1 (d\theta_1 + \cos \phi_2 d\theta_2). \quad (2.3)$$

The five-form field strength is given by

$$F_5 = 4R^4 \left(-\frac{1}{z^5} dx^+ \wedge dx^- \wedge dx^1 \wedge dx^2 \wedge dz + \text{vol}(S^5) \right). \quad (2.4)$$

¹A more general class of Schrödinger deformations of $AdS_5 \times X_5$ has been studied in [27].

We perform a TsT transformation to this geometry. We make a first T-duality along ψ , followed a shift $x^- \rightarrow x^- - \mu\psi$, and then apply a second T-duality along ψ coordinate. After this TsT transformation, the solution reads

$$ds^2 = R^2 \left(-\mu^2 \frac{(dx^+)^2}{z^4} + \frac{-2dx^+ d\tilde{x}^- + (dx^1)^2 + (dx^2)^2 + dz^2}{z^2} + (d\tilde{\psi} + A)^2 + ds_{\mathbb{CP}^2}^2 \right), \quad (2.5)$$

$$B_2 = \mu R^2 (d\tilde{\psi} + A) \wedge \frac{dx^+}{z^2}. \quad (2.6)$$

The five-form F_5 is invariant under the transformation. The TsT translation preserves the symmetries that commute with $\hat{J}$ and $\hat{P}_-$. The symmetries of $SO(4, 2)$ that commute with $\hat{P}_-$ generate the Schrödinger group. The Schrödinger group contains Galilean group as a subgroup and has two more generators corresponding to a non-relativistic scale transformation $\hat{D} + \hat{M}_{+-}$ and a special conformal transformation $\hat{K}_-$. The energy of the string is defined as the global charge associated with the symmetry $(\hat{P}_+ + \hat{K}_-)/\sqrt{2}$ which is related to the non-relativistic scale transformation $\hat{D} + \hat{M}_{+-}$ by a similarity transformation. Holography enable one to compute the non-relativistic conformal dimensions of operators at strong coupling as the energies of strings.

The relations between the original and dual coordinates are

$$d\psi = d\tilde{\psi} + * \mu \frac{dx^+}{z^2}, \quad (2.7)$$

$$dx^- = d\tilde{x}^- + * \mu \left(d\tilde{\psi} + * \mu \frac{dx^+}{z^2} + A \right). \quad (2.8)$$

Here we make a slight abuse of notation that we use the same symbols for forms on the target space and their pull-back to the worldsheet.

We consider the closed strings on the deformed background. The dual coordinates satisfy periodic boundary conditions. The original coordinates have the following twisted boundary conditions

$$x^-(2\pi) - x^-(0) = LJ, \quad (2.9)$$

$$\psi(2\pi) - \psi(0) = 2\pi m_1 - LP_-, \quad m_1 \in \mathbb{Z}, \quad (2.10)$$

where $L = 2\pi\mu/\sqrt{\lambda}$ is the deformation parameter in the dual field theory and

$$J = \frac{\sqrt{\lambda}}{2\pi} \int_0^{2\pi} d\sigma (\partial_\tau \psi + A_\tau), \quad (2.11)$$

$$P_- = \frac{\sqrt{\lambda}}{2\pi} \int_0^{2\pi} d\sigma \frac{-\partial_\tau x^+}{z^2}, \quad (2.12)$$

are global charges associated with symmetries $\hat{J}$ and $\hat{P}_-$, and $\sqrt{\lambda} = R^2/\alpha'$ is the square root of 't Hooft coupling.

2.2 Flat connection

Integrability of the string sigma model is preserved by TsT transformation, so strings in Schrödinger spacetime is integrable [16–18] (see also [28–42]).

We now construct the Lax connection for IIB superstring in $Sch_5 \times S^5$. The type IIB superstring in $AdS_5 \times S^5$ can be described by the a sigma-model in supercoset space of the super group $SU(2, 2|4)$ over $SO(4, 1) \times SO(5)$ [43]. To describe strings in Poincaré coordinates, we choose the coset representative as

$$g(x^\mu, z, \psi, \theta_i, \phi_i, \Theta) = B(x^\mu, z, \psi, \theta_i, \phi_i) e^{F(\Theta)}, \quad (2.13)$$

with

$$B(x^\mu, z, \psi, \theta_i, \phi_i) = e^{ix^+ \hat{P}_+ + ix^- \hat{P}_- + ix^1 \hat{P}_1 + ix^2 \hat{P}_2} e^{-i \log(z) \hat{D}} e^{i\psi \hat{J}} B_1(\theta_i, \phi_i), \quad (2.14)$$

and Θ represents the fermionic coordinates. We use a matrix representation such that

$$\begin{aligned} \hat{P}_- &= \left(\begin{array}{cccc|c} 0 & 0 & 0 & \sqrt{2} & \\ 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & \\ \hline & & & & 0_{4 \times 4} \end{array} \right), & \hat{P}_+ &= \left(\begin{array}{cccc|c} 0 & 0 & 0 & 0 & \\ 0 & 0 & \sqrt{2} & 0 & \\ 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & \\ \hline & & & & 0_{4 \times 4} \end{array} \right), \\ \hat{D} &= \frac{i}{2} \text{diag}(1, 1, -1, -1 | 0, 0, 0, 0), & \hat{J} &= \frac{1}{2} \text{diag}(0, 0, 0, 0 | 1, 1, 1, -3). \end{aligned} \quad (2.15)$$

The current associated with g can be defined as

$$\mathcal{J} = -g^{-1} dg = e^F \mathcal{J}_B e^{-F} - e^F d e^{-F} = \mathcal{J}_B + \mathcal{J}_F, \quad (2.16)$$

where

$$\mathcal{J}_B = -B^{-1} dB = -\frac{i}{z} dx^\mu \hat{P}_\mu + \frac{i}{z} dz \hat{D} - i d\psi B_1^{-1} \hat{J} B_1 - B_1^{-1} dB_1 \quad (2.17)$$

$$\mathcal{J}_F = \frac{\sinh \mathcal{M}}{\mathcal{M}} \nabla F + 2[F, \frac{\sinh^2 \mathcal{M}/2}{\mathcal{M}^2} \nabla F], \quad (2.18)$$

with

$$\nabla \cdot := d \cdot + [\mathcal{J}_B, \cdot], \quad \mathcal{M}^2 \cdot := [F, [F, \cdot]]. \quad (2.19)$$

The Lie superalgebra $su(2, 2|4)$ has a $\mathbb{Z}_4$ grading structure associated with a $\mathbb{Z}_4$ automorphism Ω . The automorphism Ω in this matrix representation is defined by

$$\Omega(M) = -K M^{st} K^{-1}, \quad K = \left(\begin{array}{cccc|cccc} 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array} \right) \quad (2.20)$$

We can decompose the current $\mathcal{J}$ into

$$\mathcal{J} = \mathcal{J}^{(0)} + \mathcal{J}^{(1)} + \mathcal{J}^{(2)} + \mathcal{J}^{(3)}, \quad (2.21)$$

where $\Omega(\mathcal{J}^{(i)}) = i^n \mathcal{J}$. Then the equation of motion of string in $AdS_5 \times S^5$ is equivalent to the conservation of the Noether current

$$d * k = 0, \quad k = g \left(\mathcal{J}^{(2)} - \frac{1}{2} \mathcal{J}^{(1)} + \frac{1}{2} \mathcal{J}^{(3)} \right) g^{-1}. \quad (2.22)$$

Lax connection $\mathcal{L}(x)$ for the superstring in $AdS_5 \times S^5$ has been derived in [6]

$$\mathcal{L} = \mathcal{J}^{(0)} + \frac{x^2 + 1}{x^2 - 1} \mathcal{J}^{(2)} - \frac{2x}{(x^2 - 1)} * \mathcal{J}^{(2)} + \sqrt{\frac{x+1}{x-1}} \mathcal{J}^{(1)} + \sqrt{\frac{x-1}{x+1}} \mathcal{J}^{(3)}. \quad (2.23)$$

If the current satisfies the equation of motion, the Lax connection is flat

$$d\mathcal{L} - \mathcal{L} \wedge \mathcal{L} = 0. \quad (2.24)$$

Using this flat connection, one can define the monodromy matrix

$$T(x) = \text{P exp} \left(\int_0^{2\pi} d\sigma \mathcal{L}_\sigma(x) \right). \quad (2.25)$$

The eigenvalues of $T(x)$ do not depend on τ and generate an infinite number of conserved quantities. The current components $\mathcal{J}_\alpha$ do not have an explicit dependence on x^- and ψ . Then the Lax connection $\mathcal{L}$ in the undeformed case can be used to derive a Lax connection and thus quasi-momenta for strings in $Sch_5 \times S^5$ background.

The quasi-momenta $p_i(x)$ are functions defined from the eigenvalues

$$\left\{ e^{\hat{p}_1}, e^{\hat{p}_2}, e^{\hat{p}_3}, e^{\hat{p}_4} | e^{\tilde{p}_1}, e^{\tilde{p}_2}, e^{\tilde{p}_3}, e^{\tilde{p}_4} \right\}$$

of the monodromy matrix $T(x)$. They are generating functions of conserved physical quantities. For instance, we can read the conserved global charges from the behavior at large x . Large x asymptotic properties of the quasi-momenta for strings with twisted boundary condition are more complex than those for close strings. Below we analysis the asymptotic behavior of the quasi-momenta.

At $x \rightarrow \infty$, the expansion of the Lax connection is

$$\mathcal{L} = -g^{-1} dg - \frac{2}{x} g^{-1} * k g + O\left(\frac{1}{x^2}\right). \quad (2.26)$$

Expanding the monodromy matrix, we get

$$\begin{aligned} \tilde{T} &\equiv g(\tau, 0) T g(\tau, 0)^{-1} \\ &= g(\tau, 0) g(\tau, 2\pi)^{-1} \left(1 - \frac{2}{x} \int_0^{2\pi} d\sigma k_\tau + O\left(\frac{1}{x^2}\right) \right) \\ &= e^{iL(P_- \hat{J} - J \hat{P}_-)} \left(1 - \frac{2}{x} \int_0^{2\pi} d\sigma k_\tau + O\left(\frac{1}{x^2}\right) \right) \end{aligned} \quad (2.27)$$

In our representation, it takes the form

$$\tilde{T} = \left(\begin{array}{ccc|ccc} 0 & 0 & 0 & -i\sqrt{2}LJ & & \\ 0 & 0 & 0 & 0 & & \\ 0 & 0 & 0 & 0 & & \\ 0 & 0 & 0 & 0 & & \\ \hline & & & & 0_{4 \times 4} & \\ & & & & & \exp\left(\frac{i}{2}LP_- \text{diag}(1, 1, 1, -3)\right) \end{array} \right) + \frac{1}{\sqrt{\lambda x}}Q + O\left(\frac{1}{x^2}\right), \quad (2.28)$$

where Q satisfies $Q_{4\hat{1}} = -2\pi i\sqrt{2}P_-$. Behavior of the quasi-momenta at $x \rightarrow \infty$ reads

$$\begin{pmatrix} \hat{p}_1 \\ \hat{p}_2 \\ \hat{p}_3 \\ \hat{p}_4 \\ \tilde{p}_1 \\ \tilde{p}_2 \\ \tilde{p}_3 \\ \tilde{p}_4 \end{pmatrix} = \begin{pmatrix} 2\sqrt{\frac{\pi LJP_-}{\sqrt{\lambda x}}} \\ 0 \\ 0 \\ -2\sqrt{\frac{\pi LJP_-}{\sqrt{\lambda x}}} \\ \frac{1}{2}LP_- \\ \frac{1}{2}LP_- \\ \frac{1}{2}LP_- \\ -\frac{3}{2}LP_- \end{pmatrix} + \frac{2\pi}{\sqrt{\lambda x}} \begin{pmatrix} O(1) \\ \Delta + S \\ -\Delta + S \\ O(1) \\ +J_1 + J_2 - J_3 \\ +J_1 - J_2 + J_3 \\ -J_1 + J_2 + J_3 \\ -J_1 - J_2 - J_3 \end{pmatrix} + O(x^{-2}), \quad (2.29)$$

where S and J_i are spins of the string. The quasi-momenta $\hat{p}_1$ and $\hat{p}_4$ are connected by a square root cut which has a branch point at infinity. The quasi-momenta satisfy

$$\oint_{\infty} dx (\tilde{p}_1^2 + \tilde{p}_2^2 + \tilde{p}_3^2 + \tilde{p}_4^2 - \hat{p}_1^2 - \hat{p}_2^2 - \hat{p}_3^2 - \hat{p}_4^2) = 0, \quad (2.30)$$

so we still have the constraint between length and filling fractions (see [9, 11]).

2.3 BMN string

We now exemplify the discussion above with BMN string solution in $Sch_5 \times S^5$ presented in [23]

$$\begin{aligned} \phi_i = \theta_i = x^2 = x^1 = 0, \quad \psi &= \mu m \sigma + \omega \tau, \\ x^+ &= \frac{\tan(\kappa \tau)}{\sqrt{2}}, \quad x^- = \mu \omega \sigma + \frac{\kappa \tan(\kappa \tau)}{2m}, \\ z &= \sqrt{\frac{\kappa}{\sqrt{2}m}} \sec(\kappa \tau). \end{aligned} \quad (2.31)$$

Virasoro constraint gives

$$\mu^2 m^2 - \kappa^2 + \omega^2 = 0. \quad (2.32)$$

The conserved global charges are

$$\Delta = \sqrt{\lambda} \kappa, \quad P_- \equiv -M = -\sqrt{\lambda} m, \quad J = \sqrt{\lambda} \omega, \quad (2.33)$$

where we denote Δ as the global energy of the string associated with the non-relativistic scale transformation $\hat{D} + \hat{M}_{+-}$. Then the classical energy of the BMN string is given by

$$\Delta = \sqrt{J^2 + \mu^2 M^2}. \quad (2.34)$$

The quasi-momenta of the BMN string are

$$\begin{aligned}
 \hat{p}_1 &= -\hat{p}_4 = \frac{2\pi\kappa x \sqrt{1-x} \sin 2\gamma}{x^2 - 1}, \\
 \hat{p}_2 &= -\hat{p}_3 = \frac{2\pi\kappa \sqrt{x} \sqrt{x-1} \sin 2\gamma}{x^2 - 1}, \\
 \tilde{p}_1 &= \frac{\pi\omega(2x - (1+x^2)\tan\gamma)}{x^2 - 1}, \\
 \tilde{p}_2 &= \frac{\pi\omega(2x - (1+x^2)\tan\gamma)}{x^2 - 1}, \\
 \tilde{p}_3 &= -\frac{\pi\omega(2x + (-3+x^2)\tan\gamma)}{x^2 - 1}, \\
 \tilde{p}_4 &= -\frac{\pi\omega(2x + (1-3x^2)\tan\gamma)}{x^2 - 1},
 \end{aligned} \tag{2.35}$$

with

$$\sin 2\gamma = \frac{4\pi\sqrt{\lambda}JLM}{4\pi^2 J^2 + \lambda L^2 M^2}, \quad \tan \gamma = \frac{\sqrt{\lambda}LM}{2\pi J}. \tag{2.36}$$

We find an expected square root cut $[\csc 2\gamma, \infty]$ connecting $\hat{p}_1$ and $\hat{p}_4$.

3 Semi-classical quantization of the BMN string

A powerful method for computing the semiclassical spectrum around string solutions is proposed in [44]. Here we begin with a review of this method for the reader's convenience.

The semiclassical spectrum around the BMN string is given by

$$\Delta = \Delta_{\text{cl}} + \Delta_{1\text{-loop}} + \delta\Delta, \tag{3.1}$$

where $\Delta_{\text{cl}} + \Delta_{1\text{-loop}}$ is the ground state energy, and $\delta\Delta$ is the energy of the excitations. To compute $\delta\Delta$, we add a perturbation $\delta p(x)$ to $p(x)$ associated with the classical solution. The perturbation $\delta p(x)$ has a single pole at x_n . The position x_n is determined by

$$p_i(x_n^{ij}) - p_j(x_n^{ij}) = 2\pi n. \tag{3.2}$$

The residues at the poles are

$$\text{res}_{x=x_n^{ij}} \delta\hat{p}_k = (\delta_{ik} - \delta_{jk})\alpha(x_n^{ij}) N_n^{ij}, \quad \text{res}_{x=x_n^{ij}} \delta\tilde{p}_k = (\delta_{jk} - \delta_{ik})\alpha(x_n^{ij}) N_n^{ij}, \tag{3.3}$$

where $i < j$ and N_n^{ij} is the excitation number for excitation with polarizations (ij) and mode number n and the function $\alpha(x)$ is defined as

$$\alpha(x) = \frac{4\pi}{\sqrt{\lambda}} \frac{x^2}{x^2 - 1}. \tag{3.4}$$

the residues at $x = \pm 1$ are synchronized as

$$\delta\{\hat{p}_1, \hat{p}_2, \hat{p}_3, \hat{p}_4 | \tilde{p}_1, \tilde{p}_2, \tilde{p}_3, \tilde{p}_4\} = \frac{1}{x \pm 1} \delta\{\alpha_{\pm}, \alpha_{\pm}, \beta_{\pm}, \beta_{\pm} | \alpha_{\pm}, \alpha_{\pm}, \beta_{\pm}, \beta_{\pm}\} + \dots \tag{3.5}$$

For each cut $\mathcal{C}^{ij}$ connecting $p_i(x)$ and $p_j(x)$ the perturbation $\delta p(x)$ satisfies

$$\delta p_i^+ - \delta p_j^- = 0, \quad x \in \mathcal{C}^{ij}. \tag{3.6}$$

where the superscript $\pm$ denotes above and below the cut.

The poles result in an energy shift

$$\delta\Delta = \sum_{n,ij} N_n^{ij} \Omega_n^{ij} (x_n^{ij}), \quad (3.7)$$

where Ω_n^{ij} are the off-shell fluctuation energies.

The asymptotic behaviour of the perturbation $\delta p(x)$ is

$$\delta\hat{p}_1 = O(x^{-1/2}), \quad \delta\hat{p}_4 = O(x^{-1/2}), \quad (3.8)$$

$$\begin{pmatrix} \delta\hat{p}_2 \\ \delta\hat{p}_3 \\ \delta\tilde{p}_1 \\ \delta\tilde{p}_2 \\ \delta\tilde{p}_3 \\ \delta\tilde{p}_4 \end{pmatrix} = \frac{4\pi}{\sqrt{\lambda}x} \begin{pmatrix} +\frac{1}{2}\delta\Delta + N^{\hat{2}\hat{3}} + N^{\hat{2}\hat{4}} + N^{\hat{2}\tilde{3}} + N^{\hat{2}\tilde{4}} \\ -\frac{1}{2}\delta\Delta - N^{\hat{2}\hat{3}} - N^{\hat{1}\hat{3}} - N^{\hat{1}\tilde{3}} - N^{\hat{2}\tilde{3}} \\ -N^{\hat{1}\hat{3}} - N^{\hat{1}\tilde{4}} - N^{\hat{1}\tilde{3}} - N^{\hat{1}\tilde{4}} \\ -N^{\hat{2}\hat{3}} - N^{\hat{2}\tilde{4}} - N^{\hat{2}\tilde{3}} - N^{\hat{2}\tilde{4}} \\ +N^{\hat{1}\hat{3}} + N^{\hat{2}\hat{3}} + N^{\hat{1}\tilde{3}} + N^{\hat{2}\tilde{3}} \\ +N^{\hat{1}\tilde{4}} + N^{\hat{2}\tilde{4}} + N^{\hat{1}\tilde{4}} + N^{\hat{2}\tilde{4}} \end{pmatrix} + O(x^{-2}). \quad (3.9)$$

The order $O(x^{1/2})$ terms in the $\delta\hat{p}_1$ and $\delta\hat{p}_4$ are determined by the constraint (2.30). Since the classical solution already obey the constraint (2.30), one can show these filling fractions N_n^{ij} satisfy

$$\sum_{n,ij} n N_n^{ij} = 0. \quad (3.10)$$

using Riemann bilinear identity (see eqs. (3.38) and (3.44) in [11]).

We now determine the most general form of the perturbation $\delta p(x)$ for the BMN string. When small poles are added to $\delta\hat{p}_2(x)$ with a square root cut, the branch point will be slightly displaced so $\delta\hat{p}_2(x)$ behave like $\partial_{x_0}\sqrt{x-x_0} \sim 1/\sqrt{x-x_0}$ near the branch point $x_0 = \sin 2\gamma$. The most general form for $\delta\hat{p}_2(x)$ is $f(x) + g(x)/K(x)$ where f and g are rational functions of x and $K(x) = \sqrt{x}\sqrt{x-x_0}$. From (3.6) and inversion symmetry we get

$$\begin{pmatrix} \delta\hat{p}_1 \\ \delta\hat{p}_2 \\ \delta\hat{p}_3 \\ \delta\hat{p}_4 \end{pmatrix} = \begin{pmatrix} -f(1/x) - \frac{g(1/x)}{K(1/x)} \\ f(x) + \frac{g(x)}{K(x)} \\ f(x) - \frac{g(x)}{K(x)} \\ -f(1/x) + \frac{g(1/x)}{K(1/x)} \end{pmatrix}. \quad (3.11)$$

One can obtained all other off-shell fluctuation energies from the knowledge of $\Omega^{\hat{2}\hat{3}}$ and $\Omega^{\hat{2}\tilde{3}}$ alone using the efficient method provided in [45]. From inversion symmetry we have

$$\Omega^{\hat{1}\hat{4}}(y) = -\Omega^{\hat{2}\tilde{3}}(1/y) + \Omega^{\hat{2}\tilde{3}}(0), \quad \Omega^{\hat{1}\tilde{4}}(y) = -\Omega^{\hat{2}\tilde{3}}(1/y) - 2. \quad (3.12)$$

the quasi-momenta for BMN string in $Sch_5 \times S^5$ are pairwise symmetric up to constant terms:

$$\hat{p}_1 = -\hat{p}_4, \quad \hat{p}_2 = -\hat{p}_3, \quad (3.13)$$

$$\tilde{p}_1 = -\tilde{p}_4 + 2\pi\kappa \sin \gamma, \quad \tilde{p}_2 = -\tilde{p}_3 - 2\pi\kappa \sin \gamma. \quad (3.14)$$

So the off-shell fluctuation energies satisfies

$$\Omega^{ij}(y) = \frac{1}{2} \left(\Omega^{ii'}(y) + \Omega^{j'j}(y) \right), \quad (3.15)$$

where

$$(\hat{1}, \hat{2}, \tilde{1}, \tilde{2}, \hat{3}, \hat{4}, \tilde{3}, \tilde{4}) = (\hat{4}, \hat{3}, \tilde{4}, \tilde{3}, \hat{2}, \hat{1}, \tilde{2}, \tilde{1}). \quad (3.16)$$

3.1 Fluctuation energies of excitations

We first consider the excitation $\hat{2}\hat{3}$. We have $\delta\tilde{p}_i = 0$, for $i = 1, 2, 3, 4$, and therefore $f(x) = 0$. We take the following ansatz

$$g(x) = \sum_n \frac{\alpha \left(x_n^{\hat{2}\hat{3}}\right) K \left(x_n^{\hat{2}\hat{3}}\right) N_n^{\hat{2}\hat{3}}}{x - x_n^{\hat{2}\hat{3}}} + \frac{2\pi}{\sqrt{\lambda}} \delta\Delta + \frac{4\pi}{\sqrt{\lambda}} \sum_n N_n^{\hat{2}\hat{3}}. \quad (3.17)$$

Large x asymptotic of $\delta\hat{p}_1$ gives

$$\delta\Delta = \sum_n N_n^{\hat{2}\hat{3}} \Omega^{\hat{2}\hat{3}} \left(x_n^{\hat{2}\hat{3}}\right), \quad (3.18)$$

$$\Omega^{\hat{2}\hat{3}}(x) = \frac{2 - 2x^2 + 2xK(x)}{x^2 - 1}, \quad (3.19)$$

and the level matching condition

$$\sum_n n N_n^{\hat{2}\hat{3}} = 0. \quad (3.20)$$

We next consider the S^3 excitation $\tilde{2}\tilde{3}$. We start with the following ansatz

$$g(x) = \frac{\alpha_1 K(1)}{x-1} + \frac{\alpha_2 K(-1)}{x+1} + \frac{2\pi}{\sqrt{\lambda}} \delta\Delta, \quad (3.21)$$

$$\delta\tilde{p}_2 = -\delta\tilde{p}_3 = \frac{\alpha_1}{x-1} + \frac{\alpha_2}{x+1} - \sum_n \frac{\alpha \left(x_n^{\tilde{2}\tilde{3}}\right) N_n^{\tilde{2}\tilde{3}}}{x - x_n^{\tilde{2}\tilde{3}}}. \quad (3.22)$$

Large x asymptotic of $\delta\tilde{p}_1$ and $\delta\tilde{p}_4$ give

$$\alpha_1 + \alpha_2 = \sum_n \alpha \left(x_n^{\tilde{2}\tilde{3}}\right) N_n^{\tilde{2}\tilde{3}} \frac{1}{\left(x_n^{\tilde{2}\tilde{3}}\right)^2}, \quad (3.23)$$

$$\alpha_1 - \alpha_2 = \sum_n \alpha \left(x_n^{\tilde{2}\tilde{3}}\right) N_n^{\tilde{2}\tilde{3}} \frac{1}{x_n^{\tilde{2}\tilde{3}}}. \quad (3.24)$$

Large x asymptotic of $\delta\hat{p}_1$ give

$$-\alpha_1 K(1) + \alpha_2 K(-1) + \frac{2\pi}{\sqrt{\lambda}} \delta\Delta = -\alpha_1 (\cos \gamma - \sin \gamma) - \alpha_2 (\cos \gamma + \sin \gamma) + \frac{2\pi}{\sqrt{\lambda}} \delta\Delta = 0. \quad (3.25)$$

The filling fractions satisfy

$$\sum_n n N_n^{\tilde{2}\tilde{3}} = 0, \quad (3.26)$$

we can solve these equations and obtain

$$\delta\Delta = \sum_n N_n^{\tilde{2}\tilde{3}} \Omega^{\tilde{2}\tilde{3}} \left(x_n^{\tilde{2}\tilde{3}}\right), \quad (3.27)$$

where

$$\Omega^{\hat{2}\hat{3}}(x) = \frac{2 \cos \gamma - 2x \sin \gamma}{x^2 - 1}. \quad (3.28)$$

Finally using (3.12) and (3.15) we find all the off-shell fluctuation energies

$$\begin{aligned} \Omega^{\hat{2}\hat{3}}(x) &= \frac{2 - 2x^2 + 2x\sqrt{x}\sqrt{x - \sin 2\gamma}}{x^2 - 1}, \\ \Omega^{\hat{1}\hat{4}}(x) &= \frac{2\sqrt{1 - x \sin 2\gamma}}{x^2 - 1}, \\ \Omega^{\hat{1}\hat{3}}(x) = \Omega^{\hat{2}\hat{4}}(x) &= \frac{1 - x^2 + x\sqrt{x}\sqrt{x - \sin 2\gamma} + \sqrt{1 - x \sin 2\gamma}}{x^2 - 1}, \\ \Omega^{\hat{2}\hat{3}}(x) = \Omega^{\hat{1}\hat{4}}(x) = \Omega^{\hat{1}\hat{3}}(x) = \Omega^{\hat{2}\hat{4}}(x) &= \frac{2 \cos \gamma - 2x \sin \gamma}{x^2 - 1}, \\ \Omega^{\hat{2}\hat{3}}(x) = \Omega^{\hat{2}\hat{3}}(x) = \Omega^{\hat{2}\hat{4}}(x) = \Omega^{\hat{2}\hat{4}}(x) &= \frac{1 + \cos \gamma - x \sin \gamma - x^2 + x\sqrt{x}\sqrt{x - \sin 2\gamma}}{x^2 - 1}, \\ \Omega^{\hat{1}\hat{4}}(x) = \Omega^{\hat{1}\hat{4}}(x) = \Omega^{\hat{1}\hat{3}}(x) = \Omega^{\hat{1}\hat{3}}(x) &= \frac{\cos \gamma - x \sin \gamma + \sqrt{1 - x \sin 2\gamma}}{x^2 - 1}. \end{aligned} \quad (3.29)$$

We now solve the pole position x_n . We choose the solution $|x_n| > 1$ for small LM to be physical poles. We have

$$\begin{aligned} x_n^{\hat{2}\hat{3}} &= \frac{2}{n\sqrt{\lambda}}J + O(1), \\ x_n^{\hat{1}\hat{4}} &= \frac{2\sqrt{L^2M^2\pi^{-2} + n^2} - 2LM\pi^{-1}}{\sqrt{\lambda}n^2}J + O(1), \\ x_n^{\hat{1}\hat{3}} = x_n^{\hat{2}\hat{4}} &= \frac{(2\pi n - LM)}{\pi\sqrt{\lambda}n^2}J + O(1), \\ x_n^{\hat{2}\hat{3}} = x_n^{\hat{1}\hat{3}} &= \frac{\sqrt{\pi J^2 + \lambda n(\pi n - LM)} + \sqrt{\pi}J}{\sqrt{\pi}\sqrt{\lambda}n}, \\ x_n^{\hat{1}\hat{4}} = x_n^{\hat{2}\hat{4}} &= \frac{\sqrt{\pi}\sqrt{\pi J^2 + \lambda n(LM + \pi n)} + \pi J}{\sqrt{\lambda}(LM + \pi n)}, \\ x_n^{\hat{2}\hat{3}} = x_n^{\hat{1}\hat{3}} &= \frac{\sqrt{16\pi^2 J^2 + \lambda(LM - 4\pi n)^2} + 4\pi J}{\sqrt{\lambda}(LM + 4\pi n)}, \\ x_n^{\hat{2}\hat{3}} &= \frac{\sqrt{16\pi^2 J^2 + \lambda(3LM - 4\pi n)^2} + 4\pi J}{\sqrt{\lambda}(4\pi n - LM)}, \\ x_n^{\hat{2}\hat{4}} &= \frac{\sqrt{16\pi^2 J^2 + \lambda(LM + 4\pi n)^2} + 4\pi J}{\sqrt{\lambda}(3LM + 4\pi n)}, \\ x_n^{\hat{1}\hat{4}} &= \frac{(LM + 4\pi n) \left(\sqrt{16\pi^2 J^2 + \lambda(3LM + 4\pi n)^2} + 4\pi J \right)}{\sqrt{\lambda}(3LM + 4\pi n)^2}, \\ x_n^{\hat{1}\hat{3}} &= \frac{(4\pi n - 3LM) \left(\sqrt{16\pi^2 J^2 + \lambda(LM - 4\pi n)^2} + 4\pi J \right)}{\sqrt{\lambda}(LM - 4\pi n)^2}, \\ x_n^{\hat{2}\hat{4}} = x_n^{\hat{1}\hat{4}} &= \frac{(4\pi n - LM) \left(\sqrt{16\pi^2 J^2 + \lambda(LM + 4\pi n)^2} + 4\pi J \right)}{\sqrt{\lambda}(LM + 4\pi n)^2}. \end{aligned} \quad (3.30)$$

The exact expressions of $x_n^{\hat{i}\hat{j}}$ are very complex, so we only consider the leading order terms in the large J expansion. When $LM \neq 0$, a finite number of $x_n^{\hat{1}\hat{j}}$ ($x_n^{i\hat{4}}$) will enter the cut connecting $\hat{p}_1$ and $\hat{p}_4$ and become $x_n^{\hat{4}\hat{j}}$ ($x_n^{i\hat{1}}$).

Plugging x_n into the off-shell fluctuation energies, in the large J limit we get the on-shell fluctuation frequencies

$$\begin{aligned}
 \Omega^{\hat{2}\hat{3}}(x_n^{\hat{2}\hat{3}}) &= \frac{\lambda(\pi n^2 - LMn)}{2\pi J^2} + O(J^{-3}), \\
 \Omega^{\hat{1}\hat{4}}(x_n^{\hat{1}\hat{4}}) &= \frac{\lambda}{2J^2} \left(\sqrt{\frac{L^2 M^2 n^2}{\pi^2} + n^4} + \frac{LMn}{\pi} \right) + O(J^{-3}), \\
 \Omega^{\hat{1}\hat{3}}(x_n^{\hat{1}\hat{3}}) &= \Omega^{\hat{2}\hat{4}}(x_n^{\hat{2}\hat{4}}) = \frac{\lambda n^2}{2J^2} + O(J^{-3}), \\
 \Omega^{\hat{2}\hat{3}}(x_n^{\hat{2}\hat{3}}) &= \Omega^{\hat{1}\hat{3}}(x_n^{\hat{1}\hat{3}}) = \frac{\lambda(\pi n^2 - LMn)}{2\pi J^2} + O(J^{-3}), \\
 \Omega^{\hat{2}\hat{4}}(x_n^{\hat{2}\hat{4}}) &= \Omega^{\hat{1}\hat{4}}(x_n^{\hat{1}\hat{4}}) = \frac{\lambda(\pi n^2 + LMn)}{2\pi J^2} + O(J^{-3}), \\
 \Omega^{\hat{2}\hat{3}}(x_n^{\hat{2}\hat{3}}) &= \Omega^{\hat{1}\hat{3}}(x_n^{\hat{1}\hat{3}}) = \frac{\lambda(16\pi^2 n^2 - 8\pi LMn - 3L^2 M^2)}{32\pi^2 J^2} + O(J^{-3}), \\
 \Omega^{\hat{2}\hat{3}}(x_n^{\hat{2}\hat{3}}) &= \frac{\lambda(16\pi^2 n^2 - 24\pi LMn + 5L^2 M^2)}{32\pi^2 J^2} + O(J^{-3}), \\
 \Omega^{\hat{2}\hat{4}}(x_n^{\hat{2}\hat{4}}) &= \frac{\lambda(16\pi^2 n^2 + 8\pi LMn - 3L^2 M^2)}{32\pi^2 J^2} + O(J^{-3}), \\
 \Omega^{\hat{1}\hat{4}}(x_n^{\hat{1}\hat{4}}) &= \frac{\lambda(4\pi n + 3LM)^2}{32\pi^2 J^2} + O(J^{-3}), \\
 \Omega^{\hat{1}\hat{3}}(x_n^{\hat{1}\hat{3}}) &= \frac{\lambda(4\pi n - LM)^2}{32\pi^2 J^2} + O(J^{-3}), \\
 \Omega^{\hat{2}\hat{4}}(x_n^{\hat{2}\hat{4}}) &= \Omega^{\hat{1}\hat{4}}(x_n^{\hat{1}\hat{4}}) = \frac{\lambda(4\pi n + LM)^2}{32\pi^2 J^2} + O(J^{-3}).
 \end{aligned} \tag{3.31}$$

Then we obtained the energy shift $\delta\Delta$ given by (3.7).

3.2 One-loop shift

The one loop shift is equal to one half of the graded sum of all fluctuation mode frequencies. Using zeta function regularization, we have

$$\sum_{n \in \mathbb{Z}} ((n+q)^2 + pn) = q^2 + \zeta(-2, 1+q) + \zeta(-2, 1-q) = 0. \tag{3.32}$$

Therefore when we compute the one loop shift energy at order J^{-2} , only the contribution from $\Omega^{\hat{1}\hat{4}}$ is nontrivial. Then we sum over the energies of the $sl(2)$ modes to get the

one-loop shift:

$$\begin{aligned}
\Delta_{1\text{-loop}} &= \frac{1}{2} \sum_{n \in \mathbb{Z}} \Omega^{\hat{1}\hat{4}}(x_n^{\hat{1}\hat{4}}) \\
&= \sum_{n=1}^{\infty} \left(\sum_{k=0}^{\infty} \binom{1/2}{k} n^{2-2k} \pi^{-2k} L^{2k} M^{2k} \right) \frac{\lambda}{2J^2} + O(J^{-3}) \\
&= \left(\sum_{k=0}^{\infty} \binom{1/2}{k} \zeta(2k-2) \pi^{-2k} L^{2k} M^{2k} \right) \frac{\lambda}{2J^2} + O(J^{-3}) \\
&= \left(-\frac{\lambda L^2 M^2}{8\pi^2} - \frac{\lambda L^4 M^4}{96\pi^2} + \frac{\lambda L^6 M^6}{2880\pi^2} + O(L^8 M^8) \right) \frac{1}{J^2} + O(J^{-3}). \quad (3.33)
\end{aligned}$$

4 Comparison between the string and the gauge theory results

Comparison between the results obtained in the gauge theory and string theory is possible in the large spin regime with $J \rightarrow \infty$ and λ/J^2 kept fixed and small (see e.g. [26, 46–48]). Type IIB superstring in $Sch_5 \times S_5$ is dual to null dipole deformed $\mathcal{N} = 4$ super Yang-Mills. The $sl(2)$ sector nontrivially affected by the deformation has been studied in [23]. The one-loop spectrum of the $sl(2)$ sector can be obtained by Baxter equation. It is proposed in [23] that the Baxter equation takes the same form as in the undeformed case

$$t(u)Q(u) = (u + i/2)^J Q(u + i) + (u - i/2)^J Q(u - i), \quad (4.1)$$

and

$$t(u) = 2u^J + LMJu^{J-1} + \dots \quad (4.2)$$

The 1-loop energy is given by

$$\Delta^{(1)} = \frac{i\lambda}{8\pi^2} \partial_u \log \frac{Q(u + i/2)}{Q(u - i/2)} \Big|_{u=0}. \quad (4.3)$$

We now solve the Baxter equation in the expansion in LM . At each order in LM , the Q -function is simply a polynomial. We write the ansatz

$$Q(u) = \sum_{k=0}^{\infty} p_{k+m}(u) L^k M^k, \quad (4.4)$$

where p_{k+m} is a polynomial in u of degree $k + m$. The small LM expansion of Q can be interpreted as a function with a finite number of zeros near the Bethe roots in the undeformed limit and an infinite number of zeros of order $L^{-1}M^{-1}$. In the string picture the zeros of order $L^{-1}M^{-1}$ correspond to the cut connecting $\hat{p}_1$ and $\hat{p}_4$.

Substituting the above ansatz into the Baxter equation, we can determine $Q(u)$ up to multiplication by a function in LM . We consider the following two solutions:

$$Q_0 = 1 - LMu + L^2 M^2 \frac{Ju^2}{2(J+1)} + O(L^3 M^3), \quad (4.5)$$

$$\begin{aligned} Q_1 = (u - u_n) + \frac{LM(-2Ju^2 + 2Ju_n^2 + 4u_n^2 + 1)}{2(J+2)} \\ + L^2 M^2 \frac{1}{2(J+2)^3(J+3)} \left(J^4(u^3 + u_n u^2 - 2u_n^3) + 2J^3(2u^3 + 3u_n u^2 - 9u_n^3 - u_n) \right. \\ \left. + J^2(4u^3 + 12u_n u^2 - 60u_n^3 - 11u_n) - 22J(4u_n^3 + u_n) - 12(4u_n^3 + u_n) \right) + O(L^3 M^3), \end{aligned} \quad (4.6)$$

with

$$u_n = \frac{1}{2} \cot\left(\frac{\pi n}{J}\right). \quad (4.7)$$

In the undeformed case, Q_0 and Q_1 correspond to the ground state and one particle state respectively. Although not shown here, we have computed Q_0 and Q_1 to $L^4 M^4$ order. Then the energies of the ground state and one particle state are

$$\Delta_0^{(1)} = \frac{\lambda L^2 M^2}{8\pi^2(J+1)} - \frac{\lambda L^4 M^4}{96\pi^2(J+1)^2} + O(L^5 M^5), \quad (4.8)$$

$$\begin{aligned} \Delta_1^{(1)} = \frac{\lambda}{2\pi^2(4u_n^2 + 1)} + \frac{2\lambda LM u_n}{\pi^2(J+2)(4u_n^2 + 1)} \\ + \frac{\lambda L^2 M^2 (4(J(J(J+10) + 20) + 24)u_n^2 + (J-6)(J+2)^2)}{8\pi^2(J+2)^3(J+3)(4u_n^2 + 1)} \\ - \lambda L^3 M^3 \frac{48J^3 u_n^3 + 4(J+2)^2(J(4J+7) + 12)u_n}{3\pi^2(J+2)^5(J+3)(J+4)(4u_n^2 + 1)} \\ - \lambda L^4 M^4 \frac{c_4 u_n^4 + c_2 u_n^2 + c_0}{96\pi^2(J+2)^7(J+3)^3(J+4)(J+5)(4u_n^2 + 1)} + O(L^5 M^5), \end{aligned} \quad (4.9)$$

$$c_4 = 96J^3(J^5 + 36J^4 - 92J^3 - 1808J^2 - 4656J - 2880),$$

$$c_2 = 4(J+2)^2(J^8 + 18J^7 + 495J^6 + 2474J^5 + 4520J^4 + 12304J^3 \\ + 35568J^2 + 33696J + 17280),$$

$$c_0 = (J+2)^4(J+4)(J^5 + 6J^4 - 211J^3 - 738J^2 - 756J - 1080).$$

In large J limit, we get

$$\Delta_0^{(1)} = \frac{\lambda L^2 M^2}{8\pi^2 J} + \left(-\frac{\lambda L^2 M^2}{8\pi^2 J^2} - \frac{\lambda L^4 M^4}{96\pi^2 J^2} + O(L^5 M^5) \right) + O(J^{-3}), \quad (4.10)$$

$$\Delta_1^{(1)} = \frac{\lambda L^2 M^2}{8\pi^2 J} + \left(\frac{\lambda n^2}{2J^2} + \frac{\lambda LM n}{\pi J^2} + \frac{\lambda L^2 M^2}{8\pi^2 J^2} - \frac{\lambda L^4 M^4 (\pi^2 n^2 + 6)}{96J^2 (\pi^4 n^2)} + O(L^5 M^5) \right) + O(J^{-3}). \quad (4.11)$$

Assuming that the interaction between particles can be neglected in the large J limit as in the undeformed case, the energy of an excited state above ground state energy is

$$\Delta^{(1)} - \Delta_0^{(1)} = \sum_n N_n \left(\frac{\lambda n^2}{2J^2} + \frac{\lambda L^2 M^2}{4\pi^2 J^2} - \frac{\lambda L^4 M^4}{16(\pi^4 J^2 n^2)} + O(L^5 M^5) \right) + O(J^{-3}), \quad (4.12)$$

where N_n is excitation number for mode number n , and we assume that the total momentum is zero.

To compare the spectral curve result with the spin chain result, we expand $\Omega^{\hat{1}\hat{4}}(x_n^{\hat{1}\hat{4}})$ for small LM and obtain the energy shift of the $sl(2)$ sector

$$\delta\Delta = \sum_n N_n^{\hat{1}\hat{4}} \Omega^{\hat{1}\hat{4}}(x_n^{\hat{1}\hat{4}}) = \sum_n N_n^{\hat{1}\hat{4}} \left(\frac{\lambda n^2}{2J^2} + \frac{\lambda L^2 M^2}{4\pi^2 J^2} - \frac{\lambda L^4 M^4}{16(\pi^4 J^2 n^2)} + O(L^5 M^5) \right) + O(J^{-3}). \quad (4.13)$$

where we have used the level matching condition. The result agrees with (4.12).

The energy of the spin chain ground state to order $L^6 M^6$ has been computed in [23]

$$\begin{aligned} \Delta_0^{(1)} &= \frac{\lambda}{4\pi^2} \left(\frac{L^2 M^2}{2(J+1)} - \frac{L^4 M^4}{24(J+1)^2} + \frac{L^6 M^6 (J^2 + J + 2)}{720(J+1)^3(J+2)} + O(L^8 M^8) \right), \\ &= \frac{\lambda L^2 M^2}{8\pi^2 J} + \left(-\frac{\lambda L^2 M^2}{8\pi^2} - \frac{\lambda L^4 M^4}{96\pi^2} + \frac{\lambda L^6 M^6}{2880\pi^2} + O(L^8 M^8) \right) \frac{1}{J^2} + O(J^{-3}), \end{aligned} \quad (4.14)$$

The order J^{-1} term matches the classical quantity $\Delta_{\text{cl}} - J$, and the order J^{-2} terms perfectly match the one-loop shift $\Delta_{1\text{-loop}}$ given in (3.33).

5 Conclusion and discussion

In this paper we study the algebraic curve for superstring in $Sch_5 \times S^5$ and its application to the spectral problem. The asymptotic properties of the quasi-momenta for strings in $Sch_5 \times S^5$ are nontrivial. The point at infinity is a branch point of a cut connecting two Riemann sheets. We compute the semiclassical spectrum of the BMN string. Remarkably, we show that in the large J limit the string results match the gauge field results obtained by Baxter equation. We provide a detailed test of the Schrödinger/dipole CFT duality.

Our results encourage further exploration of integrability in Schrödinger/dipole CFT duality. It would be nice to derive the full quantum spectral curve of null dipole deformed $\mathcal{N} = 4$ super Yang-Mills theory, because the quasi-momenta are related to the quantum spectral curve in the strong coupling limit. It is also worth trying to obtain higher-order corrections on the field theory side to get a precise match with string theory predictions. One can also study the three dimensional counterpart of Sch_5 , the warped AdS_3 . We hope that integrability would be a powerful tool for the spectral problem of warped AdS_3 /dipole CFT duality [49].

Acknowledgments

The author would like to thank Nikolay Bobev, Hui-Huang Chen, Fedor Levkovich-Maslyuk, Zhibin Li, Wei Song, Jun-Bao Wu and Konstantin Zarembo for very helpful discussions and comments. This work is supported in part by the National Natural Science Foundation of China (Grant No. 11575202).

Open Access. This article is distributed under the terms of the Creative Commons Attribution License ([CC-BY 4.0](https://creativecommons.org/licenses/by/4.0/)), which permits any use, distribution and reproduction in any medium, provided the original author(s) and source are credited.

References

- [1] J.M. Maldacena, *The Large- N limit of superconformal field theories and supergravity*, *Int. J. Theor. Phys.* **38** (1999) 1113 [*Adv. Theor. Math. Phys.* **2** (1998) 231] [[hep-th/9711200](#)] [[INSPIRE](#)].
- [2] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *Gauge theory correlators from noncritical string theory*, *Phys. Lett. B* **428** (1998) 105 [[hep-th/9802109](#)] [[INSPIRE](#)].
- [3] E. Witten, *Anti-de Sitter space and holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253 [[hep-th/9802150](#)] [[INSPIRE](#)].
- [4] N. Beisert et al., *Review of AdS/CFT Integrability: An Overview*, *Lett. Math. Phys.* **99** (2012) 3 [[arXiv:1012.3982](#)] [[INSPIRE](#)].
- [5] J.A. Minahan and K. Zarembo, *The Bethe ansatz for $N = 4$ super Yang-Mills*, *JHEP* **03** (2003) 013 [[hep-th/0212208](#)] [[INSPIRE](#)].
- [6] I. Bena, J. Polchinski and R. Roiban, *Hidden symmetries of the $AdS_5 \times S^5$ superstring*, *Phys. Rev. D* **69** (2004) 046002 [[hep-th/0305116](#)] [[INSPIRE](#)].
- [7] V.A. Kazakov, A. Marshakov, J.A. Minahan and K. Zarembo, *Classical/quantum integrability in AdS/CFT*, *JHEP* **05** (2004) 024 [[hep-th/0402207](#)] [[INSPIRE](#)].
- [8] V.A. Kazakov and K. Zarembo, *Classical/quantum integrability in non-compact sector of AdS/CFT*, *JHEP* **10** (2004) 060 [[hep-th/0410105](#)] [[INSPIRE](#)].
- [9] N. Beisert, V.A. Kazakov and K. Sakai, *Algebraic curve for the $SO(6)$ sector of AdS/CFT*, *Commun. Math. Phys.* **263** (2006) 611 [[hep-th/0410253](#)] [[INSPIRE](#)].
- [10] S. Schäfer-Nameki, *The Algebraic curve of 1-loop planar $N = 4$ SYM*, *Nucl. Phys. B* **714** (2005) 3 [[hep-th/0412254](#)] [[INSPIRE](#)].
- [11] N. Beisert, V.A. Kazakov, K. Sakai and K. Zarembo, *The Algebraic curve of classical superstrings on $AdS_5 \times S^5$* , *Commun. Math. Phys.* **263** (2006) 659 [[hep-th/0502226](#)] [[INSPIRE](#)].
- [12] C.P. Herzog, M. Rangamani and S.F. Ross, *Heating up Galilean holography*, *JHEP* **11** (2008) 080 [[arXiv:0807.1099](#)] [[INSPIRE](#)].
- [13] J. Maldacena, D. Martelli and Y. Tachikawa, *Comments on string theory backgrounds with non-relativistic conformal symmetry*, *JHEP* **10** (2008) 072 [[arXiv:0807.1100](#)] [[INSPIRE](#)].
- [14] A. Adams, K. Balasubramanian and J. McGreevy, *Hot Spacetimes for Cold Atoms*, *JHEP* **11** (2008) 059 [[arXiv:0807.1111](#)] [[INSPIRE](#)].
- [15] O. Lunin and J.M. Maldacena, *Deforming field theories with $U(1) \times U(1)$ global symmetry and their gravity duals*, *JHEP* **05** (2005) 033 [[hep-th/0502086](#)] [[INSPIRE](#)].
- [16] D. Orlando, S. Reffert and L.I. Uruchurtu, *Classical Integrability of the Squashed Three-sphere, Warped AdS3 and Schroedinger Spacetime via T-duality*, *J. Phys. A* **44** (2011) 115401 [[arXiv:1011.1771](#)] [[INSPIRE](#)].
- [17] I. Kawaguchi and K. Yoshida, *Classical integrability of Schrödinger σ -models and q -deformed Poincaré symmetry*, *JHEP* **11** (2011) 094 [[arXiv:1109.0872](#)] [[INSPIRE](#)].
- [18] I. Kawaguchi, T. Matsumoto and K. Yoshida, *Schroedinger σ -models and Jordanian twists*, *JHEP* **08** (2013) 013 [[arXiv:1305.6556](#)] [[INSPIRE](#)].

- [19] M. Alishahiha and O.J. Ganor, *Twisted backgrounds, PP waves and nonlocal field theories*, *JHEP* **03** (2003) 006 [[hep-th/0301080](#)] [[INSPIRE](#)].
- [20] A. Bergman and O.J. Ganor, *Dipoles, twists and noncommutative gauge theory*, *JHEP* **10** (2000) 018 [[hep-th/0008030](#)] [[INSPIRE](#)].
- [21] K. Dasgupta, O.J. Ganor and G. Rajesh, *Vector deformations of $N = 4$ super Yang-Mills theory, pinned branes and arched strings*, *JHEP* **04** (2001) 034 [[hep-th/0010072](#)] [[INSPIRE](#)].
- [22] A. Bergman, K. Dasgupta, O.J. Ganor, J.L. Karczmarek and G. Rajesh, *Nonlocal field theories and their gravity duals*, *Phys. Rev. D* **65** (2002) 066005 [[hep-th/0103090](#)] [[INSPIRE](#)].
- [23] M. Guica, F. Levkovich-Maslyuk and K. Zarembo, *Integrability in dipole-deformed $\mathcal{N} = 4$ super Yang-Mills*, *J. Phys. A* **50** (2017) 394001 [[arXiv:1706.07957](#)] [[INSPIRE](#)].
- [24] V.G. Drinfeld, *Quasi Hopf algebras*, *Alg. Anal.* **1N6** (1989) 114.
- [25] N. Reshetikhin, *Multiparameter quantum groups and twisted quasitriangular Hopf algebras*, *Lett. Math. Phys.* **20** (1990) 331 [[INSPIRE](#)].
- [26] D.E. Berenstein, J.M. Maldacena and H.S. Nastase, *Strings in flat space and pp waves from $N = 4$ super Yang-Mills*, *JHEP* **04** (2002) 013 [[hep-th/0202021](#)] [[INSPIRE](#)].
- [27] N. Bobev, A. Kundu and K. Pilch, *Supersymmetric IIB Solutions with Schrödinger Symmetry*, *JHEP* **07** (2009) 107 [[arXiv:0905.0673](#)] [[INSPIRE](#)].
- [28] C. Klimčík, *On integrability of the Yang-Baxter σ -model*, *J. Math. Phys.* **50** (2009) 043508 [[arXiv:0802.3518](#)] [[INSPIRE](#)].
- [29] C. Klimčík, *Yang-Baxter σ -models and dS/AdS T duality*, *JHEP* **12** (2002) 051 [[hep-th/0210095](#)] [[INSPIRE](#)].
- [30] F. Delduc, M. Magro and B. Vicedo, *An integrable deformation of the $AdS_5 \times S^5$ superstring action*, *Phys. Rev. Lett.* **112** (2014) 051601 [[arXiv:1309.5850](#)] [[INSPIRE](#)].
- [31] I. Kawaguchi, T. Matsumoto and K. Yoshida, *Jordanian deformations of the $AdS_5 \times S^5$ superstring*, *JHEP* **04** (2014) 153 [[arXiv:1401.4855](#)] [[INSPIRE](#)].
- [32] T. Matsumoto and K. Yoshida, *Yang-Baxter deformations and string dualities*, *JHEP* **03** (2015) 137 [[arXiv:1412.3658](#)] [[INSPIRE](#)].
- [33] T. Matsumoto and K. Yoshida, *Integrability of classical strings dual for noncommutative gauge theories*, *JHEP* **06** (2014) 163 [[arXiv:1404.3657](#)] [[INSPIRE](#)].
- [34] T. Matsumoto and K. Yoshida, *Schrödinger geometries arising from Yang-Baxter deformations*, *JHEP* **04** (2015) 180 [[arXiv:1502.00740](#)] [[INSPIRE](#)].
- [35] S.J. van Tongeren, *On classical Yang-Baxter based deformations of the $AdS_5 \times S^5$ superstring*, *JHEP* **06** (2015) 048 [[arXiv:1504.05516](#)] [[INSPIRE](#)].
- [36] S.J. van Tongeren, *Yang-Baxter deformations, AdS/CFT and twist-noncommutative gauge theory*, *Nucl. Phys. B* **904** (2016) 148 [[arXiv:1506.01023](#)] [[INSPIRE](#)].
- [37] D. Osten and S.J. van Tongeren, *Abelian Yang-Baxter deformations and TsT transformations*, *Nucl. Phys. B* **915** (2017) 184 [[arXiv:1608.08504](#)] [[INSPIRE](#)].
- [38] B. Hoare and A.A. Tseytlin, *Homogeneous Yang-Baxter deformations as non-abelian duals of the AdS_5 σ -model*, *J. Phys. A* **49** (2016) 494001 [[arXiv:1609.02550](#)] [[INSPIRE](#)].
- [39] S.J. van Tongeren, *Almost abelian twists and AdS/CFT*, *Phys. Lett. B* **765** (2017) 344 [[arXiv:1610.05677](#)] [[INSPIRE](#)].

- [40] B. Hoare and D.C. Thompson, *Marginal and non-commutative deformations via non-abelian T-duality*, *JHEP* **02** (2017) 059 [[arXiv:1611.08020](#)] [[INSPIRE](#)].
- [41] T. Araujo, I. Bakhmatov, E.Ó. Colgáin, J. Sakamoto, M.M. Sheikh-Jabbari and K. Yoshida, *Yang-Baxter σ -models, conformal twists and noncommutative Yang-Mills theory*, *Phys. Rev. D* **95** (2017) 105006 [[arXiv:1702.02861](#)] [[INSPIRE](#)].
- [42] T. Araujo, I. Bakhmatov, E.Ó. Colgáin, J.-i. Sakamoto, M.M. Sheikh-Jabbari and K. Yoshida, *Conformal Twists, Yang-Baxter σ -models & Holographic Noncommutativity*, [arXiv:1705.02063](#) [[INSPIRE](#)].
- [43] R.R. Metsaev and A.A. Tseytlin, *Type IIB superstring action in $AdS_5 \times S^5$ background*, *Nucl. Phys. B* **533** (1998) 109 [[hep-th/9805028](#)] [[INSPIRE](#)].
- [44] N. Gromov and P. Vieira, *The $AdS_5 \times S^5$ superstring quantum spectrum from the algebraic curve*, *Nucl. Phys. B* **789** (2008) 175 [[hep-th/0703191](#)] [[INSPIRE](#)].
- [45] N. Gromov, S. Schäfer-Nameki and P. Vieira, *Efficient precision quantization in AdS/CFT* , *JHEP* **12** (2008) 013 [[arXiv:0807.4752](#)] [[INSPIRE](#)].
- [46] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *A Semiclassical limit of the gauge/string correspondence*, *Nucl. Phys. B* **636** (2002) 99 [[hep-th/0204051](#)] [[INSPIRE](#)].
- [47] S. Frolov and A.A. Tseytlin, *Semiclassical quantization of rotating superstring in $AdS_5 \times S^5$* , *JHEP* **06** (2002) 007 [[hep-th/0204226](#)] [[INSPIRE](#)].
- [48] S. Frolov and A.A. Tseytlin, *Multispin string solutions in $AdS_5 \times S^5$* , *Nucl. Phys. B* **668** (2003) 77 [[hep-th/0304255](#)] [[INSPIRE](#)].
- [49] W. Song and A. Strominger, *Warped AdS_3 /Dipole-CFT Duality*, *JHEP* **05** (2012) 120 [[arXiv:1109.0544](#)] [[INSPIRE](#)].